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CERTIFICATE

This is to certify that the work reported in this thesis entitled "ON CHAOS AND FRACTALS IN GENERAL TOPOLOGICAL SPACES" is based on the bona fide work done by Vinod Kumar.P.B under my supervision at the Dept. of Mathematics, Cochin University of Science and Technology, Cochin – 682022, Kerala, India and has not been included in any other thesis submitted previously for the award of any degree or diploma.

25/9/01

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CONTENTS

CHAPTER – 0

INTRODUCTION

Chaos theory originated with the work of Edward Lorenz, a metereologist in 1960 when he was working on the problem of weather prediction with a set of twelve equations; which express the relationships between temperature and pressure, pressure and wind speed; to model the weather in his computer set up at Massuchusetts institute of technology. A Scientist considers himself lucky if he can get measurements with accuracy to three decimal places. The fourth and fifth may be difficult to measure using reasonable methods can't have a huge effect on the outcome of the experiment. But Lorenz proved that this idea is wrong. ie. approximation may lead to disorder. The consequence of Lorenz experiment is known as butterfly effect. Flapping of a butterflys wings today may produce after some period a storm. ie. a small change in the initial conditions may cause a big change in the output. This phenomenon is also known as sensitive dependence on initial conditions. The phenomenon which changes order into disorder is known as Chaos. In 1964, A.N.Sharkovsky[67] defined a partial order on the set of positive integers and proved that for a continuous map on a closed interval there is a point of period m if there is a point of period n where m preceeds n in that order[Result 1:1:14]. Existence of such periodic points is a necessary condition for a system to be Chaotic. But a precise Mathematical definition of Chaos is given by Tien-Yien Li and James. A Yorke [71] in 1975. They defined Chaotic functions [Def:1:1:5] and proved that if f is a continuous map on a closed interval such that f has a point of period 3 then f is Chaotic[Result:1:1:13]. Since there are other definitions of Chaos , let us call this definition Li Yorke (LY) Chaos . In 1977, Melvyn.B.Nathanson[56, 57] proved that the condition that there is a point of period 3 is not necessary for Li-Yorke Chaos. He proved that if f has a point of period which is divisible by 3, 5 or 7 then f is Li-Yorke Chaotic. In 1979, Frederick.J.Fuglister[40], proved that if f has a point of period p where p is not a power of 2, then f is Li-Yorke Chaotic. He also showed an example of a non Chaotic function with points of period p, for each p which is a power of 2. In 1980, Joseph Auslander and James.A.Yorke[48] defined Chaotic functions in Metric Spaces [Definition:1:1:9]. In that definition existence of a point with dense orbit is needed. This paper can be considered as first generalization of Chaos into metric spaces. Defining topological transitivity of a map, they proved that for a Compact Metric Space topological transitivity is equivalent to existence of dense orbit [Result:1:1:11]. In [31] J.Doyne Farmer, Edward ott and James.A.Yorke have illustrated some applications of Li-Yorke Chaos. Marcy Barge and Joe Martin[54.55], stated some conditions for dense periodic orbits, dense set of periodic points on a closed interval. Topological entropy defined by R.T.Adler, A.G.Konhecim and M.H.Mc Andrew[2] played an important role in the complexity of the space later[Definition:1:1:10]. J.Smital[69] gives a relationship between topological entropy and Li-Yorke Chaos. In 1986, K.Jankova and J.Smital[46] stated some equivalent conditions for Li-Yorke Chaos. In 1989, R.L.Devaney[30] defined Chaotic functions in general metric spaces, which became popular[Definition:1:1:7]. There are three conditions namely transitivity, dense set of periodic points and sensitive dependence for a function to be Chaotic. But in 1994, J.Banks, J.Brooks, G.Cairns, G.Davis and P.Stacey[14] proved a surprising result that the first two conditions of Devaney's definition imply third one. Note that third one is well known as butterfly effect. In Devaney's definition the first two conditions are topological and the third one depends on the metric. By the Result of J.Banks *etal* it has been proved that property of being Chaotic is purely topological. It is unfortunate that the property from which a theory originated is found to be unnecessary later. In 1994. Michel Vellekoop and Raoul Berglund[59] proved that if we restrict Devaney's definition in to intervals on R, a function is chaotic if it is transitive .In 1995, G.L. Forti, L. Paganmi and J. Smital [39] extended Li- Yorke chaos in to R^2 and obtained some equivalent condition .Chaotic nature of triangular maps has also been studied in that paper. To get a visual idea of chaotic nature see [27]. Jan. M. Arts and Fons. G.M.Daalderoop, in 1998[1] studied chaotic homeomorphisms on manifolds. LI. Alseda, S. Kolyada, J. Llibre and LV. Snoha in 1999 [3] obtained some connection between transitivity, density of the set of periodic points and topological entropy .In 1999 we have extended, Devaney's definition of chaos in to topological spaces.

Theory of fractals has its origin with the work of Benoit Mandelbrot in 1977[52]. Roughly saying, fractals are "irregular" objects. By 'regular' we means objects which have nice geometric properties. According to a Mandelbrot, a set is called a fractral if its Hausdorff dimension is strictly greater than topological (small inductive) dimension. But the calculation of Hausdorff dimenstion is a tedious task. So a question arises- How can we characterise fractals? In 1981, John. E. Hutcinsion [47] introduced self similar sets [Def: 1:2:2]. If every part of a set looks like the whole then it is called a self similar set. Some fractals are self similar; but not all . Some extra ordinary sets like Cantor set, Von- Koch curve, Sierpenski gasket which had their birth in early 20th century were found to be self similar fractals later. In the same way some Julia sets, due to the work of P.Fatou and G. Julia in 1920's are self similar. Self similar sets, holomorphic dynamics and dimension properties are thrust area in the study of fractals. In 1982, F.M. Dekking [28] introduced recurrent sets as a generalization of self similar sets. Christoph Bandt [6-13] published a series of papers on self similar sets (1, 2, 3, 4, 5, 6, 7, 8) from 1989 to 1992. In 1987, W.J. Gilbert introduced partial self similar sets as a generalization of self similar sets in some other direction. A set is partial self similar if it consists of copies of different parts of it. In 1995, Kenneth Falconer [34] introduced subself similar sets as a generalization of self similar sets [Def: 1: 2: 3]. A set is subself similar if it is contained in a union of similar copies of it self. In 1995, R. Daniel Mauldin [24] extended theory of finite iterated function systems into infinite iterated function systems by introducing conformal iterated function systems. James keesling[44] in 1999, studied the properties of boundaries of self similar sets. We can measure the irregularity of fractals by assigning it to a real number. Usually, Hausdorff dimension is used for this purpose. Similarity dimension measures the irregularity. Hausdorff dimension was introduced by Felix Hausdorff [41] in 1919. But its importance as a measure of irregularity was realized only after the introduction of fractals. In 1982, Claude tricot[74] introduced packing dimension which may also be a non-integer. In 1984, Curtis Mcmullen [22] calculated the Hausdorff dimension of general Sierpinski carpets. Tim Bedford[72] in 1986 calculated Hausdorff dimension of some recurrent sets. The work of James keesling [45] on Hausdorff dimension is from topological point of view. R.D. Mauldin and

S.C. Williams [25] in 1988, published a paper on Hausdorff dimension of some directed graphs, in 1992, G.A. Edgar [32] calculated Hausdorff dimension of some self affine sets. A Deliu, J.S Geronimo, R. Shankwiler and D, Herting calculated [4], Box counting dimension of some recurrent sets. In 1996, R.D Mauldin and M. Urbanski [26] calculated various dimensions of infinite iterated function systems. Dimension of some fractal families calculated by B. Solomyak in 1998 [16]. Considering the theory of complex dynamics, the study of this subject began during the first world war. Both P Fatou and G. Julia independently published a number of compete rendeus notes and then both wrote large memoires[49]. Most of the Julia sets are 'irregular', so a question arises:- which Julia sets are fractals?. Hence the interest in Complex dynamics rebegan in 1980's, after the introduction of fractals. The survey article by Paul Blanchard [64] covers almost all primary results in this area. In 1986, M. Yu Lyubich [51] published a papers in which he analyses different topological properties of Julia sets of rational transforms. Curtis T Mcmullen [22] in 1987, calculated area and Hausdorff dimension of Julia sets of some entire functions. Expository article by John Milnor [60] in one dimensional complex dynamics is very useful for a beginner. In 1987, I.N. Baker and A. Eremenko [5] classified those polynomials whose Julia sets are same. in 1990, Tan Lei [70] proved that Mandelbrot set is locally self similar at certain points and corresponding Julia sets are self similar. In 1990, John Millor [61] has studied some geometrical properties of the mandelbort set. In 1992, Michael frame and John Robertson [58] generalized the concept of Mandelbort set into higher degree polynomial ie. degree.2. The joint work of P. Fatou and G. Julia includes the study of Julia set and its complement called Fatou set. In 1994. G.M. Levin [50] studied the properties of complement of the Mandelbrot set. The progress in complex dynamics until 1994 is nicely written in the survey article by Curtis T Mc Mullen [21]. The dynamics of polynomial like maps has been studied by Nutria Fagella [63] in 1995. Hiroki Sumi [42] in 1997, has analyzed properties of Julia sets of a well defined collection of rational transforms. In his classical work Curtis T McMullen [20] in 1998 has computed Hausdorff dimension of a large class of Julia sets which include the Julia sets of $f(z) = z^2+c$ for different values of c.M

Shishikura [68] in 1998 proved that the boundary of the Mandelbrot sets has Hausdorff dimension 2; same as topological

dimension of the complex plane. Hiroki sumi [43] in 1998, studied the dimension properties of Julia sets of a collection of rational transforms. In 1999, C. McMullan proved that Mandelbrot set is universal.

There are several books which gives basic ideas in chaos theory and fractals. Books by Gullick [29] includes an account of chaos in one- dimension. The first book by B. Mandelbort [53] in the theory of fractals includes different fractals shapes in the nature. Books by Falconer [35], Edgar [33] , Bransley [17] and Tricot [74] consist of basic results in the theory of fractals and dimension.Beardon A.F. [15] and Pietgen & Ritcher [65] gives a nice account of the results in complex dynamics.

We are not claming that the above references are complete. The literature is so vast that we can't quote all the references. But the above quoted references are those which we used in our work and to prepare the thesis.

In this thesis, in chapter 1, we quote some definitions and results in the chaos theory (section 1:1) and theory of fractals (section 1:2) which are already in the literature. In chapter 2, we introduce chaos in topological spaces, we study some properties of chaos spaces which also include hyper spaces. Topological entropy is a measure to determine the complexity of the space. We study different properties of topological entropy in chaos spaces. By defining a measure on chaos spaces, we compare different chaos spaces. In chapter 3 we study some properties of self similar sets and partial self similar sets. We can associate a directed graph to each partial selfsimilar set. Dimension properties of partial self similar sets are studied using this graph. We introduce superself similar sets as a generalization of self similar sets. We also prove that chaotic self similar self are dense in the hyper space. In chapter 4, we define Julia sets and Mandelbrot set on general topological spaces. We define generalized Julia sets and Mandelbrot sets on chaos spaces. In chapter 5, we study some relationships between different kinds of dimension and fractals. By defining regular sets through packing dimension in the same way as regular sets defined by K. Falconer through Hausdorff dimension, we examine different properties of regular sets.

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CHAPTER – 1
PRELIMINARIES

In this chapter we quote some definitions and results in chaos theory and Fractal geometry which are already in the literature. Through out this chapter X denote a topological space, (X, d) denote a Metric space and f is a continuous function from X to X. Throghout the thesis K (X) denote collection of all non empty compact subsets of X.

1.1.CHAOS

Definition.1.1.1 [30]: Orbit of a point x inX under the mapping f is

$O_f(x) = \{ x, f(x), f^2(x), \ldots\ldots, \ldots\}$.

Definition.1.1.2 [30]: x in X is called a periodic point of f if $f^n(x) = x$, for some n $\in Z_+$. Smallest of these n is called period of x .

Definition.1.1.3 [30]: Let U,V be open sets in X. We say that f is transitive on X if there exist $n \in Z_+$ such that $f^n(U) \cap V \neq \phi$.

Definition.1.1.4 [30]: $x \in X$ is a non wandering point if given any open set U containing x there exist $n \in Z_+$ such that $f^n(U) \cap U \neq \phi$. A point which is not non wandering is wandering. Let $\Omega_f(X) = \{ x \in X \mid x \text{ is non wandering } \}$.

Definition.1.1.5 [71]: Let I = [0,1].Then f is said to be chaotic on I if there is an uncountable set $S \subset I$ which satisfies the following conditions .

For every $p, q \in S$ with $p \neq q$,

(i) $\lim_{n \to \infty} Sup \left| f^n(p) - f^n(q) \right| > 0$ and

$\lim_{n \to \infty} Inf \left| f^n(p) - f^n(q) \right| = 0$

(ii) For every $p \in S$ and periodic point $q \in I$,

$\lim_{n \to \infty} Sup \left| f^n(p) - f^n(q) \right| > 0$.

Definition.1.1.6 [30]: f is sensitive if for each $\delta > 0$ there exist (a) $\varepsilon > 0$ (b) $y \in X$ and (c) $n \in Z_+$ such that $d(x,y) < \delta$ and $d(f^n(x), f^n(y)) > \varepsilon$.

Definition.1.1.7 [30]: f is chaotic on (X,d) if (i)Periodic points of f are dense in X (ii)Orbit of x is dense in X for some x in X and (iii)f is sensitive.

Definition.1.1.8 [48] : f is stable at $x \in X$ if for any $\varepsilon > 0$, there is a $\delta > 0$ such that $d(x,y) < \delta$ implies $d(f^n(x), f^n(y)) < \varepsilon$, for n = 0 , 1 , 2 , ... Otherwise f is unstable at x.

Definition.1.1.9 [48] : f is chaotic on X if (i) f is unstable at every point of X and (ii) Orbit of x is dense in X for some x in X .

Definition.1.1.10 [2]: Let X be compact. To each open cover U of X associate a sequence $\{a_n\}$ of integers as , a_n = smallest cardinality of a subcover of $U \vee f^{-1}(U) \vee \ldots\ldots \vee f^{(n-1)}(U)$. Define, $h(f,U) = \lim_{n\to\infty} \frac{\log a_n}{n}$ and $top(f) = \operatorname{Sup}\{ h(f,U) \mid U$ is an open cover$\}$.Then top(f) is called topological entropy of f .

Result.1.1.11 [30] : If X is second countable and Baire without isolated points then $\overline{O_f(x)} = X$, for some $x \in X \Leftrightarrow$ f is transitive on X.

Result.1.1.12 [30] : f is transitive and periodic points are dense $\Rightarrow$ f is sensitive .

Result.1.1.13 [71] : Let $f : [0,1] \to [0,1]$. Then if f has a point of period three then f is chaotic [LY sense].

Result.1.1.14 [67] : Let $f : R \to R$. Arrange positiveintegers in the following order $3<5<7<\ldots\ldots 2.3<2.5<\ldots\ldots\ldots<2^2.3<2^2.5<\ldots\ldots\ldots\ldots<2^\infty<\ldots 2^3<2^2<2<1$. Then if f has a point of period n then f has a point of period m for every m such that n<m.

Result.1.1.15 [2] : If $\{f^n\}$ is equicontinuous on (X, d) then top(f) = 0 .

Result.1.1.16 [2] : $top(f^n) = n.\, top(f)$.

Result.1.1.17 [3] : If $f : [0,1] \to [0,1]$ is transitive then $top(f) \geq \frac{\log 2}{2}$.

Result.1.1.18 [59] : Suppose that I is a , not necessarily finite , interval and $f : I \to I$ is a continuous map . If $J \subseteq I$ is an interval which contains no periodic points of f and z, $f^m(z)$ and $f^n(z) \in J$ with $0 < m < n$, then either $z < f^m(z) < f^n(z)$ or $z > f^m(z) > f^n(z)$.

1.2.FRACTALS

Definition.1.2.1[47]: f is a similarity if there exist some c >0 such that $d(f(x),f(y)) = c.d(x,y)$. C is called similarity ratio .

Definition.1.2.2 [47]: A set $F \subseteq X$ is called self similar if there are similarities $f_1,f_2,...f_n$ such that $F = \bigcup_{i=1}^{n} f_i(F)$.

Definition.1.2.3 [34]: A set $F \subseteq X$ is called sub self similar if there are similarities $f_1,f_2,...f_n$ such that $F \subseteq \bigcup_{i=1}^{n} f_i(F)$.

Definition.1.2.4 [35]:Let $F \subseteq X$. Hausdorff dimension, $\dim_H(F)$, is defined as $\dim_H(F) = \text{Inf}\{ s \mid H^s(F) = 0 \} = \text{Sup}\{ s \mid H^s(F) = \infty \}$,where $H^s(F)$ is the s-dimensional Hausdorff measure .

Definition.1.2.5 [53] : Let $F \subseteq X$. F is called a Fractal if its Hausdorff dimension is strictly greater than its topological dimension .

Definition.1.2.6. [15] : Let $f: \bar{C} \to \bar{C}$ be a rational function. Then Julia set of f , J (f), is defined as $J(f) = \{ z \in \bar{C} \mid \{f^n\} \text{ is not normal at } z \}$.

Definition.1.2.7 [15] : Mandelbrot set M is defined as ,

$M = \{ c \in \bar{C} \mid J(f_c) \text{ is connected}\}$. Here $f_c : \bar{C} \to \bar{C}$ is defined as $f_c(z) = z^2+c$.

Result.1.2.8 [47] : If (X,d) is a complete metric space and $\{f_i\}_{i=1}^{n}$ are contractions then there is a unique F such that $F = \bigcup_{i=1}^{n} f_i(F)$.

Result.1.2.9 [35] : Self similar sets are dense in R^n .

Result.1.2.10 [34]: Let (X, d) be a complete metric space and $\{f_i\}_{i=1}^{n}$ satisfies open set condition , where f_i's are contractions . Let F be the self similar set corresponding to $\{f_n\}$. Then similarity dimension of F = Hausdorff dimension of F .

Result.1.2.11 [18] : Let f be a meromorphic function on a domain in C, then { f^n } is normal iff { f^n } is equicontinuous .

Result.1.2.12 [15] : Let f : $\bar{C} \to \bar{C}$ be a rational function. Then Julia set of f , J (f),is non empty , perfect and compact .

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CHAPTER – 2
CHAOS ON TOPOLOGICAL SPACES

There are five sections in this chapter. In the first section we define chaos spaces. Our definition of chaotic functions on topological spaces can be considered as a generalization of Devaney's definition. There are many definitions of chaotic functions in the literature. Relations between them are given in the second section. In the third section we define sharkovsky space. Topological entropy is a measure of complexity of a function. In the fourth section we give characterization of topological entropy on trees. In the fifth section we define a measure on chaos spaces.

2.1. Chaos spaces[1]

As we mentioned in the introduction the first generalization of chaotic functions into metric spaces is given by Auslander and Yorke [48] (Def.1.1.9). But the most popular is the one given by Devaney [30](Def.1.1.7). We will see relations between these definitions in the next section. For Auslander and Yorke chaos means unstability and dense orbit whereas for Devaney it is transitivity, dense periodic points and sensitivity. Unstability (Def.1.1.8) is a local property where as sensitivity (Def.1.1.6) is a uniform property. We define sensitivite functions on topological spaces which can be considered as a generalization of both these. Throughout this section $F\varepsilon\ K(X)$.

Definition.2.1.1 :

Let $(X, \Im)$ be a topological space and $f:X\rightarrow X$ be a continuos map. We say that f is sensitive at $x\in X$ if given any open set U containing x these exist (i) $y \in U$ (ii) $n\in Z_+$ and (iii) an open set V such that $f^n(x) \in V$ and $f^n(y) \notin \overline{V}$.

We say that f is sensitive on F if f/F is sensitive at every point of F.

Examples.2.1.2 :

(1) In Indiscrete spaces and discrete spaces there are no sensitive functions because in Indiscrete spaces we can't find an open set V satisfying 2.1.1 (iii) and for discrete spaces 2.1.1. (i) can't hold when U={x}

(2) Constant functions are not sensitive because of 2.1.1. (iii)

(3) Let I = [0,1] and f: I→ I be defined by

$$f(x) = \begin{cases} 2x, 0 \le x \le 1/2 \\ 2(1-x), 1/2 \le x \le 1 \end{cases}$$

then f is sensitive on I. This map f is the usual tent map on I

From the definition it is clear that if there is a sensitive function at x then x cannot be isolated.

Notation.2.1.3 :

Let $F \subseteq X$ and S(F) = {f|f is sensitive on F}

Result.2.1.4 :

If $S(F) \neq \phi$ then F has no isolated points

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[1] Some results in section 2.1 is published in Far East Journal of Mathematical Sciences Sp. Volume1 (1999) and to be published in Int. J. of Bifurcation and Chaos .

Cor.2.1.5 :

If $S(F) \neq \phi$ and F is closed in X then F is perfect.

Proof : Any closed set without isolated points is perfect.

Result follows from Result. 2.1.4

If the space is T_3 then the converse is also true.

Result.2.1.6 :

Let X be T_3 and F be perfect. If $f:F \to F$ is continuos such that f^n is non constant for some $n \geq 1$ on $F \cap G$, for every open set G in X then $f \in S(F)$.

Proof : Since F is perfect if U is an open set containing x, then $U \cap F \neq \phi$. So there exists

$y \in U \cap F$, there exists a $y \in U \cap F$ with $f^n(x) \neq f^n(y)$ for some n (otherwise f^n becomes constant on $U \cap F$). Since X is T_3 there exists an open set V such that $f^n(x) \in V$ and $f^n(y) \notin \bar{V}$.

$\therefore f \in S(F)$.

Now we have following corolories whose proofs are immediate from Result.2.1.6

Cor.2.1.7 :

Let X be T_3 and $f:X \to X$ is such that f is not constant on every open set. Then $f \in S(F)$ <=> F is Perfect in X.

Cor.2.1.8 :

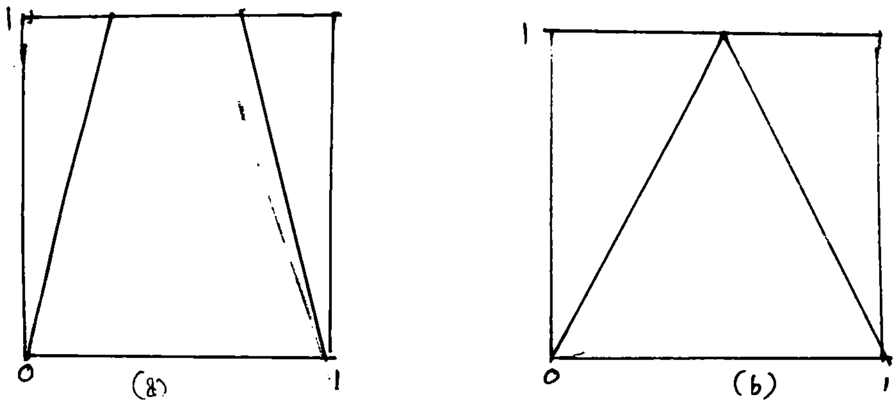
Let (X,d) be a metric space and F be perfect. If $f:F \to F$ is such that f^n is non constant, for some n, on every open set in F then $f \in S(F)$.

Cor.2.1.9 :

On a compact metric space without isolated points every continuous function which is non constant on every open set is always sensitive.

If we consider I = [0,1] then every strictly increasing or decreasing functions are sensitive. Also any piecewise linear (non constant) function is sensitive. Fig.2.1.(a) and Fig.2.1. (b) represents non sensitive and sensitive functions respectively on I.

Fig. 2.1



Now we define chaotic functions.

Definition.2.1.10 :

Let $(X, \mathfrak{I})$ be a topological space and $F \in K(X)$. Let $f: F \to F$ be continuous. Then f is chaotic on F if

(i) $\overline{O_f(x)} = F$, for some $x \in F$ (ii) Periodic points of f are dense in F and

(iii) $f \in S(F)$.

Notation.2.1.11 :

(i) $C(F) = \{ f:F \to F \mid f \text{ is chaotic on } F\}$

(ii) $CH(X) = \{F \in K(X) \mid C(F) \neq \phi \}$

Definition.2.1.12 :

X is called a 'chaos space' if $CH(X) \neq \phi$.

If X is a chaos space then elements of CH(X) are called chaotic sets.

Note that if (X, d) is a compact metric space then by Result 1.1.11 ; $\overline{O_f(x)} = X$ <=> f is transitive on X and by Cor.2.1.9, f is sensitive on X. So if (X, d) is a Compact metric space then f: X→X is chaotic if (i) and (ii) of Def.2.1.10 are satisfied which is equivalent to Devaney's definition by [14]. So on compact metric spaces our definition coincides with Devaney's definition. Let us denote our definition of chaos ie. Def.2.1.10 by 'TC'.

So we have,

<u>Result 2.1.13</u>:

On Compact metric spaces, DC <=> TC. In [14] it has been proved that there are redundancies in Devaney's definition. But in our definition in general topological spaces there are no such redundancies. We will show this in the next set of examples.

<u>Examples 2.1.14</u>:

Consider the definition 2.1.10.

(1) (i) and (ii) ≠> (iii)

Let X = {a, b} , $\Im$ = {X, {a}, ϕ}. Let f be the identify function on X. Then, $\overline{O_f(a)} = X$ and $\overline{P_f(X)} = X$. But f is not sensitive on X, since {a} is open. Note that Sierpinski space is not metrizable. In general in any topological space if {x} is open then f is not sensitive at x.

(2) Let x = {a, b, c, d}. J = {x, ϕ, {a, b}, {a, c}, {a, b, c}, {b, d}, {a, b, d}, {b}, {a}} and f:X→X be defined by f(a) = b, f(b) = a, f (c) = d f(d)= c. Then $\overline{O_f(a)} = \overline{O_f(b)} = \overline{\{a, b\}}$ = X and $\overline{P_f(X)}$ = X. But f is not sensitive. Let x = c and u = {c, d} then y = d and,

If n = 1; we have V = {b, d}. So that $f^n(x) = f^1(x) = d \in V$ and $f^n(y) = f^1(d) = c \notin \bar{V}$. (for V = {b, d} is clopen in X) if n=2, we have, V = {a, c}. So that $f^n(x) = f^2(c) = c \in V$ and $f^n(y) = f^2(d) = d \notin \bar{V}$. So if U = {c, d} then there is no problem. But if U ={a, c} then y = a and if n = 1, $f^n(x) = f^1(c) = d$ and $f^n(y) = f^1(a) = b$. But every open set which contains d also contains b. So we can't have an open set V such that $d \in V$, $b \notin \bar{V}$. If

n = 2, $f^n(x) = f^2(c) = c$ and $f^n(y) = f^2(a) = a$. Again every open set which contains c also contains a. So we can't have an open set V such that $f^n(x) \in V$ and $f^n(y) \notin \bar{V}$. n = 3, 5, 7, ... is same as the case with n =1 and n = 4, 6, 8, ... is same as the case with n = 2. So if $n \in Z_+$ there exist no open set V such that $f^n(x) \in V$ and $f^n(y) \notin \bar{V}$. ∴ f is not sensitive on X

(3) (ii) ≠> (i)

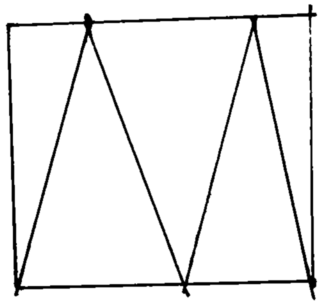
The identify function on any interval in IR with usual topology proves this.

(4) (ii) and (iii) ≠> (ii)

Let X = [0, ∞]. Define f:X→ X as,

$$f(x) = \begin{cases} 4x\ ; & 0 \le x \le \tfrac{1}{4} \\ -4x + 2\ ; & \tfrac{1}{4} \le x \le \tfrac{1}{2} \\ 4x - 2\ ; & \tfrac{1}{2} \le x \le \tfrac{3}{4} \\ 4 - 4x\ ; & \tfrac{3}{4} \le x \le 1 \\ f(x - 1)\ ; & x \ge 1. \end{cases}$$

Fig.



This example is enough to show that (iii) ≠> (i). Since $|f'(x)| = 4$; $\forall$ x ε X, every neighborhood around a point will expand under iteration. Since f is the tent map on each [n, n + ½] ; n ≥ 0 ; $\overline{P_f(x)} = X$. But f[0, 1] = [0, 1]. So $\overline{O_f(x)} \neq X$ for any x ε X.

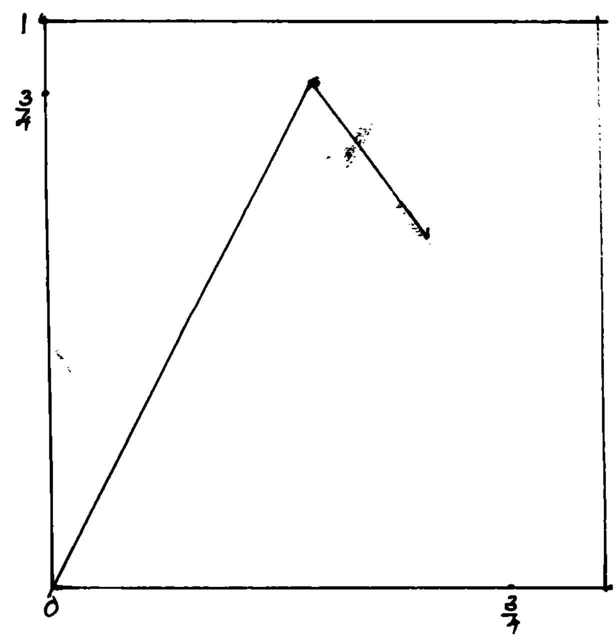
(5) (iii) ≠> (ii)

Let X = [0, ¾] and f : X→ X be defined by,

$$f(x) = \frac{3x}{2}, \text{ if } 0 \le x \le ½$$

$$= \frac{3(1-x)}{2}, \text{ if } ½ \le x \le ¾.$$

Fig.



Since $|f'(x)| > \frac{3}{2}$, $\forall$ x $\in$ [0, ¾], the function is expanding. But there is no periodic point in $(0, \frac{3}{8})$. So $P_f(X) \neq X$. This example is enough to show that (i) ≠> (ii)and (i) and

(iii) ≠> (ii).

It is clear that C(F) ⊆ S(F). The other inclusion need not be true in general. Let X = IR, F = [-1, 1] and f: F → F be defined by $f(x) = x^2$. By Cor.2.1.9, f ε S(F). But $P_f(F) = \{-1, 0, 1\}$. So $P_f(F) \neq F$. So f ∉ C(F).

If X is discrete or indiscrete, S(F) = ϕ, $\forall$ F $\in$ K(X) ∴ X is not a chaos space.

Result.2.1.15 :

IR is a chaos space.

Proof : Let F = [0, 1] and f:F → F be the map

defined by, $f(x) = \begin{cases} 2x, 0 \leq x \leq 1/2 \\ 2(1-x), 1/2 \leq x \leq 1 \end{cases}$

then f $\in$ C(F). ∴ H(F) ≠ ϕ ∴ F $\in$ CH(R). Hence R is a chaos space.

Result.2.1.16 :

If each X_i, i = 1, 2, 3,……, n is T_3 and chaos then $X = \prod_{i=1}^{n} X_i$ is a chaos space.

Proof : Since each X_i is T_3 it is T_2 also.

∴X is T_3 and T_2.

Given that each X_i is chaos. So there exist F_i ε $K(X_i)$ and $f_i : F_i \to F_i$ such that

$f_i \varepsilon C(F_i)$. Let $F = \prod_{i=1}^{n} F_i$. Then $F \varepsilon K(X)$. Since each F_i is perfect F is also perfect.

Define $f : F \rightarrow F$ by $f(x) = f(x_1, x_2, \ldots x_n) = (f_1(x_1), f_2(x_2), \ldots f_n(x_n))$ where $x = (x_1, x_2, \ldots x_n)$.

$f \in S(F)$ by Cor.2.1.8. We will prove that $f \in C(F)$. ie. We have to show that,

(i) $\overline{O_f(x)} = F$, for some $x \varepsilon F$ and (ii) $\overline{P_f(F)} = F$.

First we prove (i). Since $f_i \varepsilon C(Fi)$, there exist $x_i \varepsilon Fi$

such that $\overline{O_{f_i}(x_i)} = Fi$, for $i = 1, 2, 3, \ldots, n$. Let $x = (x_1, x_2, \ldots x_n)$.

Then we have,

$$O_f(x) \subseteq O_{f1}(x_1) \times O_{f2}(x_2) \times \ldots \times O_{fn}(x_n).$$

$$\therefore \overline{O_f(x)} \subseteq \overline{O_f(x_i)} \times \overline{O_{f_2}(x_2)} \times \ldots \times \overline{O_{f_n}(x_n)}$$

$$\therefore \overline{O_f(x)} \subseteq F \text{ (Since } \overline{O_{f_i}(x_i)} = F_i, \text{ for } i = 1, 2, \ldots n)$$

Now we prove, $F \subseteq \overline{O_f(x)}$

For that, let $y \varepsilon F$. Let $y = (y_1, y_2, \ldots, y_n)$, where $y_i \varepsilon F_i$ for $i = 1, 2, \ldots, n$. Also each F_i is such that $F_i = \overline{Of_i(x_i)}$, for some $x_i, \in F_i$. So there is a sequence in $Of_i(x_i)$ which converges to y_i, for $i = 1, 2 \ldots n$ ie. there is a subsequence of $\{f_i^k(x_i)\}_{k=1}^{\infty}$ which converges to y_i. So given a neighborhood V_{yi} of y_i, $\exists N_i \in Z_+$ Such that $\forall m \geq N_i$ $f_i^m(x_i) \in Vy_i$.

Let $N = \text{Max}\{N_1, N_2, \ldots N_n\}$. Then if $G(= \prod_{i=1}^{n} Vy_i)$ is an open neighbourhood of y, then $\forall n \geq N$, $f^n(x) \in G$. So $\{f^K(x)\}$ converges to y. So $y \in O_f(x)$. Hence, $F \subseteq \overline{O_f(x)}$.

$\therefore$ We have, $F = \overline{O_f(x)}$.

By a similar argument we have, $\overline{P_f(F)} = F$

$\therefore f \in C(F)$, hence $F \in CH(X)$

$\therefore X$ is a chaos space

Cor.2.1.17 :

IR^n, $\forall n \geq 1$, is a chaos space.

Proof : By Result 2.1.15 and 2.1.16

Result.2.1.18 :

Being a chaos space is a topological property.

Proof : We want to prove that if X is a chaos space and if X and Y are homeomorphic, then Y is a chaos space. Let $h : X \rightarrow Y$ be a homeomorphism. Since X is a chaos space, there exists $F \in CH(X)$. So $C(F) \neq \phi$. Let $f \in C(F)$. Let $g = hofoh^{-1}$. Then g is a continuous function from h(F) to h(F). We prove that $g \in C(h(F))$ so that $h(F) \in CH(Y)$. ie. We have to prove that (i) $\overline{O_g(y)} = h(F)$, for some $y \in h(F)$.

(ii) $\overline{P_g(h(F))} = h(F)$ and (iii) $g \in S(h(F))$.

To prove (i) :

Since $F \in CH(X)$, $\exists x \in F \ni \overline{O_f(x)} = F$..........(1)

If possible, suppose $\overline{O_g(y)} \neq h(F)$, for any $y \in h(F)$. ie. $\overline{O_g(h(x))} \neq h(F)$, for any $x \varepsilon F$. ie. There is an open set V in h(F) such that $O_g(h(x)) \cap V = \phi$

=> $g^n(h(x)) \notin V$, for any $n \varepsilon Z_+$

=> $(h_0 f_0 h^{-1})^n (h(x)) \notin V$, for any n

=> $(h_0 f^n{}_0 h^{-1}) (h(x)) \notin V$, for any n

=> $(hof^n)(x) \notin V$, for any n

=> $f^n(x) \notin h^{-1}(v)$, for any n, which contradicts (1) .

$\therefore \overline{Og(y)} = h(F)$, for some $y \in h(F)$.

Proof of (ii) is similar To prove (iii).

Let $y_1 \in h(F)$. Then $y_1 = h(x_1)$, for some $x_1 \in F$. Let V be an open neighborhood of y_1. Then $h^{-1}(v)$ is open in F and $h^{-1}(v)$ is a neighborhood of x_1. Since $f \in S(F)$, $\exists x_2 \in h^{-1}(v) \cap F$ and $n \in Z_+$ and an open set μ such that $f^n(x_1) \in U$ and $f^n(x_2) \notin \bar{U}$. Since

$f^n(x_1) \in U$, $(hof^n)(x_1) \in h(U)$

$\Rightarrow (hof^n oh^{-1})(h(x_1)) \in h(U)$

$\Rightarrow g^n(y_1) \in h(U)$

In the same way, $g^n(y_2) \notin \overline{h(u)}$. Hence $g \in C(h(F))$ and so, $h(F) \in CH(Y)$. $\therefore$ Y is a chaos space

<u>Notation.2.1.19</u> :

Let $D_f(F) = \{x \in F | \overline{O_f(x)} = F\}$

<u>Result.2.1.20</u> :

Let X be a T_2 space without isolated points and $f : X \to X$ be continuous. Then, $D_f(F) \neq \phi \Rightarrow \overline{D_f(F)} = F$.

Proof. Let $x \in D_f(F)$. Then $\overline{O_f(x)} = F$. Since X is T_2 having no isolated points, removing a finite set from a dense set leaves a dense set.

So, $O_f(f(x)) = (O_f(x) - \{x\})$ is dense in F.

So, $f(x) \in D_f(F)$. Also $D_f(F)$is invariant under f.

$\therefore$ If $x \in D_f(F)$, $O_f(x) \subseteq D_f(f)$. But $\overline{O_f(x)} = F$

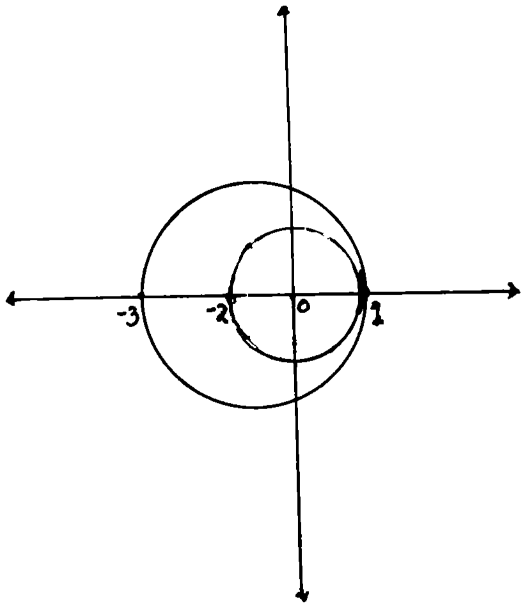
$\therefore \overline{D_f(F)} = F$

The following example shows that $\overline{Of(x)} = F \not\Rightarrow \overline{O_f{}^2(x)} = F$

Let C be the Complex plane and S_1^1 be the unit circle in C.

Let C→ C be defined by, g(z) = 2z-1. Let S_2^1 be the circle of radius 2 with centre at –1. Then $g(S_1^1) = S_2^1$.

Fig.



S_1^1 is internally tangent to S_2^1 at 1.

Let $X = S_1^1 \cup S_2^1$ with Eucledian topology. Define f:x→ x by,

$$f(z) = g(z^2) \text{ ; if } z \in S_1^1$$
$$[g^{-1}(z)]^2 \text{ ; if } z \in S_2^1$$

By Baire Category theorem, there is a point x ε x. Such that $\overline{O_f(x)} = X$. But f^2 leaves both S_1^1 and S_2^1 invariant and thus has no dense orbits. In particular, $\overline{O_{F^2}(x)} \neq$

X. Note that f is chaotic on X because any $(4^n-1)^{th}$ root of unity is periodic and f-image of that root is also periodic. But f^2 is not chaotic.

So a question aries that under what conditions f is chaotic=> f^2 is chaotic. We will answer this question through the following results.

Result.2.1.22 :

Let X be T_2 without isolated points and $f : X \to X$ be continuous such that $x \in D_f(X)$. Then, $x \notin D_{f^2}(X)$ <=> There exist a seperation D_1, D_2 of D such that each of the sets D_1 and D_2 is invariant under f^2.

Proof. Assume that $x \notin D_{f^2}(X)$. $\therefore$ $\overline{O_{f^2}(x)} \neq X$ Let $G = X - \overline{O_{f^2}(x)}$. So G is open in X. We will show that, for each non-negative integer k, $f^{-2k}(G) \subseteq G$. Let k be a non-negative integer and suppose that $[f^{-2k}(G)] \cap \overline{O_{f^2}(x)} \neq \phi$----------- (*)

G is open and f is continuous. $\therefore$ $f^{-2k}(G)$ is open. Since $\overline{O_f(x)} = X$, $[f^{-2k}(G)] \cap O_f(x) \neq \phi$.

So there exists a non negative integer m such that $f^{2m}(x) \in [f^{-2k}(G)]$. So $f^{2(k+m)}(x) \in G$. Which is a contradiction to the definition of G.

$\therefore$ (*) is not possible. ie. $f^{-2k}(G) \subseteq G$.

Claim :

$f^{-1}(G) \subseteq (X \backslash G)$.

If possible assume that $f^{-1}(G) \cap G \neq \varnothing$.

Since $\overline{O_f(x)} = X$ and $f^{-1}(G) \cap G$ is open, there is a non-negative integer j such that $f^j(x) \in f^{-1}(G) \cap G$. But this j can't be even (for $f^j(x) \in G$) and j can't be odd (since $f^j(x) \in f^{-1}(G) \Rightarrow f^{j+1}(x) \in G$ and j+1 is even).

This is a contradiction .

$\therefore f^{-1}(G) \subseteq (X \setminus G)$.

$\therefore f^{-1}(G) \cap G = \phi$.

Note that $f^{-1}(G)$ and G are open.

Let $y \in D_f(X)$. Then there is a k>0 such that $f^k(y) \in G$. ie. $y \in f^{-k}(G)$ and is either in G (if k is even) or $f^{-1}(G)$ [if k is odd, because then $f(y) \in f^{-2r}(G) \leq G$ for some r]. So $y \in D_f(x) \Rightarrow y \in G$ or $y \in f^{-1}(G)$. $\therefore D_f(X) \subseteq f^{-1}(G) \cup G$.

By hypothesis $D_f(x)$ is dense. $\therefore D_f(X) \cap f^{-1}(G) \neq \phi$ and $D_f(x) \cap G \neq \phi$.

Let $D_1 = D_f(X) \cap G$ and $D_2 = D_f(x) \cap f^{-1}(G)$.

Then D_1, D_2 is a seperation of $D_f(x)$. We complete the proof by showing that $f(D_2) \subseteq D_1$ and $f(D_1) \subseteq D_2$.

Suppose $z \in D_2$. Then $z \in f^{-1}(G)$ and $z \in D_f(x)$

$\Rightarrow f(z) \in G$ and $z \in D_f(x)$

$\Rightarrow f(z) \in G$ and $f(z) \in D_f(x)$. (By Result.2.1.20)

$\Rightarrow f(z) \in G \cap D_f(x) = D_1$

$\therefore f(D_2) \subseteq D_1$.

Now, let $z^1 \in D_1$

$\Rightarrow z^1 \in G \cap D_f(x)$

$\Rightarrow z^1 \in G$

Suppose $f(z^1) \in G$. Then $z^1 \in G \cap f^{-1}(G)$.

But this contradicts the fact that $G \cap f^{-1}(G) = \phi$.

$\therefore f(z^1) \subseteq (X \setminus G)$

So $f(z) \in$ can't be in D_1. Since $z^1 \in D_1$, $f(z^1) \in D_f(x)$. So $f(z^1) \in D_2$

$\therefore f(D_1) \subseteq D_2$.

Result.2.1.23 :

Let X be T_2 without isolated points. Let $x \in X$ be such that $x \in D_f(x)$ and $x \notin D_{f^2}(x)$. Let $U_1 = [X \setminus (\overline{O_{f^2}(x)})]$ and $U_2 = f^{-1}(U_1)$. Then ,

(i) $U_1 \cap U_2 = \phi$

(ii) $\overline{U_1 \cup U_2} = X$

(iii) $f(U_2) \subseteq U_1$

(iv) $f(U_1) \subseteq (X \backslash U_1)$

(v) $(U_1 \cup U_2)^C$ is invariant under f.

Proof : (i), (ii) and (iv) follows from Result.2.1.22. (iii)follows from definition of U_2. Now, to prove (v) :

Let $z \in (U_1 \cup U_2)^C$. If f(z) were in U_1, then $z \in f^{-1}(U_1) = U_2$. If $f(z) \in U_2$, then $f^2(z) \in U_1$, so by Result 2.1.21, $z \in U_1$.

$\therefore f(z) \in (S_1 \cup S_2)^C$.

$\therefore f((U_1 \cup U_2)^C) \subseteq (U_1 \cup U_2)^C$.

Result.2.1.24 :

Let X be a T_2 space without isolated points such that $x \in D_f(X)$ and the closure of the set of points of x having odd period under f has non-empty interior, then $x \in D_{f^2}(X)$.

Proof : Neither U_1 nor U_2 contain a point having odd period under f and $((U_1 \cup U_2)^C)^0 = \phi$ because $\overline{U_1 \cup U_2} = X$. (by Result 2.1.23). If possible assume that $x \notin D_{f2}(X)$. Then set of points having odd period is contained in $(U_1 \cup U_2)^C$, a closed set with empty interior, which is a contradiction.

$\therefore x \in D_f^2(X)$.

Result.2.1.25 :

Let X be a regular space without isolated points. If $f \in C(X)$ and the closure of the set of points of X having odd period under f has non-empty interior then $f^2 \in C(X)$.

Proof : $f \in C(X) \Rightarrow \overline{O_f(x)} = X$, for some $x \in X$

$\Rightarrow \overline{O_{f^2}(x)} = X$ (By Result.2.1.24)

$\Rightarrow x \in D_f^2(X)$ ----------------- (1)

Again, $f \in C(X) \Rightarrow \overline{P_f(X)} = X$

$$\Rightarrow \overline{P_{f^2}(x)} = X \text{ ---------------- } (2)$$

Since f $\in C(X)$, f $\in S(X)$ => X is perfect. (By Result Cor.2.1.7) because f is non-constant (If f is constant $D_f(X) = \phi$).

$\therefore f^2 \in S(X)$. So, $f^2 \in C(X)$.

Note that if $f^2 \in C(X)$ then f $\in C(X)$, for any topological space X. So we have,

Cor.2.1.26 :

Let X be regular without isolated points and f : X→ X be continuous such that closure of set of points having odd period under f has non-empty interior then,

$f \in C(X) \Leftrightarrow f^2 \in C(X)$.

Proof : By Result.2.1.25.

2.2. Several definitions of chaos

There are several definitions of chaos in the literature. Some of them are defined on [0, 1]. Some of them in metricspaces and some of them are on more general topological spaces. In this section we give generalizations of these definitions and compare their relations.

The First one is Definition 2.1.10, which we call by the name TC. The second one is Devaney's definition (in metricspaces) we call this by DC.

Next we define;

Definition.2.2.1 :

Let $(X, \Im)$ be a topological space and $f : X \to X$ be continuous. f is said to be Sharkovsky chaotic (SC) on X if f has points of period n, $\forall n \in Z_+$.

By Li-Yorke result f: $[0, 1] \to [0, 1]$ is SC if f has a point of period three.

Next, we generalize Li-Yorke's definition [Definition 1.1.5] of chaotic functions in [0, 1] into metric spaces. For that we want a notation.

Notation.2.2.2 :

Let $M = \{(x, y)\} \in X \times X \mid \limsup_{n\to\infty} d(f^n(x), f^n(y)) > o$ and $\liminf_{n\to\infty} d(f^n(x), f^n(y)) > o\}$

and $\Delta = \{(x, x) \mid x \in X\}$

Now we define,

Definition.2.2.3 :

Let (X, d) be a metricspace and f: $X \to X$ be continuous. f is Li-Yorke chaotic (LYC) if there exist an uncountable set $S \subseteq X$ such that,

$$(M \setminus \Delta)/_S = (SxS) \setminus (\Delta/_S).$$

[Here, $((M \setminus \Delta)/_S) = \{x \in M \setminus \Delta | x \in S\}$ and

$\Delta/_S = \{(x, x) \mid x \in S\}$]

Now we generalize Auslander – Yorke definition into topological spaces. [see Definition 1.1.9]. Before that,

Definition.2.2.4 :

Let X be a topological space and f: $X \to X$ be continuous f is stable at x if given any neighborhood. U of x there is a neighborhood V of x such that, $V \subseteq U$ and $f^n(V) \subseteq U$, $\forall n \geq 1$. f is unstable at x if f is not stable at x.

Definition.2.2.5 :

Let $(X, \Im)$ be a topological space and f: $X \to X$ be continuous. f is Auslander-Yorke chaotic (AYC) on X if (i) $\overline{O_f(x)} = X$, for some $x \in X$ and (ii) f is unstable at x, $\forall x \in X$.

Definition.2.2.6 :

Let (X, d) be a metricspace and f: $X \to X$ be Continuous. Then f is Generically chaotic (GC) on X if there exist. $D \subseteq M$ such that $\bar{D} = X \times X$.

(M is as defined in Notation 2.2.2.)

We have Block-Coppel definition of chaos in terms of topological entropy [Definition.1.1.10].

Definition.2.2.7 :

Let $(X, \Im)$ be a topological space and f: $X \to X$ be continuous, then f is Block-Coppel chaotic (BC) on X if topological entropy of f, ie. top (f), > 0.

Knudsen [49*] defined chaotic functions as,

Definition.2.2.8 :

Let (X, d) be a metric space and f: $X \to X$ be continuous. Then f is Knudsen chaotic (KC) iff (i) $\overline{O_f(x)} = X$, for some $x \in X$ and (ii) f is sensitive to initial conditions.

Now we define expansive chaotic functions.

Definition.2.2.9 :

Let (X, d) be a Metric space and f : $X \to X$be continuous. Then f is expansive on x if $\exists \delta > 0$ such that $\forall x, y \in x$ with $x \neq y$; $d(f^n(x), f^n(y)) \geq \delta$, for some n. (Note that δ is uniform ie. δ is independent of the points x and y).

It is clear from the definition that,

Result.2.2.10 :

If f is expansive then f is sensitive

Definition.2.2.11 :

Let (X, d) be a metric space and f: $X \to X$ be continuous. Then f is Expansively chaotic (EC) if (i) f is transitive, (ii) $\overline{P_f(x)} = X$ and (iii) f is expansive. Now we compare all definitions.

2.2.12 : TC versus other definitions

(1) TC ≠> DC

Suppose X has an isolated point, say $a \in X$, then f can't be transitive at a. But if x has no isolated points then f is TC => f is DC because, by Result.1.1.11.

If X is Second countable Baire space then f is transitive => $x \in D_f(X)$, for some $x \in X$.

So we have,

If X is a second countable Baire space without isolated points and $f: X \to X$ is continuous f^n is non constant, for some n, in every open set G, then

f is TC <=> f is DC

Proof is clear from above statement .

(3) TC ≠>SC

Let f [0, 1] → [0, 1] defined by

$$f(x) = x + \frac{1}{3}; \qquad 0 \le x \le \frac{2}{3}$$

$$= 1 - 9\left(x - \frac{2}{3}\right), \quad \frac{2}{3} < x < \frac{23}{30}$$

$$= x - \frac{2}{3}, \qquad \frac{23}{30} \le x \le 1 .$$

clearly f is TC. But there is no point of period 2. So f is not SC.

(4) TC ≠> LYC

Let S^1 be the unit circle and $f : S^1 \to S^1$ be the irrational rotation of S^1. ie. $f(e^{i\theta}) = e^{i(\theta+\alpha)}$, where $\frac{\alpha}{\pi}$ is not rational. Then $\overline{O_f(x)} = S^1$, $\forall x \in S^1$ and f is TC. But f is not LYC.

(5) TC ≠> AYC

But on metric spaces, TC => AYC. Since on metric spaces f is sensitive => f is unstable.

(6) TC ≠> BC

Let X be a Compact chaos metric space and $f \in H(X)$. Such that $\{f^n\}$ is equicontinuous then top (f) = 0 (by Result 1.1.15). So f is TC but not BC.

(7) TC ≠> KC

clear from definitions

(8) TC ≠> EC

Usual tent map is an example for TC. But it is not EC because there are points $x \neq y$ such that $f(x) = f(y)$.

2.2.13 : DC versus other definitions

(1) DC $\neq>$ TC

because, f is transitive $\neq>$ $D_f(x) \neq \phi$.

But if X is a second countable Baire space then

f is DC => f is TC

(2) DC $\neq>$ SC

Same as example 2.2.12(.3).

(3) DC => LYC

Clear from definitions

(4) DC $\neq>$ AYC

But on second countable Baire spaces DC => AYC

(5) DC => GC

Since $D_f(X) \neq \phi$, $\overline{D_f(x)} = x$, by Result 2.1.20

So $D = D_f(X) \times D_f(X)$ is such that $\bar{D} = X \times X$.

(6) DC $\neq>$ BC

Same as 2.2.12.(6)

(7) DC $\neq>$ KC

But if X is second Countable baire then DC => KC.

(8) DC => EC

2.2.14 : SC versus other definitions

(1) SC $\neq>$ Any other definitions

Just one example is enough to show this .

Let X = [0, 1] and f be the identity map on X . Note that top(f) = log ($f^1(x)$) = log 1 = 0.

2.2.15 : LYC versus other definition

(1) LYC ≠> any other definitions

Let X = [-1, 1] and f : X→ X be defined as,

$$f(x) = \begin{cases} -2x-2 \; ; & -1 \le x \le ½ \\ 2x \; ; & |x| < ½ \\ 2-2x \; ; & ½ \le x \le 1 \end{cases}$$

Note that f restricted to [0, 1] is the usual tent map. Tent map is LYC. So there exist a scrambled set M for f. but f is not transitive on [-1, 1]; Since none of the points in (0, 1) can reach any subinterval of (-1, 0). Also top (f) = 0.

But if f is transitive, then f is LYC => f is TC and f is LYC => f is DC.

2.2.16 : AYC versus other definitions

(1) AYC ≠> any other definitions

Classical sturmian systems are examples. Let Y be the 1-torus and f_α be the rotation by the irrational number $\alpha \in y$. Code the orbits of Y according to the closed cover

C= {[0, 1 - α], [1 - α, 1]}of Y. Let X be the closed set of all bi-finite C-names and let σ be the shift. Then σ : x → x is AYC but not TC. (there are no periodic points for σ). This example is enough to prove that AYC ≠> other definitions except for AYC ≠> EC.

(2) AYC ≠> EC

The test map on [0, 1] is AYC, but not expansive so not EC.

Note that if set of periodic points if f is non-empty then f is AYC => f is LYC.

2.2.17 : GC versus other definitions

(1) GC ≠> any other definitions

Let X be the set of all words with two letter and f: X→ X be full shift. Let g be the irrational rotation of the unit circle Y. Define, h : X ×Y→ X ×Y as h(x, y) = (f(x), g(y)). Then h is GC on X ×Y with product topology ; But h is not any other chaos.

2.2.18 : BC versus other definitions

All chaotic maps with zero topological entropy is an example for this. For example 2.2.15(1) .

2.2.19 : KC versus other definitions

(1)KC ≠> TC

Because periodic points need not be dense

If periodic points of fare dense then f is KC => f is TC.

(2) KC ≠> DC

If X has no isolated points and periodic points of f are dense then f is KC => f is DC.

(3) KC ≠> SC

See 2.2.12.(3).

(4) KC ≠> LYC

See 2.2.12.(4).

(5) KC => AYC

f is sensitive => f is unstable

(6) KC ≠> GC

See 2.2.16.(1).

(7) KC ≠> BC

See 2.2.15.(1).

(8) KC ≠> EC

Any example for $\overline{O_f(x)} = X$ ≠> f is transitive

2.2.20 : EC versus other definitions

(1) EC ≠> any other definitions

Examples can be taken from already given ones.

Now we represent all these implications in the following table.

For example third row last column is filled by 'T'. That means DC => EC in general.

But second row sixth column is entered with 2. That means in general,TC ≠>AYC

But if 2 is assumed, ie. in Metricspaces,TC => AYC

(see the table on next page)

Table.1

=>	TC	DC	SC	LYC	AYC	GC	BC	KC	EC
TC	T	1	F	F	2	F	F	T	F
DC	3	T	F	T	3	T	F	3	T
SC	F	F	T	F	F	F	F	F	F
LYC	4	4	F	T	4	F	F	4	4
AYC	6	1&6	F	5	T	F	F	4	F
GC	F	F	F	4	F	T	F	F	F
BC	F	F	F	F	F	F	T	F	F
KC	6	1&6	F	F	T	F	F	T	F
EC	3	T	F	F	3	F	F	3	T

Where 'T' means True, 'F' means False

1. ⇒true if X has no isolated points.
2. ⇒true if X is a Metric space.
3. ⇒true if X is second countable Baire.
4. ⇒true if F is transitive.
5. ⇒true if Set of Periodic points is non-empty.
6. ⇒true if Periodic points are dense.

Now we prove that in [0, 1] almost all the above definitions are equivalent and some more can be added to this. For that we want following notations

Notation.2.2.21 :

$$L_f(x) = \bigcap_{k=o}^{\infty} \overline{(O_f(f^K(x)))}$$

$$\Omega_f(X) = \{x \in x \mid x \text{ is non wandering}\}$$

and we define,

Definition.2.2.22 :

Let X be a topological space and f: X→ X be continuous. Then f is 'stirring' if given any pair of non empty open sets U and V in X, ∃ some k ∈ Z_+ such that $f^k(U) \cap f^k(V) \neq \phi$.

f is 'strongly stirring' if ∃ k ∈ Z_+ and an open set G in X such that, $G \subseteq f^k(U) \cap f^k(V)$.

Result.2.2.23 :

Let X be a T_2, second countable, Baire space without isolated points and f: X→ X be continuous and onto. Then the following are equivalent.

(i) f is transitive on x

(ii) There exists x ∈X such that $\overline{O_f(x)}$=X (ie $D_f(X) \neq \phi$)

(iii) There exists x ∈ X such that $L_f(x) = X$.

Proof .

(iii) => (ii)

If $L_f(x) = X$ and since $\overline{O_f(x)} = X$ $O_f(x) \cup L_f(x)$, we have

(ii) => (iii)

Suppose that these exist an x∈X with $\overline{O_f(x)} = X$ .Let x'∈ X with f(x') = x

Now, x' ∈ $\overline{O_f(x)} = O_f(x) \cup L_f(x)$.

If x' $\in L_f(x)$ then x = f(x') ∈ $L_f(x)$. Since $L_f(x)$ is closed and $f(L_f(x)) \subseteq L_f(x)$,

$\overline{O_f(x)} \subseteq L_f(x)$. So, $L_f(x) = X$.

Now, if x' ∈ $O_f(x)$ then x = f(x') ∈ $O_f(x)$ and $O_f(x) = \overline{O_f(x)} = L_f(x) = X$ is a finite set.

But X is infinite. So x' $\notin O_f(x)$.

(i) => (iii)

Suppose f is transitive. ie. given U, V open in X there exist n>0 such that $f^n(U) \cap V \neq \emptyset$.

Let B = {$U_1, U_2, \ldots, U_n, \ldots$} be a countable base for X. For j, N ∈ Z_+, Define $U_j^{(N)} = \bigcup_{k \geq n} (f^k)^{-1}(U_j)$. Since f is transitive and continuous $U_j^{(N)}$ is open and dense. Since X is Baire, $\bigcap_{j,N} U_j^{(N)}$ is dense in X.

So, $f^k(x) \in U_j$, for some $k \geq N$. Hence $L_f(x) \cap \bar{U}_j \neq \phi$.

$\therefore L_f(x) = X$.

(iii) => (i)

Suppose $L_f(x) = X$, for some $x \in X$. Let U, V be open in X. We show that $[\bigcup_k (f^k)^{-1}(U)] \cap V \neq \emptyset$. Let $k > 0$ such that $f^k(x) \in V$. Let $r > 0$ Such that $f^{r+k}(x) \in U$.

So, $f^k(x) \in V \cap (f^r)^{-1}(U) \subseteq V \cap [\bigcup_k (f^k)^{-1}(U)]$. Hence f is transitive.

From Example 2.1.14.(3) we have $\overline{P_f(x)} = X \not\Rightarrow \overline{O_f(x)} = X$.

From Result.2.2.23. on T_2, Second countable Baire spaces without isolated points $\overline{O_f(x)} = X \iff$ f is transitive. So in general, $\overline{P_f(x)} = X. \not\Rightarrow$ f is transitive But if f is strongly stirring then $\overline{P_f(x)} = X \Rightarrow$ f is transitive.

Result.2.2.24 :

Let X be topological space and $f : X \to X$ be continuous. Then $\overline{P_f(X)} = X$ and f is strongly stirring => f is transitive.

Proof. Let U and V be open sets in X. Since f is strongly stirring there is an open set G in X such that $G \subseteq f^K(U) \cap f^K(V)$, for some $K \in Z_+$.

Let $\tilde{V} = (f^k)^{-1}(G) \cap V$. Since f is continuous $\tilde{V}$ is open. Since $\overline{P_f(X)} = X$, there is a periodic point $x \in \tilde{V}$ say of period p. But since $x \in \tilde{V}$, $f^k(x) \in G$ and $f^k(x) \in V$. But $G \subseteq f^k(U) \cap f^k(V)$. So $f^k(x) \in f^k(U)$.

ie. $f^k(x) = f^K(y)$ for some $y \in U$

$\therefore f^p(y) = f^{(p-k)}(f^k(y)) = f^{(p-k)}(f^k(x)) = f^P(x) = x$, where y ∈ U and x ∈ V (∵ $\tilde{V}$ ⊆V).

$\therefore f^{P}(U) \cap V \neq \phi$.:

∴ f is transitive.

Cor.2.2.25 :

Let X be a topological space and $f : X \to X$ be continuous. Then $f \in C(X)$ if (i) $\overline{P_f(X)} =$ X (ii) f is strongly string and (iii) $f \in S(X)$.

Proof. Follow from Definition.2.1.10 and Result 2.2.24.

Result.2.2.26:

Let X be T_2 and $f : X \to X$ is either transitive or $\overline{P_f(X)} = X$, then $\Omega_f(X) = X$.

Proof. Suppose f is transitive on X. Let U be a neighborhood of $x \in X$. Then $f^n(U) \cap U \neq \phi$, for some $n \in Z_+$. So $x \in \Omega(x)$. Since $x \in X$ is arbitrary $\Omega_f(X) = X$.

If $\overline{P_f(x)} = X$ then, $x \in \overline{P_f(X)}$ => there is a sequence $\{x_1, x_2, \ldots, x_n \ldots\}$ in $P_f(X)$ which converges to x. So given a neighbourhood U of x $\exists k > 0$ such that $x_n \in U$, $\forall n \geq K$. But x_i, for i = k+1, k+2,....are periodic points. So there exist $n_i \in Z_+$ such that $f_{ni}(x_i) = x_i$. Hence $f^{ni}(U) \cap U \neq \phi$. So, $x \in \Omega_f(X)$.

ie. $\overline{P_f(X)} \subseteq \Omega_f(X)$. Since $\overline{P_f(X)} = X$, $\Omega_f(X) = X$.

Cor.2.2.27 :

If f is chaotic on X then $\Omega_f(X) = X$

Proof . Follows from Result.2.2.26

Result.2.2.28 :

Let X be a topological space and $f : X \to X$ be continuous. Then $\Omega_f(X) = X$ => f is transitive on X.

Proof. Let U and V be any two open sets in X. then (U∩V) is also open in X. Since

$\Omega_f(X) = X$, $\exists n \in Z_+$ such that, $f^n[(U \cap V)] \cap (U \cap V) \neq \phi$; which in turn implies $f^n(U) \cap V \neq \phi$.

$\therefore$ f is transitive on X.

Result.2.2.29 :

Let X be a T_2, Second Countable Baire space without isolated points and $f: X \rightarrow X$ be continuous. Then following are equivalent.

(i) f is TC

(ii) f is transitive and $\overline{P_f(X)} = X$

(iii) Given any two non-empty open sets U, V in X there exist a periodic point $p \in U$ such that $O_f(p) \cap V \neq \phi$

(iv) Let $\{U_i\}_{i=1}^n$ be a collection of open sets. Then there exists a periodic point $p_i \in U_i$. such that $O_f(p_i) \cap U_j \neq \phi, \forall i \neq j$.

Proof. Suppose f is TC.

Then $(\overline{O_f(x)}) = X$ for some $x \in X$ and $\overline{P_f(X)} = X$. But by Result.2.2.3, f is transitive and $\overline{P_f(X)} = X$. So (i) => (ii)

If f is transitive and periodic points are dense and since has no isolated points $f \in S(X)$.

$\therefore f \in S(X)$.

$\therefore f \in C(X)$ $\therefore$ (ii) => (i) .

To prove (ii) => (iii)

Suppose f is transitive and $\overline{P_f(X)}$=X. Given any pair of non-empty open sets U, V in X $\exists u \in U$ and $k \in Z_+$. Such that $f^k(u) \in V$. Let $W = (f^k)^{-1}(V) \cap U$. Since W is the intersection of two open sets and $u \in (f^k)^{-1}(V) \cap U$. W is non empty and open. Also $f^k(W) \subseteq V$. Since. $\overline{P_f(X)}$=X, there is a periodic point p in W. So. $p \in W \subseteq U$. Such that $f^k(P) \in f^k(W) \subseteq V$. So, $O_f(p) \cap V \neq \phi$ .Hence (ii) = (iii)

To Prove (iii) => (ii)

Assume (iii). Given any open set V there is a periodic point p in X such that

$O_f(p) \cap V \neq \phi$. Since p is periodic, $f^n(p)$ is periodic for all $n \in Z_+$ So V contains a periodic point. Hence $\overline{P_f(X)}=X$. Let U, V be any two non-empty open sets. Then there exists $p \in U$ such that $f^k(p) \in V$ for some $k \in Z_+$. ie. $f^k(U) \cap V \neq \phi$. So f is transitive. ie. (iii) => (ii).

Now we prove (iii) => (iv)

Assume (iii). ie. Given any two non-empty open sets U and V, there is a periodic point $p \in U$. Such that $O_f(p) \cap V \neq \phi$. We have to prove that if $\{U_i\}_{i=1}^n$ is a collection of open sets then there exist some $p_i \in U_i$ Such that $O_f(p_i) \cap U_j \neq \phi$, $\forall i \neq j$.

We prove this by induction.

Let n be the number of open sets in our collection. If n =1, Since $\overline{P_f(X)}= X$ (by (ii)) (iv) is true. If n =2, by (iii) ; (iv) is true. Assume that the assertion (iii) => (iv) holds for n = r, say. We will show that it holds for (r+1) non-empty open subsets. We can assume that the collection of (r+1) open subsets are disjoint, because if they are not disjoint then the intersection is again open so that pair can be replaced by their intersection giving r open sets for which our assertion holds. From our disjoint collection choose a subset G. the remaining r subsets, by hypothesis, has the property (iv). Let U_0 be a subset from this collection. By (iv) $\exists p_0$ with period such that $O_f(p) \cap U_i \neq \varnothing \forall i$. Since the collection is disjoint m>(r-1). Now consider the iterates of p. $\exists K_1, \in Z_+$ such that $f^{K1}(p)$ is in one of the open set in the collection. Let this subset be U_1. $K_1<m$. Again $\exists K_2 \in Z_+$ such that $f^{K2}(p) \in U_2$ for some U_2 is the collection ; $0<K_1<K_2<m$. Continuing in this way, we have $f^{Ki}(p) \in U_i$, $\forall i =0, 1, 2,\ldots..(r-1)$ where $0 = K_0< K_1<\ldots..<K_{r-1}<m$.

Let $W_0= U_{r-1}$ Then $f^{K(r-1)}(p) \in W_0$

Let $W_1 = f^{[K(r-2)-K(r-1)]}(W_0) \cap U_{r-2}$

W_1 is open. It is non-empty because,

$f^{K(r-2)}(p) \in U_{r-2}$ and $f^{K(r-1)}(p) \in W_0$.

$\therefore f^{K(r-2)}(p) = f^{[K(r-2)-K(r-D)]}(f^{K(r-1)}(p) \in f^{[K(r-2)-k(r-1)]})(W_0)$.

So, $f^{K(r-2)}(p) \in W_1$

Also $f^{[K(r-1)-K(r-2)]}(W_1) \subseteq W_0$

In general, we have,

$W_i = f^{[K(r-1)-K(r-i)]}(W_{i-1}) \cap U_{r-(i+1)}$; for i=1, 2,.....(r-1).

Also, $f^{[K(r-1)-K(r-i-1)]}(W_i) \subseteq W_{i-1}$, for i=1, 2,...., (r-1)

So there exist a periodic point $p \in G$ such that , $O_f(p) \cap W_{r-1} \neq \phi$.

ie $f^s(p^1) \in W_{r-1} \subseteq U_0$. ie. $f^s(p^1) = f^{s+k0}(p^1) \in W_{r-1} \subseteq U_0$

$f^{S+K1}(p^1) = f^{(K1-K0)}(f^{s+K0})(P^1) \in f^{(K1-K0)}(W_{r-1}) \subseteq W_{r-2} \subseteq U_1$

ie. $f^{s+k}{}_1(p^1) \in Ui$ for i=0, 1, 2,...,(r-1)

So, $O_f(p) \cap Ui \neq \phi$, $\forall$ i=0, 1, 2,...,(r-1)

$\therefore$ (iii) => (iv) .

If for any collection r, (iv) is true it is true for r = 2 also.

$\therefore$ (iv) => (iii)

So we proved (i) <=> (ii) <=> (iii) <=> (iv) .

Result.2.2.30 :

Let f be continuous on X =[0, 1]. Then the following are equivalent.

(i) f is DC

(ii) $\overline{D_f(X)} \neq \phi$

(iii) $\bigcup_{n=1}^{\infty} f^{-n}(U)$ is dense in [0, 1] for each open set U.

Proof . f is DC => f is transitive <=> $\overline{D_f(x)} \neq \phi$ (by Result.2.2.23) .

So (i) => (ii).

To prove (ii) => (i) .

Suppose f is Such that $\overline{D_f(X)} \neq \phi$. Then f is transitive.

(i) is proved if we prove that $\overline{P_f(X)} = X$.

Suppose not, ie. $\overline{P_f(X)} \neq X$. Then $\exists$ $J \subseteq X$ Containing no periodic points. Let $x \in J$ and $N \subseteq J$. Let $E \subseteq (J \setminus N)$. Since f is transitive on X there exist a m>0 such that

$f^m(N) \cap E = \phi$. So $\exists$a $y \in J$ with $f^m(y) \in E \subseteq J$. Since $P_f(J) = \phi$, $y \neq f^m(y)$. Since f is continuous there exist a neighborhood U of Y with $f^m(u) \cap u = \phi$. Since U is an open set and f is transitive there exist n>m and a $z \in U$ with $f^n(z) \in U$. But then, 0<m<n and z, $f^n(z) \in u$ while $f^n(z) \notin U$. This violates the Result 1.1.18 .

So (ii) => (i) .

To prove (ii) => (iii) .

Suppose $\overline{D_f(X)} \neq \varnothing$. Let U, W be open sets in X. We show that

$$\bigcup_{n=1}^{\infty} f^{-n}(U) \cap W \neq \phi.$$

Let K>0 such that $f^K(x) \in W$. Let l >0 such that, $f^{K+l}(x) \in U$.

So, $f^K(x) \in W \cap f^{-l}(U) \subseteq W \cap \bigcup_{n=1}^{\infty} f^{-n}(U)$.

To prove (iii) => (ii)

Suppose for each open set U, $\bigcup_{n=1}^{\infty} f^{-n}(U)$ is dense in X. Let B ={U_1, U_2,...} be a countable basis for X. Let $U_j^{(N)} = \bigcup_{k \geq N} f^{-k}(U_j)$; $\forall U_j \in B$ and $N \in Z_+$.

By Baire category theorem, $\bigcap_{j>0, N>0} U_j^{(N)} \neq \phi$.

So for every pair of positive integers j and N , $x \in U_j^{(N)}$ so $f^k(x) \in U_j$, for some $k \geq N$.

$\therefore L_f(x) \cap \bar{U}_j \neq \phi$

Hence $\overline{L_f(x)} = X \Rightarrow \overline{O_f(x)} = X$, by result 2.2.23

In 1986, K Jankova and J. Smital proved that f is LYC on I = [0,1] iff $L_f(x)$ contains two f- non separable points a and b, for some $x \in I$

Combing all the results that we have so far

<u>Result. 2.2.3.1:</u>

let f : I→I be a continuous function, Where I = [0,1]. Then the following are equivalent ,

(1) f is transitive an X

(2) f is TC

(3) f is DC

(4) f is LYC

(5) f is AYC

(6) f is KC

(7) f is EC

(8) every point of I is unstable

(9) $D_f(I) \neq \phi$

(10) $\exists x \in I$ such that $L_f(x) = I$

(11) f is transitive and periodic points are dense

(12) $\Omega_f(I) = I$

(13) given any two non- empty open sets U, V in I there exits a periodic point $p \in U$ such that $O_f(p) \cap V \neq \phi$

(14)let $\{U_i\}_{i=1}^{n}$ be a collection of open sets in I. Then there is a periodic point pi $\in$ Ui such that $O_f(p_i) \cap U_j \neq \phi$, $\forall i \neq j$

(15) $\bigcup_{n=1}^{\infty} f^{-n}(U)$ is dense in I for each open set U .

(16) $\exists$ x ε I such that $L_f(x)$ contains two f- non separable points .

From the Result1.1.13, we have if f: [0,1]→ [0,1] has a point of period 3 than f is LYC. Also by Sharkovskys theorem if f: [0,1] → [0,1] has a point of period 3 ten f is SC.

So, period 3 => LYC

and period 3 => SC

Motivated by this we consider space in which period 3 => SC. We define Sharkovsky spaces in the next section.

2.3. Sharkovsky Spaces

In this section we define Sharkovsky spaces. In Sharkovsky spaces, period three implies SC.

Definition 2.3.1.

let X be a topological space and f: X→ X be continuous. Arrange the set of positive integers in the following order

$$3<5<7<9<\ldots.<2.3<2.5<2.7<\ldots.<2^2.3<2^2.5<\ldots<2^3<2^2<2^2<2<1$$

X is called a Sharkovsky space iff for every continuos function on X existence of a point of period n => existence of a point of period m for every m such that $n<m$. From the definitions it is clear that,

Result 2.3.2.

Let X is a Sharkovsky space and f: X → X be continuos. If there is a point of period 3 for f then f is SC on X

From the Sharkovsky's theorem (Result. 1.1.14) we have

Result. 2.3.3.

R is a Sharkovsky space

Note that if f is SC on X then f must have points of all periods. In particular f has points of period 3. So, in general, the converse of Result. 2.3.2 is true so we have,

Result. 2.3.4.

Let X be a Sharkosky space and f: X→X be continuous then following are equivalent,

(i) f is SC

(ii) f has a point of period 3

Next example shows that a chaos space need not be a Sharkovsky space

Example 2.3.5.

We know that C is a chaos space since C is homeomerphic to R^2. But C is not a Sharkosky space.

Define $f:C \to C$ as , $f(z) = -z+z^2$.

Then f has a point of period 3, but those is no point of period 2(note that 3<2 in the order) .

So a question arises which properly of R makes it a sharkovsky space, which lacks for C. we will try to answer this question. For that first we have.

Definition 2.3.6.

If $(X, \leq)$ be a totally ordered set with more than one point. Then X is a linear continuum if

(i) X has the least upper bound property (equivalently the greatest lower bound property),

(ii) X is order dense ie. If $x<y$, then there exist z such that $x<z<y$.

(iii) X has the order topology .

The following result is a generalization of the Intermediate value theorem.

Result 2.3.7.

Let [a, b] be a closed interval in the linear continuum X. if f: $[a, b] \to X$ is a continuous function and y is a point of X lying between f(a) and f(b), then there exists a point $x \varepsilon [a, b]$ such that $f(x)=y$.

We will prove that,

Result. 2.3.8.

Every linear continuum is Sharkovsky space.

To prove this result, we want the following results.

Result. 2.3.9.

Let X be a linear continuum. Let I and J be closed intervals in X and let f: X→X be a continuous function. If J⊆f (I), then there exists a closed interval Q⊆I so that

f(Q)= J.

Proof. We take two points p, q ∈ I with p<q so that J= [f (p), f(q)] or J= [f (q), f(p), and define , p ≤ r<q , by r= sup [x ε [p, q] | f(x)= f(p)}. We claim that f(r)= f (p). For otherwise we can find (as X is T_2) an open set V so that f(r) ∈V and f (p) ∉V. and by continuity of f an open neighborhood Uof r such that f(U) ⊆ V. As X is order dense, we can choose p≤ r'<r with [r' r] ⊆U, hence f [r' r] ⊆V implies f (p) ∉ f ([r', r]). But this contradicts the definition of r as a sup. If we define r < s ≤ q by s = inf {x∈[r, q] | f(x)= f(q)}, then f(s)= f(q) can be proved analogously. Let Q= [r, s] ⊆I. We prove that f(Q)=J.

Result. 2.3.7 Applied to f ([r, s]) shows that J⊆f ([r, s]). But we also have f ([r, s]) ⊆J, for otherwise there exists. r<x<s with f(x) ∉J. if f(x) <f(p) <(q) or f(q) < f(p) < f(x), then result. 2.3.7 applied to f | [x, s] asserts that f(p)= f(x') for some r<x<x'<s, which contradicts the definition of r as a sup. If f(x)<f(q)<f(p) or f(p)<f(q)<f(x), then the result. 2.3.7 applied to f | [r, x] yields a contradiction to the definition of s as an infemum. Therefore no such point can exist. So, J=f(Q).

Result 2.3.10.

Let X be a linear continuum. Let I be a closed interval in X and $f:X \to X$ be a continuous function If $I \subseteq f(I)$, then f has a fixed point in I.

Proof : We use Result. 2.3.9 to choose a closed interval $Q \subseteq I$ with $f(Q)=I$, and we will show that f has a fixed point in Q.

If possible assume that f/Q has no fixed point. Than $Q \subseteq AUB$, where $A = \{x \in X | x<f(x)\}$ and $B= \{x \in X | f(x)<x\}$. To see that A is open, let $x \in A$ be a point such that $x<z<f(x)$ and an open neighborhood $U \subseteq (^{-}\infty,z)$ of x with $f(U) \subseteq (z, \infty)$. Then $U \subseteq A$, so x is an interior point of A. Similarly B is open. Hence $(Q \cap A)$and $(Q \cap B)$ are open in Q, and $Q = (Q \cap A) U (Q \cap B)$. To see that $Q \cap A \neq \phi$, let $I= [c,d]$. As $f(Q)= I$, there exists $x' \in Q$ with $f(x')= d$, and if f/Q has no fixed point then $x' \neq d$. As $Q \subseteq I$, we have $x'<f(x')= d$, so

$x' \in Q \cap A$. Analogously we can find $x'' \in Q-\{c\}$ with $f(x'')= c$ and $x'' \in Q \cap B$. Hence $Q \cap A$ and $Q \cap B$ form a separation of the connected set Q, which is impossible

$\therefore$ f must have a fixed point in Q .

Result 2.3.11.

Let X be a linear continuum and $f: X \to X$ be a continuous function. If f has a periodic point, then f has a fixed point.

Proof. Let $f: X \to X$ have a point of period $k>1$, and choose a in the sequence a, f(a), $f^2(a), \ldots, f^k(a)= a$, so that $a < f(a)$. As the orbit of a returns to a, these exists a point $b= f^i(a)$, for $i= 1,2, \ldots, k-1$, with $f(b)<b$. As in the proof of Result 2.3.10 it follows that the sets $A= \{x \in X | x<f(x)\}$ and $B= \{x \in X | f(x) < x\}$ are open. If f has no fixed point, then $X= AUB$, hence $a \in A$, $b \in B$ and $A \cap B= \phi \Rightarrow$ A and B form a separation of the connected space X, which is a contradiction. So f must have a fixed point.

Proof of Result. 2.3.8 can now he completed analogously as proof of theorem 3.3 in [67]. In that proof replace R by X. since there is nothing new in the proof we are not giving it now.

Result 2.3.12.

Every sharkovsky space is connected.

Proof. Let X be a Sharkovsky space. We have to prove that X is connected. If X is not connected, then $X = A \cup B$, where A and B are non-empty closed subsets of X with $A \cap B = \phi$, We choose points $a \in A$ and $b \in B$, and define $f: X \to X$ by,

$$f(x) = \begin{cases} b, \forall x \in A \\ a, \forall x \in B \end{cases}$$

Then f is continuous by the pasting lemma and a and b are points of period two. But f has no fixed point and 2<1. So X is not a Sharkovsky space, which is a contradiction. ∴ X must be connected.

Next set of examples show that we can not avoid the assumption of a linear continuum in Result. 2.3.8.

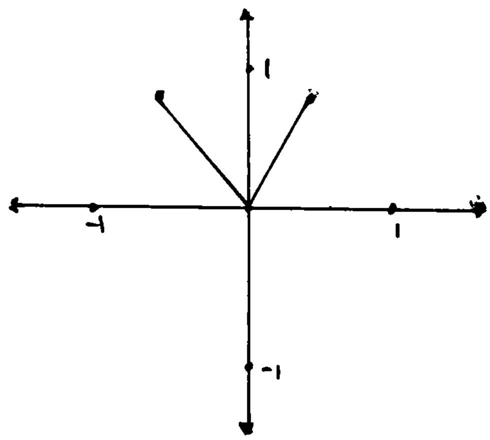
Example 2.3.13.

(i) Let $X= (-1,0) \cup (0,1)$. X is order dense but lacks the sup. property clearly X is not a sharkovsky space, as it is not connected.

(ii) Let $X= [-2, -1] \cup [1,2]$. X has sup. property but is not order dense. X is not a sharkovsky space.

(iii) Let X be the space which consist of three line segments which join the point (0,-1), $(\frac{\sqrt{3}}{2}, \frac{1}{2})$, $(\frac{-\sqrt{3}}{2}, \frac{1}{2})$ to the origin. A partial order which is order dense and has the sup. property is defined if we write $(x, y) \leq (x', y')$ when ever $y \leq y'$.

Fig.



Let f: X→X be the rotation through $\frac{2\pi}{3}$ with the origin as centre. Then all point of X \ {(0,0)} are point of period three and the origin is a fixed point .ie. f has no points of period n where n ∉ {1,3} in X, so X is not a Sharkovsky space.

Next we consider topological entropy.

2.4. Topological entropy

Let X be a chaos space and X ∈ CH(X). Then there may be so many chaotic function on X. Topological entropy measures complexity of those chaotic functions on X. In [2] Lemma 4.1.2 it has been proved that top (f^n)= n top (f) and Lemma 4.4.19 states that top (f) > 0 iff f has periodic points of period 2^n, ∀ n ε Z_+. Now, topological entropy can be considered as a function from C (X, X) to [0,∞]. So given a function from C(X,X) to [0,∞] whether it is the topological entropy or not is the question of our interest. ie. Can we characterize topological entropy? We can't give a full answer to this question. But we know that I= [0,1] ∈ CH(IR). We give a characterization of topological entropy as a function from C(I,I) to [0, ∞].

Note that the ordering we defined in definition. 2.3.1 is called sharkovsky ordering

Now we define,

Definition 2.4.1.

For $t \in N \cup \{2^{\infty}\}$, we define $N(t) = \{k \in N | k \leq t\}$.

Notation 2.4.2.

Per (f) = set of all periods of f.

Definition 2.4.3.

Let f: $X \rightarrow X$ then f is said to be of 'type t' if per (f) = N(t). The class of all maps $f \in C(X,X)$ such that type (f) = t (respectively, type (f) < t, type (f)≤t, type (f) > t, type (f) ≥ t) will be denoted by T(t) (respectively, T (<t), T (≤ t) T(>t), T(≥ t)) .

Remark 2.4.4.

Let ψ: C (X, X)→ [0,∞]. Then we say that "ψ characterizes BC" if ψ (f) > 0 is equivalent to top (f) > 0.

Result. 2.4.5

Let ψ: C(X, X) → [0,∞] satisfy the following properties.

(i) ψ is lower semicontinuos

(ii) $\psi(f^n) = n.\ \psi(f)$ for $n \geq 0$

(iii) $\psi(f) = 0$ when ever $f \in T(1)$

(iv) $\psi(f) > 0$ when ever f= gog where $g \in T(3)$

Then ψ characterizes BC.

Proof. Let $f \in T(\leq 2^{\infty})$. We will first prove that ψ (f)= 0. If $f \in T(1)$ then there is nothing to prove because of (iii) .

If $f \in T(2^n)$ then $f^{2n} \in T(1)$. So by (ii) and (iii) $\psi(f) = \frac{1}{2n}\psi(f^{2n}) = 0$.

If $f \in T(2^{\infty})$ then clearly $f \varepsilon \overline{\{f \mid \text{type}(f) < 2^{\infty}\}}$ so by (i), $\psi(f) = 0$.

Hence we have proved that $\psi(f) = \text{top}(f) = 0$ for all maps f such that type $(f) < 2^{\infty}$.

We have from [69] top(f)>0 iff $f \in T(> 2^{\infty})$. So we have proved.

2.5. A measure on chaos spaces.

Suppose X is a chaos space we can measure the measure of complexity of the space through the chaotic function on X.In this section we will find measure of some chaos space.

Definition. 2.5.1.

Let X be a chaos space and $F \varepsilon$ CH (X)

Let $\mu(F) = \text{Inf}\{\text{top}(f) \mid f \varepsilon C(F)\}$.

Definition 2.5.2.

A tree X is any space which is uniquely arc wise connected and homeomophic to the union of finitely many copies of the unit interval.

By result. 2.2.31. f is chaotic on I iff f is transitive on I. So we have,

Result. 2.5.3.

Let X be a tree and $f: X \to X$ be continuous then $f \varepsilon C(X)$ iff f is transitive on X.

In [3] it has been proved that if $f: I \to I$ is transitive, then top $(f) \geq (\text{Log}2)/2$.

So we have,

Result. 2.5.4.

$\mu(I) \geq (\text{Log}2)/2$.

In [3] it has been proved that, top (f) $\geq \frac{1}{n}\log 2$, where f: X $\rightarrow$ X, X is a tree with n end points. So more generally.

Result. 2.5.5.

Let X be a tree with n end points, then μ (x) $\geq$ (Log2) /n .

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CHAPTER 3

PARTIAL SELF SIMILARITY, SUPER SELF SIMILARITY, SUBSELF SIMILARITY AND FRACTALS

There are three sections in this chapter. In the first section we define Partial self similarity . Super self similar sets are defined in section two. In section three we prove that chaotic self-similar sets are dense.

3.1. Partial self similarity [1]

Definition 3.1.1.

Let (X,d) be a metric space and $K \subseteq X$. K is called Partial self similar if there are sets $K_1, K_2, \ldots, K_t$ such that $K = \bigcup_{i=1}^{t} K_i$ and for each K_i, there are contraction maps ϕ_{ijk}, for i= 1, ...t, j= 1,...,t and k= 1, ... w(i,j) , with w (i,j) > 0 such that $K_i = \bigcup_{j,k} \phi_{ijk}(K_j)$.

This partial self similarity can be represented by means of a directed graph. The vertices of the graph correspond to the sets $K_1, K_2, \ldots, K_t$. Each K_i is a union of contracted copies of the K_j s and there is one directed edge from K_i to K_j for each contraction map ϕ_{ijk}. Note that the contraction ϕ_{ijk} maps K_j to K_i but that the arrows goes from the node K_i to the node K_j.

If the directed graph has only one node then the set is strictly self similar.

Fig.

Suppose that each contraction map ϕ_{ijk} in a partial self similarity, is a similitude with ratio r_{ijk} suppose the set of similitudes satisfies the open set condition, so that K_i do not overlap too much. Then a similarity argument shows that the similarity dimension of the set K is the largest value of x such that the t × t matrix, whose $(i,j)^{th}$ element is $\sum_{K} r_{ijk}^{x}$, has one as an eigen value[33].

If all the similitudes have the same ratio, so that $r_{ijK} = r$ for all i, j and k then this similarity dimension is $\frac{\log \lambda}{\log\left(\frac{1}{r}\right)}$, where λ is the dominant eigen value of the matrix whose (i,j)th element is N_{ij}, the number of edges from the node K_i to node K_j. Our aim is to compare the Hausdorff dimension of the partial self similar set and its similarity dimension.

We will prove the following,

Result 3.1.2.

Let (X,d) be a complete separable metric space. Let K be a partially self similar set with similarity dimension s. then $\dim_H(K) \leq s$.

Proof . Since most of the result in this area use 'words' we use them here also. Consider the graph corresponding to a partial self similar set write $E_{uv}^{(w)}$ for the set of all infinite words, using symbols from E, where the initial vertex of the first edge is u and the terminal vertex of each edge is the initial vertex of the next edge. A Path in the graph is a finite string $\alpha = e_1, e_2, \ldots\ldots e_k$ of edges, such that terminal vertex of each edge e_i is the initial vertex of next edge e_{i+1}. The initial vertex of α is the terminal vertex of last letter e_k. We write $E_{uv}(k)$ for the set of all paths of length k that begin at u and end at v, and

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[1] Some results in section 3.1. is to be published in *Fractals* .

$E_u^{(k)}$ for the set of all paths of length K that begin at u. $E_u^{(*)}$ is the set of all finite paths of any length that begin at u and $E^{(*)}$ is the set of all finite paths. By convention, $E_u^{(0)}$ consists of a single "empty path" λ_u, of length zero, from node u to itself. If $\alpha \in E_{uv}^{(k)}$ and $\beta \in E_{vw}^{(n)}$, then we write $(\alpha\ \beta)$ for the path made by concatenation of α and β, so that

$(\alpha\ \beta) \in E_{uw}^{(k+n)}$. A partial order may be defined on $E^{(*)}$ as follows. Write $\alpha \le \beta$ iff α is a prefix of β . ie $\beta = (\alpha\ \gamma)$ for some path γ. With this ordering, $E_u(*)$ becomes a tree $\forall$ u $\in$V. α and β are incomparable iff neither is a prefix of the other $\bar{\alpha} = \alpha \mid k-1$, obtained by omitting the last letter of α. A 'cut' is a finite set $T \in E^*$ such that for every infinite word $\sigma \in E^{(w)}$ there is exactly one n with the restriction $\sigma/n \in T$.

Let $\delta > 0$ be fixed.

$T = \{ \alpha \mid u,v \in V\ \alpha \in E_{uv}^{(*)}, r_\alpha \le \delta\}$ is a cut

For each $\alpha \in T$, we have,

$\delta .\ r_{min} \le r_\alpha \le \delta$.

Let us estimate cardinalities of T_u,

$$\lambda_u = \sum_{u \in V} \sum_{\alpha \in T_{uv}} (r_\alpha)^s \lambda_v$$

$$\ge (\delta r_{min})^s .\lambda_{min} \mid T_u \mid$$

$$\therefore \mid T_u \mid \le (\delta\ r_{min})^{-s} . \frac{\lambda_{max}}{\lambda_{min}}, \forall u \in V,$$

But, $K_u \subseteq \bigcup_{\alpha \in T_u} \alpha K_u$ (Here $K = \cup K_u$ where $K_u = \bigcup_{v \in V, e \in E_{uv}} f_e(K_v)$).

and, diam $(\alpha K_v) \le r_\alpha$ diam $(K_v) \le \delta.d_{max}$. Let $\eta = \delta.d_{max}$. So considering number of boxes of K_v that covers K_v and their diameter less than $\delta.d_{max}$, we have,

$$N_\eta(K_u) \le \left(\frac{\eta . r_{min}}{d_{max}}\right)^{-s} . \frac{\lambda_{max}}{\lambda_{min}}$$

Taking logarithm and dividing by $-\log \eta$ and let $\eta \to 0$ we get,

$\dim_B(K_u) \le s$.

But from [35] $\dim_H(X) \le \dim_B(X)$.

$\therefore \dim_H (K_u) \le s$.

Since X is separable K is also separable . Assume that $\dim_H(X) < \infty$. Then $\dim_H(K) = \sup_u \{\dim_H(K_u)\}$.

$\therefore \dim_H (K_u) \le S$.

Result 3.13

Let (X, d) be a complete metric space and $K \subseteq X$ such that $K = \bigcup_u K_u$ where

$K_u = \bigcup_{v \in V, e \in E_{uv}} f_e(K_v))$ where

i) $d(f_e(x), f_e(y) \ge r_e' . d(x, y)$ and

ii) For any $u, v, v' \in V$, $e \in E_{uv}$, $e' \varepsilon E_{uv'}$, $e \ne e'$ we have $f_e(K_v) \cap f_{e'}(K_{v'}) = \phi$

Then, $\dim_H(k) \ge S$., where S is the similarity dimension of K.

Proof. Since (f_e) are disjoint there is $\eta > 0$ so that $d(f_e(k_v), f_{e'}(k_v)) > \eta, \forall u\ v, v', e, e'$.

we claim that if $\alpha$and α' are not comparable in $Eu^{(*)}$ then,

$d(\alpha K_v, \alpha' K_{v'}) > r'_{\bar{\alpha}} . \eta$

For let γ be the longest common prefix of $\alpha$and α',since $\alpha$and $\alpha'$are in comparable, they are strictly stronger than γ. So, $\gamma \le \bar{\alpha}$, hence $r'_\alpha \le r'_\gamma$ so there exist e and e' such that $\alpha = \gamma e\ \beta$, $\alpha' = \gamma e'\ \beta'$. Let $\beta\ \varepsilon\ E_{wv}^{(*)}$ and $\beta'\ \varepsilon\ E_{w'v'}^{(*)}$. Then $\beta K_v \le K_w$ and $\beta' K_{v'} \subseteq k_{w'}$. since $e \ne e'$. we have

$d(e\ \beta\ K_v, e'\beta'\ K_{v'}) \ge d(eK_w, e'\ k_{w'}) > \eta$

$\therefore d(\gamma e\ \beta\ K_v, \gamma e'\ \beta'\ K_{v'}) \ge \gamma_{r'} . \eta \ge r_{\bar{\alpha}'} . \eta$

Let $B \subseteq X$ be a broel set and $\delta = \dfrac{diam(B)}{\eta}$.

Let $T = \{\alpha \mid u \in V, v \in V, \alpha \in E_{uv}^{(*)}, r_\alpha' \le \delta \le r_{\bar{\alpha}}'\}$

Then if 'C' is a countable cover of K_u,we have

$$\sum_{B\in\mathcal{C}} (diam B)^s \geq \left(\frac{\eta^s}{\lambda_{\max}}\right) \cdot \sum \mu_u \left(h^{-1}(B)\right)$$

$$\geq \frac{\eta^s}{\eta_{\max}} \cdot \mu_u(K_u)$$

$$\therefore H^s(K_u) \geq \frac{\eta^s}{\eta_{\max}} \cdot \mu_u(k_u) > 0$$

$$\therefore \dim_H(K_u) \geq S$$

$$\therefore \dim_H(K) \geq S$$

From result 3.1.2 and 3.1.3 we have ,

Result 3.1.4

Let (X, d) be a complete separable metric space and $K \subseteq X$ be partialy self similar with $K = \cup K_i$ where $K_i = \bigcup_{jk} \phi_{ijk}(K_j)$ where each $\phi_{ijk}{}^s$ are similarities with

$\phi_{ijk}(K_j) \cap \phi_{ijk}(K_j) = \phi$; then similarity dimension of K coincides with Hausdorff dimension of K.

3.2. Super self similar sets

We know that self similar sets are introduced by Hutchinson and sub self similar sets by K. Falconer.Reversing the inclusion in the definition of subself similar sets we define super self similar sets.

Definition 3.2.1

Let (X, d) be a metric space and $E \in K(X)$. E is called super self similar if there are contractive similarities $f_i : X \to X$ for i = 1, ... , n such that $E \supseteq \bigcup_{i=1}^{n} f_i(E)$.

Result 3.2.2.

No finite set in K (R^n) with more than one element is super self similar.

Proof Let F be a finite set with more than one element and let $\{f_i\}_{i=1}^{m}$ be contractive similarities. We will show that $F \not\supseteq \bigcup_{i=1}^{m} f_i(F)$. Let x be the fixed point of f_1 and let $y \in F$ such that $|x-y| = d(x, F\setminus\{x\})$ then $f_1(y) \notin F$. so $F \not\supseteq \bigcup_{i=1}^{m} f_i(F)$.

Since finite sets are dense in $K(R^n)$ we have,

Cor. 3.2.3.

The set of non super self similar sets is dense in $K(R^n)$.

Result. 3.2.4.

The set of sub self similar sets can be expressed as the countable union of closed, no where dense subsets of $K(R^n)$.

Proof. For m, n ε N, define $S_{m,n}$ to be the set of all those sub self similar sets E such that there exists contractive similarities $\{f_i\}_{i=1}^{m}$ with $\frac{1}{n} \leq r(f_i) \leq 1-\frac{1}{n}$, $E \subseteq \bigcup_{i=1}^{m} f_i(E)$ and

$|f_i(0)| \leq n$ for every $i \in \{1,\ldots m\}$. Clearly $\bigcup_{m=1}^{\infty}\bigcup_{n=1}^{\infty} S_{m,n}$ is precisely the set of sub self similar sets. We prove that $S_{m,n}$ is closed for every m, n $\in$ N. suppose that $E_k \to E$ in the Hausdorff metric, where $E_k \in S_{m,n}$ for every k $\in$N. To each E_k corresponds $\{f_i^k\}$ such that $\frac{1}{n} \leq r(f_i^k) \leq 1-\frac{1}{n}$, $E_k \subseteq \bigcup_{i=1}^{m} f_i^k(E_k)$ and $|f_i^k(0)| \leq n$. Using the standard matrix, vector representation of an affine transformation each f_i^k may be associated with a point, x_i^k, in R^{n^2+n}. The condition on each f_i^k ensure that the set of all such points, K, is compact. By recursively choosing successively finer susbsequences, we may assume that each sequence $\{x_i^K\}_{k=1}^{\infty}$ is convergent to say $x_i \varepsilon$ K. Each point x_i in turn defines a contractive similarity $f_i: R^n \to R^n$ satisfying $\frac{1}{n} \leq r(f_i) \leq 1-\frac{1}{n}$ and $|f_i(0)| \leq n$ for ever i ε {1, .., m}. The correspondence between affine transformations on R^n and points in R^{n^2+n}, along with the continuity of the algebraic operations, implies that $f_i^k \to f_i$ pointwise as $k\to\infty$. We must now show that $E \subseteq \bigcup_{i=1}^{m} f_i(E)$. Let $x \in E$. Then for every $k \in N$, there is an $x_k \in E_k$ such that the sequence $\{x_k\}_{k=1}^{\infty}$ converges to x. Since

$E_K \subseteq \bigcup_{i=1}^{m} f_i^k(E_k)$, there is an

$i_k \in \{1,\dots,m\}$ such that $x_k \in f_{i_K}^k(E_k)$. Since there are only finitely many choices for i_k, at least one must occur infinitely often. Thus we have a subsequence $\{K_j\}_{j=1}^{\infty}$ and a fixed

I $\in$ {1,..,m} such that i_{kj} = i for every j. Along this subsequence we have $f_{i_K}^{Kj}(E_{Kj})= f_i^{Kj}(E_{Kj}) \to f_i(E)$ as $j \to \infty$, since $f_i^{Kj} \to f_i$ point wise and $E_{kj} \to E$ in the Hausdorff metric. Thus $x \in f_i(E)$ since $x_{kj} \to x$ and $x_{kj} \in f_i^{Kj}(E_{kj})$, $\forall$ j. $S_{m,n}$ is closed and contains no open set because $S_{m,n}$ is dense in R^n by cor. 3.2.3 .

Result. 3.2.5.

The set of super self similar sets can be expressed as the countable union of closed, no where dense subsets of $K(R^n)$.

Proof. The proof is similar to that of result 3.2.4.

Notation 3.2.6

Let (X, d) be a complete metric space

S= $\{F \subseteq X \mid$ F is self similar $\}$

$S^* = \{F \subseteq X \mid F$ is super self similar$\}$

$S_* = \{F \subseteq X \mid F$ is sub self similar$\}$

Result 3.2.7.

(i) $(S_* \backslash S) = S_*$, in R^n

(ii) $(S^* \backslash S) = S^*$, in R^n

Proof. Proof of (i) is simple because any finite set is subself similar and finite sets are dense in $K(R^n)$ and so dense in $(S_* \backslash S)$.

To prove (ii). It suffices to find a class of super self similar sets which are not self similar, but are dense in $K(R^n)$. Note that $\overline{S} = K(R^n)$ ([35]). Let E be self similar for the transformations $\{f_i\}_{i=1}^m$. Choose R>0 such that $f_i(B_R(0)) \subseteq B_R(0)$ for each i ε {1,2,.., m}. Let $E_1 = \bigcup_{i=1}^{m} f_i(B_R(0))$ and for n>1, let $E_n = \bigcup_{i=1}^{m} f_i(E_{n-1})$. Then each E_n is super self similar, but not self similar and $E_n \rightarrow E$ in the Hausdoff metric. So we can approximate a self similar set by a sequence of super self similar sets which are not self similar.

In the next section we prove some results on self similar sets.

3.3. Self Similar Sets[2]

We know that I=, [0,1] is a self sinmilar set. Is true for S^1? we will prove tat S^1 is a self similar set. ie. There are contractive mappings $\{f_i\}_{i=1}^n$ from S^1 to S^1such that

[2] Some results in section 3.3 is to be published in Journal of Mathematical Analysis and Applications.

Result 3.3.1.

S^1 is a self similar set.

Proof. Topologicaly $S^1 = A \cup B \cup C \cup D$ where

$A = [0,1] \times \{0\}$, $B = \{1\} \times [0,1]$, $C = [0,1] \times \{1\}$ and $D = \{0\} \times [0,1]$

Let ϕ_θ, for $\theta = -\pi/2, \pi/2, \pi$ be the antilock wise rotation centred at $(\frac{1}{2}, \frac{1}{2})$ of angle θ.

Note that ϕ_θ is a map from S^1 to S^1.

Let f_1: $S' \to B$ be defined as,

$$f_1(x,y) = \begin{cases} (1, \frac{2}{3}(\frac{1}{2} - x)) \text{ if } (x,0) \in A \text{ and } x \leq \frac{1}{2} \\ (1, \frac{2}{3}(x - \frac{1}{2})) \text{ if } (x,0) \in A \text{ and } x \geq \frac{1}{2} \\ (1, \frac{1}{3}(1+y)) \text{ if } (1,y) \in B \\ (1, 1-\frac{2}{3}(x - \frac{1}{2})) \text{ if } (x,1) \in C \text{ and } x \geq \frac{1}{2} \\ (1, \frac{2}{3}(x+1)) \text{ if } (x,1) \in C \text{ and } x \leq \frac{1}{2} \\ (1, \frac{1}{3}(1+y)) \text{ if } (0,y) \in D \end{cases}$$

let f_2: $S^1 \to C$ be defined as

$f_2 = \phi_{\pi/2} \circ f_1 \circ \phi_{-\pi/2}$

f_3: $S^1 \to D$ be defined as,

$f3 = \phi_\pi \circ f_1 \circ \phi_\pi$

and f_4: $S^1 \to A$ be defined as

$f_4 = \phi_{-\pi/2} \circ f_1 \circ \phi_{\pi/2}$

Now we define d_1, as

$$d_1((x,0)\,(x', y')) = \begin{cases} x' - x \text{ if } (x', y') \varepsilon A \text{ and } x \le x' \\ 1-(x - y') \text{ if } (x', y') \in B. \\ 3-(x + x') \text{ if } (x', y') \in C \\ 4-(x + y') \text{ if } (x', y') \in D \\ 4-(x-x') \text{ if } (x', y') \in A \text{ and } x' < x \ . \end{cases}$$

and

$$d_1((x, y), (x', y')) = \begin{cases} d_1(\phi_{\pi/2}(x,y), \phi_{-\pi/2}(x', y')) \text{ if } (x,y) \varepsilon B \\ d_1(\phi_\pi(x, y), \phi_\pi(x', y')) \text{ if } (x,y) \varepsilon C \\ d_1(\phi_{\pi/2}(x,y), \phi_{-\pi/2}(x', y')) \text{ if } (x,y) \varepsilon D \end{cases}$$

Associated wit this d, and for a fixed $v \varepsilon S^1$ we can define '$\le$' on S^1 as,

$Z \le_v w$ if $d_1(v, z) \le d_1(v, w)$

This 'd', is a quasimetric because need not be symmetric on S^1.

Let $d_2 = d_1^{-1}$ be the conjugated quasimetric of S^1

Let $d(z, w) = \min\{d_1(z, w); d_2(z,w)\}$

Now this 'd' is a metric on S^1.

(i) $d(x,y) = 0$ iff $\min\{d_1(z, w); d_2(z,w)\} = 0$ iff $z = w$

(ii) since $d_2(z,w) = d_1^{-1}(z,w) = d_1(w,z)$ then $d(z,w) = \min\{d_1(z, w); d_1(w,z)\} = d(w,z)$

(iii) suppose $d(z,w) = d_1(z,w)$

if $t \le_z w$ then $d(z,t) = d_1(z,t)$, $d(t,w) = d_1(t,w)$ and then $d(z,w) = d_1(z,w) = d_1(z,t) + d_1(t,w) = d(z,t) + d(t,w)$.

<u>Case. 1</u>

$d(z, t) = d_2(z, t)$ and $d(t, w) = d_2(t, w)$.

Then $d(z, w) \leq d_2(z, t) \leq d_2(z, t) + d_2(t, w) = d(z, t) + d(t, w)$.

Case. 2

$d(z, t) = d_2(z, t)$ and $d(t, w) = d_1(t, w)$

Since $w \leq_z t$ then $d_1(z, w) \leq d_1(t, w)$ and then

$d(z, w) = d_1(z, w) \leq d_1(t, w) = d(t,w) \leq d(z, t) + d(t, w)$.

Case. 3

$d(z, t) = d_1(z, t)$ and $d(t, w) = d_2(t, w)$.similar to case 2.

Case. 4

$d(z, t) = d_1(z, t)$ and $d(t, w) = d_1(t, w)$ similar to case 1.

In any case we have that $d(z, w) \leq d(z, t) + d(t, w)$

(iv) The case $d(z, w) = d_2(z, w)$ is analogous to(iii)

Let $A_1 = [0, \frac{1}{2}] \times \{0\}$,

$A_2 = [\frac{1}{2}, 1] \times \{0\}, C_1 = [0, \frac{1}{2}] \times \{1\}$

$C_2 = [1/2, 1] \times \{1\}$.

Let $a_1 = (0,0)$, $a_2 = (\frac{1}{2}, 0)$; $a_3 = (1,0)$

$a_4 = (0,1)$; $a_5 = (\frac{1}{2}, 1)$ and $a_6 = (0,1)$

'd' is compatible with the topology of S^1.

let $x \in S^1$, and let $r = \min \{d(x,a_1), d(x, a_3), d(x, a_4) d(x,a_6)\}$. Then $B_{du}(x, \delta) = B_d(x, \delta); \forall \delta \leq r$, were d_u is the usual metric on S^1. This proves that the identify map is a local isomety between the two metrics.

On the other hand, it is clear that ϕ_θ is d- isometric for $\theta = \pi/2, \pi, -\pi/2$ since d_1 is defined using ϕ_0 we prove that f_1 is a contraction so that all f_i s' for ϕ_θ is a d-isometry for $\theta = \pi/2, \pi, -\pi/2$.

1. If z, w are both in the same set A_1, A_2, C_1, C_2, then it is clear that $d(f_1(z), f_1(w)) = \frac{2}{d} d(z, w)$ and if z, w are both in B or D then $d(f_1(z), f_1(w)) = 1/3\, d(w, z)$.
2. Let $z \varepsilon A_1$ and $w \varepsilon A_2$. Then $d(f_1(z), f_1(w)) \le d((f_1)(z), f_1(a_2)) + d(f_1(a_2), f_1(w)) = \frac{2}{3}(d(z, a_2) + d(a_2, w)) = \frac{2}{3} d(z, w)$. The cases $z \in A_2$ and $w \in A_1$; $z \varepsilon C_1$ and $w \in C_2$; $z \in C_2$ and $w \in C_1$ are analogous to this. Therefore, if z w are both in the same set A,B,C, D then we have that $d(f_1(z), f_1(w)) \le \frac{2}{3} d(z, w)$.
3. Let $z \in A$ and $w \varepsilon\in B$. then $d(f_1(z), f_1(w)) = d(f_1(z), f_1(a_3)) + d(f_1(a_3), f_1(w)) \le \frac{2}{3}(d(z, a_3) + d(a_3, w)) = \frac{2}{3} d(z, w)$. the cases $z \in B$ and $w \in A$, $z \in C$ and $w \in B$; $z \in B$ and $w \in A$; $z \in A$ and $w \in D$; $z \in D$ and $w \in A$; $z \in C$ and $w \in D$; $z \in D$ and $w \in C$ are analogous to this.
4. Let $z \in A$ and $w \varepsilon C$. if $d(z, w) = d_1(z,w)$ then $d(f_1(z), f_1(w)) \le d(f_1(z), f_1(a_6)) + d(f_1(a_6), f_1(w)) \le \frac{2}{3}(d(z, a_6) + d(a_6, w)) = \frac{2}{3} d(z, w)$. If $d(z, w) = d_2(z, w)$ the reasoning is analogous. The cases $z \in C$ and $w \in A$; $z \in B$ and $w \in D$; $z \in D$ and $w \in B$ are analogous to this.

So f_1 is a contractive map with factor of contractivity $\frac{2}{3}$ and hence

$S^1 = f_1(S^1) \cup f_2(S^1) \cup f(S^1) \cup f_4(S^1)$ $\therefore$ S^1 is self similar.

In [35] falconer proved that self simil sets in R^n are dense ie. In $((R^n, d)$, $\overline{S} = K(R^n)$

Next, we prove that chaotic self similar sets are dense in R^n ie $\overline{(CH(IR^n)) \cap S} = K(R^n)$

Result 3.3.2.

Chaotic self similar sets are dense in R^n

Proof Consider R . $[0,1] \in CH(R)$. so any closed internal is in CH (R). Collection of closed intervals is dense in K (R) , so $\overline{CH(IR)} = K(R)$. Hence $\overline{CH(IR^n)} = K(R^n)$

Since we have $\bar{S} = K(R^n)$, $\overline{(CH(IR^n) \cap S)} = K(R^n)$.

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CHAPTER – 4

FRACTALS IN TOPOLOGICAL SPACES

We know that some Julia sets (Def.1.2.6) in C are fractals. Note that C is a compact chaos space. So it is interesting see how fractals behave in chaos spaces. But the concept of fractals can't be extended to general topological space fast it involves Hausdorff dimensions. In this chapter we define a set which can be considered as a generalization of Julia sets. We also study some chaotic properties of these sets. We also give an example to show that if $f_\lambda : \bar{C} \to \bar{C}$ are functions which converges to f then it need not be true that $J(f_\lambda) \to J(f)$ in K (C). We also define a set which is a generalization of the classical Mandelbrot set.

4.1 A Generalization of Julia Set

Julia set of a function f in $\bar{C}$ is defined as the collection of points at which $\{f^n\}$ is not normal. But according to Montel's theorem, if f is meromorphic then $\{f^n\}$ is normal at z iff $\{f^n\}$ is equicontinous at z. Motivated by this we define,

Definition 4.1.1

Let (X, d) be a compact metric space and f: X→ Xbe continuous. Let

$E(f) = \{x \in X | \{f^n\} \text{is equicontinuous at } x\}$

Let $E^*(f) = X \ (E(f))$.

From Montel's theorem we have,

Result 4.1.2:

Let f: $\bar{C} \to \bar{C}$ be meromorphic, then $J(f) = E^*(f)$

Result 4.1.3.

Let (X, d) be a compact metric space and f: X→X be continuous. If f is stable at x ε X then x ε E (f).

Proof. clear from the definition of stability and E(f) .

Cor. 4.1.4

If (X,d) is compact and f: X→X is continuos, then {x x|f} if is stable at x} ⊆ E(f).

Result 4.1.5.

Let (X, d) be a compact Metric space and f: X→X be continuous. If f is transitive and every point is stable then $D_f(X) = X$ and f is a homeomorphism.

Proof. We first prove that $D_f(X) = X$.

For that, let x ε $D_f(X)$. Let y ε X.Then there are sequence $\{K_i\}$ and $\{l_i\}$ of positive integers such that $f^{ki}(x) \to y$, and $f^{ki+li}(x) \to x$. Since y is a stable point , $d(f^{li+ki}(x), f^{li}(y))= d(f^{li}(f^{ki}(x), f^{li}(y)) \to 0$, so $d(x,f^{li}(y)) \to 0$, and $x \in O_f(x) \in \overline{O_f(x)}$, so $y \in D_f(X)$.

$\therefore X \subseteq D_f(X)$.

But always, $D_f(X) \subseteq X$

Hence $D_f(X) = X$.

To show that f is a homeomorphism ,

Consider the space C(X,X) with the topology of pointuise convergence

Let G= $\overline{\{f^n \mid n = 1,2,\ldots,,,,\}} \subseteq C(X,X)$

Since $\{f^n\}$ is equicontinuous, all elements of G are continuous. If $x,y \in X$, there is a g $\in G$ such that g(y) = x (because $\overline{O_f(y)} = X$ and C(X,X) is compact). Also gof =fog. (because, let $\{f^{ni}\}$ be a net such that $f^{ni} \to g$. Then $f^{ni+1} = fo f^{ni} \to fog$ and $f^{ni+1} = f^{ni} \to gof$) now, let $x \in X$, and let y = f(x). Let $g \in G$ be such that g(y) = x. then (fog) (x) = (gof) (x) = g (y) = x. Also, $gof^k = f^k og$, $\forall k > 0$ and (gof) $f^k(x) = f^k(gof (x) = f^k(x)$.

similarly, (fog) $f^k(x) = f^k(x)$. Hence fog = gof is the identity on $O_f(x)$. But $\overline{of(x)} = X$, fog = gof is the identity map. So f is inevitable.

Hence f is a homeomorphism.

Result 4.1.6.

Let (X, d) be a compact metric space and f: X → X be transitive. Then,

1. $E(f) \neq \phi \Rightarrow E(f) = D_f(X)$
2. $E(f) = \phi \Rightarrow$ f is sensitive

Proof (1) Let $x \varepsilon D_f(X)$. Let $G_k = \{x \in X| \exists$ a neighborhood U of x such that $x_1, x_2 \in U \Rightarrow d(f^n(x_1), f^n(x_2) \leq \frac{1}{K}; \forall n \geq 0\}$. Since G_k is open there is an n>o such that $f^n(x) \in G_k$.

Since G_k is invariant, $x \in G_k$. Thus $x \in \cap G_k = E(f)$

$\therefore D_f(X) \subseteq E(f)$.

Since f is transitive. Then, $x \notin D_f(X) \Rightarrow O_f(x) \neq X \Rightarrow \overline{Lf(x)} \neq X \Rightarrow$ f is not equicontinuous at x.

$\therefore x \notin E(f)$.

So, $E(f) \subseteq D_f(X)$

Hence $D_f(X) = E(f)$

2) Each G_k is open and $E(f) = \bigcap_K G_K$.

If $E(f) = \phi$ then some G_k is non dense in X (by Baire category theorem) and so empty. (since if A is invariant, non- empty, open then f is transitive iff $\bar{A} = X$)

Note that if we define

$d_f(x,y) - \sup_{n\in N} d(f^n(x), f^n(y))$ ten d and d_f are equivalent if $\{f^n\}$ is equicontinuous. So $x \in G_k$ iff x has a neighbourhood whose d_f diameter is ≤ 1/k. so we have $G_k = \phi$, for some k. but every non- empty open set has d_f diameter greater than 1/k. so f is sensitive on X.

Cor.4.1.7.

Let (X, d) be compact and f: X→X be continuous and transitive. Then E*(f) = X => f is sensitive on X .

ie. E*(f)= X=> f ε S (x).

4.2. Julia sets on $\bar{C}$

Result 4.2.1

Let f: $\bar{C} \to \bar{C}$ be a meromorphic function. Then for every open set $V \subseteq \bar{C}$ satisfying $V\cap J(f) \neq \phi$, there exist some integer $n \in N$ such that $f^n(V\cap J(f))$ covers J(f) except at most two points.

Proof. Fix an open set $V \subseteq \bar{C}$ satisfying $V \cap J(f) \neq \phi$. Assume $V \subseteq \bar{C}$.

Case: 1

Assume $f^n/_v$ is analytic, ∀ n ε N.

Then $\{f^n/_v\}$n N can't miss 3 points because otherwise it forms a normal family => $V \cap J(f) = \phi$. Hence $f^n(V \cap J(f))$ cover J(f) except at most two points for some $n \in N$.

Case: 2

Assume that $f^n/_v$ is not analytic for some $n \in N$. then f as an essential singularity at ∞.

Assume that $f^{(n-1)}/_V$ to be analytic and $\infty \in f^{n-1}$ (V)= U. Due to open mapping theorem, U is an open neighborhood of ∞. But ∞ is an essential singularity of f.
Hence picards theorem => f (U\{∞}) to cover except at most two points.
$f^n(v \cap J(f))$ cover J (f) except at most two points _.

<u>Result 4.2.2</u>

Let f : $\bar{C} \rightarrow \bar{C}$be a meromorpic function let $0 < \delta < ½$ diam (J(f)) and x ε J (f) such that f^n is defined at x. Then for every open neighborhood U of x contains a point z ε U $\cap$ J(f) such that d ($f^n(x)$, f^n (z)) > δ, for some n ∈ N.

Proof. Let D = diam (J(f)).
Due to result 4.2.1. ∃ some n ∈ N ∍ f^n (U ∩ J (f)) covers J(f) except at most two points. Now we choose two point x_1, x_2 ∈ J (f) such that d (x_1, x_2) = D since J (f) is perfect,
∃ $\{w_m\}_{m\varepsilon N} \subseteq$ J (f) converging to x_1.
∴ $f^n(y_1) = w_{m1}$ and d ($f^n(y_1)$, x_1) < ½ (D-δ) for some m_1 ∈ N and some
y_1 ε J (f) ∩ Usatisfying $f^n(y_1) = W_{m1}$.
Similarly, we have some y_2 ∈ J (f) ∩ Usuch that d ($f^n(y_2)$, x_2) < ½ (d-s).
Now, d ($f^n(y_1)$, $f^n(y_2)$) > 2 δ.
∴ d ($f^n(z)$, $f^n(y_1)$) > δ or d ($f^n(z)$, f^n y_2)) > δ.

This result says that f is sensitive at x and for every neighborhood U of z_0 we can choose δ ε (0, ½ diam J(f)) as sensitivity constant. Next example show that the Result 4.2.3 can't be generalized into arbitrary spaces.

<u>Example 4.2.3.</u>

There exists a family $\{f_n\}_{n\varepsilon N}$ of chaotic (TC) mappings of the unit vide S' such that $(\frac{1}{n})$ is the optimal bound for the sensitivity constant of fn for every n ε N.

Let f : $\bar{R} \rightarrow \dot{R}$ be defined as,

$f(x) = 4x^3 - 12x^2 + 9x$. $f \varepsilon C(\bar{R})$ and $J(f) = [0,2]$.

Let X= $\bar{R}/_{2z}$ Then $X \cong S^1$ and f can be viewed as self mapping of $R/_{2z}$, diam (x)= 2. For $x \in \bar{R}$, let [x] denote the remainder with respect to division by 2, ie. $[x] \in R/_{2z}$, $x \equiv [x] \bmod 2$ and $x-[x] \in 2z$. fix $n \in N$ and define,

$f_n: X \to X$ as,

$$f_n(x) = \frac{1}{n}\,[f([nx] + (nx - [nx])) + 2]$$

f_n: X→ X, and for u = o, .., n-1, f_n is a mapping of degree 3 from $\left[\frac{2u}{N}, \frac{2(u+1)}{n}\right]$ onto $\left[\frac{2(u+1)}{N}, \frac{2(u+2)}{n}\right]$. Since, f $\left(\frac{2u+1)}{N} = \frac{2(u+3)}{n}\right)$ holds the sensitivity constant has to be smaller than $(\frac{1}{n})$. Also f_X^N is chaotic on $\left[\frac{2v}{n}, \frac{2(v+1)}{n}\right]$.So if $c \in (0, (\frac{1}{n})$. Then c can be used as sensitivity constant.

4.2 Generalized Mandelbrot set

Definition 4.3.1.

let (X, d) be a compact metric space and let $\Sigma \subseteq C(X,X)$.

Define $M(\Sigma) = \{f \in \Sigma \mid E^*(f)$ is connected$\}$.

Definition 4.3.2.

The collection $\Sigma \subseteq C(X, X)$ is said to be 'distinguishable' if there is a one- one on to map from C to X.

If we consider X = $\bar{C}$ and f = { $z^2 + c$}, c ∈ C then f is distinguishable, because ψ: ⌣ → $\bar{C}$ is defined as, $\psi(f_c)$ where f_c: $\bar{C}$→ $\bar{C}$ is $f_c(z)= z^2+ c$; is one- one and onto; then M is the classical Mandelbrot set.

Not that the function space C(X, X) can be considered as the subspace of the hypespace K(X). in this direction we prove some results. Our space is (X,d) which is compact and each member of C (X, X) has a closed graph in X × X and hence such a function can be identified as an element of K (X × X) note that since X is compact , so is X × X and hence K (X × X) is also compact. Local conceitedness of the Mandelbrot set is an open problem. In [62] these are many results on K (X). but we will prove a result which is not in [62] and some to be very are in our context.
Here we topologise K(X) with the vieterious topology ie topology induced by the Hausdorff metric.

Let A1, A2....., An are subsets of X, then

(A1, A2......., An) = { E ∈K (X) | for each i= 1,2,.... N, E ∩Ai ≠ ϕ and E ⊆ $\bigcup_{i=1}^{n} Ai$. then B = { (u1, u2, ..., un)| ui's are open in X} is a basis for a topology. That topology is called viterious topology.

Notation 4.3.3. C (X) = {E ⊆ X | E is connected }

C_k (X) = { E ⊆ X| E ∈ K (X) ∩ C (X) }

Result 4.3.4. Let (X, d) be locally compact. Ten following are equivalent

1) X is locally connected
2) C (X) is locally connected
3) C_k (X) is locally connected

Proof: (1)=> (2)

Suppose X is locally connected. Let E ∈ C (X) .

Let $(u_1, u_2,\ldots, u_n) \cap C(X)$ be a basic open set containing E. for each $i = 1,2,\ldots n$ and each $x \in E \cap U_i$, let V_x be a connected neighborhood of x such that $\bar{V}_x \subseteq Ui$ and $\bar{V}_x$ is compact. Let Υ be the collection of all such V_x. Then Υ covers E and $V = U\Upsilon \subseteq \bigcup_{i=1}^{n} u_i$ and V is connected and open .Choose one $V_{xi} \in \Upsilon$ such that $V_{xi}. \subseteq U_i$, for each i = 1,2,... n. Then $E\in (V_{x1}, V_{x2}, \ldots, V_{xn}, V) \subset (u_1,u_2,\ldots u_n)$ [because $(V_1,\ldots, V_m) \subseteq (U_1, \ldots U_n)$ iff $\bigcup_{i=1}^{n} u_i \subseteq \bigcup_{i=1}^{n} u_n$ and for each U_i these is a V_j such that $V_j \subseteq U_i$ [62]] and $(V_{x1}, V_{x2} \ldots V_{xn}, V) \cap C(x) \subseteq (U_1, U_2, \ldots U_n) \cap C(x)$. Since $C_K(X)$ is open and dense in C (x) [62], $(V_{xu},\ldots V_{xn}, V) \cap C_k(X)$ is dense in $(Vx1, Vx2,\ldots Vxn, V) \cap C(X)$. so , $(Vx1, Vx2,\ldots, Vxn, V) \cap C(X)$ is connected if $(Vx1, Vx2, \ldots, Vxn, V) \cap C_k(X)$is connected .Let $F, E_0 \in (Vx1, Vx2, \ldots, Vxn, V) \cap C_k(X)$ .Since $E_0 \cup F$ is a compact subset of the connected, locally compact M containing $E_0 \cup F$ such that $M\subseteq V$. Since $E_0 \subseteq M$ implies $M \cap V_{xi} \neq \phi$ for each i and $M\subseteq V$, $M\in (Vx1, Vx2,\ldots Vxn, V) \cap C_k(X)$.

Let $L_F = (G \in C(M) \mid FCG\}$ and

$L_{E_0} = \{G \in C(M) \mid E_0 \subseteq G\}$. Then each of L_{E_0} and L_F is connected and E_0, m ε L_{E_0} and F. M ε L_F. Hence $L_{E_0} \cup L_F$ is connected.

Let G ε L_F. Then $F\subseteq G \subseteq M$ implies that $G \in (V_{x1}, V_{x2},\ldots V_{xn}, V) \cap C_k(X)$.

So $L_F \subseteq (V_{x1}, V_{x2}, \ldots, V_{xn}, V) \cap C_k(X)$ is connected

Similarly $L_{E_0} \subseteq (V_{x1}, V_{x2}, \ldots, V_{xn}, V) \cap C_k(X)$

$\therefore (V_{x1}, V_{x2}, \ldots, V_{xn}, V) \cap C_k(X)$ is connected .

Hence C(X) is locally connected at E

$\therefore$ C(X) is locally connected.

(2) => (3) .

Suppose C(X) is locally connected. But $C_k(X)$ is open in C(X) [62] . Since an open subspace of a locally connected space is locally connected, $C_k(X)$ is locally connected.

(3) => (1)

Suppose C_k (X) is locally connected. Ten for each x $\in$ X, C_k (X) is connected in Kleinen at {x}. [62]. Thus for each x ε X, X is connected im. Kleinen at x .So X is locally connected.

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CHAPTER – 5

HAUSDORFF DIMENSION, PACKING DIMENSION AND FRACTALS

In this chapter we study the relations between Hausdorff dimension and packing dimension. We will define regular sets in Metric spaces using packing measures. In [35] regular sets were defined in IR^n using Hausdorff measures.

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5.1 Regular sets[1]

Regular points are defined in IR^n [35] using Hausdorff measures. By considering Borel measures and Packing measures we generalize this concept into metric spaces. Packing measures were introduced by G.A.Edgar[33]

Definition 5.1.1.

A 'Constituent' in a metric space (X, d) is an ordered pair (x, r); where $x \in X$ and $r > 0$. Collection of constituents is denoted by C .

We can consider (x, r) as $B_r(x)$; Open ball of radius r.

Definition 5.1.2.

Let $\varepsilon > 0$. A collection of Constituents is said to be ε-fine iff $r<\varepsilon$; $\forall (x, r) \in C$

Now we define some type of packings which are generalizations of packing defined in [33]

Definition 5.1.3.

C (i) is an (a)- packing iff $d(x, y) \geq \max \{r, r'\}$; $\forall (x, r) \neq (y, r')$ inC

(ii) is a (b)- packing iff $B_r(x) \cap B_{r'}(y) = \phi$; $\forall (x, r) \neq (y, r')$ in C .

(iii) is a (c)- packing iff $d(x, y) \geq r + r'$; $\forall (x, r) \neq (y, r')$ in C.

[1] Some results in section5.1 is to be published in Bulletin of London Mathematical Society .

Definition 5.1.4.

An (a)- packing of a Set $F \subseteq X$ is an (a)- packing C Such that $x \in C$ F, $\forall$ (x, r) $\in$C. Similarly for (b) and (c) packings.

Definition 5.1.5.

(i) Let $P^a_\epsilon(\phi, F) = \text{Sup} \sum_{(x,r)\in C} \phi(r)$, where the supremum is over all ε-fine (a) packings of F.

(ii) Let $\overline{P^a}(\phi, F) = \lim_{\varepsilon \to 0} P^a_\epsilon(\phi, F)$

(iii) Let $p^a(\phi, F) = \text{Inf} \{ \sum_{U \in U} P^a(\phi, U) / U$ is a countable cover of F}.

We have the definition of Hausdorff function in the first chapter [Definition.1.2.].

Definition 5.1.6.

Let ϕ be a Hausdorff function. Then ϕ is c-blanked iff there is a constant $c < \infty$ such that. $\phi(2r) \leq c. \phi(r)$; $\forall r \leq 1$.

For example; $\phi(r)=(2r)^S$ is c-blanked for every c greater than or equal to 2^S while $\phi(r) = 2^{-M/r\propto}$ is not blanked for any c.

Notation 5.1.7.

If $\phi(r) = (2r)^S$ then we denote, $P^a(\phi, F)$ as $P^a(s, F)$.

Definition 5.1.8.

Let $P^a(F) = \text{Sup}\{s \mid P^a(S, F) = \propto\}$

$= \text{Inf}\{s \mid P^a(S, F) = 0\}$.

Then we say that $P^a(F)$ is the (a)-packing dimension of F.

All definitions for (b) and (c) packing dimensions are analogous.

Notation 5.1.9.

In this chapter g will denote a function from F to $[0, \infty]$.

Definition 5.1.10.

A collection C of Constituents is g-fine iff $r \leq g(x)$; $\forall (x, r) \in C$.

Now we define regular sets in Metric spaces via the packing dimension.

Definition 5.1.11.

Let μ be a Borel measure on X and ϕ be a Hausdorff function.

Define (i) $\underline{D}_{\mu}^{\phi}(F, x) = \liminf_{r \downarrow 0} \frac{\mu(F \cap B_r(x))}{\phi(r)}$

(ii) $\overline{D}_{\mu}^{\phi}(F, x) = \limsup_{r \downarrow 0} \frac{\mu(F \cap B_r(x))}{\phi(r)}$

If $\underline{D}_{\mu}^{\phi}(F, x) = \overline{D}_{\mu}^{\phi}(F, x)$ then we say that density of x in F exist and denote it by $\overline{D}_{\mu}^{\phi}(F, x)$. $\underline{D}_{\mu}^{\phi}(X, x)$ is denoted as $\overline{D}_{\mu}^{\phi}(x)$.

If $\underline{D}_{\mu}^{\phi}(F, x) = 1$ then x is called a (μ,ϕ) regular point of F. Otherwise, x is a (μ,ϕ) irregular point of F.

Notation 5.1.12.

Let $Reg_{\mu}^{\phi}(F) = \{x \in X \mid x \text{ is a } (\mu,\phi) \text{ regular point of } F\}$.

and $Irg_{\mu}^{\phi}(F) = \{ x \in X \mid x \text{ is a } (\mu,\phi) \text{ irregular point of } F \}$.

Definition 5.1.13.

Density of F in X is defined as,

$\overline{D}_{\mu}^{\phi}(F) = \operatorname{Inf}_{x \in F} \overline{D}_{\mu}^{\phi}(F, x)$

Definition 5.1.14.

Let $F \subseteq X$. Then F is 'regular' in X iff $\overline{D_\mu^\phi}$ (F) = 1. Collection of all regular subsets of X is denoted by R(X).

Following definition of Vitali properties for packing dimensions are natural generalizations of usual vitali properties.

Definitions 5.1.15.

Let μ be a Borel measure on X. We say that μ has (a).Vitali property (respectively (b)-Vitali, (c)-Vitali) iff for any Borel Set $E \subseteq X$ with $\mu(E) < \infty$ and any five cover U of E, these exist a countable (a)-packing (respectively (b)-packing, (c)-packing) $C \subseteq U$ of E such that $\mu\left(E - \bigcup_{(X,r)\in c} B_r(x)\right) = 0$.

Remark 5.1.16.

(1) If $X = IR^n$, d is the Eucledian metric, μ is the Hausdorff measure and ϕ (r) $= (2r)^S$ then $\overline{D_\mu^\phi}$ (F, x) coincides with the densities defined in [; P.70].

(2) If $X=IR^2$. d is the Eucledian metric, μ is the area (Lebesgue measure on IR^2) and ϕ (r) = diam ($B_r(x)$) then D_μ^ϕ (F, x) is exactly same as the Lebesgue density.

(3) Clearly Hausdorff measure satisfies (b)-vitali properly and (a)-vitali properly in IR^n.

Result 5.1.17.

Let (X. d) be a metric space and $F \in K(x)$. IF $x \notin F$ then $x \in Irg(F)$.

Proof. F is Compact, so closed

$x \notin F \Rightarrow B_r(x) \cap F = \phi$, for very small r

$\Rightarrow \mu(B_r(x)) \cap F) = 0$. $\forall r$ which is very small

$\Rightarrow D_\mu^\phi(F)$ (F, x) = 0

$\Rightarrow x \in Irg(F)$.

$\therefore (X \setminus F) \subseteq Irg\,(F)$.

Result 5.1.18.

Let (X, d) be a metric space and let μ be a finite Borel measure and $F \subseteq X$. Let ϕ be a Hausdorff function such that $D^{\phi}_{\mu}(F)$ (F) is finite and non zero; then

(i) $D^{\phi}_{\mu}(F)\ (F) = \dfrac{\mu(F)}{P^{b}(\phi, F)}$, if μ satisfies (b) vitali properly

(ii) $D^{\phi}_{\mu}(F) \geq \dfrac{\mu(F)}{C^{2}\, P^{b}(\phi, F)}$ if ϕ is c-blanketed.

Proof.

(i) We first prove that $D^{\phi}_{\mu}(F) \leq \dfrac{\mu(F)}{P^{b}(\phi, F)}$ (1)

Assume that $D^{\phi}_{\mu}(F) > 0$. Let $k \in IR$ be such that $D^{\phi}_{\mu}(F) > k$. We have to prove that,

$k > P^{b}(Q, F) \leq \mu(F)$

Let ε>0 be given. Since μ is a Borel measure then there is an open set $U \supseteq F$ such that $\mu(u) < \mu(F) + \in$ (2).

Corresponding to each $x \in F$, let g(x)>0 be so small that

$\mu(B_r(x) \cap F) > k.Q(r)$, $\forall r < g(x)$

and $g(x) < d(x, X \setminus U)$

Then $g: E \to (0, \infty)$. Let C be a δ-fine (b) packing of F.

Then $UB_r(x) \subseteq U$ and

$$\sum_{(x, r) \in C} \phi(r) \leq \frac{1}{k} \sum \mu(B_r(x) \cap F).$$

$$\leq \frac{1}{K}\, \mu(u)$$

$\therefore P^{b}(\phi, F) \leq \dfrac{1}{K}(\mu(F) + \varepsilon)$ from (2)

Let $\in \to 0$, then we have,

$P^b(\phi, F) \leq \frac{1}{k}(\mu(F)$

ie. $k.P^b(\phi, F) \leq \mu(F)$. So (1) is proved. Now we prove,

$$D_\mu^\phi(F) \geq \frac{\mu(F)}{P^b(\phi, F)} \qquad (3)$$

Here we uses the condition that μ has (b) vitali properly. Let $k' < \infty$ Such that $D_\mu^\phi(F) < k'$. We have to prove that $\mu(F) \leq k.\ P^b(\phi, F)$.

Let $g: F \to (0, \infty)$ be function

Then, $B = \{(x, r) \mid x \varepsilon F; r < g(x); \mu(Br(x) \cap F) \leq k' \phi(x)\}$ is a fine corer of F. BY (b) vitali properly, there is a (b) packing $C \subseteq B$ of F with

$$\mu(F) = \mu(F \cap (\bigcup_C B_r(x)))$$

$$\therefore \mu(F) = \mu(F \cap (\bigcup_C B_r(x)))$$

$$\leq \sum_C \mu(B_r(x))$$

$$K^1 \sum_C \phi(r)$$

$$\mu(F) \leq K\ P_g^h(\phi, F)$$

Since g is arbitrary

$$\mu(F) \leq K^1 P_g^h(\phi, F)$$

$\therefore$ (3) is proved

from (1) and (3),

$$D_\mu^\phi(F) = \frac{\mu(F)}{P^h(\phi, F)}$$

(ii) let $t < \infty$, such that $P_{\mu}^{h}(F) < t$. It is enough if we prove that $\mu(F) \leq t\, c^2 . P^b(\phi, F)$

since ϕ is C- blanketed

$\phi(4r) \leq C^2 \phi(r)$

let $g: F \to (0, \infty)$ be a function

$$B = \{(x, r) \mid x \varepsilon F, r < g(x), \frac{\mu(B_{4r}(x) \cap F)}{\phi(4_r)} \leq t\}$$

Is a fine cover of F. Now By Result. 1.2., there is a packing $\{(x, r_i) \mid i = 1, 2, \ldots\}$ B such that

$$F \subseteq \bigcup_{i=1}^{\alpha} B_{3ri}(x_i) = \bigcup_{i=1}^{\infty} B_{4ri}(x_i)$$

$$\mu(F) \leq \sum_{i=1}^{\alpha} \mu(B_{4ri}(x_i))$$

$$\leq t \sum_{i=1}^{\alpha} \phi(r_i)$$

so, $\mu(F) \leq t\, c^2\, P_g^h(\phi, F)$

since g is arbitrary

$\mu(F) \leq t\, c^2\, P^h(\phi, F)$.

in [35] Taylor, defined $F \subseteq X$ to be fractal if $\dim_H(F) = \dim_p(F)$. in that sense a (μ, ϕ) regular set F in IR^n is a factal.

<u>Cor. 5.1.9</u>

Let F be a (μ, ϕ) regular set in IR^n. then $P^b(\phi, F) = H^s(F)$.

Proof: in result 5.1.18 put $\mu = H^S$.

<u>Cor. 5.1.20</u>

If F is (μ, ϕ) regular then $\dim_H(F) = P^b(F)$

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