A Note On Some Domination Parameters in Graph Products

S. Aparna Lakshmanan and A. Vijayakumar
Department of Mathematics
Cochin University of Science and Technology
Cochin - 682 022, Kerala, India.
e-mail: aparnaren@gmail.com, vijay@cusat.ac.in

Abstract

In this paper, we study the domination number, the global domination number, the cographic domination number, the global cographic domination number and the independent domination number of all the graph products which are non-complete extended p-sums (NEPS) of two graphs.

Keywords. Domination, Non-complete extended p-sums (NEPS), Supermultiplicative graphs, Submultiplicative graphs

2000 Mathematics Subject Classification: 05C

1 Introduction

We consider only finite, simple graphs $G = (V, E)$ with $|V| = n$ and $|E| = m$.

A set $S \subseteq V$ of vertices in a graph $G$ is called a dominating set if every vertex $v \in V$ is either an element of $S$ or is adjacent to an element of $S$. A dominating set $S$ is a minimal dominating set if no proper subset of $S$ is a dominating set. The domination number $\gamma(G)$ of a graph $G$ is the minimum cardinality of a dominating set in $G$ [4]. A dominating set $S$ is global dominating if $S$ dominates both $G$ and $G^c$. The global domination number $\gamma_g(G)$ of a graph $G$ is the minimum cardinality of a global dominating set in $G$ [10].

A graph which does not have $P_4$ - the path on four vertices, as an induced subgraph is called a cograph. A set $S \subseteq V$ is called a cographic dominating set if $S$ dominates $G$ and the subgraph induced by $S$ is a cograph [9]. The minimum cardinality of a cographic dominating set is called the cographic
domination number, $\gamma_{cd}(G)$. A set $S \subseteq V$ is called a global cographic dominating set if it dominates both $G$ and $G^c$ and the subgraph induced by $S$ is a cograph. The minimum cardinality of a global cographic dominating set is called the global cographic domination number, $\gamma_{gcd}(G)$ [9]. A set $S \subseteq V$ is called an independent dominating set if $S$ is independent and dominates $G$. The minimum cardinality of an independent dominating set is called the independent domination number, $\gamma_i(G)$ [4].

A graphical invariant $\sigma$ is supermultiplicative with respect to a graph product $\times$, if given any two graphs $G$ and $H$ $\sigma(G \times H) \geq \sigma(G)\sigma(H)$ and submultiplicative if $\sigma(G \times H) \leq \sigma(G)\sigma(H)$. A class $C$ is called a universal multiplicative class for $\sigma$ on $\times$ if for every graph $H$, $\sigma(G \times H) = \sigma(G)\sigma(H)$ whenever $G \in C$ [8].

Let $\mathcal{B}$ be a non-empty subset of the collection of all binary $n$-tuples which does not include $(0,0,...,0)$. The non-complete extended p-sum (NEPS) of graphs $G_1, G_2, ..., G_p$ with basis $\mathcal{B}$ denoted by $\text{NEPS}(G_1, G_2, ..., G_p; \mathcal{B})$, is the graph with vertex set $V(G_1) \times V(G_2) \times ... \times V(G_p)$, in which two vertices $(u_1, u_2, ..., u_p)$ and $(v_1, v_2, ..., v_p)$ are adjacent if and only if there exists $(\beta_1, \beta_2, ..., \beta_p) \in \mathcal{B}$ such that $u_i$ is adjacent to $v_i$ in $G_i$ whenever $\beta_i = 1$ and $u_i = v_i$ whenever $\beta_i = 0$. The graphs $G_1, G_2, ..., G_p$ are called the factors of NEPS [2]. Thus, the NEPS of graphs generalizes the various types of graph products, as discussed in detail in the next section.

In this paper, we study the domination number, the global domination number, the cographic domination number, the global cographic domination number and the independent domination number of NEPS of two graphs.

All graph theoretic terminology and notations not mentioned here are from [1].

2 NEPS of two graphs

There are seven possible ways of choosing the basis $\mathcal{B}$ when $p = 2$.

$B_1 = \{(0,1)\}$
$B_2 = \{(1,0)\}$
$B_3 = \{(1,1)\}$
$B_4 = \{(0,1), (1,0)\}$
$B_5 = \{(0,1), (1,1)\}$
$B_6 = \{(1,0), (1,1)\}$
$B_7 = \{(0,1), (1,0), (1,1)\}$

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs with $|V_i| = n_i$ and $|E_i| = m_i$ for $i = 1, 2$. 
The NEPS($G_1, G_2; B_1$) is $n_1$ copies of $G_2$ and the NEPS($G_1, G_2; B_2$) =
NEPS($G_2, G_1; B_1$).

In the NEPS($G_1, G_2; B_j$) two vertices $(u_1, v_1)$ and $(u_2, v_2)$ are adjacent
if and only if,

- $j = 3$: $u_1$ is adjacent to $u_2$ in $G_1$ and $v_1$ is adjacent to $v_2$ in $G_2$. This
  is same as the tensor product [1] of $G_1$ and $G_2$.
- $j = 4$: $u_1 = u_2$ and $v_1$ is adjacent to $v_2$ in $G_2$ or $u_1$ is adjacent to
  $u_2$ in $G_1$ and $v_1 = v_2$. This is same as the cartesian product [1] of $G_1$
  and $G_2$.
- $j = 5$: Either $u_1 = u_2$ or $u_1$ is adjacent to $u_2$ in $G_1$ and $v_1$ is adjacent
  to $v_2$ in $G_2$.
- $j = 6$: This is same as NEPS($G_2, G_1; B_3$).
- $j = 7$: Either $u_1 = u_2$ and $v_1$ is adjacent to $v_2$ in $G_2$ or $u_1$ is adjacent
to $u_2$ in $G_1$ and $v_1 = v_2$ or $u_1$ is adjacent to $u_2$ in $G_1$ and $v_1$ is adjacent
to $v_2$ in $G_2$. This is same as the strong product [1] of $G_1$ and $G_2$.

3 Domination in NEPS of two graphs

3.1 NEPS with basis $B_1$ and $B_2$

The value of $\gamma(\text{NEPS}(G_1, G_2; B_1))$, $\gamma_\text{cd}(\text{NEPS}(G_1, G_2; B_1))$, $\gamma_\text{cd}(\text{NEPS}(G_1, G_2; B_1))$, $\gamma_\text{cd}(\text{NEPS}(G_1, G_2; B_1))$, $\gamma_\text{cd}(\text{NEPS}(G_1, G_2; B_1))$ are $n_1 \cdot \gamma(G_2)$, $n_1 \cdot \gamma(G_2)$, $n_1 \cdot \gamma(G_2)$, $n_1 \cdot \gamma(G_2)$ and $n_1 \cdot \gamma(G_2)$ respectively and the case
of NEPS($G_1, G_2; B_2$) follows similarly.

3.2 NEPS with basis $B_3$

In [3] it was conjectured that $\gamma(G \times H) \geq \gamma(G) \cdot \gamma(H)$, where $\times$ denotes the
tensor product of two graphs. But, the conjecture was disproved in [6] by
giving a realization of a graph $G$ such that $\gamma(G \times G) \not\leq \gamma(G)^2 - k$ for any
non-negative integer $k$.

Theorem 1. There exist graphs $G_1$ and $G_2$ such that $\sigma(\text{NEPS}(G_1, G_2;
B_3)) - \sigma(G_1) \cdot \sigma(G_2) = k$ for any positive integer $k$, where $\sigma$ denotes any of
the domination parameters $\gamma$, $\gamma_{cd}$ or $\gamma_i$.

Proof. Let $G_1$ be the graph defined as follows. Let $u_{i1}u_{i2}u_{i3}$, $u_{21}u_{22}u_{23}$,
\ldots, $u_{k1}u_{k2}u_{k3}$ be $k$ distinct $P_3$ s and let $u_{j1}$ be adjacent to $u_{j+1,1}$ for
$j = 1, 2, \ldots, k - 1$. Then $\sigma(G_1) = k$. Let $G_2$ be $K_2$. Then, $\sigma(G_2) = 1$. Also, $\sigma(\text{NEPS}(G_1, G_2; B_3)) = 2k$. Therefore, $\sigma(G_1) \cdot \sigma(G_2) = \sigma(G_1) \cdot \sigma(G_2) = 2k$. Therefore, $\sigma(\text{NEPS}(G_1, G_2; B_3)) - \sigma(G_1) \cdot \sigma(G_2) = 2k$.
\qed
Theorem 2. The $\gamma_g$ and $\gamma_{gcd}$ are neither submultiplicative nor supermultiplicative with respect to the NEPS with basis $B_3$. Moreover, given any integer $k$ there exist graphs $G_1$ and $G_2$ such that $\sigma(\text{NEPS}(G_1, G_2; B_3)) = \sigma(G_1)\sigma(G_2) = k$, where $\sigma$ denotes $\gamma_g$ or $\gamma_{gcd}$.

Proof. Case 1. $k \leq 0$ is even.

Let $G_1 = K_n$ and $G_2 = K_2$. Then, $\sigma(G_1) = n$ and $\sigma(G_2) = 2$. But, $\sigma(\text{NEPS}(G_1, G_2; B_3)) = 2$. Therefore, the required difference is $2 - 2n$ which can be zero or any negative even integer.

Case 2. $k < 0$ is odd or $k = 1$.

Let $G_3 = P_3$ and $G_1$ be as in Case 1. Then $\sigma(G_3) = 2$. Also, $\sigma(\text{NEPS}(G_1, G_3; B_3)) = 3$. Therefore, the required difference is $3 - 2n$ which can be one or any negative odd integer.

Case 3. $k > 1$.

Let $G_3$ be as in Case 2. Let $G_4$ be the graph defined as follows. Let $u_{11}u_{12}u_{13}$, $u_{21}u_{22}u_{23}$, ..., $u_{k1}u_{k2}u_{k3}$ be $k$ distinct $P_3$ s and let $u_{ji}$ be adjacent to $u_{j+1,i}$ for $j = 1, 2, ..., k - 1$. Then $\sigma(G_4) = k$. Also, $\sigma(\text{NEPS}(G_4, G_3; B_3)) = 3k$. Therefore, the required difference is $k$. \qed

3.3 NEPS with basis $B_4$

Vizing’s conjecture [11]. The domination number is supermultiplicative with respect to the cartesian product i.e; $\gamma(G \Box H) \geq \gamma(G)\gamma(H)$.

Remark 3. There are infinitely many pairs of graphs for which equality holds in the Vizing’s conjecture [7].

Remark 4. Vizing’s type inequality does not hold for cographic, global cographic and independent domination numbers. For example, let $G$ be the graph obtained by attaching $k$ pendant vertices to each vertex of a path on four vertices. Then, $\gamma_{cd}(G) = \gamma_{gcd}(G) = k + 3$ and $\gamma_{cd}(G \Box G) = \gamma_{gcd}(G \Box G) = 16k + 8$. For $k \geq 10$, $\gamma_{cd}(G \Box G) \leq \gamma_{cd}(G)^2$.

Theorem 5. There exist graphs $G_1$ and $G_2$ such that $\sigma(\text{NEPS}(G_1, G_2; B_4)) - \sigma(G_1)\sigma(G_2) = k$ for any positive integer $k$, where $\sigma$ denotes any of the domination parameters $\gamma$, $\gamma_{cd}$ or $\gamma_i$.

Proof. Let $G_1 = P_4$ and $G_2 = K_2$. Then, $\sigma(G_1) = \lceil \frac{n+1}{2} \rceil$ [4] and $\sigma(G_2) = 1$. Also, $\sigma(\text{NEPS}(G_1, G_2; B_4)) = \lceil \frac{n+1}{2} \rceil$ [5]. Therefore, for any positive integer $k$, if we choose $n = 6k - 2$ the claim follows. \qed

Theorem 6. The $\gamma_g$ and $\gamma_{gcd}$ are neither submultiplicative nor supermultiplicative with respect to the NEPS with basis $B_4$. Moreover, given any integer $k$ there exist graphs $G_1$ and $G_2$ such that $\sigma(\text{NEPS}(G_1, G_2; B_4)) - \sigma(G_1)\sigma(G_2) = k$, where $\sigma$ denotes $\gamma_g$ or $\gamma_{gcd}$. \qed
Proof. Case 1. $k < 0$ is even.

Let $G_1 = K_n$ and $G_2 = K_2$. Then, $\sigma(G_1) = n$ and $\sigma(G_2) = 2$. But, $\sigma(\text{NEPS}(G_1, G_2; B_4)) = 2$. Therefore, the required difference is $2 - 2n$ which can be any positive even integer.

Case 2. $k < 0$ is odd.

Let $G_3 = P_3$ and $G_1$ be as in Case 1. Then $\sigma(G_3) = 2$. Also, $\sigma(\text{NEPS}(G_1, G_3; B_4)) = 3$. Therefore, the required difference is $3 - 2n$ which can be any negative odd integer.

Case 3. $k \geq 1$.

Let $G_4 = P_n$ and $G_5 = P_4$. Then, $\sigma(G_4) = \lfloor \frac{n+2}{2} \rfloor$ and $\sigma(G_5) = 2$. For any positive integer $k$, if we choose $n = 3k+4$, then $\sigma(\text{NEPS}(G_4, G_5; B_4)) = n$. (Note that the value is $n+1$ only when $n = 1, 2, 3, 5, 6, 9 \ [5]$). Therefore the required difference is $k$.

3.4 NEPS with basis $B_5$ and $B_6$

Theorem 7. There exist graphs $G_1$ and $G_2$ such that $\sigma(\text{NEPS}(G_1, G_2; B_5)) - \sigma(\gamma_1)\sigma(\gamma_2) = k$ for any positive integer $k$, where $\sigma$ denotes any of the domination parameters $\gamma_1$, $\gamma_{cd}$ or $\gamma_g$.

Proof. Let $G_1 = P_n$ and $G_2 = K_2$. Then $\sigma(G_1) = \lfloor \frac{n+2}{2} \rfloor$ and $\sigma(G_2) = 1$. Also, $\sigma(\text{NEPS}(G_1, G_2; B_5)) = \lfloor \frac{n+2}{2} \rfloor$. For a positive integer $k$, if we choose $n = 6k - 2$ then the difference is $k$. Hence, the theorem.

Theorem 8. There exist graphs $G_1$ and $G_2$ such that $\sigma(\text{NEPS}(G_1, G_2; B_5)) - \sigma(\gamma_1)\sigma(\gamma_2) = k$ for any negative integer $k$, where $\sigma$ denotes $\gamma_9$ or $\gamma_{gcd}$.

Proof. Let $G_1 = P_n$ and $G_2 = K_2$. Then $\sigma(G_1) = \lfloor \frac{n+2}{2} \rfloor$ and $\sigma(G_2) = 2$. Also, $\sigma(\text{NEPS}(G_1, G_2; B_5)) = \lfloor \frac{n+2}{2} \rfloor$. Therefore, if we choose $n = 6k - 2$, the required difference is $-k$.

3.5 NEPS with basis $B_7$

Theorem 9. The $\gamma_1$, $\gamma_9$ and $\gamma_{gcd}$ are submultiplicative with respect to the NEPS with basis $B_7$.

Proof. Let $D_1 = \{u_1, u_2, \ldots, u_s\}$ be a dominating set of $G_1$ and $D_2 = \{v_1, v_2, \ldots, v_t\}$ be a dominating set of $G_2$. Consider the set $D = \{(u_1, v_1), (u_1, v_2), \ldots, (u_1, v_t), (u_2, v_1), (u_2, v_2), \ldots, (u_s, v_t)\}$. Let $(u, v)$ be any vertex in $\text{NEPS}(G_1, G_2; B_7)$. Since $D_1$ is a $\gamma$-set in $G_1$, there exists at least one $u_i \in D_1$ such that $u = u_i$ or $u$ is adjacent to $u_i$. Similarly, there exists at least one $v_j \in D_2$ such that $v = v_j$ or $v$ is adjacent to $v_j$. Therefore, $(u_i, v_j)$ dominates $(u, v)$ in $\text{NEPS}(G_1, G_2; B_7)$. Hence, $\gamma(\text{NEPS}(G_1, G_2; B_7)) \leq \gamma(G_1)\gamma(G_2)$.

\[ \Box \]
Similar arguments hold for the independent domination and global domination numbers also.

**Note.** The difference between $\gamma(G_1)\gamma(G_2)$ and $\gamma(\text{NEPS}(G_1,G_2;B_7))$ can be arbitrarily large. Similar is the case for $\gamma_i$ and $\gamma_g$. For, let $G_1$ be the graph, $n$ copies of $C_4$ s with exactly one common vertex. Then, $\gamma(G_1) = \gamma_i(G_1) = n + 1$. Also, $\gamma(\text{NEPS}(G_1,G_2;B_7)) \leq n^2 + 3$ and $\gamma_i(\text{NEPS}(G_1,G_2;B_7)) \leq n^2 + 5$. Also, $\gamma_g(K_n) = n$, $\gamma_g(P_3) = 2$ and $\gamma_g(\text{NEPS}(G_2,G_3;B_7)) = n + 2$, if $n > 1$.

**Theorem 10.** The $\gamma_{cd}$ and $\gamma_{gcd}$ are neither submultiplicative nor supermultiplicative with respect to the NEPS with basis $B_7$. Moreover, for any integer $k$ there exist graphs $G_1$ and $G_2$ such that $\sigma(\text{NEPS}(G_1,G_2;B_7)) - \sigma(G_1)\sigma(G_2) = k$, where $\sigma$ denotes $\gamma_{cd}$ or $\gamma_{gcd}$.

**Proof.**

**Case 1.** $k \leq 0$.

Let $G_1$ be the graph $P_3$ with $k$ pendant vertices each attached to all the three vertices of the $P_3$. Let $G_2$ be the graph $P_4$ with $k$ pendant vertices each attached to all the four vertices of the $P_4$. So, $\sigma(G_1) = 3$ and $\sigma(G_2) = k + 3$. Also, $\sigma(\text{NEPS}(G_1,G_2;B_7)) = 2k + 10$. Therefore, the required difference is $1 - k$.

**Case 2.** $k \geq 0$.

Let $G_1$ be as in Case 1 and $G_3$ be the graph $P_6$ with $k$ pendant vertices each attached to all the six vertices of the $P_6$. So, $\sigma(G_3) = k + 5$. Also, $\sigma(\text{NEPS}(G_1,G_3;B_7)) = 4k + 14$. Therefore, the required difference is $k - 1$.

\[\square\]

**References**


