A note on energy of some graphs

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Abstract

Eigenvalue of a graph is the eigenvalue of its adjacency matrix. The energy of a graph is the sum of the absolute values of its eigenvalues. In this note we obtain analytic expressions for the energy of two classes of regular graphs.

1 Introduction

Let $G$ be a graph with $|V(G)| = p$ and an adjacency matrix $A$. The eigenvalues of $A$ are called the eigenvalues of $G$ and form the spectrum of $G$ denoted by $\text{spec}(G)$ [3]. The energy [6] of $G$, $\mathcal{E}(G)$ is the sum of the absolute values of its eigenvalues.

From the pioneering work of Coulson [2] there exists a continuous interest towards the general mathematical properties of the total $\pi$-electron energy $\mathcal{E}$ as calculated within the framework of the Hückel Molecular Orbital (HMO) model [7]. These efforts enabled one to get an insight into the dependence of $\mathcal{E}$ on molecular structure. The properties of $\mathcal{E}(G)$ are discussed in detail in [6, 8, 9].

In [5] the spectra and energy of several classes of graphs containing a linear polyene fragment are obtained. In [12], we obtain the energy of cross products of some graphs. In [15], the energy of iterated

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line graphs of regular graphs and in [13], the energy of some self-complementary graphs are discussed. The energy of regular graphs are discussed in [10]. Some works pertaining to the computation of $E(G)$ can be seen in [1, 6, 4, 11, 14].

As there is no easy way to find the eigenvalues of a graph $G$, the computation of the actual value of $E(G)$ is an interesting problem in graph theory. In this note we obtain analytic expressions for the energy of two classes of regular graphs.

All graph theoretic terminology is from [3]. We use the following lemmas and definitions in this paper.

**Lemma 1.** [3] Let $M, N, P$ and $Q$ be matrices with $M$ invertible. Let
\[ S = \begin{bmatrix} M & N \\ P & Q \end{bmatrix}. \]
Then, $|S| = |M| |Q - PM^{-1}N|$ and if $M$ and $P$ commutes, then, $|S| = |MQ - PN|$ where the symbol $|.|$ denotes the determinant.

**Lemma 2.** [3] Let $G$ be an $r$− regular connected graph on $p$ vertices with $A$ as an adjacency matrix and $r = \lambda_1, \lambda_2, \ldots, \lambda_m$ as the distinct eigenvalues. Then there exists a polynomial $P(x)$ such that $P(A) = J$, where $J$ is the all one square matrix of order $p$ and $P(x)$ is given by $P(x) = p \times \frac{(x-\lambda_1)(x-\lambda_2)\ldots(x-\lambda_m)}{(r-\lambda_1)(r-\lambda_2)\ldots(r-\lambda_m)}$, so that $P(r) = p$ and $P(\lambda_i) = 0$, for all $\lambda_i \neq r$.

**Lemma 3.** [3] $\text{spec}(C_p) = \begin{pmatrix} 2 & 2 \cos \frac{2\pi}{p} j \\ 1 & 1 \end{pmatrix}$ and $\text{spec}(\overline{C_p}) = \begin{pmatrix} p - 3 & -1 - 2 \cos \frac{2\pi}{p} j \\ 1 & 1 \end{pmatrix}$, $j = 1$ to $p - 1$.

**Lemma 4.** [3] Let $G$ be an $r$− regular graph with an adjacency matrix $A$ and incidence matrix $R$. Then, $RR^T = A + rI$.

**Definition 1.** Let $G$ be a $(p,q)$ graph. The complement of the incidence matrix $R$, denoted by $\overline{R} = [\overline{r}_{ij}]$ is defined by

\[ \overline{r}_{ij} = 1 \text{ if } v_i \text{ is not incident with } e_j \]
\[ = 0, \text{ otherwise.} \]

**Definition 2.** Let $G$ be a $(p,q)$ graph. Corresponding to every edge $e$ of $G$ introduce a vertex and make it adjacent with all the vertices not incident with $e$ in $G$. Delete the edges of $G$ only. The resulting graph is called the partial complement of the subdivision graph of $G$, denoted by $\overline{S}(G)$.
2 Partial complement of the subdivision graph

In this section we obtain the spectrum of the partial complement of the subdivision graph $S(G)$ of a regular graph $G$ and the energy of $S(C_p)$.

Lemma 5. Let $G$ be an $r$--regular graph with an adjacency matrix $A$ and incidence matrix $R$. Then, $R = J_p 	imes q - R$, $R^T = J_q 	imes p - R^T$ and $RR^T = (q - 2r)J + (A + r)I$ where $J$ is the all one matrix of appropriate order.

Proof. By Definition 1, $R = J_p 	imes q - R$. Therefore

$$RR^T = (J_p 	imes q - R)(J_q 	imes p - R^T)$$

$$= qJ - rJ - rJ + A + rI$$

$$= (q - 2r)J + (A + r)I,$$ by Lemma 4.

Hence the lemma.

Lemma 6. Let $G$ be a connected $r$--regular $(p,q)$ graph. Then, $S(G)$ is regular if and only if $G$ is a cycle.

Proof. From Definition 2, we have the degree of vertices in $S(G)$ corresponding to the edges of $G$ is $p - 2$ each and of those corresponding to the vertices of $G$ is $q - r$ each. Since $G$ is $r$--regular, $q = \frac{pr}{2}$ and hence $q - r = p - 2$ if and only if $r = 2$. Thus, $S(G)$ is regular if and only if $G$ is a cycle.

Theorem 1. Let $G$ be a connected $r$--regular $(p,q)$ graph. Then,

$$\text{spec}(S(G)) = \left( \begin{array}{ccc} \pm \sqrt{p(q - 2r)} + 2r & \pm \sqrt{\lambda_i + r} & 0 \\ 1 & 1 & q - p \end{array} \right), i = 2 \text{ to } p.$$ 

Proof. The adjacency matrix of $S(G)$ can be written as $\begin{bmatrix} 0 & R \\ R^T & 0 \end{bmatrix}$. Then, the theorem follows from Lemmas 1 and 5.
Theorem 2.

\[ \mathcal{E}(\mathcal{S}(C_p)) = \begin{cases} 
2 \left( p - 4 + 2 \cot \frac{\pi}{2p} \right), & p \equiv 0(\text{mod} 2) \\
2 \left( p - 4 + 2 \cosec \frac{\pi}{2p} \right), & p \equiv 1(\text{mod} 2) 
\end{cases} \]

Proof. By Lemma 3 and Theorem 1 we have

\[ \text{spec}(\mathcal{S}(C_p)) = \begin{pmatrix} p - 2 & -(p - 2) & \pm 2 \cos \frac{\pi j}{p} \\
1 & 1 & 1 \end{pmatrix}, \ j = 1 \text{ to } p - 1. \]

We shall consider the following two cases.

Case 1. \( p \equiv 0(\text{mod} 2) \).

The cosine numbers \( 2 \cos \frac{\pi j}{p} \) are positive only for \( \frac{\pi j}{p} \leq \frac{\pi}{2} \). Then, the positive cosine numbers are

\[ 2 \cos \frac{\pi}{p}, 2 \cos \left( \frac{\pi}{p} \times 2 \right), \ldots, 2 \cos \left( \frac{\pi}{p} \times \frac{p}{2} \right). \]

Let \( C = 2 \cos \frac{\pi}{p} + 2 \cos \left( \frac{\pi}{p} \times 2 \right) + \ldots + 2 \cos \left( \frac{\pi}{p} \times \frac{p}{2} \right) \) and

\[ S = 2 \sin \frac{\pi}{p} + 2 \sin \left( \frac{\pi}{p} \times 2 \right) + \ldots + 2 \sin \left( \frac{\pi}{p} \times \frac{p}{2} \right) \]

so that

\[ C + iS = 2\gamma + 2\gamma^2 + \ldots + 2\gamma^\frac{p}{2} = 2\gamma \frac{1 - \gamma^\frac{p}{2}}{1 - \gamma} \]

where \( \gamma = \cos \frac{\pi}{p} + i \sin \frac{\pi}{p} \) and \( i = \sqrt{-1} \).

Now, equating real parts, we get \( C = \cot \frac{\pi}{2p} - 1 \). Since the spectrum of \( (\mathcal{S}(C_p)) \) is symmetric with respect to zero, the energy contribution from the cosine numbers is \( 2C \). Thus,

\[ \mathcal{E}(\mathcal{S}(C_p)) = 2 \times (p - 2 + 2C) = 2 \left( p - 4 + 2 \cot \frac{\pi}{2p} \right) \]

Case 2. \( p \equiv 1(\text{mod} 2) \).

When \( p \) is odd, the cosine numbers \( 2 \cos \frac{\pi j}{p} \) are positive for \( j \leq \frac{p - 1}{2} \). Then, by a similar argument as in Case 1, we get \( \mathcal{E}(\mathcal{S}(C_p)) = 2 \left( p - 4 + 2 \cos \frac{\pi}{2p} \right) \). Hence the theorem.
3 Energy of the complement of a cycle.

In [5], I. Gutman obtained an analytic expression for the energy of a cycle \( C_p \). In this section we derive the energy of \( \overline{C_p} \), the complement of the cycle \( C_p \).

**Theorem 3.**

\[
\mathcal{E}(\overline{C_p}) = \begin{cases} 
2 \left( \frac{2p-9}{3} + \sqrt{3} \cot \frac{\pi}{p} \right); & p \equiv 0 \text{ (mod 3)} \\
2 \left( \frac{2p-8}{3} + \frac{2 \sin \frac{2\pi j}{p}(1-\frac{j}{p})}{\sin \frac{\pi}{p}} \right); & p \equiv 1 \text{ (mod 3)} \\
2 \left( \frac{2p-10}{3} + \frac{2 \sin \frac{2\pi j}{p}(1+\frac{j}{p})}{\sin \frac{\pi}{p}} \right); & p \equiv 2 \text{ (mod 3)}
\end{cases}
\]

**Proof.** We have \( \text{spec}(\overline{C_p}) = \begin{pmatrix} p-3 & -1 \sin 2\pi j/p ; j=1 \text{ to } p-1 \end{pmatrix} \), by Lemma 3.

We shall consider the following cases.

**Case 1.** \( p \equiv 0 \text{ (mod 3)}. \)

Then, \(- \left( 1 + 2 \cos \frac{2\pi j}{p} \right) \geq 0 \) if and only if \( \frac{j}{p} \leq \frac{j}{p} \).

Let \( \sum_{j=\frac{j}{p}}^{\frac{j}{p}} \left( 1 + 2 \cos \frac{2\pi j}{p} \right) = \frac{p+3}{3} + \sum_{j=\frac{j}{p}}^{\frac{j}{p}} 2 \cos \frac{2\pi j}{p} = \frac{p+3}{3} + C \) and

\[ S = \sum_{j=\frac{j}{p}}^{\frac{j}{p}} 2 \sin \frac{2\pi j}{p}, \text{ so that } C + iS = \sum_{j=\frac{j}{p}}^{\frac{j}{p}} \gamma^j \text{ where } \gamma = \cos \frac{2\pi}{p} + i \sin \frac{2\pi}{p}. \]

Equating real parts, we get \( C = -(1 + \sqrt{3} \cot \frac{\pi}{p}). \)

The total sum of positive eigenvalues

\[
= p - 3 + \sqrt{3} \cot \frac{\pi}{p} + 1 - \left( \frac{p+3}{3} \right) \\
= \frac{2p-9}{3} + \sqrt{3} \cot \frac{\pi}{p}.
\]

Thus, \( \mathcal{E}(\overline{C_p}) = 2 \times \left[ \frac{2p-9}{3} + \sqrt{3} \cot \frac{\pi}{p} \right]. \)

The other two cases \( p \equiv 1 \text{ (mod 3)} \) and \( p \equiv 2 \text{ (mod 3)} \) can be proved similarly.

**References**


