ANALYSIS OF SOME STOCHASTIC INVENTORY MODELS WITH POOLING/RETRIAL OF CUSTOMERS

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Certificate

This is to certify that thesis entitled **Analysis of Some Stochastic Inventory Models with Pooling/ Retrial of Customers** is a bonafide record of the research work carried out by Mr. Mohammad Ekramol Islam under my supervision in the Department of Mathematics, Cochin University of Science and Technology. The result embodied in the thesis have not been included in any other thesis submitted previously for the award of any degree or diploma.

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Chapter 1

Introduction

1.1 Inventory System and its Motivation

Inventory management of physical goods or other products or elements is an integral part of logistic systems common to all sectors of the economy, such as business, industry, agriculture and defense. In an economy that is perfectly predictable, inventory may be needed to take advantage of the economic features of a particular technology, or to syncronize human tasks, or to regulate the production process to meet the changing trends in demand. When uncertainty is present, inventories are used as a protection against risk of stockout.

The existance of inventory in a system generally implies the existance of an organized complex system involving inflow, accumulation, and outflow of some commodities or goods or items or products. For example, in business the inflow of goods is generated through procurement, purchase, or production. The outflow is generated through demand for the goods. Finally, the difference between the rate of outflow and the rate of inflow generates an inventory for the goods.

The regulation and control of inventory must proceed within the context of this organized system. Thus inventories, rather than being interpreted as idle resources, should be regarded as a very essential element, the study of which may provide insight into the aggregate operation of the system. The scientific analysis of inventory systems define the degree of interrelationship between inflow and outflow and identifies economic control methods for operating such systems.

There are several factors affecting the inventory. They are demand, life-times of items stored, damage due to external disaster, production rate, the time lag between order and supply, availability of space in the store etc. If all the parameters are known beforehand, then the inventory is called deterministic. If some or all of these parameters are not known with certainty, then it is justifiable to consider them as random variables with some probability distribution and the resulting inventory is then called stochastic or probabilistic. System in which one commodity is held independent of other commodities are analyzed as single commodity inventory problems. Multi-commodity inventory problems deal with two or more commodities held together with some form of dependence. Inventory systems may again be classified as continuous review or periodic review. A continuous review policy is to check inventory level continuously in time and a periodic review policy is to monitor the system at discrete, equally spaced instants of time.

Efficient management of inventory system is done by finding out optimal values of the decision variables. The important decision variables in an inventory system are order level or maximum capacity of the inventory, re-ordering point, scheduling period and lot-size or order quantity. They are usaully represented by the letters S, s, t and q respectively. Different policies are obtained when different combinations of decision variables are selected. Existing prominent inventory policies are: (i) (s, S)-policy in which an order is placed for a quantity up to S whenever the inventory level falls to s or below. (ii) (s, q)-policy where the order is given for q quantity when the inventory level is in s or below. (iii) (t, S)-policy which places an order at scheduling periods of length t so as to bring back the inventory level up to S and (iv) (t, q)-policy that gives an order for q quantity at epochs of t interval length.

The time elapsed between an order and the consequent replenishment is termed as lead time. If the replenishment is instantaneous, then lead time is zero and the system is then called an inventory system without lead time. Inventory models with positive lead time are complex to analyze; still more complex are the models where the lead times are taken to be random variables.

Shortage of inventory occur in systems with positive lead time, in systems with negative re-ordering points, or in multi-commodity inventory systems in which an order is placed only when the inventory levels of at least two commodities fall to or below their re-ordering points. There are different methods to face the stock out periods of the inventory. One of the methods is to consider the demands during the dry periods as lost sales. The other is partial or full backlogging of the demands during these period.

In most of the analysis of inventory systems the decay and disaster factors are ignored. In several existing models, it is assumed that products have infinite shelf-life. But in several practical situations, a certain amount of decay or waste is experienced on the stocked items. For example this may arise in certain food products subjected to deterioration or radio-active materials where decay is present, or volatile fluids under evaporation. These deterioration of items in the inventory system occur due to one or many factors such as storage condition, weather condition including the nature of the particular product under study.Some items in the inventory system may deteriorate wheras other can be stored for an indefinite period without deterioration. The deterioration is usually a function of the total amount of inventory on hand. This is one of the crucial factor that affect the inventory system.

1.2 Literature survey

The mathematical analysis of inventory problem was started by Harris [30]. He proposed the famous EOQ formula that was popularized by Wilson. Pierre Masse [63] discussed the stochastic behaviour of the inventory in the case of scheduling the use of stored water to minimize the cost of supplying electric energy. He obtained a satisfactory result regarding this problem. The first paper related to (s, S)-policy is by Arrow, Harris and Marchak [3].They showed that the total expected cost incurred from the use of an (s, S)-policy satisfied a renewal equation. Dvorestzky, Kiefer and Wolfowitz [17] established sufficient conditions an (s, S)-policy for the single stage inventory problem to be optimal. Whitin [89] and Gani [21] have sumarized several results in storage systems.

A systematic account of the (s, S) inventory policy is provided Arrow, Karlin and Scarf [4] based on renewal theory. Hadley and Whitin [29] give several applications of different inventory models. In a review article Veinott [87] provided a detailed account of the work carried out in inventory theory. Naddore [56] compared different inventory policies by discussing their cost analysis. Gross and Harris [27] considered the inventory system with state dependent lead times. In a later work [26] they dealt with the idea of dependence between replenishment times and the number of outstanding orders. Tijms [86] contains a detailed analysis of the inventory system under (s, S)-policy.

Sivazlian [79] analyzed the continuous review (s, S) inventory system with arbitrarily distributed interarrival times and unit demands. He showed that the limiting distribution of the position inventory is uniform and independent of the interarrival time distribution. Richards [70] proved the same result for compound renewal demands. Later [71] he dealt with a continuous review (s, S) inventory system in which the demand for items in inventory is dependent on an external environment. Sahin [75] discussed continuous review (s, S) inventory with continuous state space and constant lead times. Srinivasan [81] extended Sivazlian's result to the case of arbitarily distributed lead times. He derived explicit expressions for probability mass function of the stock level and extracted steady state results. This was further extended by Manoharan, Krishnamoorthy and Madhusoodanan [54] to the case of non-identically distributed inter-arrival times. Sahin [74] derived the binomial moments of the transient and stationary distributions of the number of backlogs in a continuous review (s, S) inventory model with arbitrarily distributed lead time and compound renewal demand. Thangaraj and Ramanarayanan [85] deal with an inventory system with random lead time having two order levels.

Kalpakam and Sapna [38] analyze an (s, S) ordering policy in which items are procured on an emergency basis during stock out period. Again they [39] dealt with the problem of controlling the replenishment rates in a lost sales inventory system with compound Poisson demands and two re-order levels with varying order quantities. Prasad [64] developed a new classification system that compare different inventory systems. Hill [31] analyzed a continuous review lost sales inventory model in which more than one order may be outstanding. Perry et al.[62] analyzed continuous review inventory systems with exponential random yield by the techniques of level crossing theory. Sapna [77] deals with (s, S) inventory system with priority customers and arbitrary lead time distribution. Kalpakam and Sapna [40] discuss an environment dependent (s, S) inventory system with renewal demands and lost sales where the environment changes between available and unavailable periods according to a Markov chain.

A lot of work related to perishable inventory system are reported. Still more work are going on in this direction because of its influencing nature in the inventory system. Ghare and Schrader [22] introduced the concept of exponential decay in inventory problems. Nahmias and Wang [58] derive a heuristic lot size re-order policy for an inventory problem subject to exponential decay. Graves [25] apply the theory of impatient servers to some continuous review perishable inventory models. An exhustive review of the work done in perishable inventory until 1982 can be seen in Nahmias [57]. Kaspi and Perry [42, 43] deal with inventory system with constant life times applicable to blood banks.

Kalpakam and Arivarignan [34, 35] studied a continuous review inventory system having an exhibiting item subject to random failure (exponentially distributed life-times). They [36] extended the result to exhibit items having Erlangian life times under renewal demands. Again they [33] analyzed a perishable inventory model having exponential life-times for all the items. Manoharan and A.Krishnamoorthy [53] considered an inventory problem with all items subject to decay and having arbitrary interarrival time distribution. They derived the system state limiting probabilities. Srinivasan [83] investigated an inventory model of decaying items with positive lead time under (s, S) policy. Incorporating adjustable re-order size he discussed a solution procedure for inventory model of decaying items.

Liu [50] considered an inventory system with random life-times allowing backlogs, but having zero lead time. He obtained a closed form for the long run cost function and discussed its analytic properties. Raafat [66] provide a survey of decaying inventory models up to [1990]. Ravichandran [67] analyzed an (s, S) perishable inventory system with random replenishment time and Poisson demands. In that study, he assumed that the aging of the new stock begins only after exhausting the existing stock and some analytical results were obtained. Using Matrix Analytic Method, Liu and Yang [51] analyzed an (s, S) inventory model with random shelf-time, exponential replenishing time and no restriction on the number of backlogged units. Arivarignan, Elango and Arumugam [2] considered a perishable inventory system at a service facility, with arrival of customers forming a Poisson process. Each customer requires a single item which is delivered through a service of random duration having exponential distribution. Several performance measures were given.

Since this thesis provides results on retrial inventory, inventory with postponed demands and inventory with service times we first give the motivation for considering such results.

From Retrial Queues to Retrial Inventory

Queueing system in which arriving customers who find all servers and waiting position (if any) occupied , may retry for service after a period of time, are called retrial queues or queues with repeated attempts. The most obvious example is provided by a person who desires to make a phone call. If the line is busy, then he can not queue up but tries again some time later. Thus, retrial queues are characterised by the following feature: a customer arriving when all servers accessible for him are busy, leaves the service area but after some random time repeats his demand. Retrial queues are a type of network with reservicing after blocking. Thus, this network contains two nodes: the main node where blocking is possible and a delay node for repeated attempts. As for other networks with blocking, the investigation of such systems presents great analytical difficulties. Nevertheless, the main feature of the theory of retrial queueing systems as an independent part of queueing theory are quite clearly drawn. In particular, the nature of results obtained, methods of analysis and areas of applications allow us to devide retrial queues into three large groups in a natural way: Single-channel system, multi-channel fully available systems and structually complex systems. The standard queueing models do not take into account the phenomenon of retrials and therefore can not be applied in solving a number of practically important problems. Retrial queues have been introduced to solve this deficiency.

On the other hand retrial in inventory occurs as follows: Customers arrive to an establishment demanding an item. If the item is available the same is supplied (may be with negligible service time or with a positive (not necessarily random) service time). However, when at a demand epoch the item is out of stock, such customers need not be backlogged nor lost. An alternative to these is the retrial by such customers. At random epochs of time such customers retries until either the demand is met or finally the customer decides not to approach that establishment (may be he is no more in need of the item or he procures it from elsewhere).

Queues with Postponed Work and Inventory with Pooled Customers

Postponement of work is a common phenomena. This may be to attend a more important job than the one being processed at present or for a break or due to lack of quorum (in case of bulk service) and so on. Postponement of service to customers take place in different ways depending on the nature of the input and service process. For example in the case of priority queues service to customers of lower priority stands postponed when one of the higher priority calls on. In the case of preemptive service, customers of lower priority in service is pushed out the moment one with higher priority arrives. For further details on priority queues one may refer to, for example, Gross and Harris [28] Jaiswal [32], Takagi [84]. Queues with vacation to server also can be regarded as a queue where work stands postponed. For example in gated vacation, the server closes a gate behind the last customer in the system before the start of a service on return from vacation. For details refer to Takagi [84]. In the case of queues with general bulk service rule for example Neuts [60], the service of the next batch customers stands postponed until a minimum of 'a' are available at a service completion epoch. In control policies such as N.T.D. a busy cycle stands only an accumulation of N customers in the system, an elapse of T time unit, the place or the work load accumulator to D, respectively. Hence these control policies can also be regarded as postponement of service.

On certain occations postponement of work reduces partly or some time completely customer impatience, especially in the context of priority queues. There are several other means of reducing customer impatience. Of these the one introduced by Qi-Ming He and Neuts [65] deserves special mention. They devised a control machanism of a system consisting of two queues served by two different servers, by introducing transfer of customers in bulk from the larger to the shorter queue. They established that even when the queues are not separately stable the combined system can be stable. By identifying a two dimensional Markov chain with one component representing the sum of the number of customers in the two queues and the other, the difference between queue 1 and queue 2, they analyzed the resulting system as a level independent QBD. Some earlier work involving transfer of customers (jockeying) could be found in [90, 92, 93].

In this thesis we introduce postponement of suppy of the items to a demand as described below. At a demand epoch if the item is out of stock then such customers are directed to a pool. Such customers are referred to as pooled customers / postponed work (demands). On replenishment customers from the pool are selected for providing the item according to some rules as described in chapters to follow. This is an alternative to backlogging of demands where at the time of arrival of a customer the system is found to be out of stock. Whereas in backlogged case such customers are provided the item immediately on replenishment, in the case postponed demand, this facility is not extended to the customers. In the latter the system management takes the decisions as to when the 'postponed customers'be served.

Inventory with Service Time

In all works reported in inventory prior to 1993 it was assumed that the time required to serve the item to the customer is negligible. Berman, Kim and Shimshak [9] is the first attempt to introduce positive service time in inventory, where it was assumed that service time is deterministic. Latter Berman and Kim [10, 11] extended this results to random service time. Some other work reported in inventory with service time are Berman and Sapna[12, 13] investigate inventory control at a service facility, which uses one item of inventory for service provided. Assuming Poisson arrival process, arbitrarily distributed service times and zero lead time they analyze the system with the restriction that waiting space is finite. Under a specific cost structure they derive the optimum ordering quantity that minimizes the long run expected cost rate. With all these still there are only a handful of papers that deals with inventory involving service time. In a few chapters to follow we consider inventory with random service times.

1.3 Outline of the Present Work

This thesis is divided into seven chapters including this introductory chapter. Second chaper contains two models. In the first model we analyze an (s, S) production Inventory system with retrial of customers. Arrival of customers from out side the system form a Poisson process. When the inventory level reaches s due to the external demand or due to purchases made by orbital customers, the system is immediately converted to ON mode from the OFF mode i,e. production starts. The inter production times are exponentially distributed with parameter μ. When inventory level reaches zero further arriving demands are sent to the orbit which has capacity $M(< \infty)$. Customers, who find the orbit full and inventory level at zero are lost to the system.Service to the the orbital customers or external demands are provided if atleast one item is in the inventory. Demands arising from the orbital customers are exponentially distributed with parameter γ. The long run joint probability distribution of the number of customers in the orbit and the inventory level is obtained. Some measures of the system performance in the long run are derived and numerrical illustrations provided. In the model-II we extend these results to perishable inventory system assuming that the life-time of each item follows exponential distribution with parameter θ. Also it is assumed that when inventory level is zero the arriving demands choose to enter the orbit with probability β and with probability $(1 - \beta)$ it is lost for ever. All assumptions of model -I hold in this case also. Here again the long run joint probability distribution of the number of customers in the orbit and the inventory level is obtained. Some measures of the system performance in the long run are derived and numerrical illustrations provided.

Third chapter deals with an (s, S) production inventory with service times and retrial of unsatisfied customers. Primary demands occur according to a Markovian Arrival Process (MAP). In this system, there is a buffer which has finite capacity equal to inventory level in the system at any given time. When the maximum buffer size is reached, further demands proceed to an orbit of infinite capacity. Initially the system is assumed to be in S and in OFF mode. When inventory level reaches s due to service provided to customers production starts, production follow the PH- distribution. The orbital customers try their luck to access the buffer for service at a constant rate. Service times of customers are exponentially distributed. Using matrix analytic method, the steady state analysis of the system is performed. Some performance measures are obtained and a few numerical illustrations provided. Further we also discuss the particular case of the system where arrival form a MAP and production process follows exponential distribution. Based on these we list some system performance measures and finally provide some numerical illustrations.

In the fourth chapter we consider an (s, S)-retrial inventory with service time in which primary demands occur according to a Batch Markovian Arrival Process (BMAP). The inventory is controlled by the (s, S) policy. Replenishment times are assumed to follow exponential distribution with parameter β. In this system, there is a buffer which is of finite capacity equal to inventory level in the system at any given time, when the maximum buffer size is reached, further demands proceed to an orbit of infinite capacity. The orbital customers try their luck to access the buffer for service with constant retrial rate θ. Service time of the customers are exponentially distributed with parameter μ. Using matrix analytic method the steady state analysis of the system is performed. Some performance measures are listed and provide a few numerical illustrations.

Chapter five deals with an (s, S) inventory system with random service time. Primary demands occur according to Poisson process with parameter λ. In this system there is a finite buffer whose capacity varies according to the inventory level at any given time. When the maximum buffer size is reached, further demands join a pool of infinite capacity with probability γ and with probability $(1-\gamma)$ it is lost for ever. When inventory level is larger than the number of customers in the buffer, an external demand can enter the buffer for service.Two models are discussed in that chapter. In model-I, we assume that a pooled customer is transfered to the buffer for service at a service completion epoch with probability p if the inventory level exceeds $s+1$ and also larger than the number of customers in the buffer. In model-II, we extend the model-I by including the assumption that when inventory level is atleast one and no customer is in the buffer then also with probability one a pooled customer is picked up for service. It is assumed that initially the inventory level is S. When inventory level reaches to s due to service an order for replenishment is placed. The lead is exponentially distributed with parameter β.

For both of the models; we obtain the steady state system size distribution, some performance measures are obtained and a few numerrical illustrations provided.

In the sixth chapter we consider two models. In the first model we analyze an (s, S) Inventory system with postponed demands where arrivals of demands form a Poisson process. When inventory level reaches zero due to demands, further demands are sent to a pool which has capacity $M(< \infty)$. Demands of the pooled customers will be met after replenishment against the order placed. Further they are served only if the inventory level is atleast $s + 1$. The lead time is exponentially distributed.The joint probability distribution of the number of customers in the pool and the inventory level is obtained in both the transient and steady state cases. Some measures of the system performance in the steady state are derived and numerical illustrations are provided. In the second model, we extend our result to perishable inventory system assuming that the life-time of each item follows exponential distribution with parameter θ. Also it is assumed that when inventory level is zero the arriving demands choose to enter the pool with probability β and with complementary $(1 - \beta)$ it is lost for ever. All other assumptions of model-I hold in this case also.

In the seventh chapter we analyze an (s, S) production inventory system with switching time. A lot of work is reported under the assumption that the switching time is negligible but this is not the case for several real life situation. Some production system may take significant time to start the production run. We assume the switching time to be exponentially distributed. Shortages are allowed and infinite backlog permitted. Identifying a two dimensional Markov chain, we investigate the optimal switching time for the system in steady state case. Waiting time distribution is derived. A suitable cost function is defined and analyzed. Some numerrical illustrations are provided.

Chapter 2

Inventory System with Retrial of Customers

2.1 Introduction

In this chapter we discuss an (s, S) production Inventory system with retrial of customers. Two models are discussed. In the first model we examine the case in which inventoried items have infinite shelf-life time and in the second model we assume that the items have random shelf-life time which is exponentially distributed with parameter θ

To start with we provide an overview of retrial queues as it is from it (retrial queue) that the concept of retrial in inventory emerged. Retrial Queues deal with the behaviour of queueing systems of customers who could not find a position at the service station at the arrival time. It has been investigated extensively (see the survey papers by Yang and Templeton[91] and Falin [18], the monograph by Falin and Templeton [19] and also the more recent state of art in retrial queues by Artalejo [5]). Retrials of failed components for service was introduce into reliability of k-out-of-n system by Krishnamoorthy and Ushakumari [47]. Artaljo, Krishnamoorthy and Lopez herrero [6] is the first attempt to studying inventory control with positive lead time and

*The results of Model-I of this chapter will appear in the Stochastic Modelling and Application.

**The results of Model-II of this chapter are appeared in Mathematical and Computational Models (Editors : G. Arulmozhi and R. Nadarajan); Allied Pub.; p : 89-98 ; 2003.

retrial of customers who could not get their demands satisfied during their earlier attempts to access the service station.

Work so far reported in inventory with retrials is very little. Except the one mentioned above (Artaljo, Krishnamoorthy and Lopez herrero [6]), no work in this direction has come to our notice. In this chapter we investigate retrial of unsuccessful customers in accessing the service station in an (s, S) production inventory system both for the case of perishable and non-perishable inventoried items.

This chapter is organized as follows: In section 2.2 some assumptions are made for the models. Model-I is discussed in section 2.3. This section contains four subsections. Steady state analysis of the model is studied in the subsection 2.3.1. In subsection 2.3.2 we list some system performance measures and based on that measures a cost function is developed and some numericals are provided in the subsections 2.3.3 and 2.3.4 respectively. In section 2.4 we discuss the model-II. This section contain five subsections. We discuss the model in subsection 2.4.1. In subsection 2.4.2. we studied the system in steady state case for perishable inventory system. System characteristics measure is given in 2.4.3. A cost function is discussed in the subsection 2.4.4 and finally, we provided illustrative numerical examples in subsection 2.4.5.

2.2 Assumptions

1. Initially the inventory level is S.
2. Arrival of demands form a Poisson process with parameter λ.
3. Inter arrival times of items from the production process are exponentially distributed with parameter μ.
4. Production starts when the level depletes to s due to external demands or demands from retrial customers.
5. When the inventory level is zero, incoming customers go to orbit (subject to the maximum capacity) and try their luck after some time with inter-retrial times of each orbital

customer exponentially distributed with parameter γ.

6. Orbit has finite capacity M.

2.3 Model-I

In this model the inventory system starts with S units of the item on stock and production unit is in OFF mode. When the inventory level reaches s, due to demands from primary or orbital customers, the system is immediately switched on to ON mode ie. production starts. The time required to produce one unit of the item is exponentially distributed with parameter μ. When inventory level reaches zero, the incoming customers join an orbit of finite capacity M (provided it is not full) and try their luck after some time. Thus customers who encounter the system when inventory level is zero and orbit full are lost. Demands arrive according to a Poisson process with rate λ. Each orbital customer try to access the service counter such that the inter retrial times follow exponential distribution with parameter $k\gamma$ when there are k customers in the orbit. If atleast one unit of the item is available the demand will be met immediately; otherwise the customer return to the orbit. The production will remain in ON mode until the inventory level reaches to S. Let
$I(t), t \geq 0$, be the inventory level at time t.
$N(t), t \geq 0$, be the number of customers in the orbit at time t.

Define

$$X(t) = \begin{cases} 1 & \text{if the system is in ON mode} \\ 0 & \text{if the system is in OFF mode} \end{cases}$$

To get continuous time Markov process, we consider $\{(I(t), X(t), N(t)), t \geq 0\}$ whose state space is $E = E_1 \cup E_2$ where,

$$E_1 = \{(i, 0, N) : i = s+1, s+2, \cdots, S; N = 0, 1, \cdots, M\}$$
$$E_2 = \{(i, 1, N) : i = 0, 1, 2, \cdots, S-1; N = 0, 1, \cdots, M\}$$

The infinitisemal generator of the process is given by
$\tilde{A}$=$(a(i,j,k:l,m,n))$; $(i,j,k),(l,m,n) \in E$ where

$$a(i,j,k:l,m,n) = \begin{cases} \lambda & \text{if } i = s+2,\cdots,S; j=0, k=0 \\ & l=i-1; m=j, n=k \\ \lambda & \text{if } i = s+1; j=0, k=0,1,\cdots,M \\ & l=i-1; m=1; n=k \\ \lambda & \text{if } i = 1,2,\cdots,S-1; j=1; k=0,1,\cdots,M \\ & l=i-1; m=j, n=k \\ k\gamma & \text{if } i = 1,2,\cdots,S-1; j=1, k=1,2,\cdots,M \\ & l=i-1; m=j; n=k-1 \\ \lambda & \text{if } i = 0; j=1, k=0,1,\cdots,M-1 \\ & l=i; m=j; n=k+1 \\ \mu & \text{if } i = 0,1,\cdots,S-2; j=1, k=0,1,\cdots,M \\ & l=i+1; m=j; n=k \\ \mu & \text{if } i = S-1; j=1, k=0,1,\cdots,M \\ & l=i+1; m=0; n=k \\ -(\lambda+\mu) & \text{if } i = 0,1,\cdots,S-1; j=1, k=0,1,\cdots,M \\ & l=i; m=j; n=k \\ -(\lambda+\mu+k\gamma) & \text{if } i = 0,1,\cdots,S-1; j=1, k=1,\cdots,M \\ & l=i; m=j; n=k \\ -(\lambda+k\gamma) & \text{if } i = s+1,\cdots,S; j=0, k=1,\cdots,M \\ & l=i; m=1; n=k \\ k\gamma & \text{if } i = s+1; j=0, k=1,2,\cdots,M \\ & l=i-1; m=1; n=k-1 \\ k\gamma & \text{if } i = s+2,\cdots,S; j=0, k=1,2,\cdots,M \\ & l=i-1; m=j; n=k-1 \end{cases}$$

Write $A_{il} = (a(i,j,k;l,m,n))$

Then the infinitesimal generator $\tilde{A}$ can be convinientIy expressed as a pertitioned matrix $\tilde{A} = ((A_{il}))$ where A_{il} is an $(M+1) \times (M+1)$ matrix which is given by,

$$A_{il} = \begin{cases} A_1 & \text{if } i = s+2,\cdots,S; l = i-1 \text{ and production off or} \\ & i = 1,2,\cdots,S-1; l = i-1 \text{ and production on or} \\ & i = s+1; l = i-1 \text{ and production off} \\ A_2 & \text{if } i = 0,1,\cdots,S-2; l = i+1 \text{ and production on or} \\ & i = S-1; l = i+1 \text{ and production on} \\ A_3 & \text{if } i = s+1,\cdots,S; l = i \text{ and production off} \\ A_4 & \text{if } i = 1,2,\cdots,S-1; l = i \text{ and production on} \\ A_5 & \text{if } i = 0; l = i \text{ and production on} \\ 0 & \text{otherwise} \end{cases}$$

with

$$A_1 = \begin{matrix} M \\ M-1 \\ M-2 \\ \cdots \\ 2 \\ 1 \\ 0 \end{matrix} \begin{pmatrix} \lambda & M\gamma & 0 & 0 & 0 & 0 \\ 0 & \lambda & (M-1)\gamma & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 \\ \cdots & & & & & \cdots \\ 0 & 0 & 0 & \lambda & 2\gamma & 0 \\ 0 & 0 & 0 & 0 & \lambda & \gamma \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$$

$$A_2 = \begin{matrix} M \\ M-1 \\ \dots \\ 1 \\ 0 \end{matrix} \begin{pmatrix} \mu & 0 & & 0 & 0 \\ 0 & \mu & & 0 & 0 \\ \dots & & & & \dots \\ 0 & 0 & & \mu & 0 \\ 0 & 0 & & 0 & \mu \end{pmatrix}$$

$$A_3 = \begin{matrix} M \\ M-1 \\ \dots \\ 1 \\ 0 \end{matrix} \begin{pmatrix} -(\lambda + M\gamma) & 0 & 0 & 0 \\ 0 & -(\lambda + (M-1)\gamma) & 0 & 0 \\ & & & \\ 0 & 0 & -(\lambda+\gamma) & 0 \\ 0 & 0 & 0 & -\lambda \end{pmatrix}$$

$$A_4 = \begin{matrix} M \\ M-1 \\ \dots \\ 1 \\ 0 \end{matrix} \begin{pmatrix} -(\lambda + \mu + M\gamma) & 0 & 0 & 0 \\ 0 & -(\lambda + \mu + (M-1)\gamma) & 0 & 0 \\ & & & \\ 0 & 0 & -(\lambda+\mu+\gamma) & 0 \\ 0 & 0 & 0 & -(\lambda+\mu) \end{pmatrix}$$

$$A_5 = \begin{matrix} M \\ M-1 \\ M-2 \\ \dots \\ 2 \\ 1 \\ 0 \end{matrix} \begin{pmatrix} -\mu & 0 & 0 & 0 & 0 & 0 \\ \lambda & -(\lambda+\mu) & 0 & 0 & 0 & 0 \\ 0 & \lambda & -(\lambda+\mu) & 0 & 0 & 0 \\ \dots & \dots & & & \dots & \\ 0 & 0 & 0 & -(\lambda+\mu) & 0 & 0 \\ 0 & 0 & 0 & \lambda & -(\lambda+\mu) & 0 \\ 0 & 0 & 0 & 0 & \lambda & -(\lambda+\mu) \end{pmatrix}$$

Thus we can write $\tilde{A}$ in the partitioned form as

$$\tilde{A} = \begin{array}{l} (S,0) \\ (S-1,0) \\ S-2 \\ \cdots \\ (s+1,0) \\ (S-1,1) \\ \cdots \\ (s+1,1) \\ (s,1) \\ (s-1,1) \\ \\ 2 \\ 1 \\ 0 \end{array} \begin{pmatrix} A_3 & A_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_3 & A_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & A_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \cdots & & & & & & & & & & \\ 0 & 0 & 0 & A_3 & 0 & 0 & A_1 & 0 & 0 & 0 & 0 \\ A_2 & 0 & 0 & 0 & A_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ \cdots & & & & & & & & & & \cdots \\ 0 & 0 & 0 & 0 & 0 & A_4 & A_1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A_2 & A_4 & A_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_2 & A_4 & 0 & 0 & 0 \\ \cdots & & & & & & & & & \cdots & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_4 & A_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_2 & A_4 & A_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A_2 & A_5 \end{pmatrix}$$

2.3.1 Steady State Analysis

It can be seen from the structure of matrix $\tilde{A}$ that the state space E is irreducible. Let the limiting distribution be denoted by $\Pi^{(i,j,k)}$:

$$\Pi^{(i,j,k)} = \lim_{t\to\infty} Pr[(I(t), X(t), N(t)) = (i,j,k)],\ (i,j,k) \in E$$

$$\text{Write } \Pi = (\Pi^{(S,0)}, \cdots, \Pi^{(s+1,0)}, \Pi^{(S-1,1)}, \Pi^{(S-2,1)}, \cdots, \Pi^{(0,1)})$$

$$\text{and } \Pi^{(K)} = (\Pi^{(K,M)}, \Pi^{(K,M-1)}, \ldots, \Pi^{(K,1)}, \Pi^{(K,0)})$$

$$\text{for } K = (S,0), \cdots, (s+1,0), (S-1,1), \cdots, (0,1)$$

The limiting distribution exists and satisfies the following relations:

$$\Pi\tilde{A} = 0 \text{ and } \sum \Pi^{(i,j,k)} = 1$$

The first equation of the above yields the following set of relations:-

$\Pi^{(S,0)}A_3 + \Pi^{(S-1,1)}A_2 = 0$

$\Pi^{(i+1,0)}A_1 + \Pi^{(i,0)}A_3 = 0 \quad \text{if}: i = s+1, \cdots, S-1$

$\Pi^{(i+1,1)}A_1 + \Pi^{(i,1)}A_4 + \Pi^{(i-1,1)}A_2 = 0 \quad \text{if}: i = s+1, \cdots, S-2$

$\Pi^{(i+1,1)}A_1 + \Pi^{(i+1,0)}A_1 + \Pi^{(i,1)}A_4 + \Pi^{(i-1,1)}A_2 = 0 \quad \text{if}: i = s$

$\Pi^{(i+1,1)}A_1 + \Pi^{(i,1)}A_4 + \Pi^{(i-1,1)}A_2 = 0 \quad \text{if}: i = 1, 2, \cdots, s-1$

$\Pi^{(1,1)}A_1 + \Pi^{(0,1)}A_5 = 0$

The solution of the above equaticns (except the last one) can be conviniently expressed as:-

$$\Pi^{(S-i,0)} = \Pi^{(S,0)}\beta_{(S-i,0)} \text{ and}$$
$$\Pi^{(S-i,1)} = \Pi^{(S,0)}\beta_{(S-i,1)}$$

where

$$\beta_{(S-i,0)} = \begin{cases} I & \text{if } i = 0 \\ A_1A_3^{-1} & \text{if } i = 1 \\ (-1)^i(A_1A_3^{-1})^i & \text{if } i = 2, 3, \cdots, S-s-1 \end{cases}$$

and

$$\beta_{(S-i,1)} = \begin{cases} -A_3A_2^{-1} & \text{if } i = 1 \\ (-1)^{i+1}\beta_{(S-1,1)}(A_4A_2^{-1}) & \text{if } i = 2 \\ (-1)^{i-1}\beta_{(S-1,1)}(A_4A_2^{-1}) + (-1)^i\beta_{(S-1,1)}(A_1A_2^{-1}) & \text{if } i = 3 \\ -\beta_{(S-i+1)}(A_4A_2^{-1}) - \beta_{(S-i+2)}(A_1A_2^{-1}) & \text{if } i = 4, 5, \cdots, S-s \\ -\beta_s(A_4A_2^{-1}) - \beta_{s+1}(A_1A_2^{-1}) - (-1)^{S-s+1}(A_1A_3^{-1})^{S-s+1}(A_1A_2^{-1}) & \text{if } i = S-s+1 \\ -\beta_{(S-i+1)}(A_4A_2^{-1}) - \beta_{(S-i+2)}(A_1A_2^{-1}) & \text{if } i = S-s+2, \cdots, S \end{cases}$$

and to compute $\Pi^{(S,0)}$, we use the relations

$$\Pi^{(1,1)}A_1 + \Pi^{(0,1)}A_5 = 0 \text{ and } \sum \Pi^{(K)}e_{M+1} = 1$$

which yield, respectively,

$$\Pi^{(S,0)}(\beta_{(1,1)}A_1 + \beta_{(0,1)}A_5) = 0 \text{ and } \Pi^{(S,0)}(I + \sum \beta_{(i,0)} + \sum \beta_{(i,1)}) = 1$$

2.3.2 System Characteristics

Mean Inventory Level

Let α_1 denote the average inventory level in the long run. Then we have

$$\alpha_1 = \sum_{i=s+1}^{S} i \sum_{k=0}^{M} \Pi^{(i,0,k)} + \sum_{i=1}^{S-1} i \sum_{k=0}^{M} \Pi^{(i,1,k)}$$

Switching rate

Suppose α_2 is the mean switching rate. Then

$$\alpha_2 = \lambda \sum_{k=0}^{M} \Pi^{(s+1,0,k)} + \sum_{k=1}^{M} k\gamma\Pi^{(s+1,0,k)}$$

Expected Number of orbital Customers

The expected number of orbital customers α_3 is given by

$$\alpha_3 = \sum_{k=1}^{M} k(\sum_{i=0}^{S-1} \Pi^{(i,1,k)} + \sum_{i=s+1}^{S} \Pi^{(i,0,k)})$$

The average number of customer's lost

The average number α_4 of customers lost is,

$$\alpha_4 = \lambda\Pi^{(0,1,M)}$$

Expected Waiting Time

Denote by W_k the waiting time of the k^{th} customer in the orbit, $k = 1, 2, \cdots, M$. We evaluate $E(W_k)$ conditional on the system state. Figure 2.1 provides the transition diagram for computing $E(W_k)$ Thus $E(W_k) = \sum_{k=0}^{M} E(W_k|$ System state at $(0,k))$.P(system in state $(0,k)$) where $E(W_k|$ System state at $(0,k)) = [\frac{(k\gamma)^2+2k\gamma\mu+2k\gamma\lambda}{(\lambda+\mu)(\lambda+k\gamma)^2}]$ for $k = 1, 2, \cdots, M$

Now the average waiting time is

$$\alpha_5 = \sum_{k=1}^{M} \Pi^{(0,1,k)} [\frac{(k\gamma)^2 + 2k\gamma\mu + 2k\gamma\lambda}{(\lambda+\mu)(\lambda+k\gamma)^2}]$$

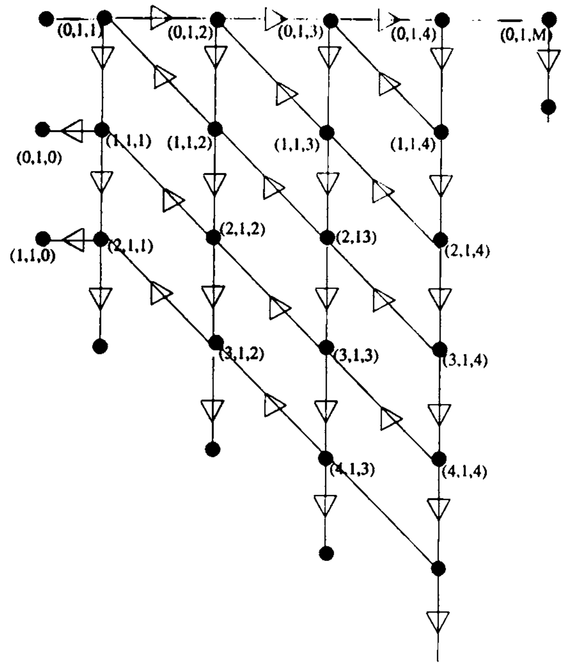

Figure 2.1:

2.3.3 Cost Function

Define

C_1 =Inventory holding cost per unit per unit time

C_2 =Switching Cost for production

C_3 =Loss due to customers lost to the system

So the total expected cost of the system is

$$E(TC) = C_1\alpha_1 + C_2\alpha_2 + C_3\alpha_4$$

2.3.4 Numerical Illustration

By giving values to the underlying parameters we provide some numerical illustrations: Take $S = 5, s = 2, M = 2, \lambda = 0.3, \mu = 0.2, \gamma = 0.1, C_1 = 1, C_2 = 10, C_3 = 2$

Table 2.1:

Average Inventory held	0.879339266
Expected Switching rate	0.103558591
Expected Number of orbital Customers	1.327324771
Expected Number of Lost customers	0.106622418
Expected Total cost of the system	1.140799882

Table 2.2:

M-Values	Expected Waiting Time	Expected total cost
$M = 1$	0.503827566	1.332502691
$M = 2$	0.84547260	1.140799882
$M = 3$	1.081066550	0.939071560
$M = 4$	1.329813020	0.811073726

Table 2.3:

s-Varying	Expected Waiting Time	Expected total cost
$s = 1$	0.853879500	1.020885630
$s = 2$	0.845457526	1.140799882
$s = 3$	0.837596164	1.169143032
S-Varying	Expected Waiting Time	Expected total cost
$S = 5$	0.845457526	1.140799882
$S = 6$	0.832381751	1.203457311
$S = 7$	0.7222560995	1.25459858
$S = 8$	0.715776549	1.26158201
$S = 9$	0.710764684	1.29799481

Then we get the measures as described in Table 2.1. In Table 2.2 the expected total cost is computed by varying over M and in Table 2.3 we vary over s and S keeping other parameter values fixed. Steady state probabilities for $M = 2$ are given in appendix-I

As expected, we see (from Table 2.2) that with M increasing, the expected waiting time of customers in orbit also increase. However expected total cost decreses with increase in value of M, as loss due to customers not admitted to orbit, for want of space, decreases. With S increasing, the expected waiting time of orbital customers decresed (Table 2.3). However the expected total cost increases due to increase in the expected inventory held.

2.4 Model-II

In this model we extended the result of model-I to an (s, S) production inventory system where items produced have random life-times which is exponentially distributed with parameter θ. Also it is assumed that when inventory level is zero the arriving demands choose to enter the orbit with probability β and with probability $(1 - \beta)$ it is lost for ever. All assumptions of model -I hold in this case also.

2.4.1 Model and Analysis

Let

$I(t), t \geq 0$, be the inventory level at time t.

$N(t), t \geq 0$, be the number of customers in the orbit at time t.

Define

$$X(t) = \begin{cases} 1 & \text{if the system is in ON mode} \\ 0 & \text{if the system is in OFF mode} \end{cases}$$

To get continuous time Markov chain, we consider $\{(I(t), X(t), N(t)), t \geq 0\}$ whose state space is $E = E_1 \cup E_2$ where,

$$E_1 = \{(i, 0, N) : i = s + 1, s + 2, \cdots, S; N = 0, 1, \cdots, M\}$$

$$E_2 = \{(i,1,N) : i = 0,1,2,3,\cdots,S-1; N = 0,1,\cdots,M\}$$

The infinitisemal generator $\tilde{A}$ of the process has entries given by,
$\tilde{A} = (a(i,j,k : l,m,n)); (i,j,k),(l,m,n) \in E$, where

$$a((i,j,k:l,m,n)) = \begin{cases} \lambda & \text{if } i = 1,2,\cdots,S-1; j=1, k=0,1,\cdots,M \\ & l=i-1; m=j, n=k \text{ or} \\ & \text{if } i = s+2,\cdots,S; j=0, k=0,1,\cdots,M \\ & l=i-1; m=j, n=k \text{ or} \\ & \text{if } i = s+1; j=0, k=0,1,\cdots,M \\ & l=i-1; m=l, n=k \text{ or} \\ \lambda\beta & \text{if } i=0; j=1, k=0,1,\cdots,M-1; l=0; m=j, n=k+1 \\ -\lambda\beta & \text{if } i=0; j=1, k=0,1,\cdots,M-1; l=i; m=j; n=k \\ \mu & \text{if } i=0,1,\cdots,S-2; j=1, k=0,1,\cdots,M \\ & l=i+1; m=j; n=k \text{ or} \\ & \text{if } i=S-1; j=1, k=0,1,\cdots,M; l=i+1; m=0; n=k \\ -\mu & \text{if } i=0; j=1, k=M; l=i; m=j; n=k \\ k\gamma & \text{if } i=1,2,\cdots,S-1; j=1, k=1,2,\cdots,M \\ & l=i-1; m=j; n=k-1 \text{ or} \\ & \text{if } i=s+2,\cdots,S; j=0, k=1,\cdots,M \\ & l=i-1; m=j; n=k-1 \text{ or} \\ & \text{if } i=s+1; j=0, k=1,\cdots,M; l=i-1; m=1; n=k-1 \\ i\theta & \text{if } i=1,2,\cdots,S-1; j=1, k=0,1,2,\cdots,M \\ & l=i-1; m=j; n=k \text{ or} \\ & \text{if } i=s+2,\cdots,S; j=0, k=1,\cdots,M\ . \\ & l=i-1; m=j; n=k \text{ or} \\ & \text{if } i=s+1; j=0, k=1,\cdots,M; l=i-1; m=1; n=k \\ -(\lambda\beta+\mu) & \text{if } i=0; j=1, k=0,1,\cdots,M-1; l=i; m=j; n=k \\ -(\lambda+i\theta) & \text{if } i=s+1,\cdots,S; j=0, k=0; l=i; m=j; n=k \\ -(\lambda+\mu+i\theta) & \text{if } i=1,\cdots,S-1; j=1, k=0; l=i; m=j; n=k \\ -(\lambda+i\theta+k\gamma) & \text{if } i=s+1,\cdots,S; j=0, k=1,\cdots,M; l=i; m=j; n=k \\ -(\lambda+\mu+i\theta+k\gamma) & \text{if } i=1,\cdots,S-1; j=1, k=1,\cdots,M; l=i; m=j; n=k \end{cases}$$

Define $A_{il} = (a(i,j,k;l,m,n))$

Then the infinitesimal generator $\tilde{A}$ can be conviniently expressed as a partitioned matrix $\tilde{A} = ((A_{il}))$ where A_{il} is a $(M+1)\times(M+1)$ matrix is given by

$$A_{il} = \begin{cases} A & \text{if } i = 0,1,\cdots,S-1; l = i+1 \text{ and the production on} \\ A_i & \text{if } i = s+1,\cdots,S; l = i-1 \text{ and the production off} \\ B_i & \text{if } i = s+1,\cdots,S; l = i \text{ and the production off or} \\ C_i & \text{if } i = 1,2,\cdots,S-1; l = i \text{ and the production on} \\ D & \text{if } i = 0; l = i \text{ and the production on} \\ D_i & \text{if } i = 1,\cdots,S-1; l = i-1 \text{ and the production on} \\ 0 & \text{otherwise} \end{cases}$$

with

$$A = \begin{array}{c} M \\ M-1 \\ M-2 \\ \ldots \\ 2 \\ 1 \\ 0 \end{array} \begin{pmatrix} \mu & 0 & 0 & & 0 & 0 & 0 \\ 0 & \mu & 0 & & 0 & 0 & 0 \\ 0 & 0 & \mu & & 0 & 0 & 0 \\ \ldots & & & & & & \ldots \\ 0 & 0 & 0 & & \mu & 0 & 0 \\ 0 & 0 & 0 & & 0 & \mu & 0 \\ 0 & 0 & 0 & & 0 & 0 & \mu \end{pmatrix}$$

$$A_i = \begin{array}{c} M \\ M-1 \\ M-2 \\ \ldots \\ 1 \\ 0 \end{array} \begin{pmatrix} (\lambda+i\theta) & M\gamma & 0 & 0 & 0 \\ 0 & (\lambda+i\theta) & (M-1)\gamma & 0 & 0 \\ 0 & 0 & (\lambda+i\theta) & 0 & 0 \\ \ldots & & & & \\ 0 & 0 & 0 & (\lambda+i\theta) & \gamma \\ 0 & 0 & 0 & 0 & (\lambda+i\theta) \end{pmatrix}$$

$i = s+1,\cdots,S; l = i-1$ and production is off

$$B_i = \begin{array}{l} M \\ M-1 \\ \ldots \\ 1 \\ 0 \end{array} \left(\begin{array}{cccc} -(\lambda + i\theta + M\gamma) & 0 & 0 & 0 \\ 0 & -(\lambda + i\theta + (M-1)\gamma) & 0 & 0 \\ & & & \\ 0 & 0 & -(\lambda + i\theta + \gamma) & 0 \\ 0 & 0 & 0 & -(\lambda + i\theta) \end{array} \right)$$

$$C_i = \begin{array}{l} M \\ M-1 \\ \ldots \\ 0 \end{array} \left(\begin{array}{ccc} -(\lambda + \mu + i\theta + M\gamma) & 0 & 0 \\ 0 & -(\lambda + \mu + i\theta + (M-1)\gamma) & 0 \\ \ldots & & \\ 0 & 0 & -(\lambda + \mu + i\theta) \end{array} \right)$$

$$D = \begin{array}{l} M \\ M-1 \\ M-2 \\ \ldots \\ 2 \\ 1 \\ 0 \end{array} \left(\begin{array}{ccccccc} -\mu & 0 & 0 & & 0 & 0 & 0 \\ \lambda\beta & -(\lambda\beta + \mu) & 0 & & 0 & 0 & 0 \\ 0 & \lambda\beta & -(\lambda\beta + \mu) & & 0 & 0 & 0 \\ \ldots & \ldots & & & & & \\ 0 & 0 & 0 & & -(\lambda\beta + \mu) & 0 & 0 \\ 0 & 0 & 0 & & \lambda\beta & -(\lambda\beta + \mu) & 0 \\ 0 & 0 & 0 & \ldots & 0 & \lambda\beta & -(\lambda\beta + \mu) \end{array} \right)$$

$$D_i = \begin{array}{l} M \\ M-1 \\ M-2 \\ \cdots \\ 2 \\ 1 \\ 0 \end{array} \begin{pmatrix} (\lambda+i\theta) & M\gamma & 0 & 0 & 0 & 0 \\ 0 & (\lambda+i\theta) & (M-1)\gamma & 0 & 0 & 0 \\ 0 & 0 & (\lambda+i\theta) & 0 & 0 & 0 \\ & & & & & \\ 0 & 0 & 0 & (\lambda+i\theta) & 2\gamma & 0 \\ 0 & 0 & 0 & 0 & (\lambda+i\theta) & \gamma \\ 0 & 0 & 0 & 0 & 0 & (\lambda+i\theta) \end{pmatrix}$$

$$i = 1, \cdots, S-1; l = i-1 \text{ and production is on}$$

So we can write the partitioned matrix as follows:

$$\tilde{A} = \begin{array}{l} (S,0) \\ (S-1,0) \\ \cdots \\ (s+1,0) \\ (S-1,1) \\ \cdots \\ (s+1,1) \\ (s,1) \\ (s-1,1) \\ \\ 2 \\ 1 \\ 0 \end{array} \begin{pmatrix} B_S & A_S & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_{S-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & & & & & & & & \\ 0 & 0 & B_{s+1} & 0 & 0 & A_{s+1} & 0 & 0 & 0 & 0 \\ A & 0 & 0 & C_{S-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & & & & & & & & \cdots \\ 0 & 0 & 0 & 0 & C_{s+1} & D_{s+1} & 0 & 0 & 0 & 0 \\ B & 0 & 0 & 0 & A & C_s & D_s & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & A & C_{s-1} & 0 & 0 & 0 \\ \cdots & \cdots & & & & & & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_2 & D_2 & 0 \\ 0 & 0 & B & 0 & 0 & 0 & 0 & A & C_1 & D_1 \\ 0 & 0 & 0 & B & 0 & 0 & 0 & 0 & A & D \end{pmatrix}$$

2.4.2 Steady State Analysis

It can be seen from the structure of matrix $\tilde{A}$ that the state space E is irreducible. Let the limiting distribution be denoted by $\Pi^{(i,j,k)}$:

$$\Pi^{(i,j,k)} = \lim_{t\to\infty} Pr[I(t), X(t), N(t) = (i, j, k)] \; (i, j, k) \in E$$

$$\text{write } \Pi = (\Pi^{(S,0)}, \cdots, \Pi^{(s+1,0)}, \Pi^{(S-1,1)}, \Pi^{(S-2,1)}, \cdots, \Pi^{(0,1)})$$

$$\text{and } \Pi^{(K)} = (\Pi^{(K,M)}, \Pi^{(K,M-1)}, \ldots, \Pi^{(K,1)}, \Pi^{(K,0)})$$

$$\text{for } K = (S,0), \cdots, (s+1,0), (S-1,1), \cdots, (0,1)$$

The limiting distribution exists and satisfies the following relations:

$$\Pi\tilde{A} = 0 \text{ and } \sum_{i=s+1}^{S} \sum_{j=0}^{M} \Pi^{(i,j,0)} + \sum_{i=0}^{S-1} \sum_{j=0}^{M} \Pi^{(i,j,1)} = 1$$

The first of the above relations yields the following set of equations:-

$\Pi^{(i+1,1)} D_{i+1} + \Pi^{(i,1)} D = 0 \quad \text{for } i = 0$

$\Pi^{(i+1,1)} D_{i+1} + \Pi^{(i,1)} C_i + \Pi^{(i-1,1)} A = 0 \quad \text{for } i = 1, \cdots, s-1 \text{ and for } i = s+1, \cdots, S-2$

$\Pi^{(i+1,0)} A_{i+1} + \Pi^{(i+1,1)} D_{i+1} + \Pi^{(i,1)} C_i + \Pi^{(i-1,1)} A = 0 \quad \text{for } i = s$

$\Pi^{(i+1,0)} A_{i+1} + \Pi^{(i,0)} B_i = 0 \quad \text{for } i = s+1, \cdots, S-1$

$\Pi^{(S,0)} B_S + \Pi^{(S-1,1)} A = 0$

The solution of the above equations (except the last one) can be conviniently expressed as:-

$$\Pi^{(S-i,0)} = \Pi^{(S,0)} \beta_{S-i,0} \text{ and}$$
$$\Pi^{(S-i,1)} = \Pi^{(S,0)} \beta_{S-i,1}$$

where

$$\beta_{S-i,0} = \begin{cases} I & \text{if } i = 0 \\ -A_S B_{S-1}^{-1} & \text{if } i = 1 \\ -\beta_{S-i+1,0} A_{S-i+1} B_{S-1}^{-1} & \text{if } i = 2, 3, \cdots, S-s-1 \end{cases}$$

and

$$\beta_{S-i,1} = \begin{cases} -B_S A^{-1} & \text{if } i = 1 \\ -\beta_{S-1,1} C_{S-1} A^{-1} & \text{if } i = 2 \\ -\beta_{S-i+2,1} D_{S-i+2} A^{-1} - \beta_{S-i+1,1} C_{S-i+1} A^{-1} & \text{if } i = 3, \cdots, S-s \\ -\beta_{s+1,0} A_{s+1} A^{-1} - \beta_{s+1,1} D_{s+1} A^{-1} - \beta_{s,1} C_s A^{-1} & \text{if } i = S-s+1 \\ -\beta_{S-i+2,1} D_{S-i+2} A^{-1} - \beta_{S-i+1,1} C_{S-i+1} A^{-1} & \text{if } i = S-s+2, \cdots, S \end{cases}$$

To compute $\Pi^{(S,0)}$, we use the relations

$\Pi^{(1,1)} D_1 + \Pi^{(0,1)} D = 0$ and $\sum \Pi^{(K)} e_{M+1} = 1$

which yield, respectively,

$\Pi^{(S,0)}(\beta_{1,1} D_1 + \beta_{0,1} D) = 0$ and

$\Pi^{(S,0)}(I + \sum_{i=s+1}^{S} \beta_{i,0} + \sum_{i=0}^{S-1} \beta_{i,1}) = 1$

2.4.3 System Characteristics

Mean Inventory Level

Let α_1 denote the average inventory level in the long run. Then we have

$$\alpha_1 = \sum_{i=s+1}^{S} i \sum_{k=0}^{M} \Pi^{(i,0,k)} + \sum_{i=1}^{S-1} i \sum_{k=0}^{M} \Pi^{(i,1,k)}$$

Switching rate

Suppose α_2 is the mean switching rate. Then we have

$$\alpha_2 = \lambda \sum_{k=0}^{M} \Pi^{(s+1,0,k)} + \sum_{k=1}^{M} k\gamma \Pi^{(s+1,0,k)} + (s+1)\theta \sum_{k=0}^{M} \Pi^{(s+1,0,k)}$$

2.4 Model-II

Expected Number of orbital Customers

The expected number α_3 of orbital customers is given by

$$\alpha_3 = \sum_{k=1}^{M} k \sum_{i=0}^{S-1} \Pi^{(i,1,k)} + \sum_{k=1}^{M} k \sum_{i=s+1}^{S} \Pi^{(i,0,k)}$$

The average number of customer's lost to the system

The average number α_4 of customers lost is

$$\alpha_4 = \lambda \Pi^{(0,1,M)} + (1-\beta)\lambda \sum_{k=0}^{M-1} \Pi^{(0,1,k)}$$

Mean Number of Perished items

The mean number of items that perish in the system is

$$\alpha_5 = \sum_{i=s+1}^{S} i\theta \sum_{k=0}^{M} \Pi^{(i,0,k)} + \sum_{i=1}^{S-1} i\theta \sum_{k=0}^{M} \Pi^{(i,1,k)}$$

The probability that an external demand will be satisfied immediately on it's arrival

The probability that an external demand will be satisfied immediately on arrival is

$$\alpha_6 = \sum_{i=s+1}^{S} \sum_{k=0}^{M} \Pi^{(i,0,k)} + \sum_{i=1}^{S-1} \sum_{k=0}^{M} \Pi^{(i,1,k)}$$

The rate that an external demand enters the orbit

The rate that an external demand enters the orbit is

$$\alpha_7 = \lambda\beta \sum_{k=0}^{M-1} \Pi^{(0,1,k)}$$

2.4.4 Cost Function

Define

L =Set up cost of production system.

C_1 =holding cost per unit per unit time

C_2 =Switching Cost for production

C_3 =Cost due to decay of items

C_4=Loss to the system due to customers not joining the system

So, the total expected cost of the system is

$$E(TC) = C_1\alpha_1 + C_2\alpha_2 + C_3\alpha_5 + C_4\alpha_4$$

2.4.5 Numerical Illustration

Since analytical expressions are impossible to arrive at we provide some numerical illustrations by giving values to the underlying parameters . Take

$$L = 3, S = 5, s = 2, M = 3, \lambda = 0.3, \mu = 0.2, \gamma = 0.2,$$
$$\beta = 0.6, \theta = 0.1, C_1 = 1, C_2 = 10, C_3 = 2, C_4 = 3$$

Thus we get the measures as described in the following table and the long run system state probabilities corresponding to the above parameters is given in the Apendix-II.

Table 2.4:

α_1 =Average Inventory held in the system	0.379025
α_2 =Expected Switching rate of the system	0.000733156
α_3 =Expected Number of orbital Customers	1.81155
α_4 =Expected Number of Lost customers	0.1233402
α_5 =Average perish items in the system	0.0379025
α_6 =Probability that an external demand will be satisfied	0.280116
α_7 =Probability that the arrival demand will enter the orbit	0.431609
Expected Total cost of the system	4.218538

Appendix-I

$\Pi^{(5,0,2)}$	0.001474239	$\Pi^{(3,1,2)}$	0.012899591
$\Pi^{(5,0,1)}$	0.002777491	$\Pi^{(3,1,1)}$	0.016664942
$\Pi^{(5,0,0)}$	0.006854095	$\Pi^{(3,1,0)}$	0.025702856
$\Pi^{(4,0,2)}$	0.000884543	$\Pi^{(2,1,2)}$	0.039620173
$\Pi^{(4,0,1)}$	0.002820237	$\Pi^{(2,1,1)}$	0.03797676
$\Pi^{(4,0,0)}$	0.007779924	$\Pi^{(2,1,0)}$	0.046057937
$\Pi^{(3,0,2)}$	0.000530726	$\Pi^{(1,1,2)}$	0.11852513
$\Pi^{(3,0,1)}$	0.004850539	$\Pi^{(1,1,1)}$	0.071944262
$\Pi^{(3,0,0)}$	0.0087200004	$\Pi^{(1,1,0)}$	0.053899355
$\Pi^{(4,1,2)}$	0.0036855977	$\Pi^{(0,1,2)}$	0.355408063
$\Pi^{(4,1,1}$	0.005554981	$\Pi^{(0,1,1)}$	0.118413949
$\Pi^{(4,1,0)}$	0.010281142	$\Pi^{(0,1,0)}$	0.046672954

Appendix-II

$\Pi^{(5,0,3)}$	0.000024924	$\Pi^{(3,1,3)}$	0.001308551
$\Pi^{(5,0,2)}$	0.000111757	$\Pi^{(3,1,2)}$	0.002179281
$\Pi^{(5,0,1)}$	0.000130351	$\Pi^{(3,1,1)}$	0.003584683
$\Pi^{(5,0,0)}$	0.000384401	$\Pi^{(3,1,0)}$	0.006919222
$\Pi^{(4,0,3)}$	0.000015338	$\Pi^{(2,1,3)}$	0.008549204
$\Pi^{(4,0,2)}$	0.000054234	$\Pi^{(2,1,2)}$	0.011378816
$\Pi^{(4,0,1)}$	0.000140703	$\Pi^{(2,1,1)}$	0.014971675
$\Pi^{(4,0,0)}$	0.000476559	$\Pi^{(2,1,0)}$	0.021643543
$\Pi^{(3,0,3)}$	0.000008947	$\Pi^{(1,1,3)}$	0.051617252
$\Pi^{(3,0,2)}$	0.000047167	$\Pi^{(1,1,2)}$	0.051951493
$\Pi^{(3,0,1)}$	0.000150232	$\Pi^{(1,1,1)}$	0.051714957
$\Pi^{(3,0,0)}$	0.000602888	$\Pi^{(1,1,0)}$	0.049451038
$\Pi^{(4,1,3)}$	0.000174473	$\Pi^{(0,1,3)}$	0.288330006
$\Pi^{(4,1,2)}$	0.000335273	$\Pi^{(0,1,2)}$	0.20566344
$\Pi^{(4,1,1)}$	0.000651758	$\Pi^{(0,1,1)}$	0.146672935
$\Pi^{(4,1,0)}$	0.001537607	$\Pi^{(0,1,0)}$	0.079272805

Chapter 3

Retrial in PH-Distribution Production Inventory System with MAP Arrivals

3.1 Introduction

In this chapter we consider an (s, S) production inventory with service time and retrial of customers who could not find a berth in the buffer during previous arrivals. Primary arrivals of demands (customers who arrive for the first time) follow a Markovian Arrival Process (MAP). Demands enter the buffer of capacity equal to the number of items held in the inventory at that instant of time. When buffer is full (equal to the number of inventoried items) further demands proceed to an orbit of infinite capacity. The orbital customers try their luck after some random length of time, exponentially distributed with parameter θ. These customers keep on trying until they succeed in finding a berth at the buffer. Service times of customers are exponentially distributed with parameter μ. We assume that initially the inventory level is S and the production mode is OFF. Inventory level decreases by one unit by providing service to each customer in the buffer. When inventory level reaches s production starts i,e. system mode is converted from OFF mode to ON mode. Production follows PH-distribution. Production

*Some results given this chapter appeared in the proceedings of the International Conference on Modern Mathematical Methods of Analysis and Optimization of Telecommunication Networks; September 23-25 '2003; Gomel; Belarus; p: 148-156.

process is continued until the inventory level reaches S.

A brief description of retrial queues was provided in 1.2 and 2.1. So we straight pass on to retrial inventory. Artalejo, Krishnamoorthy and Lopez-Herrero [6] is the first investigation on (s, S) inventory policies with positive lead time and retrial of orbital customers (linear retrial rate) who could not get the item during their earlier attempts. They assumed that the service time for providing the items to the customers is negligible. Several system performance measures were computed. No further work is reported until the present work is taken up.

Berman and Kim [10, 11] was the first effort to analyze the non-deterministic inventory model for service facilities. They analyzed the system in which customers arrive at a service facility according to a Poisson process with service times exponentially distributed where each customer demands exactly one unit of the item in the inventory; both zero lead time and positive lead-time cases were discussed. Berman and Sapna [12, 13] studied inventory control at a service facility, which uses one item if inventory for service provided. Assuming Poisson arrival process, arbitrarily distributed service times and zero lead time they analyzed the system with the restriction that, waiting space is finite. Under specific cost structure they derived the optimum ordering quantity that minimizes the long run expected cost rate.

This chapter is organized as follows: In section 3.2 we describe the mathematical model and study the stability condition. In section 3.3, we list some system performance measures . In section 3.4, we discuss the the particular case when production is exponentially distributed. Steady state analysis is done and stability condition discussed. System performance measures are derived in section 3.5 and based on these we provide numerical examples in section 3.6.

3.2 Model and Analysis

We consider a production inventory system in which initially there are S items and the production process is in OFF mode. Demands from outside occur according to Markovian Arrival Process (MAP). The demands are served singly with service times exponentially distributed with parameter μ. There is a buffer in which customers stay before getting service. The capacity of this buffer is restricted to the number of items held in the inventory at any given instant.

Then this varies from 0 to S. When inventory level reaches to s due to service provided to the customers in the buffer, production starts i,e., system is switched to ON mode. Production process follows PH- distribution. Production continues until inventory level reaches S. Customers who find no place in the buffer go to an orbit of infinite capacity and try their luck to enter the buffer at a constant rate θ.

The MAP, a special class of tractable Markov renewal process, is a rich class of point processes that includes many well known processes and Markov modulated Poisson process. One of the most significant feature of MAP is the underlying Markovian structure and fits ideally in the context of matrix analytic solutions to stochastic models. The continuous time MAP is described as follows:

Let the underlying Markov chain (on a finite set be irreducible) and let $Q^* = (q_{ij})$ be the generator of the Markov chain which is exponentially distributed with parameter $\lambda_i \geq -q_{ii}$, one of the following two events could occur: with probability $p_{ij}(1)$ the transition corresponds to an arrival of 1 customer and the underlying Markov chain is in state j with $1 \leq i, j \leq m$; with probability $p_{ij}(0)$ the transition corresponds to no arrival and the state of the Markov chain moves to j, $j \neq i$. Note that the Markov chain can go from state i to state i only through an arrival. Define matrices $D_k = (d_{ij}(k))$ for $k = 0, 1$ such that $d_{ii}(0) = -\lambda_i$, $1 \leq i, j \leq m$; $d_{ij}(0) = \lambda_i p_{ij}(0)$ for $j \neq i$, $1 \leq i, j \leq m$; and $d_{ij}(1) = \lambda_i p_{ij}(1)$. By assuming D_0 to be a non-singular matrix, the interarrival times will be finite with probability one and the arrival process does not terminate. Hence, we see that D_0 is a stable matrix. The generator Q^* is then given by

$$Q^* = D_0 + D_1$$

Thus, D_0 governing the transitions corresponding to no arrival and D_1 governing those corresponding to one arrival. For use in sequel, let

I_i denote identity matrix of order i,

$\otimes$ stands for Kronecker product of two matrices,

A' means transpose of matrix A,

e denotes column vector of 1's of appropriate order.

Let, $N_1(t)$, $I(t)$, $N_2(t)$, $X(t)$, $J_1(t)$ and $J_2(t)$ denote, respectively, the number of customers in orbit, the number of items held in the inventory, the number of customers in the buffer, the status of the production mechanism (i.e., OFF mode or ON mode), phases of the arrival process and phase of the production process at time t.

Then $\{(N_1(t), I(t), N_2(t), X(t), J_1(t), J_2(t))\}$ is a continous time Markov chain with state space given by

$$\begin{aligned}\Omega =& \{(i,j,k,1,r,l);\ i \geq 0, 0 \leq j \leq S-1, 0 \leq k \leq j, 1 \leq r \leq m, 1 \leq l \leq n\}\\ &\cup \{(i,j,k,0,r);\ i \geq 0, s+1 \leq j \leq S, 0 \leq k \leq j, 1 \leq r \leq m\}\end{aligned}$$

The level i, i $\geq$ 0, is defined as the set of states given by,

$$\begin{aligned}\mathbf{i} =& \{(i,j,k,1,r,l) : 0 \leq j \leq S-1, 0 \leq k \leq j, 1 \leq r \leq m, 1 \leq l \leq n\}\\ &\cup \{(i,j,k,0,r) : s+1 \leq j \leq S, 0 \leq k \leq j, 1 \leq r \leq m\}\end{aligned}$$

These are arranged in the lexicographic order.

Define the following auxiluary matrices for use in sequel

$$\tilde{A}_{00} = D_0 \otimes I_n + I_m \otimes S \tag{3.1}$$

$$\tilde{A}_{ii} = \begin{bmatrix} D_0 \otimes I_n + I_m \otimes S & D_1 \otimes I_n & & \\ & (D_0 \otimes I_n + I_m \otimes S) - \mu I_{mn} & & \\ & & \ddots & \\ & & & D_1 \otimes I_n \\ & & & (D_0 \otimes I_n + I_m \otimes S) - \mu I_{mn} \end{bmatrix}_{(i+1)mn \times (i+1)mn}$$

$$;1 \leq i \leq s \quad (3.2)$$

$$\tilde{A}_{ii} = \begin{bmatrix} D_0 & 0 & D_1 & & & & & \\ & D_0 \otimes I_n + I_m \otimes S & 0 & D_1 \otimes I_n & & & & \\ & & D_0 - \mu I_m & 0 & D_1 & & & \\ & & & (D_0 \otimes I_n + I_m \otimes S) - \mu I_{mn} & 0 & D_1 \otimes I_n & & \\ & & & & \ddots & \ddots & \ddots & \\ & & & & & & & D_1 \otimes I_n \\ & & & & & & D_0 - \mu I_m & 0 \\ & & & & & & & (D_0 \otimes I_n + I_m \otimes S) - \mu I_{mn} \end{bmatrix}$$

$$; s+1 \leq i \leq S-1 \tag{3.3}$$

where $\tilde{A}_{ii}$ is a $(i+1)2mn \times (i+1)2mn$ matrix.

$$\tilde{A}_{SS} = \begin{bmatrix} D_0 & D_1 & 0 & 0 & & \\ 0 & D_0 - \mu I_m & D_1 & 0 & & \\ 0 & 0 & D_0 - \mu I_m & D_1 & & \\ & & & \vdots & & \\ & & & & & D_1 \\ & & & & & D_0 - \mu I_m \end{bmatrix}_{(S+1)m \times (S+1)m} \tag{3.4}$$

$$\tilde{A}_{i,i-1} = \begin{bmatrix} 0 \\ I_i \end{bmatrix}_{(i+1) \times i} \otimes \mu I_{mn} \quad ; 1 \leq i \leq s \tag{3.5}$$

$$\tilde{A}_{s+1,s} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ I_m \otimes \mu\beta & & 0 & \cdots & 0 \\ \mu I_{mn} & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & I_m \otimes \mu\beta \\ 0 & 0 & 0 & \cdots & \mu I_{mn} \end{bmatrix}_{((s+2)mn+(s+2)m) \times (s+1)mn} \tag{3.6}$$

$$\tilde{A}_{i,i-1} = \begin{bmatrix} 0 & 0 & 0 & & 0 \\ 0 & 0 & 0 & & 0 \\ \mu I_m & 0 & 0 & & 0 \\ 0 & \mu I_{mn} & 0 & & 0 \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \mu I_m & 0 \\ 0 & 0 & 0 & 0 & \mu I_{mn} \end{bmatrix}_{((i+1)mn+(i+1)m)\times(imn+im)} \quad ; s+2 \leq i \leq S-1 \tag{3.7}$$

$$\tilde{A}_{S,S-1} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 \\ \mu I_m & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \mu I_m & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & \mu I_m & 0 \end{bmatrix}_{(S+1)m\times(Smn+Sm)} \tag{3.8}$$

$$\tilde{A}_{i,i+1} = (I_{i+1}, 0) \otimes (I_m \otimes S^0\beta) \quad ; 0 \leq i \leq s-1 \tag{3.9}$$

$$\tilde{A}_{s,s+1} = \begin{bmatrix} 0 & I_m \otimes S^0\beta & & & & & \\ & 0 & 0 & I_m \otimes S^0\beta & & & \\ & & & \ddots & & & \\ & & & 0 & I_m \otimes S^0\beta & 0 & 0 \end{bmatrix}_{((s+1)mn)\times((s+2)mn+(s+2)m)} \tag{3.10}$$

$$\tilde{A}_{i,i+1} = \begin{bmatrix} 0 & & & & & & \\ & I_m \otimes S^0\beta & & & & & \\ & & 0 & & & & \\ & & & I_m \otimes S^0\beta & & & \\ & & & & & & \\ & & & & 0 & 0 & 0 \\ & & & & I_m \otimes S^0\beta & 0 & 0 \end{bmatrix} \quad ; s+1 \leq i \leq S-2 \tag{3.11}$$

where $\tilde{A}_{i,i+1}$ is of order $((i+1)mn+(i+1)m) \times ((i+2)mn+(i+2)m)$

$$\tilde{A}_{S-1,S} = \begin{bmatrix} 0 & 0 & & & \\ I_m \otimes S^0 & 0 & & & \\ 0 & 0 & & & \\ 0 & I_m \otimes S^0 & & & \\ & & & & \\ & & & I_m \otimes S^0 & 0 \end{bmatrix}_{(Smn+Sm)\times(S+1)m} \tag{3.12}$$

$$B_0 = \begin{bmatrix} \tilde{A}_{00} & \tilde{A}_{01} & & & & & \\ \tilde{A}_{10} & \tilde{A}_{11} & \tilde{A}_{12} & & & & \\ & \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} & & & \\ & & & \ddots & & & \\ & & \tilde{A}_{s,s-1} & \tilde{A}_{ss} & \tilde{A}_{s,s+1} & & \\ & & & \ddots & & & \\ & & & & & \tilde{A}_{S-1,S-1} & \tilde{A}_{S-1,S} \\ & & & & & \tilde{A}_{S,S-1} & \tilde{A}_{SS} \end{bmatrix} \tag{3.13}$$

$$\bar{A}_{ii} = \begin{bmatrix} D_0 \otimes I_n + I_m \otimes S - \theta I_{mn} & D_1 \otimes I_n & \\ 0 & D_0 \otimes I_n + I_m \otimes S - \mu I_{mn} - \theta I_{mn} & D_1 \otimes I_n \\ & & \\ & & D_0 \otimes I_n + I_m \otimes S - \mu I_{mn} \end{bmatrix} ; 1 \leq i \leq s \tag{3.14}$$

$$\bar{A}_{00} = D_0 \otimes I_n + I_m \otimes S \tag{3.15}$$

$$\bar{A}_{ii} = \begin{bmatrix} D_0 - \theta I_m & 0 & D_1 & & & & & \\ & D_0 \otimes I_n + I_m \otimes S - \theta I_{mn} & 0 & D_1 \otimes I_n & & & & \\ & & D_0 - \mu I_m - \theta I_m & 0 & D_1 & & & \\ & & & D_0 \otimes I_n + I_m \otimes S - \mu I_{mn} - \theta I_{mn} & 0 & D_1 \otimes I_n & & \\ & & & & \ddots & & & \\ & & & & & & D_0 - \mu I_m & 0 \\ & & & & & & & D_0 \otimes I_n + I_m \otimes S - \mu I_{mn} \end{bmatrix} \quad ; s+1 \leq i \leq S-1 \tag{3.16}$$

$$\bar{A}_{SS} = \begin{bmatrix} D_0 - \theta I_m & D_1 & & \\ & D_0 - \mu I_m - \theta I_m & D_1 & \\ & & \ddots & D_1 \\ & & & D_0 - \mu I_m \end{bmatrix} \tag{3.17}$$

$$A_1 = \begin{bmatrix} \tilde{A}_{00} & \tilde{A}_{01} & & & & \\ \tilde{A}_{10} & \bar{A}_{11} & \tilde{A}_{12} & & & \\ & & \ddots & & & \\ & & & \tilde{A}_{S-1,S-2} & \bar{A}_{S-1,S-1} & A_{S-1,S} \\ & & & & \tilde{A}_{S,S-1} & \bar{A}_{SS} \end{bmatrix} \tag{3.18}$$

$$\tilde{B}_{00} = D_1 \otimes I_n \tag{3.19}$$

$$\tilde{B}_{ii} = (a_{i+1} \cdot a'_{i+1}) \otimes (D_1 \otimes I_n) \quad ; 1 \leq i \leq s \tag{3.20}$$

where a_{i+1} is a column vectors of all zeros accept last entry which is 1.

$$\tilde{B}_{ii} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & D_1 & 0 \\ 0 & 0 & \cdots & 0 & D_1 \otimes I_n \end{bmatrix} ; s+1 \leq i \leq S-1 \tag{3.21}$$

$$\tilde{B}_{SS} = (a_{i+1} \cdot a'_{i+1}) \otimes D_1 \tag{3.22}$$

$$A_0 = \begin{bmatrix} \tilde{B}_{00} & & & & & \\ & \tilde{B}_{11} & & & & \\ & & \tilde{B}_{22} & & & \\ & & & \ddots & & \\ & & & & \tilde{B}_{ss} & \\ & & & & & \ddots \\ & & & & & & \tilde{B}_{SS} \end{bmatrix} \tag{3.23}$$

$$\tilde{C}_{00} = \text{ is a zero matrix of order } mn \times mn \tag{3.24}$$

$$\tilde{C}_{ii} = \begin{bmatrix} 0 & I_i \\ 0 & 0 \end{bmatrix}_{(i+1)\times(i+1)} \otimes \theta I_{mn}; \quad 1 \leq i \leq s \tag{3.25}$$

$$\tilde{C}_{ii} = \begin{bmatrix} 0 & 0 & \theta I_m & & & \\ & & & \theta I_{mn} & & \\ & & & & \theta I_m & \\ & & & & & \ddots \\ & & & & & & \theta I_{mn} \\ & & & & & & 0 \\ & & & & & & 0 \end{bmatrix}_{((i+1)mn+(i+1)m)\times((i+1)mn+(i+1)m)} ; s+1 \leq i \leq S-1 \tag{3.26}$$

$$\tilde{C}_{SS} = \begin{bmatrix} 0 & I_i \\ 0 & 0 \end{bmatrix}_{(S+1)\times(S+1)} \otimes \theta I_m \tag{3.27}$$

$$A_2 = \begin{bmatrix} \tilde{C}_{00} & & & & \\ & \tilde{C}_{11} & & & \\ & & \tilde{C}_{22} & & \\ & & & \ddots & \\ & & & & \tilde{C}_{SS} \end{bmatrix} \tag{3.28}$$

The markov chain $\{(N_1(t), I(t), N_2(t), X(t), J_1(t), J_2(t)),\ t \in R_+\}$ has the generator Q in partitioned form given by

$$Q = \begin{bmatrix} B_0 & A_0 & 0 & 0 & 0 & \cdots \\ A_2 & A_1 & A_0 & 0 & 0 & \cdots \\ 0 & A_2 & A_1 & A_0 & 0 & \cdots \\ 0 & 0 & A_2 & A_1 & A_0 & \cdots \\ \vdots & \vdots & \vdots & & & \end{bmatrix}$$

where the entries in Q are given by (3.1) to (3.28).
Let $A = A_0 + A_1 + A_2$ and

π denote the steady-state probability vector of A, i.e,

$$\pi A = 0, \quad \pi e = 1$$

The vector π can be partitioned as

$$\begin{aligned} \pi &= (\pi(0), \pi(1) \dots \pi(s) \dots \pi(S)) \\ \pi(i) &= \pi(i,j,1) \quad ; 0 \le i \le s,\ 0 \le j \le i, \\ \pi(i) &= \pi(i,j,k) \quad ; s+1 \le i \le S-1,\ 0 \le j \le i,\ k = 0,1 \\ \pi(S) &= \pi(S,j,0) \quad ; 0 \le j \le S \end{aligned}$$

We have the following result on system stability

Lemma 3.2.1. *The system is stable if*

$$\Big(\sum_{i=1}^{S-1}\sum_{j=0}^{i-1} \pi(i,j,1)\theta e_{mn} + \sum_{i=s+1}^{S}\sum_{j=0}^{i-1} \pi(i,j,0)\theta e_m\Big) > \Big(\sum_{i=0}^{S-1} \pi(i,i,1)(D_1 e_m \otimes e_n) + \sum_{i=s+1}^{S} \pi(i,i,0)(D_1 e_m)\Big)$$

Proof. From the well-known result [Neuts [61] theorem 1.7.1] on the positive recurrence of Q, which states that

$$\pi A_0 e < \pi A_2 e$$

and by exploiting the structure of the matrices A_0 and A_2, the stated result follows. □

Theorem 3.2.2. *When the stability condition holds the steady state probability vector x of Q which satisfies $xQ = 0, xe = 1$ exists.*

The steady state probability vector

$$x = (x(0), x(1), x(2), \ldots)$$

where components are given by

$$x(i) = x(0)R^i, i \geq o$$

where R is the minimal non-negative solution of the matrix quadratic equation

$$R^2 A_2 + RA_1 + A_0 = 0$$

The vector $x(0)$ can be calculated using the equation

$$x(0)[B_0 + RA_2] = 0$$

together with the normalizing condition

$$x(0)(I - R)^{-1} e = 1$$

Proof. Follows immediately from the well-known result on matrix-geometric methods (see Neuts [61]). □

For calculating the rate matrix R we use Logarithmic Reduction Algorithm.

3.3 System Performance Measures

We partition the steady state probability vector $x = (x(0), x(1), x(2), \ldots)$as

$$\begin{aligned} x(i) &= (y(i,j)) \quad i \geq 0, 0 \leq j \leq S \\ where y(i,j) &= (z(i,j,k,l)) \quad \text{with, } l = 1 \text{ for } 0 \leq j \leq s \\ & \qquad l = 0, 1 \text{ for } s+1 \leq j \leq S-1 \\ & \qquad l = 0 \text{ for } j = S \\ & \qquad 0 \leq k \leq j \end{aligned}$$

Some of the system performance measures are given below:

1. The probability mass function of number of customer in orbit: The probability that there are i customers in the orbit is given by

$$P_i = x(i)e = x(0)R^i e; \quad i \geq 0$$

2. The rate at which the orbiting customers try to enter the buffer is given by

$$\theta_1^* = \theta(1 - x(0)e)$$

3. The rate at which the orbiting customers successfully enter the buffer is given by

$$\theta_2^* = \theta\delta$$

where

$$\delta = \sum_{i=1}^{\infty}[\{\sum_{j=1}^{S-1}\sum_{k=0}^{j-1} z(i,j,k,1)e_{mn}\} + \{\sum_{j=s+1}^{S}\sum_{k=0}^{j-1} z(i,j,k,0)e_m\}]$$

4. Probability that an orbiting customer fail to enter the buffer is,

$$\theta_3^* = 1 - \delta$$

5. Expected Inventory level:

   Expected inventory level is given by

$$E_* = \sum_{i=0}^{\infty}\Big[\{\sum_{j=1}^{S-1} j \sum_{k=0}^{j} z(i,j,k,1)\}e_{mn} + \{\sum_{j=s+1}^{S} j \sum_{k=0}^{j} z(i,j,k,0)\}e_m\Big]$$

6. Expected number of customers in the orbit

$$E_{\text{or}} = \sum_{i=0}^{\infty} ix(i) = x(0)R(I-R)^{-2}e$$

7. Expected number of customers in the buffer is,

$$E_B = \sum_{i=0}^{\infty}\Big[\sum_{j=1}^{S-1}\sum_{k=1}^{j} kz(i,j.K,1)e_{mn} + \sum_{j=s+1}^{S}\sum_{k=1}^{j} kz(i,j,k,0)e_m\Big]$$

8. The fraction of retrials that are successful

$$F = \frac{\text{The rate of retrials that are successful}}{\text{The overeall rate}} = \frac{\theta_1^*}{\theta_1^*}$$

9. The factorial moments of the orbit size is,

$$k!x_0R^k(I-R)^{-1-k}e, \quad k \geq 1$$

3.4 Exponentially Distributed Production Process

Let $N_1(t)$, $I(t)$, $N_2(t)$, $X(t)$ and $J(t)$ denote, respectively, the number of customers in orbit, the number of items held in the inventory, the number of customers in the buffer, the status of the production mechanism (ie., in OFF mode or ON mode) and phase of the arrival process at time t. Then $\{(N_1(t), I(t), N_2(t), X(t), J(t))\}$ is a continuous time Markov chain with state space given by

$$\Omega = \{(i,j,k,1,r) \,:\, i \geq 0, 0 \leq j \leq S-1, 0 \leq k \leq j, 1 \leq r \leq m\} \cup \{(i,j,k,0,r) \,:\, i \geq 0, s+1 \leq j \leq S, 0 \leq k \leq j, 1 \leq r \leq m\}$$

The level i, i ≥ 0, is defined as the set of states given by

$$\mathbf{i} = \{(i,j,k,1,r) \,:\, 0 \leq j \leq S-1, 0 \leq k \leq j, 1 \leq r \leq m\} \cup \{(i,j,k,0,r) \,:\, s+1 \leq j \leq S, 0 \leq k \leq j, 1 \leq r \leq m\}$$

These states are arranged in the lexicographic order.

Define the following auxiliary matrices for use in sequel.

$$\tilde{A}_{00} = D_0 - \beta I_m \tag{3.29}$$

$$\tilde{A}_{ii} = \begin{bmatrix} D_0 - \beta I_m & D_1 & & \\ & D_0 - (\mu+\beta) I_m & D_1 & \\ & & \ddots & \\ & & & D_1 \\ & & & D_0 - (\mu+\beta) I_m \end{bmatrix}_{(i+1)m \times (i+1)m}, \quad 1 \leq i \leq s \tag{3.30}$$

$$\tilde{A}_{ii} = \begin{bmatrix} D_0 & 0 & D_1 & & & & & & \\ & D_0 - \beta I_m & 0 & D_1 & & & & & \\ & & D_0 - \mu I_m & 0 & D_1 & & & & \\ & & & D_0 - (\beta + \mu) I_m & 0 & D_1 & & & \\ & & & & & & & D_1 & \\ & & & & & & D_0 - \mu I_m & 0 & \\ & & & & & & & D_0 - (\mu + \beta) I_m \end{bmatrix}_{(i+1)2m \times (i+1}$$

$$; s + 1 \leq i \leq S - 1 \quad (3.31)$$

$$\tilde{A}_{SS} \begin{bmatrix} D_0 & D_1 & & & \\ & D_0 - \mu I_m & D_1 & & \\ & & \ddots & & \\ & & & & D_1 \\ & & & & D_0 - \mu I_m \end{bmatrix}_{(S+1)m \times (S+1)m} \quad (3.32)$$

$$W_i = \begin{bmatrix} 0 \\ I_i \end{bmatrix}_{(i+1)\times i}, \ 1 \leq i \leq S \tag{3.33}$$

$$\tilde{A}_{i,i-1} = W_i \otimes \mu I_m, \ 1 \leq i \leq s \tag{3.34}$$

$$\tilde{A}_{s+1,s} = (W_{s+1} \otimes \begin{bmatrix} 1 \\ 1 \end{bmatrix}) \otimes \mu I_m \tag{3.35}$$

$$\tilde{A}_{i,i-1} = (W_i \otimes I_2) \otimes \mu I_m, \ s+2 \leq i \leq S-1 \tag{3.36}$$

$$\tilde{A}_{S,S-1} = (W_S \otimes [1\ 0]) \otimes \mu I_m \tag{3.37}$$

$$V_i = [I_{i+1}\ 0]_{(i+1)\times(i+2)}, \ 0 \leq i \leq S-1 \tag{3.38}$$

$$\tilde{A}_{i,i+1} = V_i \otimes \beta I_m, \ 0 \leq i \leq s-1 \tag{3.39}$$

$$\tilde{A}_{s,s+1} = (V_s \otimes [0\ 1]) \otimes \beta I_m \tag{3.40}$$

$$\tilde{A}_{i,i+1} = (V_i \otimes \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}) \otimes \beta I_m, s+1 \leq i \leq S-2 \tag{3.41}$$

$$\tilde{A}_{S-1,S} = (V_{S-1} \otimes \begin{bmatrix} 0 \\ 1 \end{bmatrix}) \otimes \beta I_m \tag{3.42}$$

$$B_0 = \begin{bmatrix} \tilde{A}_{00} & \tilde{A}_{01} & & & & \\ \tilde{A}_{10} & \tilde{A}_{11} & \tilde{A}_{12} & & & \\ & \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} & & \\ & & & \ddots & & \\ & & & \tilde{A}_{S-1,S-2} & \tilde{A}_{S-1,S-1} & \tilde{A}_{S-1,S} \\ & & & & \tilde{A}_{S,S-1} & \tilde{A}_{SS} \end{bmatrix} \tag{3.43}$$

The matrix A_1 is obtained from B_0 by replacing the matrices $\tilde{A}_{ii}$, $1 \leq i \leq S$, by $\bar{A}_{ii}$. where

$$\bar{A}_{ii} = \tilde{A}_{ii} - (\begin{bmatrix} I_i & 0 \\ 0 & 0 \end{bmatrix} \otimes \theta I_m) \tag{3.44}$$

for $1 \leq i \leq s$ and for $i = S$ and for the remaining i values, except for $i = 0$,

$$\bar{A}_{ii} = \tilde{A}_{ii} - \left(\begin{bmatrix} I_{2i} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \otimes \theta I_m \right) \tag{3.45}$$

$$A_0 = \begin{bmatrix} \tilde{B}_{00} & & & \\ & \tilde{B}_{11} & & \\ & & \ddots & \\ & & & \tilde{B}_{SS} \end{bmatrix} \tag{3.46}$$

where

$$\tilde{B}_{00} = D_1 \tag{3.47}$$

$$\tilde{B}_{ii} = (a_{i+1} \cdot a'_{i+1}) \otimes D_1; 1 \leq i \leq s \text{ and for } i = S \tag{3.48}$$

where, a_{i+1} is a column vector of zeros except last entry which is 1 and is of order $(i+1) \times 1$.

$$\tilde{B}_{ii} = \begin{bmatrix} 0 & 0 \\ 0 & I_2 \end{bmatrix}_{2(i+1)\times 2(i+1)} \otimes D_1; s+1 \leq i \leq S-1 \tag{3.49}$$

$$A_2 = \begin{bmatrix} \tilde{C}_{00} & & & \\ & \tilde{C}_{11} & & \\ & & \ddots & \\ & & & \tilde{C}_{SS} \end{bmatrix} \tag{3.50}$$

where

$$\tilde{C}_{00} \text{ is a zero matrix of order } m \times m \tag{3.51}$$

$$\tilde{C}_{ii} = \begin{bmatrix} 0 & I_i \\ 0 & 0 \end{bmatrix}_{(i+1)\times(i+1)} \otimes \theta I_m \quad ; 1 \leq i \leq s \text{ and also for } i = S \tag{3.52}$$

$$\tilde{C}_{ii} = \begin{bmatrix} 0 & 0 & I_{2i} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2(i+1)\times 2(i+1)} \otimes \theta I_m; s+1 \leq i \leq S-1 \tag{3.53}$$

The Markov chain $\{(N_1(t), I(t), N_2(t), X(t), J(t))\}$ has the generator Q in partitioned form given by

$$Q = \begin{bmatrix} B_0 & A_0 & 0 & 0 & 0 & \cdots \\ A_2 & A_1 & A_0 & 0 & 0 & \cdots \\ 0 & A_2 & A_1 & A_0 & 0 & \cdots \\ 0 & 0 & A_2 & A_1 & A_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

where the entries in Q are given by (3.29) to (3.53)

Let $A = A_0 + A_1 + A_2$, and π denote the steady-state probability vector of A i,e

$$\pi A = 0, \pi e = 1$$

The vector π can be partitioned as $\pi = \big(\pi(0), \pi(1), \ldots, \pi(s), \ldots, \pi(S)\big)$ where the vectors $\pi(i)$ are again partitioned as

$$\pi(i) = (\pi(i,j,1)); 0 \leq i \leq s, 0 \leq j \leq i$$

$$\pi(i) = (\pi(i,j,k)); s+1 \leq i \leq S-1, 0 \leq j \leq i, k = 0, 1$$

$$\pi(S) = (\pi(S,j,0)); 0 \leq j \leq S$$

We have the following result on system stability.

Lemma 3.4.1. *The system is stable if*

$$\gamma(D_1 + \theta I_m)e_m < 0$$

where γ is given by

$$\gamma = \Big[\sum_{i=0}^{S-1} \pi(i,i,1) + \sum_{i=s+1}^{S} \pi(i,i,0)\Big]$$

Proof. From the well-known result [Neuts [61] theorem 1.7.1] on the positive recurrence of Q, which states that

$$\pi A_0 e < \pi A_2 e$$

and by exploiting the structure of the matrices A_0 and A_2, the stated result follows □

If stability holds, by using theorem 3.2.2 and Logarithmic Reduction Algorithm (see Latouche and Ramaswamy [49]) we can calculate the rate matrix R.

3.5 System Performance Measures

Steady state probability vector

$x = \big(x(0), x(1), x(2), \dots\big)$ is again partition as $x(i) = \big(y(i,j)\big) \quad i \geq 0, 0 \leq j \leq S$ and

$$\begin{aligned} y(i,j) = \big(z(i,j,k,l)\big) \text{ where } & l = 1 \text{ for } 0 \leq j \leq s \\ & l = 0,1 \text{ for } s+1 \leq j \leq S-1 \\ & l = 0 \text{ for } j = S \\ & \text{and } 0 \leq k \leq j \end{aligned}$$

1. The probability mass function of number of customers in orbit :-

   The probability that there are i customers in the orbit is given by

$$P_i = x(i)e = x(0)R^i e \quad i \geq 0$$

2. The rate at which the orbiting customers try to enter the buffer is given by

$$\theta_1^* = \theta(1 - x(0)e)$$

3. The rate at which the orbiting customers successfully enter the buffer is given by

$$\theta_2^* = \theta\delta$$

where,

$$\delta = \sum_{i=1}^{\infty}\Big[\Big\{\sum_{j=1}^{S-1}\sum_{k=0}^{j-1} z(i,j,k,1)\Big\}e + \Big\{\sum_{j=s+1}^{S}\sum_{k=0}^{j-1} z(i,j,k,0)\Big\}e\Big]$$

4. Probability that an orbit customer fail to enter the buffer is,

$$\theta_3* = 1 - \delta$$

5. Expected inventory level in the system:- Expected inventory level in the system is given by

$$E_* = x(0)(I - R)^{-1}e_*$$

where, $e_* = [e_0, e_1, \ldots, e_s, \ldots, e_S]'$ and

$$e_i = [i, i, \ldots, i]_{1\times(i+1)m} \text{ for } 0 \le i \le s \text{ and for } i = S$$
$$e_i = [i, i, \ldots, i]_{1\times 2(i+1)m} \text{ for } s + 1 \le i \le S - 1$$

6. Expected number of customers in the orbit

$$E_{0r} = x(0)R(I - R)^{-2}e$$

7. Expected number of customers in the buffer is,

$$E_B = \sum_{i=0}^{\infty} \left[\left\{ \sum_{j=1}^{S-1} \sum_{k=1}^{j} kz(i,j,k,1) \right\} e + \left\{ \sum_{j=s+1}^{S} \sum_{k=1}^{j} kz(i,j,k,0) \right\} e \right]$$

3.6 Numerical Illustration

We provide an example based on our system performance measures. In Table 3.1, fixing the other parameter values involved in the system we vary over service time μ whereas in Table 3.2 and Table 3.3 we vary over production time β and retrial rate θ. For different values of these parameter μ β and θ corresponding values of the system measures are provided.

Take

$$D_0 = \begin{bmatrix} -0.21 & 0.0 \\ 0.0 & -0.20 \end{bmatrix}$$

$$D_1 = \begin{bmatrix} 0.10 & 0.11 \\ 0.20 & -0.0 \end{bmatrix}$$

and

$$\rho = \frac{\pi A_0 e}{\pi A_0 e}$$

Table 3.1: Fixed $\theta = 0.5$, $\beta = 1.0$, $s = 3$, $S = 8$

	$\mu = 1.0$	$\mu = 1.5$	$\mu = 2.0$	$\mu = 2.5$	$\mu = 3.0$	$\mu = 3.5$	$\mu = 4.0$
ρ	0.05392	0.03201	0.02648	0.02428	0.02313	0.02244	0.02198
θ_1^*	0.00056759	0.00028509	0.0002149	0.00018591	0.00017062	0.00016114	0.000015473
θ_2^*	0.00036846	0.00018988	0.00014353	0.00012413	0.00011378	0.00010742	0.00010318
E_*	5.740475	5.740343	5.740291	5.740262	5.740242	5.740231	5.740221
E_{or}	0.0014946	0.0007251	0.0005433	0.0004692	0.00043001	0.000406	0.00039003
E_B	0.259579	0.15948	0.115063	0.089992	0.073890	0.062676	0.054416

Table 3.2: Fixed $\mu = 1.0$, $\beta = 1.0$, $s = 3$, $S = 8$

	$\theta = 1.0$	$\theta = 1.5$	$\theta = 2.0$	$\theta = 2.5$	$\theta = 3.0$	$\theta = 3.5$	$\theta = 4.0$
ρ	0.11081	0.14687	0.1684	0.18216	0.19161	0.19847	0.20366
θ_1^*	0.00080955	0.0010533	0.00129664	0.00154004	0.00178492	0.00202942	0.00227404
θ_2^*	0.00039515	0.0004165	0.00043519	0.00045225	0.00046841	0.00048392	0.00049897
E_*	5.740419	5.740382	5.740357	5.740341	5.740328	5.74032	5.740316
E_{or}	0.001064	0.000922	0.00085	0.000809	0.0007809	0.0007608	0.0007458
E_B	0.259654	0.259692	0.259715	0.259730	0.259741	0.259749	0.259755

Table 3.3: Fixed $\mu = 1.0$, $\theta = 0.5$, $s = 3$, $S = 8$

	$\beta = 1.0$	$\beta = 1.5$	$\beta = 2.0$	$\beta = 2.5$	$\beta = 3.0$	$\beta = 3.5$	$\beta = 4.0$
ρ	0.05392	0.03693	0.03248	0.03064	0.02969	0.02911	0.02872
θ_1^*	0.00056759	0.00029618	0.00022808	0.00020027	0.00018549	0.00017634	0.00017029
θ_2^*	0.00036846	0.00020237	0.00015736	0.00013805	0.00012762	0.00012110	0.00011668
E_*	5.740475	5.84053	5.884952	5.910028	5.926137	5.937356	5.945621
E_{or}	0.001494	0.000752	0.000576	0.000505	0.000467	0.000444	0.000429
E_B	0.259579	0.259735	0.259793	0.259822	0.259839	0.259851	0.259859

Chapter 4

Retrial Inventory with BMAP and Service Time

4.1 Introduction

In this chapter we consider an (s, S)-retrial inventory with service time where primary arrivals of demands follow a batch Markovian arrival process (BMAP). Demands enter the buffer of capacity equal to the number of items held in the inventory at that time. When buffer is full (equal to the number of inventoried items), further demands proceed to an orbit of infinite capacity. The orbital customers will try their luck after some random time, exponentially distributed with parameter θ. These customers keep on trying until they succeed in finding a berth at the buffer. Service times of customers are *i.i.d.* exponential random variables with parameter μ. Inventory level decreases by one unit for providing service to a customer in the buffer. When inventory level reaches s an order for replenishment is given. Lead time is exponentially distributed with parameter β.

Retrial queues deal with the behaviour of queueing systems of customers who could not find a position at the service station at the arrival time. It has been investigated extensively (See the survey papers by Yang and Templeton [91] and Falin [18], the monograph by Falin and Templeton [19]) and also the more recent state of art in re-trial queues by Artalejo [5].

Artalejo, Krishnamoorthy and Lopez-Herrero [6] is the first investigation on inventory policies with positive lead time and retrial of orbital customers (linear retrial rate) who could not get service during their earlier attempts to access the service station.

Berman, Kim and Shimark [9] in which they assumed that both the demand and the service rate are deterministic and constant and as such, queues can form only during stock out period.They determined optimal order quantity that minimize the total cost per unit time. Berman and Kim [10, 11] analyzed an inventory system in which customers arrive at a service facility according to a Poisson process with service times exponentially distributed where each customer demands exactly one item in the inventory; both zero lead time and positive lead time cases were discussed. Berman and Sapna [13, 12] studied inventory control at a service facility, which uses one item of inventory for service provided.Assuming Poisson arrival process, arbitrarily distributed service times and zero lead time they analyzed the system with the restriction that the waiting space is finite. Under a specific cost structure they devised the optimum ordering quantity that minimizes the long run expected cost rate.

This chapter is organized as follows: In section 4.2 we discuss the model and provide the brief discription of BMAP. Steady state analysis of the model is studied in the section 4.3. we list some system performance measures in section 4.4 and for particular case of BMAP (when arrival of demands form Poisson process) we Provide numerical results in the sections 4.5

4.2 Model and Analysis

We consider an inventory system with service time in which demands occur according to a Batch Markovian Arrival Process (BMAP). The demands are served singly with service times exponentially distributed with parameter μ. There is a buffer in which demands can stay before getting service. The capacity of the buffer is restricted to the number of items held in the inventory at any given instant; Thus this varies from 0 to S. Customers who find no place in the buffer go to an orbit of infinite capacity. The inventory is controlled by (s, S)-policy. Replenishment time is exponentially distributed with parameter β.The customers in orbit retry for service with constant re-trial rate θ. If the inventory level is greater than the number of customers in the buffer then both retrials customers and primary customers can get into the buffer on arrival. Its overflow lead to customers being directed to the orbit.

The BMAP in the continuous time can be described as follows:-

Let the underlying MC be irreducible and let Q^* be its generator. After a sojourn in a state i which is exponentially distributed with parameter λ_i, $1 \leq i \leq m$, two things can occur

1. It can go to state j: $1 \leq j \leq m$ the transition corresponds to the arrival of a batch of size $k \geq 1$ with probability $p_{ij}(k)$

2. It can go to state j: $1 \leq j \leq m$, $j \neq i$ and the transition corresponds to no arrival with probability $p_{ij}(0)$

We have

$$\sum_{k=1}^{\infty}\sum_{j=1}^{m} p_{ij}(k) + \sum_{j=1, j\neq i} p_{ij}(0) = 1, \quad 1 \leq i \leq m$$

For $k \geq 0$ define the matrices $D_k = (d_{ij}(k))$ such that

$$d_{ij}(0) = \lambda_i p_{ij}(0) \quad j \neq i,\ 1 \leq i, j \leq m$$

$$d_{ii}(0) = -\lambda_i \quad \text{and} \quad d_{ij}(k) = \lambda_i p_{ij}(k)$$

By assuming D_0 to be non-singular matrix, the inter arrival times will be finite and the arrival process doesn't terminate. The generator is

$$Q^* = \sum_{k=0}^{\infty} D_k$$

Thus, the BMAP is governed by the matrices $\{D_k\}$ with D_0 governing the transition corresponding to no arrival and D_k governing those corresponding to arrivals of a group of size k, $k \geq 1$.

In this chapter we assume that $D_i = 0$ for $i > K$ so that the maximum possible batch size in K.

For the use in the sequel let

I_i denote identity matrix of order i

$\otimes$ stands for Kronecker product of two matrices

A' means transpose of matrix A

e denotes column vector of 1's of appropriate order.

4.3 The Steady State Analysis of the Model at an Arbitrary Time Epoch

Let $N_1(t)$, $I(t)$, $N_2(t)$ and $J(t)$ denote respectively, the number of customers in orbit, the number of items held in the inventory, the number of customers in the buffer including the one getting service and the phase of the arrival process at time t. Now $\{N_1(t), I(t), N_2(t), J(t)\}$ is a continuous time Markov chain with state space given by

$$\Omega = \{(i, j, k, l) \ : \ i \geq 0, 0 \leq j \leq S, \ 0 \leq k \leq j, 1 \leq l \leq m\}$$

Let $\hat{0}$ denote the set of states given by

$$\hat{0} = \{(r, j, k, l) \ : \ 0 \leq r \leq K, 0 \leq j \leq S, 0 \leq k \leq j, 1 \leq l \leq m\}$$

and

$$\hat{i} = \{(iK + r, j, k, l) : 1 \leq r \leq K, 0 \leq j \leq S, 0 \leq k \leq j, 1 \leq l \leq m\}, \ i \geq 1$$

The above set of states are arranged in lexicographic order. Define the following matrices for later use,

$\tilde{H}=$

$$
\begin{array}{c|cccccccccc}
 & 0 & 1 & 2 & s & s+1 & Q-1 & Q & Q+1 & S-1 & S \\
\hline
0 & \tilde{H}_{00} & & & & & & \tilde{H}_{0Q} & & & \\
1 & \tilde{H}_{10} & \tilde{H}_{11} & & & & & & \tilde{H}_{1Q+1} & & \\
2 & & \tilde{H}_{21} & \tilde{H}_{22} & & & & & & & \\
s & & & & \tilde{H}_{ss} & & & & & & \tilde{H}_{sS} \\
s+1 & & & & \tilde{H}_{s+1,s} & \tilde{H}_{s+1,\,s+1} & & & & & \\
Q-1 & & & & & & & & & & \\
Q & & & & & & \tilde{H}_{Q,\,Q-1} & \tilde{H}_{Q\,Q} & & & \\
Q+1 & & & & & & & \tilde{H}_{Q+1,\,Q} & \tilde{H}_{Q+1,\,Q+1} & & \\
S-1 & & & & & & & & & & \\
S & & & & & & & & & \tilde{H}_{S,S-1} & \tilde{H}_{S\,S}
\end{array}
\tag{4.1}
$$

where

$$\tilde{H}_{ii} = D_{ii} - E_{ii} - F_{ii} \qquad 0 \le i \le s \tag{4.2}$$

$$\tilde{H}_{ii} = D_{ii} - F_{ii} \qquad s+1 \le i \le S \tag{4.3}$$

$$\tilde{H}_{i,i-1} = \begin{bmatrix} 0 \\ I_i \otimes \theta I_m \end{bmatrix} \quad \text{0 is zero matrix of order } m \times im \quad 1 \le i \le S \tag{4.4}$$

$$\tilde{H}_{i,i+Q} = [I_{i+1} \otimes \beta I_m \; 0] \quad \text{0 is zero matrix of order } (i+1)m \times Qm \quad 0 \le i \le s \tag{4.5}$$

Where

$$D_{ii} = \begin{bmatrix} D_0 & D_1 & & D_{i-1} & D_i \\ & D_0 & \dots & D_{i-2} & D_{i-1} \\ & & & \vdots & \vdots \\ & & & D_0 & D_1 \\ & & & & D_0 \end{bmatrix}_{(i+1)m\times(i+1)m} \qquad 0 \le i \le S \tag{4.6}$$

Note that $D_i = 0$ when $i \geq K + 1$

$$E_{ii} = I_{i+1} \otimes \beta I_m \qquad 0 \le i \le s \tag{4.7}$$

$F_{00} = 0$

$$F_{ii} = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}_{(i+1)\times(i+1)} \otimes \mu I_m = \begin{bmatrix} 0 & 0 \\ 0 & I_i \end{bmatrix}_{(i+1)\times(i+1)} \otimes \mu I_m \tag{4.8}$$

where $1 \le i \le S$

$$H = \begin{bmatrix} \tilde{H}_{00} & & & & \tilde{H}_{0Q} & & & \\ \tilde{H}_{10} & H_{11} & & & & \tilde{H}_{1,Q+1} & & \\ & \tilde{H}_{21} & H_{22} & & & & & \\ & & & & & & & \tilde{H}_{sS} \\ & & & \tilde{H}_{QQ-1} & H_{QQ} & & & \\ & & & & & & \tilde{H}_{SS-1} & H_{SS} \end{bmatrix} . \tag{4.9}$$

where

$$H_{ii} = D_{ii} - E_{ii} - F_{ii} - V_{ii} \qquad 1 \le i \le s \tag{4.10}$$

$$H_{ii} = D_{ii} - F_{ii} - V_{ii} \qquad s+1 \le i \le S \tag{4.11}$$

where

$$V_{ii} = \begin{bmatrix} 1 & 0 & \dots & & 0 \\ 0 & 1 & \dots & & 0 \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 0 \end{bmatrix}_{(i+1)\times(i+1)} \otimes \theta I_m = \begin{bmatrix} I_i & 0 \\ 0 & 0 \end{bmatrix}_{(i+1)\times(i+1)} \otimes I_m \tag{4.12}$$

$1 \le i \le S$

$$L = \begin{bmatrix} 0 & & & & \\ & L_{11} & & & \\ & & L_{22} & & \\ & & & \ddots & \\ & & & & L_{SS} \end{bmatrix} \tag{4.13}$$

where

$$L_{ii} = \begin{bmatrix} 0 & I_i \\ 0 & 0 \end{bmatrix}_{(i+1)\times(i+1)} \otimes \theta I_m \quad 1 \le i \le S \tag{4.14}$$

$$B_{0j} = \begin{bmatrix} B_{0j}^{00} & & & & \\ & B_{0j}^{11} & & & \\ & & \ddots & & \\ & & & B_{0j}^{ii} & \\ & & & & \ddots \\ & & & & & B_{0j}^{SS} \end{bmatrix}, \ 1 \le j \le K \tag{4.15}$$

where

$$B_{0j}^{ii} = [e^{(i+1)}]' \otimes \begin{bmatrix} D_{i+j} \\ D_{i+j-1} \\ \vdots \\ D_j \end{bmatrix}_{(i+1)m \times m} \tag{4.16}$$

where $e^{(i+1)}$ is a column vector of $(i+1)$ entries with the $(i+1)$th entry 1 all other entries are 0's.

Here also note that if $k \geq K+1, D_k = 0$.

$$A_{10} = \begin{bmatrix} \tilde{H} & B_{01} & B_{02} & & B_{0K-1} & B_{0K} \\ L & H & B_{01} & & B_{0K-2} & B_{0K-1} \\ 0 & L & H & & B_{0K-3} & B_{0K-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & & L & H \end{bmatrix} \tag{4.17}$$

$$A_{00} = \begin{bmatrix} 0 & 0 & 0 & & 0 & 0 \\ B_{0K} & 0 & 0 & & 0 & 0 \\ B_{0K-1} & B_{0K} & 0 & & 0 & 0 \\ B_{0K-2} & B_{0K-1} & B_{0K} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ B_{01} & B_{02} & B_{03} & \cdots & B_{0K-1} & B_{0K} \end{bmatrix} \tag{4.18}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & & L \\ 0 & 0 & 0 & & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{4.19}$$

$$A_1 = \begin{bmatrix} H & B_{01} & B_{02} & & B_{0K-2} & B_{0K-1} \\ L & H & B_{01} & & B_{0K-3} & B_{0K-2} \\ 0 & L & H & & B_{0K-4} & B_{0K-3} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & & L & H \end{bmatrix} \tag{4.20}$$

$$A_0 = \begin{bmatrix} B_{0K} & 0 & 0 & & 0 \\ B_{0K-1} & B_{0K} & 0 & & 0 \\ B_{0K-2} & B_{0K-1} & B_{0K} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ B_{01} & B_{02} & B_{03} & & B_{0K} \end{bmatrix} \tag{4.21}$$

The Markov chain $(N_1(t), I(t), N_2(t), J(t))$ has the generator Q in the partitioned form

$$Q = \begin{bmatrix} A_{10} & A_{00} & & & \\ A_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & & \ddots & \ddots \\ & & & & \end{bmatrix} \tag{4.22}$$

The generator $A = A_0 + A_1 + A_2$ is given by

$$A = \begin{bmatrix} H + B_{0K} & B_{01} & B_{02} & & B_{0K-2} & B_{0K-1} + L \\ L + B_{0K-1} & H + B_{0K} & B_{01} & & B_{0K-3} & B_{0K-2} \\ B_{0K-2} & L + B_{0K-1} & H + B_{0K} & & B_{0K-4} & B_{0K-3} \\ \vdots & & & & & \\ B_{01} & B_{02} & B_{03} & \cdots & L + B_{0K-1} & H + B_{0K} \end{bmatrix} \tag{4.23}$$

Lemma 4.3.1. *The steady-state probability vector π satisfying*

$$\pi A = 0, \quad \pi e = 1$$

is given by

$$\pi = \frac{1}{K}(e' \otimes w) \tag{4.24}$$

where the vector w is the steady–state probability vector of

$$M = H + L + \sum_{j=1}^{K} B_{0j}$$

$$= \begin{bmatrix} \overline{H}_{00} & & & & & \tilde{H}_{0Q} & \\ \tilde{H}_{10} & \overline{H}_{11} & & & & & \\ & \tilde{H}_{21} & \overline{H}_{22} & & & & \\ & & & & & & \\ & & & \hat{H}_{ss} & & & \tilde{H}_{sS} \\ & & & & & & \\ & & & & \tilde{H}_{Q,Q-1} & \hat{H}_{QQ} & \\ & & & & & & \hat{H}_{SS} \end{bmatrix} \tag{4.25}$$

e' is a column vector with K entries all equal to 1 where

$$\overline{H}_{ii} = \begin{bmatrix} D_o - (\beta+\theta)I_m & D_1 + \theta I_m & D_2 & D_3 & sum_{j=1}^{K} D_{i+j} \\ & D_0 - (\beta+\theta+\mu)I_m & D_1 + \theta I_m & D_2 & \sum_{j=1}^{K} D_{i+j-1} \\ & & D_0 - (\beta+\theta+\mu)I_m & D_1 + \theta I_m & \sum_{j=1}^{K} D_{i+j-2} \\ & & & & \\ & & & & \sum_{j=1}^{K} D_{j+1} + \theta I_m \\ & & & & D_0 - (\beta+\mu)I_m + \sum_{j=1}^{K} D_j \end{bmatrix} \tag{4.26}$$

$1 \leq i \leq s$

$$\hat{H}_{ii} = \overline{H}_{ii} + I_{i+1} \otimes \beta I_m \textit{ for } s+1 \leq i \leq S$$

H_{ii-1} for $1 \leq i \leq S$ and $H_{i,i+Q}$ for $0 \leq i \leq s$ are as defined in (4.4) *and* (4.5)

Proof. Noting that A is a circulant matrix, we see that the vector π is of the form $\pi = \frac{1}{k}(e' \otimes w)$

where the vector w is the steady–state probability vector of the generator M given by

$$M = H + L + \sum_{j=1}^{K} B_{0j} \tag{4.27}$$

Partitioning the vector w as $w = (w_0, w_1, \ldots w_S)$
where $w_0 = w_0^{00}, w_i = (w_i^{i0}, w_i^{i,1}, \ldots w_i^{ii}), 1 \leq i \leq S$, the following result on stability condition is obtained. □

Lemma 4.3.2. *The system is stable if*

$$\sum_{i=0}^{S} \left\{ \left(\sum_{l=i}^{S} w_l^{l(l-i)} \right) \left(\sum_{j=i}^{K} (j-i) D_j e \right) \right\} < \theta \left(\sum_{j=1}^{S} \sum_{i=0}^{j-1} w_j^{ji} \right) e \tag{4.28}$$

Proof. From the result on the positive recurrence of Q which states that

$$\pi A_0 e < \pi A_2 e$$

and exploiting the structure of matrices A_0, A_2, π.
Let x, partitioned as $x = (x(\hat{0}), x(\hat{1}), x(\hat{2}) \ldots)$, denote the steady state probability vector of Q. Then x satisfies

$$xQ = 0xe = 1$$

□

Theorem 4.3.3. *When the stability condition holds good, the steady state probability vector x is given by*

$$x(\hat{i}) = x(\hat{1}) R^{i-1}, i \geq 1$$

where the matrix R is the minimal non-negative solution of matrix quadratic equation:

$$R^2 A_2 + R A_1 + A_0 = 0$$

and the vector $x(\hat{0})$ and $x(\hat{1})$ are obtained by solving

$$x(\hat{0})A_{10} + x(\hat{1})A_2 = 0$$

$$x(\hat{0})A_{00} + x(\hat{1})[A_1 + RA_2] = 0$$

subject to the normalizing condition

$$x(\hat{0})e + x(\hat{1})(I - R)^{-1}e = 1$$

Proof. Follows from well known results. □

Computation of the matrix R.

To compute R first we compute G matrix. The special structure of the matrix A_2 implies that the matrix G will have the following structure.

$$G = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ K \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & & K \end{array} \\ \begin{pmatrix} & & & G_1 \\ & & & G_2 \\ & & & \vdots \\ & & & G_K \end{pmatrix} \end{array} \tag{4.29}$$

where each G_i is a square matrix of order $\frac{(S+1)(S+2)}{2}m$. Now G satisfies the matrix quadratic equation:

$$A_0G^2 + A_1G + A_2 = 0$$

Using the above form of G this equation gives

$$B_{0K}G_1G_K + HG_1 + B_{01}G_2 + \cdots + B_{0K-1}G_K + L = 0$$

and

$$\sum_{j=1}^{i} B_{0,j-i+K}G_jG_K + LG_{i-1} + HG_i + \sum_{j=1}^{K-i} B_{0j}G_{i+j} = 0, \qquad 2 \leq i \leq K.$$

which gives

$$G_1 = (-H)^{-1}[B_{0K}G_1G_K + \sum_{j=1}^{K-1} B_{0j}G_{j+1} + L]$$

and for $2 \leq i \leq K$,

$$G_i = (-H)^{-1}[\sum_{j=1}^{i} B_{0,j-1+K}G_jG_K + LG_{i-1} + \sum_{j=1}^{K-i} B_{0j}G_{i+j}]$$

Now we can use Block Gauss Seidel iterative method to evaluate $G_1, \ldots G_K$ and hence G. The accuracy checks can be done using $G_{ie} = 1, 1 \leq i \leq K$.

After this R can be evaluated using the formula

$$R = A_0(-A_1 - A_0G)^{-1}$$

4.4 System Performance Measures

1. Expected inventory level

$$E_1 = \sum_{i=1}^{S} i[\sum_{j=0}^{K} x(0,j) + \sum_{l=1}^{\infty}\sum_{j=1}^{K}(x_{l,j})]e_i$$

where e_i is a column vector of $\frac{(S+1)(S+2)}{2}m$ entries of which the entries in positions $\frac{i(i+1)}{2}m + 1$, $\frac{i(i+1)}{2}m + 2, \ldots,$ $\frac{(i+1)(i+2)}{2}m$ are 1s'and rest zeros.

2. Expected number of customers in the buffer conditional on the inventory level and then remove the conditioning, is given by

$$E_2 = x(0,0)e(b) + \sum_{i=0}^{\infty}\sum_{j=1}^{K}[x(i,j)e(b)]$$

where $e(b) = a' \otimes b'$ with $a = (\{0\}, \{0,1\}, \{0,1,2\}, \ldots, \{0,1,2,\ldots,S\})$ and $b = (1,\ldots,1)_{1\times m}$

3. Expected number of customers in the orbit:

$$E_3 = \sum_{i=0}^{\infty}\sum_{j=1}^{K}(iK+j)x(i,j)e$$

4. Blocking probability (Probability that a customer (primary/orbital) finds the buffer at its permitted maximum). This will be conditional probability (conditioned on the inventory level):

$$x(0,0)e(c) + \sum_{i=0}^{\infty}\sum_{j=1}^{K} x(i,j)e(c)$$

where $e(c)$ is the column vector with 1's in the positions $1,\ldots,m; 2m+1,\ldots,3m; 5m+1,\ldots,6m; \frac{1}{2}(S+1)(S+2)m - m + 1, \ldots, \frac{(S+1)(S+2)}{2}m.$

5. Probability of encountering the system with inventory level zero.

$$x(0,0)e(d) + \sum_{i=0}^{\infty}\sum_{j=1}^{K} x(i,j)e(d)$$

where $e(d)$ is the column vector with first m entries 1s' and the rest zeros.

4.5 Numerical illustration

We provide an example based on our system performance measure in the particular case (Arrival of demands follow Poisson process). In Table 4.1, fixing the other parameter values involved in the system we vary over service time μ whereas in Table 4.2 and Table 4.3 we vary over replenishment rate β and retrial rate θ respectively. For different values of these parameter μ, β and θ corresponding values of the system measures are provided.

Table 4.1: Fixed $\lambda = 0.5, \theta = 0.7, \beta = 0.6, s = 2, S = 5$

	$\mu = 0.5$	$\mu = 0.6$	$\mu = 0.7$	$\mu = 0.8$	$\mu = 0.9$	$\mu = 1.0$
E_1	3.49621	3.48527	3.46317	3.42747	3.37993	3.32369
E_2	2.04322	1.8643	1.70886	1.57053	1.44644	1.33507
E_3	1.71564	1.14123	0.839337	0.664212	0.556632	0.488157

Table 4.2: Fixed $\lambda = 0.5, \theta = 0.7, \mu = 0.7, s = 2, S = 5$

	$\beta = 0.6$	$\beta = 0.7$	$\beta = 0.8$	$\beta = 0.9$	$\beta = 1.0$	$\beta = 1.1$
E_1	3.46317	3.31458	3.32762	3.36807	3.41198	3.45337
E_2	1.70886	1.4018	1.30959	1.27057	1.25161	1.24187
E_3	0.839337	0.67463	0.585385	0.527918	0.488025	0.458975

Table 4.3: Fixed $\lambda = 0.5, \beta = 0.6, \mu = 0.7, s = 2, S = 5$

	$\theta = 0.4$	$\theta = 0.5$	$\theta = 0.6$	$\theta = 0.7$	$\theta = 0.8$	$\theta = 0.9$
E_1	3.36839	3.40749	3.4385	3.46317	3.48285	3.49865
E_2	1.62447	1.66106	1.68846	1.70886	1.72398	1.73511
E_3	1.02356	0.952938	0.891877	0.839337	0.794146	0.73511

Chapter 5

Inventory System with Postponed Demands and Service Facilities

5.1 Introduction

In most of the inventory models it is assumed that the inventory deplete at a rate equal to demand rate (service time negligible). However, it becomes unrealistic for the service facilities where the stocked item is delivered to the customers after some service is performed. In this chapter we consider an (s, S) inventory system with service facilities. Arrival of demands form a Poisson process with parameter $\lambda(> 0)$ to a buffer of finite capacity equal to the inventory level at any given time t. When the maximum buffer size is reached, further demands join a pool of infinite capacity with probability γ and with probability $(1 - \gamma)$ it is lost for ever. In this chapter we consider two models. In the first model, pooled customers are taken to the buffer with probability p at a service completion epoch if the inventory level is atleast $s + 1$ and provided the number of customers in the buffer is less than the number of items held in the inventory.

In the second model we assume that when inventory level is atleast one and no customer is

*Some results of this chapter was presented in the Annual Conference of Kerala Mathematical Association, Payyanur College, Kannur, Kerala; 8-10 January'2004.

in the buffer then with probability one a customer is picked up from the pool for service. The other assumptions of model-I remain valid for model-II as well. The service time is assumed to be exponentially distributed with parameter μ in both models. It is also assumed that initially the inventory level is S. When inventory level reaches s an order for replenishment is placed. Lead time is exponentially distributed with parameter β.

First we have a brief review of the research reported in inventory with service. Berman,Kim and Shimshak[9] consider an inventory system with service in which they assume that both the demand and the service rates are deterministic and constant and as such queues can form only during stock out period. They determine optimal order quantity that minimize the total cost per unit time. Later Berman and Kim[10, 11] analyze the non-deterministic inventory model for service facilities. They analyze the system in which customers arrive at a service facility according to Poisson process where service times are exponentially distributed and each customer demands exactly one unit of the item in the inventory; both zero and positive lead time cases are discussed. Berman and Sapna[12, 13] investigate inventory control at a service facility, which uses one item of inventory for service provided. Assuming Poisson arrival process, arbitrarily distributed service times and zero lead time they analyze the system with the restriction that waiting space is finite. Under a specific cost structure they derive the optimum ordering quantity that minimizes the long run expected cost rate.
The notations used in this chapter in the sequel are explained below:-

$I(t)$= Inventory level at time t; this takes values $\{0, 1, \cdots, S\}$

$B(t)$= Number of customers in the buffer at time t

$N(t)$= Number of customers in the pool at time t

A'- Transpose of a matrix A

e= The column vector of 1's of appropriate order.

We have $\{(N(t), I(t), B(t)), t \geq 0\}$ is a continous time Markov chain with state space given by

$$\Omega = \{(i, j, k);\ i \geq 0, 0 \leq j \leq S, 0 \leq k \leq j\} \tag{5.1}$$

These states are arranged in the lexicographic order .

This chapter is organized as follows: Model-I is discussed in section 5.2. This section contains three subsection. In subsection 5.2.1 we descripe the model-I. we list some system performance measures and based on that some numerial examples are provided in the subsections 5.2.2 and 5.2.3. In section 5.3, we discuss the model-II. This section also contain three subsections. We discuss the model in subsection 5.3.1. System performance measure is given in 5.3.2 and finally, we provided illustrative numerical examples in section 5.3.3.

5.2 Model-I

5.2.1 Model Discription

In the present model, when inventory level is larger than the number of customers in the buffer, an external demand can enter the buffer for service. A pooled customer is transfered to the buffer for service at a service completion epoch with probability p, if the inventory level exceeds s and is also larger than the number of customers in the buffer. For convenience we define the following matrices for use in sequel

$$B_0 = \begin{bmatrix} A_{00} & & & \tilde{A}_{0,Q} & & & \\ \tilde{A}_{10} & \tilde{A}_{11} & & & \tilde{A}_{1,Q+1} & & \\ & & & & \ddots & & \\ & & & & & \tilde{A}_{S-1,S-1} & \\ & & & & & \tilde{A}_{S,S-1} & \tilde{A}_{SS} \end{bmatrix} \tag{5.2}$$

where

$$A_{00} = -\lambda\gamma - \beta \tag{5.3}$$

$$\tilde{A}_{11} = \begin{bmatrix} -\lambda-\beta & \lambda \\ 0 & -\lambda\gamma-\beta-\mu \end{bmatrix} \tag{5.4}$$

$$\tilde{A}_{ii} = \begin{bmatrix} -\lambda-\beta & \lambda & & \\ & -\lambda-\beta-\mu & \lambda & \\ & & \ddots & \\ & & & -\lambda\gamma-\beta-\mu \end{bmatrix} for 2 \leq i \leq s \tag{5.5}$$

$$\tilde{A}_{ii} = \begin{bmatrix} -\lambda & \lambda & & \\ & -\lambda-\mu & \lambda & \\ & & \ddots & \\ & & & -\lambda\gamma-\mu \end{bmatrix} for s+1 \leq i \leq S \tag{5.6}$$

$$\tilde{A}_{1,0} = \begin{bmatrix} 0 \\ \mu \end{bmatrix} \tag{5.7}$$

$$\bar{A}_{i,i-1} = \begin{bmatrix} 0 \\ I_i \end{bmatrix}_{(i+1)\times i} \mu \quad for 2 \leq i \leq S \tag{5.8}$$

$$\tilde{A}_{0,Q} = [\beta, 0, \cdots, 0] \tag{5.9}$$

$$\tilde{A}_{i,i+Q} = [I_{i+1}, 0, 0, \cdots, 0]\beta \quad for 1 \leq i \leq s \tag{5.10}$$

$$A_1 = \begin{bmatrix} A_{00} & & \tilde{A}_{0,Q} & & & \\ \tilde{A}_{10} & \tilde{A}_{11} & & \tilde{A}_{1,Q+1} & & \\ & & & \ddots & & \\ & & & & \tilde{A}_{S-1,S-1} & \\ & & & & \bar{A}_{S,S-1} & \tilde{A}_{SS} \end{bmatrix} \tag{5.11}$$

where

$$\bar{A}_{i,i-1} = \begin{bmatrix} 0 \\ I_i \end{bmatrix} \mu \quad for 2 \leq i \leq s \tag{5.12}$$

$$\bar{A}_{i,i-1} = \begin{bmatrix} 0 \\ I_i \end{bmatrix}_{(i+1)\times i} \quad (\mu(1-p)) \quad for s+1 \leq i \leq S \tag{5.13}$$

$$A_2 = \begin{bmatrix} C_{00} & & & & & \\ \tilde{C}_{10} & \tilde{C_{11}} & & & & \\ & \tilde{C}_{21} & \tilde{C_{22}} & & & \\ & & & \ddots & & \\ & & & & \tilde{C}_{SS-1} & \tilde{C_{SS}} \end{bmatrix} \tag{5.14}$$

where

$$C_{00} = 0 \tag{5.15}$$

$$\tilde{C}_{ii}\text{are matrices of all elements with zeros} \quad for s+1 \leq i \leq S \tag{5.16}$$

$$\tilde{C}_{i,i-1}\text{are matrices of all elements with zeros} \quad for 1 \leq i \leq s \tag{5.17}$$

$$\tilde{C}_{i,i-1} = \begin{bmatrix} 0 & & & \\ & p\mu & & \\ & & p\mu & \\ & & & \\ & & & p\mu \\ & & & 0 \end{bmatrix}_{(i+1)\times i} \quad for \quad s+1 \leq i \leq S \tag{5.18}$$

$$A_0 = \begin{bmatrix} B_{00} & & & \\ & \tilde{B}_{11} & & \\ & & & \\ & & & \tilde{B}_{SS} \end{bmatrix} \tag{5.19}$$

$$B_{00} = \lambda\gamma \tag{5.20}$$

$$\tilde{B}_{ii} = (a_{i+1} \cdot a'_{i+1})\lambda\gamma \qquad 1 \leq i \leq S \tag{5.21}$$

where a_{i+1} is a column vector of zeros except last entry which is 1 and
$\tilde{B}_{ii}$ is the $(i+1) \times (i+1)$ matrix.

The Markov chain $\{(N(t), I(t), B(t)), t \geq 0\}$ has the generator Q in partitioned form given by

$$Q = \begin{bmatrix} B_0 & A_0 & 0 & 0 & 0 & \cdots \\ A_2 & A_1 & A_0 & 0 & 0 & \cdots \\ 0 & A_2 & A_1 & A_0 & 0 & \cdots \\ 0 & 0 & A_2 & A_1 & A_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

where , the entries in Q are given by (5.2) to (5.22).

Let $A = A_0 + A_1 + A_2$ and π denote the steady-state probability vector of A, i.e,

$$\pi A = 0, \quad \pi e = 1$$

The vector π can be partitioned as

$$\pi(i) = (\pi(i,0), \pi(i,1), \cdots, \pi(i,i)); i = 0, 1, \cdots, S$$

Then the π's can be calculated as

$$\pi(s-i) = \pi(s)\beta_{s-i}$$
$$\pi(S-i) = \pi(s)\beta_{S-i}$$

where

$$\beta_{s-i} = \begin{cases} -\beta_{s-i+1}(\tilde{A}_{s-i+1,s-i} + \tilde{C}_{s-i+1,s-i})(\tilde{A}_{s-i,s-i} + \tilde{B}_{s-i,s-i} + \tilde{C}_{s-i,s-i})^{-1} & \text{if } i = 1, 2, \cdots, s \\ I & \text{if } i = 0 \end{cases}$$

$$\beta_{S-i} = \begin{cases} [-\beta_{S-i+1}\bar{A}_{S-i+1,S-i} - \beta_{s-i}\tilde{A}_{s-i,S-i}](\tilde{A}_{S-i,S-i} + \tilde{B}_{S-i,S-i} + \tilde{C}_{S-i,S-i})^{-1} & \text{if } i = 1, 2, \cdots, s \\ -\tilde{A}_{sS}(\tilde{A}_{S,S} + \tilde{B}_{S,S} + \tilde{C}_{S,S})^{-1} & \text{if } i = 0 \end{cases}$$

$$\pi(i) = -\beta_{i+1}(\bar{A}_{i+1,i} + \tilde{C}_{i+1,i})(\tilde{A}_{i,i} + \tilde{B}_{i,i} + \tilde{C}_{i,i})^{-1} \text{ if } i = Q-1, Q-2, \cdots, s+1$$

We have the following result on system stability

Lemma 5.2.1. *The system is stable if*

$$\sum_{i=s+2}^{S}\sum_{j=1}^{i-1}\pi(i,j) > \left(\frac{\lambda\gamma}{p\mu}\right)\sum_{i=0}^{S}\pi(i,i)$$

Proof. From the well-known result [Neuts [61] theorem 1.7.1] on the positive recurrence of Q, which states that

$$\pi A_0 e < \pi A_2 e$$

and by exploiting the structure of the matrices A_0 and A_2, the stated result follows. □

If stability holds, by using theorem 3.2.2 and Logarithmic Reduction Algorithm (see Latouche and Ramaswamy [1999]) we can calculate the rate matrix R.

5.2.2 System Performance Measures

We write the steady state probability vector $x = (x(0), x(1), x(2), \ldots)$ where

$$x(i) = (y(i,j,k)); \quad i \geq 0, 0 \leq j \leq S, 0 \leq k \leq j$$

Some of the system performance measures are given below:

1. The probability mass function of number of customer in the pool: The probability that there are i customers in the pool is given by

$$P_i = x(i)e = x(0)R^i e; \quad i \geq 0$$

2. Expected Inventory level in the system: Expected inventory level in the system is given by

$$\alpha_1 = \sum_{i=0}^{\infty}\left[\{\sum_{j=1}^{S} j \sum_{k=0}^{j} y(i,j,k)\}\right]e$$

3. Expected number of customers in the buffer is,

$$\alpha_2 = \sum_{i=0}^{\infty} \Big[\sum_{j=1}^{S} \sum_{k=1}^{j} k y(i, j.k) \Big] e$$

4. Expected number of customers in the pool

$$\alpha_3 = \sum_{i=0}^{\infty} i x(i) e = x(0) R(I - R)^{-2} e$$

5. Average Customers lost to the system is,

$$\alpha_4 = \lambda(1 - \gamma) \sum_{i=0}^{\infty} \Big[\sum_{j=k=0}^{S} y(i, j, k) \Big] e$$

6. Expected rate at which customer enter the pool is,

$$\alpha_5 = \lambda\gamma \sum_{i=0}^{\infty} \Big[\sum_{j=k=0}^{S} y(i, j, k) \Big] e$$

7. The Average rate at which the pool customers enter the buffer is given by

$$\alpha_6 = p\mu \sum_{i=1}^{\infty} [\sum_{j=s+1}^{S} \sum_{k=1}^{j} y(i, j, k)] e$$

5.2.3 Numerical illustration

Fixed $S = 5$,$s = 2$,$Q = 3$,$\lambda = 0.5$, $\mu = 0.7$,$\beta = 0.6$,$p = 0.6$,$\gamma = 0.6$

We provide a numerical illustration based on performance measures.

Table 5.1:

	$\lambda = 0.2$	$\lambda = 0.3$	$\lambda = 0.4$	$\lambda = 0.5$	$\lambda = 0.6$
α_1	3.64122	3.44974	3.27038	3.1189	3.00847
α_2	0.362165	0.593453	0.85437	1.14709	1.47747
α_3	0.0427584	0.176074	0.551752	1.71793	9.46571
α_4	0.00222627	0.00937859	0.0246719	0.050412	0.0882317
α_5	0.003394	0.0140679	0.0370079	0.075618	0.132348
α_6	0.103909	0.143281	0.175928	0.203971	0.229517

Table 5.2:

	$\mu = 0.6$	$\mu = 0.7$	$\mu = 0.8$	$\mu = 0.9$	$\mu = 1.0$
α_1	3.15381	3.1189	3.10037	3.09042	3.0852
α_2	1.44111	1.14709	0.945552	0.800566	0.692144
α_3	3.87896	1.71793	1.07129	0.778464	0.617492
α_4	0.0634834	0.050412	0.0419241	0.0361165	0.0319699
α_5	0.0952251	0.075618	0.0628862	0.0541748	0.0479548
α_6	0.125296	0.146179	0.167062	0.207452	0.208827

Table 5.3:

	$\beta = 0.4$	$\beta = 0.5$	$\beta = 0.6$	$\beta = 0.7$	$\beta = 0.8$
α_1	2.7194	2.95409	3.1189	3.24018	3.33281
α_2	1.10445	1.12528	1.14709	1.16721	1.18511
α_3	6.50534	2.68521	1.17793	1.29464	1.06335
α_4	0.0695216	0.0576941	0.050412	0.0456062	0.042261
α_5	0.104282	0.0865412	0.075618	0.0684092	0.0633917
α_6	0.178756	0.193112	0.203971	0.212473	0.219313

Table 5.4:

	$p = 0.4$	$p = 0.5$	$p = 0.6$	$p = 0.7$	$p = 0.8$
α_1	3.13578	3.12578	3.1189	3.11374	3.06936
α_2	1.17199	1.1566	1.14709	1.14089	1.00008
α_3	4.22862	2.41191	1.17793	1.35453	0.680104
α_4	0.050528	0.0504136	0.050412	0.0504763	0.0466839
α_5	0.0757921	0.0756203	0.075618	0.0757144	0.0466839
α_6	0.138115	0.171072	0.203971	0.236803	0.250033

Table 5.5:

	$\gamma = 0.4$	$\gamma = 0.5$	$\gamma = 0.6$	$\gamma = 0.7$	$\gamma = 0.8$
α_1	3.14825	3.13416	3.1189	3.10209	3.08329
α_2	1.05942	1.10032	1.14709	1.20083	1.26299
α_3	0.653433	1.05776	1.17793	2.90313	5.41881
α_4	0.067264	0.0593129	0.050412	0.0403486	0.0288498
α_5	0.0448427	0.0593129	0.075618	0.0941467	0.115399
α_6	0.196428	0.199945	0.203971	0.2086	0.213959

In Table 5.1 to 5.5,we provide measures of the system performance by fixing the parameter's values involved in the system. we vary over the parameters λ, μ, β, p and γ. For different values of these parameters corresponding values of the system performance measures are provided.

5.3 Model-II

5.3.1 Model Discription

In this model we assume that if there is atleast one unit in the inventory and no customer in the buffer, then with probability one, service to the head of the line in the pool customer will start. The rest of the assumptions are similar to model-I. In the present model, system will be affected through the matrix A_1 and A_2 in the infinitesimal generator Q of model-I. For the convenience, we are redefining the entries of matrix A_1 and A_2. The entries of the matrix A_1 can be written as:-

$$\tilde{A}_{11} = \begin{bmatrix} -\lambda - \beta - \mu & \lambda \\ 0 & -\lambda\gamma - \beta - \mu \end{bmatrix} \tag{5.22}$$

$$\tilde{A}_{ii} = \begin{bmatrix} -\lambda - \mu - \beta & \lambda & & \\ & -\lambda - \beta - \mu & \lambda & \\ & \ddots & & \\ & & & -\lambda\gamma - \beta - \mu \end{bmatrix} for 2 \leq i \leq s \tag{5.23}$$

$$\tilde{A}_{ii} = \begin{bmatrix} -\lambda - \mu & \lambda & & \\ & -\lambda - \mu & \lambda & \\ & \ddots & \ddots & \\ & & & -\lambda\gamma - \mu \end{bmatrix} for s+1 \leq i \leq S \tag{5.24}$$

whereas other entries are identical to that of the entries of matrix A_1 in model-I.

The entries of the matrix A_2 for the present model can be writen as:-

$$\tilde{C}_{i,i-1} = \begin{bmatrix} \mu & & & \\ & p\mu & & \\ & & p\mu & \\ & & & \\ & & & p\mu \\ & & & 0 \end{bmatrix}_{(i+1)\times i}, \quad for s+1 \leq i \leq S \tag{5.25}$$

$$\tilde{C}_{i,i-1} = \begin{bmatrix} \mu \\ 0 \end{bmatrix}_{(i+1)\times i}, \quad i = 1 \tag{5.26}$$

$$\tilde{C}_{i,i-1} = \begin{bmatrix} \mu & 0 \\ 0 & 0 \end{bmatrix}_{(i+1)\times i}, \quad for 2 \leq i \leq s \tag{5.27}$$

The other entries are identical to that of the entries of matrix A_2 in model-I.

we have the following condition for stability

Lemma 5.3.1. *The system is stable if*

$$\sum_{i=1}^{S} \pi(i,0)\mu + \sum_{i=s+1}^{S} \sum_{j=1}^{i-1} \pi(i,j)p\mu > \lambda\gamma \sum_{i=0}^{S} \pi(i,i)$$

Proof. From the well-known result [Neuts [61] theorem 1.7.1] on the positive recurrence of Q, which states that

$$\pi A_0 e < \pi A_2 e$$

and by exploiting the structure of the matrices A_0 and A_2, the stated result follows. □

Theorem 3.2.2 is applicable for the present model when the stability condition holds.For calculating the rate matrix R, Logarithmic Reduction Algorithm (Latouche and Ramaswami [49]) can be used. Note that the vector π of the generator A can be calculated in the same fashion as calculated in model-I.

5.3.2 System Performance Measures

We write the i^{th} component of the steady state probability vector $x = (x(0), x(1), x(2), \ldots)$

$$x(i) = (z(i,j,k)); \quad i \geq 0, 0 \leq j \leq S, 0 \leq k \leq j$$

Some of the system performance measures are given below:

1. The probability mass function of number of customers in the pool: The probability that there are i customers in the pool is given by

$$P_i = x(i)e = x(0)R^i e; \quad i \geq 0$$

2. Expected Inventory level in the system: Expected inventory level is given by

$$\beta_1 = \sum_{i=0}^{\infty} \Big[\{ \sum_{j=1}^{S} j \sum_{k=0}^{j} z(i,j,k) \} \Big] e$$

3. Expected number of customers in the buffer is

$$\beta_2 = \sum_{i=0}^{\infty} \Big[\sum_{j=1}^{S} \sum_{k=1}^{j} k z(i,j.k) \Big] e$$

4. Expected number of customers in the pool

$$\beta_3 = \sum_{i=0}^{\infty} i x(i) e = x(0) R (I - R)^{-2} e$$

5. Average Number of Customers lost to the system is

$$\beta_4 = \lambda(1-\gamma) \sum_{i=0}^{\infty} \Big[\sum_{j=k=0}^{S} z(i,j,k) \Big] e$$

6. Expected rate that a customer will enter the pool is,

$$\beta_5 = \lambda\gamma \sum_{i=0}^{\infty} \Big[\sum_{j=k=0}^{S} z(i,j,k) \Big] e$$

7. The Average rate at which the pooled customers enter the buffer is given by

$$\beta_6 = p\mu \sum_{i=1}^{\infty} [\sum_{j=s+1}^{S} \sum_{k=1}^{j} z(i,j,k)] e + \mu \sum_{i=1}^{\infty} [\sum_{j=1}^{S} z(i,j,0)] e$$

5.3.3 Numerical Illustration

We provide a numerical illustration based on performance measure.

Table 5.6:

	$\lambda = 0.3$	$\lambda = 0.4$	$\lambda = 0.5$	$\lambda = 0.6$	$\lambda = 0.7$
β_1	3.47192	3.30217	3.15116	3.02365	2.92394
β_2	0.574263	0.801166	1.03429	1.27722	1.41279
β_3	0.0743328	0.235863	0.628411	1.69963	2.47404
β_4	0.00884	0.0224701	0.0443471	0.0753339	0.107078
β_5	0.01326	0.0337052	0.0665206	0.113001	0.160617
β_6	0.553833	0.501887	0.452489	0.405742	0.376486

Table 5.7:

	$\mu = 0.5$	$\mu = 0.6$	$\mu = 0.7$	$\mu = 0.8$	$\mu = 0.9$
β_1	3.22136	3.1774	3.15116	3.1367	3.12871
β_2	1.70973	1.28765	1.03429	0.859483	0.732528
β_3	5.26143	1.22989	0.628411	0.396087	0.280761
β_4	0.0752161	0.0552871	0.0443471	0.0372473	0.0323662
β_5	0.112824	0.0829306	0.0665206	0.0558709	0.0485493
β_6	0.278375	0.365966	0.452489	0.54048	0.62965

Table 5.8:

	$\beta = 0.4$	$\beta = 0.5$	$\beta = 0.6$	$\beta = 0.7$	$\beta = 0.8$
β_1	2.74395	2.986	3.15116	3.27066	3.36096
β_2	0.913272	0.983043	1.03429	1.07372	1.10513
β_3	1.06962	0.768233	0.62841	0.551497	0.504454
β_4	0.0589525	0.0498858	0.0443471	0.0407051	0.0381752
β_5	0.0884288	0.0748287	0.0665206	0.0610576	0.0572628
β_6	0.408505	0.43567	0.452489	0.463708	0.471619

Table 5.9:

	$p = 0.4$	$p = 0.5$	$p = 0.6$	$p = 0.7$	$p = 0.8$
β_1	3.14934	3.15008	3.15116	3.15152	3.15182
β_2	0.998271	1.01151	1.0235	1.03429	1.04396
β_3	0.727222	0.691369	0.658411	0.601111	0.576402
β_4	0.0430501	0.0437241	0.0443471	0.0449207	0.0454472
β_5	0.0645752	0.0655862	0.0665206	0.067381	0.0681708
β_6	0.392067	0.421981	0.452489	0.483497	0.514921

Table 5.10:

	$\gamma = 0.4$	$\gamma = 0.5$	$\gamma = 0.6$	$\gamma = 0.7$	$\gamma = 0.8$
β_1	3.17546	3.16416	3.15116	3.1364	3.1197
β_2	0.999049	1.01605	1.03429	1.05395	1.07535
β_3	0.309636	0.448205	0.628411	0.866194	1.18667
β_4	0.0621832	0.0535514	0.0443471	0.0344916	0.023895
β_5	0.0414555	0.0535514	0.0665206	0.0804803	0.0955801
β_6	0.460935	0.456892	0.452489	0.447675	0.442377

In tables 5.6 to 5.10, we provide measures of the system performance by fixing the parameter values involved in the system. We vary over the parameters λ, μ, β, p and γ. For different values of these parameters corresponding values of the system measures are provided.

Table Analysis

We evaluated certain system performance measures based on model-I. Where we assumed that the pooled customer will be picked up to buffer at a service completion epoch only if inventory level is atleast $s+1$. In model-II we relax this particular restriction and consider that the customer from the pool will be picked up with probability one even when inventory level is atleast one and no customer is present in the buffer. So it is expected that in the latter case the expected number of pooled customers get reduced whereas the customer entering rate from the pool to buffer will be increased. And as an overall affect inventory level will be decreased in model-II. Comparing the results in tables for model-I and model-II we notice that they are in agreement with our expectation.

Chapter 6

(s, S) Inventory System with Postponed Demands

6.1 Introduction

In this chapter we discuss an (s, S) inventory system with postponed demands. Two models are discussed. In the first model we examine the case in which life time of the inventoried items is infinite and in the second model the inventoried items have random shelf-life which is exponentially distributed with parameter $\theta(\theta > 0)$ under the same assumptions except that when inventory level is zero, external demand has choice to join the pool with probability β or leave the system with probability $(1 - \beta)$.

Many researchers have considered (s, S) inventory system and examine the system characteristics. Gross and Harris [26] analyzed a continuous review (s, S) inventory model with state dependent lead-time. Srinivasan [75] analyzed an (s, S) inventory system with random lead time and unit demand. Sahin [74] discuss an (s, S) inventory model under compound renewal demand and random lead time. Beckman and Srinivasan [7] consider an inventory system with

•The results of Model-I of this chapter will appear in the Journal of Stochastic Analysis and Application, Vol.22, No.3, 2004.

•The results of Model-II of this chapter was presented in the Annual Conference of Indian Society for Probability and Statistics at Nagarjuna University, Andhrapradesh; 18-20 December'2003.

Poisson demands and exponential lead time. Ramanarayan and Jacob [69] analyze an (s, S) inventory system with random lead time and bulk demand. An (s, S) inventory system with random life-time and positive lead time is discussed in Kalpakam and Sapna [37].

On the other hand extensive work has already been done by many researchers in the field of perishable inventory systems by assuming a constant rate of deterioration and also constant demand. Goel and Giri [24] study an inventory model by considering demands as a function of selling price and three parameter Weibull rate of deterioration. Nahmias [57] is an excellent reference for literature on perishable inventories. Raafat [66] presented a complete survey of literature for the deteriorating inventory models up to [1990].

Notations

The following notations are used in this chapter

$I(t)$ = Inventory level at time t

$N(t)$ =Number of customers in the pool at time t

$\{(I(t), N(t))\} := \{(i, j) \mid 0 \leq i \leq S; 0 \leq j \leq M\}$

$f^*(\alpha)$ =Laplace Transform of $f(.)$

$E_1 = \{0, 1, 2, \cdots, S\}$

$E_2 = \{0, 1, 2, \cdots, M\}$

$E = E_1 \times E_2$

$e_{M+1} = (1, 1, \cdots, 1)^T$: an $(M + 1)$ -component column vector of 1's

This chapter is organized as follows: Model-I is discussed in section 6.2. This section contains five subsections. Some assumptions are made to study the model in 6.2.1. Model analysis both for transient and steady state cases are discussed in subsection 6.2.2. In subsection 6.2.3 we list some system performance measures and based on that measures a cost function is developed and some numericals are provided in the subsections 6.2.4 and 6.2.5 respectively. In section 6.3 we discuss the model-II. This section contain four subsections. In subsection 6.3.1 we study the system in steady state case for perishable inventory system. System characteristics measure is given in 6.3.2. A cost function is discussed in the subsection 6.3.3 and finally, we provided illustrative numerical examples in subsection 6.3.4.

6.2 Model I

In the first model we examine an (s, S) inventory system with Postponed demands.We assume that customers arrive to the system according to a Poisson process with rate $\lambda(> 0)$. When inventory level depletes to s either due to a demand or service to a pooled customer, an order for replenishment is placed. The lead time is exponentially distributed with parameter γ. When inventory level reaches zero, incoming customers are sent to a pool of capacity M. Any demand that takes place when the pool is full and inventory level zero, is assumed to be lost. After replenishment, as long as the inventory level is greater than s, the pooled customers are selected according to an exponentially distributed time lag, with rate depending on the number in the pool.The difference between the problem under discussion and classical (s, S) inventory models with lead time is that pooled customers will have to wait even when inventory level is positive whearas in the latter backlogs are cleared, partially or fully depending on availability, on replenishment of inventory, in the former this need not take place. In both cases lead time plays a crucial rule.

6.2.1 Assumptions

1. Initially the inventory level is S, i,e. $I(0) = S$
2. Inter arrival time of demands are exponentially distributed with parameter λ
3. Lead time is exponentially distributed with parameter γ
4. Demands that arrive when the inventory level is 0, is sent to a pool of capacity M. Beyond M the demand is lost provided inventory level is also zero
5. When the inventory level $I(t) > s$, demands from both pooled customers and external customers can be met, but when $I(t) \leq s$ only external demands will be met and pooled

customers have to wait until the next replenishment.

6.2.2 Model and Analysis

The maximum inventory level is fixed at S. The inter-arrival time between two successive demands is assumed to be exponentially distributed with parameter λ. Each demand is for exectly one unit of the item.When inventory level $I(t)$ depletes due to demands and reaches the re-order level s, an order for replenishment is placed. Lead time is exponentially distributed with parameter γ. When level is zero, demands that take place are sent to a pool, which has a finite capacity M. When inventory level $\geq s+1$ both external and pooled customer's demands are met. The infinitisemal generator $\tilde{A} = (a(i,j;k,l)), (i,j), (k,l) \in E$, of the process can be obtained using the following arguments:-

a. The arrival of a demand makes a transition from

$(i,j) \longrightarrow (k = i-1, l = j)$ if $1 \leq i \leq S$ and

$(i,j) \longrightarrow (k = i, l = j+1)$ if $i = 0, 0 \leq j \leq M-1$

b. When a pooled customer is picked up, it leaves the pool size and also the inventory level less by one i,e. the transition

$(i,j) \longrightarrow (k = i-1, l = j-1)$ if $s+1 \leq i \leq S$

c. Transition from (i,j) to $(i+Q, l = j)$ if $i \leq s$ and has rate γ.

Hence we get

$$a(i,j;k,l) = \begin{cases} \lambda & \text{if } k = i-1; i = s+1,\dots,S \\ & j = 0,1,2,\dots,M; l = j \\ j\mu & \text{if } k = i-1; i = s+1,\dots,S \\ & l = j-1; j = 1,2,\dots,M \\ -(\lambda + j\mu) & \text{if } k = i; i = s+1,\dots,S \\ & l = j; j = 0,1,\dots,M \\ \lambda & \text{if } k = i = 0 \\ & l = j+1; j = 0,1,\dots,M-1 \\ \gamma & \text{if } k = i+Q; i = 0,1,\dots,s \\ & l = j; j = 0,1,\dots,M \\ -(\lambda+\gamma) & \text{if } k = i; i = 0,1,\dots,s \\ & l = j; j = 0,1,\dots,M \\ \lambda & \text{if } k = i-1; i = 0,1,\dots,s \\ & l = j; j = 0,1,\dots,M \end{cases}$$

Define

$A_{ik} = (a(i,j),(k,l))_{j,l \in E_2}, i,k \in E_1$

The infinitesimal generator $\tilde{A}$ can be conveniently express as a partitioned matrix

$\tilde{A} = ((A_{ik}))$

where A_{ik} is a $(M+1) \times (M+1)$ matrix which is given by,

$$A_{ik} = \begin{cases} A_1 & \text{if } k = i-1; i = s+1,\dots,S \\ A_2 & \text{if } k = i; i = s+1,\dots,S \\ A_3 & \text{if } k = i; i = 0 \\ A_4 & \text{if } k = i+Q; i = 0,1,\dots,s \\ A_5 & \text{if } k = i; i = 1,2,\dots,s \\ A_6 & \text{if } k = i-1; i = 1,2,\dots,s \\ 0 & \text{otherwise} \end{cases}$$

with

$$A_1 = \begin{array}{l} M \\ M-1 \\ M-2 \\ \ldots \\ 2 \\ 1 \\ 0 \end{array} \begin{pmatrix} \lambda & M\mu & 0 & 0 & 0 & 0 \\ 0 & \lambda & (M-1)\mu & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 & 0 \\ \ldots & \ldots & & & & \ldots \\ 0 & 0 & 0 & \lambda & 2\mu & 0 \\ 0 & 0 & 0 & 0 & \lambda & \mu \\ 0 & 0 & 0 & 0 & 0 & \lambda \end{pmatrix}$$

for $k = i-1; i = s+1, \cdots, S$.

$$A_2 = \begin{array}{l} M \\ M-1 \\ \ldots \\ 2 \\ 1 \\ 0 \end{array} \begin{pmatrix} -(\lambda+M\mu) & 0 & 0 & 0 & 0 \\ 0 & -(\lambda+(M-1)\mu) & 0 & 0 & 0 \\ \ldots & & & & \\ 0 & 0 & -(\lambda+2\mu) & 0 & 0 \\ 0 & 0 & 0 & -(\lambda+\mu) & 0 \\ 0 & 0 & 0 & 0 & -\lambda \end{pmatrix}$$

for $k = i; i = s+1, \cdots, S$

$$A_3 = \begin{array}{l} M \\ M-1 \\ M-2 \\ \ldots \\ 2 \\ 1 \\ 0 \end{array} \begin{pmatrix} -\gamma & 0 & 0 & & 0 & 0 & 0 \\ \lambda & -(\lambda+\gamma) & 0 & & 0 & 0 & 0 \\ 0 & \lambda & -(\lambda+\gamma) & & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & & & & \\ 0 & 0 & 0 & & -(\lambda+\gamma) & 0 & 0 \\ 0 & 0 & 0 & & \lambda & -(\lambda+\gamma) & 0 \\ 0 & 0 & 0 & \ldots & 0 & \lambda & -(\lambda+\gamma) \end{pmatrix}$$

for $k = i; i = 0$

$$A_4 = \begin{array}{l} M \\ M-1 \\ M-2 \\ \ldots \\ 2 \\ 1 \\ 0 \end{array} \begin{pmatrix} \gamma & 0 & 0 & & 0 & 0 & 0 \\ 0 & \gamma & 0 & & 0 & 0 & 0 \\ 0 & 0 & \gamma & & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & & & & \ldots \\ 0 & 0 & 0 & & \gamma & 0 & 0 \\ 0 & 0 & 0 & & 0 & \gamma & 0 \\ 0 & 0 & 0 & & 0 & 0 & \gamma \end{pmatrix}$$

for $k = i + Q; i = 0, 1, \cdots, s$

$$A_5 = \begin{array}{l} M \\ M-1 \\ M-2 \\ \ldots \\ 1 \\ 0 \end{array} \begin{pmatrix} -(\lambda+\gamma) & 0 & 0 & 0 & 0 \\ 0 & -(\lambda+\gamma) & 0 & 0 & 0 \\ 0 & 0 & -(\lambda+\gamma) & 0 & 0 \\ \ldots & & & & \\ 0 & 0 & 0 & -(\lambda+\gamma) & 0 \\ 0 & 0 & 0 & 0 & -(\lambda+\gamma) \end{pmatrix}$$

for $k = i; i = 1, 2, \cdots, s$

$$A_6 = \begin{array}{l} M \\ M-1 \\ M-2 \\ \ldots \\ 2 \\ 1 \\ 0 \end{array} \begin{pmatrix} \lambda & 0 & 0 & & 0 & 0 & 0 \\ 0 & \lambda & 0 & & 0 & 0 & 0 \\ 0 & 0 & \lambda & & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & & \ldots & \ldots & \ldots \\ 0 & 0 & 0 & & \lambda & 0 & 0 \\ 0 & 0 & 0 & & 0 & \lambda & 0 \\ 0 & 0 & 0 & & 0 & 0 & \lambda \end{pmatrix}$$

for $k = i - 1; i = 1, 2, \cdots, s$

So we can write the partitioned matrix as follows:

$$\tilde{A} = \begin{array}{l} S \\ S-1 \\ \cdots \\ Q+1 \\ Q \\ \cdots \\ s+1 \\ s \\ \cdots \\ 2 \\ 1 \\ 0 \end{array} \begin{pmatrix} A_2 & A_1 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & A_2 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \cdots & & & & & & & & & \cdots \\ 0 & 0 & & A_2 & A_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & & 0 & A_2 & 0 & 0 & 0 & 0 & 0 \\ \cdots & & & & & & & & & \cdots \\ 0 & 0 & & 0 & 0 & A_2 & A_1 & 0 & 0 & 0 \\ A_4 & 0 & & 0 & 0 & 0 & A_5 & 0 & 0 & 0 \\ \cdots & & & & & & & & & \cdots \\ 0 & 0 & & 0 & 0 & 0 & 0 & A_5 & A_6 & 0 \\ 0 & 0 & \cdots & A_4 & 0 & 0 & 0 & 0 & A_5 & A_6 \\ 0 & 0 & & 0 & A_4 & 0 & 0 & 0 & 0 & A_3 \end{pmatrix}$$

Transient Analysis:

Define

$$\phi((i,j),(k,l),t) = Pr[I(t) = k, N(t) = l | I(0) = i, N(0) = j], (i,j),(k,l) \in E$$

Let, $\phi_{i,k}(t)$ denote a matrix whose $(j,l)^{th}$ element is $\phi((i,j),(k,l),t)$ and $\Phi(t)$ denotes a block partitioned matrix with the sub-matrix $\phi_{i,k}(t)$ at $(i,k)^{th}$ position. The Kolmogorov differential equation satisfied by $\phi((i,j),(k,l),t)$ in matrix form is

$$\Phi'(t) = \Phi(t)A$$

The solution of the above equation is given by

$$\Phi(t) = e^{At}$$

Now,

$$\Phi^*_\alpha = \int_0^\infty e^{-\alpha t} e^{At} dt = (\alpha I - A)^{-1}$$

where

$\Phi^*_\alpha = (\phi^*_{ik}(\alpha))$ and $\phi^*_{i,k}(\alpha) = (\phi^*((i,j),(k,l),\alpha))_{j,l\in E_2}$

with $\phi^*((i,j),(k,l),\alpha) = \int_0^\infty e^{-\alpha t}\phi((i,j),(k,l),t)dt$

The matrix $(\alpha I - A)$ has the form

$P = (\alpha I - A) =$

$$\begin{array}{c} S \\ S-1 \\ \\ Q \\ \\ s+1 \\ s \\ \\ 1 \\ 0 \end{array}
\begin{pmatrix}
D_S & -B_S & 0 & & 0 & 0 & 0 & 0 \\
0 & D_{S-1} & 0 & & 0 & 0 & 0 & 0 \\
 & & & & & & & \\
0 & 0 & D_Q & & 0 & 0 & 0 & 0 \\
\cdots & \cdots & & & & & & \\
0 & 0 & 0 & \cdots & D_{s+1} & -B_{s+1} & 0 & 0 \\
-H_s & 0 & 0 & & 0 & D_s & 0 & 0 \\
\cdots & \cdots & & & & & & \\
0 & 0 & 0 & & 0 & 0 & D_1 & -B_1 \\
0 & 0 & -H_0 & & 0 & 0 & 0 & D_0
\end{pmatrix}$$

where

$$D_i = \begin{cases} \alpha I - A_2 & \text{if } i = s+1,\dots,S \\ \alpha I - A_5 & \text{if } i = 1,2,\dots,s \\ \alpha I - A_3 & \text{if } i = 0 \end{cases}$$

$$B_i = \begin{cases} A_1 & \text{if } i = s+1,\dots,S \\ A_6 & \text{if } i = 1,2,\dots,s \end{cases}$$

$$H_i = A_4 \quad \text{if } i = 0,1,\cdots,s$$

To compute $P^{-1} = (\alpha I - A)^{-1}$ we proceed as described below:-

Consider the lower triangular matrix

$$\tilde{Q} = \begin{array}{c} S \\ S-1 \\ \cdots \\ Q \\ \cdots \\ s \\ \cdots \\ 1 \\ 0 \end{array} \begin{pmatrix} U_{S,S} & 0 & 0 & 0 & 0 & 0 \\ U_{S-1,S} & U_{S-1,S-1} & 0 & 0 & 0 & 0 \\ & & & & & \\ U_{Q,S} & U_{Q,S-1} & U_{Q,Q} & 0 & 0 & 0 \\ \cdots & & \cdots & & & \\ U_{s,S} & U_{s,S-1} & U_{s,Q} & U_{s,s} & 0 & 0 \\ \cdots & \cdots & & & & \\ U_{1,S} & U_{1,S-1} & U_{1,Q} & U_{1,s} & U_{1,1} & 0 \\ U_{0,S} & U_{0,S-1} & U_{0,Q} & U_{0,s} & U_{0,1} & U_{0,0} \end{pmatrix}$$

with $U_{i,i} = 1; i = 0, 1, \cdots, S$

And an almost lower triangular Matrix,

$$R = \begin{array}{c} S \\ \cdots \\ Q \\ \cdots \\ s \\ \cdots \\ 1 \\ 0 \end{array} \begin{pmatrix} 0 & -B_S & 0 & 0 & 0 & 0 \\ \cdots & & \cdots & & & \\ 0 & 0 & -B_Q & 0 & 0 & 0 \\ \cdots & \cdot\cdot & & & & \\ 0 & 0 & 0 & -B_s & 0 & 0 \\ \cdots & \cdots & & & & \\ 0 & 0 & 0 & 0 & -B_1 & 0 \\ R_{0,S} & R_{0,S-1} & R_{0,Q} & R_{0,s} & R_{0,1} & R_{0,0} \end{pmatrix}$$

such that $P\tilde{Q} = R$

The $(i,j)^{th}$ sub-matrix of PQ, denoted by $[PQ]_{i,j}$ is given by,

$$[PQ]_{i,j} = \begin{cases} D_iU_{i,j} - B_iU_{i-1,j} & \text{if } i = 1,2,\ldots,S; j = i \\ -B_iU_{j,j} & \text{if } i = 1,2,\ldots,S; j = i-1 \\ D_iU_{i,j} - B_iU_{i-1,j} & \text{if } i = 1,2,\ldots,S-1; j = i+1 \\ D_0U_{0,j} & \text{if } j = 0,1,\ldots,Q-1 \\ D_iU_{i,i+Q} - B_iU_{i-1,i+Q} - H_iU_{Q,j} & \text{if } i = 1,2,\ldots,s; j = i+Q \\ D_0U_{0,j} - H_0U_{Q,j} & \text{if } j = Q,Q+1,\ldots,S \end{cases}$$

By equating sub-matrix of PQ to the corresponding element of R, we get

$$U_{i,j} = \begin{cases} B_{i+1}^{-1}D_{i+1} & \text{if } i = 0,1,\ldots,S-1; j = i+1 \\ B_{i+1}^{-1}D_{i+1}U_{i+1,j} - B_{i+1}^{-1}H_{i+1}U_{i+Q+1,j} & \text{if } i = 0,1,\ldots,S-1; j = 1,2,\ldots,S \end{cases}$$

$$R_{0,j} = \begin{cases} D_0 & \text{if } j = 0 \\ D_0U_{0,j} & \text{if } j = 1,2,\ldots,Q-1 \\ D_0U_{0,j} - H_0U_{Q,j} & \text{if } j = Q+1,Q+2,\ldots,S \end{cases}$$

Determinent and Inverse of Matrix R

The $\det R$ is given by

$\det(R_{0,S})det(-B_S)det(-B_{S-1})\cdots det(-B_2)det(-B_1)$

For evaluating R^{-1}, we know

$$R^{-1} = \frac{adjR}{detR} \tag{6.1}$$

By using 6.1 we get

$$R^{-1} = \begin{matrix} S \\ S-1 \\ \dots \\ \dots \\ 0 \end{matrix} \begin{pmatrix} R_{0,S}^{-1}R_{0,S-1}B_S^{-1} & R_{0,S}^{-1}R_{0,S-2}B_{S-1}^{-1} & & R_{0,S}^{-1}R_{0,0}B_1^{-1} & R_{0,S}^{-1} \\ -B_S^{-1} & 0 & & 0 & 0 \\ & & & & \\ & \dots & & \dots & \\ 0 & 0 & & -B_1^{-1} & 0 \end{pmatrix}$$

The equation $P\tilde{Q} = R$ implies,

$$P^{-1} = tildeQR^{-1} \tag{6.2}$$

By using Q and R^{-1} in 6.2, we can easily evaluate

$$P^{-1} = (\alpha I - A)^{-1} = \Phi_\alpha^* \tag{6.3}$$

Steady State Analysis

It can be seen from the structure of matrix $\tilde{A}$ that the state space E is irreducible. Let the limiting distribution be denoted by $\Pi^{(i,j)}$:

$$\Pi^{(i,j)} = \lim_{t\to\infty} Pr[I(t), N(t) = (i,j)]\,, (i,j) \in E$$

$$\text{write } \Pi = (\Pi^{(S)}, \Pi^{(S-1)}, \dots, \Pi^{(1)}, \Pi^{(0)}) \text{ and}$$
$$\Pi^{(K)} = (\Pi^{(K,M)}, \Pi^{(K,M-1)}, \dots, \Pi^{(K,1)}, \Pi^{(K,0)}) \text{ for } K = 0, 1, \dots, S$$

These limits exists and satisfy the following equations:

$$\Pi\tilde{A} = 0 \quad and \sum \Pi^{(i,j)} = 1 \tag{6.4}$$

The first equation of the above yields the following set of equations,

$\Pi^{(i+1)}A_6 + \Pi^{(i)}A_3 = 0 \quad \text{if} : i = 0$

$\Pi^{(i+1)}A_6 + \Pi^{(i)}A_5 = 0 \quad \text{if}: i = 1, 2, \ldots, s-1$

$\Pi^{(i+1)}A_1 + \Pi^{(i)}A_5 = 0 \quad \text{if}: i = s$

$\Pi^{(i+1)}A_1 + \Pi^{(i)}A_2 = 0 \quad \text{if}: i = s+1, \ldots, Q-1$

$\Pi^{(i+1)}A_1 + \Pi^{(i)}A_2 + \Pi^{(i-Q)}A_4 = 0 \quad \text{if}: i = Q, \ldots, S-1$

$\Pi^{(S)}A_2 + \Pi^{(s)}A_4 = 0$

The solution of the above equations (except the last one) can be conviniently expressed as

$$\Pi^{(i)} = \Pi^{(0)}\beta_i \quad i = 0, 1, \ldots, S$$

where

$$\beta_i = \begin{cases} I & \text{if } i = 0 \\ -A_3A_6^{-1} & \text{if } i = 1 \\ (-1)^{i-1}\beta_1(A_5A_6^{-1})^{i-1} & \text{if } i = 2, 3, \ldots, s \\ (-1)^{s}\beta_1(A_5A_6^{-1})^{s-1}(A_5A_1^{-1}) & \text{if } i = s+1 \\ (-1)^{i-1}\beta_1(A_5A_6^{-1})^{s-1}(A_5A_1^{-1})(A_2A_1^{-1})^{i-s-1} & \text{if } i = s+1, \cdots, Q \\ -\beta_{i-1}(A_2A_1^{-1}) - (A_4A_1^{-1})\beta_{i-Q-1} & \text{if } i = Q+1, \cdots, S \end{cases}$$

To compute $\Pi^{(0)}$, we can use the following equations

$\Pi^{(S)}A_2 + \Pi^{(s)}A_4 = 0$ and $\sum \Pi^{(K)}e_{M+1} = 1$

which yield, respectively,

$\Pi^{(0)}(\beta_S A_2 + \beta_s A_4) = 0$ and $\Pi^{(0)}(I + \sum \beta_i)e_{M+1} = 1$

6.2.3 System Characteristics

Mean Inventory Level

Let L denote the average inventory level in the steady state. Then we have

$$L = \sum_{i=1}^{S} i \sum_{j=0}^{M} \Pi^{(i,j)}$$

Mean Re-order Rate

In the system the re-order is given when either external demand take place or demand from pooled customer is met, resulting in the level reaching s.

Let α_1 denote the mean order rate. Then we have:-

$$\alpha_1 = \lambda \sum_{j=0}^{M} \Pi^{(s+1,j)} + \sum_{j=1}^{M} j\mu \Pi^{(s+1,j)}$$

Expected Number of Pooled Customers

Let α_2 be the expected number of pooled customers. Then we have,

$$\alpha_2 = \sum_{j=1}^{M} j \sum_{i=0}^{S} \Pi^{(i,j)}$$

Expected Waiting Time:

Denote by W_j the waiting time of the j^{th} customer in the pool; $j = 1, 2, \cdots, M$. We evaluate $E(W_j)$ conditional on the system state. Figure 6.1 provides the transition diagram for computing $E(W_j)$ Thus $E(W_j) = \sum_{j=0}^{M} \{E(W_j|$ system state at $(0, j))\}\{P($system state at $(0, j))\}$

where, $E(W_j|$system state at $(0, j)) = [\frac{1}{\lambda+\gamma} + \frac{\gamma}{j\mu(\lambda+\gamma)} + \frac{\gamma}{(\lambda+\gamma)} + \frac{\lambda\gamma}{j\mu(\lambda+\gamma)}]$ for $k = 1, 2, \cdots, M$

Now the average waiting time

$$\alpha_3 = \sum_{j=1}^{M} \Pi^{(0,j)} [\frac{1}{\lambda+\gamma} + \frac{\gamma}{j\mu(\lambda+\gamma)} + \frac{\gamma}{(\lambda+\gamma)} + \frac{\lambda\gamma}{j\mu(\lambda+\gamma)}]$$

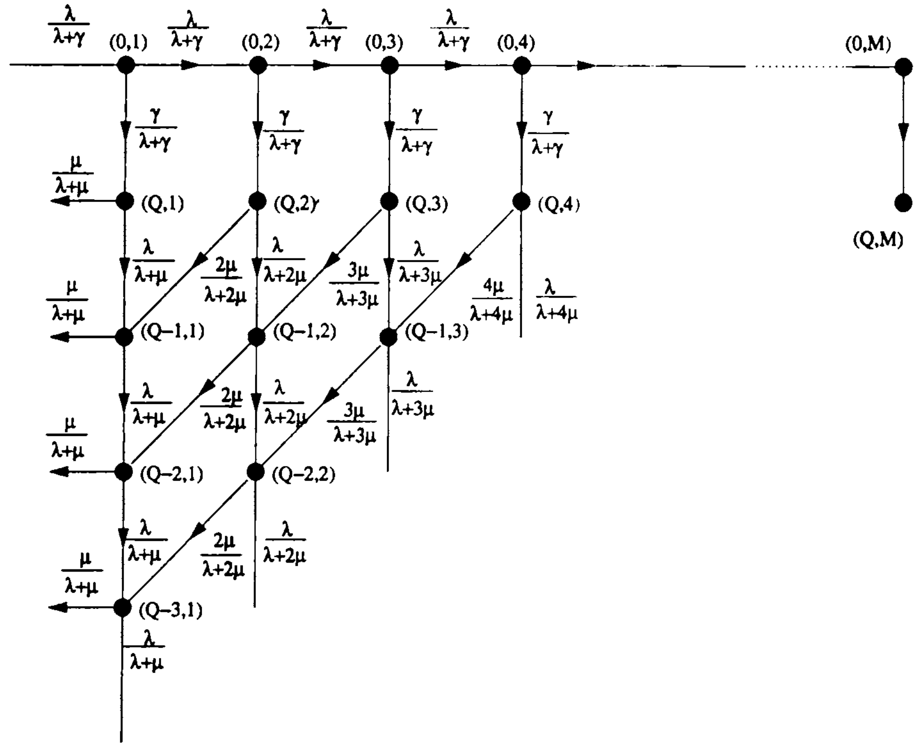

Figure 6.1:

The average number of customers lost

Let, α_4 be the average number of customers lost to the system. Then α_4 is given by,

$$\alpha_4 = \lambda \Pi^{(0,M)}$$

6.2.4 Cost Function

Define

C_1 =Inventory holding cost per unit per unit time

C_2 =Replenishment Cost

C_3 =Waiting cost of customers in the pool

C_4 =Loss due to customers not admitted to pool for want of inventory and space in the pool

So expected total cost rate of the system can be defined as:-

$$E(TC) = C_1 L + C_2 \alpha_1 + C_3 \alpha_2 + C_4 \alpha_4$$

6.2.5 Numerical Illustration

By giving values to the underlying parameters we provide some numerical illustrations. Take $S = 5, s = 2, M = 3, \lambda = 0.3, \mu = 0.2, \gamma = 0.6, Q = 3, C_1 = 5, C_2 = 2, C_3 = 3, C_4 = 2$ Then we get the measures as described in Table-6.1

Table 6.1:

Mean Inventory of the system	3.488123232
Mean Replenishment rate of the system	0.103558591
Mean lost customers of the system	0.000313942
Customer's waiting cost in the pool	0.204927851
Expected Total cost of the system	18.26315574

In the Table-6.2 and Table-6.3 we vary over M keeping the other parameter fixed at the values given in the above table. Calculated steady state probabilities for $M = 3$ are given in the Appendix-I

Table 6.2:

M-value	Expected Waiting Time
$M = 1$	0.029392763
$M = 2$	0.035886864
$M = 3$	0.041290403
$M = 4$	0.041298622

Table 6.3:

M-value	$C_1 = 3$ and$C_3 = 5$	$C_1 = 5$ and $C_3 = 3$
$M = 1$	12.05731445	19.82870805
$M = 2$	10.97402285	17.89590807
$M = 3$	11.96676498	18.26315574
$M = 4$	11.9613483	18.33277972

6.3 Model-II

In several existing models, it is assumed that products have infinite shelf-time. But in a number of practical situations, a certain amount of decay or waste is experienced on the stocked items. For example, this arises in certain food products subjected to deterioration or radio active materials where decay is present or volatile fluid under evaporation. These deterioration occur due to one or many factor viz. storage condition, weather condition including the nature of the particular product under study. The deterioration is usually a function of the total amount of inventory on hand. Hence the need to study inventory system with deterioration arises.In this model, we extended the result of model-I to a perishable (s, S) inventory system.We assume that the life-time of each item has exponential distribution with rate $\theta(> 0)$. Also it is assumed that when inventory level is zero the arriving demands choose to enter the pool with probability β and with probability $(1 - \beta)$ it is lost for ever. All assumptions of model -I hold in this case also.

6.3.1 Model and Analysis

It can be verified that $\{(I(t), N(t)), t \geq 0\}$ is a Markov process on the state space E.

The infinitisemal generator of the process

$\tilde{A} = (a(i,j;k,l)), (i,j), (k,l) \in E$, can be epressed as;-

$$a((i,j),(k,l)) = \begin{cases} \lambda & \text{if } k = i-1; i = s+1,\dots,S \\ & j = 0,1,2,\dots,M; l = j \\ j\mu & \text{if } k = i-1; i = s+1,\dots,S \\ & l = j-1; j = 1,2,\dots,M \\ i\theta & \text{if } k = i-1; i = 1,2,\dots,S \\ & l = j; j = 0,1,\dots,M \\ -(\lambda + i\theta + j\mu) & \text{if } k = i; i = s+1,\dots,S \\ & l = j; j = 0,1,\dots,M \\ \lambda\beta & \text{if } k = i = 0 \\ & l = j+1; j = 0,1,\dots,M-1 \\ \gamma & \text{if } k = i+Q; i = 0,1,\dots,s \\ & l = j; j = 0,1,\dots,M \\ -(\lambda + \gamma + i\theta) & \text{if } k = i; i = 0,1,\dots,s \\ & l = j; j = 0,1,\dots,M \\ \lambda & \text{if } k = i-1; i = 0,1,\dots,s \\ & l = j; j = 0,1,\dots,M \end{cases}$$

Define

$A_{ik} = (a(i,j),(k,l)))_{j,l \in E_2}, i,k \in E_1$

The infinitesimal generator $\tilde{A}$ can be conveniently express as a partitioned matrix

$\tilde{A} = ((A_{i,k}))$

where $A_{i,k}$ is a $(M+1) \times (M+1)$ matrix which is given by,

$$A_{i,k} = \begin{cases} A & \text{if } k = i; i = 0 \\ B & \text{if } k = i + Q; i = 0, 1, \ldots, s \\ A_i & \text{if } k = i - 1; i = s + 1 \ldots, S \\ B_i & \text{if } k = i; i = s + 1 \ldots, S \\ C_i & \text{if } k = i; i = 1, 2, \ldots, s \\ D_i & \text{if } k = i - 1; i = 1, 2, \ldots, s \end{cases}$$

with

$$A = \begin{array}{c} M \\ M-1 \\ M-2 \\ \ldots \\ 2 \\ 1 \\ 0 \end{array} \begin{pmatrix} -\gamma & 0 & 0 & 0 & 0 & 0 \\ \lambda\beta & -(\lambda\beta+\gamma) & 0 & 0 & 0 & 0 \\ 0 & \lambda\beta & -(\lambda\beta+\gamma) & 0 & 0 & 0 \\ \ldots & & & & & \\ 0 & 0 & 0 & -(\lambda\beta+\gamma) & 0 & 0 \\ 0 & 0 & 0 & \lambda\beta & -(\lambda\beta+\gamma) & 0 \\ 0 & 0 & 0 & 0 & \lambda\beta & -(\lambda\beta+\gamma) \end{pmatrix}$$

$$B = \begin{array}{c} M \\ M-1 \\ M-2 \\ \ldots \\ 2 \\ 1 \\ 0 \end{array} \begin{pmatrix} \gamma & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma & 0 & 0 & 0 \\ \ldots & \ldots & \ldots & & & \ldots \\ 0 & 0 & 0 & \gamma & 0 & 0 \\ 0 & 0 & 0 & 0 & \gamma & 0 \\ 0 & 0 & 0 & 0 & 0 & \gamma \end{pmatrix}$$

$$A_i = \begin{array}{l} M \\ M-1 \\ M-2 \\ \dots \\ 2 \\ 1 \\ 0 \end{array} \begin{pmatrix} (\lambda+i\theta) & M\mu & 0 & 0 & 0 & 0 \\ 0 & (\lambda+i\theta) & (M-1)\mu & 0 & 0 & 0 \\ 0 & 0 & (\lambda+i\theta) & 0 & 0 & 0 \\ & & & & & \dots \\ 0 & 0 & 0 & (\lambda+i\theta) & 2\mu & 0 \\ 0 & 0 & 0 & 0 & (\lambda+i\theta) & \mu \\ 0 & 0 & 0 & 0 & 0 & (\lambda+i\theta) \end{pmatrix}$$

$$B_i = \begin{array}{l} M \\ M-1 \\ \dots \\ 1 \\ 0 \end{array} \begin{pmatrix} -(\lambda+i\theta+M\mu) & 0 & 0 & 0 \\ 0 & -(\lambda+i\theta+(M-1)\mu) & 0 & 0 \\ \dots & & & \\ 0 & 0 & -(\lambda+i\theta+\mu) & 0 \\ 0 & 0 & 0 & -(\lambda+i\theta \end{pmatrix}$$

$$C_i = \begin{array}{l} M \\ M-1 \\ \dots \\ 1 \\ 0 \end{array} \begin{pmatrix} -(\lambda+i\theta+\gamma) & 0 & 0 & 0 \\ 0 & -(\lambda+i\theta+\gamma) & 0 & 0 \\ \dots & & & \dots \\ 0 & 0 & -(\lambda+i\theta+\gamma) & 0 \\ 0 & 0 & 0 & -(\lambda+i\theta+\gamma) \end{pmatrix}$$

$$D_i = \begin{array}{l} M \\ M-1 \\ M-2 \\ \dots \\ 2 \\ 1 \\ 0 \end{array} \begin{pmatrix} (\lambda+i\theta) & 0 & 0 & & 0 & 0 & 0 \\ 0 & (\lambda+i\theta) & 0 & & 0 & 0 & 0 \\ 0 & 0 & (\lambda+i\theta) & & 0 & 0 & 0 \\ \dots & & & & & \dots & \dots \\ 0 & 0 & 0 & & (\lambda+i\theta) & 0 & 0 \\ 0 & 0 & 0 & & 0 & (\lambda+i\theta) & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & (\lambda+i\theta) \end{pmatrix}$$

So, we can write the partitioned matrix as follows;

$$\tilde{A} = \begin{array}{c} S \\ S-1 \\ \cdots \\ Q+1 \\ Q \\ \\ s+1 \\ s \\ s-1 \\ \cdots \\ 2 \\ 1 \\ 0 \end{array} \begin{pmatrix} B_S & A_S & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_{S-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & & & & & & & & \\ 0 & 0 & B_{Q+1} & A_{Q+1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_Q & 0 & 0 & 0 & 0 & 0 & 0 \\ \cdots & \cdots & & & & & & & & \cdots \\ 0 & 0 & 0 & 0 & B_{s+1} & A_{s+1} & 0 & 0 & 0 & 0 \\ B & 0 & 0 & 0 & 0 & B_s & D_s & 0 & 0 & 0 \\ 0 & B & 0 & 0 & 0 & 0 & C_{s-1} & 0 & 0 & 0 \\ \cdots & & \cdots & & & & & & & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & C_2 & D_2 & 0 \\ 0 & 0 & B & 0 & 0 & 0 & 0 & 0 & C_1 & D_1 \\ 0 & 0 & 0 & B & 0 & 0 & 0 & 0 & 0 & A \end{pmatrix}$$

Steady State Analysis

It can be seen from the structure of matrix $\tilde{A}$ that the state space E is irreducible. Let the limiting distribution be denoted by $\Pi^{(i,j)}$:

$$\Pi^{(i,j)} = \lim_{t \to \infty} Pr[I(t), N(t) = (i,j),]$$

$(i,j) \in E$

write $\Pi = (\Pi^{(S)}, \Pi^{(S-1)}, \ldots, \Pi^{(1)}, \Pi^{(0)})$ and

$\Pi^{(K)} = (\Pi^{(K,M)}, \Pi^{(K,M-1)}, \ldots, \Pi^{(K,1)}, \Pi^{(K,0)})$ for $K = 0, 1, \ldots, S$

The limiting distribution exists, satisfies the following equations:

$\Pi\tilde{A} = 0$ and $\sum \Pi^{(i,j)} = 1$

The first equation of the above yields the following set of equations. We can write these equations in general,

$\Pi^{(i+1)}A_{i+1} + \Pi^{(i)}B_i + \Pi^{(i-Q)}B = 0 \ : i = Q, \ldots, S-1$

$\Pi^{(i+1)}A_{i+1} + \Pi^{(i)}B_i = 0 \ : i = s+1, \ldots, Q-1$

$\Pi^{(i+1)}A_{i+1} + \Pi^{(i)}C_i = 0 \ : i = s$

$\Pi^{(i+1)}D_{i+1} + \Pi^{(i)}C_i = 0 \ : i = 1, \ldots, s-1$

$\Pi^{(i+1)}D_{i+1} + \Pi^{(i)}A = 0 \ : i = 0$

$\Pi^{(S)}B_S + \Pi^{(s)}B = 0$

The solution of the above equations (except the last one) can be conviniently expressed as:-

$\Pi^{(i)} = \Pi^{(0)}\beta_i \ i = 0, 1, \ldots, S$

where

$$\beta_i = \begin{cases} I & \text{if } i = 0 \\ -AD_1^{-1} & \text{if } i = 1 \\ (-1)^i AD_1^{-1}C_1D_2^{-1}\ldots C_iD_{i+1}^{-1} & \text{if } i = 2, 3, \ldots, s \\ -\beta_{i-1}C_{i-1}A_i^{-1} & \text{if } i = s+1 \\ -\beta_{i-1}B_{i-1}A_i^{-1} & \text{if } i = s+2, \ldots, Q \\ -\beta_{i-1}B_{i-1}A_i^{-1} - \beta_{i-(Q+1)}BA_i^{-1} & \text{if } i = Q+1, \ldots, S \end{cases}$$

To compute $\Pi^{(0)}$, we can use the following equations

$\Pi^{(S)}B_S + \Pi^{(s)}B = 0$ and $\Pi^{(0)} \sum \Pi^{(K)} e_{M+1} = 1$

which yield, respectively,

$\Pi^{(0)}(\beta_S B_S + \beta_s B) = 0$ and $\Pi^{(0)}(I + \sum \beta_i)e_{M+1} = 1$

6.3.2 System Characteristics

Mean Inventory Level

Let μ_1 denote the average inventory level in the steady state. Then we have:-

$$\mu_1 = \sum_{i=1}^{S} i \sum_{j=0}^{M} \Pi^{(i,j)}$$

Mean Re-order Rate

Suppose μ_2 is the mean re-order rate. Then we have:-

$$\mu_2 = \lambda \sum_{j=0}^{M} \Pi^{(s+1,j)} + \sum_{j=1}^{M} j\mu\Pi^{(s+1,j)} + (s+1)\theta \sum_{j=0}^{M} \Pi^{(s+1,j)}$$

Mean Number of Perished Items

The mean number of perished items μ_3 is

$$\mu_3 = \sum_{i=1}^{S} i\theta \sum_{j=0}^{M} \Pi^{(i,j)}$$

Mean Number of Pool Customers

The expected number of pool customers μ_4 is,

$$\mu_4 = \sum_{j=1}^{M} j \sum_{i=0}^{S} \Pi^{(i,j)}$$

The average number of customer's lost

The average number of customer's lost $\mu_5$is,

$$\mu_5 = \lambda\Pi^{(0,M)} + (1-\beta)\lambda \sum_{j=0}^{M-1} \Pi^{(0,j)}$$

The probability that the external demands will be satisfied after immediately it's arrival

The probability that the external demands will be satisfied after immediately it's arrival is

$$\sum_{i=1}^{S} \sum_{j=0}^{M} \Pi^{(i,j)}$$

The probability that the external demand that's it's arrival enter the pool

The probability that the external demand that's it's arrival enter the pool is

$$\lambda\beta\sum_{j=0}^{M-1}\Pi^{(0,j)}$$

6.3.3 Cost Function

Define

C_1 =Inventory holding cost of the system

C_2 =Cost of re-order of the system

C_3 =Cost of items perished in the system

C_4 =Cost of customers lost to the system

So, the total expected cost of the system is

$$E(TC) = C_1\mu_1 + C_2\mu_2 + C_3\mu_3 + C_4\mu_5$$

6.3.4 Numerical Illustration

By giving value to the underlying parameters we provide some numerical illustrations. Take

$$S = 6, s = 2, M = 3, \lambda = 0.3, \mu = 0.2, \gamma = 0.6, Q = 3$$
$$\theta = 0.1, \beta = 0.6, C_1 = 1, C_2 = 2, C_3 = 3, C_4 = 2$$

Then we get the measures as described in belowing table and steady state probabilities for the above parameter is given in appendix-II

Mean Inventory of the system	3.20423
Mean re-order rate of the system	0.185128
Mean perished item of the system	0.320423
Mean lost custeomers to the system	0.006399
Mean pool customers in the system	0.397455
Probability that the external demand will be satisfied just after it's arrival	0.947962
Probability that the arrival demands enter the pool	0.03086
Total expected cost of the system	4.54855318

Appendix-I

$\Pi^{(0,0)}$	0.011233746	$\Pi^{(3,0)}$	0.287025356
$\Pi^{(0,1)}$	0.004941649	$\Pi^{(3,1)}$	0.024413569
$\Pi^{(0,2)}$	0.001957046	$\Pi^{(3,2)}$	0.005919092
$\Pi^{(0,3)}$	0.001046476	$\Pi^{(3,3)}$	0.001223153
$\Pi^{(1,0)}$	0.033701238	$\Pi^{(4,0)}$	0.250002555
$\Pi^{(1,1)}$	0.003591201	$\Pi^{(4,1)}$	0.02184004
$\Pi^{(1,2)}$	0.000929488	$\Pi^{(4,2)}$	0.006744105
$\Pi^{(1,3)}$	0.000135905	$\Pi^{(4,3)}$	0.001576509
$\Pi^{(2,0)}$	0.101103716	$\Pi^{(5,0)}$	0.184403905
$\Pi^{(2,1)}$	0.010773603	$\Pi^{(5,1)}$	0.012930029
$\Pi^{(2,2)}$	0.002788466	$\Pi^{(5,2)}$	0.017339342
$\Pi^{(2,3)}$	0.000407717	$\Pi^{(5,3)}$	0.013971134

Appendix-II

$\Pi^{(0,3)}$	0.000857867	$\Pi^{(3,1)}$	0.011927793
$\Pi^{(0,2)}$	0.002705345	$\Pi^{(3,0)}$	0.267052524
$\Pi^{(0,1)}$	0.010569641	$\Pi^{(4,3)}$	0.000436204
$\Pi^{(0,0)}$	0.037906091	$\Pi^{(4,2)}$	0.001981623
$\Pi^{(1,3)}$	0.000069396	$\Pi^{(4,1)}$	0.01250042
$\Pi^{(1,2)}$	0.000519085	$\Pi^{(4,0)}$	.225330124
$\Pi^{(1,1)}$	0.003553056	$\Pi^{(5,3)}$	0.00006543
$\Pi^{(1,0)}$	0.07391689	$\Pi^{(5,2)}$	0.000646654
$\Pi^{(2,3)}$	0.000138791	$\Pi^{(5,1)}$	0.005812371
$\Pi^{(2,2)}$	0.001038174	$\Pi^{(5,0)}$	0.13911527
$\Pi^{(2,1)}$	0.007106122	$\Pi^{(6,3)}$	0.000055516
$\Pi^{(2,0)}$	0.14783378	$\Pi^{(6,2)}$	0.000479138
$\Pi^{(3,3)}$	0.000254452	$\Pi^{(6,1)}$	0.003876532
$\Pi^{(3,2)}$	0.001648864	$\Pi^{(6,0)}$	0.042604115

Chapter 7

Production Inventory Model with Switching Time

7.1 Introduction

Very little investigation on production inventory in stochastic set-up had been made in the past. The analysis becomes highly complex when the items are of random life-time and the lead time are positive. Altiok[1] analyzed a production inventory system with compound Poisson demand and phase-type distribution for processing time. Berge et. al[8] deal with production inventory system with unreliable meachines , thus incorporating reliability into production inventory. They obtain some performance measures of the system. Sharafali[78] considered a production inventory operating under the (s, S) policy where demands arrive according to a Poisson process and production times are exponentially distributed. He assumed that the machine is subject to failure and repair time has general distribution. He analyzed the problem by looking at the underlying semi-regenerative process. Ching[88] considered optimal (s, S)policies with delivery time gurantees for production planning in manufacturing system with early set-up. They assumed that the inter-arrival time of the demand and the processing

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time for one unit of product are exponentially distributed and a set up time is required for the machine. Raju [68] considered N-policy for production inventory system and assumed that the machines are highly reliable and no break down take place during production process.

In this chapter we consider a production inventory system in which demand form a Poisson process and the production times have exponential distribution. The policy is (s, S) of type. When the inventory level reaches s from S production is switched on; the switching time is exponentially distributed with parameter α. During the switching time no demand is processed.

This chapter is organized as follows: In section 7.2 we describe the mathematical model. Limit distribution and waiting time distribution are discussed in section 7.3 and 7.4. In section 7.5 we evalute the expected cycle length. In section 7.6, we list some system performance measures . Steady state cost analysis is done in section 7.7. Based on the system performance measures we provide illustrative examples and sensitivity analysis in section 7.8. In this chapter following notations are used:-

λ =Demand rate

μ = Production rate

α =Switching time parameter

$H(t)$ =Inventory level at time t

7.2 Model and Analysis

The initial inventory level is S. Demands arrive as a Poisson process with rate λ. When the inventory level depletes to s the machine is switched on for the next production run. We assume that a certain amount of time which is exponentially distributed with parameter α is required for the production starts. Demands that arise during the switching time are not entertained. Shortage is allowed and infinite backlogs are permitted. The system remains in active (ON mode) until the inventory level reaches level S. The inventory level $H(t)$ at time t takes values in the set $A = \{\cdots - N, -N+1, \cdots, 0, \cdots, s, \cdots, S\}$. To get a two dimensional Markov process we incorporate the process $\{X(t), t \geq 0\}$ into $\{H(t), t \geq 0\}$ process where, $X(t)$ is define by,

$$X(t) = \begin{cases} 1 & \text{if the system is in ON mode} \\ 0 & \text{if the system is in OFF mode} \end{cases}$$

Now $\{(H(t), X(t)), t \geq 0\}$ is a two dimensional Markov chain defined on the state space $E = E_1 \cup E_2$ where,
$E_1 = \{(i, 0) : i = s, s+1, \cdots, S\}$
$E_2 = \{(i, 1) : i = -N, \cdots, s, \cdots, S-1\}$
It is to noted that $\{(H(t), X(t)), t \geq 0\}$ is a pure death process during the transition from the state $(S, 0)$ through the states $(S-1, 0), (S-2, 0), \cdots, (s, 0)$ i,e. when the production process is in off mode. When the system reaches $(s, 1)$ from $(s, 0)$ with switching time α the process is turns out to be a birth-death process till it reaches $(S, 0)$. Let us assumed that $H(0) = S$, sothat $X(0) = 0$. Consider the transition probabilities:-

$$P_{(S,0),(i,j)}(t) = P\{(H(t), X(t)) = (i, j) | ((H(0), X(0)) = (S, 0)\}$$

From now on we can write $P_{(i,j)}(t)$ for $P_{(S,0),(i,j)}(t)$.
The Kolmogorov forward differential equations satisfied by $P_{(i,j)}(t)$ are given below:-

$$\begin{aligned}
P'_{(S,0)}(t) &= -\lambda P_{(S,0)}(t) + \mu P_{(S-1,1)}(t) \\
P'_{(i,0)}(t) &= -\lambda P_{(i,0)}(t) + \lambda P_{(i+1,0)}(t) \quad &: s \leq i \leq S-1 \\
P'_{(s,1)}(t) &= -(\lambda + \mu) P_{(s,1)}(t) + \alpha P_{(s,0)}(t) + \lambda P_{(s+1,1)}(t) + \mu P_{(s-1,1)}(t) \\
P'_{(S-1,1)}(t) &= -(\lambda + \mu) P_{(S-1,1)}(t) + \mu P_{(S-2,1)}(t) \\
P'_{(i,1)}(t) &= -(\lambda + \mu) P_{(i,1)}(t) + \lambda P_{(i+1,1)}(t) + \mu P_{(i-1,1)}(t) \quad &: s+1 \leq i \leq S-2 \\
P'_{(i,1)}(t) &= -(\lambda + \mu) P_{(i,1)}(t) + \lambda P_{(i+1,1)}(t) + \mu P_{(i-1,1)}(t) \quad &: -\infty \leq i \leq s-1
\end{aligned}$$

7.3 Limit Distribution

The steady state probabilities for $(i,j) \in E$ of the system size are obtained by taking the limits as $t \longrightarrow \infty$ on both sides of the above equations and solving them recursively. Note that under steady state conditions $\lim_{t\to\infty} P'_{(i,j)}(t) = 0$

Thus we can write $q_{(i,0)} = \rho q_{(S-1,1)} \quad fors \leq i \leq S$ where $\rho = \frac{\mu}{\lambda}$

and for $j = 1$ we can write

$q_{(S-2,1)} = (1 + \frac{1}{\rho})q_{(S-1,1)}$

$q_{(S-3,1)} = (1 + \frac{1}{\rho} + \frac{1}{\rho^2})q_{(S-1,1)}$

$q_{(S-4,1)} = (1 + \frac{1}{\rho} + \frac{1}{\rho^2} + \frac{1}{\rho^3})q_{(S-1,1)}$

In general we can write

$q_{(S-i,1)} = \sum_{j=1}^{i} \frac{1}{\rho^{j-1}} q_{(S-1,1)}$ for $i = 2, 3, \cdots, S - s - 1, j = 1$

When the system is in on mode and infinite backlogs are permitted then the system may visit the state $\cdots, -1, 0, \cdots, s, \cdots, S$. To evaluate the system probabilities we consider the truncation of the system at state $-N$. After truncation we get the relations;

$q_{[-N,1]} = (\frac{\rho-\rho^{2+s-N}}{\rho^{N+s+1}(\rho-1)})q_{(S-1,1)}$

By implementing the truncation result, we can write:-

$q_{(-i,j)} = (\frac{\rho-\rho^{2+s-S}}{\rho^{i+s+1}(\rho-1)})q_{(S-1,1)} \quad : i = 1, 2, \cdots, j = 1$

$q_{(s-i,j)} = (\frac{\rho-\rho^{2+s-S}}{\rho^{i+1}(\rho-1)})q_{(S-1,1)} \quad : i = 1, 2, \cdots, s, j = 1$ and

$q_{(s,1)} = (\frac{\rho-\rho^{2+s-S}}{(\rho^2-1)} + \frac{\rho-\rho^{2+s-S}}{\rho(\rho^2-1)} + \frac{\alpha\rho}{\lambda(1+\rho)})q_{(S-1,1)}$

where $q_{(S-1,1)}$ can be be obtained by using the normalizing condition $\sum_i \sum_j q_{(i,j)} = 1$

7.4 Waiting Time Distribution

Let T be the random variable denoting the waiting time of a customer to recieve the item. Then the distribution function $F_T(.)$ of T is given by

$$F_T(t) = \begin{bmatrix} 0 & \text{if } t < 0 \\ \sum_{i=1}^{S-1} q_{(i,1)} + \sum_{i=s+1}^{S} q_{(i,0)} & \text{if } t = 0 \\ \sum_{n=0}^{\infty} q_{(-n,1)} \int_0^t \gamma_{\mu,n+1}(u)du & \text{if } t > 0 \end{bmatrix}$$

where $\gamma_{\mu,n}$ denotes the gamma density with parameter μ and n. The expression for $t = 0$ is obvious. For $t > 0$ we have two cases $X = 0$ and $X = 1$. In our system infinite backlogs are permitted. Suppose there are n backlogs at a demand epoch. Then if $X = 1$ (that is, the production process is in ON mode) the waiting time is the production time of $n + 1$ units. On the other hand if $X = 0$ (that is, the production process is off), and the number of backlogs is n, the production starts only after receiving $N - (n + 1)$ more orders and the demand of the arrival under consideration is met at the moment the $(n + 1)^{st}$ unit is produced, where N is the truncation state. Then the expected time $E(T)$ for infinite backlogs is given by,

$$E(T) = \sum_{n=0}^{\infty}\left(\frac{n+1}{\mu}\right)q_{(-n,1)} \tag{7.1}$$

7.5 Expected Cycle Length

Let us define,

$E(L_0)$ = Expected length of off mode in a cycle

$E(L_1)$ = Expected length of on mode in a cycle

$E(TL)$ = Expected total cycle length of the system

Expected length of off mode

We have

$$E(L_0) = \int_0^\infty \gamma_{\lambda,S-s}(t)dt + \int_0^\infty \alpha e^{-\alpha t}dt$$

Therefore

$$E(L_0) = (\frac{S-s}{\lambda} + \frac{1}{\alpha}) \tag{7.2}$$

Expected length of ON mode

Let $T_{(s,1)}^{(S,0)}$ = Time to reach $(S,0)$ startng from $(s,0)$ for the first time and $T_{(i,1)}^{(i+1,1)}$ = Time to reach $(i+1,1)$ startng from $(i,1)$ now,

$E(T_{(i,1)}^{(i+1,1)}) = \sum_{L,R} E(T_{(i,1)}^{(i+1,1)}|$ Transition left or right$)P($Transition left or right$)$

$$= \frac{\mu}{(\lambda+\mu)^2} + \frac{\lambda}{(\lambda+\mu)}[\frac{1}{(\lambda+\mu)} + E(T_{(i-1,1)}^{(i,1)}) + E(T_{(i,1)}^{(i+1,1)})]$$

$$E(T_{(i,1)}^{(i+1,1)}) = \frac{1}{\mu} + \frac{\lambda}{\mu}E(T_{(i-1,1)}^{(i,1)}) \tag{7.3}$$

Now

$$E[L_1] = E(T_{s,1}^{S,0}) = \sum_{i=s}^{S-1} E(T_{(i,1)}^{(i+1,1)}) \tag{7.4}$$

Using the equation (7.3) and putting the recursive relation in equation (7.4) we get,

$$E[L_1] = \sum_{i=0}^{S-s-1} [\sum_{j=1}^{i} \frac{\lambda^{j-1}}{\mu^i} + \frac{\lambda^i}{\mu^i}E(T_{(s,1)}^{(s+1,1)})] \tag{7.5}$$

Evaluation of $E(T_{(s,1)}^{(s+1,1)})$

For evaluating of $E(T_{(s,1)}^{(s+1,1)})$ let us consider the truncation occurred at the $(-N,1)$ state. So by using equation (7.3)

$E(T_{(-N,1)}^{(-N+1,1)}) = \frac{1}{\mu}$

$E(T_{(-N+1,1)}^{(-N+2,1)}) = \frac{1}{\mu}(1 + \frac{1}{\rho})$

$E(T^{(-N+3,1)}_{(-N+2,1)}) = \frac{1}{\mu}(1 + \frac{1}{\rho} + \frac{1}{\rho^2})$

Recursively we have

$E(T^{(s+1,1)}_{(s,1)}) = \frac{1}{\mu}(\sum_{r=0}^{N+s}(\frac{1}{\rho})^r)$ if $N \longrightarrow \infty$ then

$$E(T^{(s+1,1)}_{(s,1)}) = \frac{1}{\mu}(\sum_{r=0}^{\infty}(\frac{1}{\rho})^r) \tag{7.6}$$

So putting the value of (7.6) in (7.5) we get

$$E[L_1] = \sum_{i=0}^{S-s-1}[\sum_{j=1}^{i}(\frac{1}{\rho})^j\frac{1}{\lambda} + \frac{1}{\mu}(\frac{1}{\rho})^i(\sum_{r=0}^{\infty}(\frac{1}{\rho})^r)]$$

$$= \frac{1}{\lambda\mu(\rho-1)^2}\{\rho^{-S}(\mu\rho^{1+s} - \lambda\rho^{2+s} + s\mu\rho^S - S\mu\rho^S - \mu\rho^{1+S} - s\mu\rho^{1+S} + S\mu\rho^{1+S} + \lambda\rho^{2+S})\}$$

thus

$$E(TL) = [(\frac{S-s}{\lambda} + \frac{1}{\alpha}) + \frac{1}{\lambda\mu(\rho-1)^2}\{\rho^{-S}(\mu\rho^{1+s} - \lambda\rho^{2+s} + s\mu\rho^S - S\mu\rho^S - \mu\rho^{1+S} - s\mu\rho^{1+S} + S\mu\rho^{1+S} + \lambda\rho^{2+S})\}]$$

7.6 System Performance Measures

Mean Inventory Level of the system

Let $E[I]$ be expected inventory level in the steady state. Then $E[I]$ can be defined as:-

$$E[I] = \sum_{i=s}^{S} iq_{(i,0)} + \sum_{i=1}^{S-1} iq_{(i,1)}$$

$$E[I] = [\{\frac{(S-s+1)(S+s)\rho}{2}\} + \{\frac{\rho^{-2S}(\rho + (s(-1+\rho) - \rho)\rho^s)(-\rho^{1+s} + \rho^S)}{(\rho-1)^3}\} + \{\frac{\rho - \rho^{2+s-S}}{(\rho^2-1)}$$
$$+ \frac{\rho - \rho^{2+s-S}}{\rho(\rho^2-1)} + \frac{\alpha\rho}{\lambda(1+\rho)}\} + \frac{1}{\rho^s}(s(-1+\rho)^2 + s^2(-1+\rho)^2 - S^2(-1+\rho)^2 - 2\rho$$
$$+ S(-1+\rho)^2)]q_{(S-1,1)}$$

Average Backlogs in the System

Let $E[B]$ be the expected backlogs in the system in the steady state. $E[B]$can be defined as:-

$$E[B] = \frac{(\rho - \rho^{2+s-S})\rho^{-s}}{(\rho - 1)^3} q_{(S-1,1)}$$

Expected Number of Items Produced in a Cycle

Let $E[P]$ the expected number of items produced over a cycle. Then $E[P]$ can be obtained as:-
$E[P] = \mu\times$ (Expected cycle length when system is in on mode)
$E[P] = \frac{1}{\lambda(\rho-1)^2}\rho^{-S}(\mu\rho^{1+s} - \lambda\rho^{2+s} + s\mu\rho^S - S\mu\rho^S - \mu\rho^{1+S} - s\mu\rho^{1+S} + S\mu\rho^{1+S} + \lambda\rho^{2+S})$

The average number of customer's lost to the system

Demands that arise during the switching time are not entertained and hence lost. Let $E[N]$ the expected number of customers lost to the system. Then

$$E[N] = \frac{\lambda}{\alpha}$$

Evaluation of $q_{(S-1,1)}$:

$q_{(S-1,1)}$ can be obtained using the normalizing condition
$\sum_i \sum_j q_{(i,j)} = 1$ we have
$[\{\frac{((s-S)(1+s-S)+s(-s+S))\rho}{2}\}+\{\frac{(\rho-\rho^{2+s-S})\rho^{-1-s}}{(\rho-1)^2}\}+\{\frac{(\rho-\rho^{2+s-S})\rho^{-1-s}(\rho^s-\rho)}{(\rho-1)^2}\}+\{\frac{\rho-\rho^{2+s-S}}{(\rho^2-1)}+\frac{\rho-\rho^{2+s-S}}{\rho(\rho^2-1)}+\frac{\alpha\rho}{\lambda(1+\rho)}\} + \{\frac{\rho(s+S(-1+\rho)-s\rho+\rho(-1+\rho^{s-S})}{(-1+\rho)^2}\}]q_{(S-1,1)} = 1$
which gives,
$q_{(S-1,1)} = [\{\frac{((s-S)(1+s-S)+s(-s+S))\rho}{2}\}+\{\frac{(\rho-\rho^{2+s-S})\rho^{-1-s}}{(\rho-1)^2}\}+\{\frac{(\rho-\rho^{2+s-S})\rho^{-1-s}(\rho^s-\rho)}{(\rho-1)^2}\}+\{\frac{\rho-\rho^{2+s-S}}{(\rho^2-1)}+\frac{\rho-\rho^{2+s-S}}{\rho(\rho^2-1)} + \frac{\alpha\rho}{\lambda(1+\rho)}\} + \{\frac{\rho(s+S(-1+\rho)-s\rho+\rho(-1+\rho^{s-S})}{(-1+\rho)^2}\}]^{-1}$

7.7 Steady State Cost Analysis of the System

Let us considered costs under steady state as given below:-

K =The initial set-up cost of the system

C_1 =Inventory holding cost per unit per unit time

C_2 =Backlog cost per unit per unit time

C_3 =Production cost per unit per unit time

C_4 =Switching time per unit per unit time

C_5 =Cost of customers lost to the system

So, the total expected cost of the system is

$$E(TC(\alpha)) = \frac{K + C_1E[I] + C_2E[B] + C_3E[P] + \frac{c_4}{\alpha} + C_5E[N]}{E(TL)} \tag{7.7}$$

Due to large number of parameters involve in the cost function, it is not possible to prove that the above cost function is convex. By the by, for proving the convex nature of the function we adopt a numerrical search procedure. For the particular value of the parameters our calculation of $E(TC(\alpha))$ revealed a convex structure and we evaluated the optimal value of α^*. Cost rate analysis of the optimal values by varying parameters value is presented in the tables.

7.8 Numerical Illustration

The results we obtain in steady state case may be illustrated through the following numerical example:-

By giving values to the underlying parameters we illustrate the convexity of the cost function $E(TC(\alpha))$ in Table 7.1. The optimal switching time parameter is shown by indicating*.Using this optimal α value we can easily evaluate optimal switching time of the system as $E[ST] = \frac{1}{\alpha^*}$. Cost rate analysis is given in Table 7.2 and Table 7.3. Take

$S = 30, s = 9, \lambda = 1.0, \mu = 1.3, \rho = 1.3, K = 55, C_1 = 15, C_2 = 9, C_3 = 6,$

$C_4 = 10, C_5 = 5$. Then we get the measures as described in table 7.1

From Table 7.1 the optimal α value is 39. So the expected optimal switching time is 0.0256.

Table 7.1:

$\alpha-$values	Cost function	$\alpha-$values	Cost function
30	607.764	40	607.743
31	607.759	41	607.743
32	607.752	42	607.744
33	607.750	43	607.744
34	607.747	44	607.745
35	607.746	45	607.747
36	607.744	46	607.748
37	607.743	47	607.750
38	607.743	48	607.752
39	607.742	49	607.754

Cost rate Analysis

Table 7.2:

μ-Values	ρ-value	Expected total cost
1.3	1.3	607.742
1.4	1.4	504.786
1.5	1.5	443.876
1.6	1.6	403.971
1.7	1.7	376.064
1.8	1.8	355.654
1.9	1.9	340.240

In Table 7.2 and Table 7.3 we varied over certain parameters associated with the system. In Table 7.2 we vary over the values of the parameter μ and compute the corresponding cost rate we obseve that μ has the significant influence in the system behaviours. In Table 7.3 we varied over the cost parameter involve in the system i,e. C_1 (holding cost),C_2 (backlog cost), C_3 (production cost) and C_4 (switching cost). Among these parameter the table shows that the cost function is highly sensitive with respect to C_3. So we come to the conclusion that production rate and production cost parameters drastically affect the system running cost.

Table 7.3:

C_1-Varying		C_2-Varying		C_3-Varying		C_4-Varying	
C_1-Value	Cost	C_2-Value	Cost	C_3-Value	Cost	C_4-Value	Cost
15	607.742	09	607.742	06	607.742	10	607.742
16	607.742	10	607.751	07	698.742	11	607.767
17	607.742	11	607.760	08	789.742	12	607.792
18	607.743	12	607.769	09	880.742	13	607.817
19	607.743	13	607.778	10	971.742	14	607.842
20	607.744	14	607.787	11	1062.75	15	607.876
21	607.744	15	607.796	12	1153.75	16	607.892
22	607.744	16	607.805	13	1244.76	17	607.917
23	607.745	17	607.814	14	1335.76	18	607.942
24	607.745	18	607.823	15	1426.76	19	607.967
25	607.746	19	607.832	16	1517.77	20	607.992
26	607.747	20	607.841	17	1608.78	21	608.017

Conclusion

In this thesis we have presented several inventory models of utility. Of these inventory with retrial of unsatisfied demands and inventory with postponed work are quite recently introduced concepts, the latter being introduced for the first time. Inventory with service time is relatively new with a handful of research work reported. The difficulty encountered in inventory with service, unlike the queueing process, is that even the simplest case needs a 2-dimensional process for its description. Only in certain specific cases we can introduce generating function to solve for the system state distribution. However numerical procedures can be developed for solving these problem.

Retrial inventory, unlike retrial queues, also poses the same problem as discribed above. Further when an orbital customer makes a successful attempt to access the server in an inventory system with negligible service time, diagonal transitions result thereby violating the definition of a QBDP (assuming the process to be Markov). This is also the case with inventory with postponed demands (and negligible service time).

In this thesis we attempted to provide performance measures of all the models discussed. However, in most cases restricted the numerrical illustrations to the case in which the underlying distributions are exponential.

The work reported in this thesis could be extended in different directions. One among these is the introduction of arbitrarily distributed service time. It is also possible to have interarrival times of customers assumed to follow an arbitrary distribution. The interarrival time of orbital customers can also be assigned an arbitrary distribution provided we follow certain assumptions as given in Gomez Corral [23] in the case of retrial queues. All these and more are proposed in our future investigations.

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