Non-deterministic Fuzzification Using Multilattices

Thesis submitted to

COCHIN UNIVERSITY OF SCIENCE AND TECHNOLOGY

for the award of the degree of

DOCTOR OF PHILOSOPHY

under the Faculty of Science by

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September 2015

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September 2015

Certificate

This is to certify that the thesis entitled **'Non-deterministic Fuzzification Using Multilattices'** submitted to the Cochin University of Science and Technology by Mr. Gireesan K.K. for the award of the degree of Doctor of Philosophy under the Faculty of Science is a bonafide record of studies carried out by him under my supervision in the Department of Mathematics, Cochin University of Science and Technology. This report has not been submitted previously for considering the award of any degree, fellowship or similar titles elsewhere.

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Declaration

I, Gireesan K. K., hereby declare that this thesis entitled **'Non-deterministic Fuzzification Using Multilattices'** contains no material which had been accepted for any other Degree, Diploma or similar titles in any University or institution and that to the best of my knowledge and belief, it contains no material previously published by any person except where due references are made in the text of the thesis.

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Acknowledgements

I would like to set down my deep felt sense of indebtedness and gratitude to the following persons but for whose help and support I could never have accomplished this task.

I feel that words are insufficient to express my obligation to my supervising guide, Dr. T. Thrivikraman, Professor(Rtd.), Department of Mathematics, CUSAT, a man of immense knowledge and unflagging energy who stood by me as a constant source of help, support and motivation from the beginning to the end this venture.

Dr. A. Krishnamoorthy, Emeritus Professor, Department of Mathematics, CUSAT, the other member of the doctoral committee also deserves my gratitude for the encouragement and inspiration he gave me during the research activity and the submission of the thesis.

Let me register my sincere feeling of obligation and gratitude to Dr. P. G. Romeo, Dr. M. Jathavedan, Dr. M. K. Ganapathy, Dr. R. S. Chakravarthy, Dr. M. N. Narayanan Namboodiri, Dr. A. Vijayakumar, Dr. B. Lakshmi and all the teaching and Non-teaching staff of the Department of Mathematics, CUSAT for their generous help and wholehearted co-operation which enabled me to bring my research study to a successful finish. I am also grateful to the faculties of the department of Mathematics Dr. Noufel and Dr. Ambily for their timely help.

I am thankful to my fellow Research Scholars Dr. Varghese Jacob, Dr. Sreenivasan C. Dr. Manikandan, Mr. Didimos, Mr.

Pravas, Ms. Dhanya Shajin, Mr. Shinoj K. M., Ms. Savitha K. S., Ms. Jaya S., Ms. Seethu Varghese, Ms. Akhila, Ms. Smisha, Mr. Mahesh who always offered me help and support without any reservations.

I would like to acknowledge my indebtedness to Dr. Kiran Kumar, Mr. Manjunath, Mr Tijo James, Mr. Satheesh Kumar and Mr. Arun Kumar C. S. whose timely help and encouragement was extremely valuable to me in the last phase of the enterprise, the submission of the thesis.

Let me record my gratitude to my friend Dr. Sabu Sebastian who was an unfailing source of support and guidance all along the pursuit of my studies and the preparatrion of the thesis.

I am extremely grateful to Dr. Syamprakash, Principal, Govt. College of Engineering, Kannur for his understanding and positive stand in granting me the leave I required for the timely submission of the thesis.

With a deep sense of gratitude I remember my colleagues of Mathematics Prof. Johny George, Mr. Eswaran Namboodiri, Mr. Vinodan P. K. and Mr. Sivadas P. V., whose support and inspiration was of great worth to me in attaining my goal as Research Scholar.

I am extremely grateful to my parents, brothers and sisters for their support and care given to me during this prolonged venture. A special thanks to my Valyamma whose spiritual support was a constant source of strength and inspiration.

Last but not the least, my sincere love and gratitude to my wife and children who stood by me patiently throughout giving me all support and encouragement.

From the bottom of my heart I thank one and all who have directly or indirectly lent me a hand in this venture.

Gireesan K. K.

Contents

Notations used

$\leq$	Partial order relation
$\forall$	For every (universal quantifier)
$\exists$	There exist
$\sum$	Summation
$\vee$	Join
$\wedge$	Meet
$\vee_L$	Join operation on a lattice
$\wedge$	Meet operation on a lattice
$\vee_M$	Join operation on a multilattice
$\wedge_M$	Meet operation on a multilattice
$\bigvee$	Multisuprimum of arbitrary collection of sets
$\bigwedge$	Multiinfimum of arbitrary collection of sets
$\bigcup$	Set union
$\bigcap$	Set intersection
A, B, C	Arbitrary sets (crisp/fuzzy/nd-M-fuzzy subsets) or matrix (lattice or multilattice)
$A = B$	Equality of nd-M-fuzzy sets or equality of two matrices
$A \subseteq B$	Set inclusion
$A \sqsubseteq_{EM} B$	Inclusion by Egli-Milner ordering
A^0	Interior of A
$\overline{A}$	Closure of A
A^c or A'	Complement of A
$A(x)$ or $\mu(x)$	Membership grade of x in A
A_α	α level of A
μ_α	α level of μ

X, Y, Z	Universal set
I, J and K	Indexing Set
i, j, k	Elements in I, J, K respectively
L	Complete lattice
M	Complete and consistent Multilattice
L^X	Set of all L-fuzzy subset of X
$(2^M)^X$	Set of all nd-M-fuzzy subset of X
L_n	Set of all lattice matrices of order n
M_n	Set of all multilattice matrices of order n
τ, υ, ν	Fuzzy topology/ nd-M-fuzzy topologies
A^T	Transpose of a matrix.
(a_{ij}) or $(A)_{ij}$	$(i, j)^{th}$ element of A
τ_α	α level of τ
$\overline{R}$	nd-M-Fuzzy relation

Chapter 1

Introduction

The fuzzification of crisp concepts is an important topic which attracts the attention of a number of researchers. There are approaches which are based either on the structure of lattice or more restrictive structures. *Zadeh* [51] defined $[0, 1]$ valued fuzzy sets, *Goguen*[16] generalised them to the L-valued fuzzy sets, where L has the structure of a lattice. *Rosenfeld*[38] started the pioneer work in the domain of fuzzification of the algebraic objects. Also *C. L. Chang* [11] introduced the concept of fuzzy topological space and *R. Lowen* [28] introduced a more natural definition of fuzzy topological space.

Weakening the structure of the underlying set of membership functions for fuzzification has been studied extensively in recent years. One can find some attempts aiming at weakening the restrictions imposed on a lattice namely "the existence of least upper bounds and greatest lower bounds" relaxed to the existence of minimal upper bounds and maximal lower bounds. In this direction we have a structure of a multilattice.

In 1954 *M. Benado* introduced the notion of a multilattice which generalize a lattice by replacing the axiom of existence of a $l.u.b$ for two elements by that of a set of minimal upper bounds and dually. But in L-fuzzy sets introduced by *Goguen* where L is a lattice, the membership function gives unique values in L for each element of its domain. Here we are fuzzifying a crisp concept through a membership function, the membership function gives a set of values to each element of its domain. We call this type of membership function as non- deterministic (nd, for short). Thus we introduce the non-deterministic-M-fuzzy set in terms of non-deterministic membership functions, where M has the structure of a complete and consistent multilattice [21].

1.1 Summary of the thesis

The main objective of the Thesis is to study the extension of lattice theoretic works using mutilattices. In this thesis we study the non deterministic fuzzification using multilattices. The structure of the thesis is divided into five chapters. A brief chapter wise description is given below.

Chapter 2 contains brief outline of preliminary results for this thesis. So in that chapter we present a short summary of elementary notions of lattices [5], multilattices [4, 8, 12], L-fuzzy subsets and properties [18], L-fuzzy topological spaces [27], L-fuzzy lattice [2], strong L-fuzzy lattice [39] and lattice matrix [50]. All results here are quoted from existing literature.

Chapter 3 introduces the concept of $nd-M$-fuzzy subsets. Then we define the union, intersection, complementation and distributivity in $nd-M-$fuzzy subsets and also we define $nd-M-$fuzzy extensions of functions. Then we introduced $nd-M-$fuzzy topological spaces. Also defined interior and closure of $nd-M-$fuzzy topological spaces along with some properties and define the notion of $nd-M-$closure operator and $nd-M-$interior operators. Then we discussed the continuous mappings on $nd-M-$ fuzzy topological spaces .

The notion of L-fuzzy lattice was introduced by *Tepavčević* and *Goran Trajakoviski* [2]. Chapter 4 extends the concepts of L-fuzzy lattices to $nd-M-$fuzzy lattices, where M has the structure of a complete and consistent multilattice along with the *Egli-Milner* [21] ordering of subsets. As in the L-fuzzy lattice [2] here we introduced two types of $nd-M-$ fuzzy lattice. The first is obtained by assigning a singleton set or set of values to each element of the carrier of the bounded lattice. The second type is obtained by nondeterministic fuzzy relation of the order in a crisp relation. Then we define the relation between the two approaches and we prove that these two types of approaches are equivalent.

The fifth chapter introduces the concept of strong $nd-M$-fuzzy lattice which is the extension of strong L-fuzzy lattice [39]. Before introducing $nd-M$-fuzzy lattice, first we introduce the concept of $nd-M$-fuzzy meet(join)-semilattices and $nd-M-fuzzy^{*}meet(join)$ - semilattices.

In chapter six we introduces the matrices over a multilattice. We develop this concept on the basis of lattice matrices [50]. In lattice matrices the entries are elements from a complete distributive lattice. But here we use set of elements to each entries of the matrix from a complete consistent and distributive multilatice M. Later we define algebraic operations and properties of these matrices.

Chapter 2

Preliminaries

In this chapter we discuss some basic concepts needed for the study of $nd - M-$ fuzzy subsets and related concepts. We develop the concept of $nd - M-$fuzzy sets on the platform of fuzzy set theory.

2.1 Lattices

One of the important concepts in all mathematics is that of a relation. The particular interests are for equivalence relation, functions and order relations. An order relation, denoted by $\leq$ on a set X is called a partial order relation if it is reflexive ($x \leq x$ for every $x \in X$), antisymmetric (that is if x and y are such that $x \leq y$, $y \leq x$ then $x = y$, for every $x, y \in X$) and transitive (that is if x, y and z such that $x \leq y$, $y \leq z$ then $x \leq z$, for every x, y and $z \in X$). A partially ordered set (or poset) is a set in which a partial order relation is defined on it. The diagrammatic representation of a finite poset is called a Hasse diagram.

A *lattice* is a partially ordered set in which any two elements have a unique supremum (the elements least upper bound; called their *join*) and an infimum (greatest lower bound; called their *meet*). A subset A of L is called a sublattice of L if for each $x, y \in A$, $x \wedge y \in A$ and $x \vee y \in A$. An element 0 in L is called a lower bound (or least element) of L if $0 \leq x$, for every $x \in L$. An element 1 in L is called a upper bound (or greatest element) of L if $x \leq 1$ for every $x \in L$.ies. Since the two definitions are equivalent, lattice theory draws on both order theory and the universal algebra.

Definition 2.1.1. [5, 17] A poset $(L, \leq)$ is a lattice if for any two elements a and b of L, $a \vee b = sup(a, b)$ and $a \wedge b = inf(a, b)$ exist. A subset A of L is called a sub lattice of L if for each $x, y \in A$, $x \wedge y \in A$ and $x \vee y \in A$. An element 0 in L is called a lower bound (or least element) of L if $0 \leq x$, for every $x \in L$. An element 1 in L is called a upper bound (or greatest element) of L if $x \leq 1$ for every $x \in L$.

An algebraic structure $(L, \vee, \wedge)$ consisting of a set L and two binary operations $\vee$ and $\wedge$ on L is a *Lattice* if the follwing axiomatic identities hold for all elements a, b, c of L.

1. Commutative laws:

$$a \vee b = b \vee a$$

$$a \wedge b = b \wedge a$$

2. Associative laws:

$$a \vee (b \vee c) = (a \vee b) \vee c$$

$$a \wedge (b \wedge c) = (a \wedge b) \wedge c$$

3. Absorption laws:

$$a \vee (a \wedge b) = b$$

$$a \wedge (a \vee b) = b$$

The following two identities are also usually regarded as axioms, even though they follow from the two absorption laws taken together,
Idempotent laws: $a \vee a = a$ and $a \wedge a = a$

These axioms assert that both $(L, \vee)$ and $(L, \wedge)$ are respectively join-semi lattices and meet-semilattices.

Definition 2.1.1. A bounded lattice is an algebraic structure of the form $(L, \vee, \wedge, 1, 0)$ such that $(L, \vee, \wedge)$ is a lattice, and 0 (the lattice's bottom) is the identity element for the the join operation $\vee$ and 1 (the lattice's top) is the identity element for the meet operation $\wedge$.

Definition 2.1.2. A poset is called a complete lattice if all its subsets have a join and a meet.

Remark 2.1.1. Every complete lattice is a bounded lattice.

Definition 2.1.3. A lattice $(L, \vee, \wedge)$ is called distributive lattice if for all $a, b, c \in L$, one of the following is satisfied.

1. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
2. $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

Definition 2.1.2. [5, 17] A sublattice of a lattice L is a non-empty subset of L which is a lattice with the same meet and join operations as L. That is , if L is a lattice and $M \neq \emptyset$ is a subset of L such that for every pair of elements $a, b \in M$ both $a \wedge b$ and $a \vee b$ are in M, then M is a sublattice of L.

Definition 2.1.3. Let L be a lattice with 0, an element x of L is called an *atom* if $0 < x$ and there exists no element y of L such that $0 < y < x$.

Definition 2.1.4. Let L be a lattice with 0 and 1, an element x of L is called a *co-atom* if for all $y \in L$ with $x < y < 1 \Rightarrow x = y$

Definition 2.1.5. A complemented lattice is a bounded lattice (with least element 0 and greatest element 1) in which every element a has a complement, i.e., an element b such that $a \vee b = 1$ and $a \wedge b = 0$.

2.2 Multilattices

Given $(M, \leq)$ is a partially ordered set and $B \subseteq M$, multisupremum of B is a minimal element of the set of upper bounds of B and multisup(B) denote the Multisuprema of B. Dually we define the multiinfima.

Definition 2.2.1. [12] A poset $(M, \leq)$ be an ordered multilattice if and only if it satisfies the condition that for all a, b, x with $a \leq x$ and $b \leq x$, there exist $z \in$ *multisup* $\{a, b\}$ such that $z \leq x$.

When comparing with lattices, we see that least upper bound (which is a unique element) is replaced by the non empty set of all minimal (instead of least)upper bounds and dually.

Definition 2.2.2. [20] A multilattice is distributive if for each $a, b, c \in P$, the conditions $(a \vee b) \cap (a \vee c) \neq \emptyset$ and $(a \wedge b) \cap (a \wedge c) \neq \emptyset \Rightarrow b = c$ (where $\cap$ - is the usual set intersection and $\cup$- is the usual set unions).

Similarly to lattice theory, if we define $a \vee b = Multisup\{a, b\}$ and $a \wedge b = multiinf\{a, b\}$, then $(M, \wedge, \vee)$ be a algebraic multilattice and if we define $a \leq b$ if and only if $a \vee b = \{b\}$ and $a \wedge b = \{a\}$ it is possible to obtain the order version of multilattice.

Definition 2.2.3. A complete multilattice is a partially ordered set $(M, \leq)$ such that every subset $X \subseteq M$ the set of upper bounds of X has minimal (maximal) element, which are called multisuprema (multiinfima), that is for any subset A of X, multiinf(A) and Multisup(A) exists and non empty.

Definition 2.2.4. A poset $(M, \leq)$ is said to be a multisemilattice if it satisfies that for all $a, b, x \in M$ with $a \leq x, b \leq x$, there exist $z \in$ *multisup* $\{ a, b\}$ such that $z \leq x$ and dually.

Definition 2.2.5. Let (M, $\leq$) be a poset. The element $a \in M$

is called a greatest element of M if all other element are smaller. That is $a \geq x$ for every $x \in M$. Similarly $b \in M$ is called a smallest element of M if $b \leq x$ for every $x \in M$
If a multi-lattice has a greatest element and smallest element, then (M, $\leq$) is said to be bounded. Normally greatest element is taken as 1 and smallest element is taken as 0.

Definition 2.2.6. A multilattice M with 0 and 1 is called complemented if for each $x \in M$, there is at least one element y such that $x \wedge y = \{0\}$ and $x \vee y = \{1\}$.

Remark 2.2.1. Let M be complete distributive multilattice. Then every element in M has exactly one complement in M. For, if $a \in M$ has two complements say a_1 and a_2 in M. Then $a \vee a_1 = \{1\}$ and $a \wedge a_1 = \{0\}, a \vee a_2 = \{1\}$ and $a \wedge a_2 = \{0\}$ then $(a \vee a_1) \cap (a \vee a_2) = \{1\} \cap \{1\} = \{1\} \neq \phi$, and $(a \wedge a_1) \cap (a \wedge a_2) = \{0\} \cap \{0\} = \{0\} \neq \phi$. Therefore $a_1 = a_2$, two complements are equal.

Note that as by assumption our sets will not necessary have a supremum but a set of multisuprema. Then we are going to ordering between subsets of posets. Here we are considering three different orderings, the Hoare ordering, the Smyth ordering and the Egli-Milner ordering.

Definition 2.2.7. [21] consider $A, B \subseteq 2^M$, then

(i) $A \sqsubseteq_H B$ if and only if for all $a \in A$ exists $b \in B$ such that

$a \leq b$

(ii) $A \sqsubseteq_S B$ if and only if for all $b \in B$ there exists $a \in A$ such that $a \leq b$

(iii) $A \sqsubseteq_{EM} B$ if and only if $A \sqsubseteq_H B$ and $A \sqsubseteq_S B$.

Definition 2.2.8. [40] $(M, \wedge, \vee)$ - be a algebraic multilattice. Let $x \in M$ and A and B be subsets of M, then
$x \wedge A = \cup \{(x \wedge a)/a \in A\}$
$x \vee A = \cup \{(x \vee a)/a \in A\}$
Also $A \wedge B = \cup \{(a \wedge b)/a \in A, b \in B\}$
$A \vee B = \cup \{(a \vee b)/a \in A, b \in B\}$.

Definition 2.2.9. [21] A multilattice M is said to be consistent if the following set of inequalities holds for all $A \subset M$
$LB(A) \sqsubseteq_{EM}$ multiinf(A)
Multisup(A)$\sqsubseteq_{EM} UB(A)$
Where $LB(A)$ and $UB(A)$ are the lower bound of A and upper bound of A respectively.

Note 1. A multilattice should not contain infinite sets of mutually incomparable elements.

2.3 Fuzzy sets

In 1965 *L. A Zadeh* introduced the concept of fuzzy sets. He used the interval [0, 1] for describing the vagueness mathematically and used membership values in [0, 1] for solving such problems to each member of a given set.

Definition 2.3.1. [18] Let X be a non empty set. A fuzzy set A of X is a mapping $A : X \to [0, 1]$, that is
$A = \{(x, \mu_A(x)) : \mu_A(x)$ is the membership grade of x in $A, x \in A\}$

Let μ_A and μ_B be membership functions of the fuzzy subsets A and B respectively. The set of all fuzzy sets on X is denoted by $\mathcal{F}(X)$

1. $A = B \Leftrightarrow \mu_A(x) = \mu_B(x), \forall x \in X.$
2. $A \subseteq B \Leftrightarrow \mu_A(x) \leq \mu_B(x), \forall x \in X.$
3. $\mu_{A \cup B}(x) = \max\{\mu_A(x), \mu_B(x)\}, \forall x \in X.$
4. $\mu_{A \cap B}(x) = \min\{\mu_A(x), \mu_B(x)\}, \forall x \in X.$
5. $\mu_{A'}(x) = 1 - \mu_A(x), \forall x \in X$ where A' is the fuzzy complement of A.

2.3.1 Zadeh's extension of functions

Let $\mu_A(x)$ and $\mu_{f(A)}(y)$ be denoted by $A(x)$ and $f(A)(y)$ respectively, where $f : X \to Y$ be a crisp function.

Definition 2.3.2. [18] Let $f : X \to Y$ be a crisp function. The fuzzy extension of f and the inverse of the extension are f :

$\mathcal{F}(X) \to \mathcal{F}(Y)$ and $f^{-1} : \mathcal{F}(Y) \to \mathcal{F}(X)$ defined by

$$f(A)(y) = \bigvee_{y=f(x)} A(x) \ \ , A \in \mathcal{F}(X), \ \ y \in Y$$

and

$$f^{-1}(B)(x) = B(f(x)), \ \ B \in \mathcal{F}(Y), x \in X.$$

Theorem 2.3.3. *Let f be a function from X to Y then*

1. $(f^{-1}(B))' = f^{-1}(B')$ *for any fuzzy set B in Y;*
2. $(f(A))' \subseteq f(A')$ *for any fuzzy set A in X;*
3. $B_1 \subseteq B_2 \Rightarrow f^{-1}(B_1) \subseteq f^{-1}(B_2)$ *where B_1 ,B_2 are any fuzzy set in Y;*
4. $A_1 \subseteq A_2 \Rightarrow f(A_1) \subseteq f(A_2)$ *where A_1 ,A_2 are any fuzzy set in X;*
5. $A \subseteq f^{-1}(f(A))$ *for any fuzzy set A in X;*
6. $f(f^{-1}(B)) \subseteq B$, *for any fuzzy set B in Y.*

2.3.2 $L-$fuzzy sets

Definition 2.3.4. L-fuzzy set is the generalisation of Zadeh's definition of fuzzy sets. Let X be a non-empty ordinary set and L be any lattice . An L-fuzzy set on X is a mapping $A : X \to L$, The family of all the L-fuzzy set on X is denoted by L^X consisting of all the mappings from X to L.

The algebraic operations on L^X are defined by $\forall x, y \in X, A, B \in L^X$

$$\mu_{A\cup B}(x) = \mu_A(x) \vee \mu_B(x)$$

$$\mu_{A\cap B}(x) = \mu_A(x) \wedge \mu_B(x).$$

Definition 2.3.5. Let X be a non-empty set and L be a complete lattice. Let $\alpha \in L$ and $A \in L^X$. Then the α-level of A is a crisp set defined by

$$A_\alpha = \{x \in X : A(x) \geq \alpha\}$$

.

2.4 Fuzzy topology

Definition 2.4.1. [11, 27] Let X be non empty set, L a F-lattice, $\tau \subseteq L^X$, τ is called a L-fuzzy topology on X, and (L^X, τ) is called an L-fuzzy topological space, if τ satisfying the following conditions:

1. $\underline{0}, \underline{1} \in \tau$.

2. if $\mu, \gamma \in \tau$ then $\mu \wedge \gamma \in \tau$.

3. if $\mu_i \in \tau$ for each $i \in \Gamma$, then $\bigvee_{i\in\Gamma} \mu_i \in \tau$.

Where $\underline{0}$ represents null set and $\underline{1}$ represents full set.

A fuzzy set $A \in \tau$ is called $\tau - closed$ if and only if its complement is A' is $\tau - open$.

Remark 2.4.1.

1. The element in τ are $\tau - open$ fuzzy sets in X.

2. A fuzzy set $A \in \tau$ is called $\tau - closed$ if and only if its complement A' is $\tau - open$.

3. The collection of all constant fuzzy sets in X is a fuzzy topology on X.

4. Let $A \in L^X$, then interior of A is the join of all the open sets contained in A.

5. Let $A \in L^X$, then closure of A is the meet of all closed subsets containing A.

Definition 2.4.1. Let (L^X, τ) and (L^Y, ν) be $L-$fuzzy topological spaces $\overrightarrow{f} : L^X \rightarrow L^Y$ be an $L-$fuzzy mapping, we say $\overrightarrow{f}$ is an $L-$fuzzy continuous mapping from (L^X, τ) to (L^Y, ν) if its reverse mapping $\overleftarrow{f} : L^Y \rightarrow L^X$ maps every open subsets in (L^Y, ν) as an open set in (L^X, τ). i.e., $\forall\ \upsilon \in \nu,\ \overleftarrow{f}(\upsilon) \in \tau$.

Theorem 2.4.2. *Let (X, τ) and (Y, ν) be fuzzy topological spaces and let f be a function from X into Y. Then, f is fuzzy continuous if and only if $\overleftarrow{f}(C)$ is closed in X in X, for each closed fuzzy set C in Y.*

Proposition 2.4.1. If $f : (X, \tau) \longrightarrow (Y, \nu)$ and $g : (Y, \nu) \longrightarrow (Z, \upsilon)$ are fuzzy continuous, then $g \circ f : (X, \tau) \longrightarrow (Z, \upsilon)$ is fuzzy continuous.

2.5 L-fuzzy lattice

Definition 2.5.1. [2] Let X be a lattice and $(L, \vee_L, \wedge_L)$ is a complete lattice with 0_L and 1_L. Let μ be a L-fuzzy set defined on X. The p cut $(p \in L)$ of μ is defined by $\mu_p = \{x \in X : \mu(x) \geq p\}$, A fuzzy set μ defined on L is a fuzzy sub lattice of L, if

$$\mu(x \wedge y) \wedge \mu(x \vee y) \geq \min(\mu(x), \mu(y)) \;\; x, y \in X$$

$$\mu(x \wedge y) \wedge \mu(x \vee y) \geq \mu(x) \wedge \mu(y)$$

Note 2. $\mu \in L^X$ is a L-fuzzy sub lattice of X if and only if μ_p is a sublattice of X for each $p \in L$.

Proposition 2.5.1. A L-fuzzy lattice satisfies the following results. Let $L, \wedge_L, \vee_L$ be a lattice and $M, \wedge_M, \vee_M$ a complete lattice with 0_L and 1_L then the mapping $A : M \to L$ is an L-fuzzy lattice iff both of the following relations hold for all $x, y \in M$

1. $A(x) \wedge_L A(y) \leq A(x \wedge_M y)$.

2. $A(x) \wedge_L A(y) \leq A(x \wedge_M y)$.

Definition 2.5.2. [44] Let $(M, \wedge_M)$ be a meet semilattice and $(L, \vee_L, \wedge_L)$ is a complete lattice with 0_L and 1_L. Let μ be a L-fuzzy set defined on X.Then μ is called an L-fuzzy meet semi lattice of of M, if all the p $(p \in L)$ level sets of μ are sub meet semilattice of M.

Definition 2.5.3. Let $(M, \vee_M)$ be a meet- semilattice and $(L, \vee_L, \wedge_L)$ a complete lattice with 0_L and 1_L. Let μ be a L-fuzzy set defined on X. Then μ is called an L-fuzzy join- semilattice of of M, if all the p $(p \in L)$ level sets of μ are sub join-semilattice of M.

2.6 Lattice matrix

Definition 2.6.1. [50] Let L be a distributive lattice with 0_L and 1_L and let $a + b = sup(a, b)$ and $a.b = inf(a, b)$. Then L_n represents the set of all $n \times n$ matrices over a lattice L.
The algebraic operations in L_n are defined in terms of suprimum and infimum.
i.e.,

$$L_n = \{A = (a_{ij})/a_{ij} \in L\},$$

a_{ij} is the $(ij)^{th}$ element of A.

Definition 2.6.2. Let $A, B \in L_n$, then

1. $A + B = C$ if and only if $c_{ij} = a_{ij} + b_{ij}$.

2. $A \leq B$ if and only if $A + B = B$, that is $a_{ij} \leq b_{ij}$.

3. $A \wedge_M B = C$ if and only if $c_{ij} = a_{ij}.b_{ij}$.

4. $A.B = AB = C$ if and only if $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$.

5. $A^T = C$ if and only if $c_{ij} = a_{ji}$

6. For $a \in M$, $aA = a.A = C$ if and only if $c_{ij} = a.a_{ij}$.

7. $I = (a_{ij})$, where $a_{ij} \;\; = 1$ for $i = j$
   $= 0$ for $i \neq j$.

8. $A^0 = I$, $A^{k+1} = A^k A$

9. $O = (o_{ij})$ where $o_{ij} = 0$ for every i and j.

10. $E = (e_{ij})$ where $e_{ij} = 1$ for every i and j.

Definition 2.6.3. A L_n Matrix A is called a unit if and only if there is an L_n matrix B such that $AB = BA = I$ and A is called orthogonal if and only if $AA^T = A^T A = I$

Chapter 3

nd-M-Fuzzy Topological Spaces

In this chapter we introduce new concept of $nd-M-fuzzy$ subset and $nd-M-fuzzy$ topological space. Also we will study some concepts in $nd-M-fuzzy$ topological spaces.

3.1 nd- M-fuzzy subsets

Definition 3.1.1. A non deterministic $M-$Fuzzy subset of X (or nd-M-fuzzy subset) is a function from X to 2^M, where M is a complete and consistent multilattice. Then the collection of all the $nd-M-$ fuzzy subsets of X is called $nd-M-$ fuzzy space and is denoted by $(2^M)^X$.

Definition 3.1.2. A complete and consistent multilattice M is called a nd-$F-$multi-lattice if M has an order reversing involution $' : 2^M \to 2^M$.

Let X be a non empty ordinary set and M a $F-$ multilattice. and $A \in (2^M)^X$. Then $A^{'}(x) = [A(x)]^{'} = \cup\{a^{'}|a \in A(x)\}$.

If M is a complete and consistent multilattice, then $A^{'}(x) = (A(x))^{'} = \{a^{'}|a \in A(x)\}$. Now, $' : (2^M)^X \to (2^M)^X$, the pseudo complementary operation on $(2^M)^X, A^{'}$ is the pseudo complementary set of A in $(2^M)^X$.

Definition 3.1.3. **Rules of set relations on** $(2^M)^X$
Let A and B be two $nd - M-$ fuzzy subset of X. Then

1. $A = B$ if $A(x) = B(x)$ for every $x \in X$.

2. $A \sqsubseteq_{EM} B$ if $A(x) \sqsubseteq_{EM} B(x)$, for every $x \in X$.

3. $C = A \vee B$ if $C(x) =$ multisup $\{(A(x), B(x)) \mid$, for every $x \in X\}$
$= \cup\{a \vee b \mid a \in A(x), b \in B(x)\}$, for every $x \in X$.

4. $D = A \wedge B$ if $D(x) =$ multiinf $\{(A(x), B(x)) \mid$, for every $x \in X\}$
$= \cup\{(a \wedge b) \mid a \in A(x), b \in B(x)\}$, for every $x \in X$.

5. $E = X - A$ if $E(x) = \{a' \mid a \in A(x)\}$, for every $x \in X$.

Note 3. Let $A \in (2^L)^X$ and $\alpha \in 2^L$. If $A(x) = \alpha$ for every $x \in X$, then $A$is called constant $nd-M$-fuzzy subset and is denoted by $\underline{\alpha}$. But in this thesis we use $A = \alpha$ instead of using $\underline{\alpha}$ for constant $nd - M-$ fuzzy subset.

Definition 3.1.4. An $nd - M$ fuzzy subset A is said to be bounded if for each $x \in X$, there exist K and L such that $K \leq A(x) \leq L$, where K and L depends only on x.

Proposition 3.1.1. Let $A, B, C \in (2^M)^X$ be any bounded $nd - M-$fuzzy subset in X, then

1. $A \sqsubseteq_{EM} A \vee B$, $B \sqsubseteq_{EM} A \vee B$
2. $A \wedge B \sqsubseteq_{EM} A$ and $A \wedge B \sqsubseteq_{EM} B$.

Proof. 1. we have $A \vee B = \cup\{a \vee b | a \in A \text{ and } b \in B\}$. Then there exist t_1 and t_2 belongs to Multisup $\{L_1, L_2\}$ such that $t_1 \geq a$ and $t_2 \geq b$ for all $a \in A$ and $b \in B$. Thus

$$A \sqsubseteq_S A \vee B \text{ and } B \sqsubseteq_S A \vee B \tag{3.1}$$

Also all the elements in A and B are less than or equal to the Multisup $\{L_1, L_2\}$. Thus

$$A \sqsubseteq_H A \vee B \text{ and } B \sqsubseteq_H A \vee B. \tag{3.2}$$

From 3.1 and 3.2, we have the required result.

2. $A \wedge B = \sqcup\{a \wedge B)/a \in A$ and $b \in B\}$. Since A and B are bounded there exist K_1, L_1, K_2 and L_2 such that $K_1 \leq A(x) \leq L_1$ and $K_2 \leq B(x) \leq L_2$. Then there exist elements

z_1 and z_2 belongs to multiinf $\{K_1, K_2\}$ such that $z_1 \leq a$ and $z_2 \leq b$ for every $a \in A$ and $b \in B$. Thus

$$A \wedge B \sqsubseteq_S A \text{ and } A \wedge B \sqsubseteq_S B \tag{3.3}$$

Also every elements in A and B are greater than or equal to the multiinf $\{K_1, K_2\}$. Thus

$$A \wedge B \sqsubseteq_H A \text{ and } A \wedge B \sqsubseteq_H B \tag{3.4}$$

hence from 3.3 and 3.4, we have the result.

□

Example 3.1.1. $A \vee A = A$ is not generally true.
Let X be the set $\{p, q, r, s, t\}$ and M be the multilattice given in the Figure 3.1, the $nd - M$-fuzzy subset is defined by

$$A = \begin{pmatrix} p & q & r & s & t \\ \{a, b\} & \{c\} & \{d\} & \{1\} & \{0\} \end{pmatrix}$$

Then

$$\begin{aligned} (A \vee A)(p) &= A(p) \vee A(p) \\ &= \{a, b\} \vee \{a, b\} \\ &= (a \vee a) \cup (a \vee b) \cup (b \vee a) \cup (b \vee b) \\ &= \{a\} \cup \{c, d\} \cup \{c, d\} \cup \{b\} \\ &= \{a, b, c, d\} \end{aligned}$$

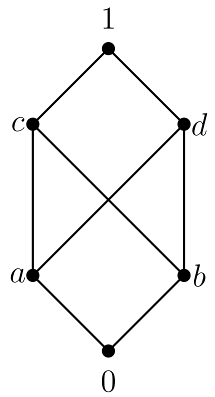

Figure 3.1: The multilattice in Example 3.1.1

thus $A \vee A \neq A$.

Remark 3.1.1. $A \sqsubseteq_{EM} B$ does not implies $A \vee B = B$ and $A \sqsubseteq_{EM} B$ does not implies $A \wedge B = A$. From the example 3.1.1, let $A = \{0, a\}, B = \{b, 1\}$, where A and B are constant $nd-M-$fuzzy subsets. Then $A \sqsubseteq_{EM} B$ but

$$A \vee B = \{o, a\} \vee \{b, 1\} = \{b, c, d, 1\} \neq A$$

and let $A = \{0, c\}$, $B = \{d, 1\}$ then $A \sqsubseteq_{EM} B$ but

$$A \wedge B = \{0, a, b, c\} \neq A$$

Proposition 3.1.2. Let $A \in (2^M)^X$ and for any $\alpha \in (2^L)$, then the set $A_\alpha = \{x \in X/\alpha \sqsubseteq_{EM} A(x)\ ,\ \alpha \in (2^M)\}$ be the α level of A . If $A, B \in (2^M)^X$, then for any $\alpha, \beta \in (2^M)^X$

1. $\alpha \sqsubseteq_{EM} \beta \Rightarrow A_\beta \subseteq A_\alpha$.

2. $A \sqsubseteq_{EM} B$ if and only if $A_\alpha \subseteq B_\alpha$.

3. $A = B$ if and only if $A_\alpha = B_\alpha$.

Proof. 1. Let $\alpha \sqsubseteq_{EM} \beta$ and $x \in A_\beta$, then $\beta \sqsubseteq_{EM} A(x)$
since $\alpha \sqsubseteq_{EM} \beta$, we have $\alpha \sqsubseteq_{EM} A(x)$
there fore $A_\beta \subseteq A_\alpha$.

2. $A \sqsubseteq_{EM} B \Longrightarrow A(x) \sqsubseteq_{EM} B(x)$, for all $x \in X$.
then for every $\alpha \in 2^M$ and $y \in A_\alpha \Longrightarrow y \in B_\alpha$
since $\alpha \sqsubseteq_{EM} A(y)$ and $A(y) \sqsubseteq_{EM} B(y)$. There fore $A_\alpha \sqsubseteq_{EM} B_\alpha$.

   Conversely assume that $A_\alpha \sqsubseteq_{EM} B_\alpha$, for every $\alpha \in 2^M$.
Then for every $x \in A_\alpha \Longrightarrow x \in B_\alpha$.
That is $\alpha \sqsubseteq_{EM} A(x) \sqsubseteq_{EM} B(x)$, for every $\alpha \in 2^M$. Hence $A \sqsubseteq_{EM} B$.

3. $A = B$ if and only if $A(x) = B(x)$ if and only if $A_\alpha = B_\alpha$, for every $\alpha \in 2^M$.

□

Proposition 3.1.3. For any family $\{A_i\}$ of $nd - M-$ fuzzy subset in X, the De Morgan's Law does not hold, but if each A_i are bounded and $\bigwedge A_i \sqsubseteq_{EM} A_i$, $A_i \sqsubseteq_{EM} \bigvee A_i$, for every $i \in I$, then

1. $(\bigwedge A_i)' \sqsubseteq_{EM} \bigvee A_i{}'$
$\bigvee A_i{}' \sqsubseteq_{EM} (\bigwedge A_i)'$.

2. $(\bigvee A_i)' \sqsubseteq_{EM} \bigwedge A_i'$
$\bigwedge A_i' \sqsubseteq_{EM} (\bigvee A_i)'$.

Proof. Since $\bigwedge A_i \sqsubseteq_{EM} A_i$ and $A_i^{'} \sqsubseteq_{EM} (\bigwedge A_i)^{'}$ by the order reversing involution .
So,

$$\bigvee A_i^{'} \sqsubseteq_{EM} (\bigwedge A_i)^{'} \tag{3.5}$$

Similarly, from the fact $A_i \sqsubseteq_{EM} \bigvee A_i$, we get $(\bigvee A_i)^{'}) \sqsubseteq_{EM} {A_i}^{'}$.
That is

$$(\bigvee A_i)^{'} \sqsubseteq_{EM} \bigwedge A_i^{'} \tag{3.6}$$

If we substitute $A_i^{'}$ for A_i in 3.6, we get $(\bigvee {A_i}^{'})^{'} \sqsubseteq_{EM} \bigwedge (A_i)^{'})'$.
Therefore $(\bigvee {A_i}')^{'} \sqsubseteq_{EM} \bigwedge (A_i)$
So

$$(\bigwedge A_i)^{'} \sqsubseteq_{EM} \bigvee A_i^{'} \tag{3.7}$$

Similarly if we replace ${A_i}'$ for A_i in 3.5, we get
$\bigvee ({A_i}^{'})^{'} \sqsubseteq_{EM} (\bigwedge A_i)^{'})^{'}$.
That is $(\bigvee A_i) \sqsubseteq_{EM} \bigwedge {A_i}^{'}$.
Thus

$$\bigwedge {A_i}^{'} \sqsubseteq_{EM} (\bigvee A_i)^{'}. \tag{3.8}$$

□

3.2 nd-M-fuzzy extensions of functions

Let $(2^M)^X$ and $(2^M)^Y$ be $Nd-M-$fuzzy spaces and $f : X \to Y$ be an ordinary mapping. Based on $f : X \to Y$ define $nd-M-$fuzzy mapping $\overrightarrow{f} : (2^M)^X \to (2^M)^Y$ by $\overrightarrow{f}(A)(y) = \bigvee_{y=f(x)}\{A(x)/x \in X\}$ for every $A \in (2^M)^X$, for every $y \in (2^M)^Y$.

Similarly $\overleftarrow{f} : (2^M)^Y \to (2^M)^X$ by

$\overleftarrow{f}(B)(x) = B(f(x))$, for every $B \in (2^M)^Y$, for every $x \in X$.

Theorem 3.2.1. *Let $(2^M)^X$, $(2^M)^Y$ be $nd-M$- fuzzy spaces, $f : X \longrightarrow Y$ an ordinary mapping. Then for every $\alpha \in 2^M$ and every $A \in (2^M)^X$, $\overrightarrow{f}(\alpha A) = \alpha\overrightarrow{f}(A)$*

Proof. For every $\alpha \in 2^M$, for every $A \in (2^M)^X$, for every $y \in Y$, we have

$$\begin{aligned}
\overrightarrow{f}(\alpha A)(y) &= \bigvee\{(\alpha A)(x) : x \in X, f(x) = y\} \\
&= \bigvee\{\alpha \wedge (A(x)) : x \in X, f(x) = y\} \\
&= \alpha \wedge \bigvee\{A(x) : x \in X, f(x) = y\} \\
&= \alpha \wedge (\overrightarrow{f}(A)(y)) \\
&= \alpha\overrightarrow{f}(A)(y).
\end{aligned}$$

Therefore $\overrightarrow{f}(\alpha A) = \alpha\overrightarrow{f}(A)$.

□

Theorem 3.2.2. *Let $(2^M)^X$, $(2^M)^Z$ and $(2^M)^Z$ be $nd-M$-*

fuzzy spaces,
$f : X \longrightarrow Y$ *and* $g : Y \longrightarrow Z$ *be ordinary mappings. Then*

1. $\overrightarrow{g}\,\overrightarrow{f} = \overrightarrow{gf}$.

2. $\overleftarrow{f}\,\overleftarrow{g} = \overleftarrow{gf}$.

Proof. 1. for every $A \in (2^M)^X, z \in Z$, then

$$\begin{aligned}\overrightarrow{g}\,\overrightarrow{f}(A)(x) &= \bigvee\{\overrightarrow{f}(A)(y) : y \in Y, g(y) = z\}\\ &= \bigvee\{\bigvee\{A(x) : x \in X, f(x) = y\} : y \in Y, g(y) = z\}\\ &= \bigvee\{A(x) : x \in X, gf(x) = z\}\\ &= \overleftarrow{gf}(A)(z).\end{aligned}$$

Therefore $\overrightarrow{g}\,\overrightarrow{f} = \overrightarrow{gf}$

2. for every $C \in (2^M)^Z$, for every $x \in X$, then

$$\begin{aligned}\overleftarrow{f}\,\overleftarrow{g}(C)(x) &= \overleftarrow{g}(C)(f(x))\\ &= C((gf)(x)\\ &= \overleftarrow{gf}(C)(x).\end{aligned}$$

Therefore $\overleftarrow{f}\,\overleftarrow{g} = \overleftarrow{gf}$

□

Theorem 3.2.3. *Let* $f : X \to Y$ *be an arbitrary crisp function. Then for any* $A_i \in (2^M)^X$ and $B_i \in (2^M)^Y$, $i \in I$, *the following properties of functions obtained by the extension principle hold .*

1. If $A_1 \sqsubseteq_{EM} A_2 \Rightarrow \overrightarrow{f}(A_1) \sqsubseteq_{EM} \overrightarrow{f}(A_2)$

2. $\overrightarrow{f}(\bigvee_{i\in I} A_i) = \bigvee_{i\in I} \overrightarrow{f}(A_i)$

3. $\overrightarrow{f}(\bigwedge_{i\in I} A_i) \sqsubseteq_{EM} \bigwedge_{i\in I} \overrightarrow{f}(A_i)$

4. $B_1 \sqsubseteq_{EM} B_2 \Rightarrow \overleftarrow{f}(B_1) \sqsubseteq_{EM} \overleftarrow{f}(B_2)$

5. $\overleftarrow{f}(\bigvee_{i\in I} B_i) = \bigvee_{i\in I} \overleftarrow{f}(B_i)$

6. $\overleftarrow{f}(\bigwedge_{i\in I} B_i) = \bigwedge_{i\in I} \overleftarrow{f}(B_i)$

Proof.

1.

$$\begin{aligned} If A_1 \sqsubseteq_{EM} A_2 &\Longrightarrow A_1(x) \sqsubseteq_{EM} A_2(x), \text{ for every } x \in X \\ &\Longrightarrow \bigvee_{y=f(x)} (A_1(x))/y = f(x) \sqsubseteq_{EM} \bigvee_{y=f(x)} (A_2(x)/y = f(x)) \\ &\Longrightarrow \overrightarrow{f}(A_1)(y) \sqsubseteq_{EM} \overrightarrow{f}(A_2(y)), \text{ for every } y \in Y \\ &\Longrightarrow \overrightarrow{f}(A_1) \sqsubseteq_{EM} \overrightarrow{f}(A_2) \end{aligned}$$

2.

$$\begin{aligned}\overrightarrow{f}(\bigvee_{i\in I} A_i)(y) &= \bigvee_{y=f(x)} (\bigvee_{i\in I} A_i(x) : x \in X, y = \overrightarrow{f}(x))\\ &= \bigvee(\bigvee_{i\in I} A_i(x) : x \in X, y = \overrightarrow{f}(x))\\ &= \bigvee_{i\in I}(\bigvee_{y=f(x)} A_i(x) : x \in X, y = \overrightarrow{f}(x))\\ &= \bigvee_{i\in I}(\overrightarrow{f}(A_i)(y) : y = \overrightarrow{f}(x))\\ &= \bigvee_{i\in I} \overrightarrow{f}(A_i)(y)\end{aligned}$$

Thus $\overrightarrow{f}(\bigvee_{i\in I} A_i) = \bigvee_{i\in I} \overrightarrow{f}(A_i)$.

3.

$$\begin{aligned}\overrightarrow{f}(\bigwedge_{i\in I} A_i)(y) &= \bigvee_{y=f(x)} (\bigwedge_{i\in I}(A_i(x)/x \in X, y = f(x))\\ &\sqsubseteq_{EM} \bigwedge_{i\in I}(\bigvee_{y=f(x)} A_i(x)/x \in X, y = \overrightarrow{f}(x))\\ &= \bigwedge_{i\in I}(\overrightarrow{f}(A_i(y))\end{aligned}$$

Thus $\overrightarrow{f}(\wedge_{i\in I} A_i) = \wedge_{i\in I} \overrightarrow{f}(A_i)$.

4. $B_1 \sqsubseteq_{EM} B_2 \longrightarrow B_1(y) \sqsubseteq_{EM} B_2(y)$, for every $y \in Y$. $\overleftarrow{f}(B_1)(x) = B_1(f(x)) \sqsubseteq_{EM} B_2(f(x)) = \overleftarrow{f}(B_2)(x)$, for every $x \in X$. There for $\overleftarrow{f}(B_1) \sqsubseteq_{EM} \overleftarrow{f}(B_2)$.

5. for every $x \in X$, we have

$$\begin{aligned}\overleftarrow{f}(\bigvee_{i\in I} B_i)(x) &= (\bigvee_{i\in I} B_i) f(x)\\ &= \bigvee_{i\in I} B_i(f(x))\\ &= \bigvee_{i\in I}(\overleftarrow{f}(B_i))(x)\\ &= (\bigvee_{i\in I}\overleftarrow{f}(B_i))(x)\end{aligned}$$

Hence $\overleftarrow{f}(\bigvee_{i\in I} B_i) = \bigvee_{i\in I}\overleftarrow{f}(B_i)$

6.

$$\begin{aligned}\overleftarrow{f}(\bigwedge_{i\in I}(B_i)(x) &= (\bigwedge_{i\in I}(B_i)(f(x))\\ &= \bigwedge_{i\in I}(B_i(f(x))\\ &= \bigwedge_{i\in I}\overleftarrow{f}(B_i)(x)\end{aligned}$$

Hence $\overleftarrow{f}(\bigwedge_{i\in I} B_i) = \bigwedge_{i\in I}\overleftarrow{f}(B_i)$

□

3.3 nd-M-fuzzy topological spaces

Definition 3.3.1. Let X be a non empty set and M be a complete and consistent F- multilattice. Let $\tau \subseteq (2^M)^X$. Then τ is called a non-deterministic M fuzzy topology on X if it satisfies the following conditions.

1. $\{\underline{0}\}, \{\underline{1}\} \in \tau$
2. If $A, B \in \tau$, then $A \wedge B \in \tau$
3. Let $\{A_i, i \in I\} \subset \tau$, where I is an index set, then $\bigvee_{i \in I} A_i \in \tau$.

where$\{\underline{0}\} \in \tau$ means the empty set and $\{\underline{1}\}$ means the whole set X. Then the pair $((2^M)^X, \tau)$ is called a non deterministic $M-$fuzzy topological space.

The elements in τ are called open elements and the elements in the complement of τ are called closed elements, and the set of complements of open sets is denoted by τ'

Example 3.3.1. 1. Every $L-$ fuzzy topological space is a $nd-M-$fuzzy topological spaces where $L = M$ is a complete distributive lattice.

2. Take $\tau = \{\underline{\alpha} : \alpha \in 2^M\} \subset (2^M)^X$ is a nd-M fuzzy topological space, where $\underline{\alpha}$ denote the constant $nd - M$ fuzzy subset. That is every element in X has the membership values α.

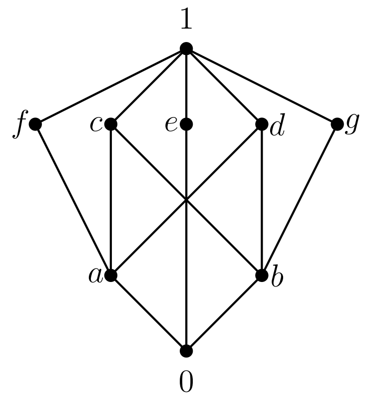

Figure 3.2: The multilattice in Example 3.3.1

3. Let X be any non empty set and M be a multi-lattice given in the Figure 3.2.

Let $\tau = \{\{\underline{0}\}, \{\underline{1}\}, \{a\}, \{b\}, \{a, b\}, \{c, d\}, \{a, c, d\}, \{0, a\}, \{0, b\}, \{0, a, b\}, \{a, b, c, d\}, \{0, a, b, c, d\}, \{0, a, b, c, d, 1\}, \{0, a, c, d\}, \{0, a, c, d, 1\}\}$ Where each sets in τ are constant $nd - M$ fuzzy subsets of X.

Then τ forms a nd-M-fuzzy topology on X

Definition 3.3.2. nd-M-Pseudo interior and nd-M-pseudo closure
For any $nd - M-$fuzzy subset, we define

1. The nd-M-Pseudo interior of A as the join of all the open nd-M-fuzzy subsets contained in A denoted by A^o, that is $A^o = \vee\{B \in \tau \mid B \leq A\}$

2. The nd-M-Pseudo closure of A as the meet of all the closed $nd - M$-fuzzy subsets containing A, denoted by $\overline{A}$, that is

$$\overline{A} = \wedge\{B \in \tau^{'} \mid A \leq B\}$$

In the above example, The nd-M-Pseudo τ-closed subsets are
$\tau' = \{\{\underline{0}\}, \{\underline{1}\}, \{e, g\}, \{e, f\}, \{e\}, \{e, g, 1\}, \{e, f, 1\}, \{e, g, f, 1\}, \{e, g, f\},$
$\{0, e, f, g, 1\}, \{0, 1, e, g\}\}$.

1. Let $A = \{0, a, c\}$. Then

$$\begin{aligned} A^0 &= \vee\{B/B \sqsubseteq_{EM} A\} \\ &= \vee\{\{0\}, \{0, a\}\} \\ &= \{o\} \vee \{0, a\} \\ &= (0 \vee 0) \cup (a \vee a) \\ &= \{0\} \cup \{a\} \\ &= \{0, a\}. \end{aligned}$$

2.

$$\begin{aligned} \overline{A} &= \wedge\{B \in \tau^{'}/A \sqsubseteq_{EM} B\} \\ &= \wedge\{1\}, \{e, g, 1\}, \{e, f, 1\}, \{0, e, f.1\}, \{e, f, g, 1\}, \{0, e, g, f, 1\}, \{0, 1, e, g\}\} \\ &= \wedge\{\{0, e, g, f, 1\} \wedge \{0, e, f, 1\}, \{e, g, f, 1\}, \{0, e, g, f, 1\}, \{0, 1, e, g\}\} \\ &= \wedge\{\{0, e, f, g, 1\} \wedge \{e, g, f, 1\}, \{0, e, f, g, 1\}, \{0, e, g, 1\}\} \\ &= \wedge\{\{0, e, f, g, 1\} \wedge \{0, e, g, f, 1\}, \{0, 1, e, g\}\} \\ &= \{\{0, e, g, f, 1\} \wedge \{0, 1, e, g\}\} \\ &= \{0, e, f, g, 1\}. \end{aligned}$$

3. from the above example of topological space, let $A = \{c\}$, then

$$\begin{aligned} A^0 &= \vee\{\{0\}, \{a\}, \{b\}, \{0, a\}, \{0, b\}, \{a, b\}, \{0, a, b\} \\ &= \vee\{\{a\} \vee \{b\}, \{0, a\}, \{0, b\}, \{a, b\}, \{0, a, b\}\} \\ &= \vee\{\{c, d\}, \{0, a\}, \{0, b\}, \{a, b\}, \{0, a, b\}\} \\ &= \vee\{\{c, d\}, \{a, b\}, \{0, a, b\}\} \\ &= \{\{c, d\} \vee \{a, b\}, \{0, a, b\}\} \\ &= \{\{c, d\} \vee \{0, a, b\}\} \\ &= \{c, d\}. \end{aligned}$$

$$\begin{aligned} (A^0)^0 &= \vee\{\{o\}, \{a\}, \{b\}, \{0, a\}, \{0, b\}, \{a, b\}, \{0, a, b\}\} \\ &= \{c, d\}. \end{aligned}$$

$$\therefore (A^0)^0 = A$$

but A^0 not a Egli- Milner subset of A

4. Now let $A = \{a\}$ and $B = \{d\}$ then

$$A^0 = \vee\{\{a\}, \{0\}\} = \{a \vee o\} = \{a\}$$

and
$B^0 = \vee\{\{0\}, \{b\}\}, \{0, a\}, \{0, b\}, \{a, b\}, \{0, a, b\}$
$=\{c, d\}$

. $A^0 \wedge B^0 = \cup\{\{a\} \wedge \{c, d\}\}$
$=\{(a \wedge c) \cup (a \wedge d)\}$
$= \{a\}$.
Now $A \wedge B = \{\{a\} \wedge \{d\}\} = \{a \wedge d\} = \{a\}$ then
$(A \wedge B)^0 = \{a\}$
$(A \wedge B)^0 = A^0 \wedge B^0$

Theorem 3.3.3. *Let $((2^M)^X, \tau)$ be an nd-M-fuzzy Topological space. Then,*

1. *(a) $\{\underline{0}\}^o = \{\underline{0}\}$ and (b) $\{\underline{1}\}^0 = \{\underline{1}\}$*

2. *$A^0 \sqsubseteq_{EM} A$ or A^0 is not compare with A by Egli-Milner ordering.*

3. *$(A^0)^0 = A^0$*

4. *Let $A^0 \sqsubseteq_{EM} A$ and $B^0 \sqsubseteq_{EM} B$.If*

$$A \sqsubseteq_{EM} B \Rightarrow A^0 \sqsubseteq_{EM} B^0$$

.

5. *Let $A^0 \sqsubseteq_{EM} A$ and $B^0 \sqsubseteq_{EM} B$.Then $(A \wedge B)^0 = A^0 \wedge B^0$*

Proof. 1. (a) and (b) are by the definition of nd-M-Pseudo interior.

2. nd-M-Pseudo interior of A is the join of all open subsets contained in A. That is interior of A contains the element of Multisup of elements in the open set contained in A.But we know that any set$A$, $Multisup(A) \sqsubseteq_{EM} UB(A)$.If the Multisup of open subset of A is a subset of A ,then $A^0 \sqsubseteq_{EM} A$.Otherwise Multisup of open subset contained in A contains elements not in A. So that A^0 may not be a Egli-Milner subset of A because some of the elements in A^0 is not compare with elements in A.

3. Since $(A^0)^0$ is the largest openset contained in A^0 and A^0 is itself open ,then $A^0)^0 = A^0$.

4. Assume $A^0 \sqsubseteq_{EM} A$ and $B^0 \sqsubseteq_{EM} B$.

   Given that $A^0 \sqsubseteq_{EM} A$. So if $A \sqsubseteq_{EM} B$, We have $A^0 \sqsubseteq_{EM} B$. Thus A^0 is an open set contained in B. So $A^0 \sqsubseteq_{EM} B^0$.

5. $(A \wedge B)^0 \sqsubseteq_{EM} A^0$ and $(A \wedge B)^0 \sqsubseteq_{EM} B^0$
   . So $(A \wedge B)^0 \sqsubseteq_{EM} A^0 \wedge B^0$.
   Since $A^0 \sqsubseteq_{EM} A$ and $B^0 \sqsubseteq_{EM} B$, $A^0 \wedge B^0 \subseteq_{EM} A \wedge B$, of which $A^0 \wedge B^0$ is an open set contained in $A \wedge B$; Hence $A^0 \wedge B^0$ must be contained in the largest open set $(A \wedge B)^0$. Thus $A^0 \wedge B^0 \sqsubseteq_{EM} (A \wedge B)^0$

□

Theorem 3.3.4. *Let $((2^M)^X, \tau)$ be an nd-M-fuzzy topological* *space. Then,*

1. *(a)* $\{\overline{0}\} = \{0\}$ *and (b)* $\{\overline{1}\} = \{1\}$.

2. $A \sqsubseteq_{EM} \overline{A}$ *or* A *not compare with* $\overline{A}$ *by Egli-Milner ordering.*

3. $\overline{(\overline{A})} = \overline{A}$

4. *Let* $A \sqsubseteq_{EM} \overline{A}$ *and* $B \sqsubseteq_{EM} \overline{B}$. *If* $A \sqsubseteq_{EM} B \Rightarrow \overline{A} \sqsubseteq_{EM} \overline{B}$.

5. *If* $A \sqsubseteq_{EM} \overline{A}$ *and* $B \sqsubseteq_{EM} \overline{B}$, *then*
   $\overline{(A \vee B)} \sqsubseteq_{EM} \overline{A} \vee \overline{B}$ *and* $\overline{A} \vee \overline{B} \sqsubseteq_{EM} \overline{(A \vee B}$.

Proof. 1. (a) and (b) are by the definition of nd-M-Pseudo closure.

2. $\overline{A}$ is the meet all the closed supersets containing A. That is, nd-M-Pseudo closure of A contains the elements of multiinf of closed superset of A. But we know that for any set A $LB(A) \sqsubseteq_{EM} multiinf(A)$. If the multiinfmum of all the supersets containing A contains all the elements in A ,then $A \sqsubseteq_{EM} \overline{A}$. Otherwise multiinf of closed superset containing A contains elements not in A, which are not comparable with the elements in A. So that A is not a Egli-Milner subset of $\overline{A}$.

3. since $\overline{A}$ is the smallest closed set containing $\overline{A}$ and $\overline{A}$ itself is closed, then $\overline{(\overline{A})} = \overline{A}$

4. Given that $A \sqsubseteq_{EM} \overline{A}$ and $B \sqsubseteq_{EM} \overline{B}$. Since $B \sqsubseteq_{EM} \overline{B}$, if $A \sqsubseteq_{EM} B$, we have $A \sqsubseteq_{EM} \overline{B}$, since $\overline{B}$ is closed, we must have $\overline{A} \sqsubseteq_{EM} \overline{B}$.

5. $\overline{A} \sqsubseteq_{EM} \overline{(A \vee B}$ and $\overline{B} \sqsubseteq_{EM} \overline{(A \vee B}$, so $\overline{A} \vee \overline{B} \sqsubseteq_{EM} \overline{(A \vee B)}$. Also since $\overline{A}$ and $\overline{B}$ are closed set containing A and B, respectively ;$\overline{A} \vee \overline{B}$ is a closed set containing $(A \vee B)$. As $\overline{(A \vee B)}$. is the smallest closed set containing $A \vee B$, hence $\overline{(A \vee B)} \sqsubseteq_{EM} \overline{A} \vee \overline{B}$.

□

Definition 3.3.5. $nd - M-$**fuzzy boundary**
The boundary of a $nd - M$ fuzzy subset of A is defined as
$\partial A = \overline{A} \wedge (\overline{A}^{'})$

Example 3.3.2. From above example, the boundary of the set $A = \{0, a, c\}$ is given by

$\partial A = \{0, e, f, g, 1\} \wedge \{a, b, c, d, f, g\}$ $=\{0, a, b, c, f, g\}$

From the above theorems and we can introduce the two concepts, which are the new directions in defining a new $nd-M-$fuzzy topology.

Definition 3.3.6. nd-M-closure operator
An operator $c : (2^M)^X \rightarrow (2^M)^X$ is a non deterministic-M- closure operator ($nd - M$ Closure operator) if the following conditions are satisfied.

1. $c(\{\underline{0}\}) = \{\underline{0}\}$
2. $A \sqsubseteq_{EM} c(A)$, for all $A \in (2^M)^X$

3. $c(A \vee B) \sqsubseteq_{EM} c(A) \vee c(B)$

4. $(c(A) \vee c(B)) \sqsubseteq_{EM} c(A \vee B)$

5. $c(c(A)) = c(A)$, for all $A \in (2^M)^X$

Definition 3.3.7. $nd - M-$**interior operator**
An operator $i : (2^M)^X \to (2^M)^X$ is a non deterministic interior operator (nd-M-interior operator) if the following conditions are satisfied.

1. $i(\{\underline{1}\}) = \{\underline{1}\}$

2. $i(A) \sqsubseteq_{EM} A$, for all $A \in (2^M)^X$

3. $i(A \wedge B) \sqsubseteq_{EM} i(A) \wedge i(B)$

4. $i(A) \wedge i(B) \sqsubseteq_{EM} i(A \wedge B)$

5. $i(i(A)) = A$,for all $A \in (2^L)^X$.

We know that the interior operator corresponds to one fuzzy topology and each closure operator corresponds to one fuzzy topology [27]. In the similar way we have each $nd-$interior operator corresponds to one $nd-M-$fuzzy topology and each $nd-$closure operator corresponds to one $nd-M-$fuzzy topology. That is, in general, if we define two operators, $nd-$closure and $nd-$ interior,separately they will define two $nd - M-$fuzzy topologies.

Let X be a non-empty set and let $I = \{\{\underline{1}\}, \{\underline{0}\}\}$ and $D = \{A \mid A \in (2^M)^X\}$. Then I and D are both $nd - M-$fuzzy topologies on X such that for any $nd - M-$ fuzzy topology τ on X , $I \leq \tau \leq D$ Where $\leq$ means the ordering of topologies on X.

Let $((2^M)^X, \tau)$ be a $nd - M-$fuzzy topological space, then $\tau_\alpha = \{A \mid A(x) \geq \alpha, \forall x \in X\}$ is called the α- level of a $nd - M-$fuzzy topological spaces X.

If $A, B \in \tau_\alpha$, then it is always not true that $\alpha \sqsubseteq_{EM} A \wedge B$. If an α-level set satisfying $\alpha \sqsubseteq_{EM} A \wedge B$,then τ_α is denoted by τ^*_α

Proposition 3.3.1. Let $((2^M)^X, \tau)$ be a $nd - M-$fuzzy topological space. Then for each $\alpha \in ((2^M))$, then τ^*_α together with $\{\underline{0}\}$ form a $nd - M-$fuzzy topology on X.

Proof. 1. $\{\underline{0}\}$ and $\{\underline{1}\} \in \tau_\alpha$

2. Let $A, B \in \tau_\alpha$, then $A(x) \geq \alpha$ and $B(x) \geq \alpha$,

   So $(A \wedge B)(x) \geq \alpha$ .Therefore $A \wedge B \in \tau_\alpha$.

3. Let $A_i \in \tau_\alpha$ for $i \in I$.

   So $\alpha \sqsubseteq_{EM} A_i(x)$,for every $i \in I$ and for all $x \in X$.
   Then, $\alpha \sqsubseteq_{EM} A_i(x)$
   $\sqsubseteq_{EM} \bigvee A_i(x)$

$\sqsubseteq_{EM} (\bigvee A_i)(x)$,for every $x \in X$. So $\bigvee A_i \in \tau$

Hence τ^*_α form a nd- M fuzzy topology on X. □

3.4 nd-M-fuzzy continuous maps

Definition 3.4.1. Let $((2^M)^X, \tau)$ and $((2^M)^Y, \upsilon)$ be two $nd-M-$ fuzzy topological spaces and $f : X \longrightarrow Y$ a map. The map $\overrightarrow{f} : (2^M)^X \longrightarrow (2^M)^Y$ is a nd-M- fuzzy mapping.We say $\overrightarrow{f}$ is an nd-M fuzzy continuous mapping from $((2^M)^X, \tau) \longrightarrow ((2^M)^Y, \upsilon)$ if for each $B \in \upsilon$, $B \sqsubseteq_{EM} \overleftarrow{f}(B)$. That is $B(f(x)) \sqsubseteq_{EM} \overleftarrow{f}(B)(x)$,for every $x \in X$.

Proposition 3.4.1. Let $((2^L)^X, \tau)$ and $((2^M)^Y, \upsilon)$ be two $nd-M-$ fuzzy topological spaces and $f : X \longrightarrow Y$ a map. Then the map $\overrightarrow{f} : ((2^M)^X, \tau) \longrightarrow ((2^M)^Y, \upsilon)$ is fuzzy continuous if and only if , for all $\alpha \in 2^M$,$\overrightarrow{f} : ((2^M)^X, \tau_\alpha) \longrightarrow ((2^M)^Y, \upsilon_\alpha)$ is fuzzy continuous.

Proof. Suppose $\overrightarrow{f} : ((2^M)^X, \tau) \longrightarrow ((2^M)^Y, \upsilon)$ is fuzzy continuous map and $\alpha \in 2^M$.
Take $B \in \upsilon_\alpha$, then $\alpha \sqsubseteq_{EM} B(f(x)) \sqsubseteq_{EM} \overleftarrow{f}(B)(x)$, for every $x \in X$
Therefore $\overleftarrow{f}(B)$ is open and so $\overleftarrow{f}(B) \in \tau_\alpha$.
That is $\overrightarrow{f} : ((2^M)^X, \tau_\alpha) \longrightarrow ((2^M)^Y, \upsilon_\alpha)$ is nd-M- fuzzy continuous.
Conversely suppose $\overrightarrow{f} : ((2^M)^X, \tau_\alpha) \longrightarrow ((2^M)^Y, \upsilon_\alpha)$ is nd-M- fuzzy continuous.
Let $B \in \upsilon$. If $B = 0$, then it is obvious that $B \sqsubseteq_{EM} \overleftarrow{f}(B)$.

Assume $B \neq 0$,$B(f(x)) = \lambda$, for every $x \in X$, Then $B \in \upsilon_\lambda$ So $\overleftarrow{f}(B) \in \tau_\lambda$, by the nd-M-fuzzy continuity of $\overrightarrow{f} : ((2^M)^X, \tau_\lambda) \longrightarrow ((2^M)^Y, \upsilon_\lambda)$.
Hence $\lambda = B(f(x)) \sqsubseteq_{EM} \overleftarrow{f}(B)(x)$, for all $x \in X$. Thus $B \sqsubseteq_{EM} \overleftarrow{f}(B)$. Therefore $\overrightarrow{f} : ((2^M)^X, \tau) \longrightarrow ((2^M)^Y, \upsilon)$ is fuzzy continuous . ☐

Proposition 3.4.2. Let $((2^M)^X, \tau)$ $((2^M)^Y, \upsilon)$ and $((2^M)^Z, \nu)$ be three $nd - M-$ fuzzy topological spaces . If $\overrightarrow{f} : ((2^M)^X, \tau) \longrightarrow ((2^L)^Y, \upsilon)$ and $\overrightarrow{g} : ((2^M)^Y, \upsilon) \longrightarrow ((2^M)^Z, \nu)$ are nd-M- fuzzy continuous maps. Then $\overrightarrow{g} \circ \overrightarrow{f} : ((2^M)^X, \tau) \longrightarrow ((2^M)^Z, \nu)$ is also nd-M- fuzzy continuous.

Proof. Obvious. ☐

Definition 3.4.1. A map $\overrightarrow{f} : ((2^M)^X, \tau) \longrightarrow ((2^M)^Y, \upsilon)$ is called a nd-M -fuzzy homomorphism if $f : X \longrightarrow Y$ is bijective and $\overrightarrow{f}$ and $\overleftarrow{f}$ are nd-M-fuzzy continuous. A map $\overrightarrow{f} : ((2^M)^X, \tau) \longrightarrow ((2^L)^Y, \upsilon)$ is said to be fuzzy open if $\mu \sqsubseteq_{EM} \overrightarrow{f}(\mu)$ for all $\mu \in (2^M)^X$. A map $\overrightarrow{f} : ((2^M)^X, \tau) \longrightarrow ((2^L)^Y, \upsilon)$ is said to be nd -M-fuzzy closed if $\mu' \sqsubseteq_{EM} \overrightarrow{f}(\mu')$, where $\mu \in (2^M)^X$.

Proposition 3.4.3. Let $((2^M)^X, \tau)$, $((2^M)^Y, \upsilon)$ be two nd-M- fuzzy topological spaces and $f : X \longrightarrow Y$ a bijection. Then the following are equivalent.

1. $\overrightarrow{f}$ is a nd-M- fuzzy homeomorphism.

2. $\overrightarrow{f}$ is nd-M-fuzzy continuous and nd-M-fuzzy open.

3. $\overrightarrow{f}$ is nd-M-fuzzy continuous and nd-M-fuzzy open.

4. $\mu \sqsubseteq_{EM} \overrightarrow{f}(\mu)$ and $\overrightarrow{f}(\mu) \sqsubseteq_{EM} \mu$ for all $\mu \in (2^M)^X$

5. $\lambda \sqsubseteq_{EM} \overleftarrow{f}(\lambda)$ and $\overleftarrow{f}(\lambda) \sqsubseteq_{EM} \lambda$ for all $\lambda \in (2^M)^Y$

Proof. Obvious □

Chapter 4

nd-M-Fuzzy Lattice

The L-fuzzy lattice was introduced by *Tepavčević* and *Goran Trajakoviski*[2], where a bounded lattice is fuzzified by using a complete lattice. They defined two types of fuzzy lattices. The first type of fuzzy lattices is obtained by fuzzifying the membership of the elements from the carrier of a crisp lattice and second type of fuzzy lattices is obtained as a result of fuzzification of the order relation in a crisp lattice. They arrived at the conclusion that these two types of fuzzy lattices are equivalent.

In this chapter first we discuss $nd - M$-fuzzy order relation. Then we extend the idea of L-fuzzy lattice[2] to the $nd - M$-fuzzy lattice using Egli-Miler ordering of subsets. Here we fuzzified a bounded lattice by using a complete and consistent multilattice M.

As in the L-fuzzy lattice, we defined two types of non-deterministic M-fuzzy lattice. The first is obtained by assigning single or set of values to each element of the carrier of the bounded lattice. The second type is obtained by non-deterministic fuzzyfication of the order relation in a lattice.We arrived at the conclusion that these type of approaches are equivalent.

4.1 nd-M-fuzzy relation

Definition 4.1.1. Let $(M, \wedge, \vee)$ be a complete and consistent multilattice with bottom element 0_M and top element 1_M. Let X be a non-empty set. Then any mapping $\bar{R} : X \times X \to 2^M$ is a $non-deterministic\ M-valued\ fuzzy\ relation$ on X called $nd-M-$fuzzy relation on X.

Definition 4.1.2. For $\alpha \in 2^M$ an $\alpha - level$ of $\bar{R}$ is a mapping $\bar{R}_\alpha : X \times X \to \{0,1\}$, such that $\bar{R}(x,y) = 1$ if and only if $\alpha \sqsubseteq_{EM} \bar{R}(x,y)$. Then

$$R_\alpha = \{(x,y) : \alpha \sqsubseteq_{EM} \bar{R}(x,y)\}$$

is the corresponding level set of $\bar{R}$,which is a crisp relation on X called α level of $\bar{R}$.

Definition 4.1.3. An $nd-M-$fuzzy relation is

1. $nd-M-\ reflexive$ if $R(x,x) = \{1\}$ for every $x \in X$.

2. $Weakly-nd-M-\ reflexive$ if

$$R(x,y) \sqsubseteq_{EM} R(x,x) \text{ and } R(y,x) \sqsubseteq_{EM} R(x,x)\ \forall\ x,y \in X$$

3. $nd - M -$ $anti - symmetric$ if

$$R(x,y) \wedge_M R(y,x) = \{0\} \ \forall \ x,y \in X \ \ with \ \ x \neq y$$

4. $nd - M -$ $transitive$ if

$$R(x,y) \wedge R(y,x) \sqsubseteq_{EM} R(x,z) \ \forall \ x,y,z \in X$$

.

An $nd - M-$ valued relation R on X is an $nd - M-$fuzzy ordering relation on X if it is an $nd - M-$reflexive $nd - M-$ anti-symmetric and $nd - M-$transitive.

Definition 4.1.4. Let $(L, \wedge_L, \vee_L, 0, 1)$ be a bounded lattice and $(M, \wedge_M, \vee_M)$ be a non-trivial complete and consistent multi-lattice. Let μ be a $nd - M-$ fuzzy subset defined on L, denoted by $\mu \in (2^M)^L$.
For $\mu \in (2^M)^L$ and $\alpha \in 2^M$, then the $\alpha-$ level of μ is defined by

$$\mu_\alpha = \{x \in L : \alpha \sqsubseteq_{EM} \mu(x)\}$$

4.2 nd-M-fuzzy lattice

Definition 4.2.1. An nd-M fuzzy subset $\mu \in (2^M)^L$ is a $nd - M-$fuzzy sub lattice of L if $\alpha \sqsubseteq_{EM} \mu(x) \wedge_M \mu(y)$ for every $x, y \in \mu_\alpha$ and μ_α is a sub lattice of L for each $\alpha \in 2^M$.

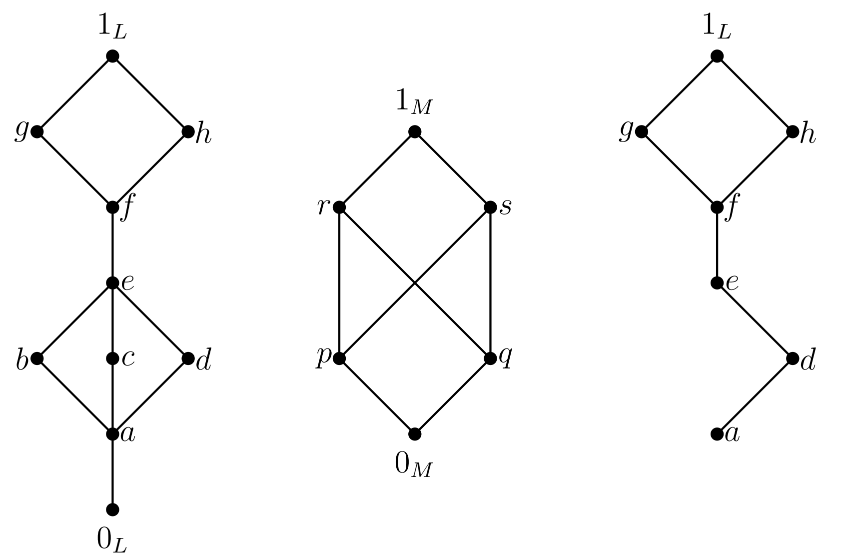

Figure 4.1: The valuated lattice L, the valuating multilattice M and the α-level μ_α where $\alpha = \{p\}$ in Example 4.2.1

Example 4.2.1. 1. For any L-fuzzy lattice is $nd - M-$ fuzzy lattice.

2. Choose $\alpha \in 2^M$ such that $\alpha \wedge \alpha = \alpha$, define $\mu : L \to 2^M$ by $\mu(x) = \alpha \ \forall \ x \in L$. Then μ is a $nd - M-$ fuzzy lattice.

3. Let $(L, \wedge_L, \vee_L)$ be a lattice and $(M, \wedge_M, \vee_M)$ be a multilattice with 0_M, 1_M where $L = \{0_L, a, b, c, d, e, f, g, h, 1_L\}$ and $M = \{0_M, p, q, r, s, 1_M\}$.

Let

$$\bar{L} = \begin{pmatrix} 0 & a & b & c & d \\ \{p\} & \{p\} & \{q\} & \{p,q\} & \{p\} \\ & & & & & e & f & g & h & 1 \\ & & & & & \{p,q\} & \{s\} & \{r,s\} & \{s\} & 1_M \end{pmatrix}$$

is an $nd - M-$ fuzzy lattice.

If $\alpha = \{p\}$ then $L_\alpha = \{a, d, e, f, g, h, 1\}$.

Theorem 4.2.2. *Let μ is an nd-M fuzzy subset defined on L and $\mu_\alpha, \alpha \in 2^M$ is a α level set of μ.Assume that μ_α satisfies $\alpha \sqsubseteq_{EM} \mu(x) \wedge \mu(y)$ for every $x, y \in \mu_\alpha$. Then μ is called a $nd - M-$fuzzy sub lattice of L (or simply $nd - M-$fuzzy lattice of L) if and only if for all $x, y \in L$,*

$$multiinf\{\mu(x), \mu(y)\} \sqsubseteq_{EM} multiinf\{\mu(x \wedge_L y), \mu(x \vee_L y)\}$$

That is,

$$\mu(x) \wedge_M \mu(y) \sqsubseteq_{EM} \mu(x \wedge_L y) \wedge_M \mu(x \vee_L y)$$

.

Proof. Let μ is an nd-M fuzzy subset and

$$\mu_\alpha = \{x \in L : \alpha \sqsubseteq_{EM} \mu(x)\}$$

. Assume that μ_α satisfies $\alpha \sqsubseteq_{EM} \mu(x) \wedge_M \mu(y)$, for every $x, y \in \mu_\alpha$

Let $T = \mu(x) \wedge_M \mu(y)$, Then $T \sqsubseteq_{EM} \mu(x)$ and $T \sqsubseteq_{EM} \mu(y)$
That is $x, y \in \mu_T$.
But by the assumption,μ_T is a sub lattice of L,then $x \wedge_L y$ and $x \vee_L y$ belongs to μ_T and so $T \sqsubseteq_{EM} x \vee_L y$ and $T \sqsubseteq_{EM} x \wedge_M y$
Since $x \wedge_L y$ and $x \vee_L y$ belongs to μ_T, $T \sqsubseteq_{EM} \mu(x \vee_L y) \wedge_M \mu(x \wedge_L y)$.
Therefore $T = \mu(x) \wedge_M \mu(y) \sqsubseteq_{EM} \mu(x \vee_L y) \wedge_M \mu(x \wedge_L y)$
$\mu(x) \wedge_M \mu(y) \sqsubseteq_{EM} \mu(x \vee_L y) \wedge_M \mu(x \wedge_L y)$

conversely assume that μ satisfies
$\mu(x) \wedge_M \mu(y) \sqsubseteq_{EM} \mu(x \vee_L y) \wedge_M \mu(x \wedge_L y)$
Let T is an arbitrary element of 2^M. For every $x, y \in \mu_T$, Then $T \sqsubseteq_{EM} \mu(x)$ and $T \sqsubseteq_{EM} \mu(y)$. Hence $T \sqsubseteq_{EM} \mu(x) \wedge_M \mu(y)$
But our assumption ,we have
$T \sqsubseteq_{EM} \mu(x) \wedge_M \mu(y) \sqsubseteq_{EM} \mu(x \wedge_L y) \wedge_M \mu(x \vee_L y)$.
Hence $T \sqsubseteq_{EM} \mu(x \wedge_L y)$ and $T \sqsubseteq_{EM} \mu(x \vee_L y)$.
Hence $x \vee_L y \in \mu_T$ and $x \wedge_L y \in \mu_T$ and thus μ_T is a sublattice of L. Therefore μ is an nd-M fuzzy lattice.

□

An $nd - M-$ fuzzy lattice satisfies the following proposition.

Theorem 4.2.3. *Let $\bar{L} : L \to 2^M$ be an $nd - M-$ fuzzy lattice and let $\alpha, \beta \in 2^M$. If $\alpha \sqsubseteq_{EM} \beta$ then $\bar{L}_\beta$ is an $nd - M-$ sub lattice of $\bar{L}_\alpha$.*

Proof. Let $x \in \bar{L}_\beta$, Then $\beta \sqsubseteq_{EM} \bar{L}(x)$. So if $\alpha \sqsubseteq_{EM} \beta$,then $\alpha \sqsubseteq_{EM} \bar{L}(x)$. There for $x \in L_\alpha$. So $\bar{L}_\beta \subseteq \bar{L}_\alpha$. Thus the collection of all level sets is closed under intersection and contains the greatest element. □

Theorem 4.2.4. *Let $(L, \wedge_L, \vee_L)$ be a lattice $(M, \wedge_M, \vee_M)$ be a complete and consistent multilattice with 0_M and 1_M. Then the mapping $\bar{L} : L \to 2^M$ is an $nd - M-$ fuzzy lattice if and only if both of the following relations hold for all $x, y \in L$*

1. $\bar{L}(x) \wedge_M \bar{L}(Y) \sqsubseteq_{EM} \bar{L}(x \wedge_L y)$
2. $\bar{L}(x) \wedge_M \bar{L}(Y) \sqsubseteq_{EM} \bar{L}(x \vee_L y)$

The following gives an idea of how to construct an $nd - M-$ fuzzy lattice having a family of lattice as its family of level sets [2]. Let P_1 and P_2 be two posets with disjoint underlying sets. The disjoint union of posets P_1 and P_2 is the poset $(P_1 \cup P_2, \leq)$ where $\leq$ is defined by $x \leq y$ if and only if $x, y \in P_1$ and $x \leq y$ in P_1
or $x, y \in P_2$ and $x \leq y$ in P_2 or $x \in P_1$ and $y \in P_2$.
The linear sum of Posets P_1 and P_2 is the poset $(P_1 \cup P_2, \leq)$,denoted by $(P_1 \oplus P_2)$ where $\leq$ is defined by

$$x \leq y \text{ if and only if } \begin{array}{l} x, y \in P_1 \text{ and } x \leq y \text{ in } P_1 \\ \text{or } x, y \in P_2 \text{ and } x \leq y \text{ in } P_2 \\ \text{or } x \in P_1,\ y \in P_2. \end{array}$$

Theorem 4.2.5. *Let $\mathcal{F}$ be a collection of lattices with disjoint elements. Then there exists an $nd - M-$ fuzzy lattice whose non-trivial α-levels are exactly the lattices from $\mathcal{F}$.*

Proof. Let $\mathcal{F}$ be a collection of lattices $(L_i, \wedge_i, \vee_i)$ with disjoint elements. bottom 0_i and top element 1_i, $(i \in I)$. Our aim is to find a $nd - M-$ fuzzy lattice using the collection of lattices, which is obtained in the following manner.

Let $\{0_M, 1_M\} \cup \{p_i, q_i : i \in I\}$ be the elements of the multilattice M where the order is defined by $p_i \geq q_i \;\forall\; i$.

$$p_1 \geq q_2, p_2 \geq q_1$$

, where p_i,'s are co atoms and q_i's are atoms.

Let L be a poset defined by
$L \;= 0_L \oplus \bigcup_{i \in I} L_i \oplus 1_L$, where 0_L and 1_L are the one element lattices, $\oplus$ is a linear sum and $\bigcup$ is disjoint union of posets. Clearly L is a lattice. Where define the mapping $\bar{L} : L \to 2^M$ by

$$\bar{L}\{x\} = \{p_i, q_i\} \text{ iff } x \in L_i, i \in I \text{ and } \bar{L}\{0_L\} = \{0_M\}, \bar{L}\{1_L\} = \{1_M\}$$

Then for each $\alpha \in 2^M$, it is clear that all the non-trivial α-levels of $\bar{L}$ are exactly the lattices from $\mathcal{F}$. □

Example 4.2.2. Let $\mathcal{F}$ consists of three lattices L_1, L_2 and L_3. Then construct M and L according to the previous theorem. Then the required $nd - M-$ fuzzy lattices given by the mapping

$$\bar{L} = \begin{pmatrix} a & b & c & d & e & f \\ \{p_1, q_1\} & \{p_1, q_1\} & \{p_1, q_3\} & \{p_2, q_2\} & \{p_2, q_3\} & \{p_2, q_3\} \\ g & h & i & 0_L & 1_L \\ \{p_2, q_2\} & \{p_3, q_3\} & \{p_3, q_3\} & 0_M & 1_M \end{pmatrix}$$

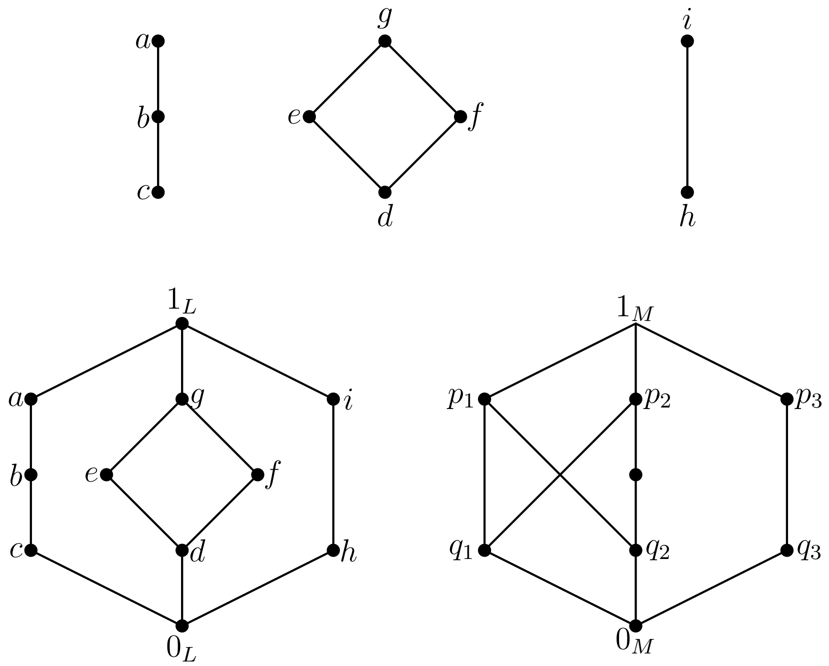

Figure 4.2: The lattices and the multilattice in Example 4.2.2

4.3 $nd - M-$ fuzzy lattices as $nd - M-$ fuzzy relations

In the previous section we defined $nd - M-$ fuzzy lattices as $nd - M-$ fuzzy algebraic structures. In this section, we introduce another approaches to nd-M- fuzzy lattices (via) $nd-M-$ fuzzification of the order relation.

Let $(M, \leq)$ is a complete multi lattice with bottom element 0_M and top element 1_M and let O be the one element lattice (which is also a multi lattice). Let $M' = O \oplus M$. Clearly $(M', \leq)$ is a complete multi lattice with bottom element 0_M and top element 1_M. Let $\bar{R} : L^2 \to 2^{M'}$ be an $nd - M-$ fuzzy relation.

Let N_α is the set defined by

$$N_\alpha = \{x \in L : \alpha \sqsubseteq_{EM} R(x,x)\}$$

Now we have the definition of a nd-M-fuzzy lattice (as an nd-M-fuzzy relation

Definition 4.3.1. Let L be a non-empty set and $M' = O \oplus M$ be a complete and consistent multilattice, then the pair $(L, \bar{R})$ where $\bar{R} : L^2 \to 2^{M'}$ is an $nd - M-$ fuzzy relation, is called an $nd - M-$ valued fuzzy lattice if (L, R_{0_M}) is a lattice and all the α-levels of $\bar{R}, \alpha \in 2^M$, satisfies $\alpha \sqsubseteq_{EM} \bar{R}(x_1, y_1) \wedge \bar{R}(x_2, y_2)$, where $(x_1, y_1), (x_2, y_2) \in R_\alpha$ and also R_α are sub lattice of it.

Note 4. We know that $\{0_M\}$ level of R equal to L^2 which is not an nd-M-fuzzy ordering relation and thus neither a nd-M-fuzzy lattice.Our aim is to find a nd-M-fuzzy sublattice of L,that is why we introduce the artificial element $\{0_L\}$.

The next theorem gives the necessary and sufficient conditions under which an $nd-M-$ fuzzy relation is an $nd-M-$fuzzy lattice.

Theorem 4.3.2. *Let L be a non-empty set and M complete and consistent multilattice. Then $M' = O \oplus M$ be a complete and consistent multi lattice with the least element 0 and a unique atom 0_M. Then the mapping*
$\bar{R} : L^2 \to 2^{M'}$ is an $nd-M-$ fuzzy lattice if and only if the following

holds

1. $\bar{R}$ *is a weak* $nd - M-$ *fuzzy ordering relation.*

2. *For all* $x, y \in L$ *there exist* $S \in L$ *such that for all* $\alpha \in \{0_M\} \cup \{\alpha \in 2^M / x, y \in N_\alpha\}$ *the following holds .*
   $\alpha \sqsubseteq_{EM} \bar{R}(x, S)$, $\alpha \sqsubseteq_{EM} \bar{R}(y, S)$ *and the following holds for all* $s \in L$*:*
   $(\alpha \sqsubseteq_{EM} \bar{R}(x, s)) \;\; \wedge_M \;\; (\alpha \sqsubseteq_{EM} \bar{R}(y, s)) \Rightarrow \alpha \sqsubseteq_{EM} \bar{R}(S, s)$.

3. *for all* $x, y \in L$ *there exist* $I \in L$ *such that for all*

$$\alpha \in \{0_L\} \cup \{\alpha \in 2^M / x, y \in N_\alpha\}$$

   the following holds
   $(\alpha \sqsubseteq_{EM} \bar{R}(I, x))$,$(\alpha \sqsubseteq_{EM} \bar{R}(I, y))$ *and the following holds for all* $i \in L$*:*

$$(\alpha \sqsubseteq_{EM} \bar{R}(i, x)) \;\; \wedge_M \;\; (\alpha \sqsubseteq_{EM} \bar{R}(i, y)) \Rightarrow \alpha \sqsubseteq_{EM} \bar{R}(i, I)$$

.

Proof. Assume that $\bar{R} : L^2 \to 2^{M'}$ be an $nd - M-$ fuzzy lattice. Let $\alpha = \{0_M\}$. Then $(L, \bar{R}_{\{0_M\}})$ is a lattice and for each $\alpha \in 2^M$,$\bar{R}$ satisfies $\alpha \sqsubseteq_{EM} \bar{R}(x_1, y_1) \wedge_M \bar{R}(x_2, y_2)$, where $(x_1, y_1), (x_2, y_2) \in R_\alpha$ and also R_α are sub lattice of it. This means that for any pair of elements $x, y \in L$, $(x \vee_L y)$ and $(x \wedge_L y)$ exists. Let $x \vee_L y = S$ and $x \wedge_L y = I$, therefore the relations in 2 to 3 holds for $\alpha = \{0_M\}$.

Suppose that $\alpha \in 2^M$, $x, y \in N_\alpha$.
Since R_α is a sublattice of $R_{\{0_M\}}$, we have that supremum and infimum for elements x and y in lattices $(N_\alpha, \bar{R}_\alpha)$ and $(L, \bar{R}_{\{0_M\}})$ are the same.
Then $\alpha \sqsubseteq_{EM} \bar{R}(x, S)$ and $\alpha \sqsubseteq_{EM} \bar{R}(y, S)$, then for all $s \in L$, the conditions in 2 and 3 holds.
since $(L, \bar{R}_{\{0_M\}}$ is a lattice and for each α levels of $\bar{R}$ satisfies $\alpha \sqsubseteq_{EM} \bar{R}(x_1, y_1) \wedge_M \bar{R}(x_2, y_2)$, where $(x_1, y_1), (x_2, y_2) \in R_\alpha$ and also R_α are sub lattice of it,they are ordering relations on subsets, that is all levels of $\bar{R}$ is an nd-M- weak ordering relations on L , condition 1 is satisfied.
Conversely suppose that the mapping $\bar{R} : L^2 \to 2^{M'}$, satisfies the conditions 1 to 3.
By weak reflexivity and condition 2, we have $\bar{R}(x, y) \sqsubseteq_{EM} \bar{R}(x, x)$ and $\bar{R}(y, x) \sqsubseteq_{EM} \bar{R}(x, x)$,for every $x, y \in L$. Since $\{0_M\} \sqsubseteq_{EM} \bar{R}(x, S)$. we have that $\bar{R}_{\{0_M\}}(x, x) = \{1\}$ for all x.
This follows that $\bar{R}_{\{0_M\}}$ is an ordering relation and by condition 2 and 3 $(L, \bar{R}_{0_M})$ is a lattice. Also from 2 and 3 ,we have $\alpha \sqsubseteq_{EM} \bar{R}(x, y) \wedge_M \bar{R}(x, y)$ whenever $\alpha \sqsubseteq_{EM} \bar{R}(x_1, y_1)$ and $\alpha \sqsubseteq_{EM} \bar{R}(x_2, y_2)$.Also we see that α level $\bar{R}_\alpha$ is an ordering relation on N_α. Thus $(N, \bar{R}_\alpha)$ is a lattice and it is a sublattice of $(L, \bar{R}_{0_M})$. □

4.4 Relation between two types of nd-M-fuzzy lattices

Theorem 4.4.1. *Let* $(L, \wedge_L, \vee_L)$ *is a lattice and* $(M, \wedge_M, \vee_M)$ *be a complete multilattice with* 0_M *and* 1_M*. Then* $M' = O \oplus M$ *be a complete multi lattice. Let* $\bar{L} : L \to 2^M$ *be an* $nd - M-$ *fuzzy lattice satisfying* $\alpha \sqsubseteq_{EM} \bar{L}(x) \wedge_M \bar{L}(y)$*, for every* $x, y \in L_\alpha$*. Then the mapping* $\bar{R} : L^2 \to 2^{M'}$ *is defined by*

$$\begin{aligned} \bar{R}(x,y) &= \bar{L}(x) \wedge_M \bar{L}(y) \ \text{ if } x \leq y \\ &= \{0_M\} \quad \text{otherwise} \end{aligned}$$

is an $nd - M-$ *fuzzy lattice (as an* $Nd - M-$ *fuzzy relation). Moreover ,*L_α *and* $(N_\alpha, \bar{R}_\alpha)$ *,for* $\alpha \in 2^M$ *are the same sub lattice of M.*

Proof. Let $\bar{L} : L \longrightarrow 2^M$ be an nd -M -fuzzy lattice, then for each $\alpha \in 2^M$,$\bar{L}$ satisfies $\alpha \sqsubseteq_{EM} \bar{L}(x) \wedge_M \bar{L}(y)$ for all $x, y \in \bar{L}_\alpha$. If $\alpha = \{0_M\}$, then $\{0_M\} \sqsubseteq_{EM} \bar{L}(x)$ for every $x \in L$. Hence $o_M \sqsubseteq_{EM} \bar{L})(x) \wedge_M \bar{L}(y)$ and so $0_M \sqsubseteq_{EM} R_{\{0_M\}}$. That is $\bar{R}_{\{o_M\}}(x,y) = 1$ for all $x \leq y$. If $\bar{R}_{\{o_M\}}(x,y) = 0$, we have that $(L, \bar{R}_{\{o_M\}})$ is the same lattice as $(L, \wedge_L, \vee_L)$.

Now let $\alpha \in 2^M$. If $x \in \bar{L}_\alpha$ if and only if $\alpha \sqsubseteq_{EM} \bar{L}(x)$ if and only if $\alpha \sqsubseteq_{EM} \bar{R}(x.x)$ if and only if $x \in N_\alpha$. Hence for all $\alpha \in 2^M$, the sets $\bar{L}_\alpha$ and N_α are equal. Now let $x.y \in \bar{L_\alpha}$ $x \leq y$, then $\alpha \sqsubseteq_{EM} \bar{L}(x)$ and $\alpha \sqsubseteq_{EM} \bar{L}(y)$, this implies $\alpha \sqsubseteq_{EM} \bar{R}(x,y)$.

That is $(x,y) \in R_\alpha$.

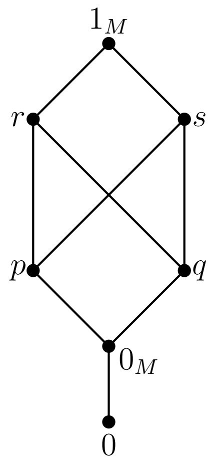

Figure 4.3: The multilattice in Example 4.4.1

If $(x, y) \in R_\alpha$, then $\alpha \sqsubseteq_{EM} \bar{R}(x, y)$, hence
$\bar{R}(x, y) \neq \{0\}$ and $\alpha \sqsubseteq_{EM} \bar{R}(x, y) = \bar{L}(x) \wedge_M \bar{L}(y)$ and $x \leq y$.
Then we have to prove that the relations $\bar{R}_\alpha$ on L_α and $\leq$ on L_α are same. Since $(L_\alpha, \leq)$ is a lattice, and also it is a sub lattice of $(L, \leq)$. This means that $(N_\alpha, \bar{R}_\alpha)$ is a sub lattice of $(L, \bar{R}\{0_M\}$. Therefore the mapping $\bar{R}$ is an nd-M-fuzzy Lattice(nd-M-fuzzy relation). □

Example 4.4.1. Consider the example 3.2,the corresponding nd-M-fuzzy lattice(as a nd-M-fuzzy relation) is mapping $\overline{R} : L^2 \longrightarrow M'$ given in the table below, where $L = \{0_M, a, b, c, d, f, g, h, 1\}$ and M' is the figure nd-M-fuzzy lattice as a nd-M-fuzzy relation from example 4.4.1

$R(x,y)$	0_L	a	b	c	d	e	f	g	h	1_L
0_L	0_M	$\{p\}$	0_M	$\{0_M,p\}$	$\{p\}$	$\{0_M,p\}$	$\{p\}$	$\{p\}$	$\{p\}$	$\{p\}$
a	0_M	$\{p\}$	0_M	$\{0_M,p\}$	$\{p\}$	$\{0_M,p\}$	$\{p\}$	$\{p\}$	$\{p\}$	$\{p\}$
b	0_M	0_M	0_M	$\{0_M,q\}$	0_M	$\{0_M,q\}$	$\{p\}$	$\{q\}$	$\{q\}$	$\{q\}$
c	0_M	0_M	0_M	$\{0_M,p,q\}$	$\{0_M,p\}$	$\{o_M,p,q\}$	$\{p,q\}$	$\{p,q\}$	$\{p,p\}$	$\{p,p\}$
d	0_M	0_M	0_M	0_M	$\{p\}$	$\{0_M,p\}$	$\{p\}$	$\{p\}$	$\{p\}$	$\{p\}$
e	0_M	0_M	0_M	0_M	0_M	$\{o_M,p,q\}$	$\{p,q\}$	$\{p,q\}$	$\{p,q\}$	$\{p,q\}$
f	0_M	0_M	0_M	0_M	0_M	0_M	$\{s\}$	$\{r,p,q\}$	$\{s\}$	$\{s\}$
g	0_M	0_M	0_M	0_M	0_M	0_M	0_M	$\{r,s\}$	$\{p,q,s\}$	$\{r,s\}$
h	0_M	0_M	0_M	0_M	0_M	0_M	0_M	0_M	$\{s\}$	$\{s\}$
1_L	0_M	0_M	0_M	0_M	0_M	0_M	0_M	0_M	0_M	1_M

Chapter 5

Strong nd-M-Fuzzy Lattice

In the previous chapter we discussed the concept of nd-M-fuzzy lattice. in this chapter we discuss the nd-M-fuzzy join and meet semilattices and also we define strong nd-M-fuzzy lattices.
Let$A : X \longrightarrow 2^M$ be a nd-M fuzzy subset of X, where X is any set and M is a complete and consistent multilattice. Let $\alpha \in 2^M$,then the α level of A is defined by $A_\alpha = \{x \in X | \alpha \sqsubseteq_{EM} A(x)\}$.

5.1 nd-M fuzzy -meet semilattice

Definition 5.1.1. Let $(L, \wedge_L)$ be a $meet - semilattice$ and $(M, \wedge_M, \vee_M)$ be a complete and consistent multi lattice. A mapping $A : L \to 2^M$ is called an nd-M-Fuzzy meet-semilattice of L, if for each $\alpha-$ level sets satisfies , $\alpha \sqsubseteq_{EM} A(x) \wedge_M A(y)$ for every $x, y \in A_\alpha$ and are sub meet-semilattices of L.

Proposition 5.1.1. Let $(L, \wedge_L)$ be a meet-semilattice and

$(M, \wedge_M, \vee_M)$ be a consistent and complete multi-lattice. Assume that A_α satisfying $\alpha \sqsubseteq_{EM} A(x) \wedge_M A(y)$ for every $x, y \in A_\alpha$ .Then A mapping $A : L \to 2^M$ is an $nd - M$- Fuzzy meet-semi lattice of L if and only if
$multi\ \ sup\ \ (A(x), A(y))\ \ \sqsubseteq_{EM}\ \ A(x \wedge_L y)\ \forall\ x, y \in L$
That is

$$A(x) \wedge_M A(y)\ \ \sqsubseteq_{EM}\ \ A(x \wedge_L y)\ \forall\ x, y \in L$$

Proof. Assume that $A : L \to 2^M$ is an $nd - M$- Fuzzy meet-semilattice of L .Then for each $\alpha \in 2^M$, A_α satisfies $\alpha \sqsubseteq_{EM} A(x) \wedge_M A(y)$ for every $x, y \in 2^M$ and are sub-meet-semilattice of L.

If $x, y \in L$ and $T = A(x) \wedge_M A(y)$,then $T \sqsubseteq_{EM} A(x)$ and $T \sqsubseteq_{EM} A(y)$.Since A_T is a sub-meet-semilattice of L, Then $x \wedge_L y \in A_\alpha$, for every $x, y \in A_T$. Hence $T \sqsubseteq_{EM} A(x \wedge_L y)$ and so
$A(x) \wedge_M A(y) \sqsubseteq_{EM} A(x \wedge y)$.

Conversely assume that $A : L \to 2^M$ satisfies the conditions

$$A(x) \wedge_M A(y)\ \ \sqsubseteq_{EM}\ \ A(x \wedge_L y)$$

Let T be an arbitrary element of 2^M. If for every $x, y \in A_T$, then $T \sqsubseteq_{EM} A(x)$ and $T \sqsubseteq_{EM} A(y)$ .Thus
$T \sqsubseteq_{EM} A(x) \wedge_M A(y)$.
By our assumption we have ,
$T \sqsubseteq_{EM} A(X) \wedge_M A(y) \sqsubseteq_{EM} A(x \wedge_L y)$
Hence $T \sqsubseteq_{EM} A(x \wedge_L y)$ and $x \wedge_L y \in A_T$. Therefore,A_T is a sub-meet-semilattice of L,and so L is an nd M-fuzzy meet semilattice.

□

Lemma 5.1.1. Let L be a meet-semilattice , M be a complete and consistent multilattice and $A_j : L \to 2^M$ be an $nd - M$- Fuzzy meet-semilattice(for each $j \in J$), then

1. $\bigwedge_{j \in J} A_j$ is an $nd - M$- Fuzzy meet-semilattice.

2. $\bigvee_{j \in J} A_j$ is an nd-M-fuzzy meet semilattice of L.

Proof. 1. We now show that each $\bigwedge A_j$ is an nd-M - Fuzzy meet-semilattice. Since each A_j is an nd-M-fuzzy meet-semilattice, each A_j satisfies
$A_j(x) \wedge A_j(y) \sqsubseteq_{EM} A_j(x \wedge_L y)$,for all $x, y \in L$.

$$\sqsubseteq_{EM} \bigwedge (A_j(x \wedge_L y))$$
$$\sqsubseteq_{EM} (\bigwedge A_j)(x \wedge_L y)$$

$((\bigwedge(A_j)(x)) \wedge_M ((\bigwedge(A_j)(x)) \sqsubseteq_{EM} (\bigwedge A_j)(x \wedge_L y)$.

2. We now show that each $\bigvee A_j$ is an $nd - M$- Fuzzy meet-semi lattice. Since each A_j is an nd-M-fuzzy meet semi lattice ,each A_j satisfies

$A_j(x) \wedge A_j(y) \sqsubseteq_{EM} A_j(x \wedge_L y)$,for all $x, y \in L$.

$$\begin{aligned} Now, (\bigvee A_j)(x) \wedge_M (\bigvee A_j)(y) &= (\bigvee A_j(x)) \wedge_M (\bigvee A_j(y)). \\ &\sqsubseteq_{EM} \bigvee(A_j(x) \wedge_M A_j(y)) \\ &\sqsubseteq_{EM} \bigvee(A_j(x \wedge_L y)) \\ &\sqsubseteq_{EM} (\bigvee A_j)(x \wedge_L y) \end{aligned}$$

$$((\bigvee(A_j)(x)) \wedge_M ((\bigvee(A_j)(x)) \sqsubseteq_{EM} (\bigvee A_j)(x \wedge_L y).$$

□

5.2 nd-M-fuzzy join-semilattice

Definition 5.2.1. Let $(L, \vee_L)$ be a $joint - semilattice$ and $(M, \wedge_M, \vee_M)$ be a complete and consistent multilattice. A mapping $A : L \to 2^M$ is called an $nd - M$- Fuzzy join-semilattice of L, if each $\alpha-$level set satisfies $\alpha \sqsubseteq_{EM} A(x) \wedge_M A(y)$ for every $x, y \in A_\alpha$ and are sub join-semilattices of L.

Proposition 5.2.1. Let $(L, \vee_L)$ be a $join - semilattice$ and $(M, \wedge_M, \vee_M)$ be a consistent and complete multilattice. Assume that A_α satisfying $\alpha \sqsubseteq_{EM} A(x) \wedge_M A(y)$ .Then a mapping $A : L \to 2^M$ is an $nd - M$- fuzzy join-semilattice of L if and only if $multi\ sup\ (A(x), A(y)) \sqsubseteq_{EM} A(x \vee_L y)\ \forall\ x, y \in L$.

That is

$$A(x) \wedge_M A(y) \sqsubseteq_{EM} A(x \vee_L y) \;\forall\; x, y \in L$$

Proof. Assume that $A : L \to 2^M$ is an $nd - M$- Fuzzy join-semilattice of L. Then for each $(\alpha \in 2^M)$,A_α satisfies $\alpha \sqsubseteq_{EM} A(x) \wedge_M A(y)$ for every $x, y \in 2^M$ and are sub meet-semilattice of L.
If $x, y \in L$ and $T = A(x) \wedge_M A(y)$, then $T \sqsubseteq_{EM} A(x)$ and $T \sqsubseteq_{EM} A(y)$.Since A_T is a sub join-semilattice of L, Then $x \vee_L y \in A_T$, for every $x, y \in A_T$. Hence $T \sqsubseteq_{EM} A(x \vee_L y)$ and so
$A(x) \wedge_M A(y) \sqsubseteq_{EM} A(x \wedge y)$.
Conversely assume that $A : L \to 2^M$ satisfies the conditions

$$A(x) \wedge_M A(y) \sqsubseteq_{EM} A(x \vee_L y)$$

Let T be an arbitrary element of 2^M. If for every $x, y \in A_T$,
then $T \sqsubseteq_{EM} A(x)$ and $T \sqsubseteq_{EM} A(y)$. Thus
$T \sqsubseteq_{EM} A(x) \wedge_M A(y)$
By our assumption we have,

$T \sqsubseteq_{EM} A(X) \wedge_M A(y) \sqsubseteq_{EM} A(x \vee_L y)$
Hence $T \sqsubseteq_{EM} A(x \wedge_L y)$ and $(x \vee_L y) \in A_T$. Therefore,A_T is a sub join-semilattice of L and so L is an nd-M-fuzzy join-semilattice. □

Lemma 5.2.1. Let L be a join-semilattice , M be a complete and consistent multilattice and $A_j : L \to 2^M$ be an $nd - M$- Fuzzy join-semilattice(for each $j \in J$), then

1. $\bigwedge_{j\in J} A_j$ is an $nd-M$- Fuzzy join-semilattice.

2. $\bigvee_{j\in J} A_j$ is an nd-M-fuzzy join semilattice of L.

Proof. 1. We now show that each $\bigwedge A_j$ is an nd-M - Fuzzy join-semilattice. Since each A_j is an nd-M-fuzzy join-semilattice, each A_j satisfies
$A_j(x) \wedge_M A_j(y) \sqsubseteq_{EM} A_j(x \vee_L y)$,for all $x, y \in L$.

$$\begin{aligned}(\bigwedge A_j)(x) \wedge_M (\bigwedge A_j)(y) &= (\bigwedge A_j(x)) \wedge_M (\bigwedge A_j(y))\\ &\sqsubseteq_{EM} \bigwedge(A_j(x) \wedge_M A_j(y))\\ &\sqsubseteq_{EM} \bigwedge(A_j(x \wedge_L y))\\ &\sqsubseteq_{EM} (\bigwedge A_j)(x \vee_L y)\end{aligned}$$

$((\bigwedge(A_j)(x)) \wedge_M ((\bigwedge(A_j)(x)) \sqsubseteq_{EM} (\bigwedge A_j)(x \vee_L y)$.

2. We now show that each $\bigvee A_j$ is an $nd-M$- Fuzzy join-semilattice. Since each A_j is an $nd-M-fuzzy$ join semilattice, each A_j satisfies
$A_j(x) \wedge_M A_j(y) \sqsubseteq_{EM} A_j(x \vee_L y)$,for all $x, y \in L$.

$$\begin{aligned}(\bigvee A_j)(x) \wedge_M (\bigvee A_j)(y) &= (\bigvee A_j(x)) \wedge_M (\bigvee A_j(y)).\\ &\sqsubseteq_{EM} \bigvee(A_j(x) \wedge_M A_j(y))\\ &\sqsubseteq_{EM} \bigvee(A_j(x \wedge_L y))\\ &\sqsubseteq_{EM} (\bigvee A_j)(x \vee_L y)\end{aligned}$$

$((\bigvee(A_j)(x)) \wedge_L ((\bigvee A_j)(y)) \sqsubseteq_{EM} (\bigvee A_j)(x \vee_L y)$.

□

Note 5. Let $(L, \wedge_L, \vee_L)$ be a lattice and $(M, \wedge_M, \vee_M)$ be a complete and consistent multilattice. A mapping $A : L \to 2^M$ is called an $nd - M$- Fuzzy lattice if and only if A is both an $nd - M$-Fuzzy meet-semilattice and an $nd - M$- Fuzzy-join-semilattice.

5.3 $nd - M - fuzzy^*$ join-semilattice and $nd - M - fuzzy^*$ meet-semilattice

Definition 5.3.1. Let $(L, \vee_L)$ be a join semilattice and $(M, \wedge_M, \vee_M)$ be a complete and consistent multilattice. A mapping $A : L \to 2^M$ is called an $nd - MFuzzy^*$ join semilattice of L if for each $\beta \in 2^M$,The set $A^\beta = \{x \in L | A(x) \sqsubseteq_{EM} \beta\}$ satisfies $A(x) \vee_M A(y) \sqsubseteq_{EM} \beta$ for every $x, y \in A^\beta$ and A^β is a sub- join semilattice of L.

Lemma 5.3.1. Let $(L, \vee_L)$ be a join semilattice and $(M, \wedge_M, \vee_M)$ be a complete and consistent multilattice .Assume thatA^β satisfies $A(x) \vee_M A(y) \sqsubseteq_{EM} \beta$ for every $x, y \in A^\beta$. Then a mapping $A : L \to 2^M$ is an $nd - M - Fuzzy^*$join semilattice of L if and only if

$$A(x \vee_L y) \sqsubseteq_{EM} A(x) \vee_M A(y) \;\; \forall \;\; x, y \in L$$

Proof. Assume that A is an $nd - M$- Fuzzy join semilattice of L. Then for each $\beta \in 2^M$,the set $A^\beta = \{x \in L | A(x) \sqsubseteq_{EM} \beta\}$ satisfies $A(x) \vee_M A(y) \sqsubseteq_{EM} \beta$ for every $x, y \in A^\beta$ and A^β is a sub- join semilattice of L.

Let $x, y \in L$ and

$$S = A(x) \vee_M A(y)$$

Then $A(x) \sqsubseteq_{EM} S$ and $A(y) \sqsubseteq_{EM} S$ and so $x, y \in A^S$.
But our assumption A^S is a sub-join semilattice of L, for any $x, y \in A^S$, $A(x \vee_L y) \sqsubseteq_{EM} S$.
Thus $A(x \vee_L y) \sqsubseteq_{EM} A(x) \vee_M A(y)$,for every $x, y \in L$.
conversely assume that, $A(x \vee_L y) \sqsubseteq_{EM} A(x) \vee_M A(y)$, for all $x, y \in L$
Let $S \in 2^M$ and $A(x) \sqsubseteq_{EM} S$ and $A(y) \sqsubseteq_{EM} S$.
Then $A(x) \vee_M A(y) \sqsubseteq_{EM} S$. Then by our assumption, we have $A(x \vee_L Y) \sqsubseteq_{EM} A(x) \vee_M A(y) \sqsubseteq_{EM} S$.
Thus $A(x \vee_L y) \sqsubseteq_{EM} S$.
That is $(x \vee_L y) \in A^S$. Hence A^S is a sub-join-semilattice of L. □

Definition 5.3.2. Let $(L, \wedge_L)$ be a meet-semilattice and $(M, \wedge_M, \vee_M)$ be a complete and consistent multilattice. A mapping $A : L \to 2^M$ is called an $nd - MFuzzy^*$ mee-semilattice of L if for each $\beta \in 2^M$,The set $A^\beta = \{x \in L | A(x) \sqsubseteq_{EM} \beta\}$ satisfies $A(x) \vee_M A(y) \sqsubseteq_{EM} \beta$, for every $x, y \in A^\beta$ and A^β is a sub- meet-semilattice of L.

Lemma 5.3.2. Let $(L, \wedge_L)$ be a meet-semilattice and $(M, \wedge_M, \vee_M)$ be a complete and consistent multilattice. Assume that A^β satisfies $A(x) \vee_M A(y) \sqsubseteq_{EM} \beta$ for every $x, y \in A^\beta$. Then a mapping $A : L \to 2^M$ is an $nd - M - Fuzzy^*$meet-semilattice of L if and only if

$$A(x \wedge_L y) \sqsubseteq_{EM} A(x) \vee_M A(y) \ \forall \ x, y \in L$$

Proof. Assume that A is an $nd - M$- Fuzzy meet-semilattice of L. Then for each $\beta \in 2^M$,the set $A^\beta = \{x \in L | A(x) \sqsubseteq_{EM} \beta\}$ satisfies $A(x) \vee_M A(y) \sqsubseteq_{EM} \beta$ for every $x, y \in A^\beta$ and A^β is a sub meet-semilattice of L. Let $x, y \in L$,and

$$S = A(x) \vee_M A(y)$$

Then $A(x) \sqsubseteq_{EM} S$ and $A(y) \sqsubseteq_{EM} S$ and so $x, y \in A^S$.
But our assumption A^S is a sub meet-semilattice of L,for any $x, y \in A^S$,
$A(x \wedge_L y) \sqsubseteq_{EM} S$.
Thus $A(x \wedge_L y) \sqsubseteq_{EM} A(x) \vee_M A(y)$,for every $x, y \in L$.
Conversely assume that,
$A(x \wedge_L y) \sqsubseteq_{EM} A(x) \vee_M A(y)$, for all $x, y \in L$
Let $S \in 2^M$ and $A(x) \sqsubseteq_{EM} S$ and $A(y) \sqsubseteq_{EM} S$.
Then $A(x) \vee_M A(y) \sqsubseteq_{EM} S$.
Then by our assumption, we have $A(x \wedge_L Y) \sqsubseteq_{EM} A(x) \vee_M A(y) \sqsubseteq_{EM} S$
. Thus $A(x \vee_L y) \sqsubseteq_{EM} S$.
That is $(x \vee_L y) \in A^S$. Hence A^S is a sub meet-semilattice of L. ☐

5.4 nd-M-fuzzy* lattice

Definition 5.4.1. Let $(L, \wedge_{(L)}, \vee_L)$ be a lattice and M be a complete and consistent multilattice with least element 0_M and the greatest element 1_M. A mapping $A : L \to 2^M$ is called an

$nd - M$-fuzzy* lattice if

$$A^\beta = \{x \in L : A(x) \sqsubseteq \beta\}$$

is a sub lattice of L, for every $\beta \in 2^M$

Theorem 5.4.1. *Let L be a lattice and M be a multilattice. Let $A : L \to 2^M$ be a nd-M fuzzy subset. Then*

1. *$A(x_1 \wedge x_2) \sqsubseteq_{EM} A(x_1) \vee_M A(x_2)$ and $A(x_1 \vee_L x_2) \sqsubseteq_{EM} A(x_1) \vee_L A(x_2)$ if and only if A is both $nd - M$ $fuzzy^*$ meet-semilattice and $nd - M$ $fuzzy^*$ join semilattice of L if and only if*
$$A^\beta = \{x \in L : A(x) \sqsubseteq_{EM} \beta\}$$
is a sub-lattice of L for every $\beta \in 2^L$ if and only if A is an nd-M-fuzzy* lattice.

2. *$A(x_1) \wedge_M A(x_2) \sqsubseteq_{EM} A(x_1 \wedge_L x_2)$ and $A(x_1) \wedge_M A(x_2) \sqsubseteq_{EM} A(x_1 \vee_L x_2)$ if and only if $A_\alpha = \{x \in L : \alpha \sqsubseteq_{EM} A(x)\}$ is a sub lattice of L for every $\alpha \in 2^M$ if and only if A is both nd-M fuzzy join semilattice and nd-M fuzzy meet semilattice if and only if A is an $nd - M$ fuzzy Lattice.*

5.5 Strong nd-M-fuzzy lattice

Definition 5.5.1. Let $(L, \wedge_L, \vee_L)$ is a lattice and $(M, \wedge_M, \vee_M)$ is a multilattice with the least element 0_M and the greatest element

1_M. The mapping $A : L \to 2^M$ is called a strong $nd - M$-fuzzy lattice if for each $\alpha, \beta \in 2^M$,the set $A_\alpha^\beta = \{x \in L | \alpha \sqsubseteq_{EM} x \sqsubseteq_{EM} \beta\}$ satisfies,

1. $\alpha \sqsubseteq_{EM} A(x) \wedge_M A(y)$ for every $x, y \in A_\alpha$
2. $A(x) \vee_M A(y) \sqsubseteq_{EM} \beta$,for every $x, y \in A_\alpha$ and $A(y) \sqsubseteq_{EM} \beta$
3. A_α^β is a sub-lattice of L, for all $\alpha, \beta \in 2^M$.

Theorem 5.5.2. *Let $(L, \wedge_L, \vee_L)$ be a lattice and $(M, \wedge_M \vee_M)$ be a complete and consistent multilattice with 0_M and 1_M. Then the mapping $A : L \to 2^M$ is a strong $nd - M$-fuzzy lattice if and only if A satisfies the following conditions, for all $x, y \in L$.*

1. $A(x) \wedge_M A(y) \sqsubseteq_{EM} A(x \wedge_L y) \sqsubseteq_{EM} A(x) \vee_M A(y)$
2. $A(x) \wedge_M A(y) \sqsubseteq_{EM} A(x \vee y) \sqsubseteq_{EM} A(x) \vee_M A(y)$

Proof. Assume that $A : L \to 2^M$ is a strong $nd - M$-fuzzy lattice of L.Then for each $\alpha, \beta \in 2^M$, A satisfies

1. $\alpha \sqsubseteq_{EM} A(x) \wedge_M A(y)$ for every $x, y \in A_\alpha$
2. $A(x) \vee_M A(y) \sqsubseteq_{EM} \beta$,for every $x, y \in A_\alpha$ and $A(y) \sqsubseteq_{EM} \beta$
3. A_α^β is a sub-lattice of L, for all $\alpha, \beta \in 2^M$.

Let $x, y \in L$, $T = A(x) \wedge_M A(y)$ and $S = A(x) \vee_M A(y)$.
Then $T \sqsubseteq_{EM} A(x) \sqsubseteq_{EM} S$ and $T \sqsubseteq_{EM} A(y) \sqsubseteq_{EM} S$.
Hence $x, y \in A_T^S$. That is
$T \sqsubseteq_{EM} A(x \wedge_L y) \sqsubseteq_{EM} S$ and $T \sqsubseteq_{EM} A(x \vee_L y) \sqsubseteq_{EM} S$

Hence $A(x) \wedge_M A(y) \sqsubseteq_{EM} A(x \wedge_L y) \sqsubseteq_{EM} A(x) \vee_M A(y)$
and $A(x) \wedge_M A(y) \sqsubseteq_{EM} A(x \vee_L y) \sqsubseteq_{EM} A(x) \vee_M A(y)$..
Conversely assume that $A : L \longrightarrow 2^M$ satisfies the conditions
$A(X) \wedge_M A(y) \sqsubseteq_{EM} A(x \wedge_L y) \sqsubseteq_{EM} A(x) \vee_M A(Y)$ and
$A(X) \wedge_M A(y) \sqsubseteq_{EM} A(x \vee_L y) \sqsubseteq_{EM} A(x) \vee_M A(Y)$
for every $x, y \in A_T^S$, then
$T \sqsubseteq_{EM} A(x) \sqsubseteq_{EM} S$ and $T \sqsubseteq_{EM} A(x) \sqsubseteq_{EM} S$
Hence $T \sqsubseteq_{EM} A(x) \wedge_M A(y) \sqsubseteq_{EM} S$ and $T \sqsubseteq_{EM} A(x) \vee_M \sqsubseteq_{EM} S$
That is, from our assumption,we have
$T \sqsubseteq_{EM} A(x) \wedge_M A(y) \sqsubseteq_{EM} A(x \wedge_L y) \sqsubseteq_{EM} A(x) \vee_M A(y) \sqsubseteq_{EM} S$
and
$T \sqsubseteq_{EM} A(x) \wedge_M A(y) \sqsubseteq_{EM} A(x \vee_L y) \sqsubseteq_{EM} A(x) \vee_M A(y) \sqsubseteq_{EM} S$
Hence $T \sqsubseteq_{EM} A(x \wedge_L y) \sqsubseteq_{EM} S$ and $T \sqsubseteq_{EM} A(x \vee_L y) \sqsubseteq_{EM} S$.
That is $x \wedge_L y \in A_T^S$ and $x \vee_L y \in A_T^S$,Hence A_T^S is a sub lattice of L and so L is a strong sub lattice of L. □

Theorem 5.5.3. *Let L be a lattice, M be a complete and consistent multilattice and $A_j : L \to 2^M$ be a strong $nd-L$-fuzzy lattice, for each $j \in J$, then $\bigvee_{j \in J}$ and $\bigwedge_{j \in J}$ are $nd-L$-fuzzy lattices.*

Proof.

$$\begin{aligned}
((\bigvee A_j)(x) \wedge_M ((\bigvee A_j)(y)) &= (\bigvee A_j(x)) \wedge_M (\bigvee A_j(y)) \\
&= \bigvee (A_j(x) \wedge_M A_j(y)) \\
&\sqsubseteq_{EM} \bigvee (A_j(x \wedge_L y)) \\
&= (\bigvee A_j)(x \wedge y) \\
&= \bigvee (A_j(x \wedge y)) \\
&\sqsubseteq_{EM} \bigvee ((A_j)(x) \vee_M A_j(y)) \\
&= (\bigvee ((A_j)(x)) \vee_M (\bigvee ((A_j)(y))
\end{aligned}$$

Therefore $((\bigvee A_j)(x) \wedge_M ((\bigvee A_j)(y)) \sqsubseteq_{EM} \bigvee (A_j(x \wedge_L y))$
$\sqsubseteq_{EM} ((\bigvee A_j)(x)) \vee_M ((\vee(A_j)(y))$.
Similarly

$$\begin{aligned}
((\bigvee A_j)(x) \wedge_M (\bigvee A_j)(y) &= (\bigvee A_j)(x)) \wedge_M (\bigvee A_j(y)) \\
&= \bigvee (A_j)(x) \wedge_M A_j(y)) \\
&\sqsubseteq_{EM} (\bigvee A_j)(x \vee_L y) \\
&= \bigvee A_j)(x \vee_M y) \\
&= \bigvee (A_j(x \vee y)) \\
&\sqsubseteq_{EM} \vee (A_j(x) \vee_M A_J(y)) \\
&= (\bigvee ((A_j)(x)) \vee_M (\bigvee ((A_j)(y)) \\
&= ((\bigvee A_j)(x)) \vee_M ((\bigvee A_j)(y))
\end{aligned}$$

Therefore

$$((\bigvee A_j)(x)) \wedge_M ((\bigvee A_j)(y)) \sqsubseteq_{EM} ((\bigvee A_j)(x \vee_L Y)$$

$$\sqsubseteq_{EM} ((\bigvee A_j)(x)) \vee_M ((\vee A_j)(y))$$

Hence the expressions from (1) and (2) together implies $(\bigvee(A_j)$ is a $nd - M-$fuzzy lattices,
Similarly,

$$\begin{aligned} ((\bigwedge A_j)(x)) \wedge_M ((\wedge A_j)(y)) &= (\bigwedge A_j(x)) \wedge_M (\bigwedge A_j(y)) \\ &= \bigwedge(A_j(x) \wedge_L A_j(y)) \\ &\sqsubseteq_{EM} \bigwedge(A_j(x \wedge_L y)) \\ &=\sqsubseteq_{EM} \bigwedge A_j(x \vee_L y) \\ &= \bigwedge(A_j(x \wedge_L y)) \\ &\sqsubseteq_{EM} \bigwedge(A_j(x) \vee_M A_j(y)) \\ &\sqsubseteq_{EM} (\bigwedge A_j(x)) \vee_M (\bigwedge A_j(y)) \\ &= ((\bigwedge(A_j)(x)) \vee_M ((\bigwedge A_j)(y)) \end{aligned}$$

Therefore

$$\begin{aligned} ((\bigwedge(A_j)(x)) \wedge_M ((\bigwedge A_j)(y)) &\sqsubseteq_{EM} (\bigwedge A_j)(x \wedge_L y) \\ &\sqsubseteq_{EM} (\bigwedge A_j)(x)) \vee_M ((\bigwedge A_j)(y)) \end{aligned}$$

similarly

$$\begin{aligned}((\bigwedge A_j)(x)) \wedge_M ((\bigwedge A_j)(y)) &= (\bigwedge A_j(x)) \wedge_M (\bigwedge A_j(y)) \\ &= \bigwedge(A_j(x) \wedge_M A_j(y)) \\ &\sqsubseteq_{EM} \bigwedge(A_j(x \vee_L y)) \\ &= (\bigwedge A_j)(x \vee_L y)) \\ &= \bigwedge(A_j(x \vee_L y) \\ &\sqsubseteq_{EM} \bigwedge(A_j(x)) \vee_M (A_j(y)) \\ &= (\bigwedge A_j(x)) \vee_M (\bigwedge A_j(y)) \\ &= ((\bigwedge A_j)(x)) \vee_M ((\bigwedge A_j)(y))\end{aligned}$$

Therefore

$$((\bigwedge A_j)(x)) \wedge_M ((\bigwedge A_j)(y)) \sqsubseteq_{EM} (\bigwedge A_j)(x \vee_L y)$$

$$\sqsubseteq_{EM} ((\bigwedge A_j)(x)) \vee_L ((\bigwedge A_j)(y))$$

Hence from the expression above we have $(\wedge A_j)$ is an $nd-M-$fuzzy lattices. □

Chapter 6

Matrices over Multilattices

Let M be a complete ,consistent and distributive multilattice with 0 and 1. The multisup(a,b) is denoted by $a + b$ and multiinf(a,b) is denoted by $a.b$. Recall that multisuprimum and multiinfimum of elements are set of elements in M. In a lattice matrix each entries of a matrix are single elements. Here we are taking a set of elements to each entry of a matrix from a multilattice M instead of taking a single elements.As defined in the lattice matrix [50], here we are defining matrices over a Multilattice along with some basic concepts and properties of these matrices are studied.

In this chapter we use 0 and 1 for bottom and top element respectively in a multilattice M instead of using 0_M and 1_M.

6.1 Definition and some properties

Definition 6.1.1. Let M be a complete, consistent and distributive multilattice with 0 and 1.The multisup(a,b) is denoted by $a+b$ and multiinf(a,b) is denoted by $a.b$ Let M_n (for $n > o$) be the set of $n \times n$ matrices over M.
ie,

$$M_n = \{A = (a_{ij})/a_{ij} \in 2^M\}$$

, a_{ij} is the $(ij)^{th}$ element of A.

Definition 6.1.2. Let $A, B \in M_n$, we define

1. $A + B = C$ if and only if $c_{ij} = a_{ij} + b_{ij}$
2. $A \sqsubseteq_{EM} B$ if and only if $a_{ij} \sqsubseteq_{EM} b_{ij}$
3. $A \wedge_M B = C$ if and only if $c_{ij} = a_{ij}.b_{ij}$
4. $A.B = AB = C$ If and only $c_{ij} = \sum\limits_{k=1}^{n} a_{ik}b_{kj}$
5. $A^T = C$ if and only if $c_{ij} = a_{ji}$
6. For $a \in M$, $aA = a.A = C$ if and only if $c_{ij} = a.a_{ij}$
7. $I = (a_{ij})$, where $a_{ij} \;= \{1\}$ for $i = j$
   and $\qquad = \{0\}$ for $i \neq j$
8. $A^0 = I$,$A^{k+1} = A^k.A$,

9. $O = (o_{ij})$,where $o_{ij} = 0$ for every i and j.

10. $E = (e_{ij})$, where $e_{ij} = \{1\}$ for every i and j

Example 6.1.1. Consider the multilattice in Figure 3.1. Let

$$A = \begin{bmatrix} \{a\} & \{1\} \\ \{b\} & \{0\} \end{bmatrix}, B = \begin{bmatrix} \{b\} & \{d\} \\ \{a\} & \{1\} \end{bmatrix}$$

$$A + B = \begin{bmatrix} \{a+b\} & \{1+d\} \\ \{b+a\} & \{0+1\} \end{bmatrix}$$

$$= \begin{bmatrix} \{c,d\} & \{1\} \\ \{c,d\} & \{1\} \end{bmatrix}$$

$$A \Lambda B = \begin{bmatrix} \{a.b\} & \{1.d\} \\ \{b.a\} & \{0.1\} \end{bmatrix} = \begin{bmatrix} \{0\} & \{d\} \\ \{0\} & \{0\} \end{bmatrix}$$

$$AB = \begin{bmatrix} \{0+a\} & \{a+1\} \\ \{b+0\} & \{b+0\} \end{bmatrix} = \begin{bmatrix} \{a\} & \{1\} \\ \{b\} & \{b\} \end{bmatrix}$$

properties with respect to addition and multiplication:

1. $A + A \neq A$

2. $A + B = B + A$

3. $(A + B) + C = A + (B + C)$

4. $AB \neq BA$

5. $(AB)C = A(BC)$

6. $A.I = I.A = A$

7. $A.O = O.A = O$

8. $A^p.A^q = A^{p+q}$

9. $(A^p)^q = A^{pq}$

10. $A(B + C) = AB + AC$

11. $(A + B)C = AC + BC$

12. if $A \sqsubseteq_{EM} B$ and $C \sqsubseteq_{EM} D$ then $AC \sqsubseteq_{EM} BD$

13. Let $E = (e_{ij})$, where $e_{ij} = \{1\}$ for every i and j and
    $I = (a_{ij})$, where $a_{ij} = \{1\}$ for $i = j$ and
    $= \{0\}$ for $i \neq j$
    Let $A = (a_{ij}$ be any matrix over a multilattice.
    Now if $I \sqsubseteq_{EM} A$ and $A \sqsubseteq_{EM} I$ then $I = A$.
    Also if $A \sqsubseteq_{EM} E$ and $E \sqsubseteq_{EM} A$, then $E = A$.

properties of transposition

1. $(A + B)^T = A^T + B^T$

2. if $A \sqsubseteq_{EM} B$ then $A^T \sqsubseteq_{EM} B^T$

3. $(A \wedge_M B)^T = A^T \wedge_M B^T$

4. $(A^T)^T = A$

Definition 6.1.3. For $\alpha \in 2^M$ we shall use the notation

$\alpha \hookrightarrow (A^k)_{ij}$, the ij^{th} entry of A^k

whenever $\alpha = a_{i_0 i_1}.a_{i_1 i_2}.\cdots.a_{i_{k-1} i_k}$,

where $i_0 = i$ and $i_k = j$ for some $i_1, i_2, \cdots, i_{k-1}$

Note 6.

$$(A^k)_{ij} = \sum_{\alpha \hookrightarrow (A^k)_{ij}} \alpha$$

Proposition 6.1.1. If $\alpha \hookrightarrow (A^k)_{ij}$, where $k \geq n$, then there are integers m_1 ,m_2 ,m_3 and ν (all of them dependent on α) such that
$0 \leq m_2 \leq n$, $m_1 + m_2 + m_3 = k$, $1 \leq \gamma \leq n$ and such that for each positive integer m:

$$\alpha \sqsubseteq_{EM} (A^{m_1})_{i\gamma}.(A^{m.m_2})_{\gamma\gamma}.(A^{m_3})_{\gamma j}$$

Proof. Let $\alpha = a_{i_0 i_1}.a_{i_1 i_2}.\cdots.a_{i_{k-1} i_k}$, Where $\alpha \in 2^M$.
Since $n \leq k$, Then $n \leq k+1$, two indices among the k+1 indices $i_0, i_1, \cdots, i_k$ must be equal. Let $i_r = i_s$, where $r < s$.
Also we can find such r and s such that $i_r = i_s, r < s$ and $s - r \leq n$.
So let $m_1 = r, m_2 = s - r, m_3 = k - s$ and $\nu = i_r = i_s$

□

Corollary 6.1.4. *If* $\alpha \hookrightarrow (A^k)_{ij}$ *where* $k \geq n$ *then there are natural numbers* m_1, m_2, m_3 *and* γ *such that* $m_1 + m_2 \leq n$ *,* $0 \leq m_2 \leq n$*,* $1 \leq \gamma \leq n$ *and such that for each* m

$$\alpha \sqsubseteq_{EM} (A^{m_1})_{i\gamma}.(A^{m.m_2})_{\gamma\gamma}.(A^{m_3})_{\gamma j}$$

Theorem 6.1.5. *If* $k \geq n$ *then* $(A^k)_{ij} \sqsubseteq_{EM}$ *multisup*$(A^{k+(p.n!)})_{ij}$ *where* p *is an arbitrary number.*

Proof. suppose $\alpha \hookrightarrow (A^k)_{ij}$.Then by the above proposition , there are natural numbers m_1, m_2, m_3 and γ(all of them dependent on α) such that

$0 < m_2 \leq n$, $m_1 + m_2 + m_3 = k$,$1 \leq \gamma \leq n$ and such that for each m,

$\alpha \sqsubseteq_{EM} (A^{m_1})_{i\gamma}.((A^{m.m_1})_{\gamma\gamma}.(A^{m_2})_{\nu j}$

Hence $\alpha \sqsubseteq_{EM} (A^{m_1+m.m_2+m_3})_{ij}$

$$= (A^{k+(m-1).m_2})_{ij}$$

Replace $(m-1)$ by $(p.n!/m_2)$ where p is an arbitrary natural number.

Then $\alpha \sqsubseteq_{EM} (A^{k+(p.n!/m_2).m_2})_{ij}$

$$= (A^{K+pn!})_{ij}$$

Then all $\alpha' s$ such that

$$\sum_{\alpha \hookrightarrow (A^k)_{ij}} \alpha = (A^k)_{ij}$$

Then $\sum_{\alpha \hookrightarrow (A^k)_{ij}} \alpha \sqsubseteq_{EM} Multisup(A^{k+pn!})_{ij}$

This implies $(A^k)_{ij} \sqsubseteq_{EM} Multisup(A^{k+pn!})_{ij}$

□

6.2 Orthogonal Matrices

Definition 6.2.1. A M_n Matrix A is called a unit if and only if there is an M_n matrix B such that $AB = BA = I$, and A is called orthogonal if and only if $AA^T = A^TA = I$

Proposition 6.2.1. 1. If $CB = E$ then $EB = E$

2. If $EAB = E$ then $EB = E$

3. Assume $A \wedge_M A = A$,If $EA = E$ if and only if $I \sqsubseteq_{EM} A^TA$

Proof. 1. For any matrix $EB \sqsubseteq_{EM} E$ and $C \sqsubseteq_{EM} E$ are always true. Therefore by the property 12, $CB \sqsubseteq_{EM} EB$.

But $CB = E$ implies $E \sqsubseteq_{EM} EB$. Thus $EB \sqsubseteq_{EM} E$ and $E \sqsubseteq_{EM} EB$,this implies $E = B$.

2. This proof is a particular case of 1

3. Let $A \wedge_M A = A$ $.EA = E$ holds if and only if for each i and j,
   $\{1\} = (EA)_{ij} = \sum_{k=1}^{n} e_{ik} a_{kj}$

   $= \sum_{k-1}^{n} a_{kj}$, since $e_{ik} = \{1\}$

   $= \sum_{k=1}^{n} a_{kj} . a_{kj}$

   $= \sum_{k=1}^{n} (A^T)_{jk} . A_{kj}$

   $= (A^T.A)_{jj}$, that is each diagonal entries are $\{1\}$.

   Hence $EA = E$ holds if and only if $I \sqsubseteq_{EM} (A^T.A)$ holds.

□

Note 7. from the above proposition we have ,$I \sqsubseteq_{EM} A^T A \Longrightarrow EA = E$, since $I \sqsubseteq_{EM} A^T A \Longrightarrow EI \sqsubseteq_{EM} EA^T A$,
that is $I \sqsubseteq_{EM} A^T A$ implies $EA^T A = E$.

Proposition 6.2.2. If A is a unit then A is orthogonal.

Proof. If A is a unit then there is a B such that $AB = BA = I$ .This implies $B^T A^T = A^T B^T = I$.
Hence $E = EAB = EBA = EB^T A^T = EA^T B^T$ and therefore by above proposition, we have $I \sqsubseteq_{EM} A^T A$,$I \sqsubseteq_{EM} AA^T$, $I \sqsubseteq_{EM} B^T B$, $I \sqsubseteq_{EM} BB^T$

Then to show That $A^T A \sqsubseteq_{EM} I$ and $AA^T \sqsubseteq_{EM} I$
That is to show that $A^T A \sqsubseteq_{EM} BA$ and $AA^T \sqsubseteq_{EM} AB$ since $AB = I$ and $BA = I$
for this, it is suffices to show that $A^T \sqsubseteq_{EM} B$ holds.

but $I \sqsubseteq_{EM} B^T B \Longrightarrow A^T \sqsubseteq_{EM} A^T B^T B$

since $A^T B^T = I$ and therefore $A^T \sqsubseteq_{EM} B$ holds.

Therefore $A^T A \sqsubseteq_{EM} I$ and $AA^T \sqsubseteq_{EM} I$.

This implies $A^T A = I$, A is orthogonal.

□

Definition 6.2.2. 1. A set $\{S_1, S_2, ..., S_n\}$ of subsets of M is a decomposition of $\{1\}$ in 2^M if and only if $\sum_{k=1}^{n} S_k = \{1\}$.

That is $Multisup\{\{S_1, S_2, ..., S_n\} = \{1\}$

2. A set $\{S_1, S_2, ..., S_n\}$ of subsets of M is said to be orthogonal if and only if $S_i Sj = \{0\}$
That is $multiinf\{SiSj\} = 0$

3. A set of subsets of M is an orthogonal decomposition of $\{ 1 \}$ in 2^M if and only if it is orthogonal and a decomposition of $\{1\}$ in 2^M

We know that $I \sqsubseteq_{EM} A^T A$, $I \sqsubseteq_{EM} A^T A$ implies $EA = E$.

Since A is orthogonal $AA^T = A^T A = I$ implies

$AA^T \sqsubseteq_{EM} I$, $A^T A \sqsubseteq_{EM} I$ $I \sqsubseteq_{EM} A^T A$ and $I \sqsubseteq_{EM} AA^T$.

Also $EA = E \Longrightarrow EA^T = E$

From this the following proposition follows.

Proposition 6.2.3. A M_n is orthogonal if and only if each row and each column of it is an orthogonal decomposition of $\{1\}$ in 2^M

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