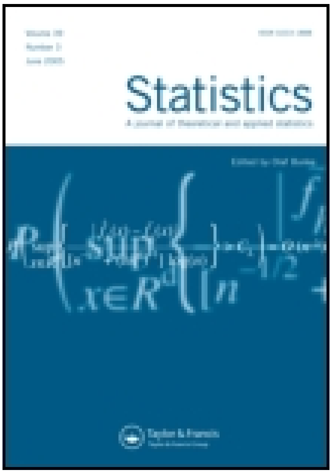

Statistics: A Journal of Theoretical and Applied Statistics
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/gsta20

Dynamic cumulative residual Renyi's entropy

S. M. Sunoj [a] & M. N. Linu [a]
[a] Department of Statistics , Cochin University of Science and Technology , Cochin , 682 022 , Kerala , India
Published online: 12 Jul 2010.

Statistics, Vol. 46, No. 1, February 2012, 41–56

Taylor & Francis
Taylor & Francis Group

Dynamic cumulative residual Renyi's entropy

S.M. Sunoj* and M.N. Linu

Department of Statistics, Cochin University of Science and Technology, Cochin 682 022, Kerala, India

(*Received 3 September 2009; final version received 2 May 2010*)

Recently, cumulative residual entropy (CRE) has been found to be a new measure of information that parallels Shannon's entropy (see Rao *et al*. [*Cumulative residual entropy: A new measure of information*, IEEE Trans. Inform. Theory. 50(6) (2004), pp. 1220–1228] and Asadi and Zohrevand [*On the dynamic cumulative residual entropy*, J. Stat. Plann. Inference 137 (2007), pp. 1931–1941]). Motivated by this finding, in this paper, we introduce a generalized measure of it, namely cumulative residual Renyi's entropy, and study its properties. We also examine it in relation to some applied problems such as weighted and equilibrium models. Finally, we extend this measure into the bivariate set-up and prove certain characterizing relationships to identify different bivariate lifetime models.

Keywords: cumulative residual entropy; Renyi's entropy; weighted distributions; characterization

AMS Subject Classifications: 62N05; 90B25

1. Introduction

In recent years, there has been a great interest in the measurement of uncertainty of probability distributions. It is well known that Shannon's entropy plays an important role in this as a quantitative measure of it and has been widely used in many areas of research. Let X be a non-negative random variable (rv) having an absolutely continuous cumulative distribution function (cdf) $F(x)$ with probability density function (pdf) $f(x)$. Then, Shannon's entropy of the rv X is defined as

$$H(X) = H(f) = -\int_0^\infty f(x)\log f(x)\,\mathrm{d}x. \tag{1}$$

Suppose X represents the lifetime of a unit, then $H(f)$ can be useful for measuring the associated uncertainty. However, if a unit has survived to an age t, then information about the remaining lifetime is an important component in many fields such as reliability, survival analysis, economics, business, etc., where $H(f)$ is not a useful tool for measuring the uncertainty about remaining lifetime of the unit. Accordingly, Ebrahimi and Pellerey [1] proposed a new measure of uncertainty

*Corresponding author. Email: smsunoj@cusat.ac.in

ISSN 0233-1888 print/ISSN 1029-4910 online

http://dx.doi.org/10.1080/02331888.2010.494730
http://www.tandfonline.com

called residual entropy, given by

$$H(f;t) = H(X;t) = -\int_t^{\infty} \frac{f(x)}{\bar{F}(t)} \left(\log \frac{f(x)}{\bar{F}(t)} \right) \mathrm{d}x, \tag{2}$$

where $\bar{F}(t) = 1 - F(t)$ is the survival function (sf). After the unit has elapsed time t, $H(f;t)$ measures the expected uncertainty contained in the conditional density of $X - t$ given $X > t$ about the predictability of remaining lifetime of the unit. For a survey of the literature on residual entropy and its applications, we refer to Asadi and Ebrahimi [2], Asadi *et al.* [3] and Sunoj *et al.* [4].

Even if Shannon's entropy (1) finds applications in many areas of research, recently, Rao *et al.* [5] identified some limitations of the use of Equation (1) in measuring the randomness of certain systems (see also Rao [6]) and introduced an alternate measure of uncertainty called cumulative residual entropy (CRE). This measure is based on the cdf $F(x)$ and is defined in the univariate case and for the non-negative rvs as follows:

$$\xi(X) = -\int_0^{\infty} \bar{F}(x) \log \bar{F}(x)\, \mathrm{d}x. \tag{3}$$

Clearly, $\xi(X)$ measures the uncertainty contained in the cdf of X. Motivated with the salient features of Equation (3) proposed by Rao *et al.* [5], Asadi and Zohrevand [7] further studied it to the residual set-up called a dynamic cumulative residual entropy (DCRE), given by

$$\xi(X;t) = -\int_t^{\infty} \frac{\bar{F}(x)}{\bar{F}(t)} \log \frac{\bar{F}(x)}{\bar{F}(t)}\, \mathrm{d}x.$$

Like $H(f;t)$, $\xi(X;t)$ measures the uncertainty or randomness contained in the conditional sf of $X - t$ given $X > t$ about the predictability of remaining lifetime of the unit.

There are several generalizations on the information measures such as Shannon's entropy (1). Of these, an important one being Renyi's entropy introduced by Renyi [8], which is also a measure to quantify diversity, uncertainty or randomness of a system. If X is an absolutely continuous rv with a pdf $f(x)$, then Renyi's entropy of order β is defined as

$$I_{\mathrm{R}}(\beta) = \frac{1}{1-\beta} \log \left(\int f^{\beta}(x)\, \mathrm{d}x \right) \quad \text{for} \quad \begin{matrix} \beta \neq 1 \\ \beta > 0 \end{matrix}$$

and

$$I_{\mathrm{R}}(1) = \lim_{\beta \to 1} I_{\mathrm{R}}(\beta) = -\int f(x) \log f(x)\, \mathrm{d}x.$$

$I_R(\beta)$ plays a vital role in different areas such as physics, electronics, engineering, ecology and statistics as a measure of uncertainty and diversity.

Similar to the definition of residual entropy (2), Abraham and Sankaran [9] recently extended Renyi's entropy of order β for the residual lifetime $X - t | X > t$ as

$$I_{\mathrm{R}}(\beta;t) = \frac{1}{1-\beta} \log \int_t^{\infty} \frac{f^{\beta}(x)}{\bar{F}^{\beta}(t)}\, \mathrm{d}x \quad \text{for} \quad \begin{matrix} \beta \neq 1 \\ \beta > 0 \end{matrix}. \tag{4}$$

When the system has the age t, for different values of β, $I_{\mathrm{R}}(\beta;t)$ provides the spectrum of Renyi's information of the remaining life of the system. Obviously, $I_{\mathrm{R}}(\beta;0) = I_{\mathrm{R}}(\beta)$. For more properties and applications and recent developments of Equation (4), we refer to Abraham and Sankaran [9], Asadi *et al.* [3] and Maya and Sunoj [10].

Motivated with the usefulness of Renyi's entropy of order β and CRE for measuring uncertainty, in this paper, we introduce a new measure of uncertainty, namely cumulative Renyi's entropy of order β. We further extend it to the residual time and study its various properties useful in reliability modelling. The rest of the paper is organized as follows. Section 2 includes the definition and properties of dynamic cumulative residual Renyi's entropy (DCRRE) and some characterization theorems arising out of it. In Section 3, we examine DCRRE in the context of weighted and equilibrium distributions and study its various relationships. Finally, Section 4 introduces DCRRE in the bivariate case and proves certain characterizations based on it.

2. Dynamic cumulative residual Renyi's entropy

Analogous to the definition of cumulative entropy (3) by Rao *et al.* [5], in present section, we define the cumulative Renyi's entropy and DCRRE.

DEFINITION 2.1 *For a non-negative rv X with an absolutely continuous sf $\bar{F}(x)$, the cumulative Renyi's entropy of order β is defined as*

$$\gamma(\beta) = \frac{1}{1-\beta} \log\left(\int_0^\infty \bar{F}^\beta(x)\,dx\right) \quad \text{for} \quad \begin{matrix} \beta \neq 1 \\ \beta > 0 \end{matrix}. \tag{5}$$

When $\beta \to 1$, Equation (5) reduces to

$$\gamma(1) = \lim_{\beta\to 1} \gamma(\beta) = -\int_0^\infty \bar{F}(x) \log \bar{F}(x)\,dx,$$

which is the cumulative entropy (3) *and hence possess all the properties discussed in Rao et al.* [5]. *However, in many life-testing experiments, frequently, one has information about the current age of the systems under consideration. Studying the effects of the age t of an individual or an item under study on the information about the remaining lifetime is important in many applications. In such situations, either Equation* (5) *or* (3) *is not suitable and therefore it should be modified to take the current age into account. Information measures that include the age are functions of t and hence called as dynamic. So, we define a DCRRE as follows.*

DEFINITION 2.2 *For a non-negative rv X with an absolutely continuous sf $\bar{F}(x)$, DCRRE of order β denoted by $\gamma(\beta; t)$ is defined as*

$$\gamma(\beta; t) = \frac{1}{1-\beta} \log\left(\int_t^\infty \frac{\bar{F}^\beta(x)}{\bar{F}^\beta(t)}\,\mathrm{d}x\right) \quad \text{and} \quad \begin{matrix} \beta \neq 1 \\ \beta > 0 \end{matrix}, \tag{6}$$

which can be written as

$$(1-\beta)\gamma(\beta; t) = \log\left(\int_t^\infty \bar{F}^\beta(x)\,\mathrm{d}x\right) - \beta \log \bar{F}(t). \tag{7}$$

Differentiating Equation (7) *with respect to t, we have*

$$(1-\beta)\gamma'(\beta; t) = \beta h(t) - \mathrm{e}^{-(1-\beta)\gamma(\beta;t)}, \tag{8}$$

where $\gamma'(\beta; t)$ denotes the derivative of $\gamma(\beta; t)$ with respect to t and $h(t) = f(t)/\bar{F}(t)$ is the hazard rate of X. Obviously, when a system has completed t units of time, for different values of β, $\gamma(\beta; t)$ gives Renyi's information for the remaining life of the system. Also, $\gamma(\beta; 0) = \gamma(\beta)$.

Remark 2.1 The variation of DCRRE of order β can be obtained from the following example.

Suppose that X follows exponential distribution with mean $\frac{1}{2}$. Then, $\gamma(\beta; t) = (1/(\beta - 1))\log 2\beta$. Clearly, for $\beta > 1$, $\gamma(\beta; t)$ is positive, whereas for $0.5 < \beta < 1$, $\gamma(\beta; t)$ is negative. When $\beta = 1/2$, $\gamma(\beta; t)$ is zero.

Examples (a) If X is distributed uniformly on $(0, a)$, then it can be easily shown that $(1 - \beta)\gamma(\beta; t) = \log((a - t)/(\beta + 1))$.

(b) When X follows Pareto I with sf $\bar{F}(t) = (k/t)^c, t > k, c, k > 0$, then $(1 - \beta)\gamma(\beta; t) = \log(t/(c\beta - 1))$.

(c) When X is distributed as Weibull distribution with sf $\bar{F}(t) = \mathrm{e}^{-t^p}, t > 0, p > 0$, then it can be shown that $(1 - \beta)\gamma(\beta; t) = \log((\beta^{-1/p}/p\mathrm{e}^{-\beta t^p})\Gamma((1/p), \beta t^p))$.

In the following theorem, we show that DCRRE determines $\bar{F}(t)$ uniquely.

Theorem 2.1 *Let X be a non-negative rv having an absolutely continuous sf $\bar{F}(t)$ and hazard rate $h(t)$ with $\gamma(\beta; t) < \infty; t \geq 0;\ \beta > 0, \beta \neq 1$. Then for each β, $\gamma(\beta; t)$ uniquely determines $\bar{F}(t)$.*

Proof Let $\bar{F}_1(t)$ and $\bar{F}_2(t)$ be two sfs with DCRRE $\gamma_1(\beta; t)$ and $\gamma_2(\beta; t)$ and failure rates $h_1(t)$ and $h_2(t)$, respectively. Now $\gamma_1(\beta; t) = \gamma_2(\beta; t)$ implies that

$$\gamma_1'(\beta; t) = \gamma_2'(\beta; t),$$

which is equivalent to

$$(1 - \beta)\gamma_1'(\beta; t) = (1 - \beta)\gamma_2'(\beta; t) \tag{9}$$

Using Equation (8), Equation (9) becomes

$$\beta h_1(t) - \mathrm{e}^{-(1-\beta)\gamma_1(\beta;t)} = \beta h_2(t) - \mathrm{e}^{-(1-\beta)\gamma_2(\beta;t)}. \tag{10}$$

But $\gamma_1(\beta; t) = \gamma_2(\beta; t)$, Equation (10) then reduces to

$$\beta h_1(t) = \beta h_2(t),$$

which implies that $h_1(t) = h_2(t)$, or equivalently $\bar{F}_1(t) = \bar{F}_2(t)$. ■

Theorem 2.2 *For the rv X considered in Theorem* 2.1, *the relationship*

$$(1 - \beta)\gamma'(\beta; t) = Ch(t), \tag{11}$$

where C is a constant, holds if and only if X is distributed as

(a) Pareto II distribution with sf

$$\bar{F}(t) = (1 + pt)^{-q};\quad p > 0,\ q > 0,\ t > 0, \tag{12}$$

(b) exponential distribution with sf

$$\bar{F}(t) = \mathrm{e}^{-\lambda t};\quad \lambda > 0,\ t > 0, \tag{13}$$

(c) finite range distribution with sf

$$\bar{F}(t) = (1 - at)^b;\quad a > 0,\ b > 0,\ 0 < t < \frac{1}{a}, \tag{14}$$

according as $C \gtreqless 0$.

Proof Assume that the relationship (11) holds. Using Equation (8), Equation (11) becomes

$$\beta h(t) - \mathrm{e}^{-(1-\beta)\gamma(\beta;t)} = Ch(t),$$

which is equivalent to

$$(\beta - C)h(t) = \mathrm{e}^{-(1-\beta)\gamma(\beta;t)}. \tag{15}$$

Using the expression of DCRRE in Equation (6), Equation (15) becomes

$$(\beta - C)f(t)\int_t^\infty \bar{F}^\beta(x)\,\mathrm{d}x = \bar{F}^{\beta+1}(t). \tag{16}$$

Differentiating Equation (16) with respect to t, we get

$$(\beta - C)f'(t)\int_t^\infty \bar{F}^\beta(x)\,\mathrm{d}x - (\beta - C)f(t)\bar{F}^\beta(t) = -(\beta+1)\bar{F}^\beta(t)\,f(t). \tag{17}$$

Using Equation (16), Equation (17) becomes

$$f'(t)\frac{\bar{F}^{\beta+1}(t)}{f(t)} - (\beta - C)\,f(t)\,\bar{F}^\beta(t) = -(\beta+1)\bar{F}^\beta(t)\,f(t). \tag{18}$$

Dividing Equation (18) by $f(t)\bar{F}^\beta(t)$ and simplifying yield $(\mathrm{d}/\mathrm{d}t)\log f(t) = (C+1)(\mathrm{d}/\mathrm{d}t)\log \bar{F}(t)$, which implies

$$\frac{\mathrm{d}}{\mathrm{d}t}\log h(t) = C\frac{\mathrm{d}}{\mathrm{d}t}\log \bar{F}(t). \tag{19}$$

Integrating Equation (19) with respect to t, we get

$$\log h(t) = C\log \bar{F}(t) + K, \tag{20}$$

where K is the constant of integration. Now differentiating Equation (20) with respect to t, we obtain $h'(t)/h(t) = -C(f(t)/\bar{F}(t))$, or

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{1}{h(t)}\right] = C. \tag{21}$$

Integrating Equation (21) with respect to t, we obtain $1/h(t) = Ct + l$, where $l > 0$ is the constant of integration, or equivalently $h(t) = 1/(Ct + l)$. Since the hazard rate uniquely determines sf using the relationship $\bar{F}(t) = \exp\left(-\int_0^t h(x)\,\mathrm{d}x\right)$, the models (12)–(14) follows according as $C \gtreqless 0$.

To prove the converse part, first assume that X is distributed as Pareto II with sf (12), now using Equation (7), we have

$$(1-\beta)\gamma(\beta;t) = \log\left[\frac{1+pt}{p(q\beta-1)}\right] = \log(1+pt) + \log\left[\frac{1}{p(q\beta-1)}\right],$$

which on differentiation yields $(1-\beta)\gamma'(\beta;t) = p/(1+pt) = Ch(t)$ with $h(t) = pq/(1+pt)$and $C = 1/q > 0$, which follows Equation (11). When $X$is distributed as exponential with sf (13), we have $(1-\beta)\gamma(\beta;t) = \log(1/\lambda\beta)$ from which Equation (11) follows with $C = 0$.When X is distributed as finite range with sf (14), we get $(1-\beta)\gamma'(\beta;t) = -a/(1-at) = Ch(t)$, with $h(t) = ab/(1-at)$ and $C = -1/b < 0$, which yield Equation (11). ■

THEOREM 2.3 *For a non-negative rv X with an absolutely continuous sf $\bar{F}(t)$ and mean residual life function $r(t) = E(X - t|X > t)$, the relationship*

$$(1-\beta)\gamma(\beta; t) = \log[Cr(t)], \tag{22}$$

holds if and only if X is distributed as with sf (12), (13) or (14) according as $(C\beta - 1)/(C(1-\beta)) \gtreqless 0$.

Proof Assume that the relationship (22) holds, then

$$(1-\beta)\gamma(\beta; t) = \log C + \log r(t). \tag{23}$$

Differentiating Equation (23) with respect to t, we get

$$(1-\beta)\gamma'(\beta; t) = \frac{r'(t)}{r(t)}. \tag{24}$$

Using Equation (8), Equation (24) becomes

$$\frac{r'(t)}{r(t)} = \beta h(t) - \mathrm{e}^{-(1-\beta)\gamma(\beta;t)}. \tag{25}$$

Using Equation (22), Equation (25) becomes

$$\frac{r'(t)}{r(t)} = \beta h(t) - \frac{1}{Cr(t)},$$

which is equivalently

$$Cr'(t) = \beta Cr(t)h(t) - 1.$$

Now using the relationship between $h(t)$ and $r(t)$, the above expression becomes $Cr'(t) = \beta C(r'(t)+1) - 1$. Equivalently, $r'(t) = (C\beta - 1)/(C(1-\beta)) = P$, a constant. This implies that $r(t) = Pt + Q$, where Q is the constant of integration, which is a characterization to the models (12)–(14) according as $P \gtreqless 0$. The converse part is quite straightforward. ■

DEFINITION 2.3 *The sf $\bar{F}(t)$ is increasing (decreasing) β-order dynamic cumulative residual Renyi's entropy IDCRRE (DDCRRE) if $\gamma(\beta; t)$ is increasing (decreasing) in t; $t > 0$, i.e. $\bar{F}(t)$ have IDCRRE (DDCRRE) if $\gamma'(\beta; t) \geq (\leq) 0$. $\bar{F}(t)$ is both ICRRE and DCRRE if $\gamma'(\beta; t) = 0$.*

Examples If X is distributed uniformly on $(0, a)$, then $\bar{F}(t)$ is IDCRRE for $\beta > 1$ and DDCRRE for $0 < \beta < 1$. When X is distributed as Pareto II with sf (12), then $\bar{F}(t)$ is IDCRRE for $0 < \beta < 1$ and DDCRRE for $\beta > 1$.

THEOREM 2.4 *$\bar{F}(t)$ is both IDCRRE and DDCRRE if and only if X follows exponential distribution.*

COROLLARY 2.1 *DCRRE is constant if and only if X is exponentially distributed.*

3. Weighted dynamic cumulative residual Renyi's entropy

The concept of weighted distributions is usually considered in connection with modelling statistical data, where the usual practice of employing standard distributions is not found appropriate. A survey of research in various fields of applications is available in Di Crescenzo and Longobardi [11], Nair and Sunoj [12], Sunoj and Maya [13] and Maya and Sunoj [10]. If X is an absolutely continuous non-negative rv with pdf $f(t)$ and sf $\bar{F}(t)$, then the pdf $f_{\mathrm{w}}(t)$ and sf $\bar{F}_{\mathrm{w}}(t)$ for the weighted rv X_{w} associated to X and to a positive real function $w(\cdot)$ are defined by

$$f_{\mathrm{w}}(t) = \frac{w(t)f(t)}{E(w(X))} \tag{26}$$

and

$$\bar{F}_{\mathrm{w}}(t) = \frac{E(w(X)\,|X > t)}{E(w(X))}\bar{F}(t), \tag{27}$$

where $Ew(X) < \infty$. When the weight function is proportional to lengths of units of interest (i.e. $w(t) = t$), then the model (26) is known as length-biased model with rv denoted by X_L. Analogous to the definition of DCRRE in Equation (6), the weighted cumulative residual Renyi's entropy denoted by $\gamma_{\mathrm{w}}(\beta; t)$ is defined as

$$\gamma_{\mathrm{w}}(\beta; t) = \frac{1}{1-\beta}\log\left(\int_t^\infty \frac{\bar{F}_{\mathrm{w}}^{\beta}(x)}{\bar{F}_{\mathrm{w}}^{\beta}(t)}\mathrm{d}x\right) \quad \text{for} \quad \begin{matrix}\beta \neq 1\\ \beta > 0\text{'}\end{matrix} \tag{28}$$

For the length-biased rv X_L, DCRRE is then $\gamma_L(\beta; t) = 1/(1-\beta)\log\left(\int_t^\infty \bar{F}_L^{\beta}(x)/\bar{F}_L^{\beta}(t)\,\mathrm{d}x\right)$, where $\bar{F}_L(t) = (m(t)/\mu)F(t)$ with $m(t) = E(X|X > t)$ denoting the vitality function.

Theorem 3.1 *If $E(w(X)|X > x) \leq E(w(X)|X > t)$ for all $x \geq t$, then $\gamma_{\mathrm{w}}(\beta; t) \leq (\geq)\gamma(\beta; t)$ for $0 < \beta < 1(\beta > 1)$. If $E(w(X)|X > x) \geq E(w(X)|X > t)$ for all $x \geq t$, then $\gamma_{\mathrm{w}}(\beta; t) \geq (\leq)$ $\gamma(\beta; t)$ for $0 < \beta < 1(\beta > 1)$.*

Proof If $E(w(X)|X > x) \leq E(w(X)|X > t)$ for all $x \geq t$, then using Equations (27) and (28) we have

$$\begin{aligned}\gamma_{\mathrm{w}}(\beta; t) &= \frac{1}{1-\beta}\log\left(\int_t^\infty \frac{[E(w(X)\,\big|X > x)\bar{F}(x)]^{\beta}}{[E(w(X)\,\big|X > t)\bar{F}(t)]^{\beta}}\,\mathrm{d}x\right)\\ &\leq (\geq)\frac{1}{1-\beta}\log\left(\int_t^\infty \frac{\bar{F}^{\beta}(x)}{\bar{F}^{\beta}(t)}\,\mathrm{d}x\right) = \gamma(\beta; t) \quad \text{for } 0 < \beta < 1(\beta > 1). \qquad \blacksquare\end{aligned}$$

Corollary 3.1 *If $m(x) \leq m(t)$ for all $x \geq t$, then $\gamma_L(\beta; t) \leq (\geq)\gamma(\beta; t)$ for $0 < \beta < 1(\beta > 1)$. If $m(x) \geq m(t)$ for all $x \geq t$, then $\gamma_L(\beta; t) \geq (\leq)\gamma(\beta; t)$ for $0 < \beta < 1(\beta > 1)$.*

When the weight function $w(t) = \bar{F}(t)/f(t)$ (also called Mill's ratio), the corresponding weighted distribution is called the equilibrium distribution. The equilibrium distribution arises naturally in renewal theory and it is the distribution of the backward or forward recurrence time in the limiting case. For a recent survey of research on various applications of equilibrium distribution, we refer to Gupta and Sankaran [14], Gupta [15], Sunoj and Maya [16] and Nair and Preeth [17].

Let X_{E} be a rv corresponding to equilibrium distribution with pdf $f_{\mathrm{E}}(t) = \bar{F}(t)/\mu, t > 0$, where $\mu = E(X) < \infty$, then DCRRE of X_{E} is obtained as

$$\gamma_{\mathrm{E}}(\beta; t) = \frac{1}{(1-\beta)} \log\left(\int_t^{\infty} \frac{\bar{F}_{\mathrm{E}}^{\beta}(x)}{\bar{F}_{\mathrm{E}}^{\beta}(t)}\, \mathrm{d}x\right) \quad \text{for} \quad \begin{matrix} \beta \neq 1 \\ \beta > 0 \end{matrix}, \tag{29}$$

where $\bar{F}_{\mathrm{E}}(t) = (r(t)/\mu)\bar{F}(t)$.

THEOREM 3.2 *If $\bar{F}(t)$ is increasing mean residual life (IMRL), then $\gamma_{\mathrm{E}}(\beta; t) \geq (\leq)\gamma(\beta; t)$ for $0 < \beta < 1(\beta > 1)$. If $\bar{F}(t)$ is decreasing mean residual life (DMRL), then $\gamma_{\mathrm{E}}(\beta; t) \leq (\geq)\gamma(\beta; t)$ for $0 < \beta < 1(\beta > 1)$.*

Proof Since $F(t)$ is IMRL (DMRL), we have $r(x) \geq (\leq)r(t)$ for all $x \geq t$, the remaining part is similar to the proof of Theorem 3.1. ■

THEOREM 3.3 *The relationship*

$$(1-\beta)\gamma_{\mathrm{E}}(\beta; t) = (1-\beta)\gamma_{\mathrm{L}}(\beta; t) = \log(Ct) \tag{30}$$

where $C(> 0)$ is a constant, holds if and only if X follows Pareto I distribution with sf $\bar{F}(t) = (k/t)^c, t > k, k > 0, c > 1$.

Proof Assume that Equation (30) holds, now using Equation (29), we obtain

$$\log\left(\int_t^{\infty} \frac{\bar{F}_{\mathrm{E}}^{\beta}(x)}{\bar{F}_{\mathrm{E}}^{\beta}(t)} \mathrm{d}x\right) = \log(Ct),$$

equivalently,

$$\int_t^{\infty} \left(\frac{\bar{F}_{\mathrm{E}}^{\beta}(x)}{\bar{F}_{\mathrm{E}}^{\beta}(t)}\right) \mathrm{d}x = Ct. \tag{31}$$

Differentiating Equation (31) with respect to t, we get

$$\frac{\beta h_{\mathrm{E}}(t)}{\bar{F}_{\mathrm{E}}^{\beta}(t)} \int_t^{\infty} \bar{F}_{\mathrm{E}}^{\beta}(x)\, \mathrm{d}x - 1 = C,$$

where $h_{\mathrm{E}}(t) = f_{\mathrm{E}}(t)/\bar{F}_{\mathrm{E}}(t) = 1/r(t)$ is the failure rate of X_{E}. Now using Equation (31) and simplifying, we get $r(t) = (\beta C/(C+1))t = Pt$, where $P(> 0)$ is a constant, follows Pareto I. The converse part is quite straightforward. ■

4. Conditional dynamic cumulative residual Renyi's Entropy

Specification of the joint distribution through its component densities, namely marginals and conditionals has been a problem dealt with by many researchers in the past. It is well known that in general, the marginal densities cannot determine the joint density uniquely unless the variables are independent. Apart from the marginal distribution of X_i and the conditional distribution of X_j given$X_i = t_i,\ i = 1, 2,\ i \neq j$, from which the joint distribution can always be found, the other quantities that are of relevance to the problem are (a) marginal and conditional distributions of the same component viz. X_1 and the X_1 given $X_2 = t_2$ or X_2 and the X_2 given $X_1 = t_1$ (b) the two conditional distributions. Characterization of the bivariate density given the forms of the marginal density of X_1 (X_2) and the conditional density of X_1 given $X_2 = t_2$ (X_2 given$X_1 = t_1$) for certain classes of distributions, have been considered by Seshadri and Patil [18], Nair and Nair [19] and Hitha and Nair [20]. On the other hand, Gourieroux and Monfort [21] have developed conditions under which the conditional densities determine the joint density $f(t_1, t_2)$ uniquely. For more recent works on conditional densities we refer to Sankaran and Nair [22], Sunoj and Sankaran [23] and Kotz *et al*. [24]. Accordingly in following Sections 4.1 and 4.2, we consider conditional dynamic residual Renyi's entropies of X_i given $X_j = t_j$ and X_i given $X_j > t_j, i, j = 1, 2, \ i \neq j$ respectively and study some characteristics relationships in the context of reliability modelling.

4.1. *Conditional dynamic cumulative residual Renyi's entropy for X_i given $X_j = t_j$*

Let $X = (X_1, X_2)$ be a bivariate random vector admitting an absolutely continuous pdf $f(t_1, t_2)$ and cdf $F(t_1, t_2)$ with respect to Lesbegue measure in the positive octant $R_2^+ = \{(t_1, t_2)|t_i > 0, i = 1, 2\}$ of the two-dimensional Euclidean space R_2. Let $\bar{F}_i(t_i|t_j), i, j = 1, 2, i \neq j$ denote the sf of X_i given $X_j = t_j$. Then, the conditional dynamic cumulative residual Renyi's entropy (CDCRRE) of X_i given $X_j = t_j$ is defined as

$$\gamma_i(\beta; t_1, t_2) = \frac{1}{(1-\beta)} \log \left(\int_{t_i}^{\infty} \frac{\bar{F}_i^{\beta}(x_i|t_j)}{\bar{F}_i^{\beta}(t_i|t_j)} \, \mathrm{d}x_i \right), \quad i, j = 1, 2, \ i \neq j, \tag{32}$$

which can be written as

$$(1-\beta)\gamma_i(\beta; t_1, t_2) = \log \left(\int_{t_i}^{\infty} \bar{F}_i^{\beta}(x_i|t_j) \, \mathrm{d}x_i \right) - \beta \log \bar{F}_i(t_i|t_j). \tag{33}$$

Differentiating Equation (33) with respect to t_i, we have

$$(1-\beta)\frac{\partial}{\partial t_i}\gamma_i(\beta; t_1, t_2) = \beta h_i(t_i|t_j) - \mathrm{e}^{-(1-\beta)\gamma_i(\beta; t_1, t_2)}, \tag{34}$$

where $h_i(t_i|t_j), i, j = 1, 2, i \neq j$, is the failure rate of X_i given $X_j = t_j$.

THEOREM 4.1 *The relationship*

$$(1-\beta)\gamma_i(\beta; t_1, t_2) = \log[Cr_i(t_i|t_j)], \quad i, j = 1, 2, \ i \neq j, \tag{35}$$

holds for all t_i and t_j, where C is a constant independent of t_i and t_j, $i \neq j, i, j = 1, 2$, and $r_i(t_i|t_j) = E(X_i - t_i|X_i > t_i, X_j = t_j)$ is the mean residual life function (MRLF) of X_i given $X_j = t_j$, if and only if X follows either bivariate distribution with Pareto conditionals given in Arnold [25] *with pdf*

$$f(t_1, t_2) = K_1(1 + a_1 t_1 + a_2 t_2 + b t_1 t_2)^{-c}, \quad K_1, a_1, a_2, b > 0, \ c > 2, \ t_1, t_2 > 0, \tag{36}$$

or bivariate distribution with exponential conditionals of Arnold and Strauss [26] *with pdf*

$$f(t_1, t_2) = K_2 \exp(-\lambda_1 t_1 - \lambda_2 t_2 - \theta t_1 t_2), \quad K_2, \lambda_1, \lambda_2, \theta > 0, \ t_1, t_2 > 0, \tag{37}$$

or bivariate distribution with beta conditionals with pdf

$$\begin{aligned} &f(t_1, t_2) = K_3(1 - p_1 t_1 - p_2 t_2 + q t_1 t_2)^d, \\ &K_3, p_1, p_2, q, d > 0, \ 0 < t_1 < \frac{1}{p_1}, \ 0 < t_2 < \frac{1 - p_1 t_1}{p_2 - q t_1}, \end{aligned} \tag{38}$$

according as $P \gtreqless 0$, where $P = ((C\beta - 1)/(C(1-\beta)))$.

Proof Suppose that Equation (35) holds, then for $i = 1$, we have

$$\log[Cr_1(t_1|t_2)] = (1-\beta)\gamma_1(\beta; t_1, t_2),$$

which is equivalent to

$$Cr_1(t_1|t_2) = \int_{t_1}^{\infty} \frac{\bar{F}_1^{\beta}(x_1\,|t_2)}{\bar{F}_1^{\beta}(t_1\,|t_2)}\,\mathrm{d}x_1. \tag{39}$$

Differentiating with respect to t_1, Equation (39) becomes

$$C\frac{\partial}{\partial t_1} r_1(t_1|t_2) = \frac{\beta h_1(t_1|t_2)}{\bar{F}_1^{\beta}(t_1|t_2)} \int_{t_1}^{\infty} \bar{F}_1^{\beta}(x_1|t_2)\,\mathrm{d}x_1 - 1. \tag{40}$$

Using Equation (39) and the relationship between failure rate and MRLF, Equation (40) reduces to

$$C\frac{\partial}{\partial t_1} r_1(t_1|t_2) = C\beta\left[\frac{\partial}{\partial t_1} r_1(t_1|t_2) + 1\right] - 1,$$

which implies that $(\partial/\partial t_1)r_1(t_1|t_2) = (C\beta-1)/(C(1-\beta))$. Now integrating with respect to t_1, we have $r_1(t_1|t_2) = ((C\beta-1)/(C(1-\beta)))t_1 + B_1(t_2) = At_1 + B_1(t_2)$, where $A = (C\beta - 1)/(C(1-\beta))$. Similarly, for $i = 2$, we have $r_2(t_2|t_1) = At_2 + B_2(t_1)$. Hence, $r_i(t_i|t_j) = At_i + B_i(t_j)$, $i \neq j, i, j = 1, 2$, where $B_i(t_j)$ is a function of t_j only. Now using Sankaran and Nair [22], the proof of the theorem follows, according as $A \gtreqless 0$.

Conversely, when X follows the model (36) and using Equation (32), we get

$$\begin{aligned}(1-\beta)\gamma_i(\beta; t_1, t_2) &= \log\left[\frac{(c-2)}{(c\beta-\beta-1)}\frac{(1+a_1t_1+a_2t_2+bt_1t_2)}{(c-2)(a_i+bt_j)}\right]\\ &= \log[Cr_i(t_i|t_j)], \quad i\neq j,\ i, j = 1, 2\end{aligned}$$

with $C = (c-2)/(c\beta-\beta-1)$ such that $(C\beta-1)/(C(1-\beta)) > 0$. When X follows the model (37), we have $(1-\beta)\gamma_i(\beta; t_1, t_2) = \log(1/\beta(\lambda_i+\theta t_j)) = \log[Cr(t_i|t_j)]$, $i \neq j, i, j = 1, 2$ with $C = 1/\beta$, so that $(C\beta-1)/(C(1-\beta)) = 0$. Finally, when X follows the model (38), results

$$\begin{aligned}(1-\beta)\gamma_i(\beta; t_1, t_2) &= \log\left[\frac{(d+2)}{(d\beta+\beta+1)}\frac{(1-p_1t_1-p_2t_2+qt_1t_2)}{(d+2)(p_i-qt_j)}\right]\\ &= \log[Cr_i(t_i|t_j)], \quad i\neq j,\ i, j = 1, 2\end{aligned}$$

with $C = (d+2)/(d\beta+\beta+1)$ implies that $(C\beta-1)/(C(1-\beta)) < 0$ proves the theorem. ■

THEOREM 4.2 *The relationship*

$$(1-\beta)\frac{\partial}{\partial t_i}\gamma_i(\beta; t_1, t_2) = Ch_i(t_i|t_j), \tag{41}$$

for all t_i and t_j, where C is a constant independent of t_i and t_j, $i \neq j, i, j = 1, 2$ holds if and only if X is distributed as Equation (36) *when $C > 0$, Equation* (37) *when $C = 0$ and Equation* (38) *when $-1 < C < 0$.*

Proof Suppose Equation (41) holds, now using Equation (34), we get

$$\beta h_i(t_i|t_j) - \mathrm{e}^{-(1-\beta)\gamma_i(\beta;t_1,t_2)} = C h_i(t_i|t_j), \quad i \neq j,\ i,j = 1,2.$$

From the definition of CDCRRE (32), the above expression becomes

$$(\beta - C) h_i(t_i|t_j) = \frac{\bar{F}_i^{\beta}(t_i|t_j)}{\int_{t_i}^{\infty} \bar{F}_i^{\beta}(x_i|t_j)\,\mathrm{d}x_i}.$$

Equivalently,

$$(\beta - C)\int_{t_i}^{\infty} \bar{F}_i^{\beta}(x_i|t_j)\,\mathrm{d}x_i = \frac{\bar{F}_i^{\beta+1}(t_i|t_j)}{f_i(t_i|t_j)}, \tag{42}$$

$$(\beta - C) f_i(t_i|t_j)\int_{t_i}^{\infty} \bar{F}_i^{\beta}(x_i|t_j)\mathrm{d}x_i = \bar{F}_i^{\beta+1}(t_i|t_j). \tag{43}$$

Differentiating Equation (43) with respect to t_i and using Equation (42), we have

$$\frac{\partial}{\partial t_i} f_i(t_i|t_j)\frac{\bar{F}_i^{\beta+1}(t_i|t_j)}{f_i(t_i|t_j)} - (\beta - C) f_i(t_i|t_j)\bar{F}_i^{\beta}(t_i|t_j) = -(\beta+1)\bar{F}_i^{\beta}(t_i|t_j) f_i(t_i|t_j). \tag{44}$$

Dividing Equation (44) with $\bar{F}_i^{\beta}(t_i|t_j) f_i(t_i|t_j)$ and simplifying, we obtain

$$\frac{\partial}{\partial t_i}\log f_i(t_i|t_j) = (C+1)\frac{\partial}{\partial t_i}\log \bar{F}_i(t_i|t_j),$$

Equivalently

$$\frac{\partial}{\partial t_i}\log h_i(t_i|t_j) = C\frac{\partial}{\partial t_i}\log \bar{F}_i(t_i|t_j). \tag{45}$$

Integrating Equation (45) with respect to t_i, we get

$$\log h_i(t_i|t_j) = C\log \bar{F}_i(t_i|t_j) + K_i(t_j).$$

Differentiating the above equation with respect to t_i and rearranging, the above equation becomes $\partial/\partial t_i[1/h_i(t_i|t_j)] = C$, which on integration with respect to t_i gives

$$\frac{1}{h_i(t_i|t_j)} = Ct_i + D_i(t_j). \tag{46}$$

From the definition of $h_i(t_i|t_j) = (f_i(t_i|t_j))/(\bar{F}_i(t_i|t_j)) = -(f(t_1,t_2))/((\partial/\partial t_j)\bar{F}(t_1,t_2))$, Equation (46) becomes $-(\partial/\partial t_j)\bar{F}(t_1,t_2) = f(t_1,t_2)[Ct_i + D_i(t_j)]$. Differentiating with respect to t_i and simplifying, we get $(\partial/\partial t_i)\log f(t_1,t_2) = -(C+1)/[Ct_i + D_i(t_j)]$. Now on integrating with respect to t_i, we have $\log f(t_1,t_2) = -((C+1)/C)\log[Ct_i + D_i(t_j)] + \log m_i(t_j)$. Equivalently,

$$f(t_1,t_2) = m_i(t_j)[Ct_i + D_i(t_j)]^{-(C+1)/C}, \quad C \neq 0,\ i \neq j,\ i,j = 1,2. \tag{47}$$

Applying for $i = 1,2$ and equating, we obtain

$$m_1(t_2)[Ct_1 + D_1(t_2)]^{-(C+1)/C} = m_2(t_1)[Ct_2 + D_2(t_1)]^{-(C+1)/C}. \tag{48}$$

As $t_1 \to 0$, Equation (48) becomes

$$m_1(t_2) = \frac{m_2(0)[Ct_2 + D_2(0)]^{-(C+1)/C}}{[D_1(t_2)]^{-(C+1)/C}}.$$

Similarly, as $t_2 \to 0$, Equation (48) becomes

$$m_2(t_1) = \frac{m_1(0)[Ct_1 + D_1(0)]^{-(C+1)/C}}{[D_2(t_1)]^{-(C+1)/C}}.$$

Substituting for $m_1(t_2)$and $m_2(t_1)$, Equation (48) becomes

$$\frac{m_2(0)[Ct_2 + D_2(0)]^{-(C+1)/C}}{[D_1(t_2)]^{-(C+1)/C}}[Ct_1 + D_1(t_2)]^{-(C+1)/C} = \frac{m_1(0)[Ct_1 + D_1(0)]^{-(C+1)/C}}{[D_2(t_1)]^{-(C+1)/C}}[Ct_2 + D_2(t_1)]^{-(C+1)/C}. \tag{49}$$

For $i = 1$ in Equation (47) and as $t_1 \to 0$, we get

$$\lim_{t_1 \to 0} f(t_1, t_2) = m_1(t_2)[D_1(t_2)]^{-(C+1)/C}. \tag{50}$$

Similarly for $i = 2$ and as $t_1 \to 0$, we have

$$\lim_{t_1 \to 0} f(t_1, t_2) = m_2(0)[Ct_2 + D_2(0)]^{-(C+1)/C}. \tag{51}$$

Equating Equations (50) and (51), we obtain

$$m_1(t_2)[D_1(t_2)]^{-(C+1)/C} = m_2(0)[Ct_2 + D_2(0)]^{-(C+1)/C}. \tag{52}$$

As $t_2 \to 0$, Equation (52) becomes

$$\frac{m_1(0)}{m_2(0)} = \frac{[D_2(0)]^{-(C+1)/C}}{[D_1(0)]^{-(C+1)/C}}.$$

Then, Equation (49) becomes

$$\frac{[Ct_2 + D_2(0)]^{-(C+1)/C}[Ct_1 + D_1(t_2)]^{-(C+1)/C}}{[D_1(t_2)]^{-(C+1)/C}} = \frac{[D_2(0)]^{-(C+1)/C}}{[D_1(0)]^{-(C+1)/C}} \frac{[Ct_1 + D_1(0)]^{-(C+1)/C}\,[Ct_2 + D_2(t_1)]^{-(C+1)/C}}{[D_2(t_1)]^{-(C+1)/C}}.$$

Equivalently, we get

$$\frac{1}{t_1 D_2(t_1)} - \frac{1}{t_1 D_2(0)} + \frac{C}{D_2(t_1) D_1(0)} = \frac{1}{t_2 D_1(t_2)} - \frac{1}{t_2 D_1(0)} + \frac{C}{D_1(t_2) D_2(0)}. \tag{53}$$

Since Equation (53) is true for all $t_1, t_2 \geq 0$, we may take both sides of Equation (53) equal to n, where n is a constant. Using the expression of $m_1(t_2)$, the joint pdf $f(t_1, t_2)$ in Equation (47) for

$i = 1$ becomes

$$f(t_1, t_2) = \frac{m_2(0)[Ct_2 + D_2(0)]^{-(C+1)/C}}{[D_1(t_2)]^{-(C+1)/C}} [Ct_1 + D_1(t_2)]^{-(C+1)/C},$$

or

$$f(t_1, t_2) = m_2(0)(D_2(0))^{-(C+1)/C} \left(1 + \frac{Ct_2}{D_2(0)}\right)^{-(C+1)/C} \left(1 + \frac{Ct_1}{D_1(t_2)}\right)^{-(C+1)/C}. \tag{54}$$

Now using Equation (53) and substituting for $1 + (Ct_1/D_1(t_2))$, the joint pdf $f(t_1, t_2)$ in Equation (54) becomes

$$f(t_1, t_2) = m_2(0)[D_2(0)]^{-(C+1)/C} \left[1 + \frac{Ct_1}{D_1(0)} + \frac{Ct_2}{D_2(0)} + nCt_1t_2\right]^{-(C+1)/C}, \tag{55}$$

which is of the form Equation (36) with $K_1 = m_2(0)[D_2(0)]^{-(C+1)/C}$, $a_1 = C/D_1(0)$, $a_2 = C/D_2(0)$, $b = nC$ and $c = (C + 1)/C$. If $C > 0$, since $D_i(t_j)$ are non-negative functions of t_j, we have $K_1, a_1, a_2, b > 0$. Similarly, if $-1 < C < 0$, Equation (55) takes the form Equation (38) with $K_3, p_1, p_2 > 0$, $d > 0$, $0 < t_1 < 1/p_1$ and $0 < t_2 < (1 - p_1t_1)/(p_2 - qt_1)$. When $C = 0$ from Equation (46), we get $h_i(t_i|t_j) = 1/D_i(t_j)$, following the similar steps, we obtain $-\log f(t_1, t_2) = (t_i/D_i(t_j)) + Q_i(t_j)$, where $Q_i(t_j)$ is a function of t_j only $i \neq j, i, j = 1, 2$. Equivalently, we have

$$f(t_1, t_2) = \mathrm{e}^{-[(t_i/D_i(t_j))+Q_i(t_j)]}, \quad i \neq j,\ i, j = 1, 2. \tag{56}$$

For $i = 1, 2$ and equating, Equation (56) becomes a functional equation $(t_1/D_1(t_2)) + Q_1(t_2) = (t_2/D_2(t_1)) + Q_2(t_1)$, which gives the solution as $D_1(t_2) = 1/(\lambda_1 + \theta t_2)$ and $D_2(t_1) = 1/(\lambda_2 + \theta t_1)$. Then $Q_1(t_2) = Q_2 + \lambda_2 t_2$ and $Q_2(t_1) = Q_1 + \lambda_1 t_1$, where $\lambda_1, \lambda_2, \theta$ are non-negative constants and $Q_i = Q_i(0)$, $i = 1, 2$. Substituting these in Equation (56), we have Equation (37). The converse part is straightforward. ■

THEOREM 4.3 $\gamma_i(\beta; t_1, t_2)$, $i = 1, 2$, *is locally constant (i.e.,* $\gamma_i(\beta; t_1, t_2)$ *function of* t_j *only) if and only if X follows bivariate distribution with exponential conditionals of Arnold and Strauss* [26] *with pdf* (37).

Proof Let $\gamma_i(\beta; t_1, t_2)$, $i = 1, 2$, be locally constant, which implies that $(\partial/\partial t_i)\gamma_i(\beta; t_1, t_2) = 0$, or $(1 - \beta)(\partial/\partial t_i)\gamma_i(\beta; t_1, t_2) = 0$. Now using Theorem 4.2, completes the proof. ■

4.2. *Conditional dynamic cumulative residual Renyi's entropy for X_i given $X_j > t_j$*

Let $X = (X_1, X_2)$ be a bivariate random vector admitting an absolutely continuous pdf $f(t_1, t_2)$ and cdf $F(t_1, t_2)$ with respect to Lesbegue measure in the positive octant $R_2^+ = \{(t_1, t_2)|t_i > 0, i = 1, 2\}$ of the two-dimensional Euclidean space R_2. Let the sf of X_i given $X_j > t_j$ be $\bar{F}_i^*(t_i|t_j)$, $i, j = 1, 2, i \neq j$. Now using Equation (6), the CDRRE of the conditional distribution of X_i given

$X_j > t_j$ turns out to be

$$\gamma_i^*(\beta; t_1, t_2) = \frac{1}{1-\beta} \log\left(\int_{t_i}^{\infty} \frac{\bar{F}_i^{*\beta}(x_i|t_j)}{\bar{F}_i^{*\beta}(t_i|t_j)}\, \mathrm{d}x_i\right), \tag{57}$$

which can be written as

$$(1-\beta)\gamma_i^*(\beta; t_1, t_2) = \log\left(\int_{t_i}^{\infty} \bar{F}_i^{*\beta}(x_i\,|t_j)\, \mathrm{d}x_i\right) - \beta \log \bar{F}_i^*(t_i|t_j). \tag{58}$$

Differentiating with respect to t_i, Equation (58) becomes

$$(1-\beta)\frac{\partial}{\partial t_i}\gamma_i^*(\beta; t_1, t_2) = \beta h_i^*(t_i|t_j) - \mathrm{e}^{-(1-\beta)\gamma_i^*(\beta; t_1, t_2)},$$

where $h_i^*(t_i|t_j) = -(\partial/\partial t_i)\log \bar{F}_i^*(t_i|t_j) = -(\partial/\partial t_i)\log \bar{F}(t_1, t_2) = h_i(t_1, t_2), i, j = 1, 2, i \neq j$, the vector-valued failure rate due to Johnson and Kotz [27].

Examples (a) If X is distributed as bivariate Pareto I with joint sf $\bar{F}(t_1, t_2) = t_1^{-\alpha_1} t_2^{-\alpha_2} t_1^{-\theta \log t_2}$; $t_1, t_2 > 1$, then $(1-\beta)\gamma_i^*(\beta; t_1, t_2) = \log t_i - \log(\beta(\alpha_i + \theta \log t_j) - 1)$, $i, j = 1, 2, i \neq j$.

(b) If X follows bivariate Weibull $\bar{F}(t_1, t_2) = \mathrm{e}^{-\alpha_1 t_1^a - \alpha_2 t_2^a - \theta t_1^a t_2^a}$; $t_1, t_2 > 0, \alpha_1, \alpha_2, a > 0$, then $(1-\beta)\gamma_i^*(\beta; t_1, t_2) = \log((1/a)(\beta(\alpha_i + \theta t_j^a))^{-1/a}\Gamma((1/a), \beta(\alpha_i + \theta t_j^a)t_i^a)) - \beta(\alpha_i + \theta t_j^a)t_i^a, i, j = 1, 2, i \neq j$.

THEOREM 4.4 *The relationship*

$$(1-\beta)\gamma_i^*(\beta; t_1, t_2) = \log[C^* r_i^*(t_i|t_j)], \tag{59}$$

holds for all t_i and t_j, where C^* is a constant independent of t_i and t_j, $i \neq j, i, j = 1, 2$, and $r_i^*(t_i|t_j) = E(X_i - t_i|X_i > t_i, X_j > t_j) = r_i(t_1, t_2)$ is the ith component of vector-valued MRLF in the bivariate case, if and only if X follows either bivariate Pareto II with joint sf

$$\bar{F}(t_1, t_2) = (1 + a_1 t_1 + a_2 t_2 + b t_1 t_2)^{-c}; \quad a_1, a_2, c, t_1, t_2 > 0;\ 0 < b \leq (c+1)a_1 a_2, \tag{60}$$

or Gumbel's bivariate exponential with joint sf

$$\bar{F}(t_1, t_2) = e^{-\lambda_1 t_1 - \lambda_2 t_2 - \theta t_1 t_2}; \quad \lambda_1, \lambda_2, t_1, t_2 > 0;\ 0 < \theta < \lambda_1 \lambda_2, \tag{61}$$

or bivariate finite range with joint sf

$$\bar{F}(t_1, t_2) = (1 - p_1 t_1 - p_2 t_2 + q t_1 t_2)^d; \quad p_1, p_2, d > 0; 0 < t_1 < \frac{1}{p_1};$$
$$0 < t_2 < \frac{1 - p_1 t_1}{p_2 - q t_1}; \quad 1 - d \leq \frac{q}{p_1 p_2} \leq 1, \tag{62}$$

according as $P^* \gtreqless 0$, where $P^* = (C^*\beta - 1)/(C^*(1-\beta))$.

Proof Assume that Equation (59) holds, then using Equation (57) and applying the similar steps as in Theorem 4.1, we obtain $r_i^*(t_i|t_j) = ((C^*\beta - 1)/(C^*(1-\beta)))t_i + B_i(t_j) = At_i + B_i(t_j)$, where $A = (C^*\beta - 1)/(C^*(1-\beta))$ and $B_i(t_j)$is a function of t_j only, $i \neq j, i, j = 1, 2$. Now using a characterization theorem in Sankaran and Nair [28], X follows bivariate Pareto II with sf

(60) when $A > 0$, Gumbel's exponential with sf (61) when $A = 0$ and bivariate finite range with sf (62) when $A < 0$.

Conversely, when X follows bivariate Pareto II with sf (60), using Equation (57) we have

$$(1-\beta)\gamma_i^*(\beta; t_1, t_2) = \log\left(\int_{t_i}^{\infty} \frac{(1+a_i x_i + a_j t_j + b x_i t_j)^{-c\beta}}{(1+a_1t_1+a_2t_2+bt_1t_2)^{-c\beta}}\,\mathrm{d}x_i\right)$$
$$= \log\left[\frac{(c-1)}{(c\beta-1)}\frac{(1+a_1t_1+a_2t_2+bt_1t_2)}{(c-1)(a_i+bt_j)}\right] = \log(C^* r_i^*(t_i|t_j)),$$

where $C^* = (c-1)/(c\beta-1)$, so that $P^* = (C^*\beta - 1)/(C^*(1-\beta)) > 0$. When X follows Gumbel's exponential with sf (61), then

$$(1-\beta)\gamma_i^*(\beta; t_1, t_2) = \log\left(\int_{t_i}^{\infty} \frac{(\mathrm{e}^{-\lambda_i x_i - \lambda_j t_j - \theta x_i t_j})^{\beta}}{(\mathrm{e}^{-\lambda_1 t_1 - \lambda_2 t_2 - \theta t_1 t_2})^{\beta}}\,\mathrm{d}x_i\right) = \log\left(\frac{1}{\beta(\lambda_i + \theta t_j)}\right)$$
$$= \log(C^* r_i^*(t_i|t_j))$$

where $C^* = 1/\beta$, such that $P^* = (C^*\beta - 1)/(C^*(1-\beta)) = 0$. Finally, when X follows bivariate finite range with sf (62), we have

$$(1-\beta)\gamma_i^*(\beta; t_1, t_2) = \log\left(\int_{t_i}^{(1-p_jt_j)/(p_i-qt_j)} \frac{(1-p_i x_i - p_j t_j + q x_i t_j)^{d\beta}}{(1-p_1t_1-p_2t_2+qt_1t_2)^{d\beta}}\,\mathrm{d}x_i\right)$$
$$= \log\left[\frac{(d+1)}{(d\beta+1)}\frac{(1-p_1t_1-p_2t_2+qt_1t_2)}{(d+1)(p_i-qt_j)}\right] = \log(C^* r_i^*(t_i|t_j)),$$

where $C^* = (d+1)/(d\beta+1)$ such that $P^* = (C^*\beta - 1)/(C^*(1-\beta)) < 0$, proves the theorem. ■

THEOREM 4.5 *The relationship*

$$(1-\beta)\frac{\partial}{\partial t_i}\gamma_i^*(\beta; t_1, t_2) = C h_i^*(t_i|t_j), \tag{63}$$

for all t_i and t_j, where C is a constant independent of t_i and t_j, $i \neq j, i, j = 1, 2$, holds if and only if X is distributed as bivariate Pareto II with sf (60) *when $C > 0$, Gumbel's exponential with sf* (61) *when $C = 0$ and bivariate finite range with sf* (62) *when $C < 0$.*

Proof Assume that Equation (63) holds, then using Equation (57) and applying the similar steps as in Theorem 4.2, we get $h_i^*(t_i|t_j) = 1/(Ct_i + D_i(t_j))$, or $h_i(t_1, t_2) = 1/(Ct_i + D_i(t_j))$. Now characterization to Equations (60)–(62) follows from Roy [29]. The converse of the theorem can be easily proved. ■

THEOREM 4.6 $\gamma_i^*(\beta; t_1, t_2)$ *is locally constant if and only if X follows Gumbel's bivariate exponential with sf* (61).

Proof Proof is similar to that of Theorem 4.3. ■

Acknowledgements

The authors thank the referees and the editor for the very useful and constructive comments on the earlier version of this paper, which have led to a substantial improvement of the paper.

References

[1] N. Ebrahimi and F. Pellerey, *New partial ordering of survival functions based on notion of uncertainty*, J. Appl. Probab. 32(1) (1995), pp. 202–211.

[2] M. Asadi and N. Ebrahimi, *Residual entropy and its characterizations in terms of hazard function and mean residual life function*, Stat. Probab. Lett. 49(3) (2000), pp. 263–269.

[3] M. Asadi, N. Ebrahimi, and E.S. Soofi, *Dynamic generalized information measures*, Stat. Probab. Lett. 71(1) (2005), pp. 85–98.

[4] S.M. Sunoj, P.G. Sankaran, and S.S. Maya, *Characterization of life distributions using conditional expectations of doubly (interval) truncated random variables*, Commun. Stat. Theory Methods 38(9) (2009), pp. 1441–1452.

[5] M. Rao, Y. Chen, B.C. Vemuri, and F. Wang, *Cumulative residual entropy: A new measure of information*, IEEE Trans. Inform. Theory. 50(6) (2004), pp. 1220–1228.

[6] M. Rao, *More on a new concept of entropy and information*, J. Theoret. Probab. 18(14) (2005), pp. 967–981.

[7] M. Asadi and Y. Zohrevand, *On the dynamic cumulative residual entropy*, J. Stat. Plann. Inference 137(6) (2007), pp. 1931–1941.

[8] A. Renyi, *On measures of Entropy and Information*, Proceeding of the Fourth Berkeley Symposium on Mathematical Statistics and Probability Vol. 1, University of California Press, Berkeley, CA, 1961, pp. 547–561.

[9] B. Abraham and P.G. Sankaran, *Renyi's entropy for residual lifetime distribution*, Statist. Papers 46(1) (2005), pp. 17–30.

[10] S.S. Maya and S.M. Sunoj, *Some dynamic generalized information measures in the context of weighted models*, Statistica LXVIII(1) (2008), pp. 71–84.

[11] A. Di Crescenzo and M. Longobardi, *On weighted residual and past entropies*, Scientiae Mathematicae Japonicae 64(3) (2006), pp. 255–266.

[12] N.U. Nair and S.M. Sunoj, *Form-invariant bivariate weighted models*, Statistics 37(3) (2003), pp. 259–269.

[13] S.M. Sunoj and S.S. Maya, *Some properties of weighted distribution in the context of repairable systems*, Commun. Stat. Theory Methods 35(2) (2006), pp. 223–228.

[14] R.P. Gupta and P.G. Sankaran, *Bivariate equilibrium distribution and its applications to reliability*, Commun. Stat. Theory Methods 27(2) (1998), pp. 385–394.

[15] R.C. Gupta, *Role of equilibrium distributions in reliability studies*, Probab. Eng. Inform. Sci. 21(2) (2007), pp. 315–334.

[16] S.M. Sunoj and S.S. Maya, *The role of lower partial moments in stochastic modeling*, Metron LXVI(3) (2008), pp. 219–238.

[17] N.U. Nair and M. Preeth, *Multivariate equilibrium distributions of order n*, Stat. Probab. Lett. 78(18) (2008), pp. 3312–3320.

[18] V. Seshadri and G.P. Patil, *A characterization of bivariate distributions by the marginal and conditional distributions of the same component*, Ann Inst. Statist. Math. 15(1) (1964), pp. 215–221.

[19] N.U. Nair and V.K.R. Nair, *A characterization of the bivariate exponential distribution*, Biometrical J. 30(1) (1988), pp. 107–112.

[20] N. Hitha and N.U. Nair, *Characterizations of bivariate Lomax and finite range distribution*, Calcutta Stat. Assoc. Bull. 41 (1991), pp. 163–167.

[21] C. Gourieroux and M. Monfort, *On the characterization of a joint distribution by conditional distributions*, J. Econom. 10 (1979), pp. 115–118.

[22] P.G. Sankaran and N.U. Nair, *Conditional mean residual life functions*, Commun. Stat. Theory Methods 29(7) (2000), pp. 1663–1675.

[23] S.M. Sunoj and P.G. Sankaran, *Bivariate weighted distributions in the context of reliability modeling*, Calcutta Statist. Assoc. Bull. 57 (2005), pp. 227–228.

[24] S. Kotz, J. Navarro, and J.M. Ruiz, *Characterizations of Arnold and Strauss' and related bivariate exponential models*, J. Multivariate Anal. 98(7) (2007), pp. 1494–1507.

[25] B.C. Arnold, *Bivariate distributions with Pareto conditionals*, Stat. Probab. Lett. 5(4) (1987), pp. 263–266.

[26] B.C. Arnold and D. Strauss, *Bivariate distributions with exponential conditionals*, J. Amer. Stat. Assoc. 83 (1988), pp. 522–527.

[27] N.L. Johnson and S. Kotz, *A vector valued multivariate hazard rate*, J. Multivariate Anal. 5(1) (1975), pp. 53–66.

[28] P.G. Sankaran and N.U. Nair, *A bivariate Pareto model and its application to reliability*, Nav. Res. Log. 40(7) (1993), pp. 1013–1020.

[29] D. Roy, *A characterization of Gumbel's bivariate exponential and Lindley and Singpurwalla's bivariate Lomax distributions*, J. Appl. Probab. 26 (1989), pp. 886–891.