

## Research Article

# Median Sets and Median Number of a Graph

**R. Ram Kumar and B. Kannan**

*Department of Computer Applications, Cochin University of Science and Technology, India*

Correspondence should be addressed to R. Ram Kumar, ram.k.mail@gmail.com

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A *profile* is a finite sequence of vertices of a graph. The set of all vertices of the graph which minimises the sum of the distances to the vertices of the profile is the *median* of the profile. Any subset of the vertex set such that it is the *median* of some profile is called a *median set*. The number of *median sets* of a graph is defined to be the *median number* of the graph. In this paper, we identify the median sets of various classes of graphs such as  $K_p - e$ ,  $K_{p,q}$  for  $P > 2$ , and wheel graph and so forth. The median numbers of these graphs and hypercubes are found out, and an upper bound for the median number of even cycles is established. We also express the median number of a product graph in terms of the median number of their factors.

## 1. Introduction

We consider only nonempty finite simple undirected connected graphs. For the graph  $G$ ,  $V(G)$ , and  $E(G)$  denote its vertex set and edge set, respectively. When the underlying graph is obvious, we will use  $V$  and  $E$  for  $V(G)$  and  $E(G)$ , respectively. A finite sequence of vertices  $\pi = (v_1, \dots, v_k) \in V^k$  is called a *profile*. For the profile  $\pi = (v_1, \dots, v_k)$  and  $x \in V$ , the *remoteness*  $D(x, \pi)$  is  $\sum_{1 \leq i \leq k} d(x, v_i)$  [1]. The set of all vertices  $x$  for which  $D(x, \pi)$  is minimum is the *median* of  $\pi$  in  $G$  and is denoted by  $M(\pi)$ . A set  $S$  such that  $S = M(\pi)$  for some profile  $\pi$  is called a *Median set* of  $G$ . The *Interval*  $I(u, v)$  between vertices  $u$  and  $v$  of  $G$  consists of all vertices which lie in some shortest path between  $u$  and  $v$ . The number of intervals of a graph  $G$  is denoted by  $\text{in}(G)$ . The *Hypercube*  $Q_n$  is the graph with vertex set  $\{0, 1\}^n$ , two vertices being adjacent if they differ exactly in one coordinate. A *Subcube* of the hypercube  $Q_n$  is an induced subgraph of  $Q_n$ , isomorphic to  $Q_m$  for some  $m \leq n$ . A graph  $G$  is a *Median graph* if, for every  $(x, y, z) \in V^3$ ,  $I(x, y) \cap I(y, z) \cap I(x, z)$  contains a unique vertex, see Avann [2]. Trees, hypercubes, and grid graphs are all median graphs. For further references concerning median graphs, see Mulder and Schrijver [3], Mulder [4], and Bandelt and Hedlíková [5]. The graph on  $p$  vertices formed by joining all the vertices of a  $(p - 1)$ -cycle to a vertex is a *wheel graph*

and is denoted by  $W_p$ . The vertex which is joined to all the vertices of the  $(p-1)$ -cycle is the *universal vertex*. The *Cartesian product*  $G \square H$  of two graphs  $G$  and  $H$  has vertex set  $V(G) \times V(H)$ , two vertices  $(u, v)$  and  $(x, y)$  being adjacent if either  $u = x$  and  $vy \in E(H)$  or  $ux \in E(G)$  and  $v = y$ . The *eccentricity*  $e(u)$  of a vertex  $u$  is  $\max_{v \in V(G)} d(u, v)$ . A vertex  $v$  is an *eccentric vertex* of  $u$  if  $e(u) = d(u, v)$ . Graph products form a well-studied area in graph theory, see [6, 7]. The problem of finding the median of a profile is very much significant in location theory, particularly in efficiency-oriented model [8], as it corresponds to finding the most desirable service point for a group of customers when the minimisation of the total cost is a primary objective. Here we use profiles of vertices instead of a set of vertices which gives us the liberty to repeat a vertex any number of times. This makes the problem more relevant in location theory as it enables us to assign different weights to different customers. The problem of finding the median of a profile has been thoroughly investigated by many authors; see [9–14]. The objective of this paper is to identify median sets of some classes of graphs and to enumerate them.

## 2. Median Number

On the set of all profiles on  $V$ ,  $V^* = \bigcup_{k \in \mathbb{Z}} V^k$ , we define a relation  $M$  by  $\pi_1 M \pi_2$  if  $M(\pi_1) = M(\pi_2)$ . This is an equivalence relation. This equivalence relation gives a partition of the whole set of profiles. The number of equivalence classes in this partition is defined as the *Median number* of graph  $G$  and is denoted by  $mn(G)$ . That is, it is the number of distinct median sets in  $G$ .

**Proposition 2.1.** *For any graph  $G = (V, E)$  on  $p$  vertices,  $in(G) \leq mn(G) \leq 2^p - 1$ .*

*Proof.* The upper bound is obvious as it is the number of nonempty subsets of the vertex set. For every  $v \in V$ ,  $v$  is a median set of the profile  $(v)$ . For every  $u, v \in V$ , the set  $I(u, v)$  is the median set of the profile  $(u, v)$ . Therefore  $in(G) \leq mn(G) \leq 2^p - 1$ .  $\square$

**Proposition 2.2.**  *$mn(K_p) = 2^p - 1$ , where  $K_p$  is the complete graph on  $p$  vertices.*

*Proof.* In  $K_p$ , each nonempty subset of the vertex set is a median set, namely, of the profile formed by taking all the elements of the set exactly once. Therefore the number of distinct median sets is the number of nonempty subsets of  $V$  which is  $2^p - 1$ .  $\square$

**Proposition 2.3.** *If  $e$  is an edge of  $K_p$ ,  $p \geq 3$ ,  $mn(K_p - e) = 2^p - 2^{p-2}$ .*

*Proof.* Let  $e = (u, v) \in E$ . For every vertex set  $S$  such that  $\{u, v\} \not\subseteq S$ , there exists a profile which has  $S$  as its median set, namely, the profile formed by taking the vertices of  $S$  exactly once. Let  $\pi$  be a profile which does not simultaneously contain  $u$  and  $v$ . Then  $M(\pi)$  is a subset of the set of vertices corresponding to the profile  $\pi$  and hence does not contain  $u$  and  $v$ . Now, let  $\pi$  be profile which contains both  $u$  and  $v$ . Then if  $u$  or  $v$  is repeated more than the other in the profile, then  $D(u, \pi) \neq D(v, \pi)$  and so they cannot appear together in the  $M(\pi)$ . Assume that  $\pi$  contains both  $u$  and  $v$  where both are repeated the same number of times. Let the profile be  $(x_1, \dots, x_k, \underbrace{u, \dots, u}_{m \text{ times}}, \underbrace{v, \dots, v}_{m \text{ times}})$ ,  $m \geq 1$ . For  $x_i$ ,  $1 \leq i \leq k$ ,  $D(x_i, \pi) \leq k - 1 + 2m$ . Also,  $D(u, \pi) = k + 2m$  and  $D(v, \pi) = k + 2m$ . Therefore  $M(\pi)$  does not contains both  $u$  and  $v$ . Now the profile  $(u, v)$  has  $V$  as its median set. Hence  $V$  is the only median set which contains both

$u$  and  $v$ . Therefore the class of all median sets of the graph consists of  $V$  and all subsets of  $V$  which do not simultaneously contain  $u$  and  $v$ . Hence,

$$\begin{aligned} mn(K_p - e) &= \binom{p}{1} + \binom{p}{2} - \binom{p-2}{0} + \binom{p}{3} - \binom{p-2}{1} \\ &\quad + \cdots + \binom{p}{p-1} - \binom{p-2}{p-3} + \binom{p}{p} \\ &= 2^p - 1 - (2^{p-2} - 1) \\ &= 2^p - 2^{p-2}. \end{aligned} \tag{2.1}$$

□

**Proposition 2.4** (Bandelt and Barthélemy [10]). *Let  $G = (V, E)$  be a median graph. For any profile  $\pi$  in  $G$  the median set is an interval  $I(u, v)$  in  $G$ .*

**Proposition 2.5.** *The median number of a tree  $T$  on  $p$  vertices is  $\binom{p}{2} + p$ .*

*Proof.* Since  $T$  is a median graph, by the above proposition all median sets are intervals. As observed in the proof of Proposition 2.1, all intervals are median sets. Therefore, class of median sets of  $T$  is precisely the class of intervals of  $T$  which is the class of all paths in  $T$ . Hence the median number is the number of distinct paths in  $T$  which is  $\binom{p}{2} + p$ . □

**Proposition 2.6** (Imrich and Klavžar [6]). *Let  $Q_r$  be a hypercube. Then, for any pair of vertices  $u, v \in Q_r$  the subgraph induced by the interval  $I(u, v)$  is a hypercube of dimension  $d(u, v)$ .*

**Theorem 2.7.** *For the hypercube  $Q_r$ ,  $mn(Q_r) = 3^r$*

*Proof.* Since  $Q_r$  is a median graph, by Propositions 2.4 and 2.6, every median set of  $Q_r$  is a subcube. Also in any graph  $G$ ,  $I(u, v)$  is the median set of the profile  $(u, v)$ , where  $u, v \in V(G)$ . Thus in a hypercube every subcube is a median set. Therefore, the median sets of  $Q_r$  are precisely the induced subcubes. So the Median number of  $Q_r$  is the number of subcubes of  $Q_r$ . Every vertex of  $Q_r$  contains  $r$  coordinates where each coordinate is either 0 or 1. Keeping  $k$  co-ordinates fixed and varying 0 and 1 over the other  $r - k$  positions, we get a subcube of dimension  $r - k$ . By varying 0's and 1's over these  $k$  positions we get  $2^k$  such subcubes. The  $k$  positions, to be fixed can be chosen in  $\binom{r}{k}$  ways. So the total number of subcubes of dimension  $r - k$  is  $2^k \times \binom{r}{k}$ . Therefore the total number of subcubes of  $Q_r$  is  $\sum_{0 \leq k \leq r} \binom{r}{k} \times 2^k = 3^r$ . □

**Theorem 2.8.** *For the wheel graph  $W_p$ ,  $p \geq 6$ ,  $mn(W_p) = (p^2 + 3p - 2)/2$ .*

*Proof.* Let  $\{v_1, v_2, \dots, v_{p-1}, v_p\}$  be the vertex set of  $W_p$  with  $v_p$  as the universal vertex, and let  $C_{p-1}$  be the cycle  $v_1, v_2, \dots, v_{p-1}, v_1$ . Each singleton set  $\{v_i\}$ ,  $1 \leq i \leq p - 1$ , is a median set. The sets  $\{v_i, v_j\}$ , where  $v_i$  and  $v_j$  are adjacent, are also median sets. The profile  $(v_i, v_{i+(p-1)1}, v_p)$ ,  $1 \leq i \leq p - 1$  ( $+_{p-1}$  means addition modulo  $p - 1$ ) has  $\{v_i, v_{i+(p-1)1}, v_p\}$  as median set. The set  $\{v_i, v_{i+(p-1)1}, v_{i+(p-1)2}, v_p\}$  is the median set of the profile  $(v_i, v_{i+(p-1)2})$ . Let  $\pi = (x_1, x_2, \dots, x_k)$  be a profile of  $W_p$  which contains the universal vertex  $v_p$ . Then since  $\pi$  contains the vertex  $v_p$ ,  $D(v_p, \pi) \leq k - 1$ . If some  $v_i$ ,  $1 \leq i \leq p - 1$ , belongs to  $M(\pi)$ , then  $D(v_i, \pi) \leq k - 1$  and this implies  $x_j = v_i$  at least for some  $j$ . Also, the number of  $x_j$ 's with  $d(v_i, x_j) = 2$  is less than the number of repetitions of  $v_i$  in  $\pi$ . Let  $v_k$  be such that  $d(v_k, v_i) = 2$ . Then  $v_k$  belongs to  $M(\pi)$

that implies number of repetitions of  $v_k$  is greater than the number of repetitions of  $v_i$  in the profile  $\pi$ . But these two statements are contradictory. Thus for a profile which contains the universal vertex the median set cannot contain two vertices which are at distance 2. Hence the only possible median sets for such a profile are

- (i) sets of type  $\{v_i, v_{i+(p-1)1}\}$ ,
- (ii) sets of type  $\{v_i, v_p\}$ ,
- (iii) sets of type  $\{v_i, v_{i+(p-1)1}, v_p\}$ .

Now, let  $\pi = (x_1, \dots, x_k)$  be a profile which does not contain  $v_p$ . Then  $D(v_p, \pi) = k$ . If some  $v_i$ ,  $1 \leq i \leq p-1$ , belongs to  $M(\pi)$ , then  $D(v_i, \pi) \leq k$ . Let  $v_j$  be such that  $v_j \in M(\pi)$  and  $d(v_i, v_j) = 2$ . Then  $D(v_j, \pi) \leq k$ . since  $D(v_i, \pi) \leq k$ , the number of zeroes in  $\{d(v_i, x_1), \dots, d(v_i, x_k)\} \geq$  number of twos in  $\{d(v_i, x_1), \dots, d(v_i, x_k)\}$ . Similarly, number of zeroes in  $\{d(v_j, x_1), \dots, d(v_j, x_k)\} \geq$  number of twos in  $\{d(v_j, x_1), \dots, d(v_j, x_k)\}$ .

Thus, number of repetitions of  $v_i$  in  $\pi =$  number of repetitions of  $v_j$  in  $\pi$ .

Now, let  $d_{C_{p-1}}(v_i, v_j) = 2$ . Without loss of generality, we may assume that  $j = i+(p-1)2$ . If some vertex other than  $v_i, v_{i+(p-1)1}, v_{i+(p-1)2}$  belongs to  $\pi$ , then  $D(v_i, \pi) = D(v_j, \pi) > D(v_p, \pi)$ . If  $v_{i+(p-1)1} \in \pi$ , then  $D(v_{i+(p-1)1}, \pi) < D(v_i, \pi)$ . Therefore  $\pi$  can only be  $(v_i, \dots, v_i, v_j, \dots, v_j)$ , where  $v_i$  and  $v_j$  are repeated the same number of times. Since  $j = i+(p-1)2$ , we have  $D(v_i, \pi) = D(v_j, \pi) = D(v_p, \pi) = D(v_{i+(p-1)1}, \pi)$  and for all other  $x \in V$ ,  $D(x, \pi) > k$ . Hence  $M(\pi) = \{v_i, v_{i+(p-1)1}, v_{i+(p-1)2}, v_p\}$ . If  $d_{C_{p-1}}(v_i, v_j) \neq 2$  then some vertex other than  $v_i$  and  $v_j$  that belong to  $\pi$  will contradict the fact that  $v_i$  and  $v_j$  belong to  $M(\pi)$ . Therefore, in this case also  $\pi = (v_i, \dots, v_i, v_j, \dots, v_j)$ , where  $v_i$  and  $v_j$  are repeated the same number of times. Here  $D(v_i, \pi) = k$ ,  $D(v_j, \pi) = k$ ,  $D(v_p, \pi) = k$  and for all other  $x \in V$ ,  $D(x, \pi) > k$ . In other words,  $M(\pi) = \{v_i, v_j, v_p\}$ .

Hence the only median sets are

- (1)  $\{v_i\}$ ,  $1 \leq i \leq p$ ,
- (2)  $\{v_i, v_{i+(p-1)1}\}$ ,  $1 \leq i \leq p-1$ ,
- (3)  $\{v_i, v_p\}$ ,  $1 \leq i \leq p-1$ ,
- (4)  $\{v_i, v_p, v_{i+(p-1)1}\}$ ,  $1 \leq i \leq p-1$ ,
- (5)  $\{v_i, v_j, v_p\}$   $1 \leq i, j \leq p-1, (i-j) \bmod (p-1) \geq 3$  (since  $p \geq 6$ , such vertices do exist),
- (6)  $\{v_i, v_{i+(p-1)1}, v_{i+(p-1)2}, v_p\}$ ,  $1 \leq i \leq p-1$ .

Thus,  $mn(W_p) = p + p - 1 + p - 1 + p - 1 + ((p-1)(p-6))/2 + p - 1 = (p^2 + 3p - 2)/2$ .  $\square$

**Theorem 2.9.**  $mn(K_{p,q}) = 2^{p+q} - 1$ , where  $K_{p,q}$  is the complete bipartite graph with  $p \leq q$ ,  $p > 2$ .

*Proof.* Let  $(X, Y)$  be a bipartition of  $K_{p,q}$  with  $|X| = p$  and  $|Y| = q$ . Let  $X = \{x_1, \dots, x_p\}$  and  $Y = \{y_1, \dots, y_q\}$ . Let  $A$  be a  $k$ -element subset of  $X$  with  $k \leq p$ . Without loss of generality, we may assume that  $A = \{x_1, \dots, x_k\}$ .

If  $k < q$ , take  $\pi = (x_1, \dots, x_k, y_1, \dots, y_q)$ . For each  $x_i$ ,  $1 \leq i \leq k$ ,  $D(x_i, \pi) = 2(k-1) + q$ . For each  $x_i$ ,  $k+1 \leq i \leq p$ ,  $D(x_i, \pi) = 2k + q$ . For each  $y_i$ ,  $1 \leq i \leq q$ ,  $D(y_i, \pi) = 2(q-1) + k$ . Therefore,  $A = \{x_1, \dots, x_k\} = M(\pi)$ .

If  $k = q$ , then  $\pi = (y_1, \dots, y_q)$  has median set  $A = \{x_1, \dots, x_k\}$ . Therefore, every subset of  $X$  is a median set.

Now, let  $B \subseteq Y$  with  $B = \{y_1, \dots, y_k\}$ .

If  $k < p$  then as in the previous case  $\pi = (x_1, \dots, x_p, y_1, \dots, y_k)$  has median set  $B$ .

Now, let  $k \geq p$  and let  $\pi$  be the profile  $(x_1, \dots, x_1, \dots, x_p, \dots, x_p, y_1, \dots, y_k)$ , where each  $x_i$  is repeated the same number of times, (say)  $r$ .

For each  $y_i, 1 \leq i \leq k, D(y_i, \pi) = 2(k-1) + pr$ , for each  $y_i, k+1 \leq i \leq q, D(y_i, \pi) = 2k + pr$ , and for each  $x_i, 1 \leq i \leq q, D(x_i, \pi) = 2r(p-1) + k$ . Moreover,  $2(k-1) + pr < 2r(p-1) + k \Leftrightarrow k-2 < (p-2)r \Leftrightarrow r > ((k-2)/(p-2)) (p > 2)$ . That is, if each  $x_i$  is repeated  $r$  times where  $r > (k-2)/(p-2)$ , then  $M(\pi) = B$ .

Now, let  $C = \{x_1, \dots, x_k, y_1, \dots, y_r\}, 1 \leq k \leq p, 1 \leq r \leq q$ .

Take  $\pi = (x_1, \dots, x_1, \dots, x_k, \dots, x_k, y_1, \dots, y_1, \dots, y_r, \dots, y_r)$ , where each  $x_i$  is repeated  $s_x$  times and  $y_i$  is repeated  $s_y$  times.

For each  $x_i, 1 \leq i \leq k, D(x_i, \pi) = 2(k-1)s_x + rs_y$ , for each  $y_i, 1 \leq i \leq r, D(y_i, \pi) = 2(r-1)s_y + ks_x$ , for each  $x_i, k+1 \leq i \leq p, D(x_i, \pi) = 2ks_x + rs_y$ , and for each  $y_i, r+1 \leq i \leq q, D(y_i, \pi) = 2rs_y + ks_x$ . Any  $x_i, k+1 \leq i \leq p$  or  $y_i, r+1 \leq i \leq q$  cannot be in  $M(\pi)$ .

Now  $2(k-1)s_x + rs_y = 2(r-1)s_y + ks_x \Leftrightarrow (k-2)s_x = (r-2)s_y$ . Hence for any  $s_x$  and  $s_y$  such that  $(k-2)s_x = (r-2)s_y$ , the profile  $(x_1, \dots, x_1, \dots, x_k, \dots, x_k, y_1, \dots, y_1, \dots, y_r, \dots, y_r)$ , where each  $x_i$  is repeated  $s_x$  times and  $y_i$  is repeated  $s_y$  times, has  $C$  as its median. Therefore, every nonempty subset of  $X \cup Y$  is a median set. Hence,  $mn(K_{p,q}) = 2^{p+q} - 1$ .  $\square$

**Lemma 2.10.** *Let  $\pi_1$  and  $\pi_2$  be profiles in the graphs  $G_1$  and  $G_2$ , respectively. If  $M(\pi_1) = M_1$  and  $M(\pi_2) = M_2$ , then  $\pi_1 \times \pi_2$  is a profile in the graph  $G_1 \square G_2$  with  $M(\pi_1 \times \pi_2) = M_1 \times M_2$ .*

*Proof.* Let  $\pi_1 = (u_1, u_2, \dots, u_m), \pi_2 = (v_1, v_2, \dots, v_n)$ , and  $M = M(\pi_1 \times \pi_2)$ .

Consider  $\pi_1 \times \pi_2 = ((u_1, v_1), \dots, (u_1, v_n), \dots, (u_m, v_1), \dots, (u_m, v_n))$ .

If  $(x_1, y_1), (x_2, y_2) \in V(G_1 \square G_2)$ , then  $d_{G_1 \square G_2}((x_1, y_1), (x_2, y_2)) = d_{G_1}(x_1, y_1) + d_{G_2}(x_2, y_2)$ , see [9]. For any  $(x, y) \in V(G_1 \square G_2), D((x, y), \pi_1 \times \pi_2) = n \sum_{1 \leq i \leq m} d(x, u_i) + m \sum_{1 \leq i \leq n} d(y, v_i)$ .

Let  $(a, b) \in M_1 \times M_2$ , that is,  $a \in M_1$  and  $b \in M_2$ . Then,

$$\begin{aligned} \sum_{1 \leq i \leq m} d(a, u_i) &\leq \sum_{1 \leq i \leq m} d(x, u_i), \quad \forall x \in V(G_1), \\ \sum_{1 \leq i \leq n} d(b, v_i) &\leq \sum_{1 \leq i \leq n} d(y, v_i), \quad \forall y \in V(G_2). \end{aligned} \tag{2.2}$$

Therefore,

$$n \sum_{1 \leq i \leq m} d(a, u_i) + m \sum_{1 \leq i \leq n} d(b, v_i) \leq n \sum_{1 \leq i \leq m} d(x, u_i) + m \sum_{1 \leq i \leq n} d(y, v_i), \quad \forall (x, y) \in V(G_1 \square G_2). \tag{2.3}$$

Hence,  $D((a, b), \pi_1 \times \pi_2) \leq D((x, y), \pi_1 \times \pi_2)$ , for all  $(x, y) \in V(G_1 \square G_2)$ .

Thus,  $(a, b) \in M_1 \times M_2 \Rightarrow (a, b) \in M$  or  $M_1 \times M_2 \subseteq M$ .

Now, let  $(a, b) \in M$

$$\begin{aligned} D((a, b), \pi_1 \times \pi_2) &= n \sum_{1 \leq i \leq m} d(a, u_i) + m \sum_{1 \leq i \leq n} d(b, v_i) \\ &\leq n \sum_{1 \leq i \leq m} d(x, u_i) + m \sum_{1 \leq i \leq n} d(y, v_i), \quad \forall (x, y) \in V(G_1 \times G_2). \end{aligned} \tag{2.4}$$

If for some  $x' \in V(G_1)$ ,

$$\begin{aligned} \sum_{1 \leq i \leq m} d(x', u_i) &< \sum_{1 \leq i \leq m} d(a, u_i), \text{ then} \\ n \sum_{1 \leq i \leq m} d(x', u_i) + m \sum_{1 \leq i \leq n} d(b, v_i) &< n \sum_{1 \leq i \leq m} d(a, u_i) + m \sum_{1 \leq i \leq n} d(b, v_i). \end{aligned} \quad (2.5)$$

This contradicts  $(a, b) \in M = M(\pi_1 \times \pi_2)$ .

Therefore

$$\begin{aligned} \sum_{1 \leq i \leq m} d(a, u_i) &\leq \sum_{1 \leq i \leq m} d(x, u_i), \quad \forall x \in V(G_1), \\ \sum_{1 \leq i \leq m} d(b, v_i) &\leq \sum_{1 \leq i \leq m} d(y, v_i), \quad \forall y \in V(G_2). \end{aligned} \quad (2.6)$$

Hence,  $a \in M_1$  and  $b \in M_2$  or  $(a, b) \in M_1 \times M_2$ . That is,  $M = M_1 \times M_2$ .  $\square$

**Theorem 2.11.**  $mn(G_1 \square G_2) = mn(G_1) \times mn(G_2)$ .

*Proof.* By the above lemma the product of median sets of  $G_1$  and  $G_2$  is again a median set of  $G_1 \square G_2$ . Now, let  $M$  be a median set of  $G_1 \square G_2$ , with  $M = M(\pi)$ , where  $\pi = ((u_1, v_1), \dots, (u_k, v_k))$ . Let  $\pi_1 = (u_1, \dots, u_k)$ ,  $\pi_2 = (v_1, \dots, v_k)$ ,  $M_1 = M(\pi_1)$ , and  $M_2 = M(\pi_2)$ . Let  $(a, b) \in M$ .

We have

$$\begin{aligned} \sum_{1 \leq i \leq k} d(a, u_i) + \sum_{1 \leq i \leq k} d(b, v_i) &\leq \sum_{1 \leq i \leq k} d(x, u_i) + \sum_{1 \leq i \leq k} d(y, v_i), \quad \forall x \in V(G_1), \quad \forall y \in V(G_2), \\ \therefore k \sum_{1 \leq i \leq k} d(a, u_i) + k \sum_{1 \leq i \leq k} d(b, v_i) &\leq k \sum_{1 \leq i \leq k} d(x, u_i) + k \sum_{1 \leq i \leq k} d(y, v_i), \quad \forall (x, y) \in V(G_1 \square G_2). \end{aligned} \quad (2.7)$$

In other words,  $D((a, b), \pi_1 \times \pi_2) \leq D((x, y), \pi_1 \times \pi_2)$ , for all  $(x, y) \in V(G_1 \square G_2)$ .

$\therefore (a, b) \in M(\pi_1 \times \pi_2)$  or  $M \subseteq M(\pi_1 \times \pi_2)$ .

Let  $(a, b) \in M(\pi_1 \times \pi_2)$ . Then  $D((a, b), \pi_1 \times \pi_2) \leq D((x, y), \pi_1 \times \pi_2)$ , for all  $x \in V(G_1)$ , for all  $y \in V(G_2)$ .

That is

$$\begin{aligned} k \sum_{1 \leq i \leq k} d(a, u_i) + k \sum_{1 \leq i \leq k} d(b, v_i) &\leq k \sum_{1 \leq i \leq k} d(x, u_i) + k \sum_{1 \leq i \leq k} d(y, v_i), \\ &\forall x \in V(G_1), \quad \forall y \in V(G_2), \\ \sum_{1 \leq i \leq k} d(a, u_i) + \sum_{1 \leq i \leq k} d(b, v_i) &\leq \sum_{1 \leq i \leq k} d(x, u_i) + \sum_{1 \leq i \leq k} d(y, v_i), \\ &\forall x \in V(G_1), \quad \forall y \in V(G_2), \\ \therefore \sum_{1 \leq i \leq k} d((a, b), (u_i, v_i)) &\leq \sum_{1 \leq i \leq k} d((x, y), (u_i, v_i)), \quad \forall (x, y) \in V(G_1 \square G_2). \end{aligned} \quad (2.8)$$

Therefore,  $(a, b) \in M$  which implies  $M(\pi_1 \times \pi_2) \subseteq M$  or  $M = M(\pi_1 \times \pi_2) = M(\pi_1) \times M(\pi_2)$ . Thus, the class of all median sets of  $G_1 \square G_2$  is the same as the class of all Cartesian products of median sets of  $G_1$  and  $G_2$ .

Hence,  $mn(G_1 \square G_2) = mn(G_1) \times mn(G_2)$ . □

The above theorem can be used to find the median number of various classes of graphs. The first of the following corollaries gives another technique to find out the Median number of a hypercube.

**Corollary 2.12.** *If  $G_1, \dots, G_k$  are  $k$  graphs, then  $mn(G_1 \square \dots \square G_k) = mn(G_1) \times \dots \times mn(G_k)$ .*

**Corollary 2.13.** *For the hypercube  $Q_r$ ,  $mn(Q_r) = 3^r$ .*

*Proof.* Since  $Q_r = \underbrace{K_2 \square \dots \square K_2}_{r \text{ times}}$ ,  $mn(Q_r) = \underbrace{mn(K_2) \times \dots \times mn(K_2)}_{r \text{ times}} = \underbrace{3 \times \dots \times 3}_{r \text{ times}} = 3^r$ . □

**Corollary 2.14.** *If  $G$  is the Grid graph  $P_r \square P_s$ ,  $mn(G) = \binom{r}{2} + r \times \binom{s}{2} + s$ .*

**Corollary 2.15.** *If  $G$  is the Hamming graph  $K_{p_1} \square K_{p_2} \square \dots \square K_{p_r}$ ,  $mn(G) = (2^{p_1} - 1) \times (2^{p_2} - 1) \times \dots \times (2^{p_r} - 1)$ .*

**Lemma 2.16.** *The only median set of the cycle  $C_{2r}$  which contains a vertex and its eccentric vertex is  $V$ .*

*Proof.* Let  $a$  and  $b$  be two eccentric vertices of the cycle  $C_{2r}$  which belongs to  $M(\pi)$  where  $\pi = (x_1, \dots, x_k)$ . Let  $D(a, \pi) = D(b, \pi) = s$ . Then

$$\begin{aligned} D(a, \pi) + D(b, \pi) &= d(a, x_1) + \dots + d(a, x_k) + d(b, x_1) + \dots + d(b, x_k) \\ &= d(a, x_1) + d(b, x_1) + \dots + d(a, x_k) + d(b, x_k) \\ &= \underbrace{d(a, b) + \dots + d(a, b)}_{k \text{ times}} \\ &= kr. \end{aligned} \tag{2.9}$$

Hence  $2s = kr$ . Now, suppose  $M(\pi) \neq V$ . Then there exists an  $x \in V$  such that  $D(x, \pi) > s$ . That is,  $d(x, v_1) + \dots + d(x, v_k) > s$ . Let  $y$  be the eccentric vertex of  $x$ .  $d(y, v_1) + \dots + d(y, v_k) \geq s$ . Therefore,  $d(x, v_1) + \dots + d(x, v_k) + d(y, v_1) + \dots + d(y, v_k) > 2s$ . That is,  $d(x, y) + \dots + d(x, y)$  ( $k$  times)  $> 2s$  or  $kr > 2s$ , a contradiction. Therefore, any set distinct from  $V$  which is a median set cannot contain two eccentric vertices. Also,  $M((a, b)) = V$ , since  $a$  and  $b$  are diametrical and  $I(a, b) = V$ . Hence the only median set of  $C_{2r}$  which contains a vertex and its eccentric vertex is  $V$ . □

**Theorem 2.17.** *For the even cycle  $C_{2r}$ ,  $mn(C_{2r}) \leq 3^r$ .*

*Proof.* Let  $V$  be the vertex set of  $C_{2r}$  with  $V = \{v_1, \dots, v_{2r}\}$ . Let  $A = \{S : S \subseteq V \text{ and } S \text{ does not contain any pair of eccentric vertices}\}$ . By the above lemma, the set of all median sets is a subset of  $A \cup \{V\}$ . Hence  $mn(C_{2r}) \leq |A| + 1$ . Let  $B_i = \{v_i, v_{i+r}\}$ ,  $1 \leq i \leq r$ . Now  $A$  consists of all subsets of  $V$  which does not simultaneously contain both the elements from the same  $B_i$ ,  $1 \leq i \leq r$ . The number of ways of choosing a  $k$ -element subset of  $V$  so that it belongs



to  $A$  is the product of the number of ways of choosing  $kB_i$ 's from the  $rB_i$ 's and the number of ways of choosing one element from each of these chosen  $B_i$ 's. That is,  $\binom{r}{k} \times 2^k$ . Therefore  $|A| = \sum_{k=1}^r \binom{r}{k} \times 2^k$ . Hence  $mn(C_{2r}) \leq (\sum_{1 \leq k \leq r} \binom{r}{k} \times 2^k) + 1 = \sum_{0 \leq k \leq r} \binom{r}{k} \times 2^k = 3^r$ .  $\square$

### 3. Conclusion

In this paper, we could identify the median set of certain classes of graphs and evaluate their median numbers. Identifying the median sets of more classes of graphs and finding their median numbers will be interesting and challenging. Another possibility is the realisation problem; given a number of finding a graph whose median number is that number.

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