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Strongly distance-balanced graphs and graph products[☆]

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ABSTRACT

A graph G is strongly distance-balanced if for every edge uv of G and every $i \geq 0$ the number of vertices x with $d(x, u) = d(x, v) - 1 = i$ equals the number of vertices y with $d(y, v) = d(y, u) - 1 = i$. It is proved that the strong product of graphs is strongly distance-balanced if and only if both factors are strongly distance-balanced. It is also proved that connected components of the direct product of two bipartite graphs are strongly distance-balanced if and only if both factors are strongly distance-balanced. Additionally, a new characterization of distance-balanced graphs and an algorithm of time complexity $O(mn)$ for their recognition, where m is the number of edges and n the number of vertices of the graph in question, are given.

1. Introduction

Let G be a simple undirected graph. The *distance* $d_G(u, v)$ between vertices $u, v \in V(G)$ is the length of a shortest path between u and v in G. (If the graph G is clear from the context, we simply write $d(u, v)$.) For a pair of adjacent vertices $a, b \in V(G)$ let W_{ab} denote the set of all vertices of G closer to a than to b and let ${}_aW_b$ denote the set of all vertices of G that are at the same distance to a and b. For each $i \geq 0$ let ℓ_i^{ab} and ${}^a\ell_i^b$ be the subsets of W_{ab} and ${}_aW_b$, resp., of all the vertices at distance

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i to a. That is

$$W_{ab} = \{x \in V(G) \mid d(a,x) < d(b,x)\}, \qquad {}_aW_b = \{x \in V(G) \mid d(a,x) = d(b,x)\},$$
$$\ell_i^{ab} = \{x \in W_{ab} \mid d(a,x) = i\}, \qquad {}^a\ell_i^b = \{x \in {}_aW_b \mid d(a,x) = i\}.$$

Distance-balanced graphs were introduced in [5] as graphs for which $|W_{ab}| = |W_{ba}|$ for every pair of adjacent vertices $a, b \in V(G)$. The authors were motivated by [3] and focused on some basic properties, local operations, and their connection to product graphs. In particular, they proved that the Cartesian product of graphs is distance-balanced if and only if both factors are distance-balanced and that the lexicographic product $G \circ H$ is distance-balanced precisely when G is distance-balanced and H is regular and connected. Additionally, an example showing that the direct and the strong product of distance-balanced graphs is not necessarily distance-balanced was given.

The study of distance-balanced graphs was continued in [8], where strongly distance-balanced graphs were introduced. A graph G is *strongly distance-balanced* (for short SDB) if $|\ell_i^{ab}| = |\ell_i^{ba}|$ holds for every pair of adjacent vertices a, b of G and for every $i \geq 0$. Note that every strongly distance-balanced graph is distance-balanced. It was shown in [8] that a graph G of diameter d is strongly distance-balanced if and only if $|S_i(a)| = |S_i(b)|$, where $S_i(a) = \{x \in V(G) \mid d(a,x) = i\}$, holds for every pair of adjacent vertices $a, b \in V(G)$ and every $i \in \{0, 1, \ldots, d\}$. It is thus clear that every vertex-transitive graph is strongly distance-balanced. SDB property of semisymmetric graphs and generalized Petersen graphs was also studied in [8] and later continued in [9]. As for graph products, it was shown in [8] that for the Cartesian and the lexicographic products, analogous results as in the case of distance-balanced graphs [5] hold also for SDB graphs. On the other hand, the strong and the direct products of graphs were not dealt with. It is the aim of the next section to fill in this gap.

In the last section a simple distance condition which is characteristic of distance-balanced graphs is given. Let G be a connected graph. The *median* $M(G)$ of G is the set of all vertices x of G for which the number

$$d(x, V(G)) = \sum_{v \in V(G)} d(x, v)$$

is minimal among all vertices of G. The concept of the median of a graph is one of the basic centrality concepts in graphs and various generalized notions of medians are studied by many authors, see e.g. [1,10]. We show that the condition $M(V(G)) = V(G)$ is characteristic of distance-balanced graphs. Note that in such a case $d(u, V(G)) = d(v, V(G))$ holds for all vertices $u, v \in V(G)$. Graphs fulfilling the above condition are of interest for studies on social networks, since all people in such graphs are 'equal'. Distance-balanced graphs having trivial automorphism group are of particular interest, since in such graphs people are not only 'equal', but also 'unique'. Two families of such graphs have been introduced in [7].

In the remainder of this section we define the Cartesian, strong and direct product and mention some of their properties.

For all three products of graphs G and H the vertex set of the product is $V(G) \times V(H)$. Their edge sets are defined as follows. In the *Cartesian product* $G \square H$ two vertices are adjacent if they are adjacent in one coordinate and equal in the other. In the *direct product* $G \times H$ two vertices are adjacent if they are adjacent in both coordinates. Finally, the edge set $E(G \boxtimes H)$ of the *strong product* $G \boxtimes H$ is the union of $E(G \square H)$ and $E(G \times H)$. Note that all three products are commutative and associative, cf. [4]. It is well-known that

$$d_{G \square H}((u, v), (x, y)) = d_G(u, x) + d_H(v, y)$$

and

$$d_{G \boxtimes H}((u, v), (x, y)) = \max\{d_G(u, x), d_H(v, y)\}.$$

The distance formula for the direct product was first shown in [6]. In this note, we use an equivalent version from [2] stated in the following lemma.

Lemma 1.1. *Let G and H be two graphs, and $(x_1, y_1), (x_2, y_2)$ be two vertices of $G \times H$. Then the distance*

$$d((x_1, y_1), (x_2, y_2))$$

in $G \times H$ is the least k such that there is an x_1x_2-walk of length k in G and a y_1y_2-walk of length k in H.

2. Strongly distance-balanced graphs and the products

It was proved in [5] that the Cartesian product of graphs is distance-balanced if and only if both factors are distance-balanced. An analogous result for SDB graphs was proved in [8]. In this section we show that the property of being SDB is invariant also under the strong product (Theorem 2.1). Moreover, we show that each component of the direct product of bipartite graphs is SDB if and only if both factors are SDB (Theorem 2.2).

Theorem 2.1. *Let G and H be graphs. Then $G \boxtimes H$ is strongly distance-balanced if and only if G and H are strongly distance-balanced.*

Proof. Suppose that G and H are strongly distance-balanced. Let $(a, b)(c, d)$ be an edge of $G \boxtimes H$. If ac and bd are edges of G and H, resp., there exist bijections

$$f_G : V(G) \to V(G) \quad \text{and} \quad f_H : V(H) \to V(H),$$

such that $f_G({}^{a}\ell_i^{c}) = {}^{a}\ell_i^{c}, f_G(\ell_i^{ac}) = \ell_i^{ca}, f_G(\ell_i^{ca}) = \ell_i^{ac}$ and $f_H({}^{b}\ell_i^{d}) = {}^{b}\ell_i^{d}, f_H(\ell_i^{bd}) = \ell_i^{db}, f_H(\ell_i^{db}) = \ell_i^{bd}$ hold for all $i \geq 0$. Otherwise $a = c$ or $b = d$; in this case let f_G resp. f_H be the identity on G resp. H. It follows that for every $u \in V(G)$ and $v \in V(H)$ we have that

$$d_G(u, a) = d_G(f_G(u), c) \quad \text{and} \quad d_H(v, b) = d_H(f_H(v), d).$$

Consider the function $f_{G\boxtimes H} : V(G \boxtimes H) \to V(G \boxtimes H)$ defined by

$$(u, v) \mapsto (f_G(u), f_H(v)).$$

Since f_G and f_H are both bijections, so is $f_{G\boxtimes H}$. Observe that for any $(u, v) \in V(G \boxtimes H)$ we have that

$$\begin{aligned} d_{G\boxtimes H}((u, v), (a, b)) &= \max\{d_G(u, a), d_H(v, b)\} = \max\{d_G(f_G(u), c), d_H(f_H(v), d)\} \\ &= d_{G\boxtimes H}((f_G(u), f_H(v)), (c, d)) = d_{G\boxtimes H}(f_{G\boxtimes H}((u, v)), (c, d)) \end{aligned}$$

and similarly $d_{G\boxtimes H}((u, v), (c, d)) = d_{G\boxtimes H}(f_{G\boxtimes H}((u, v)), (a, b))$. It follows that for every $i \geq 0$

$$\begin{array}{rcl} & (u, v) \in \ell_i^{(a,b)(c,d)} & \iff \\ d_{G\boxtimes H}((u, v), (a, b)) = i & \text{and} \quad d_{G\boxtimes H}((u, v), (c, d)) = i + 1 & \iff \\ d_{G\boxtimes H}(f_{G\boxtimes H}((u, v)), (c, d)) = i & \text{and} \quad d_{G\boxtimes H}(f_{G\boxtimes H}((u, v)), (a, b)) = i + 1 & \iff \\ & f_{G\boxtimes H}((u, v)) \in \ell_i^{(c,d)(a,b)} & \end{array}$$

hold, which implies $f_{G\boxtimes H}(\ell_i^{(a,b)(c,d)}) = \ell_i^{(c,d)(a,b)}$. Similarly $f_{G\boxtimes H}(\ell_i^{(c,d)(a,b)}) = \ell_i^{(a,b)(c,d)}$, and so $G \boxtimes H$ is a SDB graph.

Now suppose that G or H is not a SDB graph. Assume without loss of generality that H is not SDB. Then there exists an edge $xy \in E(H)$ with $|\ell_i^{xy}| \neq |\ell_i^{yx}|$ for some $i \geq 1$. Observe that for any vertex $a \in V(G)$ we have that $(u, v) \in W_{(a,x)(a,y)}$ if and only if

$$\max\{d_G(u, a), d_H(v, x)\} < \max\{d_G(u, a), d_H(v, y)\}.$$

However, since xy is an edge of H this occurs if and only if

$$\max\{d_G(u, a), d_H(v, y)\} = d_H(v, y) = d_H(v, x) + 1 \geq d_G(u, a) + 1.$$

It follows that $(u, v) \in \ell_i^{(a,x)(a,y)}$ if and only if $v \in \ell_i^{xy}$ and $d_G(u, a) \leq i$. Therefore for an edge $(a, x)(a, y) \in E(G \boxtimes H)$ of $G \boxtimes H$ we have that

$$\left|\ell_i^{(a,x)(a,y)}\right| = k\left|\ell_i^{xy}\right| \neq k\left|\ell_i^{yx}\right| = \left|\ell_i^{(a,y)(a,x)}\right|,$$

where

$$k = |\{u \in V(G) \mid d_G(u, a) \leq i\}|.$$

This proves that $G \boxtimes H$ is not SDB. $\square$

Note that the direct product of vertex-transitive graphs is vertex-transitive and hence SDB. However, the direct product of arbitrary SDB graphs is not necessarily SDB. For example, it is easy to see that the generalized Petersen graph $GPG(7,2)$ is a SDB graph [8], whereas the graph $K_2 \times GPG(7,2)$ is not SDB. We leave the details to the reader. Nevertheless, if both factors are bipartite, then their direct product is SDB if and only if they both are SDB, as the next theorem shows. Let G and H be bipartite graphs. It follows from Lemma 1.1 that then $G \times H$ is not connected and that (x_1, x_2) and (y_1, y_2) are in the same connected component of $G \times H$ if and only if $d_G(x_1, y_1)$ and $d_H(x_2, y_2)$ are either both even or both odd.

Theorem 2.2. *Let G and H be connected bipartite graphs. Then both connected components of $G \times H$ are strongly distance-balanced if and only if G and H are strongly distance-balanced.*

Proof. Let $a, b \in V(G)$. Since G is bipartite either all ab-walks are of odd length or they all are of even length. A similar observation can be made for the graph H. Thus, by Lemma 1.1, for every pair of vertices $(a, c), (b, d) \in V(G \times H)$ we have that $d_{G\times H}((a, c), (b, d)) = i$ if and only if either $d_G(a, b) = i$ and $i - d_H(c, d)$ is a nonnegative even number or $d_H(c, d) = i$ and $i - d_G(a, b)$ is a nonnegative even number. Now let $x_1x_2 \in E(G)$ and $y_1y_2 \in E(H)$. Then $(u, v) \in \ell_i^{(x_1,y_1)(x_2,y_2)}$ if and only if $u \in \ell_i^{x_1x_2}$ and $i - d_H(v, y_1)$ is a nonnegative even number or $v \in \ell_i^{y_1y_2}$ and $i - d_G(u, x_1)$ is a nonnegative even number.

Now suppose that G and H are both SDB bipartite graphs. It follows that $|\ell_i^{x_1x_2}| = |\ell_i^{x_2x_1}|$ and that the number of vertices v such that $i - d_H(v, y_1)$ is nonnegative and even is equal to the number of vertices v such that $i - d_H(v, y_2)$ is nonnegative and even. Similarly, $|\ell_i^{y_1y_2}| = |\ell_i^{y_2y_1}|$ and there is bijection between the set of vertices u with $i - d_G(u, x_1)$ nonnegative and even and the set of vertices u with $i - d_G(u, x_2)$ nonnegative and even. Therefore

$$|\ell_i^{(x_1,y_1)(x_2,y_2)}| = |\ell_i^{(x_2,y_2)(x_1,y_1)}|$$

holds for all i, and thus $G \times H$ is a SDB graph.

To prove the converse, now suppose that G and H are not both SDB. By the commutativity of the direct product we can assume that there exists a pair of adjacent vertices x_1, x_2 of G such that for some integer $i \geq 1$ we have that $|\ell_i^{x_1x_2}| \neq |\ell_i^{x_2x_1}|$. Fix an arbitrary edge y_1y_2 of H and let i be the smallest integer for which $|\ell_i^{x_1x_2}| \neq |\ell_i^{x_2x_1}|$ or $|\ell_i^{y_1y_2}| \neq |\ell_i^{y_2y_1}|$. With no loss of generality we can assume that $|\ell_i^{x_1x_2}| \geq |\ell_i^{x_2x_1}|$ and $|\ell_i^{y_1y_2}| \geq |\ell_i^{y_2y_1}|$. We claim that

$$|\ell_i^{(x_1,y_1)(x_2,y_2)}| > |\ell_i^{(x_2,y_2)(x_1,y_1)}|. \tag{1}$$

By the remarks from the first paragraph of this proof the left side of (1) equals

$$|\ell_i^{x_1x_2}| \left(\sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} |\ell_{i-2k}^{y_1y_2}| + \sum_{k=0}^{\lfloor \frac{i-1}{2} \rfloor} |\ell_{i-1-2k}^{y_2y_1}| \right) + |\ell_i^{y_1y_2}| \left(\sum_{k=1}^{\lfloor \frac{i}{2} \rfloor} |\ell_{i-2k}^{x_1x_2}| + \sum_{k=0}^{\lfloor \frac{i-1}{2} \rfloor} |\ell_{i-1-2k}^{x_2x_1}| \right).$$

Note that the first sum above is taken from $k = 0$ and the third from $k = 1$ so that the vertices $(u, v) \in \ell_i^{x_1x_2} \times \ell_i^{y_1y_2}$ are not counted twice. Analogously, the right side of (1) equals

$$|\ell_i^{x_2x_1}| \left(\sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} |\ell_{i-2k}^{y_2y_1}| + \sum_{k=0}^{\lfloor \frac{i-1}{2} \rfloor} |\ell_{i-1-2k}^{y_1y_2}| \right) + |\ell_i^{y_2y_1}| \left(\sum_{k=1}^{\lfloor \frac{i}{2} \rfloor} |\ell_{i-2k}^{x_2x_1}| + \sum_{k=0}^{\lfloor \frac{i-1}{2} \rfloor} |\ell_{i-1-2k}^{x_1x_2}| \right).$$

By minimality of i we have that

$$\sum_{k=0}^{\lfloor \frac{i-1}{2} \rfloor} |\ell_{i-1-2k}^{y_2y_1}| = \sum_{k=0}^{\lfloor \frac{i-1}{2} \rfloor} |\ell_{i-1-2k}^{y_1y_2}| \quad \text{and} \quad \sum_{k=0}^{\lfloor \frac{i-1}{2} \rfloor} |\ell_{i-1-2k}^{x_2x_1}| = \sum_{k=0}^{\lfloor \frac{i-1}{2} \rfloor} |\ell_{i-1-2k}^{x_1x_2}|$$

and since $|\ell_i^{x_1x_2}| \geq |\ell_i^{x_2x_1}|$ and $|\ell_i^{y_1y_2}| \geq |\ell_i^{y_2y_1}|$ we also find that

$$\sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} |\ell_{i-2k}^{y_1y_2}| \geq \sum_{k=0}^{\lfloor \frac{i}{2} \rfloor} |\ell_{i-2k}^{y_2y_1}| \quad \text{and} \quad \sum_{k=1}^{\lfloor \frac{i}{2} \rfloor} |\ell_{i-2k}^{x_1x_2}| \geq \sum_{k=1}^{\lfloor \frac{i}{2} \rfloor} |\ell_{i-2k}^{x_2x_1}|.$$

Since at least one of $|\ell_i^{x_1x_2}| > |\ell_i^{x_2x_1}|$ and $|\ell_i^{y_1y_2}| > |\ell_i^{y_2y_1}|$ holds, inequality (1) follows. Consequently, $G \times H$ is not a SDB graph. $\square$

3. Distance-balanced graphs and their recognition

In this section we show that for a connected graph G the condition $M(V(G)) = V(G)$ is characteristic of distance-balanced graphs, which yields a simple recognition algorithm for such graphs.

Theorem 3.1. *A connected graph G is distance-balanced if and only if $M(V(G)) = V(G)$.*

Proof. Let u, v be a pair of adjacent vertices of G. We claim that $d(u, V(G)) = d(v, V(G))$ if and only if $|W_{uv}| = |W_{vu}|$. Indeed, this follows from the equivalence of the following equations:

$$\begin{aligned}
& d(u, V(G)) = d(v, V(G)) \\
& d(u, W_{uv}) + d(u, {}_uW_v) + d(u, W_{vu}) = d(v, W_{uv}) + d(v, {}_uW_v) + d(v, W_{vu}) \\
& \sum_{x \in W_{uv}} d(u, x) + \sum_{x \in W_{vu}} d(u, x) = \sum_{x \in W_{uv}} d(v, x) + \sum_{x \in W_{vu}} d(v, x) \\
& \sum_{x \in W_{uv}} d(u, x) - \sum_{x \in W_{uv}} d(v, x) = \sum_{x \in W_{vu}} d(v, x) - \sum_{x \in W_{vu}} d(u, x) \\
& \sum_{x \in W_{uv}} (d(u, x) - d(v, x)) = \sum_{x \in W_{vu}} (d(v, x) - d(u, x)) \\
& \sum_{x \in W_{uv}} (-1) = \sum_{x \in W_{vu}} (-1)\,.
\end{aligned}$$

Since the graph G is connected, the result follows. $\square$

Is there a similar distance condition for strongly distance-balanced graphs? It is clear that in a strongly distance-balanced graph G for every pair of adjacent vertices u, v we have that $d(u, W_{uv}) = d(v, W_{vu})$. We conjecture that the converse is also true.

Conjecture 3.2. *A graph G is strongly distance-balanced if and only if $d(u, W_{uv}) = d(v, W_{vu})$ holds for every pair of adjacent vertices u, v of G.*

Corollary 3.3. *A distance-balanced graph G can be recognized in $O(mn)$ time, where n is the number of vertices and m the number of edges of G.*

Proof. Let G be a graph. For each vertex $u \in V(G)$ we have to compute

$$d(u, V(G)) = \sum_{v \in V(G)} d(u, v).$$

Clearly, this can be done in $O(m)$ time with a BFS algorithm. Doing this for each vertex of G requires $O(mn)$ operations which, by Theorem 3.1, is enough to test whether G is distance-balanced or not. $\square$

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