Stochastic Modelling: Analysis and Applications

DISCRETE TIME INVENTORY MODELS WITH/WITHOUT POSITIVE SERVICE TIME

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Certificate

This is to certify that the thesis entitled ‘ **Discrete Time Inventory Models with/without Positive Service Time** ’ submitted to the Cochin University of Science and Technology by Mrs. Deepthi C. P. for the award of the degree of Doctor of Philosophy under the Faculty of Science is a bonafide record of studies carried out by her under my supervision in the Department of Mathematics, Cochin University of Science and Technology. This report has not been submitted previously for considering the award of any degree, fellowship or similar titles elsewhere.

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Declaration

I, Deepthi C.P, hereby declare that this thesis entitled '**Discrete Time Inventory Models with/without Positive Service Time**' contains no material which had been accepted for any other Degree, Diploma or similar titles in any University or institution and that to the best of my knowledge and belief, it contains no material previously published by any person except where due references are made in the text of the thesis.

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Deepthi. C. P

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Chapter 1

Introduction

In everyday life different flows of customers to avail some service facility or other at some service station are experienced. In some of these situations, congestion of items arriving for service, because an item can not be serviced immediately on arrival, is unavoidable. A queueing system can be described as customers arriving for service, waiting for service if it is not immediate, and if having waited for service, leaving the system after being served. Examples include shoppers waiting in front of check out stands in a supermarket, programs waiting to be processed by a digital computer, ships in the harbour waiting to be unloaded, persons waiting at railway booking office etc.

A queueing system is specified completely by the following characteristics: input or arrival pattern, service pattern, number of service channels, system capacity, queue discipline and number of service stages. The ultimate objective of solving queueing models is to determine the characteristics that measure the performance of the system.

1.1 Inventory system

Inventories deal with maintaining sufficient stocks of goods that will ensure a smooth and efficient running of a system or a business activity. Inventory may include raw materials, finished goods awaiting shipment from the factory, a group of personnel undergoing training for a firm, space available for books in a library, water kept in a dam etc. Inventory models have a wide range of application in industries, hospitals, banks, agriculture, educational institutions etc.

The ultimate objective of any inventory model is to answer two basic questions: how much to order and when to order. The answer to the first question is expressed in terms of what we call the order quantity and that of the second, the reorder level. Order quantity is the optimum amount that should be ordered every time an order is placed so as to minimize the total system running cost. Reorder level depends on the type of inventory model.

The objective of inventory control is often to balance conflicting goal of making available the required item at a time of need and minimizing the related costs. In inventory models, the availability of items has also to be taken into consideration along with features of queueing theory. In inventory models with negligible service times, queue of customers is formed only when the system is out of stock and unsatisfied customers are permitted to wait. On the contrary for the case of inventory with positive service time, queue is formed even when inventoried items are available because new customers can join while a service is going on. If either service time or lead time or both are taken to be positive, then also a queue is formed, depending on assumptions on backlogging of demands/on other factors.

The real need for inventory analysis was first recognized in industries that had a combination of production scheduling problems and inventory problems. The analysis of inventory problem was started by Harris in 1915. He proposed the EOQ (Economic Order Quantity) formula and was popularized by Wilson and is usually referred to as Harris- Wilson economic lot size or simply the EOQ. It is the ordering quantity which minimizes the total inventory cost. Some of the inventory related costs are holding cost, reorder cost, procurement cost, shortage cost etc. The cost analysis of different inventory policies is given by Naddor [42]. The book by Hadley and Whitin [19] provides inventory theory and applications.

While dealing with inventory systems, there are several factors which have to be taken into consideration. These include demand process, lead time, review policy, backlog, perishability of stored items etc.

Demand process

The number of units required per period is called demand rate. The demand pattern of a commodity may be either deterministic or probabilistic.

Lead Time

Sometimes, when an order is placed, it may take some time before delivery is effected. The time between the placement of an order and its receipt is known as lead time (delivery lag). It may be deterministic or probabilistic. If the replenishment is instantaneous, then the lead time is zero, otherwise, the system is said to have positive lead time.

Review Policy (Periodic review and continuous review)

In periodic review, the level of inventory is monitored at prefixed equal time points (every week or month etc). At any point in time the amount of inventory stored is not known exactly. In continuous review, the level of inventory is monitored continuously. In this case the inventory level at any point in time is known exactly.

Backlog

The demands that arrive when the inventory is out of stock, may be backlogged partially or fully or in some cases not entertained. These demands would be satisfied as and when the replenishment is received or through subsequent replenishments. Backlog generally refers to an accumulation of work over time, waiting to be done or orders to be fulfilled. Cases of full backlog, partial backlog and no backlog are considered in the literature.

Perishability of stored items

Perishable inventory systems are studied as queues with impatient customers. The perishing of many products like fish, vegetables etc are continuous and depends upon many factors including heat, humidity etc. Several attempts have been made to study some aspects of perishable inventories. A review of the work on perishable inventory is provided by Nahmias and Stevens [45], Baker [13] ; besides many researchers have contributed to the development of such a study.

Ordering Policy

Inventory system based on (s, S) policy have been studied quite extensively by many researchers during the last three decades. In an (s, S) policy, if x is the amount of inventory on hand before an order is placed, then the order quantity is such that
if $x \leqslant s$, then order $S - x$ and if $x > s$, do not order any quantity.
Here s is such that order for replenishment placed each time the inventory level drops to s or below for the first time after the previous replenishment and S is the maximum inventory. Efficient management of inventory systems is to determine the optimal values of s and S, that minimizes the long run expected cost rate. In randomized order size, the decision of the order size is according to a discrete probability function u on the set $\{1, 2, \ldots, S\}$.

The size of a replenishment order is k with probability u_k, with $\sum_{k=1}^{S} u_k = 1$. In fixed quantity ordering policy, whenever the inventory level falls to s, an order for a fixed quantity Q, where $Q = S - s$, is placed.

1.1.1 Discrete Time Inventory Systems

Discrete time queueing system has been found to be more appropriate in modelling computer systems and communication network. It can be used to approximate the corresponding continuous system in practice. The earliest work on discrete time queue is due to Meisling [41]. Since then, discrete time queues have been studied extensively by many researchers. A few books on discrete time queues are by Bruneel and Kim [13], Takagi [55], Woodward [58]. In discrete time inventory system, the time axis is divided into equal intervals called slots. All inventory activities are assumed to occur at the epochs numbered $0, 1, \ldots$ only. We describe the discrete time system as defined by Dafermos and Neuts [15]. They consider the arrivals and the service commencements and completion which occur between time epochs n and $n+1$, to occur at time $n+1$. Service times are at least one unit of time long. In discrete time systems, more than one different events can occur simultaneously in a slot with positive probability. So in order to resolve conflicts, a rule has to be formulated in advance about the order in which the arrivals and the departures take place in case of simultaneity. Such rules come to play mainly at the boundaries. In dealing with such conflicts, there are essentially two rules :(i) Late Arrival System (LAS) in which an arrival takes precedence over a departure and (ii) Early Arrival System (EAS) in which a departure takes precedence over an arrival.They are also known as Arrival First (AF) and Departure First (DF) policies respectively. If the server is idle and a customer arrives, then either his service starts immediately

(Immediate Access (IA)) or in the following slot (Delayed Access (DA)). LAS-IA corresponds to EAS. For more details, see Gravey and Hebuterne [17]. Hunter [21] considers n^- and n^+ and then defines discrete system based on these.

Perishable inventory problems with constant lifetime have been studied quite extensively using the periodic review policy. Periodic review models fit the constant lifetime well, but they usually lead to numerically difficult dynamic optimization problems. Fries [16] and Nahmias [43] use dynamic programming in a perishable inventory model with a lifetime m, zero lead time and zero ordering cost.

Bernoulli Process

Let $E := \{0, 1, 2, \ldots\}$ and choose any parameter $p \in (0, 1)$. The definitions $X_0 := 0$ together with the transition probabilities

$$p_{ij} = \begin{cases} p & j = i + 1 \\ 1 - p & j = i \\ 0 & \text{otherwise} \end{cases}$$

for $i \in E$ determine a homogeneous Markov Chain $\mathcal{X} = \{X_n : n \in E\}$. It is called Bernoulli process with parameter p.

Geometric Distribution

Let the random variable X denote the number of trials of a random experiment required to obtain the first success. It can assume the values $1, 2, \ldots$. Now $X = r$ if and only if the first $r - 1$ trials result in failure and the r^{th} trial results in success. Hence $P(X = r) = (1 - p)^{r-1}p$; $r = 1, 2, \ldots$ where p is the probability of success and $1 - p$ that of failure. Thus X has the geometric distribution. It is the discrete time analogue of exponential distribution. Memoryless property characterizes geometric distribution among all distributions of discrete non negative integer-valued random variables.

Birth and Death Process

A Discrete Time Birth and Death Process is a Markov Chain $\{X_t : t \in N\}$ on the nonnegative integers characterized by the property that whenever a transition occurs from one state to another, then this transition can be to a neighboring state only. Let $S = \{0, 1, \ldots, i, \ldots\}$ be the state space; transitions occur from i to $i + 1$ or to $i - 1$ only.

1.2 Quasi-Birth-Death Processes

Consider a two dimensional Markov Chain $\{X_t : t \in N\}$ with state space$\{(n, j) : n \geqslant 0; 1 \leqslant j \leqslant m\}$, which we partition as $\bigcup\limits_{n \geqslant 0} l(n)$, where $l(n)$=$\{(n, 1), (n, 2), \ldots (n, m)\}$ for $n \geqslant 0$. The first coordinate n is called the level and the second coordinate j is called the phase of the state (n, j).
The Markov chain is called a QBD if one-step transitions from a state are restricted to states in the same level or in the two adjacent levels: it is possible to move in one step from (n, j) to (n', j') only if $n' = n, n{+}1$ or $n{-}1$ (provided in the last case that $n \geqslant 1$). If $n = 0$, then $n' = 0$ or 1. If the transition rates are level independent, then the QBD process is called Level Independent Quasi-Birth-Death process (LIQBD). If the transition rates depend on the level, then the QBD process is called Level Dependent Quasi-Birth-Death process (LDQBD). The transition matrix is block tridiagonal and has the following form

$$\mathbf{P} = \begin{bmatrix} C_0 & C_1 & & & \\ C_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

where entries are all matrices. A_0 represents the arrival of a customer to the system; that is transition from $l(n)$ to $l(n+1)$. A_2 represents departure of a customer after completing service: $l(n)$ to $l(n-1)$, where $l(n)$ is the set of phases in level n. A_1 describes all transitions in which the level does not change (transitions within levels).
QBDs are matrix generalizations of Birth and Death processes.

1.3 Matrix Analytic Methods

During the late 1970's Neuts introduced matrix analytic methods, subsequently it was developed by his students and collaborators. It is a tool to construct and analyze a vide class of stochastic models, particularly queueing systems or inventory systems, using a matrix formalism to develop algorithmically tractable solution. For a detailed description of this method see Neuts [46] or Latouche and Ramaswamy [34]. Assume that the QBD is aperiodic and positive recurrent. Denote by $\boldsymbol{x}$ its stationary probability vector. It is the unique solution of the system $\boldsymbol{x}P = \boldsymbol{x}$ and $\boldsymbol{x}\mathbf{e} = 1$, where $\mathbf{e}$ is a column vector of ones of appropriate order. Let $\boldsymbol{x}$ be partitioned by levels as $\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots)$. Then $\boldsymbol{x}_i$ has the matrix geometric form $\boldsymbol{x}_i = \boldsymbol{x}_1 R^{i-1}, i \geqslant 2$ where R is the minimal non negative solution of the matrix quadratic equation $R^2 A_2 + RA_1 + A_0 = R$. The vectors $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$

are obtained by solving the equations

$$\boldsymbol{x}_0(C_0 - I) + \boldsymbol{x}_1 A_2 = 0$$

and

$$\boldsymbol{x}_0 C_1 + \boldsymbol{x}_1(A_1 + RA_2 - I) = 0$$

with the normalizing condition

$$\boldsymbol{x}_0 \boldsymbol{e} + \boldsymbol{x}_1 (I - R)^{-1} \boldsymbol{e} = 1.$$

1.4 Review of related work

A systematic approach to (s, S) inventory policy is provided by Arrow, Karlin and Scarf [3] using renewal theory. One of the recent contributions of significance to inventory with positive service time is due to Schwarz et alia (et al.) [51]. They assume Poisson arrival of demands, exponential service time and balking of customers when the inventory level is zero. They derived joint stationary distributions of the queue length and inventory level in explicit product form under continuous review of inventory level and different inventory management policies (see [27], [49]).
Krishnamoorthy and Viswanath [31] analyzed production inventory system with service time wherein Schwarz et al. [51] is subsumed. By assuming that no customer joins the queue when the inventory level is zero, they obtained the long run system state probability in product form. It is the first reported work on production inventory with positive service time in the continuous case providing product form solution. The main difference between (s, S) inventory system with positive lead time and (s, S) production inventory system is that in the former case, once the order is placed, it takes a random amount of time for the replenishment, whereas in the later case once the

production process is switched on consequent to inventory level decreasing to s, it is switched off only when the inventory level reaches S. Sreenivasan [54] examined (s, S) inventory systems with adjustable reorder sizes. Jose [28] compared three (s, S) inventory models with positive service time and lead time and with retrial of customers. One of the works of the queueing theory has been carried out by Yang and Li [60] who extended the queues with repeated attempts to the discrete time systems. The survey paper by Krishnamoorthy et al. [29] discussed in details various inventory models with positive service time. Lalitha [32] studied five distinct (s, S) inventory models with positive service time and lead time where arrival of demands is according to a Poisson process, service time and lead time following distinct exponential distributions and obtained performance measures, constructed cost functions for each model and numerically analyzed them. Sajeev [50] analyzed a single server inventory system where service process is subject to interruptions.

Certain type of inventories undergo change while in storage so that with passage of time they may become partially or entirely unfit for consumption. e.g. drugs, food products, etc. become unusable after a certain time has elapsed. Perishable inventory problems with constant lifetime have been studied quite extensively using the periodic review policy. Periodic review models fit the constant lifetime well, but they usually lead to numerically difficult dynamic optimization problems. Fries [16] and Nahmias [43] use dynamic programming in a perishable inventory model with a lifetime m, zero lead time and zero ordering cost. Lian and Liu [35] developed a discrete time inventory model with geometric inter demand times and constant life time.

1.5 Summary of the thesis

The thesis is divided into seven chapters including the introductory chapter. In chapter 1 we have the pre-requisites that are needed for the development of the remaining chapters. It includes descriptions of Discrete Time inventory systems, Quasi-Birth-Death Process, Matrix Analytic Methods etc.

In chapter 2 we analyze and compare three (s, S) inventory models with different replenishment policies. In all these models the arrival of demands follow a Bernoulli process, service time and lead time follow independent and distinct geometric distributions. In the (s, S) policy, when the inventory level depletes to s, an order is placed. In model 1, we place order up to S where the replenishment quantity is $S - i$ when the inventory level is $i, 0 \leqslant i \leqslant s$, just prior to replenishment. In model 2, replenishment order is for a fixed quantity Q where $Q = S - s$. In model 3, the order size is governed by a discrete probability mass function u on the set $\{1, 2, \ldots, S\}$. Here the reorder level is fixed as 0. The size of a replenishment order is k with probability u_k, with $\sum_{k=1}^{S} u_k = 1$. In all these models, we assume that no customer joins when the inventory level is zero. Stability condition for each model is derived. Some measures of performance in the steady-state are calculated and appropriate cost functions are constructed and analyzed.

In chapter 3, we consider three perishable inventory models with positive service time and positive lifetime. In model I, when the inventory level reaches $\leqslant s$ for the first time after each replenishment, an order is placed to bring back the level up to S. When the replenishment occurs, we discard all the old items so that the remaining (fresh) items have common life time. Model 2 is a modified form of Lian et al. [36] extended to positive service time case. In this model, we place replenishment order when the inventory level reaches zero at a service completion epoch if the number of customers waiting at this epoch is at least s. Else place the order when the number of waiting customers reaches s. In model 3, we assume that the items fail one by one. In all the models, demand arrival is according to a Bernoulli process, service time and life time are distributed geometrically. In models 1 and 3, we assume that the lead time is positive and customers do not join when the inventory level is zero. In model 2, lead time is assumed to be zero and customers join even when the inventory level is zero. System stability is discussed and some performance measures are evaluated. Numerical illustrations of the system behavior are also given. Relative performance of the models are then compared.

In chapter 4, we discuss two inventory models with positive service time and lead time where the arrival of customers depend on the level of inventory. Depending on the number of items and number of customers in the queue at an epoch, the arriving customer decides to join or not to join the system: if the number of customers in the queue is less than the number of stocked items at that epoch, then necessarily he joins. If the inventory level is $\geqslant s+1$ at an arrival epoch, then also the arriving customer joins. However if it is $\leqslant s$ (but larger than zero) then he joins only if the number of customers present is less than the on hand inventory. Stability condition is derived. Steady state analysis is made. Some measures of performance are obtained. Numerical illustrations of the system behavior are also provided.

In the fifth chapter, we discuss a discrete time production inventory system where the processing of inventory requires a positive random amount of time (discrete). This leads to the formation of a queue of demands. In this system, when the inventory level falls to s, the production process is immediately 'switched on'. It is 'switched off' when the inventory level reaches S. Exactly one unit is added at a production epoch. When the inventory is in between $s+1$ and $S-1$, the production process can be either in 'on' mode or in 'off' mode. We consider the production inventory system with a single server. Demands occur according to a Bernoulli process with parameter p. Processing of inventory requires a positive random amount of time, which is distributed geometrically with parameter q. When the inventory level reaches s, the production process is 'switched on' and stays in that mode until the inventory level reaches S. Inter-production times (time between addition of items to the inventory) are geometrically distributed with parameter r. No customer is allowed to join the system when the inventory level is zero. Steady-state analysis is made and performance measures are obtained. Numerical illustrations of the system behavior are also given.

In chapter 6, we consider discrete time inventory models with arbitrarily distributed service time. Here we discuss two models. Both the models follow (s, S) policy. Arrival of demands follow a Bernoulli process with parameter p. The service times are independent identically distributed with general distribution $\{w_i\}_{i=1}^{\infty}$, generating function $W(x) = \sum_{i=1}^{\infty} w_i x^i$ and the n^{th} factorial moments of the total time spent in the service station be $\beta_n, n = 1, 2, \ldots$. In model 1, we assume that a positive random amount of time elapses between placing an order and its receipt, which is distributed geometrically with parameter r. Also assume that no customer joins when the inventory level is zero. In model 2, we assume that replenishment is instantaneous. Further no shortage is permitted. We investigate optimal values of s, S and order quantity Q.

In chapter 7, we introduce Discrete Time Markov Decision Process approach to an (s, S) inventory problem. At the time of replenishment, the following decisions or actions are made: Replenishment take places when inventory level is $i = s, s-1, s-2, \ldots, 1, 0$. We consider a replenishment policy in which quantity replenished varies according to the on hand inventory. In this situation we have to take decisions on how much to buy at the time of replenishment. We use Markov Decision theory for the solution.

Chapter 2

Discrete time inventory models with positive service time and lead time

2.1 Introduction

There is a growing research interest in discrete time queues mainly motivated by their applications in computer and communication systems because the basic time unit in these systems is a binary code (See [1], [4]). Also the discrete time system can be used to approximate the continuous system. Recently, due to the fast progress of computer and telecommunication network technologies, the discrete time models have received more attention from researchers. BISDN (Broadband Integrated Service Digital Network) has been of significant interest because it can provide a common interface for future communication needs including video, voice and data communication signals through high speed Local Area Network (LAN), on-demand video distribution and video telephony communications (see [59]). The Asynchronous Transfer Mode (ATM) is a key technology for accommodating such a wide area of services. In these systems, all the information is segmented into small packets, represented as cells. The time is slotted and in each slot the data units (packets) are transmitted. Applications in detail are discussed in the paper [12] and in the books [13], [58], [59]. By a discrete time analysis, we mean analysis in which the system is observed for analysis, only at specific points in time which are equally spaced points on the time axis. e.g., a system in which observation is made only at points of event occurrences such as arrivals or departures at specified points which are equally spaced and numbered sequentially as $0, 1, 2, \ldots$.

In this chapter, we analyze three discrete time (s, S) inventory models with positive service time and lead time. These models differ by their respective replenishment policies. Model 1 is based on replenishment of order upto S policy. That is whenever the inventory level reaches s, an order is placed to bring the level to S, where s is the reorder level and S is the maximum inventory level permitted. Model 2 is based on order placement by a fixed quantity Q, where $Q = S - s$, whenever the inventory level falls to s and in the third model it is assumed that when the inventory level reaches 0 for the first time, order for replenishment is placed and at the time of realization the quantities of units purchased is a random variable with support $\{1, 2, \ldots, S\}$. The decision of the order size is according to a discrete probability function. In all the three models we assume that demands are according to a Bernoulli process. Service times and lead times are geometrically distributed. We can construct a multidimensional Markov chain to model the joint queue length and inventory process to obtain a product form solution for these models.

2.2 Mathematical Modelling and Analysis of model 1

We consider a $Geo/Geo/1$ (s, S) inventory system with positive lead time in which demands arrive according to a Bernoulli process with parameter p. The demand quantity at an epoch is for one unit of the item with probability p and is 0 with probability $1 - p$. Thus a demand takes place at a slot boundary with probability p and no demand with probability $1 - p$. The service time and lead time for replenishment of inventory follow independent geometric distributions with parameters q and r, respectively.

It is assumed that all inventory activities (demand arrival, replenishment, departure) take place around the slot boundaries. We assume that a departure or replenishment occurs in the interval (m^-, m) and an arrival in (m, m^+). Whenever the inventory level falls to s, an order is placed to bring the level to S. It is assumed that S is greater than $2s$. This assumption is made to avoid perpetual reordering. It requires a random amount of time for the fulfillment of orders placed and the inventory level can be reduced to zero during this period due to demand. The lead time takes at least one time slot to complete, hence an order can not be received at the epoch it is placed.

There exists a rich variety of different inventory models depending on the combination of different assumptions. Some common assumptions are as follows. Continuous versus periodic review of the inventory, individual versus batch arrivals, different replenishment policies (fixed, random size, order upto level S), constant or random lead time etc. The inventory model in discussion is based on replenishment of order up to S policy. We assume that customers are not allowed to join in the system when the inventory level is zero.

Let N_m denote the number of customers in the system and I_m, the inventory level at m^+. We denote the joint queue length and inventory process by $(N_m, I_m) : m \in N$. Then $\chi = \{(N_m, I_m) : m \in N\}$ is a Markov Chain whose state space is $E = \{0, 1, 2, \ldots\} \times \{0, 1, 2, \ldots s, s+1, \ldots S\}$.
The state space of the Markov chain is partitioned into levels defined as $\hat{i} = \{(i, 0), (i, 1), \ldots (i, s), (i, s+1), \ldots (i, S)\}$. The one step transition probability matrix P of the Markov chain χ is given by

$$\mathbf{P} = \begin{bmatrix} C_0 & C_1 & & & \\ A_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

, where each entry is a square matrix of order $S+1$. In the above matrix C_0 denotes the probability of transitions among states within level 0; C_1 is those from level 0 to level 1.

The transitions from level i to level $i+1$ are represented by elements of the matrix A_0, those from level i to $i-1$ by those of A_2 and transitions within the level i are represented by that in A_1. They are given by

$$[C_0]_{ij} = \begin{cases} \overline{r}, & j = i, \quad i = 0 \\ \overline{p}\,\overline{r}, & j = i, \quad i = 1, 2, \ldots, s \\ \overline{p}, & j = i, \quad i = s+1, s+2, \ldots, S \\ \overline{p}r, & j = S, \quad i = 0, 1, \ldots, s \\ 0, & \text{otherwise} \end{cases}$$

$$[C_1]_{ij} = \begin{cases} p\,\overline{r}, & j = i, \quad i = 1, 2, \ldots, s \\ p, & j = i, \quad i = s+1, s+2, \ldots, S \\ p\,r, & j = S, \quad i = 0, 1, \ldots, s \\ 0, & \text{otherwise} \end{cases}$$

$$[A_2]_{ij} = \begin{cases} q\,\overline{r}, & j = i-1, \quad i = 1 \\ \overline{p}q\overline{r}, & j = i-1, \quad i = 2, 3, \ldots, s \\ \overline{p}q, & j = i-1, \quad i = s+1, s+2, \ldots, S \\ \overline{p}qr, & j = S-1, \quad i = 1, 2, \ldots, s \\ 0, & \text{otherwise} \end{cases}$$

$$[A_0]_{ij} = \begin{cases} p\overline{q}\,\overline{r}, & j = i \quad\quad i = 1, 2, \ldots, s \\ p, & j = i, \quad\quad i = s+1, s+2, \ldots, S \\ p\,r, & j = S \quad\quad i = 0 \\ p\overline{q}r, & j = S, \quad\quad i = 1, 2, \ldots, s \\ 0, & \text{otherwise} \end{cases}$$

$$[A_1]_{ij} = \begin{cases} \overline{r}, & j = i, \quad\quad i = 0 \\ \overline{p}\,\overline{q}\,\overline{r}, & j = i, \quad\quad i = 1, 2, \ldots, s \\ \overline{p}\,\overline{q}, & j = i, \quad\quad i = s+1, s+2, \ldots, S \\ p\;q\overline{r}, & j = i-1, \quad i = 2, 3, \ldots, s \\ p\;\;q\,r, & j = S-1, \quad i = 1, 2, \ldots, s \\ \overline{p}r, & j = S, \quad\quad i = 0 \\ \overline{p}\,\overline{q}r, & j = S, \quad\quad i = 1, 2, \ldots, s \\ 0, & \text{otherwise} \end{cases}$$

where $\overline{p} = 1 - p$, $\overline{q} = 1 - q$, $\overline{r} = 1 - r$.

2.2.1 Stability Condition

For determining the stability condition for the system, consider the transition matrix $A = A_0 + A_1 + A_2$ given by

$$[A]_{ij} = \begin{cases} \overline{r}, & j = i, \quad i = 0 \\ q\,\overline{r}, & j = i-1, \quad i = 1,2,\ldots,s \\ \overline{p}q, & j = i-1, \quad i = s+1, s+2, \ldots, S \\ \overline{q}\,\overline{r}, & j = i, \quad i = 1,2,\ldots,s \\ p+\overline{p}\,\overline{q}, & j = i, \quad i = s+1, s+2, \ldots, S \\ r, & j = S, \quad i = 0 \\ q\,r, & j = S-1, \quad i = 1,2,\ldots,s \\ \overline{q}r, & j = S, \quad i = 1,2,\ldots,s \\ 0, & \text{otherwise} \end{cases}$$

The Markov chain χ is stable if and only if $\boldsymbol{\pi} A_0 \boldsymbol{e} \ < \boldsymbol{\pi} A_2 \boldsymbol{e}$ where $\boldsymbol{\pi}$ is the stationary probability vector of A satisfying $\boldsymbol{\pi} A = \boldsymbol{\pi}$ and $\boldsymbol{\pi}\boldsymbol{e} = 1$, where $\boldsymbol{e}$ is a column vector of 1's of appropriate order. Write $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots \pi_s, \ldots, \pi_S)$. Then $\boldsymbol{\pi} A = \boldsymbol{\pi}$ gives

$$\pi_j = \begin{cases} \frac{(1-\overline{r})(1-\overline{q}\,\overline{r})^{j-1}}{(q\,\overline{r})^{j}}\pi_0, & j = 1,2,\ldots,s \\ \frac{(1-\overline{r})(1-\overline{q}\,\overline{r})^{j-1}}{\overline{p}\,q(q\,\overline{r})^{j-1}}\pi_0, & j = s+1 \end{cases}$$

$$\pi_{s+1} = \pi_{s+2} = \cdots = \pi_{S-1};$$

$$\pi_S = \frac{(1-\overline{r})[q(q\,\overline{r})^s + \overline{q}(1-\overline{q}\,\overline{r})^s]}{\overline{p}\,q(q\,\overline{r})^s}\pi_0.$$

Further $\boldsymbol{\pi}\boldsymbol{e} = 1$ gives

$$\pi_0 = \frac{\overline{p}\,q(q\,\overline{r})^s}{(1-\overline{q}\,\overline{r})^s[\overline{p}\,q + (S-s-1)r + r\,\overline{q}] + rq(q\,\overline{r})^s}$$

and a bit of algebra gives

$$\boldsymbol{\pi} A_0 \boldsymbol{e} \ = \left\{ p\,\overline{q}\left[\frac{(1-\overline{q}\,\overline{r})^s - (q\,\overline{r})^s}{(q\,\overline{r})^s}\right] + \frac{pr(S-s-1)(1-\overline{q}\,\overline{r})^s}{\overline{p}\,q(q\,\overline{r})^s} + pr + \frac{prq}{\overline{p}q} + \frac{pr\,\overline{q}(1-\overline{q}\,\overline{r})^s}{\overline{p}\,q(q\,\overline{r})^s} \right\} \pi_0$$

and

$$\boldsymbol{\pi} A_2 \boldsymbol{e} = \left\{ \overline{p}q \left[\frac{(1-\overline{q}\,\overline{r})^s - (q\,\overline{r})^s}{(q\,\overline{r})^s} \right] + \frac{r(S-s-1)(1-\overline{q}\,\overline{r})^s}{(q\,\overline{r})^s} - \overline{p}r + q + \frac{r\,\overline{q}(1-\overline{q}\,\overline{r})^s}{(q\,\overline{r})^s} \right\} \pi_0.$$

Hence we have

Theorem 2.2.1. The system χ is stable if and only if

$$\frac{q(q\,\overline{r})^s \left[pr + (\overline{p}q)^2 - p\overline{p}\,\overline{r}\right]}{(1-\overline{q}\,\overline{r})^s \left[pqr + (\overline{p}q)^2 + \overline{p}qr(S-s-1) + \overline{p}q\overline{q}(r-p) + pr(S-s)\right]} < 1.$$

2.2.2 Steady-state analysis

Assume that the stability condition is satisfied. Let $\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2, \ldots)$ be the steady-state probability vector of the Markov chain χ satisfying $\boldsymbol{x}P = \boldsymbol{x}$ and $\boldsymbol{x}\boldsymbol{e} = 1$. Then $\boldsymbol{x}_i$ has the matrix geometric form $\boldsymbol{x}_i = \boldsymbol{x}_1 R^{i-1}, i \geqslant 2$ where R is the minimal solution of the matrix quadratic equation $R^2 A_2 + RA_1 + A_0 = R$.

$\boldsymbol{x}P = \boldsymbol{x}$ leads us to

$$\boldsymbol{x}_0 C_0 + \boldsymbol{x}_1 A_2 = \boldsymbol{x}_0 \tag{2.1}$$

$$\boldsymbol{x}_0 C_1 + \boldsymbol{x}_1 A_1 + \boldsymbol{x}_2 A_2 = \boldsymbol{x}_1 \tag{2.2}$$

$$\boldsymbol{x}_{i-1} A_0 + \boldsymbol{x}_i A_1 + \boldsymbol{x}_{i+1} A_2 = \boldsymbol{x}_i, \quad i \geqslant 2 \tag{2.3}$$

Also $\boldsymbol{x}\boldsymbol{e} = 1$ gives

$$\boldsymbol{x}_0 \boldsymbol{e} + \boldsymbol{x}_1 (I-R)^{-1} \boldsymbol{e} = 1. \tag{2.4}$$

The rate matrix R can be obtained using the successive iterative method $R(n+1) := (A_0 + R(n)^2 A_2)(I - A_1)^{-1}$, with $R(0) = 0$ and $R(n)$ is the value of R at the n^{th} iteration. The iteration is usually stopped when $|R(n) - R(n+1)|_{ij} < \epsilon, \forall i, j$. Another way to solve for R is to use the Logarithmic reduction method due to Latouche and Ramaswami [33]. The steps of this algorithm are as given below.

$H := (I - A_1)^{-1}A_0$; $L := (I - A_1)^{-1}A_2$; $G := L$; and $T := H$;
and repeat
$U := HL + LH$; $M := H^2$; $H := (I - U)^{-1}M$; $M := L^2$;
$L := (I - U)^{-1}M$; $G := G + TL$; $T := TH$
until $\| 1 - G.e \|_\infty < \epsilon$.
Then $R = A_0(I - A_1 - A_0G)^{-1}$.

For finding the steady-state vector of the process $\chi = \{(N_m, I_m) : m \in N\}$, consider the system where service time is negligible and where no customer joins when inventory is out of stock. This means that if the item is available at the epoch of demand, then it would be immediately delivered. As a consequence the customer need not have to wait. Hence the system has only inventory and is of finite state space.

The corresponding Markov chain is designated as $\hat{\chi} = \{I_m : m \in N\}$ where I_m denote the inventory level. The state space of the process is given by $\hat{E} = \{0, 1, 2, \ldots, S\}$. The transition probability matrix corresponding to $\hat{\chi}$ is given by

$$[\hat{P}]_{ij} = \begin{cases} \overline{r}, & j = i, & i = 0 \\ p\,\overline{r}, & j = i-1, & i = 1, 2, \ldots, s \\ p, & j = i-1, & i = s+1, s+2, \ldots, S \\ \overline{p}\,\overline{r}, & j = i, & i = 1, 2, \ldots, s \\ \overline{p}, & j = i, & i = s+1, s+2, \ldots, S \\ r, & j = S, & i = 0, 1, \ldots, s \\ 0, & \text{otherwise} \end{cases}$$

Let $\hat{\boldsymbol{\pi}} = (\hat{\pi}_0, \hat{\pi}_1, \ldots, \hat{\pi}_S)$ be the steady-state vector of the process $\hat{\chi}$. Then $\hat{\boldsymbol{\pi}}\hat{P} = \hat{\boldsymbol{\pi}}$ and $\hat{\boldsymbol{\pi}}\boldsymbol{e} = 1$. It can be seen that

$$\hat{\pi}_j = \begin{cases} \frac{(1-\overline{r})(1-\overline{p}\,\overline{r})^{j-1}}{(p\,\overline{r})^j}\hat{\pi}_0, & j = 1, 2, \ldots, s \\ \frac{(1-\overline{r})(1-\overline{p}\,\overline{r})^{j-1}}{p(p\,\overline{r})^{j-1}}\hat{\pi}_0, & j = s+1 \end{cases}$$

$\hat{\pi}_{s+1} = \hat{\pi}_{s+2} = \cdots = \hat{\pi}_S.$

Also $\hat{\boldsymbol{\pi}} \boldsymbol{e} = 1$ gives

$\hat{\pi}_0 = \frac{p(p\bar{r})^s}{(1-\bar{p}\ \bar{r})^s[p+(S-s)r]}.$

Now using $\hat{\boldsymbol{\pi}}$, we shall find the steady-state probability vector of χ. Let $\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots)$ be the steady-state probability vector of the Markov Chain χ. Then $\boldsymbol{x}P = \boldsymbol{x}$ and $\boldsymbol{x}\boldsymbol{e} = 1$. The above system reduces to

$$\boldsymbol{x}_0 C_0 + \boldsymbol{x}_1 A_2 = \boldsymbol{x}_0$$

$$\boldsymbol{x}_0 C_1 + \boldsymbol{x}_1 A_1 + \boldsymbol{x}_2 A_2 = \boldsymbol{x}_1$$

$$\boldsymbol{x}_{i-1} A_0 + \boldsymbol{x}_i A_1 + \boldsymbol{x}_{i+1} A_2 = \boldsymbol{x}_i, \quad i \geqslant 2$$

Now let $\boldsymbol{x}_0 = \rho\hat{\boldsymbol{\pi}}$ and $\boldsymbol{x}_i = \rho(\frac{p}{\bar{p}q})^i \hat{\boldsymbol{\pi}}$, for $i \geqslant 1$, where ρ is a constant to be determined. This will satisfy the above equations. For,

$$\begin{aligned}
\boldsymbol{x}_{i-1} A_0 + \boldsymbol{x}_i A_1 + \boldsymbol{x}_{i+1} A_2 &= \boldsymbol{x}_{i-1} A_0 + \boldsymbol{x}_i [C_0 - \tfrac{\bar{p}q}{p} A_0] + \boldsymbol{x}_{i+1} A_2 \\
&= \rho(\tfrac{p}{\bar{p}q})^{i-1} \boldsymbol{\pi} A_0 + \rho(\tfrac{p}{\bar{p}q})^i \boldsymbol{\pi} [C_0 - \tfrac{\bar{p}q}{p} A_0] + \rho(\tfrac{p}{\bar{p}q})^{i+1} \boldsymbol{\pi} A_2 \\
&= \rho(\tfrac{p}{\bar{p}q})^i \boldsymbol{\pi} [C_0 + \tfrac{p}{\bar{p}q} A_2] \\
&= \rho(\tfrac{p}{\bar{p}q})^i \hat{\boldsymbol{\pi}} \\
&= \boldsymbol{x}_i
\end{aligned}$$

Also $\boldsymbol{x}\boldsymbol{e} = 1$ gives $\rho = 1 - \frac{p}{\bar{p}q}$

This leads to the following

Theorem 2.2.2. Under the necessary and sufficient condition that $p < \bar{p}q$, the steady-state vector of the process χ with transition probability matrix P is given by $\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots,)$ where $\boldsymbol{x}_0 = \rho\hat{\boldsymbol{\pi}}$ and $\boldsymbol{x}_i = \rho(\frac{p}{\bar{p}q})^i \hat{\boldsymbol{\pi}}$, for $i \geqslant 1$ $\rho = 1 - \frac{p}{\bar{p}q}$ and the finite probability vector $\hat{\boldsymbol{\pi}}$ is given by $\hat{\boldsymbol{\pi}} = (\hat{\pi}_0, \hat{\pi}_1, \ldots, \hat{\pi}_S)$ where

$$\hat{\pi}_j = \begin{cases} \frac{(1-\overline{r})(1-\overline{p}\ \overline{r})^{j-1}}{(p\overline{r})^j}\pi_0, & j = 1, 2, \ldots, s \\ \frac{(1-\overline{r})(1-\overline{p}\ \overline{r})^{s}}{p(p\overline{r})^s}\pi_0, & j = s+1, s+2, \ldots, S \end{cases} \quad \text{and}$$

$$\pi_0 = \frac{\overline{p}\, q(q\,\overline{r})^s}{(1-\overline{q}\,\overline{r})^s[\overline{p}\, q+(S-s-1)r+r\,\overline{q}]+rq(q\,\overline{r})^s}.$$

2.2.3 System Performance Measures

Let $\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots)$ be the steady-state probability vector and $\boldsymbol{x}_i$, $i \geqslant 0$ is partitioned as $\boldsymbol{x}_i = (x_{i0}, x_{i1}, \ldots, x_{iS})$. We have the following measures for evaluating performance of the system.

1. Expected number of customers EC, in the system is given by
   $\text{EC} = \sum_{i=0}^{\infty} i\ \boldsymbol{x}_i \boldsymbol{e}.$
2. Expected inventory level EI, is given by
   $\text{EI} = \sum_{i=0}^{\infty} \sum_{j=1}^{S} j\ x_{ij}.$
3. Expected reorder rate ER, is given by
   $\text{ER} = q \sum_{i=0}^{\infty} x_{i,s+1}.$
4. Expected replenishment rate ERR, is given by
   $\text{ERR} = r \sum_{i=0}^{\infty} \sum_{j=0}^{s} x_{ij}.$
5. Probability that the inventory level is zero is $\sum_{i=0}^{\infty} x_{i0}.$
6. Expected loss rate EL, of customers is given by
   $\text{EL} = p \sum_{i=0}^{\infty} x_{i0}.$

7. Expected number of customers EW, waiting in the system when the inventory level is zero is given by
   $\text{EW} = \sum_{i=0}^{\infty} i \; x_{i0}.$

8. Expected rate ED, of departure after completing service is given by
   $\text{ED} = q \sum_{i=1}^{\infty} \sum_{j=1}^{S} \; x_{ij}.$

2.3 Mathematical Formulation of model 2 and its analysis

We consider a discrete time (s, S) inventory system with positive lead time in which demands arrive according to a Bernoulli process with parameter p. The service times and lead times follow geometric distributions with parameters q and r respectively. Whenever the inventory level falls to s, place an order for replenishment by a fixed quantity Q, where $Q = S - s$. S is the maximum inventory level and s is the reorder level. There is a positive lead time for replenishment. We assume that no customer joins when the inventory level is zero. Those who are already present in the system do not renege. Exactly one item is demanded by each customer.

We denote the joint queue length and inventory process by $\{X_m\} = \{(N_m, I_m) : m \in N\}$ where N_m denotes the number of customers in the system and I_m denotes the inventory level at time m^+. Then $\{X_m\} = \{(N_m, I_m) : m \in N\}$ provides a Markov description of the inventory system whose state space is $E = \{0, 1, 2, \ldots\} \times \{0, 1, ..s, \ldots, Q, Q+1, \ldots, S\}$. The one step transition probability matrix of the process is given by

$$\mathbf{P} = \begin{bmatrix} C_0 & C_1 & & & \\ A_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

where each entry is a square matrix of order $S+1$ are given by

$$C_0 = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ s \\ s+1 \\ \vdots \\ Q \\ Q+1 \\ \vdots \\ S \end{array} \begin{array}{c} \begin{array}{cccccccccc} 0 & 1 & \cdots & s & s+1 & \cdots & Q & Q+1 & \cdots & S \end{array} \\ \left(\begin{array}{cccccccccc} \bar{r} & & & & & & \bar{p}r & & & \\ & \bar{p}\,\bar{r} & & & & & & \bar{p}r & & \\ & & \ddots & & & & & & \ddots & \\ & & & \bar{p}\,\bar{r} & & & & & & \bar{p}r \\ & & & & \bar{p} & & & & & \\ & & & & & \ddots & & & & \\ & & & & & & \bar{p} & & & \\ & & & & & & & \bar{p} & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & \bar{p} \end{array} \right) \end{array}$$

$$C_1 = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ s \\ s+1 \\ \vdots \\ Q \\ Q+1 \\ \vdots \\ S \end{array} \begin{array}{c} \begin{array}{cccccccccc} 0 & 1 & \cdots & s & s+1 & \cdots & Q & Q+1 & \cdots & S \end{array} \\ \left(\begin{array}{cccccccccc} & & & & & & pr & & & \\ & p\bar{r} & & & & & & pr & & \\ & & \ddots & & & & & & \ddots & \\ & & & p\bar{r} & & & & & & pr \\ & & & & p & & & & & \\ & & & & & \ddots & & & & \\ & & & & & & p & & & \\ & & & & & & & p & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & p \end{array} \right) \end{array}$$

For $i \geqslant 2$, the transitions from level i to level i+1, transitions within the level i and transitions from level i to level i-1 are represented by the matrices A_0 ,A_1 and A_2 respectively, and are given by

$$A_0 = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ s \\ s+1 \\ \vdots \\ Q \\ Q+1 \\ \vdots \\ S \end{array} \begin{array}{c} \begin{array}{ccccccccc} 0 & 1 & \cdots & s & s+1 & \cdots & Q & Q+1 & \cdots & S \end{array} \\ \left(\begin{array}{cccccccccc} & & & & & & pr & & & \\ & p\bar{q}\,\bar{r} & & & & & & p\bar{q}r & & \\ & & \ddots & & & & & & \ddots & \\ & & & p\bar{q}\,\bar{r} & & & & & & p\bar{q}r \\ & & & & p & & & & & \\ & & & & & \ddots & & & & \\ & & & & & & p & & & \\ & & & & & & & p & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & p \end{array} \right) \end{array}$$

$$A_1 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ s \\ s+1 \\ \vdots \\ S \end{array} \begin{array}{c} \begin{array}{ccccccccccc} 0 & 1 & \cdots & s-1 & s & \cdots & Q & Q+1 & \cdots & S \end{array} \\ \left(\begin{array}{cccccccccc} \bar{r} & & & & & & \bar{p}r & & & \\ & \bar{p}\,\bar{q}\,\bar{r} & & & & & pqr & \bar{p}\,\bar{q}r & & \\ & pq\bar{r} & \bar{p}\,\bar{q}\,\bar{r} & & & & & pqr & \bar{p}\,\bar{q}r & \\ & & \ddots & \ddots & & & & \ddots & \ddots & \\ & & & pq\bar{r} & \bar{p}\,\bar{q}\,\bar{r} & & & & pqr & \bar{p}\,\bar{q}r \\ & & & & & \overline{pq} & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & & & \overline{pq} \end{array} \right) \end{array}$$

$$
A_2 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ s \\ s+1 \\ \vdots \\ S-1 \\ S \end{array}
\begin{array}{c}
\begin{array}{cccccccccc} 0 & 1 & \cdots & s-1 & s & \cdots & Q & \cdots & S-1 & S \end{array} \\
\left(\begin{array}{cccccccccc}
 & & & & & & & & & \\
q\overline{r} & & & & & \overline{p}qr & & & & \\
 & \overline{p}q\overline{r} & & & & & & & & \\
 & & \ddots & & & & & \ddots & & \\
 & & & \overline{p}q\overline{r} & & & & & \overline{p}qr & \\
 & & & & \overline{p}q & & & & & \\
 & & & & & \ddots & & & & \\
 & & & & & & & & & \\
 & & & & & & & & \overline{p}q &
\end{array} \right)
\end{array}
$$

where, $\overline{p} = 1-p$, $\overline{q} = 1-q$, $\overline{r} = 1-r$

2.3.1 Stability Condition

For determining the stability condition for the system, consider the transition matrix $A = A_0 + A_1 + A_2$ by

$$
A = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ s \\ s+1 \\ \vdots \\ S-1 \\ S \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} 0 & 1 & \cdots & s & \cdots & Q & Q+1 & \cdots & S \end{array} \\
\left(\begin{array}{ccccccccc}
\overline{r} & & & & & r & & & \\
q\overline{r} & \overline{q}\,\overline{r} & & & & qr & \overline{q}r & & \\
 & \ddots & \ddots & & & & \ddots & \ddots & \\
 & & q\overline{r} & \overline{q}\,\overline{r} & & & & qr & \overline{q}r \\
 & & & \overline{p}q & p+\overline{p}\,\overline{q} & & & & \\
 & & & & \ddots & \ddots & & & \\
 & & & & & \ddots & \ddots & & \\
 & & & & & & & \overline{p}q & p+\overline{p}\,\overline{q}
\end{array} \right)
\end{array}
$$

The chain $\{X_m\}$ is stable if and only if the left drift rate is higher than the rate of drift to the right. That is, $\boldsymbol{\pi} A_0 \boldsymbol{e} < \boldsymbol{\pi} A_2 \boldsymbol{e}$ where $\boldsymbol{\pi}$ is the stationary probability vector of A satisfying $\boldsymbol{\pi} A = \boldsymbol{\pi}$ and $\boldsymbol{\pi}\boldsymbol{e} = 1$, where $\boldsymbol{e}$ is a column vector of 1's of appropriate order. Let $\boldsymbol{\pi} = (\pi_0, \dots \pi_s, \pi_{s+1}, \dots, \pi_Q, \dots, \pi_S)$.

Then

$$\begin{aligned}
\pi_1 &= \left(\frac{1-\overline{r}}{q\,\overline{r}}\right)\pi_0 \\
\pi_2 &= \frac{(1-\overline{r})(1-\overline{q}\,\overline{r})}{(q\,\overline{r})^2}\pi_0 \\
&\vdots \\
\pi_s &= \frac{(1-\overline{r})(1-\overline{q}\,\overline{r})^{s-1}}{(q\,\overline{r})^s}\pi_0 \\
\pi_i &= \frac{(1-\overline{r})(1-\overline{q}\,\overline{r})^s}{\overline{p}q(q\overline{r})^s}\pi_0, \quad i = s+1, s+2, \dots, Q \\
\pi_{Q+1} &= \left(\frac{(1-\overline{r})(1-\overline{q}\,\overline{r})^s}{\overline{p}\,q\,(q\,\overline{r})^s} - \frac{qr}{\overline{p}\,q\,(q\,\overline{r})}\right)\pi_0 \\
&\vdots \\
\pi_{S-2} &= \frac{r(1-\overline{r})(1-\overline{q}\,\overline{r})^{s-3}}{\overline{p}\,q(q\,\overline{r})^{s-2}}\left[\overline{q} + \frac{1-\overline{q}\,\overline{r}}{q\,\overline{r}} + \frac{(1-\overline{q}\,\overline{r})^2}{(q\,\overline{r})^2}\right]\pi_0 \\
\pi_{S-1} &= \frac{r(1-\overline{r})(1-\overline{q}\,\overline{r})^{s-2}}{\overline{p}\,q(q\,\overline{r})^{s-1}}\left[\overline{q} + \frac{1-\overline{q}\,\overline{r}}{q\,\overline{r}}\right]\pi_0 \\
\pi_S &= \frac{r(1-\overline{r})(1-\overline{q}\,\overline{r})^{s-1}}{\overline{p}\,q(q\,\overline{r})^s}\;\overline{q}\pi_0
\end{aligned}$$

and $\boldsymbol{\pi}\boldsymbol{e} = 1$ gives $\pi_0 = \dfrac{\overline{p}\,q(q\,\overline{r})^s}{(1-\overline{q}\,\overline{r})^s[Qr - pq] + q(q\,\overline{r})^s}$.

$\boldsymbol{\pi} A_0 \boldsymbol{e} = [-pq\frac{(1-\overline{q}\,\overline{r})^s}{(q\,\overline{r})^s} - p\,\overline{q} + pr]\pi_0 + p.$

$\boldsymbol{\pi} A_2 \boldsymbol{e} = [pr - \overline{p}q]\pi_0 + \overline{p}q.$

Hence we have the theorem

Theorem 2.3.1. The system $\{X_m\}$ is stable if and only if

$$\frac{\overline{p}\,q[(q-p)(q\,\overline{r})^s - pq(1-\overline{q}\,\overline{r})^s]}{(1-\overline{q}\,\overline{r})^s[Qr-pq]+q(q\,\overline{r})^s} < \overline{p}\,q - p$$

2.3.2 Steady-state analysis

Let $\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots,)$ be the steady-state probability vector of the Markov process $\{X_m\}$ satisfying $\boldsymbol{x}P = \boldsymbol{x}$ and $\boldsymbol{x}\boldsymbol{e} = 1$. Then $\boldsymbol{x}_i$ has the matrix geometric form $\boldsymbol{x}_i = \boldsymbol{x}_1 R^{i-1}, i \geqslant 2$ where R is the minimal solution of the matrix quadratic equation $R^2A_2 + RA_1 + A_0 = R$. The vectors $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$ can be obtained by solving the equations

$$\boldsymbol{x}_0 C_0 + \boldsymbol{x}_1 A_2 = \boldsymbol{x}_0 \tag{2.5}$$

$$\boldsymbol{x}_0 C_1 + \boldsymbol{x}_1 A_1 + \boldsymbol{x}_1 R A_2 = \boldsymbol{x}_1 \tag{2.6}$$

and the normalizing condition

$$\boldsymbol{x}_0\boldsymbol{e} + \boldsymbol{x}_1(I-R)^{-1}\boldsymbol{e} = 1.$$

From the above equations, to determine $\boldsymbol{x}$, we have to compute the rate matrix R. This is solved numerically. In some special cases the matrix R could be explicitly obtained.

Now we analyze the system with negligible service time where no customer joins when inventory is out of stock. The corresponding Markov chain is $\{\hat{X}_m\} = \{I_m : m \in N\}$ where I_m denotes the inventory level. The state space of the process is given by $\hat{E} = \{0, 1, 2, \ldots, S\}$. The transition probability matrix corresponding to $\{\hat{X}_m\}$ is given by

$$\hat{P} = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ s \\ s+1 \\ \vdots \\ S-1 \\ S \end{array} \begin{array}{c} \begin{array}{cccccccccc} 0 & 1 & \cdots & s-1 & s & \cdots & Q & \cdots & S-1 & S \end{array} \\ \left(\begin{array}{cccccccccc} \overline{r} & & & & & & r & & & \\ p\overline{r} & \overline{p}\,\overline{r} & & & & & & r & & \\ & \ddots & \ddots & & & & & & \ddots & \\ & & & p\overline{r} & \overline{p}\,\overline{r} & & & & & r \\ & & & & p & \overline{p} & & & & \\ & & & & & \ddots & \ddots & & & \\ & & & & & & & & & \\ & & & & & & & & p & \overline{p} \end{array} \right) \end{array}$$

Let $\hat{\boldsymbol{\pi}} = (\hat{\pi}_0, \hat{\pi}_1, \ldots, \hat{\pi}_S)$ be the steady-state vector of the process $\hat{X}_m$. Then $\hat{\boldsymbol{\pi}}\hat{P} = \hat{\boldsymbol{\pi}}$ and $\hat{\boldsymbol{\pi}}\boldsymbol{e} = 1$. It can be seen that

$$\hat{\pi}_j = \begin{cases} \frac{(1-\overline{r})(1-\overline{p}\ \overline{r})^{j-1}}{(p\overline{r})^j}\hat{\pi}_0, & j = 1, 2, \ldots, s \\ \frac{(1-\overline{r})(1-\overline{p}\ \overline{r})^{s}}{p(p\overline{r})^s}\hat{\pi}_0, & j = s+1, s+2, \ldots, Q \end{cases}$$

$$\hat{\pi}_{Q+1} = \frac{r\,(1-\overline{r})}{p(p\,\overline{r})}\left[1 + \frac{1-\overline{p}\,\overline{r}}{p\,\overline{r}} + \left(\frac{1-\overline{p}\,\overline{r}}{p\,\overline{r}}\right)^2 + \cdots + \left(\frac{1-\overline{p}\,\overline{r}}{p\,\overline{r}}\right)^{s-1}\right]\hat{\pi}_0$$

$$\vdots$$

$$\hat{\pi}_{S-1} = \frac{r(1-\overline{r})(1-\overline{p}\,\overline{r})^{s-2}}{p(p\overline{r})^{s-1}}\left[1 + \frac{1-\overline{p}\,\overline{r}}{p\,\overline{r}}\right]\hat{\pi}_0$$

$$\hat{\pi}_S = \frac{r(1-\overline{r})(1-\overline{p}\,\overline{r})^{s-1}}{p(p\overline{r})^s}\hat{\pi}_0$$

Also $\hat{\boldsymbol{\pi}}\boldsymbol{e} = 1$ gives

$$\hat{\pi}_0 = \frac{p(p\overline{r})^s}{r(1-\overline{p}\,\overline{r})^s[p + (S-s)] + (p\overline{r})^{s+1}}.$$

2.3.3 System Performance Measures

Let $\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots,)$ be the steady-state probability vector and $\boldsymbol{x}_i$, $i \geqslant 0$, be partitioned as $\boldsymbol{x}_i = (x_{i0}, x_{i1}, \ldots, x_{iS})$. We have then the following measures for evaluating performance of the system.

1. Expected number of customers in the system is given by
   $$\text{EC} = \sum_{i=0}^{\infty} i \ \boldsymbol{x}_i \boldsymbol{e}.$$

2. Expected inventory level is given by
   $$\text{EI} = \sum_{i=0}^{\infty} \sum_{j=1}^{S} j \ x_{ij}.$$

3. Expected reorder rate is given by
   $$\text{ER} = q \sum_{i=1}^{\infty} x_{i,s+1}.$$

4. Expected replenishment rate is given by
   $$\text{ERR} = r \sum_{i=0}^{\infty} \sum_{j=0}^{s} x_{ij}.$$

5. Probability that the inventory level is zero is $\sum_{i=0}^{\infty} x_{i0}$.

6. Expected loss rate of fresh arrivals is given by
   $$\text{EL} = p \sum_{i=0}^{\infty} x_{i0}.$$

7. Expected number of customers waiting in the system when the inventory level is zero is given by
   $$\text{EW} = \sum_{i=1}^{\infty} i x_{i0}.$$

8. Expected rate of departure after completing service is ED $=q\sum_{i=1}^{\infty}\sum_{j=1}^{S} x_{ij}$.

2.4 Mathematical Formulation of model 3 and its analysis

We discuss the $Geo/Geo/1$ system in which the probability of an arrival during at an epoch is p with $\overline{p} = 1 - p$ the probability of a service completion at an epoch be q with $\overline{q} = 1 - q$.
The maximum capacity of the store is fixed as S units. Due to demands that take place over time , the level of the inventory falls and when the level reaches 0, for the first time, an order is placed for replenishment. Here we fix the reorder point as 0 and allow general randomized order size. The decision of the order size is according to a discrete probability mass function on the integers $\{1, 2, \ldots, S\}$ where S is the maximum capacity of the inventory. So the size of a replenishment order is k with probability u_k where $\sum_{k=1}^{S} u_k = 1$.
The probability for replenishment at a slot end point be r with $\overline{r} = 1 - r$.
We assume that no customer joins when the inventory level is zero. The inter demand times and inter replenishment times are assumed to be independent of each other.

Let N_m denote the number of customers in the system and I_m, the inventory level at time m^+. We denote the joint queue length and inventory process by $X_m = (N_m, I_m) : m \in N$. Then
$\{X_m\} = \{(N_m, I_m) : m \in N\}$ is a Discrete Time Markov Chain whose state space is $E = \{(n, k) : n \in N_0, k \in \{0, 1, \ldots, S\}\}$.
The transition probability matrix of the process is given by

$$\mathbf{P} = \begin{bmatrix} C_0 & C_1 & & & \\ A_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$ where C_0, C_1, A_0, A_1, A_2 are given by

$$C_0 = \begin{bmatrix} \overline{r} & \overline{p}r\overline{V}_S \\ \underline{0} & \overline{p}I_S \end{bmatrix}, C_1 = p\begin{bmatrix} 0 & r\overline{V}_S \\ \underline{0} & I_S \end{bmatrix},$$

$$A_0 = p\begin{bmatrix} 0 & r\overline{V}_S \\ \underline{0} & \overline{q}I_S \end{bmatrix},$$

$$A_1 = \begin{bmatrix} \overline{r} & \overline{p}r\overline{V}_S \\ pqe_1 & O \end{bmatrix} + \overline{p}\,\overline{q}\begin{bmatrix} 0 & \underline{0} \\ \underline{0} & I_S \end{bmatrix} + pq\begin{bmatrix} 0 & \underline{0} \\ \underline{0} & \overline{D}_S \end{bmatrix},$$

$$A_2 = \overline{p}q\begin{bmatrix} \underline{0} & 0 \\ I_S & \underline{0} \end{bmatrix}$$

with $I_S = diag(1, 1, \ldots, 1)$ of order S.
$\boldsymbol{e}_j = \begin{pmatrix} 0, & \ldots, & 0, & 1, & 0, & \ldots & 0 \end{pmatrix}'$ of order S with 1 at the j^{th} position, $j = 1, 2, ..., S$; $\overline{V}_S = (u_1, u_2, \ldots, u_S)$ is the probability vector corresponding to quantity for which order is placed at the time of replenishment. O is a null matrix of order $S \times S$; $\overline{D}_S = \begin{bmatrix} \underline{0} & 0 \\ I_{S-1} & \underline{0} \end{bmatrix}$.

2.4.1 Stability condition

For determining the stability condition for the system, consider the transition matrix $A = A_0 + A_1 + A_2$ given by

$$A = \begin{bmatrix} \overline{r} & r\overline{V}_S \\ qe_1 & O \end{bmatrix} + \overline{q}\begin{bmatrix} 0 & \underline{0} \\ \underline{0} & D_S \end{bmatrix} + q\begin{bmatrix} 0 & \underline{0} \\ \underline{0} & \overline{D}_S \end{bmatrix}.$$

The Markov chain $\{X_m\}$ is stable if and only if $\boldsymbol{\pi} A_0 \boldsymbol{e} < \boldsymbol{\pi} A_2 \boldsymbol{e}$ where $\boldsymbol{\pi}$ is the stationary probability vector of A satisfying $\boldsymbol{\pi} A = \boldsymbol{\pi}$ and $\boldsymbol{\pi}\boldsymbol{e} = 1$, where $\boldsymbol{e}$ is a column vector of 1 's of appropriate order. Let $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots \pi_s, \ldots, \pi_S)$. Then $\boldsymbol{\pi} A = \boldsymbol{\pi}$ gives

$$\pi_j = \begin{cases} \frac{r}{q}\pi_0, & j = 1 \\ \frac{r(1-u_1-u_2-\ldots-u_{j-1})}{q}\pi_0, & j = 2, 3, \ldots, S \end{cases}$$

$\boldsymbol{\pi}\boldsymbol{e} = 1$ gives $\pi_0 = \left[1 + \frac{r}{q}\sum_{k=1}^{S} ku_k\right]^{-1}$.

$$\boldsymbol{\pi} A_0 \boldsymbol{e} = \left\{pr + \left[\frac{pr\,\overline{q}}{q}\sum_{k=1}^{S} ku_k\right]\right\}\pi_0$$

and

$$\boldsymbol{\pi} A_2 \boldsymbol{e} = \overline{p}\,r\left(\sum_{k=1}^{S} ku_k\right)\pi_0.$$

With these the relation $\boldsymbol{\pi} A_0 \boldsymbol{e} \; < \boldsymbol{\pi} A_2 \boldsymbol{e}$ gives

Theorem 2.4.1. The system $\{X_m\}$ is stable if and only if

$$\frac{pq}{(q-p)r\left(\displaystyle\sum_{k=1}^{S} ku_k\right)} < 1. \tag{2.7}$$

2.4.2 Steady-state analysis

Assume that the stability condition (2.7) is satisfied. Let $\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots)$ be the steady-state probability vector of the Markov chain $\{X_m\}$. Then

$$\boldsymbol{x}P = \boldsymbol{x} \text{ and } \boldsymbol{x}\boldsymbol{e} = 1.$$

The $\boldsymbol{x}_i$'s have the matrix geometric form $\boldsymbol{x}_i = \boldsymbol{x}_1 R^{i-1}, i \geqslant 2$ where R is the minimal nonnegative solution of the matrix quadratic equation $R^2A_2 + RA_1 +$

$A_0 = R$. The vectors $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$ can be obtained by solving the equations

$$\boldsymbol{x}_0 C_0 + \boldsymbol{x}_1 C_2 = \boldsymbol{x}_0 \tag{2.8}$$

$$\boldsymbol{x}_0 C_1 + \boldsymbol{x}_1 A_1 + \boldsymbol{x}_2 R A_2 = \boldsymbol{x}_1 \tag{2.9}$$

and the normalizing condition

$$\boldsymbol{x}_0 \boldsymbol{e} + \boldsymbol{x}_1 (I - R)^{-1} \boldsymbol{e} = 1. \tag{2.10}$$

Next we analyze the system where service time is negligible. In this case no queue is formed. Queue of customers is formed only when the system is out of stock and unsatisfied customers are permitted to wait. Hence the system has only inventory and is of finite state space. i.e., we do not encounter simultaneously a queue of inventoried items and one of customers. The corresponding Markov chain is denoted as $\{\hat{X}_m\} = \{I_m : m \in N\}$ where I_m is the inventory level at epoch m.
The state space of the process is given by $E = \{0, 1, \ldots, S\}$ and its transition probability matrix is given by

$$\hat{P} = \begin{bmatrix} \overline{r} & r\overline{V_S} \\ pe_1 & O \end{bmatrix} + \overline{p} \begin{bmatrix} 0 & \underline{0} \\ \underline{0} & D_S \end{bmatrix} + p \begin{bmatrix} 0 & \underline{0} \\ \underline{0} & \overline{D_S} \end{bmatrix}$$

Let $\hat{\boldsymbol{\pi}} = (\hat{\pi}_0, \hat{\pi}_1, \ldots, \hat{\pi}_S)$ be the steady-state vector of the process $\{\hat{X}_m\}$. Then $\hat{\boldsymbol{\pi}} \hat{P} = \hat{\boldsymbol{\pi}}$ and $\hat{\boldsymbol{\pi}} \boldsymbol{e} = 1$. It can be seen that

$$\hat{\pi}_j = \begin{cases} \frac{r}{p}\hat{\pi}_0, & j = 1 \\ \frac{r(1-u_1-u_2-\ldots-u_{j-1})}{p}\hat{\pi}_0, & j = 2, 3, \ldots, S. \end{cases}$$

Also the normalizing condition $\hat{\pi}\boldsymbol{e} = 1$ gives

$$\hat{\pi}_0 = \left[1 + \frac{r}{p} \sum_{k=1}^{S} k u_k\right]^{-1}.$$

2.4.3 System Performance Measures

Let $\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots)$ be the steady-state probability vector and $\boldsymbol{x}_i$, $i \geqslant 0$ is partitioned as $\boldsymbol{x}_i = (x_{i0}, x_{i1}, \ldots, x_{iS})$

1. Expected number of customers in the system EC, is given by
   $\text{EC} = \sum_{i=0}^{\infty} i \ \boldsymbol{x}_i \boldsymbol{e}$
2. Expected inventory level EI, is given by
   $\text{EI} = \sum_{i=0}^{\infty} \sum_{j=1}^{S} j \ x_{ij}$
3. Expected reorder rate ER, is given by
   $\text{ER} = q \sum_{i=0}^{\infty} x_{i,1}$
4. Expected replenishment rate is given by
   $\text{ERR} = r \sum_{i=0}^{\infty} \sum_{j=0}^{1} x_{ij}$
5. Probability that the inventory level is zero is $\sum_{i=0}^{\infty} x_{i0}$
6. Probability that the inventory level is greater than $m(\leqslant s)$ is $\sum_{i=1}^{\infty} \sum_{j=m+1}^{s} x_{ij}$
7. Expected loss rate of customers EL is given by
   $\text{EL} = p \sum_{i=0}^{\infty} x_{i0}$
8. Expected number of customers waiting in the system when the inventory level is zero is given by $\text{EW} = \sum_{i=0}^{\infty} i \ x_{i0}$

2.5 Cost Analysis

We analyze numerically the steady-state expected cost rate under the following assumptions
Let c_0 denote the fixed ordering cost
c_1 -procurement cost/ unit
c_2 -holding cost of inventory /unit/unit time
c_3 -holding cost of customers/unit/unit time
c_4 -cost due to the loss of customers /unit/unit time
For Model 1, the Expected Total Cost

$$ETC = \left[c_0 + \sum_{i=0}^{s} r(S-i)c_1\right] ER + c_2 EI + c_3 EW + c_4 EL.$$

For Model 2

$$ETC = [c_0 + Qc_1]\, ER + c_2 EI + c_3 EW + c_4 EL.$$

For Model 3

$$ETC = \left[c_0 + \sum_{k=1}^{S} k u_k c_1\right] ER + c_2 EI + c_3 EW + c_4 EL.$$

2.6 Numerical illustration

Tables 2.1, 2.2, 2.3 show that in all the three models, as the arrival rate p increases expected number of customers increases. Consequently inventory level decreases and the expected reorder rate increases. Also expected number of departure after service completion increases. As the service rate q increases, expected number of customers decreases and consequently inventory level increases and expected reorder rate also increases. Expected loss rate of customers decreases and further expected number of departure after

Table 2.1: Effect of p on Model-1. $q = 0.7$,$s = 5$,$S = 20$

p	ρ	EC	EI	ER	EL	ED
$r = 0.7$						
0.30	0.5923	1.51322	12.61249	0.04624	0.00000	0.42343
0.32	0.6509	1.93918	12.57741	0.04627	0.00000	0.46400
0.34	0.7132	2.58100	12.53914	0.04632	0.00000	0.50679
0.36	0.7795	3.65980	12.49714	0.04642	0.00000	0.55198
0.38	0.8501	5.85612	12.45079	0.04657	0.00001	0.59978
0.40	0.9256	12.79414	12.39939	0.04678	0.00001	0.65042
$r = 0.8$						
0.30	0.5946	1.52041	12.63677	0.04645	0.00000	0.42404
0.32	0.6535	1.95134	12.60361	0.04649	0.00000	0.46477
0.34	0.7159	2.60308	12.56745	0.04656	0.00000	0.50777
0.36	0.7824	3.70501	12.52778	0.04667	0.00000	0.55321
0.38	0.8532	5.97344	12.48401	0.04682	0.00000	0.60131
0.40	0.9288	13.36549	12.43544	0.04703	0.00000	0.65230
$r = 0.9$						
0.30	0.5965	1.52620	12.65553	0.04662	0.00000	0.42352
0.32	0.6555	1.96115	12.62386	0.04668	0.00000	0.46538
0.34	0.7181	2.62093	12.58934	0.04678	0.00000	0.50854
0.36	0.7846	3.74174	12.55148	0.04687	0.00000	0.55418
0.38	0.8556	6.06988	12.50970	0.04702	0.00001	0.60252
0.40	0.9313	13.85470	12.46333	0.04722	0.00001	0.65379

service completion increases. As the rate of leadtime for replenishment increases the inventory level increases, as expected. Also there is an increment in the reorder rate. As seen from table 2.4, in all the models, as the arrival rate increases, probability that the server is idle for want of customers decreases and hence probability that inventory level is zero increases. From tables 2.5, 2.6 and 2.7 we can see that in all models as service rate increases, expected number of cutomers decreases. Also reorder rate increases and expected loss rate of customers decreases. Tables 2.8, 2.9 and 2.10 show that as replenishment rate r increases, the inventory level in all models increases. Here expected number of customers also increases. Also the expected loss

Table 2.2: Effect of p on Model-2. $q = 0.7$,$s = 5$,$S = 20$

p	ρ	EC	EI	ER	EL	ED
$r = 0.7$						
0.30	0.5919	1.51244	11.53074	0.04680	0.00000	0.42337
0.32	0.6505	1.93772	11.48785	0.04688	0.00000	0.46391
0.34	0.7128	2.57813	11.44129	0.04700	0.00000	0.50667
0.36	0.7790	3.65347	11.39047	0.04717	0.00000	0.55182
0.38	0.8496	5.83861	11.33467	0.04738	0.00001	0.59956
0.40	0.9250	12.70578	11.27310	0.04767	0.00001	0.65012
$r = 0.8$						
0.30	0.5944	1.52000	11.58886	0.04678	0.00000	0.42400
0.32	0.6532	1.95058	11.55109	0.04686	0.00000	0.46473
0.34	0.7157	2.60156	11.51002	0.04696	0.00000	0.50770
0.36	0.7821	3.70164	11.46510	0.04711	0.00000	0.55312
0.38	0.8529	5.96400	11.41567	0.04730	0.00000	0.60119
0.40	0.9285	13.31558	11.36099	0.04755	0.00000	0.65215
$r = 0.9$						
0.30	0.5964	1.52603	11.63418	0.04677	0.00000	0.42450
0.32	0.6554	1.96084	11.60044	0.04684	0.00000	0.46537
0.34	0.7180	2.62032	11.56372	0.04693	0.00000	0.50852
0.36	0.7845	3.74037	11.52348	0.04706	0.00000	0.55415
0.38	0.8555	6.06598	11.47913	0.04723	0.00000	0.60247
0.40	0.9312	13.83330	11.42996	0.04746	0.00000	0.65373

rate of customers decreases.

We compute the expected total cost per unit time for the models by varying different parameters one at a time while keeping others fixed and find the most profitable one by comparing the costs.

Figure 2.1 shows that the cost functions for all the models are convex and the expected total cost is minimum for model-3. Again as the maximum inventory level S is increased, the cost function behaves as above and then

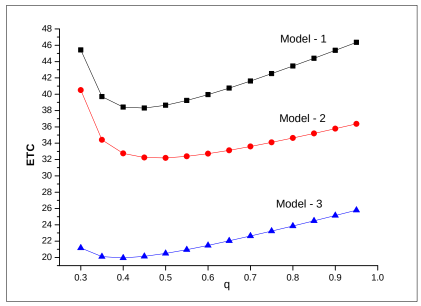

Figure 2.1: q versus ETC when $S = 20$, $s = 5$, $p = 0.2$, $r = 0.3$, $c_0 = 50, c_1 = 10, c_2 = 2, c_3 = 3, c_4 = 5$

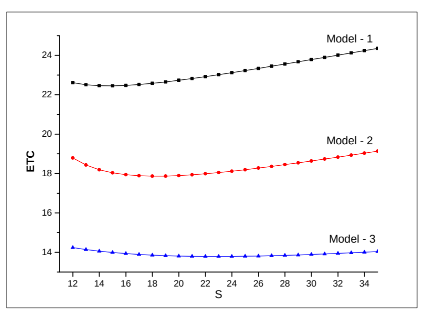

Figure 2.2: S versus ETC when $s = 4$, $p = 0.4$, $q = 0.7$, $r = 0.3$, $c_0 = 50, c_1 = 15, c_2 = 0.2, c_3 = 0.3, c_4 = 0.5$

Table 2.3: Effect of p on Model-3. $q = 0.7$, $S = 20$

p	ρ	EC	EI	ER	EL	ED
$r = 0.2$						
0.1	0.0582	0.15165	6.01452	0.06390	0.00459	0.09143
0.2	0.1310	0.32838	5.72050	0.06131	0.01770	0.18921
0.3	0.2245	0.55080	5.45076	0.05889	0.03846	0.28095
0.4	0.3492	0.87135	5.20712	0.05657	0.06612	0.35836
0.42	0.3793	0.95807	5.16193	0.05611	0.07241	0.37539
0.4205	0.3801	0.96039	5.16082	0.05609	0.07257	0.39702
$r = 0.3$						
0.1	0.0582	0.15166	6.10791	0.06489	0.00311	0.09783
0.2	0.1310	0.32849	5.89437	0.06318	0.01216	0.19149
0.3	0.2245	0.55127	5.69410	0.06152	0.02679	0.28125
0.4	0.3492	0.87282	5.51075	0.05986	0.04665	0.36735
0.42	0.3793	0.95989	5.47668	0.05953	0.05122	0.38415
0.4205	0.3801	0.96221	5.47584	0.05952	0.05134	0.39946
$r = 0.4$						
0.1	0.0582	0.15167	6.15570	0.06540	0.00235	0.09859
0.2	0.1310	0.32855	5.98534	0.06415	0.00926	0.19444
0.3	0.2245	0.55153	5.82410	0.06292	0.02055	0.28767
0.4	0.3492	0.87363	5.67623	0.06166	0.03604	0.37838
0.42	0.3793	0.96088	5.64891	0.06140	0.03962	0.39623
0.4205	0.3801	0.96221	5.47584	0.05952	0.05134	0.41275

also the expected total cost is minimum for model-3. (See figure 2.2). Hence model-3 is more profitable. That is when the inventory level reaches 0, for the first time, allow general randomized order size.

Table 2.4: Variations in arrival rate p. $r = 0.4$, $s = 5$, $S = 20$

$q = 0.7$						
	model 1		model 2		model 3	
p	P_{idle}	P_{INL-0}	P_{idle}	P_{INL-0}	P_{idle}	P_{INL-0}
0.1	0.8416	0.00000	0.8416	0.00000	0.8558	0.02347
0.2	0.6457	0.00005	0.6457	0.00005	0.7089	0.04630
0.3	0.3998	0.00044	0.4003	0.00045	0.5592	0.06849
0.4	0.0848	0.00179	0.0870	0.00189	0.4068	0.09009
0.42	0.0109	0.00225	0.0138	0.00238	0.3759	0.09434

Table 2.5: Effect of q on Model-1. $p = 0.2$,$s = 5$,$S = 20$

q	ρ	EC	EI	ER	EL	ED
$r = 0.3$						
0.4	0.6085	1.60571	12.55240	0.02575	0.00006	0.24732
0.5	0.4802	0.97057	12.56163	0.03215	0.00006	0.24735
0.6	0.3938	0.69561	12.56613	0.03858	0.00006	0.24738
0.7	0.3315	0.54211	12.56800	0.04501	0.00005	0.24741
0.8	0.2844	0.44413	12.56846	0.05146	0.00005	0.24744
0.9	0.2475	0.37616	12.56850	0.05791	0.00005	0.24747

Table 2.6: Effect of q on Model-2. $p = 0.2$,$s = 5$,$S = 20$

q	ρ	EC	EI	ER	EL	ED
$r = 0.3$						
0.4	0.6075	1.60345	11.29826	0.02673	0.00006	0.24722
0.5	0.4786	0.96948	11.30968	0.03336	0.00006	0.24725
0.6	0.3917	0.69492	11.31660	0.04001	0.00006	0.24729
0.7	0.3289	0.54162	11.32091	0.04666	0.00005	0.24732
0.8	0.2812	0.44375	11.32370	0.05332	0.00005	0.24735
0.9	0.2438	0.37586	11.32560	0.05999	0.00005	0.24738

Table 2.7: Effect of q on Model-3. $p = 0.2$, $S = 20$

q	ρ	EC	EI	ER	EL
$r = 0.3$					
0.4	0.3988	0.83068	5.90223	0.03608	0.01216
0.5	0.2738	0.55026	5.89773	0.04511	0.01216
0.6	0.1905	0.41139	5.89558	0.05415	0.01216
0.7	0.1310	0.32849	5.89437	0.06318	0.01216
0.8	0.0863	0.27340	5.89362	0.07221	0.01216
0.9	0.0516	0.23413	5.89311	0.08123	0.01216

Table 2.8: Effect of r on Model-1. $p = 0.2$, $s = 5$, $S = 20$

r	ρ	EC	EI	ER	EL	ED
$q = 0.7$						
0.30	0.3315	0.54211	12.56800	0.04501	0.00005	0.24741
0.35	0.3338	0.54371	12.61708	0.04533	0.00002	0.24776
0.40	0.3359	0.54496	12.65311	0.04557	0.00001	0.24803
0.45	0.3377	0.54596	12.68061	0.04576	0.00000	0.24823
0.50	0.3393	0.54678	12.70223	0.04592	0.00000	0.24839
0.55	0.3407	0.54747	12.71966	0.04605	0.00000	0.24853
0.60	0.3419	0.54806	12.73401	0.04616	0.00000	0.24864

Table 2.9: Effect of r on Model-2. $p = 0.2$, $s = 5$, $S = 20$

r	ρ	EC	EI	ER	EL	ED
$q = 0.7$						
0.30	0.3289	0.54162	11.32091	0.04666	0.00005	0.24732
0.35	0.3319	0.54336	11.41722	0.04667	0.00002	0.24770
0.40	0.3344	0.54470	11.48969	0.04667	0.00001	0.24798
0.45	0.3366	0.54577	11.54617	0.04667	0.00001	0.24819
0.50	0.3385	0.54664	11.59140	0.04667	0.00000	0.24837
0.55	0.3400	0.54736	11.62845	0.04667	0.00000	0.24851
0.60	0.3414	0.54798	11.65933	0.04667	0.00000	0.24863

Table 2.10: Effect of r on Model-3. $p = 0.2$, $S = 20$

r	ρ	EC	EI	ER	EL
$q = 0.7$					
0.30	0.1310	0.32849	5.89437	0.06318	0.01216
0.35	0.1310	0.32852	5.94601	0.06373	0.01051
0.40	0.1310	0.32855	5.98534	0.06415	0.00926
0.45	0.1310	0.32857	6.01629	0.06448	0.00827
0.50	0.1310	0.32858	6.04127	0.06478	0.00748
0.55	0.1310	0.32860	6.06188	0.06497	0.00682
0.60	0.1310	0.32861	6.07915	0.06516	0.00627

Chapter 3

Discrete Time inventory models with common life time and positive service time

3.1 Introduction

In the previous chapter we considered inventory with unlimited life time. In this chapter we restrict the life time to be a random variable with finite mean value. Further it is assumed that all items perish simultaneously (common life time). In most of the inventory models, it is assumed that items can be stored indefinitely to meet future demands. However, certain types of inventories undergo change during storage with the result that with passage of time they may become partially or entirely unfit for consumption. For example milk products, meat and other food stuffs, medicines, blood stored in blood banks etc become unusable after a certain time has elapsed. Also sometimes the item may become obsolete.

Inventory models for perishable or deteriorating items are of considerable importance. Perishable items have a deterministic usable life after which they become unusable. e.g., Chemicals produced by a processing plant. There is a large amount of research papers dealing with such models. Nahmias and Shah [44] studied the models where demand was assumed random in each period and products were assumed to have a certain life time which may be random. Various optimal characteristics were obtained under different conditions on the demand and the life time processes. Lian and Liu [35] studied a discrete time (s, S) perishable inventory model with geometric inter-demand times and batch demands. With zero lead time and allowing backlogs, they constructed a multidimensional Markov chain to model the inventory level process and obtained a closed form expression for average cost function. They also concluded that discrete time models may be used to approximate their continuous time counterparts effectively. Lian et al. [36] discussed a discrete time model for common life time inventory systems where demands are in batches following a discrete PH renewal process. With zero lead time and also allowing backlogs, they constructed a multidimensional Markov chain to model the inventory level process. They obtained a closed form expected cost function. Compared with the results for the constant life time model, they proved that the variance of the lifetime significantly affects the system behavior. Kaspi and Perry [26], Bar-Lev and Perry [6] have obtained the characteristics that measure the performance of perishable inventory systems by applying the results from queueing models with impatient customers. Perishable inventory problems with constant life time have been studied quite extensively using the periodic review policy. Fries [16] and Nahmias [43] use dynamic programming in a perishable inventory model with life time m and zero lead time.

In this chapter we model and analyze three discrete time perishable inventory systems with positive service time. First we discuss an inventory model in which the stored items have a common life time. Assume that lead time is positive. Second model is a modified form of Lian et al. [36] extended to positive service time case. In this model order for replenishment is placed when the inventory level reaches zero at a service completion epoch provided the number of customers waiting at that epoch is at least s. Else the order is placed as and when the number of waiting customers reaches s. We assume that the lead time is zero. In the third model, we assume that the items perish one by one and that life time follows geometric distribution. In the first and third models we assume that customers do not join the system when the inventory level is zero whereas in the second model, customers join even when the inventory level is zero.

3.2 Description of Model-1

We consider a discrete time (s, S) inventory model in which the stored items have a common life time. Demands arrive according to a Bernoulli process with parameter p. Service time and lead time follow independent geometric distributions with parameters q and r, respectively. When the inventory level reaches s for the first time after each replenishment, an order is placed to bring back the level up to S. Since lead time is positive, it will take a random amount of time to bring the items back to level S. When the replenishment occurs, we discard all the old items so that the remaining (fresh) items have common life time. Assume that life time follows geometric distribution with parameter t and that no customer joins when the inventory level is zero.

3.2.1 Analysis of the model

Let N_m denote the number of customers in the system and I_m the inventory level at epoch m. We denote the joint queue length and inventory level by $\chi = \{(N_m, I_m) : m \in N\}$. Then χ is a Discrete Time Markov Chain with state space

$$E = \{(i, j) : i \geqslant 0; 0 \leqslant j \leqslant S\}.$$

The one step transition probability matrix of the Markov chain χ is given by

$$\mathbf{P} = \begin{bmatrix} C_0 & C_1 & & & \\ A_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

where each entry is a square matrix of order $S + 1$.
They are obtained as (only transitions with positive probabilities are indicated below):

$$[C_0]_{ij} = \begin{cases} \bar{r}, & j = i, \quad i = 0 \\ \bar{p}\,\bar{r}t, & j = 0, \quad i = 1, 2, \ldots, s \\ \bar{p}\,\bar{r}\,\bar{t}, & j = i, \quad i = 1, 2, \ldots, s \\ \bar{p}\,\bar{t}, & j = i, \quad i = s+1, s+2, \ldots, S \\ \bar{p}r, & j = S, \quad i = 0, 1, \ldots, s \\ t, & j = 0, \quad i = s+1, s+2, \ldots, S \\ 0, & \text{otherwise} \end{cases}$$

$$[C_1]_{ij} = \begin{cases} p\bar{r}t, & j = 0, \quad i = 1, 2, \ldots, s \\ p\bar{r}\,\bar{t}, & j = i, \quad i = 1, 2, \ldots, s \\ p\bar{t}, & j = i, \quad i = s+1, s+2, \ldots, S \\ p\,r, & j = S, \quad i = 0, 1, \ldots, s \\ 0, & \text{otherwise} \end{cases}$$

$$[A_2]_{ij} = \begin{cases} q\overline{r}, & j = i-1, \quad i = 1,2,\ldots,s \\ \overline{p}q, & j = i-1, \quad i = s+1, s+2,\ldots,S \\ \overline{p}qr, & j = S-1, \quad i = 1,2,\ldots,s \\ 0, & \text{otherwise} \end{cases}$$

$$[A_0]_{ij} = \begin{cases} p\overline{q}\,\overline{r}, & j = i \qquad i = 1,2,\ldots,s \\ p, & j = i, \qquad i = s+1, s+2,\ldots,S \\ p\,r, & j = S \qquad i = 0 \\ p\overline{q}r, & j = S, \qquad i = 1,2,\ldots,s \\ 0, & \text{otherwise} \end{cases}$$

$$[A_1]_{ij} = \begin{cases} \overline{r}, & j = i, \qquad i = 0 \\ \overline{p}\,\overline{q}\,\overline{r}t, & j = 0, \qquad i = 1,2,\ldots,s \\ \overline{p}\,\overline{q}\,\overline{r}\,\overline{t}, & j = i, \qquad i = 1,2,\ldots,s \\ \overline{p}\,\overline{q}, & j = i, \qquad i = s+1, s+2,\ldots,S \\ \overline{p}r, & j = S, \qquad i = 0 \\ \overline{p}\,\overline{q}r + p\,q\,r, & j = S, \qquad i = 1,2,\ldots,s \\ 0, & \text{otherwise} \end{cases}$$

where $\overline{x} = 1 - x$, $x = p, q, r, t$.

3.2.2 Stability Condition

For determining the stability condition for the system, consider the transition matrix $A = A_0 + A_1 + A_2$ given by

$$[A]_{ij} = \begin{cases} \overline{r}, & j=i, & i=0 \\ q\,\overline{r}+\overline{p}\,\overline{q}\,\overline{r}t, & j=0, & i=1 \\ \overline{p}\,\overline{q}\,\overline{r}t, & j=0, & i=2,3,\dots,s \\ q\overline{r}, & j=i-1, & i=2,3,\dots,s \\ \overline{p}\,\overline{q}\,\overline{r}\,\overline{t}+p\,\overline{q}\,\overline{r}, & j=i, & i=1,2,\dots,s \\ \overline{p}q, & j=i-1, & i=s+1,s+2,\dots,S \\ p+\overline{p}\,\overline{q}, & j=i, & i=s+1,s+2,\dots,S \\ \overline{p}qr, & j=S-1, & i=1,2,\dots,s \\ r, & j=S, & i=0 \\ \overline{p}\,\overline{q}r+p\,r, & j=S, & i=1,2,\dots,s \end{cases}$$

The system χ is stable if and only if $\boldsymbol{\pi}A_0\boldsymbol{e} < \boldsymbol{\pi}A_2\boldsymbol{e}$ where $\boldsymbol{\pi}$ is the stationary probability vector of A satisfying $\boldsymbol{\pi}A = \boldsymbol{\pi}$ and $\boldsymbol{\pi}\boldsymbol{e} = 1$, where $\boldsymbol{e}$ is a column vector of 1's of appropriate order. Let $\boldsymbol{\pi} = (\pi_0, \pi_1, \dots \pi_s, \dots, \pi_S)$. Then $\boldsymbol{\pi}A = \boldsymbol{\pi}$ gives

$$\pi_j = \begin{cases} \delta\theta^{j-1}\eta^{s-j}\pi_0, & j=1,2,\dots,s \\ \frac{\delta\theta^{j-1}}{\overline{p}\,q}\pi_0, & j=s+1,s+2,\dots,S-1 \end{cases}$$

$\pi_S = \frac{\delta}{\overline{p}q}\left(\theta^s - \overline{p}qr\frac{(\eta^s-\theta^s)}{(\eta-\theta)}\right)\pi_0$, where $\theta = 1 - p\overline{q}\,\overline{r} - \overline{p}\,\overline{q}\,\overline{r}\,\overline{t}$, $\eta = q\overline{r}$,

$\delta = \frac{r(\eta-\theta)}{(\eta-\theta)+\overline{p}\,\overline{q}\,\overline{r}t(\eta^s-\theta^s)}$.

Normalizing condition $\pi\boldsymbol{e} = 1$ gives

$\pi_0 = \gamma^{-1}$ where $\gamma = 1 + \frac{\delta\theta^s(S-s)}{\overline{p}q} + \delta(1-r\delta)\frac{\eta^s-\theta^s}{\eta-\theta}$.

Also, $\boldsymbol{\pi}A_0\boldsymbol{e} = \left[(p\overline{q}\delta - pr\delta)\frac{\eta^s-\theta^s}{\eta-\theta} + (S-s)\frac{p\delta\theta^s}{\overline{p}q} + pr\right]\pi_0$ and

$\boldsymbol{\pi}A_2\boldsymbol{e} = \left[q\overline{r}\delta\frac{\eta^s-\theta^s}{\eta-\theta} + (S-s)\delta\theta^s\right]\pi_0$.

Thus we have

Theorem 3.2.1. The Markov chain χ is stable if and only if

$$\sum_{i=0}^{s-1}\eta^{s-1-i}\theta^{i} < \frac{(S-s)(\overline{p}q-p)\delta\theta^{s}-\overline{p}qpr}{\overline{p}q\delta(p\overline{q}-pr-q\overline{r})} \tag{3.1}$$

3.2.3 Steady-state analysis

Now we proceed to the computation of the steady-state probabilities of the system state. Assume that stability condition (3.1) holds. Let $\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots,)$ be the steady-state probability vector of the Markov chain χ. Thus $\boldsymbol{x}P = \boldsymbol{x}$ and $\boldsymbol{x}\boldsymbol{e} = 1$. Then $\boldsymbol{x}_i = \boldsymbol{x}_1 R^{i-1}, i \geqslant 2$ where R is the minimal solution of the matrix quadratic equation $R^2A_2 + RA_1 + A_0 = R$.
$\boldsymbol{x}P = \boldsymbol{x}$ leads us to

$$\boldsymbol{x}_0 C_0 + \boldsymbol{x}_1 A_2 = \boldsymbol{x}_0$$

$$\boldsymbol{x}_0 C_1 + \boldsymbol{x}_1 A_1 + \boldsymbol{x}_2 R A_2 = x_1.$$

In general

$$\boldsymbol{x}_{i-1}A_0 + \boldsymbol{x}_i A_1 + \boldsymbol{x}_{i+1}A_2 = \boldsymbol{x}_i, \quad i \geqslant 2.$$

Normalizing condition $\boldsymbol{x}\boldsymbol{e} = 1$ gives $\boldsymbol{x}_0\boldsymbol{e} + \boldsymbol{x}_1(I-R)^{-1}\boldsymbol{e} = 1$.

3.2.4 System Performance Measures

Let $\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots,)$ be the steady-state probability vector. Partition $\boldsymbol{x}_i$, for $i \geqslant 0$ as $\boldsymbol{x}_i = (x_{i0}, x_{i1}, \ldots, x_{iS})$

1. Expected number of customers in the system EC $= \sum_{i=0}^{\infty} i\ \boldsymbol{x}_i \boldsymbol{e}$
2. Expected inventory level EI $= \sum_{i=0}^{\infty} \sum_{j=1}^{S} j\ x_{ij}$
3. Expected reorder rate ER $= q \sum_{i=1}^{\infty} x_{i,s+1}$
4. Expected replenishment rate ERR $= r \sum_{i=0}^{\infty} \sum_{j=0}^{s} x_{ij}$
5. Probability that the inventory level is zero $= \sum_{i=0}^{\infty} x_{i0}$
6. Expected loss rate of customers EL $= p \sum_{i=0}^{\infty} x_{i0}$
7. Expected number of customers waiting in the system when the inventory level is zero EW $= \sum_{i=0}^{\infty} i\ x_{i0}$
8. Expected rate of departure after completing service ED $= q \sum_{i=1}^{\infty} \sum_{j=1}^{S} x_{ij}$
9. Expected perishability rate EP$= t \sum_{i=0}^{\infty} \sum_{j=1}^{S} j\ x_{ij}$

3.3 Description of Model-2

We consider a discrete time (s, S) inventory model in which stored items have a common life time. We modify Lian et al. [36] as follows: Place replenishment order when the inventory level reaches zero at a service completion epoch if the number of customers waiting at this epoch is at least s. Else place the order when the number of waiting customers reaches s. The lead time is assumed to be zero. The inventory control is governed by the (s, S) policy with $s \leqslant 0$ (finite). Assume that the lifetime of inventoried items follows geometric distribution with parameter t and life completion precedes arrival.

3.3.1 Analysis of the model

Let N_m denote the number of customers in the system and I_m, the inventory level at time m^+. We construct a 2-dimensional Markov chain $\psi = \{(N_m, I_m) : m \in N\}$ with state space $E = \{(i, j) : i \geqslant 0; 0 \leqslant j \leqslant S\}$ to model the joint queue length and inventory level. The transition probability matrix of ψ is given by

$$\mathbf{P} = \begin{bmatrix} C_0 & C_1 & & & \\ A_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

where C_0, C_1, A_0, A_1, A_2 are given by

$$C_0 = \begin{bmatrix} \overline{p} & \underline{0} \\ \overline{p}t\boldsymbol{e} & \overline{p}\,\bar{t}I_S \end{bmatrix}, C_1 = \begin{bmatrix} p & \underline{0} \\ pt\boldsymbol{e} & p\bar{t}I_S \end{bmatrix}$$

$$A_0 = \begin{bmatrix} 0 & p\boldsymbol{e}_S{}^T \\ \underline{0} & p\overline{q}I_S \end{bmatrix}$$

$$A_1 = \begin{bmatrix} \overline{p} & \underline{0} \\ \overline{p}\,\overline{q}t\boldsymbol{e} & \overline{p}\,\overline{q}\,\bar{t}I_S \end{bmatrix}$$

$A_2 = \begin{bmatrix} 0 & \underline{0} \\ q\boldsymbol{e}_1 & C_3 \end{bmatrix}$ with $C_3 = \ (c_{ij})_{S\times S}$ where

$$c_{ij} = \begin{cases} q, \text{ if } j = i-1, i \geqslant 2 \\ 0, \text{ otherwise} \end{cases}$$

and $\boldsymbol{e}_i{}^T = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ where 1 is at the i^{th} place, $I_S = diag(1, 1, \ldots, 1)$ of order S and $\boldsymbol{e}$ denotes the column vector of $1's$ of appropriate order.

3.3.2 Stability condition

For obtaining the stability of the system, consider the transition probability matrix A defined by $A = A_0 + A_1 + A_2$ whose entries are

$$[A]_{ij} = \begin{cases} \overline{p}, & j = i, & i = 0 \\ q + \overline{p}\,\overline{q}t, & j = i-1, & i = 1 \\ \overline{p}\,\overline{q}t, & j = 0, & i = 2, 3, \ldots, S \\ p\overline{q} + \overline{p}\,\overline{q}\,\overline{t}, & j = i, & i = 1, 2, \ldots, S \\ q, & j = i-1, & i = 1, 2, \ldots, S \\ 0, \text{otherwise} \end{cases}$$

Let $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots \pi_s, \ldots, \pi_S)$ be the stationary probability vector associated with the matrix A. Then $\boldsymbol{\pi}A = \boldsymbol{\pi}$ and $\boldsymbol{\pi}\boldsymbol{e} = 1$.

$\boldsymbol{\pi}A = \boldsymbol{\pi}$ gives

$$\pi_j = \begin{cases} \frac{(p+\overline{p}\,\overline{q}t)\pi_0 - \overline{p}\,\overline{q}t}{q}, & j = 1 \\ \left(\frac{1-p\overline{q}-\overline{p}\,\overline{q}\,\overline{t}}{q}\right)^{j-1} \left[\frac{(p+\overline{p}\,\overline{q}t)\pi_0 - \overline{p}\,\overline{q}t}{q}\right], & j = 2, 3, \ldots, S \end{cases}$$

$\boldsymbol{\pi}\boldsymbol{e} = 1$ gives

$$\pi_0 = \frac{\overline{p}\,\overline{q}t(q+\overline{p}\,\overline{q}t)^S}{(p+\overline{p}\,\overline{q}t)(q+\overline{p}\,\overline{q}t)^S - pq^S}.$$

$\boldsymbol{\pi} A_0 \boldsymbol{e} = pq\pi_0 + p\bar{q}.$

$\boldsymbol{\pi} A_2 \boldsymbol{e} = q(1 - \pi_0).$

Hence we have the theorem

Theorem 3.3.1. The system ψ is stable if and only if

$$p < q(1 - \pi_0)(1 + p) \tag{3.2}$$

where

$$\pi_0 = \frac{\overline{p}\,\overline{q}t(q + \overline{p}\,\overline{q}t)^S}{(p + \overline{p}\,\overline{q}t)(q + \overline{p}\,\overline{q}t)^S - pq^S}$$

3.3.3 Steady-state analysis

Assume that stability condition (3.2) is satisfied. Let $\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots)$ be the steady-state probability vector of the Markov process ψ; then $\boldsymbol{x}$ satisfies $\boldsymbol{x}P = \boldsymbol{x}$ and $\boldsymbol{xe} = 1$. Then $\boldsymbol{x}_i$ has the matrix geometric form

$$\boldsymbol{x}_i = \boldsymbol{x}_1 R^{i-1}, \quad i \geqslant 2 \tag{3.3}$$

where R is the minimal nonnegative solution of the matrix quadratic equation

$$R^2 A_2 + RA_1 + A_0 = R$$

. The vectors $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$ can be obtained by solving the equations.

$$\boldsymbol{x}_0 C_0 + \boldsymbol{x}_1 A_2 = \boldsymbol{x}_0 \tag{3.4}$$

$$\boldsymbol{x}_0 C_1 + \boldsymbol{x}_1 A_1 + \boldsymbol{x}_2 R A_2 = \boldsymbol{x}_1 \tag{3.5}$$

and the normalizing condition

$$\boldsymbol{x}_0\boldsymbol{e} + \boldsymbol{x}_1(I-R)^{-1}\boldsymbol{e} = 1 \tag{3.6}$$

Then $\boldsymbol{x}_i$, $i \geqslant 2$ can be obtained from (3.3)

We analyze the system in which the time required to serve the inventory is negligible. Thus the system has only inventory and is of finite state space. i.e., we do not encounter simultaneously a queue of inventoried items and one of customers. Also assume that $t < \bar{t}$. The corresponding Markov chain is labeled as $\hat{\psi} = \{I_m : m \in N\}$ where I_m denotes the inventory level at epoch m.

The state space of the process is $E = \{-(s-1), -(s-2), \ldots, 0, 1, \ldots, S\}$ and its transition probability has entries given by

$$[\hat{P}]_{ij} = \begin{cases} \overline{p}, & j = i, \quad i = -(s-1), -(s-2), \ldots, 0 \\ p, & j = S, \quad i = -(s-1) \\ p, & j = i-1, \quad i = -(s-2), -(s-3), \ldots, 0 \\ pt, & j = -1, \quad i = 1, 2, \ldots, S \\ \overline{p}\,\overline{t} & j = i, \quad i = 1, 2, \ldots, S \\ t\overline{p} + p\bar{t}, & j = 0, \quad i = 1 \\ t\overline{p}, & j = 0, \quad i = 2, 3, \ldots, S \\ p\bar{t}, & j = i-1, \quad i = 2, 3, \ldots, S \\ 0 & \text{otherwise} \end{cases}$$

Let $\hat{\boldsymbol{\pi}} = (\hat{\pi}_{-(s-1)}, \hat{\pi}_{-(s-2)}, \ldots, \hat{\pi}_0, \hat{\pi}_1, \ldots, \hat{\pi}_S)$ be the steady-state vector of the process $\hat{\psi}$. Then $\hat{\boldsymbol{\pi}}\hat{P} = \hat{\boldsymbol{\pi}}$ and $\hat{\boldsymbol{\pi}}\boldsymbol{e} = 1$. On solving these

$$\hat{\pi}_j = \frac{p(p\bar{t})^{S-j}}{(1-\bar{p}\ \bar{t})^{S-j+1}}\ \hat{\pi}_{-(s-1)}, j = 1, 2, \ldots, S;$$

$$\hat{\pi}_0 = \frac{(1+(s-1)t)}{1-t}\hat{\pi}_{-(s-1)} - \frac{t}{1-t};$$

$$\hat{\pi}_{-1} = \hat{\pi}_{-2} = \ldots = \hat{\pi}_{-(s-2)} = \hat{\pi}_{-(s-1)}.$$

Also normalizing condition $\hat{\boldsymbol{\pi}}\boldsymbol{e} = 1$ gives

$$\hat{\pi}_{-(s-1)} = \frac{t(1-\bar{p}\,\bar{t})^S}{st(1-\bar{p}\,\bar{t})^S + p\bar{t}\left[(1-\bar{p}\,\bar{t})^S - (p\bar{t})^S\right]}.$$

3.3.4 System Performance Measures

Let $\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots)$ be the steady-state probability vector and $\boldsymbol{x}_i$, $i \geqslant 0$, be partitioned as $\boldsymbol{x}_i = (x_{i0}, x_{i1}, \ldots, x_{iS})$.
Then we have expression for following performance meassures :

1. Expected number of customers in the system EC $= \sum_{i=0}^{\infty} i\ \boldsymbol{x}_i\mathbf{e}$
2. Expected inventory level EI $=\sum_{i=0}^{\infty}\sum_{j=1}^{S} j\ x_{ij}$
3. Expected reorder rate ER $= q\ \sum_{i=0}^{\infty} x_{i,0}$
4. Expected number of customers waiting in the system when the inventory level is zero EW $=\sum_{i=0}^{\infty} i\ x_{i0}$
5. Expected perishability rate EP$= t\ \sum_{i=0}^{\infty}\sum_{j=1}^{S} j\ x_{ij}$

3.4 Description of Model-3

We consider a discrete time (s, S) perishable inventory system in which items fail one by one and that life time follows geometric distribution with parameter t. No customer joins when the inventory level is zero.

3.4.1 Analysis of the model

Let N_m denote the number of customers in the system and I_m, the inventory level at time m^+. Then consider the Markov chain $\varphi = \{(N_m, I_m) : m \in N\}$ with state space $E = \{(i, j) : i \geqslant 0; 0 \leqslant j \leqslant S\}$.
The one step transition probability matrix P of this Markov chain is given by

$$\mathbf{P} = \begin{bmatrix} C_0 & C_1 & 0 & 0 & \dots \\ A_2 & A_1 & A_0 & 0 & \dots \\ 0 & A_2 & A_1 & A_0 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where each entry is a square matrix of order $S + 1$. The entries of these matrices are described below:

$$[C_0]_{ij} = \begin{cases} \overline{r}, & j = i, \quad i = 0 \\ \overline{p}\,\overline{r}t, & j = i-1, \quad i = 1, 2, \dots, s \\ \overline{p}t, & j = i-1, \quad i = s+1, s+2, \dots, S \\ \overline{p}\,\overline{r}\,\overline{t}, & j = i, \quad i = 1, 2, \dots, s \\ \overline{p}\,\overline{t}, & j = i, \quad i = s+1, s+2, \dots, S \\ \overline{p}r, & j = S, \quad i = 0, 1, \dots, s \\ 0, & \text{in all other cases} \end{cases}$$

$$[C_1]_{ij} = \begin{cases} p\overline{r}t, & j = i-1, & i = 1,2,\ldots,s \\ p\,t, & j = i-1, & i = s+1, s+2,\ldots,S \\ p\overline{r}\,\overline{t}, & j = i, & i = 1,2,\ldots,s \\ p\,\overline{t}, & j = i, & i = s+1, s+2,\ldots,S \\ p\,r, & j = S, & i = 0,1,\ldots,s \\ 0, & \text{in all other cases} & \end{cases}$$

$$[A_2]_{ij} = \begin{cases} q\overline{r}, & j = i-1, & i = 1,2,\ldots,s \\ \overline{p}q, & j = i-1, & i = s+1, s+2,\ldots,S \\ \overline{p}qr, & j = S-1, & i = 1,2,\ldots,s \\ 0, & \text{in all other cases} & \end{cases}$$

$$[A_0]_{ij} = \begin{cases} p\overline{q}\,\overline{r}, & j = i & i = 1,2,\ldots,s \\ p, & j = i, & i = s+1, s+2,\ldots,S \\ p\,r, & j = S & i = 0 \\ p\overline{q}r, & j = S, & i = 1,2,\ldots,s \\ 0, & \text{in all other cases} & \end{cases}$$

$$[A_1]_{ij} = \begin{cases} \overline{r}, & j = i, & i = 0 \\ \overline{p}\,\overline{q}\,\overline{r}t, & j = i-1, & i = 1,2,\ldots,s \\ \overline{p}\,\overline{q}t, & j = i-1, & i = s+1, s+2,\ldots,S \\ \overline{p}\,\overline{q}\,\overline{r}\,\overline{t}, & j = i, & i = 1,2,\ldots,s \\ \overline{p}\,\overline{q}\,\overline{t} & j = i, & i = s+1, s+2,\ldots,S \\ \overline{p}r, & j = S, & i = 0 \\ \overline{p}\,\overline{q}r + p\,q\,r, & j = S, & i = 1,2,\ldots,s \\ 0, & \text{in all other cases} & \end{cases}$$

where $\overline{x} = 1 - x$, $x = p, q, r, t$.

3.4.2 Stability Condition

For determining the stability condition of the system, consider the transition matrix $A = A_0 + A_1 + A_2$. Its entries are given below:

$$[A]_{ij} = \begin{cases} \overline{r}, & j = i, & i = 0 \\ q\,\overline{r} + \overline{p}\,\overline{q}\,\overline{r}t, & j = i-1, & i = 1,2,\ldots,s \\ \overline{p}\,\overline{q}\,\overline{r}\,\overline{t} + p\,\overline{q}\,\overline{r}, & j = i, & i = 1,2,\ldots,s \\ \overline{p}q + \overline{p}\,\overline{q}t, & j = i-1, & i = s+1, s+2,\ldots,S \\ p + \overline{p}\,\overline{q}\,\overline{t}, & j = i, & i = s+1, s+2,\ldots,S \\ \overline{p}qr, & j = S-1, & i = 1,2,\ldots,s \\ r, & j = S, & i = 0 \\ \overline{q}r + p\,q\,r, & j = S, & i = 1,2,\ldots,s \\ 0, & \text{in all other cases.} \end{cases}$$

The process φ is stable if and only if $\boldsymbol{\pi} A_0 \mathbf{e} < \boldsymbol{\pi} A_2 \mathbf{e}$ where $\boldsymbol{\pi}$ is the stationary probability vector of A satisfying $\boldsymbol{\pi} A = \boldsymbol{\pi}$ and $\boldsymbol{\pi}\mathbf{e} = 1$, with $\mathbf{e}$, a column vector of 1's of appropriate order. Let $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots \pi_s, \ldots, \pi_S)$. Then $\boldsymbol{\pi} A = \boldsymbol{\pi}$ gives

$$\pi_j = \left(\frac{(1-\overline{r})\left[1-\overline{q}\,\overline{r}(p+\overline{p}\,\overline{t})\right]^{j-1}}{(q\,\overline{r}+\overline{p}\,\overline{q}\,\overline{r}t)^j} \right) \pi_0, \quad j = 1,2,\ldots,s$$

$$\pi_{s+j} = \frac{1-\overline{r}}{\overline{p}} \left(\frac{1-\overline{q}\,\overline{r}(p+\overline{p}\,\overline{t})}{(q\,\overline{r}+\overline{p}\,\overline{q}\,\overline{r}t)} \right)^s \frac{(1-\overline{q}\,\overline{t})^{j-1}}{(q+\overline{q}t)^j} \pi_0, \quad j = 1,2,\ldots,S-s-1$$

$$\pi_S = (1-\overline{r}) \left(\frac{\overline{p}q(q\,\overline{r}+\overline{p}\,\overline{q}\,\overline{r}t)^s + (\overline{q}+pq)[1-\overline{q}\,\overline{r}(p+\overline{p}\,\overline{t})]^s}{\overline{p}(1-\overline{q}\,\overline{t})(q\,\overline{r}+\overline{p}\,\overline{q}\,\overline{r}t)^s} \right) \pi_0$$

Normalizing condition $\boldsymbol{\pi}\mathbf{e} = 1$ gives

$$\pi_0 = \left(\frac{\overline{p}q(q\,\overline{r}+\overline{p}\,\overline{q}\,\overline{r}t)^s}{(\overline{p}q\overline{r}+(S-s)r)[1-\overline{q}\,\overline{r}(p+\overline{p}\,\overline{t})]^s + \overline{p}qr(q\,\overline{r}+\overline{p}\,\overline{q}\,\overline{r}t)^s} \right).$$

$$\boldsymbol{\pi} A_0 \mathbf{e} = \left\{ -pq \left[\frac{[1-\overline{q}\,\overline{r}(p+\overline{p}\,\overline{t})]^s}{(q\,\overline{r}+\overline{p}\,\overline{q}\,\overline{r}t)^s} \right] - p\overline{q} + pr \right\} \pi_0 + p.$$

$$\boldsymbol{\pi} A_2\mathbf{e} = \left\{(q\overline{r} + \overline{p}qr - \overline{p}q)\left[\frac{[1-\overline{q}\,\overline{r}(p+\overline{p}\,\overline{t})]^s}{(q\,\overline{r}+\overline{p}\,\overline{q}\,\overline{r}t)^s}\right] - q\overline{r} - \overline{p}qr\right\}\pi_0 + \overline{p}q.$$

Theorem 3.4.1. The Markov Chain φ is stable if and only if

$$\left[\frac{q\overline{r} + \overline{p}\,\overline{q}\,\overline{r}t}{1 - \overline{q}\,\overline{r}(\overline{p}\,\overline{t} + p)}\right]^s < \frac{pq + \overline{p}q\overline{r} + (S-s)r - p\overline{q}\,\overline{r} - \frac{p(S-s)r}{\overline{p}q}}{2pr + q\overline{r} - p\overline{q}}.$$

3.4.3 Steady-state analysis

Now, we proceed to the computation of the steady-state probabilities of the system state. Let $\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots)$ be the steady-state probability vector of the Markov process φ. We assume that $\boldsymbol{x}_i = \boldsymbol{x}_0 R^i, i \geqslant 1$ where R is the minimal solution of the matrix quadratic equation $R^2A_2 + RA_1 + A_0 = R$. $\boldsymbol{x}P = \boldsymbol{x}$ leads us to

$$\boldsymbol{x}_0 C_0 + \boldsymbol{x}_1 A_2 = \boldsymbol{x}_0$$

$$\boldsymbol{x}_0 C_1 + \boldsymbol{x}_1 A_1 + \boldsymbol{x}_2 A_2 = \boldsymbol{x}_1.$$

$$\boldsymbol{x}_{i-1} A_0 + \boldsymbol{x}_i A_1 + \boldsymbol{x}_{i+1} A_2 = \boldsymbol{x}_i, \quad i \geqslant 2.$$

Also $\boldsymbol{x}e = 1$ gives $\boldsymbol{x}_0\boldsymbol{e} + \boldsymbol{x}_1(I-R)^{-1}\boldsymbol{e} = 1$.

3.4.4 System Performance Measures

To get a complete picture of the system it is essential to compute the long run characteristics of the system state. Let $\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots)$ be the steady-state probability vector and $\boldsymbol{x}_i$, $i \geqslant 0$ be partitioned as $\boldsymbol{x}_i = (x_{i0}, x_{i1}, \ldots, x_{iS})$.

1. Expected number of customers in the system EC $= \sum_{i=0}^{\infty} i\ \boldsymbol{x}_i\boldsymbol{e}$.

2. Expected inventory level EI $=\sum_{i=0}^{\infty}\sum_{j=1}^{S} j\ x_{ij}$.

3. Expected reorder rate ER $= q\sum_{i=1}^{\infty} x_{i,s+1} + t(s+1)\sum_{i=0}^{\infty} x_{i,s+1}$.

4. Expected replenishment rate ERR $= \ r\sum_{i=0}^{\infty}\sum_{j=0}^{s}\ x_{ij}$.

5. Expected loss rate of customers EL $= p\sum_{i=0}^{\infty} x_{i0}$.

6. Expected number of customers waiting in the system when the inventory level is zero EW $=\sum_{i=0}^{\infty} i\ x_{i0}$.

7. Expected rate of departure after completing service ED $= q\sum_{i=1}^{\infty}\sum_{j=1}^{S}\ x_{ij}$.

8. Expected perishability rate EP$= t\sum_{i=0}^{\infty}\sum_{j=1}^{S} j\ x_{ij}$.

3.5 Cost Analysis

We analyze numerically the steady-state expected cost rate under the following parameters.
Let c_0 denote the fixed ordering cost
c_1 - procurement cost/ unit
c_2 - holding cost of inventory /unit/unit time
c_3 - holding cost of customers/unit/unit time
c_4 - cost due to the loss of customers /unit/unit time
c_5 - the replacement (disposal) cost/unit decayed(perished)

Then

For Model 1, the Expected Total Cost

$$ETC = [c_0 + (S - s)c_1]\, ER + c_2 EI + c_3 EW + c_4 EL + c_5 EP.$$

For model 2

$$ETC = [c_0 + Sc_1]\, ER + c_2 EI + c_3 EW + c_5 EP.$$

For Model 3

$$ETC = \left[c_0 + \sum_{i=0}^{s} r(S - i)c_1\right] ER + c_2 EI + c_3 EW + c_4 EL + c_5 EP.$$

3.6 Numerical illustration and comparison of the performance of the different models

Table 3.1: Effect of p on Model-1 $q = 0.8$, $r = 0.7$,$s = 5$,$S = 20$

p	EC	EI	ER	EL	EP
$t = 0.1$					
0.1	0.1436	16.7681	0.0000	0.0110	0.2321
0.2	0.4134	15.6945	0.0014	0.0179	0.4838
0.3	1.0601	14.5163	0.0101	0.0187	0.7531
0.4	4.0861	13.2924	0.0323	0.0193	1.0571
0.42	6.7442	13.0006	0.0388	0.0203	1.1226
$t = 0.2$					
0.1	0.1294	15.577	0.0000	0.0197	0.4313
0.2	0.3798	15.4276	0.0003	0.0328	0.9513
0.3	1.0050	14.8326	0.0052	0.0352	1.5336
0.4	4.0273	13.5903	0.0272	0.0382	2.1474
0.42	6.6973	13.2176	0.0350	0.0391	2.2704

Tables 3.1, 3.2 and 3.3 indicate that in all the models, expected number of customers and expected perishability rate increase and inventory level

Table 3.2: Effect of p on Model-2

p	EC	EI	ER	EP
$q = 0.8$,$t = 0.1$,$s = 5$,$S = 20$				
0.1	0.1551	11.4101	0.0582	0.1579
0.2	0.4044	10.3014	0.0570	0.3161
0.3	0.9218	9.5506	0.0537	0.4889
0.4	2.8875	9.1915	0.0503	0.7047
0.42	4.1783	9.1756	0.0497	0.7544
q=0.8,t=0.2,s=5,S=20				
0.1	0.1592	12.2137	0.0562	0.3383
0.2	0.4331	11.5592	0.0560	0.7125
0.3	1.0220	11.0350	0.0559	1.1413
0.4	3.3286	10.6827	0.0547	1.6660
0.42	4.9502	10.6407	0.0546	1.7892

decreases as arrival rate p increases. Expected reorder rate also increases for models 1 and 3 but decreases in model 2. This is because the lead time is zero for model 2. In all models as maximum level of inventory is increased, the expected inventory level and expected number of customers increase. There is a decrease in the expected reorder rate. (See tables 3.4, 3.5 and 3.6). From tables 3.7, 3.8 and 3.9 as service rate q increases expected inventory level and expected customers also decrease in all models. From tables 3.10, 3.11 and 3.12 we can see that as s increases, expected inventory level increases and expected number of customers decreases. Table 3.15 shows that as arrival rate p increases, expected cost also increases in model-1 and model-3 whereas it is a strictly convex function for model-2.

Figures 3.1 and 3.2 show that expected cost increase as S increases. As q increases, cost function is convex for all models. (See figures 3.3, 3.4). The expected total cost is minimum for model-2.

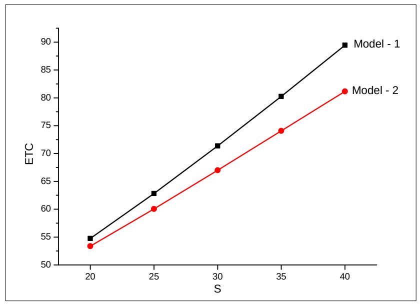

Figure 3.1: S verses ETC for Model-1 and Model-2 when $s = 5$, $p = 0.4$, $q = 0.8$, $r = 0.7$, $t = 0.2$, $c_0 = 100, c_1 = 10, c_2 = 2, c_3 = 3, c_4 = 5, c_5 = 4$

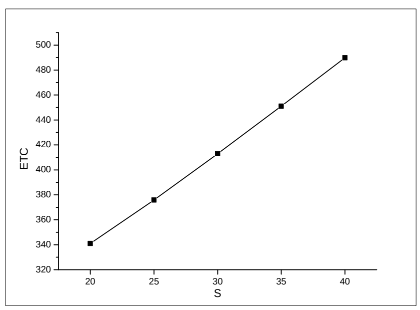

Figure 3.2: Model-3 : S versus ETC when $s = 5$, $p = 0.4$, $q = 0.8$, $r = 0.7$, $t = 0.2$, $c_0 = 100, c_1 = 10, c_2 = 2, c_3 = 3, c_4 = 5, c_5 = 4$

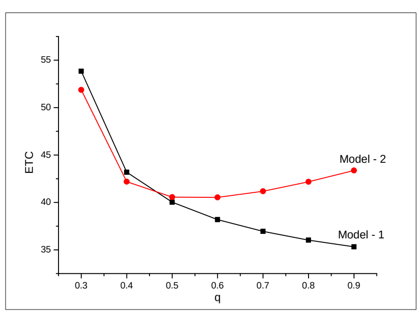

Figure 3.3: q versus ETC Model-1 and model-2 when $s = 5$, $S = 20$, $p = 0.2$, $r = 0.7$, $t = 0.2$, $c_0 = 100, c_1 = 10, c_2 = 2, c_3 = 3, c_4 = 5, c_5 = 4$

Table 3.3: Effect of p on Model-3

p	EC	EI	ER	EP
$t = 0.1$				
0.1	0.1606	12.8165	0.0527	0.1776
0.2	0.4473	12.7243	0.0534	0.3944
0.3	1.1011	12.6396	0.0541	0.6644
0.4	4.0879	12.5338	0.0621	1.0092
0.42	6.6965	12.4965	0.0692	1.0889
$t = 0.2$				
0.1	0.1603	12.7498	0.0523	0.3533
0.2	0.4459	12.6665	0.0531	0.7847
0.3	1.0965	12.5926	0.0546	1.3226
0.4	4.0617	12.5065	0.0601	2.0121
0.42	6.6384	12.4759	0.0622	2.1723

Table 3.4: Effect of S on Model-1 $p = 0.4$, $q = 0.8$,$r = 0.7$,$t = 0.2$

S	EC	EI	ER	EL	EP
$s = 5$					
20	4.0273	13.5903	0.0272	0.0200	2.1474
25	4.2123	16.8719	0.0188	0.0195	2.6738
30	4.3317	20.2636	0.0139	0.0191	3.2172
35	4.4148	23.7516	0.0108	0.0189	3.7763
40	4.4757	27.3252	0.0086	0.0188	4.3500

Table 3.5: Effect of S on Model-2

S	EC	EI	ER	EP
$p = 0.4$,$q = 0.8$,$t = 0.2$,$s = 5$				
20	3.328	10.6827	0.0547	1.6660
25	3.6540	13.2296	0.0405	2.0923
30	3.8737	15.7639	0.0322	2.5150
35	4.0320	18.2906	0.0268	2.9357
40	4.1516	20.8126	0.0229	3.3554

Table 3.6: Effect of S on Model-3

S	EC	EI	ER	EP
$p = 0.4$,$q = 0.8$,$r = 0.7$,$t = 0.2$,$s = 5$				
20	4.0617	12.5065	0.0521	2.0121
25	4.2585	15.0120	0.0392	2.4338
30	4.3873	17.5167	0.0315	2.8533
35	4.4781	20.0208	0.0263	3.2718
40	4.5455	22.5245	0.0226	3.6897

Table 3.7: Effect of q on Model-1 $p = 0.2$, $r = 0.2$, $s = 5$, $S = 20$

q	EC	EI	ER	EL	EP
$t = 0.2$					
0.5	0.8722	15.4927	0.0018	0.0250	1.4897
0.6	0.6114	15.4821	0.0009	0.0286	1.2576
0.7	0.4691	15.4654	0.0005	0.0310	1.0844
0.8	0.3798	15.4276	0.0003	0.0328	0.9513
0.9	0.3188	15.3850	0.0002	0.0342	0.8463

Table 3.8: Effect of q on Model-2

q	EC	EI	ER	EP
p=0.2, ,t=0.2,s=5,S=20				
0.3	4.4758	11.8175	0.0205	1.9380
0.4	1.5624	11.6950	0.0277	1.4368
0.5	0.9457	11.6349	0.0349	1.1435
0.6	0.6780	11.5993	0.0422	0.9508
0.7	0.5285	11.5758	0.0495	0.8143
0.8	0.4331	11.5592	0.0568	0.7125
0.9	0.3669	11.5471	0.0641	0.6335

Table 3.9: Effect of q on Model-3

q	EC	EI	ER	EP
p=0.2, ,r=0.2,s=5,S=20				
0.3	4.4720	12.4965	0.0192	3.0699
0.4	1.5774	12.4915	0.0256	2.3029
0.5	0.9580	12.4888	0.0319	1.8432
0.6	0.6881	12.4869	0.0382	1.5372
0.7	0.5369	12.4852	0.0446	1.3187
0.8	0.4402	12.4834	0.0509	1.1548
0.9	0.3731	12.4818	0.0573	1.0273

Table 3.10: Effect of s on Model-1 $p = 0.4$,$q = 0.8$,$r = 0.7$,$t = 0.2$

s	EC	EI	ER	EL	EP
$S = 50$					
5	4.5581	34.6949	0.0058	0.0186	5.5359
10	4.5217	35.7418	0.0070	0.0187	5.7265
15	4.4754	36.8552	0.0086	0.0188	5.9268
20	4.4145	38.0412	0.0108	0.0189	6.1365
25	4.3314	39.3062	0.0139	0.0191	6.3544

Table 3.11: Effect of s on Model-2

s	EC	EI	ER	EP
p=0.4,q=0.8,t=0.2,S=50				
5	4.3201	25.8475	0.0178	4.1928
10	4.1212	27.5870	0.0208	4.4511
15	3.9763	29.6050	0.0250	4.7650
20	3.8098	31.9197	0.0314	5.1163
25	3.6407	35.0729	0.0308	5.5661

Table 3.12: Effect of s on Model-3

s	EC	EI	ER	EP
$p = 0.4, q = 0.8, r = 0.7, t = 0.2, S = 50$				
5	4.6390	27.5308	0.0176	4.5245
10	4.5973	30.0275	0.0198	4.9296
15	4.5452	32.5240	0.0226	5.3316
20	4.4777	35.0203	0.0263	5.7287
25	4.3868	37.5161	0.0315	6.1183

Table 3.13: Effect of r on Model-1 $p = 0.4$, $q = 0.8$, $s = 5$, $S = 20$

r	EI	ER	EL	EW	EP
$t = 0.2$					
0.2	11.6383	0.0202	0.0733	0.0435	1.7738
0.4	12.9973	0.0243	0.0364	0.0095	2.0177
0.6	13.4627	0.0265	0.0236	0.0026	2.1171
0.8	13.6833	0.0278	0.0173	0.0007	2.1707
0.9	13.7535	0.0283	0.0152	0.0003	2.1891

Table 3.14: Effect of r on Model-3 $p = 0.4$, $q = 0.8$, $s = 5$, $S = 20$

r	EI	ER	EL	EW	EP
$t = 0.2$					
0.2	11.6734	0.0464	0.0084	0.0349	1.7694
0.4	12.2710	0.0501	0.0070	0.0043	1.9196
0.6	12.4561	0.0516	0.0000	0.0004	1.9892
0.8	12.5435	0.0524	0.0000	0.0000	2.0303
0.9	12.5717	0.0527	0.0000	0.0000	2.0451

Table 3.15: Variations in arrival rate p with ETC. $q = 0.8$, $r = 0.7$, $t = 0.1$, $s = 5$, $S = 20$, $c_0 = 100$, $c_1 = 10$, $c_2 = 2$, $c_3 = 3$, $c_4 = 5$, $c_5 = 4$

$c_6 = 2$			
	model 1	model 2	model 3
p	ETC	ETC	ETC
0.1	34.9534	38.6341	231.2867
0.2	34.9990	37.6558	240.1999
0.3	37.8456	37.7351	256.9755
0.4	51.1865	43.1437	317.7525

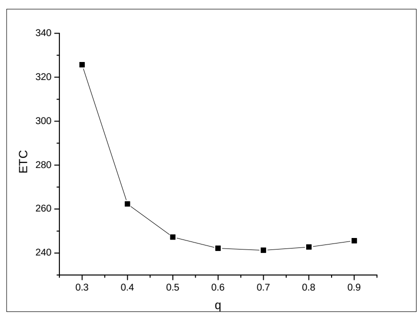

Figure 3.4: Model-3 : q versus ETC when $s = 5$, $S = 20$, $p = 0.2$, $r = 0.5$, $t = 0.3$, $c_0 = 100, c_1 = 10, c_2 = 2, c_3 = 3, c_4 = 5, c_5 = 4$

Chapter 4

(s, S) policy with inventory dependent customer arrival

4.1 Introduction

In chapters 2 and 3 we assumed that customers do not join the queue if there is no item in the inventory. In other an arriving customer joins the system if there is atleast one item in the inventory. However during lead time customers may be discouraged to join the system in case inventory on hand is less than or at most equal to the number of customers present in the system. In this situation such an arrival will get service only after the commodity is replenished. In the extreme case a new arrival does not join if the inventory level is less than or equal to the number of customers in the system. This leads to a finite system where as the system described at the beginning of this paragraph turns out to be a countably infinite system.

In this chapter we discuss two inventory models with positive service time and lead time. In model 1 we assume that an arriving customer joins the system only if the number in the queue is less than the number of items in the inventory at that epoch. In model 2 it is assumed that if the inventory level is greater than or equal to $s+1$ at the time of arrival of a customer, then he necessarily joins. However if it is less than or equal to s (but larger than zero) then he joins only if the number of customers present is less than the on hand inventory since this guaranties that he gets service without waiting for replenishment. In real life situation, if the quantity of life saving medicine in a medical shop is less than the number of customers waiting for that medicine, the newly arriving customer decides not to join the system and goes elsewhere.

4.2 Mathematical Formulation of Model 1

Consider a single product (s, S) inventory system in which customers asking for the product arrive according to a Bernoulli process with parameter p. Each demand is for exactly one unit. Here we assume that an arriving customer joins the system only if the number in the queue is less than the number of items in the inventory at that epoch. The service time follows geometric distribution with parameter q. Whenever the inventory level depletes to s due to demand, an order is placed for replenishment up to S. Lead time for replenishment of the inventory has a geometric distribution with parameter r. Each time a replenishment is done, the on hand inventory is raised up to the maximum level S.

4.3 Analysis of the model

At time m^+ the system can be described by $\{X_m : m \in N\} = \{(N_m, I_m) : m \in N\}$ where N_m is the number of customers in the system and I_m is the inventory level.
It can be shown that $\{(N_m, I_m) : m \in N\}$ is a Markov chain with finite state space

$$E = \{(i,j) : 0 \leqslant i < j; \quad 0 \leqslant j \leqslant S\}.$$

The corresponding one-step transition probability matrix P is given by

$$\mathbf{P} = \begin{bmatrix} B_{0,0} & B_{0,1} & & & & & \\ B_{1,0} & A_{1,1} & A_{1,2} & & & & \\ & A_{2,1} & A_{2,2} & A_{2,3} & & & \\ & & \ddots & \ddots & \ddots & & \\ & & & A_{S-2,S-3} & A_{S-2,S-2} & A_{S-2,S-1} & \\ & & & & A_{S-1,S-2} & A_{S-1,S-1} & A_{S-1,S} \\ & & & & & A_{S,S-1} & A_{S,S} \end{bmatrix}$$

where
$[B_{0,0}]$ is of dimension $(S+1) \times (S+1)$ and is given by

$$[B_{0,0}]_{ij} = \begin{cases} \overline{r}, & j = i, \quad i = 0 \\ \overline{p}\,\overline{r}, & j = i, \quad i = 1, 2, \ldots, s \\ \overline{p}, & j = i, \quad i = s+1, s+2, \ldots, S \\ \overline{p}r, & j = S, \quad i = 0, 1, \ldots, s \\ 0, & \text{otherwise} \end{cases}$$

$[B_{0,1}]$ is of dimension $(S+1) \times S$ and is given by

$$[B_{0,1}]_{ij} = \begin{cases} p\,\overline{r}, & j = i, \quad i = 1, 2, \ldots, s \\ p\,r, & j = S, \quad i = 0, 1, \ldots, s \\ p, & j = i, \quad i = s+1, s+2, \ldots, S \\ 0, & \text{otherwise} \end{cases}$$

$[B_{1,0}]$ is of dimension $S \times (S+1)$ and is given by

$$[B_{1,0}]_{ij} = \begin{cases} q\,\overline{r}, & j = i-1, \quad i = 1, 2, \ldots, s \\ \overline{p}q, & j = i-1, \quad i = s+1, s+2, \ldots, S \\ \overline{p}qr, & j = S-1, \quad i = 1, 2, \ldots, s \\ 0, & \text{otherwise} \end{cases}$$

$[A_{1,1}]$ is of dimension $S \times S$ and is given by

$$[A_{1,1}]_{ij} = \begin{cases} \overline{q}\,\overline{r}, & j = i, \quad i = 1 \\ \overline{p}\,\overline{q}\,\overline{r}, & j = i, \quad i = 2, 3, \ldots, s \\ \overline{p}\,\overline{q}, & j = i, \quad i = s+1, s+2, \ldots, S \\ p\ \ q\,r, & j = S-1, \quad i = 1, 2, \ldots, s \\ \overline{p}\,\overline{q}r, & j = S, \quad i = 1, 2, \ldots, s \\ 0, & \text{otherwise} \end{cases}$$

$[A_{1,2}]$ is of dimension $S \times (S-1)$ and is given by

$$[A_{1,2}]_{ij} = \begin{cases} p\overline{q}\,\overline{r}, & j = i, \quad i = 2, 3, \ldots, s \\ p, & j = i, \quad i = s+1, s+2, \ldots, S \\ p\overline{q}r, & j = S, \quad i = 1, 2, \ldots, s \\ 0, & \text{otherwise} \end{cases}$$

$[A_{2,1}]$ is of dimension $(S-1) \times S$ and is given by

$$[A_{2,1}]_{ij} = \begin{cases} q\,\overline{r}, & j = i-1, \quad i = 2, 3, \ldots, s \\ \overline{p}q, & j = i-1, \quad i = s+1, s+2, \ldots, S \\ \overline{p}qr, & j = S-1, \quad i = 1, 2, \ldots, s \\ 0, & \text{otherwise} \end{cases}$$

$[A_{2,2}]$ is of dimension $(S-1) \times (S-1)$ and is given by

$$[A_{2,2}]_{ij} = \begin{cases} \overline{q}\,\overline{r}, & j = i, \quad i = s-1 \\ \overline{p}\,\overline{q}\,\overline{r}, & j = i, \quad i = s \\ \overline{p}\,\overline{q}, & j = i, \quad i = s+1, s+2, \ldots, S \\ \overline{p}\,\overline{q}r, & j = S, \quad i = 2, \ldots, s \\ pqr, & j = S-1, \quad i = 2, 3, \ldots, s \\ 0, & \text{otherwise} \end{cases}$$

$[A_{2,3}]$ is of dimension $(S-1)\times(S-2)$ and is given by

$$[A_{2,3}]_{ij} = \begin{cases} p\overline{q}\,\overline{r}, & j=i, \quad i=s \\ p, & j=i, \quad i=s+1,s+2,\dots,S \\ p\overline{q}r, & j=S, \quad i=1,2,\dots,s \\ 0, & \text{otherwise} \end{cases}$$

$\vdots$

$$A_{S-1,S} = \begin{bmatrix} 0 \\ p \end{bmatrix};$$

$$A_{S,S-1} = \begin{bmatrix} q & 0 \end{bmatrix};$$

$$A_{S,S} = \overline{q}.$$

4.4 Long run System behaviour

Assuming $p,q,r \in (0,1)$ the Markov chain $\{X_m\}$ is seen to be irreducible and positive recurrent. An irreducible Markov Chain on finite state space is stable.

Let $\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \dots, \boldsymbol{x}_{S-1}, x_S)$ be the steady-state vector of $\{X_m\}$. Then $\boldsymbol{x}P = \boldsymbol{x}$ and $\boldsymbol{x}\mathbf{e} = 1$ gives

$\boldsymbol{x}_0 = \boldsymbol{x}_1 D_0$ where $D_0 = B_{1,0}(I - B_{0,0})^{-1}$;

$\boldsymbol{x}_1 = \boldsymbol{x}_2 A_{2,1}(I - A_{1,1})^{-1} D_1$ where $D_1 = [I - D_0 B_{0,1}(I - A_{1,1})^{-1}]^{-1}$;

$\boldsymbol{x}_2 = \boldsymbol{x}_3 A_{3,2}(I - A_{2,2})^{-1} D_2$ where $D_2 = [I - A_{2,1}(I - A_{1,1})^{-1} D_1 A_{1,2}(I - A_{2,2})^{-1}]^{-1}$;

$\boldsymbol{x}_3 = \boldsymbol{x}_4 A_{4,3}(I - A_{3,3})^{-1} D_3$ where $D_3 = [I - A_{3,2}(I - A_{2,2})^{-1} D_2 A_{2,3}(I - A_{3,3})^{-1}]^{-1}$;

$\vdots$

$\boldsymbol{x}_{S-1} = x_S A_{S,S-1}(I - A_{S-1,S-1})^{-1} D_{S-1}$

where $D_{S-1} = [I - A_{S-1,S-2}(I - A_{S-2,S-2})^{-1} D_{S-2} A_{S-2,S-1}(I - A_{S-1,S-1})^{-1}]^{-1}$.

x_S can be found using the normalizing condition

$$\boldsymbol{x}_0\mathbf{e} + \boldsymbol{x}_1\mathbf{e} + \ldots + \boldsymbol{x}_{S-1}\mathbf{e} + x_S = 1.$$

4.5 System Performance Measures

Let the steady-state probability vector $\boldsymbol{x}$ be partitioned as $\boldsymbol{x}_0 = (x_{0,0}, x_{0,1}, \ldots, x_{0,S})$; $\boldsymbol{x}_1 = (x_{1,1}, x_{1,2}, \ldots, x_{1,S})$; $\boldsymbol{x}_2 = (x_{2,2}, x_{2,3}, \ldots, x_{2,S})$; $\ldots$; $\boldsymbol{x}_{S-1} = (x_{S-1,S-1}, x_{S-1,S})$; $x_S = x_{S,S}$.

We have then the following measures for evaluating performance of the system.

1. Expected number of customers in the system is given by
   $\text{EC} = \sum_{i=0}^{S} i\boldsymbol{x}_i e.$
2. Expected inventory level is given by
   $\text{EI} = \sum_{j=1}^{S} \sum_{i=0}^{j} j\ x_{i,j}.$
3. Expected reorder rate is given by
   $\text{ER} = q \sum_{i=1}^{s} x_{i,s+1}.$
4. Expected replenishment rate is given by
   $\text{ERR} = r \sum_{j=1}^{s} \sum_{i=0}^{j} x_{i,j}.$
5. Probability that the inventory level is zero is $\sum_{i=0}^{S} x_{i,0}.$

6. Expected loss rate of fresh arrivals is given by
   $$\text{EL} = p \sum_{i=1}^{S} x_{i,i}.$$

7. Expected rate of departure after completing service is $\text{ED} = q \sum_{j=1}^{S} \sum_{i=1}^{j} x_{i,j}.$

4.6 Model 2

In this model we assume that at the time of arrival of a customer, if the inventory level is $\geqslant s+1$, then he joins. However if it is $\leqslant s$ (but larger than zero) then he joins only if the number of customers present is less than the on hand inventory.

4.7 Analysis

The state of the system can be described by $\{X_m\} = \{(N_m, I_m) : m \in N\}$ where N_m is the number of customers in the system and I_m is the inventory level at epoch after the occurence of probable events. Then $\{X_m\}$ is a Markov Chain with countably infinite state space
$\{(i,j) : i \geqslant 0; 0 \leqslant j \leqslant S\}$. It is partitioned into levels as $\{(0,0), \ldots, (0,S), (1,0), \ldots, (1,S), \ldots, (s,0), \ldots, (s,S), (s+1,0), \ldots, (s+1,S), \ldots\}$. When the inventory level j satisfies $s+1 \leqslant j \leqslant S$, transition from (i,j) to $(i+1,j)$ is possible with probability p which is not possible when $j \leqslant s$ and $i \geqslant j$.

The corresponding one step transition probability matrix P is

$$
\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ \vdots \\ s \\ s+1 \\ \vdots \end{array}
\begin{array}{c}
\begin{array}{ccccccccc} 0 & 1 & 2 & \cdots & s-1 & s & s+1 & s+2 & \cdots \end{array} \\
\begin{pmatrix}
E_0 & C_0 & & & & & & & \\
B_1 & E_1 & C_1 & & & & & & \\
 & B_2 & E_2 & C_2 & & & & & \\
 & & \ddots & \ddots & \ddots & & & & \\
 & & & & B_s & E_s & C_s & & \\
 & & & & & A_2 & A_1 & A_0 & \\
 & & & & & & \ddots & \ddots & \ddots
\end{pmatrix}
\end{array}
$$

where each sub-matrix is of order $(S+1) \times (S+1)$. They are given by

$$
[E_0]_{ij} = \begin{cases}
\overline{r}, & j = i, \quad i = 0 \\
\overline{p}\,\overline{r}, & j = i, \quad i = 1, 2, \ldots, s \\
\overline{p}, & j = i, \quad i = s+1, s+2, \ldots, S \\
\overline{p}r, & j = S, \quad i = 0, 1, \ldots, s \\
0, & \text{otherwise}
\end{cases}
$$

$$
[E_k]_{ij} = \begin{cases}
\overline{r}, & j = i, \quad i = 0, \quad k = 1, 2, \ldots, s \\
\overline{q}\,\overline{r}, & j = i, \quad i = 1, \ldots, k \quad k = 1, \ldots, s \\
\overline{p}\,\overline{q}\,\overline{r}, & j = i, \quad i = k+1, \ldots, s, \quad k = 1, \ldots, s-1 \\
pq\overline{r}, & j = i-1, \quad i = k+1, \ldots, s, \quad k = 1, \ldots, s-1 \\
pq, & j = i-1, \quad i = s+1, s+2, \ldots, S \quad k = 1, 2, \ldots, s \\
\overline{p}r, & j = S, \quad i = 0, \quad k = 1, 2, \ldots, s \\
\overline{p}\,\overline{q}, & j = i, \quad i = s+1, s+2, \ldots, S \quad k = 1, 2, \ldots, s \\
p\ \ q\,r, & j = S-1, \quad i = 1, 2, \ldots, s \quad k = 1, 2, \ldots, s \\
\overline{p}\,\overline{q}r, & j = S, \quad i = 1, 2, \ldots, s \quad k = 1, 2, \ldots, s \\
0, & \text{otherwise}
\end{cases}
$$

$$[C_0]_{ij} = \begin{cases} p\,\overline{r}, & j = i, \quad i = 1, 2, \ldots, s \\ p\,r, & j = S, \quad i = 0, 1, \ldots, s \\ p, & j = i, \quad i = s+1, s+2, \ldots, S \\ 0, & \text{otherwise} \end{cases}$$

$$[C_k]_{ij} = \begin{cases} pr, & j = S, \quad i = 0 \quad k = 1, 2, \ldots, s \\ p\overline{q}r, & j = S, \quad i = 1, 2, \ldots, s \quad k = 1, 2, \ldots, s \\ p\overline{q}\,\overline{r}, & j = i, \quad i = k+1, \ldots, s, \quad k = 1, \ldots, s-1 \\ p\overline{q} & j = i, \quad i = s+1, s+2, \ldots, S \quad k = 1, 2, \ldots, s \\ 0, & \text{otherwise} \end{cases}$$

$$[B_k]_{ij} = \begin{cases} q\overline{r}, & j = i-1, \quad i = 1, \ldots, k \quad k = 1, \ldots, s \\ \overline{p}q\overline{r}, & j = i-1, \quad i = k+1, \ldots, s \quad k = 1, \ldots, s-1 \\ \overline{p}q, & j = i-1, \quad i = s+1, s+2, \ldots, S \quad k = 1, 2, \ldots, s \\ \overline{p}qr, & j = S-1, \quad i = 1, 2, \ldots, s \quad k = 1, 2, \ldots, s \\ 0, & \text{otherwise} \end{cases}$$

$$[A_1]_{ij} = \begin{cases} \overline{r}, & j = i, \quad i = 0 \\ \overline{p}r, & j = S, \quad i = 0 \\ \overline{q}\,\overline{r}, & j = i, \quad i = 1, 2, \ldots, s \\ \overline{p}\,\overline{q}, & j = i, \quad i = s+1, s+2, \ldots, S \\ pqr, & j = S-1, \quad i = 1, 2, \ldots, s \\ \overline{p}\,\overline{q}r, & j = S, \quad i = 1, 2, \ldots, s \\ pq, & j = i-1, \quad i = s+2, s+3, \ldots, S \\ 0, & \text{otherwise} \end{cases}$$

$$[A_2]_{ij} = \begin{cases} q\,\overline{r}, & j = i-1, \quad i = 1, 2, \ldots, s \\ \overline{p}q, & j = i-1, \quad i = s+1, s+2, \ldots, S \\ \overline{p}qr, & j = S-1, \quad i = 1, 2, \ldots, s \\ 0, & \text{otherwise} \end{cases}$$

$$[A_0]_{ij} = \begin{cases} pr, & j = S, \quad i = 0 \\ p\overline{q}r, & j = S, \quad i = 1, 2, \ldots, s \\ p, & j = i, \quad i = s+1, s+2, \ldots, S \\ p\overline{q}, & j = i, \quad i = s+1, s+2, \ldots, S \\ 0, & \text{otherwise} \end{cases}$$

4.8 Stability condition

For determining the stability condition for the system under study, consider the transition probability matrix $A = A_0 + A_1 + A_2$, which is obtained as

$$[A]_{ij} = \begin{cases} \overline{r}, & j = i, \quad i = 0 \\ r, & j = S, \quad i = 0 \\ q\overline{r}, & j = i-1, \quad i = 1, 2, \ldots, s \\ \overline{q}\,\overline{r}, & j = i, \quad i = 1, 2, \ldots, s \\ qr, & j = S-1, \quad i = 1, 2, \ldots, s \\ \overline{q}r, & j = S, \quad i = 1, 2, \ldots, s \\ \overline{p}q, & j = i-1, \quad i = s+1 \\ p + \overline{p}\,\overline{q}, & j = i, \quad i = s+1 \\ q, & j = i-1, \quad i = s+2, s+3, \ldots, S \\ \overline{q}, & j = i, \quad i = s+2, s+3, \ldots, S \\ 0, & \text{otherwise} \end{cases}$$

Let $\boldsymbol{\pi} = (\pi_0, \pi_1, \ldots \pi_s, \pi_{s+1}, \ldots, \pi_S)$ be the stationary probability vector associated with A. Then $\boldsymbol{\pi} A = \boldsymbol{\pi}$ and $\boldsymbol{\pi} e = 1$, where e is a column vector of 1's of appropriate order. $\boldsymbol{\pi} A = \boldsymbol{\pi}$ gives

$$\pi_j = \begin{cases} \frac{(1-\overline{r})(1-\overline{q}\,\overline{r})^{j-1}}{(q\,\overline{r})^j}\pi_0, & j = 1, 2, \ldots, s; \\ \frac{(1-\overline{r})(1-\overline{q}\,\overline{r})^{j-1}}{\overline{p}\,q(q\,\overline{r})^{j-1}}\pi_0, & j = s+1; \\ \frac{(1-\overline{r})(1-\overline{q}\,\overline{r})^{j-2}}{q(q\,\overline{r})^{j-2}}\pi_0, & j = s+2, s+3, \ldots, S-1. \end{cases}$$

$\pi_S = \frac{(1-\overline{r})[q(q\,\overline{r})^s+\overline{q}(1-\overline{q}\,\overline{r})^s]}{q(q\,\overline{r})^s}\pi_0.$

Further $\boldsymbol{\pi}\,\boldsymbol{e} = 1$ gives

$\pi_0 = \frac{q(q\,\overline{r})^s}{(1-\overline{q}\,\overline{r})^s[\overline{p}\,q+(S-s-1)r+r\,\overline{q}]+rq(q\,\overline{r})^s}$

$\boldsymbol{\pi}A_0e = \frac{(1-\overline{q}\,\overline{r})^s[pr+(S-s-1)p\overline{p}\,\overline{q}r]+p\overline{p}qr(q\overline{r})^s}{\overline{p}q(q\,\overline{r})^s}\pi_0.$

$\boldsymbol{\pi}A_2e = \frac{(1-\overline{q}\,\overline{r})^s[q\overline{r}+r+(S-s-1)\overline{p}r]-(q\,\overline{r})^{s+1}}{(q\,\overline{r})^s}\pi_0.$

Theorem 4.8.1. The Markov chain under study is stable if and only if

$$\frac{(q\,\overline{r})^s(\overline{p}q^2\overline{r}+p\overline{p}qr)}{(1-\overline{q}\,\overline{r})^s\left[\overline{p}q^2\overline{r}+\overline{p}qr-pr+(S-s-1)\overline{p}r(\overline{p}-\overline{q})\right]} < 1 \tag{4.1}$$

4.9 Steady-state analysis

Let $\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_s,\ \boldsymbol{x}_{s+1},\ \boldsymbol{x}_{s+2}, \ldots)$ be the steady-state vector of X_m, where $\boldsymbol{x}_i = (x_{i,0}, x_{i,1}, \ldots, x_{i,S})$ for $i \geqslant 0$.
Then

$$\boldsymbol{x}P = \boldsymbol{x}, \quad \boldsymbol{x}\mathbf{e} = 1. \tag{4.2}$$

Under the stability condition (4.1), the steady-state probability vector $\boldsymbol{x}$ is obtained as $\boldsymbol{x}_i = \boldsymbol{x}_{s+1}R^{i-(s+1)};\ i \geqslant s+2$, where R is the minimal non negative solution to the matrix equation $R^2A_2 + RA_1 + A_0 = R$. The vectors $\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots, \boldsymbol{x}_s,\ \boldsymbol{x}_{s+1}$ are obtained from the equations

$$\boldsymbol{x}_0E_0 + \boldsymbol{x}_1B_1 = \boldsymbol{x}_0 \tag{4.3}$$

$$\boldsymbol{x}_{i-1}C_{i-1} + \boldsymbol{x}_iE_i + \boldsymbol{x}_{i+1}B_{i+1} = \boldsymbol{x}_i,\ 1 \leqslant i \leqslant s-1 \tag{4.4}$$

$$\boldsymbol{x}_{s-1}C_{s-1} + \boldsymbol{x}_sE_s + \boldsymbol{x}_{s+1}A_2 = \boldsymbol{x}_s \tag{4.5}$$

$$\boldsymbol{x}_s C_s + \boldsymbol{x}_{s+1} A_1 + \boldsymbol{x}_{s+2} A_2 = \boldsymbol{x}_{s+1} \tag{4.6}$$

From (4.6), we get $\boldsymbol{x}_s C_s + \boldsymbol{x}_{s+1}(A_1 - I - RA_2) = 0$.
Thus $\boldsymbol{x}_{s+1} = \boldsymbol{x}_s C_s (I - A_1 - RA_2)^{-1} = \boldsymbol{x}_s R_s$, where $R_s = C_s(I - A_1 - RA_2)^{-1}$.
From (4.5), we have
$\boldsymbol{x}_{s-1} C_{s-1} + \boldsymbol{x}_s(E_s - I - R_s A_2) = 0$.
Thus $\boldsymbol{x}_s = \boldsymbol{x}_{s-1} C_{s-1}(I - E_s - R_s A_2)^{-1} = \boldsymbol{x}_{s-1} R_{s-1}$, where
$R_{s-1} = C_{s-1}(I - E_s - R_s A_2)^{-1}$.
From (4.4) we have
$\boldsymbol{x}_i = \boldsymbol{x}_{i-1} R_{i-1}$; $1 \leqslant i \leqslant s-1$ where $R_{i-1} = C_{i-1}(I - E_i - R_{s-1} B_{i+1})^{-1}$.
Finally $\boldsymbol{x}_0$ can be found from the normalizing condition
$\boldsymbol{x}_0\ \mathbf{e} + \boldsymbol{x}_1\ \mathbf{e} + \ldots + \boldsymbol{x}_s\ \mathbf{e} + \boldsymbol{x}_{s+1}(I - R)^{-1}\mathbf{e} = 1$.
That is, $\boldsymbol{x}_0 \left(I + \sum_{i=0}^{s-1} \prod_{j=0}^{i} R_j + \prod_{j=0}^{s} R_j (I - R)^{-1} \right) \mathbf{e} = 1$.

4.10 System Performance Measures

1. Expected number of customers in the system is given by
   $\mathrm{EC} = \sum_{i=1}^{\infty} i\ \boldsymbol{x}_i \mathbf{e} = \sum_{i=1}^{s} i\ \boldsymbol{x}_i \mathbf{e} + \boldsymbol{x}_{s+1} R(I - R)^{-2}\mathbf{e} + (s+1)\boldsymbol{x}_{s+1}(I - R)^{-1}\mathbf{e}$.

2. Expected inventory level is $\mathrm{EI} = \sum_{i=0}^{\infty} \sum_{j=1}^{S} j\ x_{i,j}$.

3. Expected reorder rate is $\mathrm{ER} = q \sum_{i=1}^{\infty} x_{i,s+1}$.

4. Expected replenishment rate is given by $\mathrm{ERR} = r \sum_{i=0}^{\infty} \sum_{j=0}^{s} x_{i,j}$.

5. Expected loss rate of fresh arrivals is given by $\mathrm{EL} = p \sum_{i=0}^{\infty} x_{i,0}$.

6. Expected rate of departure ED after completing service = $q\sum_{i=0}^{\infty}\sum_{j=1}^{S} x_{i,j}$.

4.11 Cost Analysis

We analyze numerically the steady-state expected total cost with the following.
Let c_0 denote the fixed ordering cost
c_1 - procurement cost/unit
c_2 - holding cost of inventory /unit/unit time
c_3 - holding cost of customers/unit/unit time
c_4 - cost due to the loss of customers /unit/unit time
Then
For Model 1 and Model 2, the Expected Total Cost

$$ETC = [c_0 + (S - s)c_1]\, ER + c_2 EI + c_3 EC + c_4 EL.$$

Table 4.1: Effect of S on Model-1. $p = 0.3$, $q = 0.7$, $s = 5$

S	P_{idle}	EC	EI	ER	EL	ED
$r = 0.4$						
20	0.41041	1.35927	12.51734	0.02506	0.00251	0.41272
25	0.40548	1.40085	15.04177	0.01894	0.00208	0.41616
30	0.40230	1.42963	17.56195	0.01523	0.00175	0.41839
35	0.40007	1.45077	20.07878	0.01273	0.00151	0.41995
40	0.39843	1.46692	22.59291	0.01095	0.00132	0.42110
45	0.39716	1.47962	25.10484	0.00960	0.00118	0.42199

Table 4.3 shows that as the arrival rate p increases, expected number of customers as well as expected inventory level increases. Expected reorder rate decreases which is completely against our expectation. This may be due

Table 4.2: Effect of S on Model-2. $p = 0.3$, $q = 0.7$, $s = 5$

S	P_{idle}	EC	EI	ER	EL	ED
$r = 0.4$						
20	0.5716	0.52443	12.45552	0.01941	0.00004	0.01361
25	0.5715	0.52477	14.95997	0.01467	0.00003	0.01028
30	0.5715	0.52485	17.46261	0.01179	0.00003	0.00826
35	0.5715	0.52491	19.96405	0.00985	0.00002	0.00690
40	0.5715	0.52495	22.46473	0.00847	0.00002	0.00593
45	0.5714	0.52498	24.96490	0.00742	0.00002	0.00519

to the increase in the number of customers above the reorder level. Expected rate of departure after completion of service increases. From table 4.4 we notice that the expected number of customers increases and the expected inventory level decreases as p increases which is as expected. Here expected reorder rate and ED also increase with increasing value of p.

Table 4.3: Effect of p on Model-1. $q = 0.7$,$s = 5$,$S = 20$

p	P_{idle}	EC	EI	ER	EL	ED
$r = 0.3$						
0.400	0.18429	3.10560	12.30936	0.02388	0.02734	0.57100
0.425	0.13985	3.69822	12.33383	0.02117	0.04100	0.60210
0.450	0.10388	4.30922	12.38669	0.01779	0.05797	0.62728
0.475	0.07627	4.91289	12.46673	0.01416	0.07776	0.64661
0.500	0.05602	5.48963	12.56962	0.01068	0.09967	0.66078
0.525	0.04170	6.02814	12.68933	0.00764	0.12302	0.67081
0.550	0.03181	6.52450	12.81963	0.00517	0.14723	0.67773
0.575	0.02507	6.97973	12.95500	0.00330	0.17191	0.68245
0.600	0.02047	7.39737	13.09107	0.00199	0.19681	0.68567
$r = 0.4$						
0.400	0.17118	3.24947	12.44302	0.02443	0.02636	0.58017
0.425	0.12573	3.88555	12.47622	0.02156	0.04004	0.61199
0.450	0.08945	4.53842	12.53896	0.01800	0.05715	0.63738
0.475	0.06219	5.17810	12.62933	0.01422	0.07718	0.65647
0.500	0.04272	5.78265	12.74208	0.01063	0.09938	0.67009
0.525	0.02939	6.34053	12.87043	0.00752	0.12302	0.67943
0.550	0.02052	6.84898	13.00772	0.00504	0.14751	0.68564
0.575	0.01471	7.31071	13.14836	0.00319	0.17245	0.68970
0.600	0.01093	7.73089	13.28813	0.00191	0.19760	0.69235
$r = 0.5$						
0.400	0.16281	3.35611	12.52091	0.02476	0.02569	0.58604
0.425	0.11690	4.02446	12.56128	0.02176	0.03937	0.61817
0.450	0.08072	4.70769	12.63205	0.01807	0.05657	0.64349
0.475	0.05401	5.37252	12.73057	0.01417	0.07673	0.66220
0.500	0.03536	5.99544	12.85078	0.01050	0.09909	0.67525
0.525	0.02293	6.56511	12.98532	0.00737	0.12288	0.68395
0.550	0.01490	7.07996	13.12730	0.0049	0.14750	0.68957
0.575	0.00983	7.54417	13.27118	0.00308	0.17256	0.69312
0.600	0.00666	7.96420	13.41294	0.00182	0.19782	0.69534

Table 4.4: Some measures of Model-2 $q = 0.7$,$s = 5$,$S = 20$

p	P_{idle}	EC	EI	ER	EL	ED
$r = 0.3$						
0.400	0.4312	0.78998	12.09634	0.02503	0.00087	0.01534
0.425	0.3958	0.87849	12.03603	0.02655	0.00115	0.0156
0.450	0.3610	0.97460	11.98145	0.02803	0.00147	0.01583
0.475	0.3266	1.08523	11.92748	0.02953	0.00185	0.01600
0.500	0.2925	1.21591	11.87408	0.03109	0.0023	0.01612
0.525	0.2588	1.37615	11.82097	0.03278	0.00281	0.01620
0.550	0.2255	1.58437	11.76729	0.03473	0.00337	0.01622
0.575	0.1925	1.88187	11.71082	0.03724	0.00400	0.01624
0.600	0.1592	2.38153	11.64574	0.04096	0.00469	0.01635
$r = 0.4$						
0.400	0.4275	0.81025	12.25618	0.02583	0.00020	0.01535
0.425	0.3938	0.88429	12.22904	0.02720	0.00026	0.01572
0.450	0.3586	0.98327	12.18449	0.02876	0.00034	0.01592
0.475	0.3234	1.09829	12.14019	0.03036	0.00044	0.01605
0.500	0.2885	1.23601	12.09601	0.03203	0.00057	0.01611
0.525	0.2537	1.40836	12.05153	0.03385	0.00071	0.01612
0.550	0.2191	1.63981	12.00552	0.03601	0.00088	0.01626
0.575	0.1845	1.99035	11.95475	0.03889	0.00107	0.01645
0.600	0.1489	2.64823	11.89015	0.04345	0.00129	0.01656
$r = 0.5$						
0.400	0.4268	0.81367	12.36443	0.02621	0.00004	0.01545
0.425	0.3933	0.88673	12.34523	0.02759	0.00006	0.01587
0.450	0.3578	0.98719	12.30706	0.02920	0.00008	0.01607
0.475	0.3223	1.10471	12.26897	0.03086	0.00010	0.01618
0.500	0.2870	1.24682	12.23077	0.03260	0.00013	0.01623
0.525	0.2517	1.42754	12.19189	0.03452	0.00017	0.01631
0.550	0.2164	1.67716	12.15079	0.03683	0.00021	0.01642
0.575	0.1807	2.07595	12.10320	0.04002	0.00027	0.01656
0.600	0.1433	2.91097	12.03631	0.04538	0.00033	0.01669

Chapter 5

Discrete time (s, S) production inventory system with positive service time

5.1 Introduction

In earlier chapters we studied (s, S) inventory system with positive service time. Order for replenishment was placed when inventory level depletes to s. Study of inventory system where the processing of inventory requires some positive amount of time was started by Sigman and Levy [52] in continuous time set up. Later Bruneel and Kim [13] introduced positive service time in inventory where service time is assumed to be constant and obtained optimal order quantity that minimizes the total cost using Dynamic Programme technique. They assumed that new customers can join the system while a service is going on. Hence a queue of demands can be formed even when the inventory is available. Berman and Kim [8] assumed probabilistic service time and obtained steady state behavior of the system. Krishnamoorthy and Viswanath [30] were the first to study production inventory with positive service time in the continuous case. In (s, S) production inventory system, when the inventory level reaches s, production process is switched on and when the inventory level reaches S, production process is switched off. Each production is of one unit. When the inventory level is between $s + 1$ and $S - 1$, the production status is in either 'on' or 'off' mode. Hence to describe the system both the inventory level and the production status should be taken into consideration thereby obtaining a discrete time Markov chain by providing required additional information such as residual/elapsed service time.

5.2 The Mathematical model and its analysis

We discuss an (s, S) production inventory system where the processing of inventory requires a positive random amount of time (discrete). This leads to the formation of a queue of demands. Demands arrive according to a Bernoulli process. Service time and lead time are distributed geometrically.

In an (s, S) production inventory system, when the inventory level falls to s, the production process is immediately 'switched on'. It is 'switched off' when the inventory level reaches S. Each production is of one unit. When the inventory level is in between $s + 1$ and $S - 1$, the production process can be either in 'on' mode or in 'off' mode.

We consider the production inventory system with a single server. Demands occur according to Bernoulli process with parameter p. Processing of inventory requires a positive random amount of time, which is distributed geometrically with parameter q. When the inventory level reaches s, the production is 'switched on' and stays in that mode until the inventory reaches S.

Production time for each unit follows geometric distribution with parameter r. No customer is allowed to join the system when the inventory level is zero.

Let N_m denote the number of customers in the system, I_m, the inventory level and C_m, the status of the production process at epoch m. Then the corresponding DTMC is $\chi = \{(N_m, I_m, C_m) : m \in N\}$. When the inventory level is such that $0 \leqslant I_m \leqslant s$, the production process is in 'on' mode and it is in 'off' mode if $I_m = S$.
If $s + 1 \leqslant I_m \leqslant S - 1$, define

$$C_m = \begin{cases} 0, & \text{if the production is 'off' at epoch } m \\ 1, & \text{if the production is 'on' at epoch } m \end{cases}$$

The state space of the Markov Chain χ is given by

$$E = \bigcup_{i \geqslant 0} \{\{(i,j) : 0 \leqslant j \leqslant s\} \cup \{(i,j,k) : s+1 \leqslant j \leqslant S-1; k=0,1\} \cup \{(i,S)\}\}$$

The transition probability matrix of the process χ is given by

$$\mathbf{P} = \begin{bmatrix} C_0 & C_1 & & & \\ A_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

where each entry is a square matrix of order $2S - s$ which are given by

$$C_0 = \begin{bmatrix} \bar{r} & \bar{p}r & & & & & & & & & \\ & B_1 & \bar{p}r & & & & & & & & \\ & & \ddots & \ddots & & & & & & & \\ & & & B_1 & \bar{p}r & & & & & & \\ & & & & B_1 & D_1 & & & & & \\ & & & & & B_2 & D_2 & & & & \\ & & & & & & \ddots & \ddots & & & \\ & & & & & & & B_2 & D_2 & & \\ & & & & & & & & B_2 & D_3 & \\ & & & & & & & & & B_3 & \end{bmatrix}$$

with

$B_1 = \bar{p}\,\bar{r}$; $B_2 = \begin{bmatrix} \bar{p} & 0 \\ 0 & \bar{p}\,\bar{r} \end{bmatrix}$; $B_3 = \bar{p}$;

$D_1 = \begin{bmatrix} 0 & \bar{p}r \end{bmatrix}$; $D_2 = \begin{bmatrix} 0 & 0 \\ 0 & \bar{p}r \end{bmatrix}$; $D_3 = \begin{bmatrix} 0 \\ \bar{p}r \end{bmatrix}$.

$$C_1 = \begin{bmatrix} 0 & pr & & & & & & & & \\ & p\bar{r} & pr & & & & & & & \\ & & \ddots & \ddots & & & & & & \\ & & & p\bar{r} & pr & & & & & \\ & & & & p & 0 & & & & \\ & & & & & p\bar{r} & pr & & & \\ & & & & & & \ddots & \ddots & & \\ & & & & & & & p & 0 & \\ & & & & & & & & p\bar{r} & pr \\ & & & & & & & & & p \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & & & & & & & & & & \\ q\overline{r} & \overline{p}qr & & & & & & & & & \\ & \overline{p}q\overline{r} & \overline{p}qr & & & & & & & & \\ & & \ddots & \ddots & & & & & & & \\ & & & \overline{p}q\overline{r} & \overline{p}qr & & & & & & \\ & & & & E_1 & F_1 & & & & & \\ & & & & & E_2 & F_1 & & & & \\ & & & & & & \ddots & \ddots & & & \\ & & & & & & & E_2 & F_1 & & \\ & & & & & & & & E_2 & F_2 & \\ & & & & & & & & & E_3 & 0 \end{bmatrix}$$

with $E_1 = \begin{bmatrix} \overline{p}q \\ \overline{p}q\overline{r} \end{bmatrix}$; $E_2 = \begin{bmatrix} \overline{p}q & 0 \\ 0 & \overline{p}q\overline{r} \end{bmatrix}$; $E_3 = \begin{bmatrix} \overline{p}q & 0 \end{bmatrix}$;

$F_1 = \begin{bmatrix} 0 & 0 \\ \overline{p}qr & 0 \end{bmatrix}$; $F_2 = \begin{bmatrix} 0 & 0 \\ \overline{p}qr & 0 \end{bmatrix}$.

$$A_1 = \begin{bmatrix} \overline{r} & \overline{p}r & & & & & & & & \\ & \alpha & \beta & & & & & & & \\ & \gamma & \alpha & \beta & & & & & & \\ & & \ddots & \ddots & \ddots & & & & & \\ & & & \gamma & \alpha & \beta & & & & \\ & & & & \gamma & \alpha & G_0 & & & \\ & & & & & & G_1 & G_2 & & \\ & & & & & & & \ddots & \ddots & \\ & & & & & & & & G_1 & G_2 \\ & & & & & & & & & G_1 & G_3 \\ & & & & & & & & & & G_4 \end{bmatrix}$$

with $G_0 = \begin{bmatrix} \beta & 0 \end{bmatrix}$; $G_1 = \begin{bmatrix} \delta & 0 \\ \gamma & \alpha \end{bmatrix}$; $G_2 = \begin{bmatrix} 0 & 0 \\ \beta & 0 \end{bmatrix}$; $G_3 = \begin{bmatrix} 0 \\ \beta \end{bmatrix}$; $G_4 = \delta$

where $\alpha = \overline{pqr} + pqr$; $\beta = \overline{pq}r$; $\gamma = pq\overline{r}$; $\delta = \overline{pq}$.

$$A_0 = \begin{bmatrix} 0 & pr & & & & & & & \\ & p\overline{q}\,\overline{r} & p\overline{q}r & & & & & & \\ & & \ddots & \ddots & & & & & \\ & & & p\overline{q}\,\overline{r} & p\overline{q}r & & & & \\ & & & & p & & & & \\ & & & & & \ddots & \ddots & & \\ & & & & & & p\overline{q}\,\overline{r} & p\overline{q}r \\ & & & & & & & p \end{bmatrix},$$

where $\overline{p} = 1 - p$, $\overline{q} = 1 - q$, $\overline{r} = 1 - r$.

5.3 Steady-state analysis

For finding the steady-state vector of the process $\chi = \{(N_m, I_m, C_m) : m \in N\}$, consider the production inventory system where the time in serving the inventory is negligible and no backlogs are allowed once the inventory level reaches zero.

The corresponding Markov chain may be defined as $\hat{\chi} = \{(I_m, C_m) : m \in N\}$, where I_m and C_m are as defined earlier. The state space of the process $\hat{\chi}$ is given by $\hat{E} = \bigcup_{i=0}^{s}\{i\} \bigcup_{i=s+1}^{S-1} \{(i,k) : k = 0, 1\} \bigcup \{S\}$ and the corresponding transition probability matrix is given by

$$\hat{P} = \begin{bmatrix} \overline{r} & r & & & & & & & & & & \\ p\overline{r} & B_1 & r & & & & & & & & & \\ & \ddots & \ddots & \ddots & & & & & & & & \\ & & p\overline{r} & B_1 & r & & & & & & & \\ & & & p\overline{r} & B_1 & \hat{D}_1 & & & & & & \\ & & & & \hat{E}_1 & B_2 & \hat{D}_2 & & & & & \\ & & & & & \hat{E}_2 & B_2 & \hat{D}_2 & & & & \\ & & & & & & \ddots & \ddots & \ddots & & & \\ & & & & & & & \hat{E}_2 & B_2 & \hat{D}_2 & & \\ & & & & & & & & \hat{E}_2 & B_2 & \hat{D}_3 & \\ & & & & & & & & & \hat{E}_3 & B_3 & \end{bmatrix}$$

where $\hat{D}_1 = \begin{bmatrix} 0 & r \end{bmatrix}$; $\hat{D}_2 = \begin{bmatrix} 0 & 0 \\ 0 & r \end{bmatrix}$; $\hat{D}_3 = \begin{bmatrix} 0 \\ r \end{bmatrix}$;
$\hat{E}_1 = \frac{p}{\overline{p}q}\ E_1$; $\hat{E}_2 = \frac{p}{\overline{p}q}\ E_2$; $\hat{E}_3 = \frac{p}{\overline{p}q}\ E_3$; and B_1, B_2, B_3 are as defined earlier.
For determining the steady-state vector of the process $\hat{\chi}$, rearrange the transition probability matrix $\hat{P}$ as

$$\hat{P} = \begin{bmatrix} P_{11} & O & \underline{R_1} \\ P_{01} & P_{00} & \underline{0} \\ \underline{0} & \underline{R_2} & \overline{p} \end{bmatrix}, \text{ where}$$

$$[P_{11}]_{ij} = \begin{cases} \overline{r}, & j = i, \quad i = 0 \\ \overline{p}\,\overline{r}, & j = i, \quad i = 1, 2, \ldots, S \\ p\overline{r} & j = i-1, \quad i = 1, 2, \ldots, S-1 \\ r & j = i+1, \quad i = 0, 1, \ldots, S-1 \\ 0, & \text{otherwise} \end{cases}$$

$$\underline{R_1} = \begin{bmatrix} \underline{0} \\ r \end{bmatrix}_{S\times 1}$$

$$P_{01} = \begin{bmatrix} 0 & 0 & \dots & p & 0 & \dots & 0 \\ \underline{0} & \underline{0} & \dots & \underline{0} & \underline{0} & \dots & \underline{0} \end{bmatrix}_{(Q-1)\times S}$$

$$P_{00} = \begin{bmatrix} \overline{p} & & & \\ p & \overline{p} & & \\ & \ddots & \ddots & \\ & & p & \overline{p} \end{bmatrix}_{(Q-1)\times(Q-1)}$$

$$\underline{R_2} = \begin{bmatrix} \underline{0} & p \end{bmatrix}_{1\times(Q-1)}$$

Let $\hat{\boldsymbol{\pi}} = (\boldsymbol{\pi}_{\text{on}},\ \boldsymbol{\pi}_{\text{off}},\ \pi_S)$ be the steady-state vector of the process $\hat{\chi}$, where

$\boldsymbol{\pi}_{\text{on}} = (\pi_{1,0},\ \pi_{1,1},\ \dots,\ \pi_{1,s},\ \pi_{1,s+1},\ \dots,\ \pi_{1,S-1})$,

$\boldsymbol{\pi}_{\text{off}} = (\pi_{0,s+1},\ \pi_{0,s+2},\ \dots,\ \pi_{0,S-1})$

$\hat{\boldsymbol{\pi}}$ satisfies $\hat{\boldsymbol{\pi}}\hat{P} = \hat{\boldsymbol{\pi}}$ and $\hat{\boldsymbol{\pi}}e = 1$

$\hat{\boldsymbol{\pi}}\hat{P} = \hat{\boldsymbol{\pi}}$ gives

$\boldsymbol{\pi}_{\text{on}}\ P_{11} + \boldsymbol{\pi}_{\text{off}}\ P_{01} = \boldsymbol{\pi}_{\text{on}}$

$\boldsymbol{\pi}_{\text{off}}\ P_{00} + \pi_S\ R_2 = \boldsymbol{\pi}_{\text{off}}$

$\boldsymbol{\pi}_{\text{on}}\ \underline{R_1} + \pi_S\ \overline{p} = \pi_S$

By solving the above system of equations, we get

$\boldsymbol{\pi}_{\text{on}} = \underline{R_2}\ (I - P_{00})^{-1}\ P_{01}\ (I - P_{11})^{-1}\ \pi_S$ and $\boldsymbol{\pi}_{\text{off}} = \underline{R_2}\ (I - P_{00})^{-1}\ \pi_S$.

From the normalizing condition $\boldsymbol{\pi}e = 1$, we can find π_S which is obtained as $\pi_S = \frac{1}{1+\underline{R_2}\ (I-P_{00})^{-1}(\boldsymbol{e}+P_{01}\ (I-P_{11})^{-1}\boldsymbol{e})}$.

Now, using the vector $\hat{\boldsymbol{\pi}}$, we shall find the steady-state probability vector of the original system. Let $\boldsymbol{x}$ be the steady-state distribution of the original system. Now let $\boldsymbol{x}_0 = \rho\hat{\boldsymbol{\pi}}$ and $\boldsymbol{x}_i = \rho(\frac{p}{\overline{p}q})^i\hat{\boldsymbol{\pi}}$, for $i \geqslant 1$, where ρ is a constant to be determined. This will satisfies $\boldsymbol{x}P = \boldsymbol{x}$ and $\boldsymbol{xe} = 1$. Normalizing condition gives $\rho = 1 - \frac{p}{\overline{p}q}$. Now we have the following theorem

Theorem 5.3.1. Under the necessary and sufficient condition that $p < \overline{p}q$, the steady-state vector of the Markov chain $\{X_m : m \geqslant 0\}$ with transition probability matrix P, is given by $\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots,)$ where $\boldsymbol{x}_0 = \rho\hat{\boldsymbol{\pi}}$ and $\boldsymbol{x}_i = \rho(\frac{p}{\bar{p}q})^i \hat{\boldsymbol{\pi}}$ for $i \geqslant 1$, $\rho = 1 - \frac{p}{\bar{p}q}$ and the finite probability vector $\hat{\boldsymbol{\pi}}$ is given by $\hat{\boldsymbol{\pi}} = (\boldsymbol{\pi}_{\text{on}},\ \boldsymbol{\pi}_{\text{off}},\ \pi_S)$ where

$\boldsymbol{\pi}_{\text{on}} = \underline{R_2}\ (I - P_{00})^{-1}\ P_{01}\ (I - P_{11})^{-1}\ \pi_S,$

$\boldsymbol{\pi}_{\text{off}} = \underline{R_2}\ (I - P_{00})^{-1}\ \pi_S$

and $\pi_S = \frac{1}{1+\underline{R_2}\ (I-P_{00})^{-1}(\boldsymbol{e}+P_{01}\ (I-P_{11})^{-1}\boldsymbol{e})}.$

Above theorem indicates that there exists a decomposition of the state space of the system state. Let $M(z)$ denote the probability generating function of the number of customers in the system and $N(z)$, that of the number of items in the inventory when the production is in 'on' mode. Then the joint partial generating function can be written as the product of the individual generating functions. This result holds when the production is in 'off' mode as well. Thus we have the following decomposition result.

Theorem 5.3.2. Under the condition of stability, the generating function for the system state is the product of generating functions of the number of customers in a *Geo*/*Geo*/1 queue and that of the number of items in the inventory.

5.4 System Performance Measures

1. Mean number EC of customers in the system is given by
   $\text{EC} = \frac{p}{\bar{p}q - p}.$

2. Expected inventory level EI in the system is given by
   $$\text{EI} = \sum_{i=0}^{s} i\pi_{1,i} + \sum_{i=s+1}^{S-1} i(\pi_{0,i} + \pi_{1,i}) + S\pi_S.$$

3. Expected production rate EPR is given by
$$\text{EPR} = r\left(\sum_{i=0}^{s}\pi_{1,i} + \sum_{i=s+1}^{S-1}\pi_{1,i}\right).$$

4. Expected loss rate of customers ELR when the inventory level is zero is given by
$$\text{ELR} = p\ \pi_{1,0}.$$

5. Expected rate at which production process is switched 'on', is given by
$$E_{on} = q\ \left[\sum_{i=1}^{\infty}\rho(\frac{p}{\overline{p}q}^{i})\pi_{0,s+1}\right].$$

5.5 Production cycle

A production cycle is defined as the time between a switch 'on' and the next switch 'off' of the production process. When the inventory level is at $s+1$ and the production process in 'off' mode at a service completion epoch T_0, the production process is switched 'on'. Then the production process is turned 'off' only at an epoch T_1 where the inventory level reaches the maximum S. The length $T_1 - T_0$ is the time until absorption in the Markov Chain $\{Y_m : m \geqslant 0\} = \{(N_m, I_m) : m \in N\}$ where N_m and I_m are the number of customers in the system and the inventory level respectively, during the production cycle. The state space of the process Y_m is given by $\hat{E} = \{(i, j) : i \geqslant 0, 0 \leqslant j \leqslant S-1\}\bigcup\{\star\}$, where $\star$ denotes the single absorbing state, representing the switch 'off' mode in the production cycle.
The transition probability matrix of the process Y_m is given by
$$P_m = \begin{bmatrix} H & \mathbf{e} - H\mathbf{e} \\ \underline{0} & 1 \end{bmatrix}$$
where

$$H = \begin{bmatrix} D_4 & D_5 & & & & \\ D_6 & B_5 & B_4 & & & \\ & D_6 & B_5 & B_4 & & \\ & & D_6 & B_5 & B_4 & \\ & & & \ddots & \ddots & \ddots \end{bmatrix}$$

with

$$[D_4]_{ij} = \begin{cases} \overline{r}, & j = i, \quad i = 0 \\ \overline{p}\,\overline{r}, & j = i, \quad i = 1, 2, \ldots, s \\ \overline{p}r & j = i+1, \quad i = 0, 1, \ldots, s-1 \\ 0, \text{ otherwise} \end{cases}$$

$$[D_5]_{ij} = \begin{cases} pr, & j = i+1, \quad i = 0, 1, \ldots, s-1 \\ p\overline{r}, & j = i, \quad i = 1, 2, \ldots, s \\ 0, \text{ otherwise} \end{cases}$$

$$[D_6]_{ij} = \begin{cases} q\overline{r}, & j = i-1, \quad i = 1 \\ \overline{p}q\overline{r}, & j = i-1, \quad i = 2, 3, \ldots, s \\ \overline{p}qr & j = i, \quad i = 1, 2, \ldots, s \\ 0, \text{ otherwise} \end{cases}$$

$$[B_5]_{ij} = \begin{cases} \overline{r}, & j = i, \quad i = 0 \\ \overline{p}r, & j = i+1, \quad i = 0 \\ \overline{p}\,\overline{q}\,\overline{r} + pqr, & j = i, \quad i = 1, 2, \ldots, s \\ \overline{p}\,\overline{q}r, & j = i+1, \quad i = 1, 2, \ldots, s-1 \\ pq\overline{r}, & j = i-1, \quad i = 2, 3, \ldots, s \\ 0, \text{ otherwise} \end{cases}$$

$$[B_4]_{ij} = \begin{cases} pr, & j = i+1, \quad i = 0 \\ p\overline{q}\,\overline{r}, & j = i, \quad i = 1, 2, \ldots, s \\ p\overline{q}r, & j = i+1, \quad i = 0, 1, \ldots, s-1 \\ 0, \text{ otherwise.} \end{cases}$$

Now define $\boldsymbol{y} = (y_0, y_1, \ldots)$ where $\boldsymbol{y_i} = (y_{i1}, y_{i2}, \ldots, y_{iS})$ with y_{ij} standing for the expected time until absorption in the process, given the process has reached the state (i, j). Also define the probability vector $\boldsymbol{\tau} = (\tau_0, \tau_1, \ldots)$, where each τ_i is of dimension $1 \times S$ such that τ_{ij} is the probability of i customers and j inventory in the system, the production process being in the switched 'on' mode. Then the expected length of a production cycle is given by $ELP = \sum_{i=0}^{\infty} \tau_{is} y_{is}$.

5.6 Cost Analysis

Let c_k be the fixed cost for starting a production run;
c_h -holding cost of inventory /unit/unit time;
c_p -the cost of production per inventory;
c_l -the cost incurred due to loss of customers.
Consider the cost function $ETC = c_k E_{on} + c_l \text{ELR} + c_h \text{EI} + c_p \text{EPR}$.
The following tables are constructed by fixing values of input parameters and then varying over one or more of them.

5.7 Numerical illustration

Table 5.1: Effect of S on various measures. $p = 0.3$, $q = 0.7$, $r = 0.3$, $s = 5$, $c_k = 1000$, $c_l = 25$, $c_h = 2$, $c_p = 100$.

S	EI	EPR	ELR	E_{on}	ETC
15	7.4694	0.4642	0.0039	0.0166	78.0475
20	10.0530	0.3887	0.0026	0.0110	70.0410
25	12.5916	0.3491	0.0020	0.0082	68.3430
30	15.1141	0.3252	0.0016	0.0065	69.2880
35	17.6289	0.3093	0.0013	0.0054	71.6210
40	20.1395	0.2980	0.0011	0.0047	74.8060

From table 5.1 we see that, as the maximum inventory level increases, the

Table 5.2: Effect of s on various measures. $p = 0.3$, $q = 0.7$, $r = 0.3$, $S = 25$, $c_k = 500$, $c_l = 25$, $c_h = 2$, $c_p = 100$.

s	EI	EPR	ELR	E_{on}	ETC
2	12.2614	0.2562	0.0051	0.0037	52.1210
4	12.4647	0.3126	0.0027	0.0079	60.2070
6	12.7204	0.3912	0.0014	0.0086	68.8960
8	12.9495	0.4938	0.0008	0.0096	80.1000
10	13.0878	0.6265	0.0004	0.0108	94.2400

expected inventory level also increases, expected production rate and loss rate of customers decreases. Expected total cost (ETC) is convex in nature as S increases. Table 5.2 shows that as the reorder level increases, the expected inventory level and the expected production rate increases, expected loss rate of customers decreases. Expected total cost increases with increase in s.

Chapter 6

Discrete time inventory system with arbitrarily distributed service time

6.1 Introduction

In previous chapters we considered (s, S) inventory system where service time follows geometric distribution. In this chapter we consider inventory system with arbitrarily distributed service time. Here we analyze two inventory models with geometric inter arrival time with parameter p and general service time with distribution function $B(.)$. In both models we use (s, S) inventory control policy. When the inventory level depletes to s due to demands, an order is placed to bring the inventory level back to S. Also one unit from the inventory is used to serve one customer. In the first model, we assume that materialization of order for replenishment takes positive amount of time and that no customer joins when the inventory level is zero. In the

second model we assume that materialization of replenishment order is instantaneous.
At time $m = 0, 1, \ldots$, one or more of the following events may occur: a demand arrival, materialization of a replenishment order and a service completion. We assume that departure occurs in (m^-, m) and customer arrival in (m, m^+). The service times are i.i.d random variables with distribution $\{w_i\}_{i=1}^{\infty}$, where w_i = probability of the service time duration equal to i, i=1,2,..., having generating function $W(x) = \sum_{i=1}^{\infty} w_i x^i$, with mean service time β_1 (assumed to be finite). After service completion the served customer leaves the system for ever with one item from the inventory and will have no further effect on the system. The load of the system is $\rho = p\beta_1$. The inter-demand times and service times are mutually independent.

Below we analyze the two models separately.

6.2 Model 1

We consider an (s, S) inventory system in which demands arrive according to a Bernoulli process with parameter p. i.e., inter-demand times follow geometric distribution with parameter p. The service time duration are independent and identically distributed with distribution function $B(.)$. When the inventory level depletes to s due to demands, we place an order to bring back the inventory level to S. Assume that materialization of order for replenishment takes a positive (discrete) amount of time. Let the lead time for replenishment be distributed geometrically with parameter r.

6.2.1 Analysis of the model

At time m^+, the system can be described by $\{Y_m : m \in N\}$ with $Y_m = (J_m, I_m, N_m)$, where J_m denotes the remaining service time of the customer currently being served, I_m denotes the number of items in the inventory and N_m denotes the number of customers in the system. Note that for convenience for analysing the system, here we use the notation (J_m, I_m, N_m) in this order unlike previous chapters.

It can be shown that $\{Y_m : m \in N\}$ is a Markov Chain with state space $E = \{(i, j, k) : i \geqslant 0; j = 0, 1, \ldots, S; k \geqslant 0\}$.

Now we find stationary distribution

$$\pi_{i,j,k} = \lim_{m\to\infty} P\{J_m = i, I_m = j, N_m = k\}, \;\; i \geqslant 0; j = 0, 1, \ldots, S; k \geqslant 0.$$

The one step transition probabilities are given by

$$\begin{aligned} P_{(0,0,0)(0,0,0)} &= \overline{p}\,\overline{r}, \\ P_{(1,1,1)(0,0,0)} &= \overline{p}\,\overline{r}, \end{aligned}$$

and for $i = 0, k \geqslant 0$

$$\begin{aligned}
P_{(i+1,j,k)(i,j,k)} &= \overline{p}, && j \leqslant S \\
P_{(i+1,j,k-1)(i,j,k)} &= p, && j \leqslant S \\
P_{(i+1,j_1,k)(i,j,k)} &= \overline{p}r, && j_1 \leqslant s, j = S \\
P_{(i+1,j_1,k-1)(i,j,k)} &= pr, && j_1 \leqslant s, j = S \\
P_{(1,j+1,k+1)(i,j,k)} &= \overline{p}w_i, && j \leqslant S \\
P_{(1,j+1,k)(i,j,k)} &= pw_i, && s \leqslant j \leqslant S \\
P_{(1,j_1+1,k+1)(i,j,k)} &= \overline{p}rw_i, && j_1 \leqslant s, j = S \\
P_{(1,j_1+1,k)(i,j,k)} &= prw_i, && j_1 \leqslant s, j = S
\end{aligned}$$

where $\overline{p} = 1 - p$.

The Kolmogorov equations for the stationary distribution are

$$\overline{p}\,\overline{r}\ \pi_{1,1,1} = (1-\overline{p}\,\overline{r})\ \pi_{0,0,0} \tag{6.1}$$

and

$$\begin{aligned}\pi_{i,j,k} &= \overline{p}\pi_{i+1,j,k} + p\pi_{i+1,j,k-1}(1-\delta_{0k}) + \overline{p}r\pi_{i+1,j,k} \\ &\quad + pr\pi_{i+1,j,k-1}(1-\delta_{0k}) + \overline{p}w_i\pi_{1,j+1,k+1} \\ &\quad + pw_i\pi_{1,j+1,k} + prw_i\pi_{1,j+1,k} + \overline{p}rw_i\pi_{1,j+1,k+1}\end{aligned} \tag{6.2}$$

for $i > 0;\ j = 0,\ldots,S;\ k \geqslant 0$.

The normalizing condition is
$\sum_{i=0}^{\infty}\sum_{j=0}^{S}\sum_{k=0}^{\infty}\pi_{i,j,k} = 1$

To solve equations(6.1) and (6.2) we define generating function
$\phi(x,y,z) = \sum_{i=0}^{\infty}\sum_{j=0}^{S}\sum_{k=0}^{\infty}\pi_{i,j,k}\ x^i y^j z^k$
The following theorem gives the solution of the Kolmogorov equations in terms of $\phi(\cdot,\cdot,\cdot)$.

Theorem 6.2.1. The stationary distribution of the Markov chain $\{Y_m : m \in N\}$ has the generating function

$$\phi(x,y,z) = \frac{xyzr(\overline{p}+pz)\{W(x) - W[(1+r)(\overline{p}+pz)]\}}{[x-(1+r)(\overline{p}+pz)]\{W[(1+r)(\overline{p}+pz)] - zy\}}.$$

Proof. Multiplying both sides of (6.2) by z^k and summing over k, we get

$$\begin{aligned}\sum_{k=0}^{\infty}\pi_{i,j,k}\ z^k &= (1+r)(\overline{p}+pz)\sum_{k=0}^{\infty}\pi_{i+1,j,k}\ z^k \\ &\quad + \tfrac{(1+r)(\overline{p}+pz)}{z}w_i\sum_{k=0}^{\infty}\pi_{1,j+1,k}\ z^k - (1+r)\overline{p}w_i\ \pi_{1,j+1,1}\end{aligned} \tag{6.3}$$

Next we multiply both sides of (6.3) by y^j and sum over j, to get

$$\begin{aligned}\sum_{j=0}^{S}\sum_{k=0}^{\infty}\pi_{i,j,k}\ y^j z^k &= (1+r)(\overline{p}+pz)\sum_{j=0}^{S}\sum_{k=0}^{\infty}\pi_{i+1,j,k}\ z^k y^j \\ &+\tfrac{(1+r)(\overline{p}+pz)}{zy}w_i\sum_{j=0}^{S}\sum_{k=0}^{\infty}\pi_{1,j,k}\ z^k y^j \\ &-(1+r)(\overline{p}+pz)w_i\ \pi_{1,1,1}\end{aligned} \tag{6.4}$$

Finally multiplying both sides of (6.4) by x^i and summing over i, we get

$$\sum_{i=0}^{\infty}\sum_{j=0}^{S}\sum_{k=0}^{\infty}\pi_{i,j,k}\ x^i y^j z^k$$

$$\begin{aligned}&= \tfrac{(1+r)(\overline{p}+pz)}{x}\left[\sum_{i=0}^{\infty}\sum_{j=0}^{S}\sum_{k=0}^{\infty}\pi_{i,j,k}\ x^i y^j z^k - x\sum_{j=0}^{S}\sum_{k=0}^{\infty}\pi_{1,j,k}\ y^j z^k\right] \\ &+\tfrac{(1+r)(\overline{p}+pz)}{zy}W(x)\sum_{j=0}^{S}\sum_{k=0}^{\infty}\pi_{1,j,k}\ y^j z^k - (1+r)(\overline{p}+pz)W(x)\ \pi_{1,1,1}\end{aligned} \tag{6.5}$$

Let $\phi(x,y,z) = \sum_{i=0}^{\infty}\sum_{j=0}^{S}\sum_{k=0}^{\infty}\pi_{i,j,k}\ x^i y^j z^k$

Then from the above equation, we have

$$\begin{aligned}\left[\tfrac{x-(1+r)(\overline{p}+pz)}{x}\right]\phi(x,y,z) &= \left[\tfrac{(1+r)(\overline{p}+pz)}{z}\right][W(x)-zy]\sum_{k=0}^{\infty}\pi_{1,1,k}\ z^k \\ &-\tfrac{(1+r)(\overline{p}+pz)(1-\overline{p}\,\overline{r})}{\overline{p}\,\overline{r}}W(x)\pi_{0,0,0}\end{aligned} \tag{6.6}$$

put $x = (1+r)(\overline{p}+pz)$ in (6.6) we get

$$\sum_{k=0}^{\infty}\pi_{1,1,k}\ z^k = \frac{z(1-\overline{p}\,\overline{r})W[(1+r)(\overline{p}+pz)]}{\overline{p}\,\overline{r}\{W[(1+r)(\overline{p}+pz)]-zy\}}\pi_{0,0,0} \tag{6.7}$$

Substituting(6.7) in (6.6) we get

$$\phi(x,y,z) = \frac{xyz(1+r)(\overline{p}+pz)(1-\overline{p}\,\overline{r})\,[W(x)-W((1+r)(\overline{p}+pz))]}{\overline{p}\,\overline{r}[x-(1+r)(\overline{p}+pz)][W((1+r)(\overline{p}+pz))-zy]}\pi_{0,0,0} \tag{6.8}$$

with

$$\pi_{0,0,0} = \frac{\overline{p}\,\overline{r}r}{(1+r)(1-\overline{p}\,\overline{r})} \tag{6.9}$$

Substituting (6.9) in (6.8) we get

$$\phi(x,y,z) = \frac{xyzr(\overline{p}+pz)\{W(x)-W[(1+r)(\overline{p}+pz)]\}}{[x-(1+r)(\overline{p}+pz)]\{W[(1+r)(\overline{p}+pz)]-zy\}}. \tag{6.10}$$

□

Next we look for the system stability.

6.2.2 Stability condition

Theorem 6.2.2. The system is stable if and only if $\rho < 1$ where $\rho = p\beta_1$.

We have the generating function of the number of customers in the system and its mean value as given below:

- The probability generating function of the number of customers in the system is given by $\psi(z) =$

$$\phi(1,1,z) = \frac{r(1+r)(1-\overline{p}\,\overline{r})(\overline{p}+pz)\{1-W[(1+r)(\overline{p}+pz)]\}}{(1+r-\overline{p}\,\overline{r})[1-(1+r)(\overline{p}+pz)]\{W[(1+r)(\overline{p}+pz)]-z\}}$$

- Mean number of customers in the system is given by

$$EC = [\frac{d}{dz}\psi(z)]_{z=1} = \frac{(1+r)(1-\overline{p}\,\overline{r})[pW(1+r)+prW'(1+r)-\overline{p}]}{(1+r-\overline{p}\,\overline{r})[1-W(1+r)]}$$

6.3 Model 2

This is a particular case of model 1 wherein we assume the lead time to be negligible. As a result we expect sharper results. We consider a discrete time $Geo/G/1$ queue with inventory under (s, S) policy in which demands arrive according to a Bernoulli process with parameter p. i.e., inter-demand times follow geometric distribution with parameter p. The service times are independent and identically distributed with distribution function $B(.)$ having probability w_i for service time to have duration of i slots, $i \geqslant 1$. When the inventory level reaches s $(< \frac{S}{2})$, an order is placed so that the inventory level is brought up to S at the time of replenishment. The assumption $2s < S$ is made to avoid perpetual order placement. Assume that the lead time is zero. Further no shortage is permitted.

6.3.1 Analysis of the Markov chain

At time m^+, the system can be described by $\{X_m\} : m \in N\}$ with the triplet $X_m = (J_m, I_m, N_m)$, where J_m denotes the remaining service time of the customer currently being served, I_m denotes the number of items in the inventory and N_m denotes the number of customers in the system.
It can be shown that $\{X_m\} : m \in N\}$ is a Markov Chain whose state space is $E = \{0, 1, 2, \ldots\} \times \{s+1, s+2, \ldots S\} \times \{0, 1, 2, \ldots\}$.
To find the stationary distribution

$$\pi_{j,0} = \lim_{m\to\infty} P\{J_m = 0, I_m = j, N_m = 0\}, \quad j \geqslant s+1$$

$$\pi_{i,j,k} = \lim_{m\to\infty} P\{J_m = i, I_m = j, N_m = k\} \quad \text{for } i \geqslant 0; \;\; j = 0, \ldots, S; \;\; k \geqslant 0.$$

The one step transition probabilities are given by

$$
\begin{array}{lll}
P_{(j,0)(j,0)} & = & \overline{p}, \quad j = 0 \\
P_{(1,j+1,1)(j,0)} & = & \overline{p}, \quad j \geqslant s+1
\end{array}
$$

If $i \geqslant 1, k \geqslant 1$, then

$$
\begin{array}{llll}
P_{(i+1,j,k)(i,j,k)} & = & \overline{p}, & s+1 \leqslant j \leqslant S \\
P_{(i+1,j,k-1)(i,j,k)} & = & p, & k \geqslant 2 \\
P_{(1,j+1,k+1)(i,j,k)} & = & \overline{p}w_i, & s+1 \leqslant j \leqslant S \\
P_{(1,j+1,k)(i,j,k)} & = & pw_i, & s+1 \leqslant j \leqslant S \\
P_{(j,0)(i,j,k)} & = & pw_i, & k = 1, j = S
\end{array}
$$

The Kolmogorov equations for the stationary distribution are

$$p \ \pi_{0,0} = \overline{p} \ \pi_{1,1,1} \tag{6.11}$$

$$p \ \pi_{j,0} = \overline{p} \ \pi_{1,j+1,1} \text{ for } s \leqslant j \leqslant S-1. \tag{6.12}$$

and

$$\pi_{i,j,k} = \overline{p} \ \pi_{i+1,j,k} + p \ \pi_{i+1,j,k-1} + \overline{p}w_i \ \pi_{1,j+1,k+1} + pw_i \ \pi_{1,j+1,k} + pw_i \ \pi_{j,0} \ \delta_{1,k} \tag{6.13}$$

Since the lead time is assumed to be zero, the replenishment is instantaneous. Thus the number of customers in the system (level of the Markov chain) and the number of items in the inventory (phase of the chain) are independent. Therefore
$\pi_{i,j,k} = \frac{1}{Q} P_{i,k}$, where probability of number of items in the inventory is $\frac{1}{Q}$, $Q = S - s$.(See [53])
Hence equations (6.12) and (6.13) become

$$p \ P_{0,0} = \overline{p} \ P_{1,1} \tag{6.14}$$

and

$$P_{i,k} = \overline{p}\ P_{i+1,k} + p\ P_{i+1,k-1} + \overline{p}w_i\ P_{1,k+1} + pw_i\ P_{1,k} + pw_i\ P_{0,0} \qquad (6.15)$$

The normalizing condition is

$$P_{0,0} + \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} P_{i,k} = 1$$

To solve equations(6.14) and (6.15) we define generating function

$$\phi(x,z) = \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} P_{i,k}\ x^i z^k$$

The following theorem gives the solution of the Kolmogorov equations in terms of the preceding generating function

Theorem 6.3.1. If $\rho < 1$, the stationary distribution of the Markov chain $\{X_m : m \in N\}$ has the generating function

$$\phi(x,z) = \frac{pxz(1-z)[(W(x) - W(\overline{p}+pz)]}{[x-(\overline{p}+pz)][W(\overline{p}+pz)-z]} P_{0,0}$$

where $P_{0,0} = 1 - \rho$.

Proof. Multiplying both sides of (6.15) by z^k and summing over k, we get

$$\sum_{k=1}^{\infty} P_{i,k}\ z^k =$$

$$\overline{p}\sum_{k=1}^{\infty} P_{i+1,k}\ z^k + p\sum_{k=1}^{\infty} P_{i+1,k-1}\ z^k + \overline{p}w_i\sum_{k=1}^{\infty} P_{1,k+1}\ z^k + pw_i\sum_{k=1}^{\infty} P_{1,k}\ z^k + pw_i\sum_{k=1}^{\infty} P_{0,0} z^k$$

Substituting (6.14), we get

$$\sum_{k=1}^{\infty} P_{i,k}\ z^k = (\overline{p}+pz)\sum_{k=1}^{\infty} P_{i+1,k}\ z^k + (\frac{\overline{p}+pz}{z})w_i\sum_{k=1}^{\infty} P_{1,k}\ z^k - pw_i(1-z)\ P_{0,0} \qquad (6.16)$$

Multiplying both sides of (6.16) by x^i and summing over i, we get

$$
\begin{aligned}
\phi(x,z) &= \sum_{i=1}^{\infty}\sum_{k=1}^{\infty} P_{i,k}\ x^i z^k \\
&= (\overline{p}+pz)\sum_{i=1}^{\infty}\sum_{k=1}^{\infty} P_{i+1,k}\ x^i z^k + (\frac{\overline{p}+pz}{z})\sum_{i=1}^{\infty} w_i\ x^i \sum_{k=1}^{\infty} P_{1,k}\ z^k - p\sum_{i=1}^{\infty} w_i\ x^i(1-z)P_{0,0} \\
&= (\tfrac{\overline{p}+pz}{x})\phi(x,z) + (\tfrac{\overline{p}+pz}{z})[W(x)-z]\sum_{k=1}^{\infty} P_{1,k}\ z^k - p(1-z)W(x)P_{0,0}.
\end{aligned}
$$

$$
\therefore \left(\frac{x-(\overline{p}+pz)}{x}\right)\phi(x,z) = \left(\frac{\overline{p}+pz}{z}\right)[W(x)-z]\sum_{k=1}^{\infty} P_{1,k}\ z^k - p(1-z)W(x)P_{0,0}. \tag{6.17}
$$

Put $x = \overline{p}+pz$ in (6.17) we get

$$
\sum_{k=1}^{\infty} P_{1,k}\ z^k = \frac{pz(1-z)W(\overline{p}+pz)}{(\overline{p}+pz)[W(\overline{p}+pz)-z]}P_{0,0}. \tag{6.18}
$$

Substituting(6.18) in (6.17) we get

$$
\phi(x,z) = \frac{pxz(1-z)[(W(x)-W(\overline{p}+pz)]}{[x-(\overline{p}+pz)][W(\overline{p}+pz)-z]}P_{0,0}.
$$

□

Normalizing condition is

$$
P_{0,0} + \phi(1,1) = 1 \tag{6.19}
$$

Now

$$
\phi(1,z) = \frac{z[1-W(\overline{p}+pz)]}{W(\overline{p}+pz)-z}P_{0,0} \tag{6.20}
$$

$$
\lim_{z\to 1}\phi(1,z) = \frac{pW'(1)}{1-pW'(1)}P_{0,0}
$$

$$
\text{i.e., } \phi(1,1) = \frac{p\beta_1}{1-p\beta_1}P_{0,0} \tag{6.21}
$$

Substituting (6.21) in (6.19) we get

$$\begin{aligned} P_{0,0} &= 1 - p\beta_1 \\ &= 1 - \rho \end{aligned}$$

Since $P_{0,0} > 0$, we get $\rho < 1$, which is a necessary condition for the ergodicity of the Markov chain.

Therefore (6.20) becomes

$$\phi(1, z) = \frac{z[1 - W(\overline{p} + pz)]}{W(\overline{p} + pz) - z}(1 - \rho). \tag{6.22}$$

6.3.2 Stability condition

Theorem 6.3.2. The system is stable if and only if $p\beta_1 < 1$.

- The probability generating function of the number of customers in the system is given by

$$\phi(1, z) = \frac{z[(1 - W(\overline{p} + pz)]}{W(\overline{p} + pz) - z} P_{0,0}$$

- The server is idle with probability $P_{0,0} = 1 - p\beta_1$
- The server is busy with probability, PSB given by $\phi(1, 1) = \rho$
- Average inventory level $EL = \sum_{j=s+1}^{S} \frac{1}{Q} j = s + \frac{(Q + 1)}{2}$
- Mean number of customers in the system

$$EC = [\frac{d}{dz}\phi(1, z)]_{z=1} = \left(\frac{p^2 W''(1) - 2p^2[W'(1)]^2 + 2pW'(1)}{2[pW'(1) - 1]^2} \right) (1 - p\beta_1)$$

- Mean time a customer spends in the system (including the service time),

$$ET = \frac{EC}{p} = \left(\frac{pW''(1) - 2p[W'(1)]^2 + 2W'(1)}{2[pW'(1) - 1]^2}\right)(1 - p\beta_1)$$

6.4 Cost Analysis

Introduce the following costs:
c_1 - the fixed ordering cost
c_2 - procurement cost/unit
c_3 - holding cost of inventory /unit/unit time
c_4 - holding cost of customers /unit/unit time.
Then

holding cost of customers per unit time= $\frac{c_4}{2}\left(\frac{p^2W''(1)-2p^2[W'(1)]^2+2pW'(1)}{[pW'(1)-1]^2}\right)P_0$

holding cost of inventory=$c_3\left[s + \frac{(Q+1)}{2}\right]$

Total expected cost per unit time, ETC=Holding cost+Reorder cost i.e., $ETC = c_3[s + \frac{(Q+1)}{2}] + (c_1 + Qc_2)\frac{D}{Q}$, where D is the expected number of demands per unit time.

Then ETC is a separable convex function of both s and Q and the optimal value of Q is obtained as $Q = \sqrt{\frac{2c_1D}{c_3}}$.
Since no shortage is permitted, optimal value of s is zero. Hence the optimal value of S is same as that of Q.
Thus the expected minimum cost of the system is $\sqrt{\frac{c_1c_2D}{2}} + c_2D + \frac{c_3}{2} + c_3\sqrt{\frac{c_1D}{2c_2}}$

6.5 Numerical illustration

Table 6.1: Effect of p on Model-1.$w_1 = 0.4$, $w_2 = 0.3$,$w_3 = 0.2$,$w_4 = 0.1$

p	EC	ET	PSB
$r = 0.3$			
0.05	0.7499	14.9889	0.6858
0.10	0.6154	6.1540	0.7179
0.15	0.4637	3.0913	0.7468
0.20	0.2973	1.4864	0.7857
0.25	0.1182	0.4727	0.7968

Table 6.2: Effect of p on Model-2.$w_1 = 0.4$, $w_2 = 0.3$,$w_3 = 0.2$,$w_4 = 0.1$

p	EC	ET	PSB
0.05	0.1042	2.0833	0.1000
0.10	0.2188	2.1875	0.2000
0.15	0.3482	2.3214	0.3000
0.20	0.5000	2.5000	0.4000
0.25	0.6875	2.7500	0.5000

Table 6.1 shows that as the arrival rate p increases, the expected number of customers and the mean time the customer spends in the system decrease. The probability that the server is busy increases with increase in the arrival rate p. Table 6.2 shows that as the arrival rate p increases, the expected number of customers and the mean time the customer spends in the system increases. Also probability that the server is busy increases with increase in p when the lead time is assumed to be zero.

Chapter 7

Solution of (s, S) inventory problems: A Markov Decision Theory Approach

7.1 Introduction

This chapter deviates from the theme of previous chapters in that Markov decision approach to certain classes of inventory problems is discussed here. A large class of problems of sequential decision making under uncertainty can be modeled as stochastic dynamic programs, which, in general, is referred to as Markov Decision Problems. The Markov Decision model is a powerful tool for analyzing probabilistic sequential decision processes. It is a five tuple (T, I, A, p, c) where T is a point of time known as decision epoch; I the state space; A the action space; p the state transition probability distribution function and c the instantaneous cost. Decisions or actions are made at certain event occurrence epochs. When we choose an action in a state,

then an immediate cost is incurred and the system moves to another state according to certain transition probability. A solution to a Markov Decision Process is a policy, which is a function from states to actions that minimizes the long-run average costs.
We proceed to model a few inventory models as Markov decision problems. First we formulate those problems. Then the Markov decision approach is employed to compute the optimal solution.

7.2 Model Description

Consider an (s, S) inventory system, where demands follow a Bernoulli process with parameter p. Order for replenishment is placed when inventory level drops to s. The time between placing an order and its receipt is distributed geometrically with parameter r. Assume that service time is negligible. At the time of replenishment, the following decisions or actions are made: Replenishment can take place when inventory level is in any one of the states $i = s, s-1, s-2, \ldots, 1, 0$. We consider the model in which replenishment quantity varies according to the on hand inventory. In this situation we have to take decisions on how much to buy at the time of replenishment. We use Markov Decision Process for the solution.

Let $Q, Q+1, Q+2, \ldots, Q+(s-i)$ be the possible replenishment quantities when the inventory level is i at the replenishment epoch $0 \leqslant i \leqslant s$. Here $Q = S - s$. When the inventory level is s, the replenishment quantity is Q with probability $p_Q^{(s)}$ which is equal to one. Assign probabilities $p_Q^{(i)}, p_{Q+1}^{(i)}, \ldots, p_{Q+s-i}^{(i)}$ for the replenishment quantity to be $Q, Q+1, \ldots, Q+s-i, i = 0, 1, \ldots, s$ where replenishment occurs at inventory level i. Note that $\sum_{j=Q}^{Q+s-i} p_j^{(i)} = 1$, $i \in \{0, 1, \ldots s\}$.
The set of possible states of the inventory level process is denoted by $I =$

$\{0, 1, 2, \ldots, s, s+1, \ldots, S\}$.

When the inventory level is $\geqslant s+1$, no action is taken. For each state in I, a set of decisions can be made. Let $A(s)$, $A(s-1)$,..., $A(1)$, $A(0)$ be the set of possible actions associated with the states s, $s-1$,...,1, 0 respectively. Then $A(s) = a_{s,1}$, where the replenishment quantity (r.q) is Q having probability $p_Q^{(s)}$(=1 ; since there is only one choice for purchase quantity).

If replenishment takes place when inventory level is $s-1$, then the actions are:

$$A(s-1) = \begin{cases} a_{s-1,1}, \text{ where the r.q is } Q \text{ with probability } p_Q^{(s-1)} \\ a_{s-1,2}, \text{ where the r.q is } Q{+}1 \text{ with probability } p_{Q+1}^{(s-1)}. \end{cases}$$

For state $s-2$ the actions are:

$$A(s{-}2) = \begin{cases} a_{s-2,1}, \text{ where the r.q is } Q \text{ with probability } p_Q^{(s-2)} \\ a_{s-2,2}, \text{ where the r.q is } Q+1 \text{ with probability } p_{Q+1}^{(s-2)} \\ a_{s-2,3}, \text{ where the r.q is } Q+2 \text{ with probability } p_{Q+2}^{(s-2)}. \end{cases}$$

$\vdots$

Finally for state 0 the possible actions and the corresponding probabilities are:

$$A(0) = \begin{cases} a_{0,1}, \text{ where the r.q is } Q \text{ with probability } p_Q^{(0)} \\ a_{0,2}, \text{ where the r.q is } Q+1 \text{ with probability } p_{Q+1}^{(0)} \\ \vdots \\ a_{0,s+1}, \text{ where the r.q is } Q+s \text{ with probability } p_{Q+s}^{(0)}. \end{cases}$$

One step transition probabilities are given by

$$p_{ij}^{(k)}(a_{k,l}) = p_{Q+l-1}^{(k)}, \text{ where } k = 0, 1, \ldots, s-1, s \text{ and } l = 1, 2, \ldots, s{+}1{-}k$$

such that $\sum_{l=1}^{s+1-k} p_{Q+l-1}^{(k)} = 1$.

Let the stationary policies corresponding to states $s, s-1, \ldots, 1, 0$ be $R_s, R_{s-1}, \ldots, R_1, R_0$ respectively. Then $R_j = \{a_{j,k} : k = 1, 2, \ldots, s+1-j\}$; $j = s, s-1, \ldots, 0$.

7.3 Description of the problem

Let X_n be the state of the system at time n; and let D_n be the decision or action chosen. Then under a given policy R, $Y_n = (X_n, D_n)$ is a two-dimensional Markov chain with the transition probabilities

$$P\{X_{n+1} = j, D_{n+1} = d' | X_n = i, D_n = d\} = p(j|i, d)p(d'|j) \qquad (7.1)$$

where $p(j|i, d)$ is the conditional probability of the chain moving to the state j at time $n + 1$, given the current state is $X_n = i$ and a decision $D_n = d$ is taken and $p(d'|j)$ is the probability of a decision $D_{n+1} = d'$ being chosen at state $X_{n+1} = j$. Suppose demand arrival is according to a geometric process with parameter p and lead time is geometric with parameter r. Also assume that the service time is negligible. Then the one step transition probability matrix of the inventory level process is given by

$$
\mathbf{P} = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ s-1 \\ s \\ s+1 \\ \vdots \\ Q+s \end{array}
\begin{array}{c}
\begin{array}{ccccccccccc} 0 & 1 & \cdots & s & s+1 & \cdots & Q & Q+1 & \cdots & Q+s-1 & Q+s \end{array} \\
\left(\begin{array}{ccccccccccc}
\overline{r} & & & & & & rp_Q^{(0)} & rp_{Q+1}^{(0)} & \cdots & rp_{Q+s-1}^{(0)} & rp_{Q+s}^{(0)} \\
p\overline{r} & \overline{p}\,\overline{r} & & & & & & rp_Q^{(1)} & \cdots & rp_{Q+s-2}^{(1)} & rp_{Q+s-1}^{(1)} \\
 & \ddots & \ddots & & & & & & \ddots & \ddots & \vdots \\
 & p\overline{r} & \overline{p}\,\overline{r} & & & & & & & rp_Q^{(s-1)} & rp_{Q+1}^{(s-1)} \\
 & & p\overline{r} & \overline{p}\,\overline{r} & & & & & & & rp_Q^{(s)} \\
 & & & p & \overline{p} & & & & & & \\
 & & & & \ddots & \ddots & & & & & \\
 & & & & & & & & & p & \overline{p}
\end{array}\right)
\end{array}
$$

7.4 The long run average cost per unit time

Since the state space and action sets are finite stationary policies exist. (See Tijms [57]). Among different stationary policies, we look for the optimal one, which minimizes the long run average cost per unit time. Suppose a cost C_{X_n,D_n} is incurred when the process is in state X_n and a decision D_n is made. Being a function of both $X_n \in \{0, 1, \dots, s\}$ and $D_n \in \{a_{s,1}, \dots, a_{0,s+1}\}$, C_{X_n,D_n} is also a random variable. Its long-run average cost per unit time averaging over N periods is

$$
\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} E\left[C_{X_n D_n}\right] = \sum_{i=0}^{s} \sum_{j\in\{Q,Q+1,\dots,Q+s\}} \pi_{ij} c_{ij}
$$

where π_{ij} is the stationary probability distribution associated with the transition probabilities in (7.1). For an irreducible Markov chain, $\pi_{ij} \geqslant 0, \forall i, j$ and $\displaystyle\sum_{i=0}^{s} \sum_{j\in\{Q,Q+1,\dots,Q+s\}} \pi_{ij} = 1$. (see [56]).

7.5 The Optimal Policy and the Policy improvement Algorithm

Our objective is to find a policy that minimizes the long run average cost. For that purpose, we need to introduce the set of feasible policies and the associated Markov chains, the action sets associated with each state and the immediate cost associated with each state.
Assume that the Markov chain $Y_n = (X_n, D_n)$ is irreducible. Then there exists a unique equilibrium distribution $\{\pi_j(R), j \in I\}$. (see [57])
For any $j \in I$,

$$\lim_{m\to\infty} \frac{1}{m} \sum_{n=1}^{m} p_{ij}^{(n)}(R) = \pi_j(R),$$

which is independent of initial state i. The $\pi_j(R)$ are the unique solution to the system of equilibrium equations

$$\pi_j(R) = \sum_{i\in\{0,1,\ldots,s\}} p_{ij}(R_i)\pi_i(R), \ j \in \{Q, Q+1, \ldots, Q+s\}$$

with $\displaystyle\sum_{j\in\{Q,Q+1,\ldots,Q+s\}} \pi_j(R) = 1.$

Let $g(R)$ represent the long run expected average cost per unit time under any given policy R.
Then,

$$g(R) = \sum_{j\in\{Q,Q+1,\ldots,Q+s\}} c_j(R_j)\pi_j(R).$$

Let $V^n(i, R)$ denote the total expected cost with i as the initial state, R as the stationary policy and evolving over a period of length n. Then we have the recursive formula

$$V^n(i,R) = c_i(R_i) + \sum_{j\in\{Q,Q+1,\ldots,Q+s\}} p_{ij}(R_i)V^{n-1}(j,R) \tag{7.2}$$

It follows that the total expected cost $V^n(i,R)$ consists of the cost incurred when action $a = R_i$ is taken in state i at the first decision epoch and the remaining $n-1$ decision epochs, when the next state is j.

Since the Markov chain under consideration is irreducible, the average cost function $g_i(R)$ defined by $g_i(R) = \lim_{n\to\infty} \frac{1}{n} V^n(i,R)$ is equal to $g(R)$, independently of the initial state $i \in \{0,1,\ldots,s\}$. This relation motivates the heuristic assumption that bias value $v_i(R), i \in I$, exists such that, for each $i \in I$,

$$V^n(i,R) \approx ng(R) + v_i(R) \text{ for large values of n} \tag{7.3}$$

Substituting (7.3) in (7.2) , we get
$ng(R) + v_i(R)$

$$\begin{aligned}
&= c_i(R_i) + \sum_{j\in\{Q,Q+1,\ldots,Q+s\}} p_{ij}(R_i)[(n-1)g(R) + v_j(R)] \\
&= c_i(R_i) + (n-1)g(R) \sum_{j\in\{Q,Q+1,\ldots,Q+s\}} p_{ij}(R_i) + \sum_{j\in\{Q,Q+1,\ldots,Q+s\}} p_{ij}(R_i)v_j(R) \\
&= c_i(R_i) + (n-1)g(R) + \sum_{j\in\{Q,Q+1,\ldots,Q+s\}} p_{ij}(R_i)v_j(R).
\end{aligned}$$

i.e.,

$$g(R) = c_i(R_i) - v_i(R) + \sum_{j\in\{Q,Q+1,\ldots,Q+s\}} p_{ij}(R_i)v_j(R),$$

for $i = 0,1,\ldots,s$, with $V^0(i,R) = 0$.
Solving this system of equations, we get the long run average cost per unit time $g(R)$ if policy R is used. An optimal policy is that of the lowest cost

$g(R^{\star})$. To obtain the optimal policy, we use an iterative procedure, called policy-improvement algorithm (see [57]). This procedure begins by choosing an arbitrary stationary policy R. Then compute the unique solution $\{g(R), v_i(R)\}$ to the following system of linear equations:

$$v_i = c_i(R_i) - g + \sum_{j \in \{Q, Q+1, \ldots, Q+s\}} p_{ij}(R_i) v_j, \quad i \in I$$

with normalizing equation, $v_k = 0$, where k is an arbitrarily chosen state. In the second step, we can find an improved policy $\overline{R}$. For that, determine an action a_i , for each state $i \in I$,which yields the minimum in

$$\min_{a \in A(i)} \{c_i(a) - g(R) + \sum_{j \in \{Q, Q+1, \ldots, Q+s\}} p_{ij}(a) v_j(R)\}.$$

Then $\overline{R}$ is obtained by choosing $\overline{R}_i = a_i$, $\forall i \in I$ with the convention that $\overline{R}_i$ is chosen equal to the old action R_i when this action minimizes the policy-improvement quantity.
In the third step, if $\overline{R} = R$, then the algorithm is stopped with policy R. Otherwise, go to the beginning step with R replaced by $\overline{R}$.
Since the state space is finite, there are only a finite number of possible stationary policies. Hence after a finite number of iterations, we will be able to reach the optimal policy.

7.5.1 Performance measures

We have then the following measures for evaluating performance of the system.

1. Expected replenishment quantity when the inventory level is i
   $= \frac{1}{s-i+1}[Q + (Q+1) + \ldots + (Q+s-i)]$

$= Q + \frac{s-i}{2}$
where $\frac{1}{s-i+1}$ is the uniform probability that the replenishment quantity is $Q, Q+1, Q+2, \ldots, Q+(s-i), i \in \{0, 1, \ldots, s\}$.

2. Expected replenishment quantity is given by
$$\text{ERQ} = (Q + \tfrac{s}{2})(\tfrac{p\bar{r}}{1-\bar{p}\,\bar{r}})^s + r\sum_{i=1}^{s}(Q + \frac{s-i}{2})\frac{(p\bar{r})^{s-i}}{(1-\bar{p}\,\bar{r})^{s-i+1}}.$$

3. Mean time required for an arrival $= \sum_{k=1}^{\infty} k(1-p)^{k-1}p$.

4. Mean number of demands lost EL, when the inventory level is zero is given by
$$\text{EL} = \sum_{k=1}^{\infty}\sum_{n=k}^{\infty}\sum_{l=n}^{\infty} k\ nC_k(1-p)^{n-k}p^k q^l,$$
$$= \tfrac{pq}{(1-q)^3}, \quad 0 < p, q < 1.$$

7.5.2 Cost analysis

We define the following costs:
c_0 - fixed cost for order placement
c_1- cost per unit item of inventory
c_2 - revenue loss due to unit customer lost when inventory is empty.
$\alpha, 0 < \alpha < 1$ - is the discount factor.
For calculating the Expected Total Cost (ETC) at different states and the respective actions, we need expected cycle length from replenishment to replenishment.
Let $E_{i,j}$ be the expected duration of the cycle with replenishment at state i and replenishment quantity j, where $i = \{0, 1, \ldots, s-1, s\}$ and $j = \{Q, Q+1, \ldots, Q+s\}$.
Then Expected Total Cost (ETC) can be calculated as:

State	Action	ETC	
s	$a_{s,1}$	$\frac{c_0}{E_{i,j}} + c_1 Q p_Q^{(s)}$,	$i = s, j = Q$
$s-1$	$a_{s-1,1}$	$\frac{c_0}{E_{i,j}} + c_1 Q p_Q^{(s-1)}$,	$i = s-1, j = Q$
	$a_{s-1,2}$	$\frac{c_0}{E_{i,j}} + c_1 p_{Q+1}^{(s-1)}[Q+\alpha]$,	$i = s-1, j = Q+1$,
$\vdots$	$\vdots$	$\vdots$	
$\vdots$	$\vdots$	$\vdots$	
0	$a_{0,1}$	$\frac{c_0}{E_{i,j}} + c_1 Q p_Q^{(0)} + c_2 EL$,	$i = 0, j = Q$
	$a_{0,2}$	$\frac{c_0}{E_{i,j}} + c_1 p_{Q+1}^{(0)}[Q+\alpha] + c_2 EL$,	$i = 0, j = Q+1$
	$a_{0,3}$	$\frac{c_0}{E_{i,j}} + c_1 p_{Q+2}^{(0)}[Q+2\alpha] + c_2 EL$,	$i = 0, j = Q+2$
	$\vdots$	$\vdots$	
	$a_{0,s+1}$	$\frac{c_0}{E_{i,j}} + c_1 p_{Q+s}^{(0)}[Q+s\alpha] + c_2 EL$,	$i = 0, j = Q+s$

7.6 Numerical Illustration

Let us consider $s = 3$ and $S = 7$ so that $Q = 4$. The states are 0, 1, 2 and 3. The set of all possible actions or decisions on the states are defined as : $\{A(i) : i = 0, 1, 2, 3\}$, where $A(i) = \{a_{i,l} : l = 1, \ldots, 4 - i\}$. Here replenishment quantity is $3 + l$ having probability $p_{3+l}^{(i)}$.

Table 7.1: $c_0 = 100,\ c_1 = 10,\ c_2 = 1,\ p = \frac{1}{2},\ q = \frac{2}{3},\ \alpha = \frac{1}{3},\ p_4^{(3)} = 1,$ $p_4^{(2)} = p_5^{(2)} = \frac{1}{2},\ p_4^{(1)} = p_5^{(1)} = p_6^{(1)} = \frac{1}{3},\ p_4^{(0)} = p_5^{(0)} = p_6^{(0)} = p_7^{(0)} = \frac{1}{4}$

States	Actions with costs				Minimum cost	Action
0	$a_{0,1}$ 37.50	$a_{0,2}$ 29.83	$a_{0,3}$ 29.00	$a_{0,4}$ 28.64	28.64	$a_{0,4}$
1	$a_{1,1}$ 25.80	$a_{1,2}$ 24.20	$a_{1,3}$ 23.80		23.80	$a_{1,3}$
2	$a_{2,1}$ 32.50	$a_{2,2}$ 31.67			31.67	$a_{2,2}$
3	$a_{3,1}$ 52.50				52.5	$a_{3,1}$

Table 7.2: $c_0 = 100,\ c_1 = 10,\ c_2 = 1,\ p = \frac{1}{2},\ q = \frac{2}{3},\ \alpha = \frac{1}{3},\ p_4^{(3)} = 1,$ $p_4^{(2)} = \frac{2}{3}, p_5^{(2)} = \frac{1}{3},\ p_4^{(1)} = \frac{1}{2}, p_5^{(1)} = p_6^{(1)} = \frac{1}{4},\ p_4^{(0)} = \frac{3}{8}, p_5^{(0)} = p_6^{(0)} = \frac{1}{4}, p_7^{(0)} = \frac{1}{8}$

States	Actions with costs				Minimum cost	Action
0	$a_{0,1}$ 36.50	$a_{0,2}$ 29.82	$a_{0,3}$ 29.00	$a_{0,4}$ 22.39	22.39	$a_{0,4}$
1	$a_{1,1}$ 32.50	$a_{1,2}$ 20.83	$a_{1,3}$ 20.00		20.00	$a_{1,3}$
2	$a_{2,1}$ 39.17	$a_{2,2}$ 24.40			24.40	$a_{2,2}$
3	$a_{3,1}$ 52.50				52.5	$a_{3,1}$

Table 7.3: $c_0 = 100$, $c_1 = 10$, $c_2 = 1$, $p = \frac{1}{2}$, $q = \frac{2}{3}$, $\alpha = \frac{1}{3}$, $p_4^{(3)} = 1$, $p_4^{(2)} = \frac{1}{3}, p_5^{(2)} = \frac{2}{3}$, $p_4^{(1)} = p_5^{(1)} = \frac{1}{4}, p_6^{(1)} = \frac{1}{2}$, $p_4^{(0)} = \frac{1}{8}, p_5^{(0)} = p_6^{(0)} = \frac{1}{4}, p_7^{(0)} = \frac{3}{8}$.

States	Actions with costs				Minimum cost	Action
0	$a_{0,1}$ 26.50	$a_{0,2}$ 29.82	$a_{0,3}$ 29.00	$a_{0,4}$ 34.89	26.50	$a_{0,1}$
1	$a_{1,1}$ 22.50	$a_{1,2}$ 20.83	$a_{1,3}$ 31.67		20.83	$a_{1,2}$
2	$a_{2,1}$ 25.83	$a_{2,2}$ 38.89			25.83	$a_{2,1}$
3	$a_{3,1}$ 52.50				52.5	$a_{3,1}$

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