ON INFINITE GRAPHS AND RELATED MATRICES

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By

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TO

MY

HUSBAND

DECLARATION

This thesis contains no material which has been accepted for the award of any other Degree or Diploma in any University and to the best of my knowledge and belief, it contains no material previously published by any other person except where due references are made in the text of the thesis.

Kochi-22

Ancykutty Joseph

CERTIFICATE

Certified that the thesis entitled " **ON INFINITE GRAPHS AND RELATED MATRICES**" is a bona fide record of work done by Ancykutty Joseph under our guidance in the Department of Mathematics, Cochin University of Science and Technology, and that no part of it has been included anywhere previously for the award of any degree.

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Ancykutty Joseph.

CONTENTS

CHAPTER 0

INTRODUCTION

In the last two decades Graph theory has captured wide attraction as a Mathematical model for any system involving a binary relation. The theory is intimately related to many other branches of Mathematics including Matrix Theory, Group theory, Probability, Topology and Combinatorics , and has applications in many other disciplines.

Of the numerous books and research notes that we come across, only a handful concentrate on infinite graphs; the remaining deal with finite graphs. As a model, finite graphs may give more intuitive and aesthetic appeal. The transition from a finite graph to an infinite graph evolves when either the set of vertices or the set of edges (arcs for digraphs) becomes infinite. Based on the degree of the vertices, infinite graphs are widely classified into two: (i) infinite graphs of infinite degree and (ii) infinite graphs having vertices of finite degree [Konig].The latter were later called locally finite infinite graphs. They form the intermediate link between finite graphs and infinite graphs of infinite degree. In this work, we concentrate on locally finite, infinite graphs. Results on infinite graphs having only a finite number of vertices of infinite degree are available in the literature [29].

Any sort of study on infinite graphs, naturally involves an attempt to extend the well known results on the much familiar finite graphs. A graph is completely determined by either its adjacencies or its incidences. A matrix can convey this information completely. This makes a proper labelling of the vertices, edges and any other elements considered, an inevitable process. Many types of labelling of finite graphs as Cordial labelling, Egyptian labelling, Arithmetic labeling and Magical labelling are available in the literature. In Chapter II we have considered logical numbering (topological ordering) of infinite digraphs, motivated by the following observation.

Among the various matrices associated with a finite graph or digraph, the adjacency matrix is the most popular and widely investigated one [24], [25], [35],etc. In 1982, Mohar B [33] defined an adjacency operator A(G) for a locally finite, infinite graph G. A(G) acts as a linear operator on $l^2(V)$. Later, in 1989, Fujii, Sasaoka and Watatani [18] extended Mohar's definition to locally finite, infinite directed graphs. For a finite graph (digraph) the adjacency matrix is nilpotent if and only if the graph(digraph) contains no cycles (directed cycles). In other words, for a finite digraph, the adjacency matrix is nilpotent if and only if the digraph has a logical numbering. Eventhough [18] has extended most of the results in [33], [34], and [35] to infinite directed graphs, the nilpotency of the adjacency operator is strictly restricted to finite digraphs.

The shift from a finite graph to an infinite graph brings forth a radical difference in the treatment of matrices. For a finite graph or digraph, results and methods of algebra can be used, especially for matrices. In the finite matrix theory determinants play a fundamental role, but their value is lost, to a very large extent, in the theory of infinite matrices. Existence problems frequently arise for infinite matrices, which have no counterpart in the finite theory. For example, even though two infinite matrices A and B may both exist, their product AB may not exist, since $\sum_{k=1}^{\infty} a_{ik} b_{kj}$ may diverge for some or all values of i, j. Owing to convergence problems and other difficulties, the extension of theorems, established for finite matrices, to infinite matrices rarely happens. But there are some exceptions, [19], [47] and [48].

A product of infinite matrices is associative if (a) the product exists for every succession of the multiplication involved, the order of the factors remaining the same; (b) all the products so obtained are equal. Multiplication of infinite matrices is not in general associative. But the products of lower and upper-semi matrices, diagonal matrices and row-finite and column-finite matrices are associative. Also diagonal matrices and row-finite and column-finite matrices are closed for finite sum and finite products. These basic properties of infinite matrices justify the choice of locally finite, infinite graphs. The main reference is Cooke R.G [14].

The number of matrices associated with a finite graph are too many for a study of this type to be exhaustive. A large number of theorems have been established by various authors for finite matrices. The extension of these results to infinite matrices associated with infinite graphs is neither obvious nor always possible due to convergence problems.

In this thesis our attempt is to obtain theorems of a similar nature on infinite graphs and infinite matrices. We consider the three most commonly used matrices . or operators, namely, the adjacency matrix. the incidence matrix and the Laplacian which is closely related to the incidence matrix and the cycle matrix. Besides, we have defined another matrix with its entries in {-1,0,1} based on the asymmetric adjacency relation in a directed graph.

In the last decade, many important results between Laplace eigenvalues and eigen vectors of finite graphs and several other graph parameters were obtained by Biggs [10], Woess [37], Shawe-Taylor [32] etc. An extension of these results –on Laplace Spectrum, Amenability and Random walks- to locally finite infinite graphs are given in [36]. Bapat, Grossman and Kulkarni [8] have recently (2000) considered the edge version of the Laplacian matrix for a finite graph G, in particular a tree. We consider the same matrix for an infinite graph, denoted by Q_E and called the Arc Laplacian. Analysis of Q_E in comparison with Q, the Laplacian, has given another matrix $\mathcal{A}_E$ indexed by the arcs.

In a directed graph, the adjacency relation defines a partial order on the set of its vertices. On the other hand every partially ordered set has a directed graph representing it. This intrinsic relation between partially ordered sets and digraphs helps in interpreting structures defined on partially ordered sets in a "graph" sense. We have chosen the algebraic structure " Incidence Algebra" of a locally finite partially ordered set over a commutative ring with identity. This topic has a place in a study on matrices and graphs for (i) every partially ordered set has a graph representing it. (ii) every member of the incidence algebra has a matrix representation.

Concerning the applications, somewhere in between probability, harmonic analysis, geometry, graph theory and algebra we come across Random Walks. They are time-homogeneous Markov Chains whose transition probabilities are in some way adapted to a structure of the underlying state space. If the structure is discrete and infinite, it can be viewed as an infinite locally finite graph.[51]

A brief summary of the thesis is given below.

CHAPTER I -THE PRELIMINARIES

This chapter contains the basic definitions, results and notations- in infinite graphs and digraphs, infinite matrices, linear operators, partially ordered sets and rings- used in the following chapters.

CHAPTER II -THE ADJACENCY MATRIX

2.1 The Adjacency Matrix of a Finite Digraph

A logical numbering (topological ordering) of the finite digraph D = (V,E), on n vertices, is a function f: $V \rightarrow \{1,2,3,\ldots,m : m \leq n\}$ which assigns to each vertex v_i of D an integer $f(v_i)$ such that each integer is assigned to a vertex exactly once, and if $(v_i, v_j) \in E$, then $f(v_i) < f(v_j)$. **(Definition 2.1.4)**

We define analogously.

When D is infinite and locally finite, an injection $f: V \rightarrow N$ such that if $(v_i, v_j) \in E(D)$, then $f(v_i) < f(v_j)$ is a logical numbering of D.**(Definition 2.1.5)**

The following results are well known.[11,20,41,42]

Proposition 2.1.3

Any **finite** acyclic digraph has at least one source(i.e. v: $d^-(v) = 0$) and at least one sink (i.e. v: $d^+(v) = 0$).

Theorem 2.1.8

The following are equivalent for a **finite** digraph D.

(i) D is acyclic

(ii) D has a logical numbering.

(iii) The adjacency matrix A of D is upper triangular.

(iv) A is nilpotent.

2.2 Adjacency Matrix (Operator) of Infinite Digraphs

Proposition 2.1.3 and Theorem 2.1.8 are not generally true for locally finite infinite acyclic digraphs.

As an example, we have the doubly infinite directed path D = (V, E) where

$$V = \{v_i : i = 0, \pm 1, \pm 2, \pm 3 \ldots\}$$

$$E = \{ (v_i, v_{i+1}) : i = 0, \pm 1, \pm 2, \pm 3, \ldots\}$$

D is acyclic; but it has neither a source nor a sink.

In section 2.2 the mutual implications of the statements (i) ,(ii) and (iii) of theorem 2.1.8 are discussed in the case of locally finite infinite acyclic digraphs .

The main results in this section are

(a) Theorem 2.2.3

Let D = (V, E) be a locally finite, infinite acyclic digraph whose vertex set V is countably infinite. If (i) $S = \{ v \in V : d^-(v) = 0\}$ is non-empty and finite and (ii) D contains no in-rays, then D has a logical numbering of its vertices.

(b) Remark 2.2.4

The converse of Theorem 2.2.3 is not true. An example is given.

2.3 Nilpotent adjacency Operator of Infinite Digraphs

A necessary and sufficient condition for the nilpotency of the adjacency operator is given in section 2.3.

Theorem 2.3.4

Let A be the non-zero adjacency operator of a locally finite acyclic digraph D = (V, E) satisfying the conditions given in theorem 2.2.3. Then A is nilpotent if and only if D contains no out- rays.

Theorems 2.2.3, and 2.3.4 provide a complete extension of Theorem 2.1.8 on finite directed graphs to locally finite, infinite digraphs.

CHAPTER III -ANOTHER ADJACENCY OPERATOR OF A DIGRAPH

The adjacency operator considered in the previous chapter is expressed as a finite matrix or a row-column finite infinite matrix, whose entries are 0 and 1.The same adjacency relation is used here to define another matrix denoted by $\tilde{A}$(G) with entries in {-1,0,1}. The corresponding linear operator T_A which is closed on a Hilbert space H defined on $l^2(V)$ is analysed.

3.1 The Operator T_A

The new matrix $\tilde{A}$ is defined as follows.

Definition 3.1.2

Let $V = \{ v_1, v_2, v_3, \ldots \}$ be the vertex set of a digraph D. $\tilde{A} = [a_{ij}]$ where

$$a_{ij} = \begin{cases} 1, \text{ if } (v_i, v_j) \in E \\ -1, \text{ if } (v_j, v_i) \in E \\ 0, \text{ otherwise} \end{cases} \qquad (*)$$

Definition 3.1.3

The i^{th} row of the adjacency matrix A associates with each $v_i \in V$, a sequence (x_i) whose entries belong to $\{-1,0,1\}$. Since G is locally finite $\{(x_i)\}$ is a closed subspace of $l^2(V)$. Denote the Hilbert space $\{(x_i)\}$ by H. Let $\{e_v : v \in V\}$ be the canonical basis of H, where $e_v(u) = \delta_u^v$. For any

$x = \sum_{u \in V} x_u e_u$ in H, $x_u \in \{-1,0,1\}$, define

$$T_A(x) = \sum_{u \in V} \sum_{v \in D^-(u)} x_v e_u, \qquad \text{where}$$

$$\mathrm{Dom}(T_A) = \{x = \sum_{u \in V} x_u e_u : \sum_{u \in V} \Big| \sum_{v \in D^-(u)} x_v e_u \Big|^2 < \infty\}.$$

Properties of T_A are listed below.

(i) T_A is a closed operator on H . (Proposition 3.1.5)

(ii) T_A is bounded if and only if D is of bounded degree. (Proposition 3.1.6)

(iii) $\|T_A e_v\|^2 = \|T_A^* e_v\|^2 = d(v)$ (Result 3.1.9.(a))

(iv) $\langle T_A e_u, e_v \rangle = \begin{cases} 1, & \text{if } v \in D^-(u) \\ -1, & \text{if } v \in D^+(u) \\ 0, & \text{otherwise} \end{cases}$ (Result 3.1.9(b))

(v) $\langle T_A^* e_u, e_v \rangle = \begin{cases} 1, & \text{if } v \in D^+(u) \\ -1, & \text{if } v \in D^-(u) \\ 0, & \text{otherwise} \end{cases}$ (Result 3.1.9 (c))

(vi) $\left\langle T_A^* T_A\, e_u, e_v \right\rangle = d^+(u,v) + d^-(u,v) - d(u^+,v^-) - d(u^-,v^+)$ (Result 3.1.9.(d))

(vii) T_A is always normal. (Proposition 3.1.10)

CHAPTER IV – THE INCIDENCE ALGEBRAS I(G, Z) AND I(G_∞, Z)

The directed graphs considered in this chapter are the "graphs" of partially ordered sets, (V, ≤), finite or infinite and bounded or unbounded.

4.1 Incidence Algebra of a Partially Ordered Set

In section 4.1 the basic ideas on the Incidence Algebra of a partially ordered set (X,≤), over a commutative ring R are given.

4.2 Graph of a Partially Ordered Set

The graph (G, ≤) associated with a partially ordered set (V , ≤) is defined as (G, ≤) = (V,E), where V = (V,≤) and E ={(u,v) : u<v in (V,≤) }(4.2.1). The graph (G, ≤) has no cycles and multiple arcs (4.2.2). If the partially ordered set (V,≤) is infinite, the corresponding graph is denoted by (G_∞, ≤).

An ideal $\mathcal{J}$ of (G, ≤) is an induced subdigraph of G such that all directed paths with its terminal vertex in $\mathcal{J}$ are contained in $\mathcal{J}$ (Definition 4.2.5).

If I_v is a principal ideal of (V, ≤) then $\langle I_v \rangle$, the subdigraph induced by the vertices in I_v is the principal ideal generated by v in (G, ≤). $\langle I_v \rangle$ is denoted by $\mathcal{J}_v$ (Definition 4.2.6).

4.3 The Incidence Algebra I(G, Z) of (G, ≤)

A discussion of the incidence algebra I(G,Z) of the graph (G,≤) of a finite partially ordered set (V,≤) is the content of section 4.3. The main definitions as well as results are the following.

The incidence algebra I(G, Z) of (G,≤) over the commutative ring Z with identity is defined by $I(G, Z) = \{ f_i, f_i^* : V \times V \rightarrow Z, i = 0,1,2,\ldots,n-1 \}$ with operations defined by

(i) $(f_i + f_j)(u,v) = f_i(u,v) + f_j(u,v)$

(ii) $(f_i . f_j)(u,v) = \sum_w f_i(u,w) . f_j(w,v)$

(iii) $(z f_i)(u,v) = z f_i(u,v)$ for $f_i, f_j \in I(G, Z)$, $z \in Z$ and $u,v,w \in V$

where $f_i(u,v)$ denotes the number of directed paths of length 'i' from u to v and $f_i^*(u,v) = - f_i(u,v)$ (Definition 4.3.2).

With each ideal $\mathcal{J}_v = \langle I_v \rangle$ of (G, ≤) we associate an incidence algebra $I(\mathcal{J}_v, Z) = \{ f \in I(G, Z) : f : I_v \times I_v \rightarrow Z \}$ such that $f(v_i, v_j) = 0$, for all $(v_i, v_j) \notin I_v \times I_v$ (Definition 4.3.4). $I(\mathcal{J}_v, Z)$ is a subalgebra of I(G, Z).

$I(\mathcal{J}_v, Z)$ is called the subalgebra generated by the vertex v. For each principal ideal $\mathcal{J}_v$ of (G, ≤), $I(\mathcal{J}_v, Z)$ is an ideal of the ring I(G, Z) (Proposition 4.3.7). Also, every ideal of I(G, Z) has the form $I(\mathcal{J}_v, Z)$ for some principal ideal $\mathcal{J}_v$ of (G, ≤) (Proposition 4.3.9).

4.4 A Subalgebra of I (G, Z)

The construction of a subalgebra for I(G, Z) is given in section 4.4. In (G, ≤), an equivalence relation $\mathcal{R}$ is defined on $\mathcal{P}$, the set of all directed paths. The functions in I(G, Z), which are $\mathcal{R}$- compatible as well as the cases of $\mathcal{R}$ being ≤ - compatible are examined and a subalgebra is defined.

4. 5 The Incidence Algebra $I(G_\infty, Z)$

Section 4.5 contains the extension of the concepts given in section 4.3 to graphs of locally finite, infinite partially ordered sets. The partially ordered set may be bounded or unbounded. In general, the graph of a locally finite , infinite partially ordered set need not be locally finite. The extensions are obtained on weak partially ordered sets.

A locally finite partially ordered set (V, ≤) is weak if only finitely many chains intersect at every element $v \in V$ (Definition 4.5.1). Let

$(G_\infty, \leq)$ be the graph representing a locally finite, weak partially ordered set. The incidence algebra $I(G_\infty, Z)$ of the graph $(G_\infty, \leq)$ over the ring Z of integers is defined (Definition 4.5.2) as in the finite case (Notation 4.3.1 & Definition 4.3.2).

In a bounded, locally finite weak partially ordered set (V, ≤) the principal ideal generated by $v \in V$ is defined as $I_v = \{u \in V : u \leq v\}$. The corresponding principal ideal of $(G_\infty, \leq)$ is given by $\mathcal{J}_V = \langle I_v \rangle$ (Definition 4.5.5). Also, $I(\mathcal{J}_V, Z)$ is an ideal of $I(G_\infty, Z)$ (Proposition 4.5.6), extending Proposition 4.3.7 to a locally finite, infinite graph.

When (V, ≤) is unbounded, the principal ideals of (V, ≤) as well as $(G_\infty, \leq)$ are not well-defined. Also Proposition 2.2.3 is not true for unbounded partially ordered sets, in general. But, there are unbounded locally finite partially ordered sets which satisfy Proposition 2.2.3. In such cases, $I(G_\infty, Z)$ is isomorphic to a subring of the ring of upper triangular matrices. Then,

(a) principal ideals of $(G_\infty, \leq)$ are defined as $\mathcal{J}_V = \langle I_v \rangle$ where

$I_v = \{u \in V : u \leq v\}$

(b) for each $\mathcal{J}_V$, $I(\mathcal{J}_V, Z)$ is a subalgebra of $I(G_\infty, Z)$.

(c) for every $\mathcal{J}_V$, $I(\mathcal{J}_V, Z)$ is an ideal of $I(G_\infty, Z)$.

CHAPTER V- THE ARC LAPLACIAN Q_E

The Laplacian matrix $Q = BB'$,where B represents the incidence matrix for some orientation of a finite or locally finite, infinite graph is well- known. Here the matrix $B'B$ is considered as a linear operator on l^2 (E). $B'B$ is called the Arc Laplacian of G and is denoted by Q_E .

For any finite or locally finite infinite graph G = (V,E), we define its incidence matrix B as given by Biggs in [10]. B represents an operator $B: l^2(E) \to l^2(V)$, such that, for $x \in l^2(E)$, $(Bx)_u = \sum_{u(e)=u} x_e - \sum_{v(e)=u} x_e$, where u (e) and v (e) are respectively referred to as the positive and negative ends of e.

5.1 The Arc Adjacency Operator $\mathcal{A}_E$

In section 5.1.the arc adjacency operator $\mathcal{A}_E$ is defined and a few properties of $\mathcal{A}_E$ are also given.

Two arcs e_i and e_j of a digraph D are adjacent if they form a 2-path in D (a directed path of length 2). Arcs e_i and e_j are weakly adjacent (w-adjacent), if they are not adjacent in D. but form a 2- path in the underlying graph.(Definition.5.1.4).

With every arc e of D, we associate two sets S(e) and W(e) as follows.

$S(e) = \{f \in E(D) \ : \ u(f) = v(e) \text{ or } v(f) = u(e)\}$

$W(e) = \{f \in E(D) \ : \ u(f) = u(e) \text{ or } v(f) = v(e)\}$

S (e) and W(e) are disjoint for all $e \in E(D)$. (Definition 5.1.5).

The arc adjacency matrix $\mathcal{A}_E$ of a directed graph D is defined by

$$[\mathcal{A}_E]_{ij} = \begin{cases} 1, \text{if } e_i \text{ and } e_j \text{ are adjacent} \\ -1, \text{if } e_i \text{ and } e_j \text{ are w-adjacent} \\ 0, \text{ otherwise.} \end{cases}$$ (Definition 5.1.6)

The arc adjacency operator $\mathcal{A}_E$ is defined on $l^2(E)$ by

$\mathcal{A}_E(x) = \sum_{e \in E} \{ \sum_{f \in S(e)} x_f - \sum_{f \in W(e)} x_f \} \beta_e$, where $\{\beta_e : e \in E\}$ is the standard basis of $l^2(E)$ and any $x \in l^2(E)$ is represented as $x = \sum_{e \in E} x_e \beta_e$ where $x_e \in \{-1,0,1\}$.

Properties of $\mathcal{A}_E$

(a) $\|\mathcal{A}_E\| \leq 2[\Delta(G) - 1]$ (5.1.9).

(b) If D has only finitely many arcs then $\mathcal{A}_E$ is a compact linear operator on $l^2(E)$. (Proposition 5.1.10)

(c) If $d(v) \leq k$, then $\|\mathcal{A}_E\| \leq 2(k-1)$. (Proposition 5.1.11)

5.2 The Arc Laplacian Q_E

Let D be an orientation of a finite or countably infinite, locally finite graph G and B the incidence matrix of D with respect to this orientation. The arc Laplacian matrix Q_E is defined by $Q_E = B^tB$. The matrix Q_E is closely related to $\mathcal{A}_E$. The following are the main results in this section.

(a) $Q_E = 2\, I_{|E|} - \mathcal{A}_E$ where $I_{|E|}$ is the identity matrix of order $|E|$. (Proposition 5.2.6)

(b) Let Q and Q_E denote the Laplacian and Arc Laplacian matrix, respectively, of a locally finite, countably infinite graph G. The matrices Q and Q_E are equal if and only if the orientation D, of G, considered is a disjoint union of directed cycles. (Proposition 5.2.8)

(c) If μ is an eigen-value of $\mathcal{A}_E$, then $\mu \leq 2$. (Proposition 5.2.9)

(d) Let A (G), A_L(G) and $\mathcal{A}_E$ (D) denote the adjacency matrix of G, the adjacency matrix of the line graph L(G) of G and the arc adjacency matrix for some orientation D of G respectively. The eigen values of A, A_L and $\mathcal{A}_E$ coincide if and only if G is a cycle. (Proposition 5.2.12)

5. 3 Eigenvalues of Q_E

As matrices, Q and Q_E are transposes to each other. Some results on Q_E in comparison with those of Q, are given here.

(a) If λ_i, $1 \leq i \leq m$ are the eigen values of the arc adjacency matrix $\mathcal{A}_E(D)$ of a finite digraph D, then $\mu_i = 2 - \lambda_i$ are the eigen values of Q_E. (Proposition 5.3.2)

(b) $\lambda_i \leq 2$ $\mu_i \geq 0$ (Proposition 5.3.3)

(c) Let $\{ \mu_i : i \in I \}$ denote the eigen values of the arc Laplacian Q_E of a locally finite, connected graph G. $\mu_i = 0$ if and only if G contains a cycle. The multiplicity of zero as an eigen value of Q_E is the dimension of the cycle subspace of G. (Proposition 5.3.5)

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CHAPTER I

THE PRELIMINARIES

This chapter contains the basic definitions, key concepts and notations that are used in the forthcoming chapters.

1.1 Graphs and Digraphs

For graphs and directed graphs the references are mainly Bondy and Murty [11], Harary.F [20], Parthasarathy K.R[41] and Robinson & Foulds [42] whereas for infinite graphs we refer to Konig .D [28].

Definition 1.1.1

A graph is a pair G = (V, E) of two disjoint sets V and E, where V is non-empty and E is a set of 2-element subsets of V.

A multigraph may have several edges between the same vertices and edges whose end vertices coincide. Such edges are called multiple edges and loops respectively.

Definition 1.1.2

The degree or valency $d_G(v) = d(v)$ of a vertex v of G is the number of edges at v. The number $\delta(G) = \min\{d(v) / v \in V\}$ is the minimum degree of G. $\Delta(G) = \max\{d(v) / v \in V\}$ is the maximum degree of G. If all the vertices have the same degree k then G is k- regular.

Definition 1.1.3

Let $G = (V, E)$ and $G_1 = (V_1, E_1)$ be two graphs such that $V_1 \subseteq V$ and $E_1 \subseteq E$. Then G_1 is a subgraph of G (and G is a supergraph of G_1) written as $G_1 \subseteq G$.

Definition 1.1.4

If $G_1 \subseteq G$ and G_1 contains all edges $uv \in E$ with $u, v \in V_1$ then $G_1$is an induced subgraph of G.

Definition 1.1.5

A subgraph G_1 of G is a spanning subgraph of G if $V = V_1$.

Definition 1.1.6

A path P in a graph G is a non-empty subgraph $P = (V, E)$ of the form $V = \{v_0, v_1, v_2 \ldots v_k\}$, $E = \{e_1, e_2, \ldots e_k\}$ where $e = v_{i-1}v_i$, for $i \in \{1,2, \ldots k\}$ and v_i are all distinct. The number of edges of a path is its length. $P + v_k v_0$ is called a cycle. A graph without any cycles is called an acyclic graph.

Definition 1.1.7

A non-empty graph G is called connected if any two of its vertices are linked by a path in G.

Definition.1.1.8

Let $G = (V, E)$ be a graph. A maximal connected subgraph of G is called a component.

Definition 1.1.9

In definition 1.1.1, if either V or E is infinite, then G is an infinite graph.

Definition 1.1.10

The infinite graph G = (V, E) is locally finite or G is of finite degree if d (v) is finite for all $v \in V$. G is of bounded degree if there is a positive integer k, such that, $d(v) \leq k$ for all $v \in V$.

Theorem 1.1.11 [28]

The set of vertices as well as the set of edges of a connected (infinite) graph G of finite degree is finite or countably infinite.

Remark 1.1.12

The infinite graphs considered in this thesis are in general assumed to be locally finite and the vertex sets to be countably infinite.

Definition 1.1.13

The graph formed by an infinite set of edges $\{v_i v_{i+1} : i = 0, 1, 2, \ldots\}$ is called a singly infinite path. It is also called a one-way infinite path or a ray.

Under the same conditions the edges $\{v_i v_{i+1} : i = 0, \pm 1, \pm 2, \ldots\}$ form a doubly infinite path.

Remark 1.1.14 [28]

A singly infinite path has a unique endpoint.

Definition 1.1.15

A directed graph or digraph D is a pair (V, E) of disjoint sets of vertices and arcs where $E \subseteq V \times V$. The arcs "e" of D are defined by two maps $u : E \rightarrow V$ and $v : E \rightarrow V$ assigning to every arc e an initial vertex u(e) (tail) and a terminal vertex v(e) (head). If u(e) = x and v(e) = y, then e = (x, y), is directed from x to y.

Arcs of D are represented by ordered pairs throughout this work.

D may have several arcs between the same two vertices x and y. Such arcs are called multiple arcs; if they have the same direction, they are parallel. If u(e) = v(e) the arc e is called a loop.

Definition 1.1.16

A directed graph D is an orientation of an (undirected) graph or multigraph G, if V(D) = V(G) and $E(D) \approx E(G)$ such that, for every arc e = (x,y) of D there is an edge x y in G for which $\{x,y\} = \{u(e), v(e)\}$.

Definition 1.1.17

The outdegree $d^+(v)$ of a vertex v in a digraph D is the number of arcs having v as its tail. The indegree $d^-(v)$ is defined similarly and the degree d(v) of a vertex is given by $d(v) = d^-(v) + d^+(v)$.

Definition 1.1.18

A vertex v of a digraph having zero indegree, i.e. $d^-(v) = 0$ is a called a source. If $d^+(v) = 0$, v is called a sink.

Definition 1.1.19

A directed walk in a digraph D is a sequence $v_0 e_1 v_1 \ldots v_{k-1} e_k v_k$ of vertices and arcs such that $e_i = (v_{i-1}, v_i)$. The number of arcs in a walk is called its length.

A (directed) walk through distinct vertices is a (directed) path.

Directed cycles and acyclic digraphs are defined as in 1.1.8.

Theorem 1.1.20

(a) A (finite) acyclic digraph has at least one vertex of outdegree zero

(b) A (finite) acyclic digraph has at least one vertex of indegree zero.

Definition 1.1.21

Let $V = \{ v_1, v_2, v_3, \ldots \}$. A directed graph $D = \{V, E\}$ where $E = \{(v_i v_{i+1}), i = 1, 2, 3, \ldots\}$ is called an out-ray and denoted by $(\overrightarrow{v_1})$. Similarly, if $E = \{ (v_i v_{i-1}) : i = 2, 3, \ldots \}$ then $D = (V, E)$ is the in-ray $(\overleftarrow{v_1})$.

If $E = \{(v_i v_{i+1}) : i = 0, \pm 1, \pm 2, \ldots\}$, $D = (V, E)$ is called a doubly infinite directed path.

Remark 1.1.22

In-rays and out-rays are acyclic digraphs. An in(out)-ray has a unique vertex of out(in)degree zero.

Definition 1.1.23

With every directed graph $D = (V, E)$ there is an associated graph $G = (V, E)$ called the underlying graph, which is obtained by replacing each arc of D by an edge (undirected).

Definition 1.1.24

A digraph is weakly connected if the underlying graph is connected.

1.2 MATRICES

Several matrices are defined in association with graphs and digraphs both in the finite and infinite cases. In this work, the emphasis is on infinite matrices, the main reference being Cooke.R.G [14].

Definition 1.2.1

An infinite matrix is a twofold table $A = (a_{ij})$, $i, j = 1, 2, \ldots$. of real or complex numbers with addition and multiplication defined by

$A + B = (a_{ij} + b_{ij})$, $\lambda A = (\lambda\, a_{ij})$ where λ is a scalar and

$$AB = [\ \sum_{k=1}^{\infty} a_{ik} b_{kj}\ ].$$

If $AB = (c_{ij})$, then $c_{ij} = \sum_{k=1}^{\infty} a_{ik} b_{kj}$ whenever this sum exists.

The sum of two infinite matrices always exists and is commutative and associative. The distributive laws $A(B+C) = AB + AC$ and $(B+C)A = BA + CA$ hold in the sense that if AB and AC exist, then $A(B + C)$ also exists and is equal to $AB + AC$. But $A(B + C)$ may exist even when AB and AC do not exist.

Definition 1.2.2

If every row of an infinite matrix A contains only a finite number of non-zero elements, A is said to be row-finite; if the same is true for every column, A is said to be column- finite.

The matrix A is a diagonal matrix if all of its entries, except those in the main diagonal, are zeros.

If $a_{ij} = 0$ for $j < i$, A is called upper-semi matrix or upper triangular matrix. If $a_{ij} = 0$ for $i < j$, A is called a lower-semi matrix or lower-triangular matrix.

Definition 1.2.3

We say that a product of infinite matrices is associative if (a) the product exists for every succession of the multiplications involved, the order of the factors remaining the same and (b) all the products so obtained are equal.

Multiplication of infinite matrices is not in general associative. But the product of any number of diagonal matrices, lower semi matrices and upper-semi matrices is associative.

Definition 1.2.4

If in a set S of matrices, (a) S contains the scalar matrices, (b) every finite product of matrices belonging to S exists and is associative (c) S is closed under finite sum and finite product (i.e every finite sum and finite product of matrices of S belongs to S) then, S is called an associative field.

Note 1.2.5

It should be noted that, the *field* so defined for infinite matrices is not the same as the field defined in the algebraic sense.

Remark. 1.2.6

Diagonal matrices. and row-finite and column-finite matrices form associative fields. For any such matrix A, positive integral powers of A are defined as , $A^2 = A.A$, $A^3 = A.A^2$, ..., $A^n = A.A^{n-1}$.

Definition 1.2.7

If $A^2 = A \neq 0$, A is said to be idempotent.

If r is the least positive integer such that $A^r = 0$, $(A \neq 0)$, then A is said to be nilpotent with index r.

Certain well known matrices, associated with graphs and digraphs, which are considered in the present work are given below.

Definition 1.2.8

Let G = (V, E) be a graph. The adjacency matrix $A(G) = A = (a_{ij})$, where a_{ij} is 1 or 0 according as E contains an edge joining v_i and v_j or not.

The adjacency matrix A is a symmetric matrix .

Proposition 1.2.9

If A is the adjacency matrix of a finite graph G, then $[A^k]_{ij}$ is the number of walks of length k between v_i and v_j.

Remark 1.2.10

If the graph G is infinite and locally finite, A is treated as an operator on G.

Definition 1.2.11

The adjacency matrix A of a digraph D = (V, E) is defined as $A = (a_{ij})$ where $a_{ij} = 1$, if $(v_i, v_j) \in E$ and $a_{ij} = 0$, otherwise.

Remark 1.2.12

In 3.1.2 we associate another matrix $\tilde{A}$ (based on the same adjacency relation) with the digraph D.

Definition 1.2.13

Let D = (V, E) be a finite digraph where $V = \{v_1, v_2, .. v_n\}$ and $E = \{e_1, e_2, ... e_m\}$. The Incidence matrix $B = (b_{ij})$ of D has entries $b_{ij}=1$, if $e_j = (v_i, w)$, $b_{ij} = -1$, if $e_j = (w, v_i)$ for some vertex w and $b_{ij} = 0$ otherwise.

Definition 1.2.14

Let c be any cycle in the graph G and D some orientation of G. Fix some orientation for the cycle c. The cycle matrix C has a row for each cycle c and a column for each arc e such that $C_{ce} = +1$, if $e \in c$ and its cycle orientation coincides with its orientation in D, $C_{ce} = -1$, if $e \in c$ and its cycle orientation is the reverse of its orientation in D and $C_{ce} = 0$, if e is not in c.

Definition 1.2.15 [10]

The cycle subspace of D is the kernel of the incidence mapping of D.

1.3 LINEAR OPERATORS (TRANSFORMATIONS)

The term linear operator is used as a synonym for linear ransformation so that the terminology followed agrees with that given in the nain references [9], [10], [33] etc. For infinite graphs and digraphs the associated matrices are treated as operators. Two Hilbert spaces $l^2(V)$ and $l^2(E)$ are considered for infinite digraphs and certain operators are defined on these spaces. The basic results given here, for further use, are available in any standard book on Functional Analysis.

Definition 1.3.1

An operator $T : X \to Y$ is closed if and only if whenever $x_n \in Dom(T)$, $x_n \to x$ in X, $T(x_n) \to y$ in Y, then $x \in Dom(T)$ and $T(x) = y$.

Definition 1.3.2

(a) T is bounded, if $\exists$ k>0 : $\|Tx\| \le k\|x\|, \forall x \in Dom(T)$.

(b) T is unitary, if $T^* T = TT^* = I$

(c) T is normal, if $T^*T = TT^*$.

(d) T is an isometry if $T^*T = I$.

(e) T is a projection if $T = T^* = T^2$.

(f) T is nilpotent if $\exists n: T^n = 0$.

(g) T is idempotent if $T = T^2$

(h) T is positive if $\langle Tx, x\rangle \ge 0$, $\forall x \in Dom(T)$.

1.4. PARTIALLY ORDERED SETS

This section contains the basic definition and results used in chapter IV-The Incidence Algebras I(G, Z) and $I(G_\infty, Z)$. The main references are Aigner M [1], Spiegel and O'Donnel [45] and Stanley P [46].

Definition 1.4.1

A set X with a binary relation $\le$ is a partially ordered set if,

(i) $\leq$ is reflexive (i.e $x \leq x, \forall x \in X$)

(ii) antisymmetric ($x \leq y$ and $y \leq x \Rightarrow x = y$)

(iii) transitive ($x \leq y, y \leq z \Rightarrow x \leq z$)

$\leq$ is a pre-order on X if it is reflexive and transitive.

Definition 1.4.2

An element x of a partially ordered set X is maximal if whenever $x \leq y$ then $x = y$. If X has an element x such that $y \leq x$ for every y in X then x is the maximum element of X and denoted by 1. The minimum element 0 of X is defined dually.

Definition 1.4.3

A subset C of a partially ordered set is a chain if for any $x, y \in C$ either $x \leq y$ or $y \leq x$ holds. The chain C has length n if C has n elements.

Remark 1.4.4

The length of a chain is defined to be one less than its cardinality.

Definition 1.4.5

Two elements x, y of X are comparable if either $x \leq y$ or $y \leq x$

Hence, a chain is a partially ordered set in which any two elements are comparable and an antichain is a subset in which any two elements are incomparable.

Definition 1.4.6

An element z covers the element x if $x < z$ and if $x \leq y < z$, then $y = x$.

Definition 1.4.7

The atoms of a partially ordered set are the elements covering 0, if 0 exists.

Definition 1.4.8

Given x and z in a pre-ordered set, the interval or segment from x to z is the subset $\{y \in X: x \leq y \leq z\}$ and is denoted by $[x,z]$.

A pre-ordered set X is locally finite if every interval of X is finite.

Definition 1.4.9

An interval $[x,y]$ in a partially ordered set X is said to have length n, if there is a chain of length n in $[x,y]$ and any chain in this interval has length less than or equal to n.

Definition 1.4.10

The partially ordered set X is bounded if there is an integer n such that each interval $[x,y]$ of X has length atmost n.

X is unbounded, if it is not bounded.

Definition 1.4.11

Two partially ordered sets X and Y are isomorphic if there exists an order- preserving bijection $\phi: X \to Y$ whose inverse is order preserving.

i.e., $x \leq y$ in X if and only if $\phi(x) \leq \phi(y)$ in Y.

1.5 RINGS

A few results from the theory of rings are given here. [44]

Definition 1.5.1

A non-empty set R on which there are defined two operations '+' and '.' is called a ring if the following axioms hold.

(a) R is an abelian group under '+'

(b) '.' is associative: a.(b.c) = (a.b).c for all a,b,c in R

(c) The two distributive laws hold: a.(b+c) = a.b + a.c &(b+c).a = b.a + c.a

If, in addition, R contains an element 1 called the identity element or unity, such that a.1 = 1.a = a for all a in R, then R is called a ring with unity.

If the operation '.' is commutative, then R is a commutative ring.

Definition 1.5.2

A non-empty subset S of a ring R is a subring of R if S itself is a ring for the operations defined in R.

Definition 1.5.3

A non-empty subset I of a ring R is said to be an ideal of R if I is a subgroup of R under '+ 'and for every $r \in R$ and $x \in I$, both $xr, rx \in I$.

Definition 1.5.4

If R is a commutative ring with identity and $a \in R$, the ideal $\{ra / r \in R\}$ is called the principal ideal generated by a.

CHAPTER II

THE ADJACENCY MATRIX

A finite graph G is defined as a pair of sets (V,E) where V is finite and non-empty and E is a set of unordered pairs of elements of V. If $\{v_i, v_j\} \in E$, then we say that v_i and v_j are adjacent, and adjacency defines a binary relation A on V. Hence in any study on matrices related to graphs, the adjacency matrix must have its place at its outset. Instead of the graph G = (V,E) if a directed graph D is considered on the same vertex set V and the edges in E are replaced by arcs according to some orientation, the adjacency relation in D is not required to be symmetric, for $E \subseteq V \times V$.

In the graph G = (V, E) if either V or E is infinite, then G is an infinite graph. In the study of a finite graph or digraph, results and methods of algebra can be used especially for matrices associated with the graph/digraph [10]. The shift from a finite graph to an infinite graph brings forth a radical difference in the treatment of the same matrices [14]. With infinite graphs, graphs of finite degree or locally finite graphs play a distinguished role as they form an intermediate link between finite graphs and infinite graphs of infinite degree.

Mohar B. (1982) [33] has defined an adjacency operator for

Some results given in Sec.2.2 have been published in Journal of the Tripura Mathematical Society, 3 (2001) 21-28.

infinite graphs followed by Biggs [10] Woess [37] Shawe-Taylor [9] and many others. In 1989, Fuji Sasaoka and Watatani [18] have extended the definition of Mohar to locally finite, infinite directed graphs.

The results contained in this chapter are mainly based on [18] in which the nilpotency of the adjacency operator is strictly restricted to a finite digraph whose adjacency operator has an upper triangular representation (section 2.2) The main result is the necessary conditions for the adjacency matrix of an acyclic, infinite and locally finite, digraph to have an upper triangular representation. This result is used in section 2.3 to characterize nilpotent adjacency operator of infinite, locally finite, digraphs.

2.1 THE ADJACENCY MATRIX OF A FINITE DIGRAPH

Throughout this section, D = (V,E) represents a digraph whose vertex set is V and the arc set E is a subset of VxV. From definition 1.2.11, it is evident that the matrix A of a digraph need not be symmetric. A has a diagonal of zeros if D has no loops. If A_1 and A_2 are the adjacency matrices corresponding to two different labellings of the same digraph D then for some permutation matrix P we have $A_2 = P^{-1} A_1 P$.

The preliminary definitions and results given in this section are mainly from [41] & [42].

Theorem 2.1.1

Let A be the adjacency matrix of a digraph D. Then the (i,j) entry of A^k is the number of directed walks of length k from v_i to v_j.

Note 2.1.2

If D is an acyclic digraph, every directed walk from v_i to v_j is a directed path.

Proposition 2.1.3

Any finite acyclic digraph has at least one source and at least one sink.

Definition 2.1.4

A logical numbering (topological ordering) of the finite digraph D on n vertices is a function $f : V \rightarrow \{1,2,3,\ldots,m : m \leq n\}$ which assigns to each vertex v_i of D an integer $f(v_i)$ such that each integer is assigned to a vertex exactly once and if $(v_i,v_j) \in E$, then $f(v_i) < f(v_j)$.

Definition 2.1.5

When D is infinite and locally finite, an injection $f : V \rightarrow N$ such that if $(v_i,v_j) \in E(D)$, then $f(v_i) < f(v_j)$ is a logical numbering of D.

Remark 2.1.6

A finite digraph D on n vertices, n >2, may have different logical numberings, if it has any.

Note 2.1.7

The function f in 2.1.5 is a rearrangement of the rows and columns of the adjacency matrix A by the same permutation, so that A becomes upper triangular.

Theorem 2.1.8

The following are equivalent for a finite digraph D.

(i) The digraph D is acyclic.

(ii) The digraph D has a logical numbering.

(iii) The adjacency matrix A of D is upper triangular.

(iv) The matrix A is nilpotent.

2.2 ADJACENCY MATRIX (OPERATOR) OF INFINITE DIGRAPHS.

An infinite digraph D is a pair (V,E), V being countably infinite and $E \subseteq V \times V$. The digraphs considered are locally finite; i.e d (v) is finite for all $v \in V$. We begin this section with the following observation.

Let D = (V, E) represent the doubly infinite directed path, for which $V = \{ v_i : i = 0, \pm 1, \pm 2, ,\ldots\}$ and $E = \{ (v_i , v_{i+1}), i = 0, \pm 1, \pm 2, ,\ldots\}$

The digraph D is represented by the diagram

$\ldots\ldots\ldots\ o \to o \to o \to o \to o \ldots\ldots.$

D is acyclic and has neither a source nor a sink, which is obviously a case of failure of the extension of Proposition 2.1.3 and Theorem 2.1.8 to the infinite case.

Theorem 2.2.1 [28]

The vertex set as well as the edge set of a connected infinite graph of finite degree is countable.

Remark 2.2.2

Since every weakly connected, infinite digraph, which is locally finite, is an orientation of a connected infinite graph of finite degree, Theorem 2.2.1 has an extension to such graphs.

Theorem 2.2.3

Let D = (V, E) be a locally finite, infinite, acyclic digraph, whose vertex set V is countable. If, the set $S = \{ v \in V : d^{-}(v) = 0 \}$ is non-empty and finite, and D contains no in-rays, then D has a logical numbering of its vertices.

Proof:

Without loss of generality we can assume that the digraph D is weakly connected, so that both V and E are countable. Otherwise we consider the components of the digraph D, (i.e the components of the underlying graph) each of which is a weakly connected digraph, whose vertices as well as arcs are countable.

Let $S = \{ s_1, s_2, s_3, \ldots, s_n \}$

Let D_0 be the subdigraph of D, formed by the vertices and arcs of D, which lie on any path joining any two vertices in S, in the graph associated with D. D_0 is a nontrivial, weakly connected acyclic subdigraph of D. Also D_0 is finite, since D is locally finite. By Theorem 2.1.8, D_0 has a logical numbering f_0 of its vertices.

Let $\mathcal{D}$ denote the collection of all weakly connected subdigraphs of D, which admit a logical numbering and contains D_0 as a subdigraph. $\mathcal{D}$ is non-empty as $D_0 \in \mathcal{D}$.

For $(D_i, f_i), (D_j, f_j) \in \mathcal{D}$, define $(D_i, f_i) < (D_j, f_j)$ if D_i is a subdigraph of D_j and f_i is the restriction of f_j to D_i.

i.e for any $u \in V(D_i)$ and $v \in V(D_j) - V(D_i)$, we have $f_i(u) = f_j(u)$, and $f_i(u) < f_j(v)$, whenever $(D_i, f_i) < (D_j, f_j)$.

With this definition of '<' in $\mathcal{D}$, $(\mathcal{D}, <)$ is a partially ordered set. Let $K = \{ (D_i, f_i) : i \in I \}$, (I being some index set), be any chain in $(\mathcal{D}, <)$.

We claim the following for K

(a) $\bigcup_I D_i$ is weakly connected for $(D_i, f_i) \in K$.

For, let u, v be any two vertices in $\bigcup_I D_i$. Then $u \in V(D_i)$ and $v \in V(D_j)$, for some i and j. Without loss of generality, assume $(D_i, f_i) < (D_j, f_j)$. Then, $u, v \in D_j$, a weakly connected subdigraph of D. Hence, there is a u-v path in the graph associated with D_j, which is a sub graph of the graph associated with $\bigcup_I D_i$.

(b) $\bigcup_I D_i$ has a logical numbering.

Clearly, $V(\bigcup_I D_i)$ is countable and $\bigcup_I D_i$ is locally finite, being a subdigraph of D.

For any $u \in V(\bigcup_I D_i)$, we have $u \in V(D_i)$ for some i.

Let $f_\cup(u) = f_i(u)$, so that $f_\cup$ is well defined.

Let (u,v) be any arc in $\bigcup_I D_i$.

Then, $u \in V(D_i)$ and $v \in V(D_j)$ for some i, j.

Without loss of generality suppose $(D_i, f_i) < (D_j, f_j)$.

If i=j, then $f_i(u) < f_i(v)$, by the definition of logical numbering f_i.

If $i \neq j$, then $f_i(u) < f_j(v)$, since $f_i = f_j|_{D_i}$.

In either case, $f_\cup(u) < f_\cup(v)$, for any arc $(u,v) \in E(\cup_i D_i)$

Hence, $f_\cup$ is a logical numbering of $\cup_i D_i$.

(c) $(\cup_i D_i, f_\cup) \in \mathcal{D}$

This follows from the definition of $\mathcal{D}$.

(d) For any $(D_i, f_i) \in K$, we have $(D_i, f_i) < (\cup_i D_i, f_\cup)$

This is implied by (a) and (b) and the definition of $f_\cup$.

Hence, $(\cup_i D_i, f_\cup)$ is an upperbound for K.

By Zorn's lemma, $(\mathcal{D}, <)$ contains a maximal element (D^*, f^*).

Suppose $D^* \neq D$.

D^* is a weakly connected subdigraph of D.

The vertices and arcs in $D - D^*$ form, either subpaths of directed paths from some $v \in V(D^*)$ to $w \in V(D) - V(D^*)$ or subdigraphs of some out-ray $(\vec{v})$ from some $v \in V(D^*)$.

In either case, for any such $w \in V(D)-V(D^*)$, there are v-w paths from the same v or different v,s in $V(D^*)$, the lengths of the paths being equal or different . For a fixed w in $V(D) - V(D^*)$, choose v_i from all such v,s such that $f^*(v_i) > f^*(v)$, for every other v in D^*. Let $\{P_i\}$ denote the set of all directed paths from v_i to w. Let W_1 denote the set of all vertices on $\{P_i\}$

which are at a distance one from v_i. Since D is locally finite, W_1 is finite. Hence the vertices in W_1 can be arranged as $w_{11}, w_{12}, w_{13}, \ldots, w_{1r}$ for a finite r.

Consider the subdigraph $D_1=(V_1,E_1)$ of D, where

$V_1=V(D_1) = V(D^*) \cup W_1$ and

$E_1 = E(D_1) = E(D^*) \cup \{ (v_i, w_{1s}) : 1 \leq s \leq r \}$,

Define f_1 on D_1 such that

(1) $f_1(w_{1s}) > f^*(v)$, $\forall v \in V(D^*)$,

(2) $\forall$ arc $(w_{1p}, w_{1q}) \in E(D_1)$, $f_1(w_{1p}) < f_1(w_{1q})$

(3) $f_1(w_{1s}) < f_1(w)$, for $1 \leq s \leq r$, whenever $w \in V(D_1)$ and

(4) $f_1(u) = f^*(u)$, $\forall u \in V(D^*)$

f_1 is a logical numbering of D_1. Also, $(D^*, f^*) < (D_1, f_1)$, contradicting the maximality of (D^*, f^*).

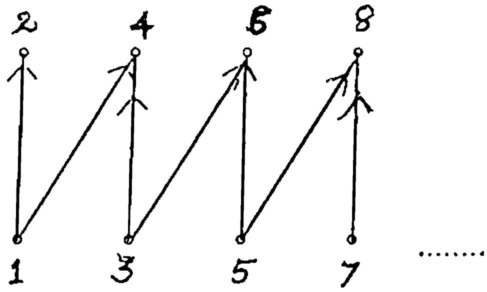
Thus $D^* = D$ and hence the theorem.

Remark 2.2.4

(a) In the statement of Theorem 2.2.3, the condition "S is finite" is not essential.

Example: The directed graph D having the vertex set $V = \{1,2,3,\ldots\}$

and (i,j) is an arc of D if and only if j=i+1 or i+3, has a logical numbering as shown in the diagram.

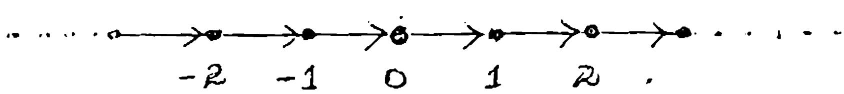


Here, the set $S = \{ v \in V : d^-(v) = 0 \}$ is infinite.

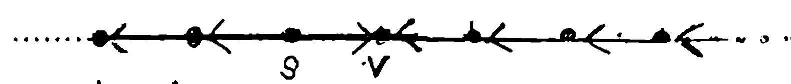
(b) If S is empty, then D has no logical numbering.

Example: The set S is empty for the doubly infinite path D=(V,E), where

$V = \{ 0, \pm 1, \pm 2, \pm 3, \ldots \}$ and $E = \{(i, i+1), i \in V\}$.



(c) If D contains an in-ray, then D has no logical numbering, even if S is finite.

Example: 

Here $S = \{s\}$, and $(\overleftarrow{v}) \subset D$

Remark 2.2.5

Theorem 2.2.3 partially extends the equivalent conditions (i) and (ii) of theorem 2.1.8 on finite digraphs to locally finite, infinite acyclic digraphs.

These results are of immediate use in discussing the nilpotency of the adjacency matrix of an infinite digraph, which is given in the next section.

2.3 NILPOTENT ADJACENCY OPERATOR OF INFINITE DIGRAPHS

When the digraph D= (V, E) is infinite, the adjacency matrix A is treated as a linear operator on vectors in $l^2(V)$. Since D is locally finite, its action is well defined on all vectors in $l^2(V)$.

Definition 2.3.1

An infinite matrix $A = [a_{ij}]$ in which $a_{ij} \neq 0$ for only finitely many j (respectively i) for each i(respectively j) is called a row finite(respectively column finite) matrix.

Remark 2.3. 2

The adjacency matrix of a locally finite digraph is both row finite and column finite.

Definition 2.3.3

Let D = (V, E) be a locally finite infinite acyclic digraph which has a logical numbering. Then every arc of D is of the form (v_i, v_j) where $i<j$, so that the adjacency matrix A is upper triangular. Powers of A are defined by,

$$[A^n]_{ij} = \begin{cases} \sum_{r_1} \sum_{r_2} \cdots \sum_{r_{n-1}} a_{ir_1} \; a_{r_1 r_2} \; . \; . \; . \; a_{r_{n-1} j}, & \text{for } \; i < r_1 < r_2 < . \; . < j \\ 0 \text{ for } i \geq j & \end{cases}$$

for $n = 1,2,3,\ldots.$

Theorem 2.3.4

Let A be the non-zero adjacency operator of a locally finite acyclic digraph, D = (V, E), where (i) V is countable (ii) $S = \{v \in V : d^-(v) = 0\}$ is non-empty and finite and (iii) D contains no in-rays. Then, A is nilpotent if and only if D contains no out-rays.

Proof :

D admits a logical numbering by Theorem 2.2.3 and hence powers of A are defined.

Assume that A is nilpotent with index n.

Then we have, $A^n = 0$, and $A^r \neq 0$, for, $r = 1, 2, \ldots, n-1$.

The operator A is represented by an upper semi matrix.

Hence, powers of are also upper semi.

Let $A^{n-1} = [b_{ij}]$ and $A^n = [0] = [c_{ij}]$, where ,

$$c_{ij} = \sum_{r=i+1}^{j-1} a_{ir} b_{rj} \text{ for } i < j \quad \text{and} \quad c_{ij} = 0 \text{, otherwise.}$$

For, $i + 1 \leq r \leq j-1$, $a_{ir} \in \{0, 1\}$ and $b_{rj} \geq 0$.

Hence, $c_{ij} = 0 \Rightarrow a_{ir} = 0$ or $b_{rj} = 0$

Case (i) $a_{ir} = 0$, $b_{rj} \neq 0$

Then the arc $(v_i, v_r) \notin E(D)$ and D contains b_{rj} paths from v_r to v_j each of length n−1. None of these paths can be extended to a $v_i - v_j$ path in D.

Hence if D contains a directed path from v_i to v_j, then $d(v_i, v_j) \leq n-1$.

Case (ii) $a_{ir} = 1, \quad b_{rj} = 0$.

Here also, $d(v_i, v_j) \leq n-1$ as in case (i).

Since i, j are arbitrary, we conclude that, if D contains a directed path from v_i to v_j, then $d(v_i, v_j) \leq n-1$. Therefore D contains no out-rays.

Conversely,

let, D be an acyclic digraph, satisfying conditions (i), (ii) (iii) and having no out-rays. Then $d(v_i, v_j)$ is finite for every v_i, v_j in V and the adjacency operator A of D is represented by an upper semi matrix.

Let $k = \max_i \max_j d(v_i, v_j)$

Then $d(v_i, v_j) \leq k$ for all v_i, v_j in V.

$\therefore A^k \neq 0$ and $[A^{k+1}]_{ij} = 0, \quad \forall$ i and j.

$\therefore$ A is nilpotent with index $k+1$.

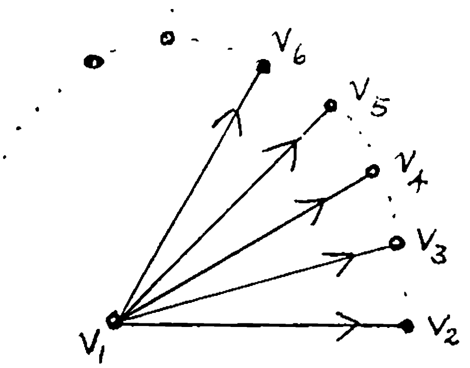
Remark 2.3.5

Theorem 2.3.4 is restricted to locally finite acyclic digraphs. There are infinite acyclic digraphs, having vertices of infinite degree whose adjacency operator is nilpotent.

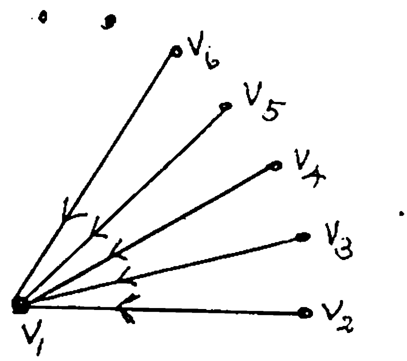
The following digraphs are examples.

(1) D = (V, E) where V={1,2,3,...} and E ={(v_1, v_j); j = 2, 3, 4,...}

(2) D*= (V*,D*) where V*={1,2,3,...}and E*= {(v_j,v_1); j=2,3,4,...}



D



D*

Their adjacency matrices A and A* ,respectively, where,

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots \\ \cdots \end{bmatrix} \quad \text{and } A^* = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 0 & 0 & 0 & \cdots \\ \cdots \end{bmatrix}$$

satisfy $A^2 = (A^*)^2 = 0$.

Theorems 2.2.3, 2.2.4 and 2.3.5 give an extension of the four equivalent conditions of theorem 2.1.8 to locally finite, infinite digraphs.

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CHAPTER III

ANOTHER ADJACENCY OPERATOR OF A DIGRAPH

The adjacency operator considered in the previous chapter is expressed as a finite matrix or row-column finite infinite matrix, whose entries are 0 and 1.The same adjacency relation is used here to define another matrix viz. another adjacency matrix denoted by $\tilde{A}(D)$ with entries in {-1, 0, 1}.The corresponding linear operator $T_{\tilde{A}}$, which is closed on a Hilbert space H defined on $l^2(V)$ is analysed.

3.1 THE OPERATOR $T_{\tilde{A}}$

The directed graph D = (V, E) is locally finite, countably infinite, having atmost one arc between any two vertices and no loops.

Definition3.1.1 [18]

We associate the following sets with vertices of D.

1. $D^+(v) = \{w \in V : (v, w) \in E\}$

2. $D^-(v) = \{u \in V : (u, v) \in E\}$

3. $D^+(u, v) = D^+(u) \cap D^+(v)$

Some results are published in the proceedings of the International Conference on Analysis and Applications and III Annual conference of K.M.A(2000),Allied Pub.New Delhi.

4. $D^-(u, v) = D^-(u) \cap D^-(v)$

5. $D(u^+, v^-) = D^+(u) \cap D^-(v)$

Their cardinalities are denoted by $d^+(v)$, $d^-(v)$, $d^+(u, v)$, $d^-(u, v)$ and $d(u^+, v^-)$ respectively.

Definition 3.1.2

Let $V = \{v_1, v_2, v_3, \ldots\}$ be the vertex set of D. The adjacency matrix $\tilde{A}$ is defined as $\tilde{A} = [a_{ij}]$ where

$$a_{ij} = \begin{cases} 1, \text{ if } (v_i, v_j) \in E \\ -1, \text{ if } (v_j, v_i) \in E \\ 0, \text{ otherwise.} \end{cases} \qquad (*)$$

Definition 3.1.3

The i^{th} row of the adjacency matrix $\tilde{A}$ associates with each $v_i \in V$ a sequence (x_i) whose entries belong to $\{-1, 0, 1\}$. Since G is locally finite, $\{(x_i)\}$ is a closed subspace of l^2. Denote the Hilbert space $\{(x_i)\}$ by H.

Let $\{e_v / v \in V\}$ be the canonical basis of H, where $e_v(u) = \delta_u^v$

$$T_A(x) = \sum_{u \in V} \sum_{v \in D^-(u)} x_v e_u \qquad \text{where}$$

$$\text{Dom}(T_A) = \{x = \sum_{v \in V} x_v e_v \in H : \sum_u |\sum_{v \in D^-(u)} x_v|^2 < \infty\} \qquad (1)$$

Remark 3.1.4

$(T_A e_u)_v = a_{vu}$ and hence the matrix representing T_A is the transpose of the matrix given by (*)

Proposition 3.1.5

T_A is a closed operator on H.

Proof:

Let (x_n) be any sequence in Dom (T_A)

$$x_n = \sum_u x_u^n e_u \quad \text{where} \quad \sum_u \Big| \sum_{v \in D^-(u)} x_v^n \Big|^2 < \infty \tag{2}$$

Suppose $x_n \to x$. Then $x = \sum_u x_u e_u$

$$\text{and } \sum_u (x_u^n - x_u)^2 \to 0 \text{ for all } v \in V(D). \tag{3}$$

$$\sum_u \Big| \sum_{v \in D^-(u)} x_v \Big|^2 = \sum_u \Big| \sum_{v \in D^-(u)} (x_v - x_v^n + x_v^n) \Big|^2$$

$$\leq \sum_u \Big\{ \Big| \sum_{v \in D^-(u)} (x_v - x_v^n) \Big| + \Big| \sum_{v \in D^-(u)} x_v^n \Big| \Big\}^2$$

$$\leq 2 \Big\{ \sum_u \Big| \sum_{v \in D^-(u)} (x_v - x_v^n)^2 \Big| + \sum_u \Big| \sum_{v \in D^-(u)} x_v^n \Big|^2 \Big\}$$

By (*) $-2 \leq x_v - x_v^n \leq 2$ and by the local finiteness of G, these extreme values are attained only for finitely many v.

Hence $\sum_u \Big| \sum_{v \in D^-(u)} x_v \Big|^2 < \infty$ and $x \in$ Dom (T_A)

Assume, $T_A x_n \to y$. i.e. $\sum_u \sum_{v \in D^-(u)} x_v^n e_u \to \sum_u y_u e_u$

Then $\sum_u \Big\{ \sum_{v \in D^-(u)} x_v^n - y_u \Big\}^2 \to 0$. i.e. $\sum_{v \in D^-(u)} x_v^n \to y_u$

Using (3), $\sum_{v \in D^-(u)} x_v = y_u$ and $\sum_u \sum_{v \in D^-(u)} x_v e_u = \sum_u y_u e_u$

Hence $T_A x = y$ i.e . T_A is a closed operator on H.

Proposition 3.1.6

T_A is bounded if and only if G is of bounded degree.

Proof :

Suppose T_A is bounded.

Then, $\exists$ a constant $k > 0$,such that, $\|T_A x\| \leq k \| x \|$, $\forall x \in \text{Dom}(T_A)$.

Also, $e_v \in \text{Dom}(T_A)$ for $v \in V$.

$$\|T_A e_v\|^2 = \langle T_A e_v, T_A e_v \rangle = \sum_u |a_{uv}|^2 = d(v) \leq k^2 \| e_v\|^2 = k^2, \quad \forall v \in V$$

Hence, D is of bounded degree.

Conversely,

assume that, D is of bounded degree.

Then we have a constant $k > 0$, such that, $d(v) \leq k$ for all $v \in V(D)$.

By the definition of d(v), $d^+(v) \leq k$ and $d^-(v) \leq k$.

$$\|T_A x\|^2 = \left\| \sum_{u \in V} \sum_{v \in D^-(u)} x_v e_u \right\|^2 = \sum_{u \in V} \left| \sum_{v \in D^-(u)} x_v \right|^2 ,$$

$$\leq \sum_{u \in V} \left[\left(\sum_{v \in D^-(u)} 1^2 \right) \left(\sum_{v \in D^-(u)} |x_v|^2 \right) \right]$$

$$\leq \sum_{u \in V} \left[d^-(u) \left(\sum_{v \in D^-(u)} |x_v|^2 \right) \right]$$

$$\leq k \sum_{u \in V} \left(\sum_{v \in D^-(u)} |x_v|^2 \right) = k \sum_{v \in V} d(u) |x_v|^2$$

$$\leq k^2 \sum_{u \in V} |x_v|^2 = k^2 \|x\|^2 \quad \Rightarrow \quad \|T_A x\| \leq k \|x\|$$

T_A is boundrd.

Definition 3.1.7

We consider another closed operator T_B, closely related to T_A, as follows.

For $x = \sum_{u \in V} x_u e_u$ in H, $\quad T_B x = \sum_{u \in V} \sum_{v \in D^+(u)} x_v e_u$

and $\quad \mathrm{Dom}(T_B) = \left\{ x \in H : \sum_u | \sum_{v \in D^+(u)} x_v |^2 < \infty \right\}$ (4)

Result 3.1.8 $\quad T_B \subseteq T_A^*$

For, let $x_i = \sum_{u \in V} x_u^i e_u \in \mathrm{Dom}(T_A)$ and $x_j = \sum_{u \in V} x_u^j e_u \in \mathrm{Dom}(T_B)$.

$$T_A x_i = \sum_r \left[\sum_{k, a_{kr}=1} a_{ik} \right] e_r \text{ and } T_B x_j = \sum_r \left[\sum_{s, a_{rs}=1} a_{js} \right] e_r \quad \text{by } (*)$$

$$\langle T_A x_i, x_j \rangle = \sum_r a_{jr} \left[\sum_{k, a_{kr}=1} a_{ik} \right] \text{ and } \tag{5}$$

$$\langle x_i, T_B x_j \rangle = \sum_r a_{ir} \left[\sum_{s, a_{rs}=1} a_{js} \right] = \sum_s a_{js} \left[\sum_{r, a_{rs}=1} a_{ir} \right] \tag{6}$$

by the local finiteness of G

Also, $\mathrm{Dom}(T_B) \subseteq \mathrm{Dom}(T_A^*)$ Hence $T_B \subseteq T_A^*$.

Proposition 3.1.9

Let T_A be defined on D as in (1)

(a) $\| T_A e_v\|^2 = \|T_A^* e_v\|^2 = d(v)$

(b) $\langle T_A e_u, e_v \rangle = \begin{cases} 1, & \text{if } v\in D^-(u) \\ -1, & \text{if } v\in D^+(u) \\ 0, & \text{otherwise} \end{cases}$ (c) $\langle T_A^* e_u, e_v \rangle = \begin{cases} 1, & \text{if } v\in D^+(u) \\ -1, & \text{if } v\in D^-(u) \\ 0, & \text{otherwise} \end{cases}$

(d) $\langle T_A^* T_A e_u, e_v \rangle = d^+(u,v) + d^-(u,v) - d(u^+, v^-) - d(u^-, v^+)$

Proof :

(a) $\| T_A e_v\|^2 = \langle T_A e_v, T_A e_v \rangle = \sum_{u\in V} |a_{uv}|^2 = d^+(v) + d^-(v) = d(v)$

The second part follows from $T_B \subseteq T_A^*$

(b) $\langle T_A e_u, e_v \rangle = \langle \sum_{w\in V} a_{wu} e_w, e_v \rangle = a_{vu} = \begin{cases} 1, & \text{if } v\in D^-(u) \\ -1, & \text{if } v\in D^+(u) \\ 0, & \text{otherwise} \end{cases}$

(c) $\langle T_A^* e_u, e_v \rangle = \langle e_u, T_A e_v \rangle = \langle e_u, \sum_{w\in V} a_{wv} e_w \rangle$

$$= a_{uv} = \begin{cases} 1, & \text{if } v\in D^+(u) \\ -1, & \text{if } v\in D^-(u) \\ 0, & \text{otherwise} \end{cases}$$

(d) $\langle T_A^* T_A e_u, e_v \rangle = \langle T_A e_u, T_A e_v \rangle = \langle \sum_r a_{ru} e_r, \sum_s a_{sv} e_s \rangle$

$$= \sum_t a_{tu} a_{tv}$$

$$a_{tu} a_{tv} = \begin{cases} 1, & \text{if } t\in (D^-(u)\cap D^-(v)) \cup (D^+(u)\cap D^+(v)) \\ -1, & \text{if } t\in (D^-(u)\cap D^+(v)) \cup (D^+(u)\cap D^-(v)) \\ 0 & \text{otherwise} \end{cases}$$

Hence, $\langle T_A^* T_A e_u, e_v \rangle = d^+(u, v) + d^-(u, v) - d(u^+, v^-) - d(u^-, v^+)$

Proposition 3.1.10

T_A is always normal.

Proof :

$$\langle T_A T_A^* e_u, e_v \rangle = \langle T_A^* e_u, T_A^* e_v \rangle$$

Also, $e_u, e_v \in \text{Dom}(T_B)$ and $T_B \subseteq T_A^*$

Hence, $\langle T_A T_A^* e_u, e_v \rangle = \langle T_B e_u, T_B e_v \rangle = \langle \sum_{w \in V} a_{uw} e_w, \sum_w a_{vw} e_w \rangle$

$$= \sum_{w \in V} a_{uw} a_{vw} = \langle T_A^* T_A e_u, e_v \rangle$$

from proposition 3.1 .9(d)

$\therefore T_A$ is normal.

Proposition 3.1.11

Let the adjacency operator T_A be defined on a digraph $D = (V,E)$.

The following are equivalent

(a) T_A is an isometry.

(b) $d(v) = 1$ for any vertex $v \in V$.

(c) The non-trivial components of G are $\bullet \!\!\rightarrow\!\! \bullet$.

Proof:

Assume (a) i.e. T_A is an isometry. Then $T_A^* T_A = I$

For any $v \in V$, $\langle T_A^* T_A e_v, e_v \rangle = \langle e_v, e_v \rangle = \|e_v\|^2 = 1$

Also $\langle T_A{}^* T_A e_v, e_v \rangle = \langle T_A e_v, T_A e_v \rangle = ||T_A e_v||^2$

$= d(v)$ from Result 3.1.9(a)

Hence, $d(v) = 1$, $\forall v \in V$.

Now suppose, $d(v) = 1$, $\forall v \in V$. i.e. $d^-(v) + d^+(v) = 1, \forall v \in V$

$\Rightarrow d^-(v) = 0, d^+(v) = 1$ or $d^-(v) = 1, d^+(v) = 0$

$\Rightarrow$ The nontrivial components of G are $\bullet\!\rightarrow\!\bullet$

Given (c), we have,

$d(v) = 1, \forall v \in V \Rightarrow ||T_A e_v||^2 = d(v) = ||e_v||, \forall v \in V$

$\Rightarrow T_A$ is an isometry.

Hence the proof.

Remark 3.1.12

Since T_A is normal, T_A has no non-unitary isometry.

Corollary 3.1.13

T_A is a projection if and only if D has no arcs.

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CHAPTER IV

THE INCIDENCE ALGEBRAS I(G, Z) AND I(G_∞, Z)

In this chapter, an algebraic object called Incidence Algebra is defined in association with a directed graph and its structure as well as subobjects are studied. This association leads to results which have a pure algebraic nature. The digraphs considered are the "Graphs" of posets $(V, \leq)$, finite or infinite and bounded or unbounded. This justifies the symbol "$(G, \leq)$" for the representation of $(V, \leq)$.

4.1 INCIDENCE ALGEBRA OF A POSET

Incidence Algebras were defined over fields, when introduced. Later a more general setup was considered over commutative rings with identity. The general setup as well as the basic concepts given in Spiegel & O'Donnel[45] are quoted below.

Definition 4.1.1

The incidence algebra I(X, R) of a locally finite partially ordered set (poset) $(X, \leq)$ over the commutative ring R with identity is defined as

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Some results given in this chapter are published in International Journal of Mathematics and Mathematical Sciences ,31:5,(2002) 301-305.

$I(X,R) = \{ f: X \times X \rightarrow R\ ;\ f(x,y) = 0 \text{ if } x \leq y \}$ with operations given by

$(f+g)(x,y) = f(x,y) + g(x,y)$

$$(f \cdot g)(x,y) = \sum_{x \leq z \leq y} f(x,z) \cdot g(z,y)$$

$(r.f)(x,y) = r \cdot f(x,y)$, for $f,g \in I(X,R)$, $r \in R$ and $x, y, z \in X$.

Proposition 4.1.2

Let X be a locally finite partially ordered set and R a commutative ring with identity. Then, I(X, R) is isomorphic to a subring of $M_{|X|}(R)$, where $|X|$ denotes the cardinality of X.

Lemma 4.1.3

A finite partially ordered set X can be labelled as $X=\{x_1,x_2,x_3,\ldots\}$ so that $x_i \leq x_j$ implies $i \leq j$.

Remark 4.1.4

Many countable partially ordered sets can be labelled as in lemma 4.1.3. Any such partially ordered set is necessarily locally finite. Applying Proposition 4.1.2 to these posets, we have the following result.

Proposition 4.1.5

Let $X = \{ x_i: i=1,2,3,\ldots\}$ be a partially ordered set. If X is so labelled that $x_i \leq x_j$ implies $i \leq j$, then there is an R-algebra isomorphism $\Psi: I(X,R) \rightarrow T_{|X|}(R)$.

Proposition 4.1.6

If X^1 is a partially ordered subset of X, then $I(X^1,R)$ is a subalgebra of I(X,R). This subalgebra consists of those functions $f \in I(X,R)$ such that $f(x,y) = 0$, if $(x,y) \notin X^1 \times X^1$.

Proposition 4.1.7

For an ideal S of R, I(S, R) is a subalgebra of I(X, R).

Proposition 4.1.8

Let E be an equivalence relation on the set of non-empty intervals of X. A function $f \in I(X, R)$ is an E – function, if $[x,y]\, E\, [u,v]$ implies $f(x,y) = f(u,v)$.

$I(X_E, R)$ is the collection of all E – functions.

Definition 4.1.9

Let E be an equivalence relation on the set of non-empty intervals of X. E is order – compatible, if $f.g \in I(X_E, R)$, whenever $f, g \in I(X_E, R)$.

Definition 4.1.10

Let x be an element of a partially ordered set $(X, \leq)$. The collection $I_x = \{ y \in X : y \leq x \}$ is the principal ideal of X, generated by the element x.

4.2 GRAPH OF A PARTIALLY ORDERED SET

Every partially ordered set $(V, \leq)$ has a graphical representation as a directed graph in which a pair of elements $u, v \in V$.

satisfying $u < v$ is represented by the ordered pair (u, v). In terms of notations, we deviate a little from the previous chapters and use $(G, \leq)$ for the digraph representing $(V, \leq)$.

Definition 4.2.1

The graph $(G, \leq)$ associated with a partially ordered set $(V, \leq)$ is defined as $(G, \leq) = (V, E)$ where $V = (V, \leq)$ and $E = \{(u, v);\ u<v \text{ in} (V, \leq)\}$.

Note 4.2.2

$(G, \leq)$ has no cycles and multiple arcs.

Remark 4.2.3

Proposition 2.1.7 and Lemma 4.1.3 have motivated the choice of the arcs in $(G, \leq)$.

Remark 4.2.4

$(V, \leq)$ may be a finite or infinite and locally finite partially ordered set with respective graphs $(G, \leq)$ or $(G_\infty, \leq)$.

Definition 4.2.5

An ideal $\mathcal{J}$ of $(G, \leq)$ is an induced subdigraph of G such that all directed paths with its terminal vertex in $\mathcal{J}$ are contained in $\mathcal{J}$.

Definition 4.2.6

If I_v is a principal ideal of $(V, \leq)$ then $\langle I_v \rangle$, the subdigraph induced by the vertices in I_v is the principal ideal generated by v in $(G, \leq)$. $\langle I_v \rangle$ is denoted by $\mathcal{J}_v$.

4.3 THE INCIDENCE ALGEBRA I(G, Z) OF (G, ≤)

Any partially ordered set (V, ≤) has a graphical representation (G, ≤) and any directed graph G = (V, E) without any cycles and multiple edges can be represented by a partially ordered set V_G. Hence, it is natural for (G, ≤) to have an incidence algebra, the properties of which depend on those of G. Here (G, ≤) represents the finite partially ordered set (V, ≤), where $V=\{v_1, v_2, \dots, v_n\}$. The ring R in definition 4.1.1 is replaced by Z, the ring of integers.

Notation 4.3.1

For u, v ∈ V, let $p_i(u, v)$ denote the number of directed paths of length 'i' from u to v.

For i = 1, 2, 3,..., n-1, define f_i, $f_i^* : V \times V \to Z$ by

$f_i(u,v) = p_i(u,v)$, $f_i^*(u,v) = - p_i(u,v)$

Definition 4.3.2

The incidence algebra I(G, Z) of (G, ≤) over the commutative ring Z with identity is defined by $I(G, Z) = \{ f_i, f_i^*: V \times V \to Z \}$, i = 0, 1, 2,..., n-1 with operations defined by

(i) $(g_i + g_j)(u,v) = g_i(u,v) + g_j(u,v)$

(ii) $(g_i \cdot g_j)(u,v) = \sum g_i(u,w) \cdot f_j(w,v)$

(iii) $(z g_i)(u,v) = z\, g_i(u,v)$ for $g_i, g_j, \in I(G, Z)$, $z \in Z$ and $u, v, w \in V$.

Remark 4.3.3

(i) f_1 is the graph analogue of $\chi \in I(X, R)$ [45] or ς-δ $\in A_K(P)$ [1].

(ii) The matrix $[f_1(v_i, v_j)]$ is the adjacency matrix of $(G, \leq)$ and

$f_1^k(v_i, v_j) = f_k(v_i, v_j)$.

(iii) For any interval $[u, v]$ of $(V, \leq)$ with length k, $f_1^k(u, v) = f_k(u, v) = 0$.

Hence for every $f \in I(G, Z)$ there is a constant $m \in Z$, such that $f_m(u, v) = 0$, for all $(u, v) \in V \times V$.

Definition 4.3.4

With each ideal $\mathcal{J}_V = \langle I_v \rangle$ of $(G, \leq)$ we associate an incidence algebra, $I(\mathcal{J}_V, Z) = \{ f \in I(G, Z) : f : I_v \times I_v \rightarrow Z \}$ such that $f(v_i, v_j) = 0$, for all $(v_i, v_j) \notin I_v \times I_v$

Remark 4.3.5

If $(H, \leq)$ is a subdigraph of $(G, \leq)$ then $I(H, Z)$ is a subalgebra of $I(G, Z)$. In particular $I(\mathcal{J}_V, Z)$ is a subalgebra of $I(G, Z)$. $I(\mathcal{J}_V, Z)$ is called the subalgebra generated by the vertex v.

Remark 4.3.6

As the graph analogue of proposition 4.1.7 , we have, if S is an ideal of Z then, $I(G, S) = \{ f \in I(G, Z) : f(u,v) \in S \}$ is a subalgebra of $I(G, Z)$.

Proposition 4.3.7

For each principal ideal $\mathcal{J}_V$ of $(G, \leq)$, $I(\mathcal{J}_V, Z)$ is an ideal of the ring $I(G, Z)$

Proof:

Let $\mathcal{J}_V$ be a proper principal ideal of $(G, \leq)$. Denote the elements of $I(G, Z)$ and $I(\mathcal{J}_V, Z)$ by f and f_1 respectively. For every $f_1 \in I(\mathcal{J}_V, Z)$ there is a unique $f \in I(G, Z)$ such that $f_1(u,w) = f(u,w)$, $\forall (u,w) \in I_v \times I_v$ and $f_1(u,w) = 0$, $\forall (u,w) \notin I_v \times I_v$.

Hence for any $(u,w) \in V \times V$, $f \in I(G, Z)$, $g_1 \in I(\mathcal{J}_V, Z)$

$$(f.g_1)(u,w) = \begin{cases} (f.g)(u,v), & \text{if } (u,v) \in I_v \times I_v \\ 0, & \text{otherwise.} \end{cases}$$

with similar values for $(g_1.f)(u, v)$ also.

Hence. $f.g_1$ and $g_1.f \in I(\mathcal{J}_V, Z)$. Therefore, $I(\mathcal{J}_V, Z)$ is an ideal of $I(G, Z)$.

Remark 4.3.8

I (G, Z) is isomorphic to a subring of the ring of upper triangular matrices over Z. In general every ideal of $M_n(Z)$ has the form $M_n(S)$ for some ideal S of Z.

Proposition 4.3.9

Every ideal of I (G, Z) has the form I ($\mathcal{J}_v$, Z) for some principal ideal $\mathcal{J}_v$ of (G, ≤).

Proof:

Let S be a proper ideal of the ring I (G, Z) . Then S = I (H, Z) for some subdigraph (H, ≤) of (G, ≤). For all f∈I (G, Z) and $g_1 \in S$, $f g_1 = g_1 f \in S$. Hence there is some $h_1 \in S$ such that, $f g_1 = h_1$, and $(f g_1)(u,v) = h_1(u,v) = p_k(u,v)$ for some k. i.e ∀ (u,v) ∈ V(H) × V(H), the number of directed paths of length k from u to v in H is the same as that in G. Hence for any v∈V(H), the subdigraph H contains all the directed paths terminating in v. Thus, H=$\mathcal{J}_v$ for some v ∈ V(G).

4.4 A SUBALGEBRA OF I(G, Z)

Here we define an equivalence relation $\mathcal{R}$ on the set of all directed paths in a partially ordered graph (G, ≤). The functions in I (G, Z) which are $\mathcal{R}$-compatible as well as the cases of $\mathcal{P}$ being $\leq$ - compatible are examined and a subalgebra is defined.

Notations 4.4.1

P_{uv} denotes a path directed from u to v in $(G, \leq)$. With any P_{uv}, a subset $\{u_0, u_1, u_2, \ldots, u_k\}$ of V is associated such that $u = u_0 < u_1 < u_2 < \ldots < u_k = v$, for some $k \in Z^+$. For any u, v $\in$ V let $\{P_{uv}\}$ be the set of all paths directed from u to v. Then $V(\{P_{uv}\}) = [u, v]$ of $(G, \leq)$.

Definition 4.4.2

Let $\mathcal{P}$ denote the set of all non-trivial paths in $(G, \leq)$ and $\mathcal{R}$ an equivalence relation on $\mathcal{P}$. $f \in I(G, Z)$ is called an $\mathcal{R}$-function if $\{P_{xy}\}\ \mathcal{R}\ \{P_{uv}\} \Rightarrow f(x,y) = f(u,v)$.

$I(G_{\mathcal{R}}, Z)$ is the set of all R-functions in $I(G, Z)$.

Definition 4.4.3

Let $\mathcal{R}$ be an equivalence relation on $\mathcal{P}$. $\mathcal{R}$ is $\leq$-compatible, if $f.g \in I(G_{\mathcal{R}}, Z)$, whenever $f, g \in I(G_{\mathcal{R}}, Z)$. $\mathcal{R}$ is an S-equivalence relation if for $\{P_{xy}\}\mathcal{R}\{P_{uv}\}$ there is a bijection map $\varphi:[x,y] \rightarrow [u,v]$ such that $\{P_{xy}\}\mathcal{R}\{P_{u\varphi(z)}\}$ and $\{P_{zy}\}\ \mathcal{R}\{P_{\varphi(z)v}\}$, $\forall z \in [x, y]$.

Proposition 4.4.4

Every equivalence relation $\mathcal{R}$ defined on $\mathcal{P}$ is an S-equivalence relation.

Proof:

Let $\mathcal{R}$ be an equivalence relation on $\mathcal{P}$.

Let $l(P_{xy})$ denote the length of P_{xy} and

$k = \max(\, l(P) : P \in \{P_{xy}\})$.

Assume $\{P_{xy}\}\ \mathcal{R}\ \{P_{uv}\}$.

For any $v \in V$, let $N_k(v) = \{w : P_{vw} \in \mathcal{P}$ and $l(P_{vw}) = k\}$.

Then $[x, y] = N_i(x)$, where each $N_i(x)$ is a partially ordered set. Also $N_i(x)$ is finite. Let $N_i(x) = \{w_{ij} : j \in$ some $\mathcal{A}\}$ such that, $w_{il} < w_{im}$ if $l < m$ and $w_{ij} < w_{lm}$ if $i < l$.

Then, for every path P_{xy} in $(G, \leq)$ there is a path P_{uv} isomorphic to it in such a way that, $[x,y]$ and $[u,v]$ are isomorphic partially ordered sets (of the same cardinality). Hence there is a bijection $\varphi : [x,y] \to [u,v]$ such that $\{P_{xy}\}\mathcal{R}\{P_{u\varphi(z)}\}$ and $\{P_{zy}\}\ \mathcal{R}\{P_{\varphi(z)v}\}$, $\forall z \in [x,y]$.

Hence, $\mathcal{R}$ is an S-equivalence relation.

Proposition 4.4.5

Let $\mathcal{R}$ be an equivalence relation on $\mathcal{P}$, the set of all non-trivial paths in $(G, \leq)$. Then $I(G_{\mathcal{R}}, Z)$ is a subalgebra of $I(G, Z)$.

Proof:

Let $\{P_{xy}\}\ \mathcal{R}\ \{P_{uv}\}$. By proposition 4.4.4, $\mathcal{R}$ is an

S-equivalence relation on $(G, \leq)$. Hence there is a bijection $\varphi : [x,y] \to [u,v]$ such that $\{P_{xy}\}\ \mathcal{R}\ \{P_{u\varphi(z)}\}$ and $\{P_{zy}\}\ \mathcal{R}\{P_{\varphi(z)v}\}$, $\forall z \in [x,y]$.

Let $f, g \in I(G_{\mathcal{R}}, Z) \Rightarrow \mathcal{P}\ f(x,y) = f(u,v)$ and $g(x,y) = g(u,v)$.

$$(f.g)(x,y) = \sum_{x \leq z \leq y} f(x,z).g(z,y) = \sum_{u \leq \phi(z) \leq v} f(u, \varphi(z)).g(\varphi(z),v)$$

$$= \sum_{u \leq w \pi v} f(u,w)\ g(w,v) = f.g(u,v)$$

$$\Rightarrow f.g \in I(G_{\mathcal{R}}, Z) \Rightarrow I(G_{\mathcal{R}}, Z) \text{ is a subalgebra of } I(G, Z).$$

Proposition 4.4.6

Let R be a relation defined on $\mathcal{P}$, the set of all non-trivial paths in $(G, \leq)$. For $[x,y]\ R\ [u,v]$, let $I_R(G, Z)$ denote the set of all $f \in I(G, Z)$ such that $f(x,y) = f(u,v)$. If $I_R(G, Z)$ is a subalgebra of $I(G,Z)$, then R is an equivalence relation on $\mathcal{P}$.

Proof:

Let $I_R(G, Z)$ be a subalgebra of $I(G,Z)$ and $f,g \in I_R(G, Z)$

Then, $f.g \in I_R(G, Z) \Rightarrow f.g(x,y) = f.g(u,v)$

$$\Rightarrow \sum_{x \leq z \leq y} f(x,z).g(z,y) = \sum_{u \leq w \pi v} f(u,w).g(w,v)$$

$$\Rightarrow \sum_{x \leq z} f(x,z).g(z,y) = \sum_{u \leq w} f(u,w).g(w,v) \qquad (1)$$

By the definition of (G, ≤), $\sum_{x \leq z} f(x,z).g(z,y)$ depends on the number of paths from x to some z<y. Hence (1) defines an equivalence relation $\mathcal{R}$ on $\mathcal{P}$ as follows.

$\{P_{xy}\}$ and $\{P_{uv}\}$ are related if there is a bijection $\psi:\{P_{xy}\} \rightarrow \{P_{uv}\}$ such that $l(P_{xy}) = l(\psi(P_{xy}))$ and [x,y] and [u,v] are isomorphic posets.

Then $I_R(G, Z) = I(G_{\mathcal{R}}, Z)$.

Proposition 4.4.7

If $\mathcal{R}$ is any equivalence relation on $\mathcal{P}$, then $\mathcal{R}$ is ≤- compatible.

Proof:

By proposition 4.4.4, $\mathcal{R}$ is an S-equivalence relation .

Hence, if $\{P_{xy}\}\ \mathcal{R}\{P_{uv}\}$, then there is a bijection $\varphi:[x,y] \rightarrow [u,v]$ such that

$\{P_{xy}\}\ \mathcal{R}\{P_{u\varphi(z)}\}$ and $\{P_{zy}\}\ \mathcal{R}\{P_{\varphi(z)v}\}$, $\forall z \in [x,y]$.

Let f,g $\in I(G_{\mathcal{R}}, Z)$. Then we have

$$f(x,y) = f(u,v) \qquad f(x,z) = f(u, \varphi(z)) \qquad f(z,y) = f(\varphi(z),v)$$

$$g(x,y) = g(u,v) \qquad g(x,z) = g(u, \varphi(z)) \qquad g(z,y) = g(\varphi(z),v)$$

$$\therefore (f.g)(x,y) = \sum_{x \leq z \leq y} f(x,z).g(z,y) = \sum_{x \leq z \leq y} f(x, \varphi(z)).g(\varphi(z),y)$$

φ is a bijective map from the poset [x,y] to the poset [u,v]. Hence we can define φ in such a way that for $x \leq z \leq y$, $u = \varphi(x) \leq \varphi(z) \leq \varphi(y) = v$.

Then $f.g \in I(G_{\mathcal{R}}, Z) \Rightarrow \mathcal{R}$ is ≤ -compatible.

4.5 THE INCIDENCE ALGEBRA $I(G_\infty, Z)$

The graph $(G_\infty, \leq)$ of an infinite partially ordered set $(V, \leq)$ may not be locally finite, even when $(V, \leq)$ is a locally finite partially ordered set. Also $(V, \leq)$ can be bounded or unbounded. By introducing weak partially ordered sets, the results given in section 4. 4 are extended to infinite weak partially ordered sets, bounded or unbounded. It is assumed that $(V, \leq)$ is countable.

Definition 4.5.1

A locally finite partially ordered set $(V, \leq)$ is weak, if only finitely many chains intersect at every element $v \in V$.

Definition 4.5.2

Let $(V, \leq)$ be a locally finite weak partially ordered set and $(G_\infty, \leq)$ the graph representing $(V, \leq)$. The incidence algebra $I(G_\infty, Z)$ of (G_∞, Z) over the ring Z of integers is given by $I(G_\infty, Z) = \{f_i, f_i^* : V \times V \rightarrow Z\}$ where, f_i, f_i^* are as given in notation 4.3.1.and satisfy the operations in definition 4.3.2.

Note 4.5.3

Here also, each $f \in I(G, Z)$ has a matrix representation [f] where $[f]_{u,v} = f(u, v)$ and $[f]$ is both row and column finite. If $M_\infty(Z)$ denotes the ring of row and column finite matrices over Z, then $I(G_\infty, Z)$ is isomorphic to a subring of $M_\infty(Z)$.

Proposition 4.5.4

Let $(V, \le)$ be a locally finite weak partially orderd set satisfying (i) the set S of its atoms is non-empty and finite and (ii) no infinite chain in $(V, \le)$ has a maximal element in $(V, \le)$, then $I(G_\infty, Z)$ is isomorphic to subring of the ring of upper triangular matrices over Z.

Proof:

The graph $(G_\infty, \le)$ is locally finite. By proposition 2.2.3, $f(v_r, v_s) \ge 0$ for $r \le s$, giving every f a representation as an upper triangular matrix over Z. Hence the result follows from definition 4.5.2.

Definition 4.5.5

In a bounded, locally finite weak partially ordered set $(V, \le)$ the principal ideal generated by $v \in V$ is defined as $I_v = \{u \in V : u \le v\}$. The corresponding principal ideal of $(G_\infty, \le)$ is given by $\mathcal{J}_V = \langle I_v \rangle$.

Proposition 4.5.6

Let $(V, \le)$ be a bounded locally finite, weak partially ordered set and $\mathcal{J}_{\nu}$ a principal ideal of $(G_{\infty}, \leq)$. Then

(a) $I(\mathcal{J}_{\nu}, Z)$ is a subalgebra of $I(G_{\infty}, Z)$.

(b) $I(\mathcal{J}_{\nu}, Z)$ is an ideal of $I(G_{\infty}, Z)$.

The proof is similar to that in proposition 4.3.7.

Proposition 4.5.7 [45]

Let $(V, \leq)$ be an unbounded partially ordered set. Then V contains a partially ordered set isomorphic to Z^{+}, Z^{-} or $\cup C_n$ where C_n denotes a chain of length n.

Remark 4.5.8

When $(V, \leq)$ is unbounded, the principal ideals of $(V, \leq)$ as well as $(G_{\infty}, \leq)$ are not well-defined. Also Proposition 2.2.3 is not true for unbounded partially ordered sets, in general.

But, there are unbounded locally finite partially ordered sets which satisfy Proposition 2.2.3. As an example, we have Z^{+} under the usual ordering. For such partially ordered sets we have the following result.

Let $(V, \leq)$ be a locally finite unbounded partially ordered set such that $I(G_{\infty}, Z)$ is isomorphic to a subring of the ring of upper triangular matrices. Then

(i) principal ideals of $(G_{\infty}, \leq)$ are defined as $\mathcal{J}_{\nu} = \langle I_{\nu} \rangle$ where

$I_v = \{u \in V : u \leq v\}$

(ii) for each $\mathcal{I}_v$, $I(\mathcal{I}_v, Z)$ is a subalgebra of $I(G_\infty, Z)$.

(iii) for every $\mathcal{I}_v$, $I(\mathcal{I}_v, Z)$ is an ideal of $I(G_\infty, Z)$.

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CHAPTER V

THE ARC LAPLACIAN Q_E

The Laplacian matrix $Q = BB'$, where B represents the incidence matrix for some orientation of a finite or locally finite infinite, graph is well- known and has been subjected to considerable investigation. Here the matrix $B'B$ is considered as a linear operator on $l^2(E)$. $B'B$ is called the Arc Laplacian of G and is denoted by Q_E. Bapat, Kulkarni and Grossman [8] have considered a similar matrix for a finite mixed graph G, which they call the edge version of the Laplacian matrix and denote by K.

5.1 THE ARC ADJACENCY OPERATOR $\mathcal{A}_E$

This section contains the preliminary definitions and some results on the arc adjacency operator $\mathcal{A}_E$, an operator which is closely related to Q_E.

Definition 5.1.1

For any finite or locally finite and countably infinite graph $G = (V,E)$, we define the incidence matrix B as given in Biggs [10].

Let D be the directed graph formed by fixing an arbitrary orientation to the edges of G. For an arc e of D, u(e) denotes its tail and v(e) its head. In other words, the arc e emanates from u(e) and terminates in v(e). Also u(e) and v(e) are referred to as, respectively, the positive and negative ends of the arc e.

The incidence matrix $B = [b_{ij}]$ where $b_{ij} = \pm 1$, according as $u(e_j) = v_i$ or $v(e_j) = v_i$ and $b_{ij} = 0$ otherwise.

Remark 5.1.2

B defines an operator $B : l^2(E) \to l^2(V)$ such that for $x \in l^2(E)$, $(Bx)_u = \sum_{u(e)=u} x_e - \sum_{v(e)=u} x_e$.

Note 5.1.3

By an n-path we mean a path of length n, which is directed in D and undirected in G.

Definition 5.1.4

Two arcs e_i and e_j of D are adjacent if they form a 2-path. The arcs e_i and e_j are weakly adjacent - w-adjacent for short - if they are not adjacent in D, but form a 2-path in G.

Definition 5.1.5

With every arc e of D we associate two sets S (e) and W(e) as follows.

$S(e) = \{f \in E(D) : u(f) = v(e) \text{ or } v(f) = u(e)\}$

$W(e) = \{f \in E(D) : u(f) = u(e) \text{ or } v(f) = v(e)\}$

S (e) and W(e) are disjoint for all $e \in E(D)$.

Definition 5.1.6

The arc adjacency matrix $\mathcal{A}_E$ of a directed graph D is defined by

$$[\mathcal{A}_E]_{ij} = \begin{cases} 1, & \text{if } e_i \text{ and } e_j \text{ are adjacent} \\ -1, & \text{if } e_i \text{ and } e_j \text{ are w-adjacent} \\ 0, & \text{otherwise.} \end{cases}$$

$\mathcal{A}_E$ is a real symmetric matrix, indexed by arcs of D, whose columns /rows give the vectors in $l^2(E)$.

Definition 5.1.7 The operator $\mathcal{A}_E$

Let $\{\beta_e : e \in E\}$ be the standard basis of $l^2(E)$. Any $x \in l^2(E)$ is represented as $x = \sum_{e \in E} x_e \beta_e$, where $x_e \in \{-1,0,1\}$.

The arc adjacency operator $\mathcal{A}_E$ is defined on $l^2(E)$ by

$$\mathcal{A}_E(x) = \sum_{e \in E} \{ \sum_{f \in S(e)} x_f - \sum_{f \in W(e)} x_f \} \beta_e ,$$

Remark 5.1.8

For a finite digraph $\mathcal{A}_E$ represents a bounded, self-adjoint operator on $l^2(E)$

5.1.9 **Computation of** $\|\mathcal{A}_E\|$

For $x \in l^2(E)$, $\langle \mathcal{A}_E(x). \mathcal{A}_E(x) \rangle = \sum_{e \in E} \{ \sum_{f \in S(e)} x_f - \sum_{f \in W(e)} x_f \}^2$

Also $|\mathcal{A}_E(x)_e| \leq 2\,[\Delta(G)-1]^2$, where $\Delta(G) = \max d(v)$.

Hence, $\|\mathcal{A}_E(x)\|^2 \leq 4\,[\Delta(G)-1]^2 \sum_{e\in E} |x_e|^2 = 4[\Delta(G)-1]^2 \|x\|^2$

And $\|A_E\| \leq 2\,[\Delta(G)-1]$, which is finite since G is locally finite.

Proposition 5.1.10

If D has only finitely many arcs, then $\mathcal{A}_E$ is a compact linear operator on $l^2(E)$.

Proof:

If D has only finitely many arcs then the matrix $\mathcal{A}_E = [a_{ij}]$ is finite. Then we have a constant k, $0 < k < \infty$, such that $\sum_{i,j} |a_{ij}|^2 \leq k$ and $\|\mathcal{A}_E\| = k^{1/2}$. Hence $\mathcal{A}_E$ is a compact linear operator on $l^2(E)$.

Proposition 5.1.11

If $d(v) \leq k$, then $\|\mathcal{A}_E\| \leq 2(k-1)$

Proof:

E is countable. Let $E = \{e_1, e_2, e_3, \ldots\}$

For $x = \sum_i x_i \beta_i$ in $l^2(E)$, with x_i denoting $(x)_{e_i}$ and β_i denoting β_e

$$|\mathcal{A}_E(x)_i| = \left|\sum_{j=1}^{\infty} a_{ij}\right| x_j$$

$$\leq \sum_{j=1}^{\infty} |a_{ij}||x_j| = \sum_{j=1}^{\infty} |a_{ij}|^{1/2} |x_j| |a_{ij}|^{1/2}$$

$$\leq \left\{ \sum_{j=1}^{\infty} | a_{ij} || x_j |^2 \right\}^{1/2} \left\{ \sum_{j=1}^{\infty} | a_{ij} | \right\}^{1/2} \qquad (1)$$

$$\sum_{j=1}^{\infty} | a_{ij} | = d(u(e_i)) + d(v(e_i)) - 2$$

Hence, $\sup_i \sum_{j=1}^{\infty} | a_{ij} | \leq 2k-2 = 2(k-1)$

From (1) $\quad |\mathcal{A}_E(x)_i|^2 \leq 2(k-1) \left\{ \sum_{j=1}^{\infty} | a_{ij} || x_j |^2 \right\}$

$$||\mathcal{A}_E(x)||^2 = \sum_{i=1}^{\infty} |\mathcal{A}_E(x)|^2$$

$$\leq 2(k-1) \sum_{j=1}^{\infty} \sum_{i=1}^{\infty} | a_{ij} || x_{ij} |^2$$

$$\leq 2(k-1)\, 2(k-1) \sum_{j=1}^{\infty} | x_{ij} |^2$$

$$\leq 4(k-1)^2 \|x\|^2$$

$||\mathcal{A}_E|| \leq 2(k-1).$

Proposition 5.1.12

If $d(v) \leq k$, then $\|B\| \leq \sqrt{2}\,k$.

Proof :

Every column of B contains exactly two non-zero entries, each of absolute value one. The proof is similar to that in Proposition 5.1.11.

5. 2 THE ARC LAPLACIAN Q_E.

We recall the definition and some results on the Laplacian matrix.

Definition. 5.2.1 [10]

Let B denote the incidence matrix for some orientation of a graph G. The Laplacian matrix Q of G is defined as $Q = BB^t$, where B^t denotes the transpose of B. Q acts on $l^2(V)$ as a linear operator.

Proposition 5.2.2 [10]

Let A be the adjacency matrix of G and Δ the diagonal matrix whose i^{th} diagonal entry is $d(v_i)$. Then $Q = \Delta - A$.

Remark 5.2.3

This result gives a simple relationship between Q and A which is true for finite as well as locally finite infinite graphs, connected or disconnected.

Definition 5.2.4

Let D be an orientation of a finite or countably infinite, locally finite graph G and B the incidence matrix of D with respect to this orientation. The arc Laplacian matrix Q_E is defined by $Q_E = B^tB$.

Note 5.2.5

Q_E is indexed by the edges of G whereas Q is indexed by the vertices.

Proposition 5.2.6

$Q_E = 2\, I_{|E|} - \mathcal{A}_E$, where $I_{|E|}$ is the identity matrix of order $|E|$.

Proof:

From the computation of B^tB, it follows that

$$[Q_E]_{ij} = \begin{cases} 2, & \text{if } i=j \\ 1, & \text{if } e_i \text{ and } e_j \text{ are w-adjacent} \\ -1, & \text{if } e_i \text{ and } e_j \text{ are adjacent} \\ 0, & \text{otherwise} \end{cases}$$

Remark 5.2.7

Since Q and Q_E act on different domains we can not claim any equality between them as operators. But regarding their representation as matrices, we have the following result.

Proposition 5.2.8

Let Q and Q_E denote the Laplacian and Arc Laplacian matrix, respectively, of a locally finite and countably infinite graph G. The matrices Q and Q_E are equal if and only if the orientation D of G considered is a disjoint union of directed cycles.

Proof:

If the matrices Q and Q_E are equal, then (i) $d(v) = 2$ for every vertex v of G (ii) no two arcs of D have a common head or tail (iii) if e_i and e_j are adjacent arcs in D then either e_i or e_j joins the vertices v_i and v_j. Hence every component of D is a directed cycle. Moreover, no two cycles of D have a common vertex, for if there is one such vertex v, then $d(v) \geq 3$.

Hence, D is a disjoint union of directed cycles.

Proposition 5.2.9

If μ is an eigen-value of $\mathcal{A}_E$, then $\mu \leq 2$.

Proof.

From Proposition 5.2.6, $\mathcal{A}_E = 2 I_{|E|} - B^tB$.

Also, the eigen values of B^tB are non- negative, since the matrix B^tB is non-negative definite Hence, eigen values of $\mathcal{A}_E$ are atmost two.

Proposition 5.2.10 [10]

If λ is an eigen value of the line graph L(G) of G, then $\lambda \geq -2$.

Proposition 5.2.11 [20]

A connected graph G is isomorphic to its line graph L (G) if and only if G is a cycle.

Proposition 5.2.12

Let A (G), $A_L(G)$ and $\mathcal{A}_E$ (D) denote the adjacency matrix of G, the adjacency matrix of the line graph L(G) of G and the arc adjacency matrix for some orientation D of G, respectively. The eigen values of A, A_L and $\mathcal{A}_E$ coincide if and only if G is a cycle.

Proof:-

Let G be the cycle C_n on n vertices. By Proposition 5.2.11, G and L(G) are isomorphic. Then A(G) = A_L and hence A(G) and A_L have the same characteristic roots. If the orientation D of G is a directed cycle D_n then A(G) and $\mathcal{A}_E(D)$ are equal.

If not , let D* be an orientation of G which is different from the directed cycle D_n. The change in the direction of an arc of D_n, produces the matrix $\mathcal{A}_E(D^*)$ with a row and column having entries opposite to those in $\mathcal{A}_E(D_n)$, whereas det $(\lambda I - \mathcal{A}_E(D))$ and det $(\lambda I - \mathcal{A}_E(D_n))$ have the same polynomial expansion. Hence the proof.

The converse follows from the same results.

Corollary 5.2.13 [10]

Spec $(C_n) \subseteq [-2,2]$.

5.3 EIGENVALUES OF Q_E

As matrices Q and Q_E are transposes of each other, which may reflect in their properties also. Together with the results on Q_E, the corresponding results on Q are also included in this section for a comparative study.

Proposition 5.3.1 [10]

Let $\mu_0 \leq \mu_1 \leq \ldots\ldots \leq \mu_{n-1}$ be the eigen values of the Laplacian matrix Q of a finite graph G on vertices. Then

(a) $\mu_0 = 0$ with eigen vector [1,1,......1]

(b) If G is connected, then $\mu_i > 0$

(c) If G is regular of degree k , then $\mu_i = k - \lambda_i$, where λ_i are the ordinary eigen values of G

Proposition 5.3.2.

If λ_i, $1 \leq i \leq m$ are the eigen values of the arc adjacency matrix $\mathcal{A}_E(D)$ of a finite digraph D, then $\mu_i = 2 - \lambda_i$ are the eigen values of Q_E.

This result follows from $Q_E = 2 I_{|E|} - \mathcal{A}_E$.

Corollary 5.3.3.

$$\lambda_i \leq 2 \quad \Rightarrow \quad \mu_i \geq 0 .$$

Corollary 5.3.4

If $\mathrm{Spec}_{Q_E}(C_n)$ denotes the arc Laplacian spectrum of the cycle C_n on n vertices, then

$$\mathrm{Spec}_{Q_E}(C_n) = \begin{pmatrix} 0 & 2(1-\cos 2\pi/n) \ldots\ldots\ldots\ldots 2[1-\cos(n-1)\pi/n] \\ 1 & 2 \qquad\qquad\qquad\qquad 2 \end{pmatrix} \quad \text{if n is odd}$$

$$\mathrm{Spec}_{Q_E}(C_n) = \begin{pmatrix} 0 & 2(1-\cos 2\pi/n) \ldots\ldots\ldots\ldots 2[1-\cos(n-2)\pi/n] & -4 \\ 1 & 2 \qquad\qquad\qquad\qquad 2 & 1 \end{pmatrix} \quad \text{if n is even}$$

Proposition 5.3.5

Let $\{ \mu_i : i \in I \}$ denote the eigen values of the arc Laplacian Q_E of a locally finite, connected graph G. $\mu_i = 0$ if and only if G contains a cycle. The multiplicity of zero as an eigen value of Q_E is the dimension of the cycle subspace of G.

Proof:

The cycle subspace $\mathcal{C}$ is the kernel of the incidence mapping. Hence for any (column) vector x in $\mathcal{C}$, we have $B_x = 0 \Rightarrow B^t Bx = Q_E x = 0$ $\Rightarrow$ zero is an eigenvalue of Q_E.

Conversely,

let H be any connected subgraph G, such that H does not contain any cycles and let zero be an eigen value of Q_E.

Define $x_H : E(G) \rightarrow R$ by $x_H(e) = \chi_H(e)$, where χ_H denotes the characterestic function. Let B_i denote the i^{th} row of the incidence matrix B.

$(B\, x_H)_i = \langle Bi, x'_H \rangle = d^+_H(v_i) - d^-_H(v_i)$

$(B\, x_H)_i = 0 \Rightarrow d^+_H(v_i) = d^-_H(v_i)$

If $d^+_H(v_i) = d^-_H(v_i) = 0$, then $v_i \notin V(H)$,since H is connected..

Otherwise, $d^+_H(v_i) = d^-_H(v_i)$ for all $v_i \in V(H)$, giving an Eulerian subgraph H of G. if H is finite. Hence we have a partition of E (H) into edge-disjoint union of cycles. contradicting the choice of H.

Hence, $\langle Bi, x'_H \rangle \neq 0$, for at least one i.

$\Rightarrow B^t B x_H \neq 0 \quad \Rightarrow Q_E x_H \neq 0$

$Q_E x = 0 \Rightarrow$ x is a cycle vector.

If H is infinite and $d^+(v_i) = d^-(v_i)$ for all $v_i \in V(H)$, then H is either an arc-disjoint union of cycles or an arc-disjoint union of doubly infinite directed paths or both, where the first case is quite similar to the finite case given above.

In the second case, let H_1 be any doubly infinite directed path in H. We have a one-to-one correspondence between the vertex set V (H_1) of H_1 and the set E (H_1) of its edges such that every vertex is an end vertex of the edge corresponding to it [28]. For any i, B_i has the entry +1 in the (i-1) column and -1 in the i column with zeros elsewhere and $x_{H_1}(e_i)$ has a non-zero entry in the i^{th} column only.

Hence, $(Bx_{H_1})_i = \langle Bi, x'_H \rangle = -1 \neq 0$.

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