A characterization of fuzzy trees

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Abstract

In this paper some properties of fuzzy bridges and fuzzy cutnodes are studied. A characterization of fuzzy trees is obtained using these concepts. © 1999 Elsevier Science Inc. All rights reserved.

1. Introduction

The theory of fuzzy sets finds its origin in the pioneering paper of Zadeh [11]. Since then this philosophy of "gray mathematics" [6] had tremendous impact on logic, information theory, etc. and finds its applications in many branches of engineering and technology [5].

A fuzzy subset [9] of a nonempty set $S$ is a mapping $\sigma: S \to [0, 1]$. A fuzzy relation on $S$ is a fuzzy subset of $S \times S$. If $\mu$ and $\nu$ are fuzzy relations, then $\mu \vee (u, w) = \operatorname{Sup}\{\mu(u, v) \Lambda \nu(v, w) : v \in S\}$ and $\mu^\wedge (u, v) = \operatorname{Sup}\{\mu(u, u) \Lambda \mu (u_1, u_2) \Lambda \cdots \Lambda \mu (u_n, u) : u_1, u_2, \ldots, u_n \in S\}$, where $\Lambda$ stands for minimum.

The theory of fuzzy graphs was independently developed by Rosenfeld [9] and Yeh and Bang [10] in 1975. A fuzzy graph is a pair $G: (\sigma, \mu)$, where $\sigma$ is a fuzzy subset of $S$ and $\mu$ is a fuzzy relation on $S$ such that $\mu(u, v) \leq \sigma(u) \Lambda \sigma(v)$ for all $u, v$ in $S$. A fuzzy graph $H: (\tau, \pi)$ is called a fuzzy subgraph of $G: (\sigma, \mu)$ if $\tau(u) \leq \sigma(u)$ and $\pi(u, v) \leq \mu(u, v)$ for all $u, v$. Further, $H$ is a spanning subgraph if $\tau(u) = \sigma(u)$ for all $u$. A path $p$ of length $n$ is a sequence of distinct nodes $u_0, u_1, u_2, \ldots, u_n$ such that $\mu(u_{i-1}, u_i) > 0$, $i = 1, 2, 3, \ldots, n$ and the weight of the
weakest arc is defined as its strength. If \( u_0 = u_n \) and \( n \geq 3 \) then \( \rho \) is called a cycle. Also, \( \sup \{ \mu^k(u, v) : k = 1, 2, 3, \ldots \} \) gives the strength of connectedness between any two nodes \( u \) and \( v \), denoted by \( \mu^2(u, v) \). A fuzzy graph \( G : (\sigma, \mu) \) is connected if \( \mu^2(u, v) > 0 \) for all \( u, v \).

Recently, automorphisms of fuzzy graphs [3], fuzzy interval graphs [4], fuzzy line graphs [7], cycles and cocycles of fuzzy graphs [8], etc., have also been studied.

In this paper some properties of fuzzy bridges and fuzzy cutnodes are studied and a characterization of fuzzy trees is obtained using them.

Throughout, we assume that \( S \) is finite, \( \mu \) is reflexive and symmetric [9]. In all the examples \( \sigma \) can be chosen in any manner satisfying the definition of a fuzzy graph. Also, we denote the underlying crisp graph by \( G' : (\sigma', \mu') \), where \( \sigma' = \{ u \in S : \sigma(u) > 0 \} \) and \( \mu' = \{ (u, v) \in S \times S : \mu(u, v) > 0 \} \).

2. Fuzzy bridges and fuzzy cutnodes

**Definition 1 [9].** An arc \((u, v)\) is a fuzzy bridge of \( G : (\sigma, \mu) \) if deletion of \((u, v)\) reduces the strength of connectedness between some pair of nodes.

Equivalently, \((u, v)\) is a fuzzy bridge if and only if there exist \( x, y \) such that \((u, v)\) is an arc of every strongest \( x-y \) path.

**Definition 2 [9].** A node is a fuzzy cutnode of \( G : (\sigma, \mu) \) if removal of it reduces the strength of connectedness between some other pair of nodes.

Equivalently, \( w \) is a fuzzy cutnode if and only if there exist \( u, v \) distinct from \( w \) such that \( w \) is on every strongest \( u-v \) path.

**Theorem 1** [9]. The following statements are equivalent.
1. \((u, v)\) is a fuzzy bridge.
2. \((u, v)\) is not a weakest arc of any cycle.

**Remark 1.** Let \( G : (\sigma, \mu) \) be a fuzzy graph such that \( G^* : (\sigma^*, \mu^*) \) is a cycle and let \( t = \min \{ \mu(u, v) : \mu(u, v) > 0 \} \). Then all arcs \((u, v)\) such that \( \mu(u, v) > t \) are fuzzy bridges of \( G \).

**Theorem 2.** Let \( G : (\sigma, \mu) \) be fuzzy graph such that \( G^* : (\sigma^*, \mu^*) \) is a cycle. Then a node is a fuzzy cutnode of \( G \) if and only if it is a common node of two fuzzy bridges.

**Proof.** Let \( w \) be a fuzzy cutnode of \( G \). Then there exist \( u \) and \( v \), other than \( w \), such that \( w \) is on every strongest \( u-v \) path. Now \( G^* : (\sigma^*, \mu^*) \) being a cycle, there exits only one strongest \( u-v \) path containing \( w \) and by Remark 1, all its arcs are fuzzy bridges. Thus \( w \) is a common node of two fuzzy bridges. Conversely, let
Let \( w \) be a common node of two fuzzy bridges \((u, w)\) and \((w, r)\). Then both \((u, w)\) and \((w, r)\) are not the weakest arcs of \( G \) (Theorem 1). Also the path from \( u \) to \( v \) not containing the arcs \((u, w)\) and \((w, r)\) has strength less than \( \mu(u, w) \land \mu(w, r) \). Thus the strongest \( u-v \) path is the path \( u, w, r \) and \( \mu^*(u, r) = \mu(u, w) \land \mu(w, r) \). Hence \( w \) is a fuzzy cutnode.

**Theorem 3.** If \( w \) is a common node of at least two fuzzy bridges, then \( w \) is a fuzzy cutnode.

**Proof.** Let \((u_1, w)\) and \((w, u_2)\) be two fuzzy bridges. Then there exist some \( u, r \) such that \((u_1, r)\) is on every strongest \( u-v \) path. If \( w \) is distinct from \( u \) and \( r \) it follows that \( w \) is a fuzzy cutnode. Next, suppose one of \( v, u \) is \( w \) so that \((u_1, w)\) is on every strongest \( u-v \) path or \((w, u_2)\) is on every strongest \( w-v \) path. If possible let \( w \) be not a fuzzy cutnode. Then between every two nodes there exist, at least one strongest path not containing \( w \). In particular, there exist at least one strongest path \( \rho \), joining \( u_1 \) and \( u_2 \), not containing \( w \). This path together with \((u_1, w)\) and \((w, u_2)\) forms a cycle.

**Case 1.** If \((u_1, w, u_2)\) is not a strongest path, then clearly one of \((u_1, w), (w, u_2)\) or both become the weakest arcs of the cycle which contradicts that \((u_1, w)\) and \((w, u_2)\) are fuzzy bridges.

**Case 2.** If \((u_1, w, u_2)\) is also a strongest path joining \( u_1 \) to \( u_2 \), then \( \mu^*(u_1, u_2) = \mu(u_1, w) \land \mu(w, u_2) \), the strength of \( \rho \). Thus arcs of \( \rho \) are at least as strong as \( \mu(u_1, w) \) and \( \mu(w, u_2) \) which implies that \((u_1, w), (w, u_2)\) or both are the weakest arcs of the cycle, which again is a contradiction.

**Remark 2.** The condition in the above theorem is not necessary. In Fig. 1, \( w \) is a fuzzy cutnode; \((u, w)\) and \((v, w)\) are the only fuzzy bridges.

**Remark 3.** In the following fuzzy graph (Fig. 2), \((u_1, u_2)\) and \((u_3, u_4)\) are the fuzzy bridges and no node is a fuzzy cutnode. This is a significant difference from the crisp graph theory.

![Fig. 1](image.png)
Theorem 4. If \((u, v)\) is a fuzzy bridge, then \(\mu^3(u, v) = \mu(u, v)\).

Proof. Suppose that \((u, v)\) is fuzzy bridge and that \(\mu^3(u, v)\) exceeds \(\mu(u, v)\). Then there exists a strongest \(u-v\) path with strength greater than \(\mu(u, v)\) and all arcs of this strongest path have strength greater than \(\mu(u, v)\). Now, this path together with the arc \((u, v)\) forms a cycle in which \((u, v)\) is the weakest arc, contradicting that \((u, v)\) is a fuzzy bridge. \(\square\)

Remark 4. The converse of the above theorem is not true. The condition for the converse to be true is discussed in Theorem 9.

3. Fuzzy trees

Definition 3 [9]. A connected fuzzy graph \(G : (\sigma, \mu)\) is a fuzzy tree if it has a fuzzy spanning subgraph \(F : (\sigma, \nu)\), which is a tree, where for all arcs \((u, v)\) not in \(F\), \(\nu(u, v) < \nu^3(u, v)\).

Equivalently, there is a path in \(F\) between \(u\) and \(v\) whose strength exceeds \(\mu(u, v)\).

Lemma 1 [9]. If \((\tau, v)\) is a fuzzy subgraph of \((\sigma, \mu)\), then for all \(u, v\), \(\nu^3(u, v) \leq \mu^3(u, v)\).

Theorem 5. If \(G : (\sigma, \mu)\) is a fuzzy tree and \(G' : (\sigma', \mu')\) is not a tree, then there exists at least one arc \((u, v)\) in \(G'\) for which \(\mu(u, v) < \mu^3(u, v)\).

Proof. If \(G\) is a fuzzy tree, then by definition there exists a fuzzy spanning subgraph \(F : (\sigma, \nu)\), which is a tree and \(\nu(u, v) < \nu^3(u, v)\) for all arcs \((u, v)\) not in \(F\). Also \(\nu^3(u, v) \leq \nu^3(u, v)\) by Lemma 1. Thus \(\mu(u, v) < \mu^3(u, v)\) for all \((u, v)\) not in \(F\) and by hypothesis there exist at least one arc \((u, v)\) not in \(F\), which completes the proof. \(\square\)

[Diagram of a fuzzy tree with labeled vertices and arcs, indicating the structure and relationships.]
Definition 4 [3]. A complete fuzzy graph is a fuzzy graph $G: (\sigma, \mu)$ such that $\mu(u, v) = \sigma(u) \wedge \sigma(v)$ for all $u$ and $v$.

Lemma 2 [3]. If $G$ is a complete fuzzy graph, then $\mu^2(u, v) = \mu(u, v)$.

Lemma 3 [3]. A complete fuzzy graph has no fuzzy cutnodes.

Remark 5. The converse of lemma 2 is not true (Fig. 3). Also, a complete fuzzy graph may have a fuzzy bridge (Fig. 4).

Theorem 6. If $G: (\sigma, \mu)$ is a fuzzy tree, then $G$ is not complete.

Proof: If possible let $G$ be a complete fuzzy graph. Then $\mu^2(u, v) = \mu(u, v)$ for all $u, v$ [lemma 2]. Now $G$ being a fuzzy tree, $\mu(u, v) < v^2(u, v)$ for all $(u, v)$ not in $F$. Thus $\mu^2(u, v) < v^2(u, v)$ contradicting lemma 1. □

Theorem 7 [9]. If $G$ is a fuzzy tree, then arcs of $F$ are the fuzzy bridges of $G$.

Theorem 8. If $G$ is a fuzzy tree, then internal nodes of $F$ are the fuzzy cutnodes of $G$.

Proof. Let $w$ be any node in $G$ which is not an end node of $F$. Then by Theorem 7, it is the common node of at least two arcs in $F$ which are fuzzy bridges of $G$ and by Theorem 3, $w$ is a fuzzy cutnode. Also, if $w$ is an end node of $F$, then $w$ is not a fuzzy cutnode; for, if so, there exist $u, v$ distinct from $w$ such that $w$ is on every strongest $u, v$ path and one such path certainly lies in $F$. But $w$ being an end node of $F$, this is not possible. □

Corollary: A fuzzy cutnode of a fuzzy tree is the common node of at least two fuzzy bridges.
4. Main result

**Theorem 9.** $G : (\sigma, \mu)$ is a fuzzy tree if and only if the following are equivalent.

1. $(u, v)$ is a fuzzy bridge.
2. $\mu^2(u, v) = \mu(u, v)$.

**Proof.** Let $G : (\sigma, \mu)$ be a fuzzy tree and let $(u, v)$ be a fuzzy bridge. Then $\mu^2(u, v) = \mu(u, v)$ (Theorem 4). Now, let $(u, v)$ be an arc in $G$ such that $\mu^2(u, v) = \mu(u, v)$. If $G$ is a tree, then clearly $(u, v)$ is a fuzzy bridge; otherwise, it follows from theorem 5 that $(u, v)$ is in $F$ and $(u, v)$ is a fuzzy bridge (Theorem 7).

Conversely, assume that $(1) \iff (2)$. Construct a maximum spanning tree $T : (\sigma, \nu)$ for $G$ [2]. If $(u, v)$ is in $T$, by an algorithm in [2], $\nu^2(u, v) = \mu(u, v)$ and hence $(u, v)$ is a fuzzy bridge. Now, these are the only fuzzy bridges of $G$; for, if possible let $(u', v')$ be a fuzzy bridge of $G$ which is not in $T$. Consider a cycle $C$ consisting of $(u', v')$ and the unique $u'-v'$ path in $T$. Now arcs of this $u'-v'$ path being fuzzy bridges they are not weakest arcs of $C$ and hence $(u', v')$ must be the weakest arc of $C$ and hence cannot be a fuzzy bridge (Theorem 1).

Moreover, for all arcs $(u', v')$ not in $T$, we have $\mu(u', v') < v^2(u', v')$: for, if possible let $\mu(u', v') \geq v^2(u', v')$. But $\mu(u', v') < \nu^2(u', v')$ (strict inequality holds, since $(u', v')$ is not a fuzzy bridge). So, $v^2(u', v') < \mu^2(u', v')$ which gives a contradiction since $v^2(u', v')$ is the strength of the unique $u'-v'$ path in $T$ and by an algorithm in [2], $\mu^2(u', v') = v^2(u', v')$. Thus $T$ is the required spanning subgraph $F$, which is a tree and hence $G$ is a fuzzy tree. \(\Box\)

**Remark 6.** For a fuzzy tree $G$, the spanning subgraph $F$ is unique (Theorem 7). It follows from the proof of the above theorem that $F$ is nothing but the maximum spanning tree $T$ of $G$. 
Theorem 10. A fuzzy graph is a fuzzy tree if and only if it has a unique maximum spanning tree.

Remark 7. For a fuzzy graph which is not a fuzzy tree there is at least one arc in T which is not a fuzzy bridge and arcs not in T are not fuzzy bridges of G. This observation leads to the following theorem.

Theorem 11. If G : (σ, μ) is a fuzzy graph with σ* = S and |S| = p then G has at most p − 1 fuzzy bridges.

Theorem 12. Let G(σ, μ) be a fuzzy graph and let T be a maximum spanning tree of G. Then end nodes of T are not fuzzy cut nodes of G.

Corollary: Every fuzzy graph has at least two nodes which are not fuzzy cut nodes.

However, there are fuzzy graphs with diametrical nodes, nodes which have maximum eccentricity [1], as fuzzy cutnodes, distinct from crisp graph theory. See u3 and u5 of Fig. 5.

References