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Existence of multiple attractors and the nature of bifurcations in a discontinuous logistic map

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Abstract. We present the analytical investigations on a logistic map with a discontinuity at the centre. An explanation for the bifurcation phenomenon in discontinuous systems is presented. We establish that whenever the elements of an n-cycle ($n > 1$) approach the discontinuities of the nth iterate of the map, a bifurcation other than the usual period-doubling one takes place. The periods of the cycles decrease in an arithmetic progression, as the control parameter is varied. The system also shows the presence of multiple attractors. Our results are verified by numerical experiments as well.

Keywords. Discontinuous maps; bifurcations; chaos; Lyapunov exponents; multiple attractors.

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1. Introduction

One-dimensional endomorphisms defined in an interval on the real axis have been used in modelling a wide variety of nonlinear systems. One of the extensively studied maps in this context is the logistic map [1–4]. Investigations on the system by modulating the control parameter have resulted in a class of modulated logistic maps [5–8]. A large number of numerical investigations on discontinuous logistic map have also been reported [9–12]. The existence of inverse cascades in which the period changes arithmetically is a novel aspect in these systems. The phenomenon of border-collision bifurcations and the formation of inverse and direct cascades in one-dimensional piecewise smooth maps have also been investigated [13–15]. Similar bifurcations have been observed for other piecewise continuous quadratic maps like the circle map [16] and the logistic-like *sawtooth map* [17]. Most of the studies in these systems have been numerical. In what follows, we present an analytical study of the logistic map with a discontinuity at the centre. We have observed that the system possesses multiple attractors with different basins of attraction. We give expressions for the basin boundaries and an explanation for the bifurcation phenomenon in this discontinuous system. Numerical findings in support of these results are also included.

The paper is organised as follows. In § 2, we give the details of our investigations on the map function. The fixed points of the system and their respective basins of attraction are identified. A local stability analysis and an explanation for the new bifurcation scenario are presented in § 3. Section 4 deals with the numerical simulations of the map. The concluding remarks and comments are given in § 5.

2. Analysis of the discontinuous logistic map; co-existence of multiple attractors and their basins of attraction

We consider the logistic map of the following form with a discontinuity at the centre:

$$\begin{aligned} x_{n+1} &= 4\lambda x_n(1-x_n); \quad \text{for } 0 < x_n \leq 1/2, \\ x_{n+1} &= 4\lambda x_n(1-x_n) + C; \quad \text{for } 1/2 < x_n < 1, \end{aligned} \tag{1}$$

where λ is the control parameter and C is a constant characterising the strength of discontinuity. For convenience, we write the mapping (1) in the form, $x_{n+1} = T(x_n)$, where the function T is such that $T(x) = f(x) = 4\lambda x(1-x)$ for $0 < x \leq 1/2$ and $T(x) = \phi(x) = 4\lambda x(1-x) + C$ for $1/2 < x < 1$. The shape of the map function is shown in figure 1. For each value of C, the control parameter λ is varied from 0 to $(1-C)$ so as to confine the iterates within the unit interval. The fixed point of the left part is $x^* = 0$ for values of λ ranging from 0 to 1/4. When $\lambda > 1/4$, the fixed point from the left part is

$$x_l^* = 1 - \frac{1}{4\lambda}. \tag{2}$$

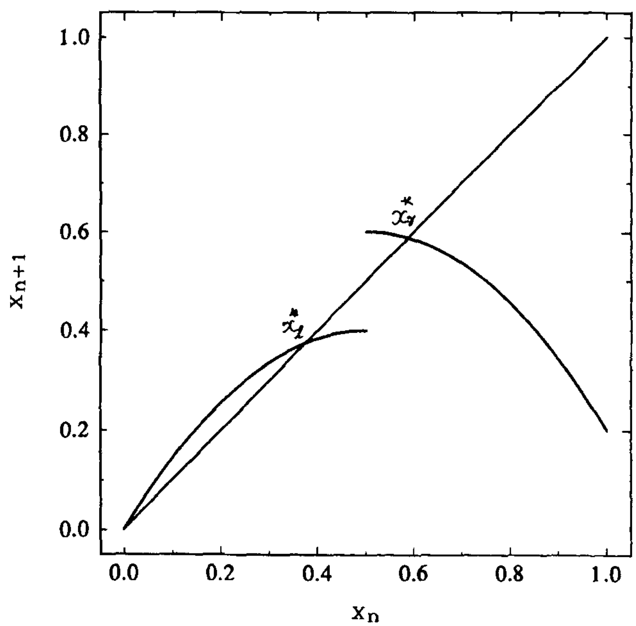


Figure 1. The map function defined in eqn. (1). The discontinuity parameter $C = 0.2$ and the control parameter $\lambda = 0.4$. Note that the two fixed points x_l^* and x_r^* co-exist in this case.

The fixed point arising from the right part is given by

$$x_r^* = \frac{1}{2} + \frac{\sqrt{(4\lambda - 1)^2 + 16\lambda C} - 1}{8\lambda}. \tag{3}$$

The fixed point (x_r^*) and its stability properties depend on the two parameters λ and C so that one can have a desired dynamics for the system by proper choice of λ and C as in the case of combination maps [18,19]. Since $x_r^* > (1/2)$, we have from eq. (3), $\lambda + C > 1/2$, i.e., $\lambda > (1/2 - C)$. Thus x_r^* exists for all $\lambda > (1/2 - C)$ and x_l^* appears for $1/4 < \lambda \leq 1/2$. Hence the two fixed points co-exist in the parameter range $(1/2 - C) < \lambda \leq 1/2$.

The basin of attraction for x_r^* is the set $R = \{x_0 | x_0 \in (1/2, x_r)\}$ where x_r is the value of x at which $\phi(x) = 1/2$. This gives,

$$x_r = 1/2 + \frac{\sqrt{16\lambda(\lambda + C - 1/2)}}{8\lambda}. \tag{4}$$

The basin of attraction for x_l^* is the set of points on the unit interval complementary to the set R. The fixed point x_r^* can co-exist with the zero fixed point, if $\lambda < 1/4$ and $(\lambda + C) > 1/2$. For this, the value of C must be $> 1/4$. Since x_l^* can never co-exist with the zero fixed point, the possibility of the co-existence of the three fixed points is ruled out.

3. Stability analysis and bifurcation scenario for the discontinuous map

We now consider the stability of the fixed points as we move on the parameter space (λ, C). Keeping C fixed, let λ be varied from 0 to $(1 - C)$. The fixed point x_r^* exhibits period doubling when the stability determining slope becomes equal to -1. The corresponding value of λ is given by,

$$\lambda_1 = \frac{(1 - 2C) + \sqrt{(2C - 1)^2 + 3}}{4}. \tag{5}$$

In the limit, $C \to 0$, $\lambda_1 \to 0.75$, is the value for the logistic map.

In the usual period doubling route to chaos, the 2-cycle bifurcates to a 4-cycle at a parameter value $\lambda = \lambda_2$ and remains stable for a range of λ and then the 4-cycle bifurcates to an 8-cycle at $\lambda = \lambda_3$ and so on and these period doublings take place ad infinitum. However, in the discontinuous map, a different type of bifurcation takes place when λ is increased. Both the elements x_1^* and x_2^* of the 2-cycle lie within the interval $R = (1/2, x_r)$. As λ increases, the cycle elements move out. The lower element x_1^* moves towards $1/2$ and the upper element x_2^* approaches x_r. Let $x_1^* = (1/2 + \epsilon)$. Then $x_2^* = \phi(x_1^*) = (\lambda + C - 4\lambda\epsilon^2)$. In the limit $\epsilon \to 0$, $x_1^* \to (1/2)_+$ and $x_2^* \to (\lambda + C)_-$. Now, the slope of $\phi^2(x)|_{x_1^*, x_2^*} = 64\lambda^2(x_1^* - 1/2)(x_2^* - 1/2)$. In the limit $\epsilon \to 0$, the slope of $\phi^2(x)|_{x_1^*, x_2^*} \to 0$. Thus the limiting 2-cycle $\{(1/2)_+, (\lambda + C)_-\}$ is a stable one. This 2-cycle continues upto $\lambda = \lambda_r$, at which the right element $(\lambda + C) = x_r$. Using eq. (4),

$$\lambda_r = \frac{(\frac{1}{2} - C) + \sqrt{1 + (\frac{1}{2} - C)^2}}{2}. \tag{6}$$

The next iterate of x_r, say, $x_1 = T(x_r) = \phi(x_r) = 1/2$, falls on the left branch. Consequently, the second iterate of x_r is decided not by the function $\phi(x)$, but by $f(x)$, i.e., $x_2 = T^2(x_r) = T(1/2) = f(1/2) = \lambda_r$. Now, since λ_r is greater than 1/2, (for $C < 1/2$, as is usually the case), the second iterate of x_r comes back to the interval $(1/2, x_r)$. The subsequent iterates are decided by the function ϕ. Thus we get a sequence of iterates, $\{x_2, \phi(x_2), \phi^2(x_2), \ldots, \phi^r(x_2), \ldots\}$ and the 2-cycle behaviour is lost. These iterates are attracted towards a 'virtual' 2-cycle $\{(1/2)_+, x_r\}$ (i.e., the 2-cycle that the system would have, if the mapping were $\phi(x)$ on both sides of the extremum). Once x_r is reached, the process is repeated. The system thus exhibits a large periodicity n, which is highly dependent on the precision of the computer. The period n will be even or odd depending on whether $x_2 > x_r^*$ or $x_2 < x_r^*$, where x_r^* is the fixed point (unstable) of $\phi(x)$ and x_2 is the second iterate of x_r. Let us consider the two cases separately. Case (1): $x_2 > x_r^*$. Since x_2 lies to the right of x_r^*, its iterate, $\phi(x_2) < \phi(x_r^*)$. (For the function ϕ is a monotonically decreasing one). But, $\phi(x_r^*) = x_r^*$. Therefore $\phi(x_2) < x_r^*$. Now, since $\phi(x_2) < x_r^*$, its iterate $\phi^2(x_2) = \phi[\phi(x_2)]$ lies to the right of x_r^*. Continuing like this, we see that the odd iterates of x_2 (viz., $\phi(x_2), \phi^3(x_2), \phi^5(x_2), \phi^7(x_2), \ldots$ lie to the left of x_r^* and the even iterates $\phi^2(x_2), \phi^4(x_2), \phi^6(x_2), \ldots$ lie to the right of x_r^*. Since $\phi(x)$ is a decreasing function of x for any $x \in (1/2, 1), \phi^2(x)$ is an increasing function of x for all x for which $\phi(x)$ is greater than $1/2$. Thus $\phi^2(x)$ is an increasing function of x for all x in $(1/2, x_r)$. Also, since x_r^* is an unstable fixed point of $\phi(x)$, the behaviour of $\phi^2(x)$ on either side of x_r^* is such that $\phi^2(x) > x$ for $x > x_r^*; \phi^2(x) < x$ for $x < x_r^*$ and $\phi^2(x) = x$ for $x = x_r^*$. Thus since $x_2 > x_r^*$, its second iterate $\phi^2(x_2) > x_2$. Again, since $\phi^2(x_2) > x_r^*, \phi^4(x_2) > \phi^2(x_2)$. Similarly, $\phi^6(x_2) > \phi^4(x_2)$ and so on. Likewise, since $\phi(x_2) < x_r^*, \phi^3(x_2) < \phi(x_2); \phi^5(x_2) < \phi^3(x_2)$ and so on. Thus we have an ordering for the iterates as,

$$x_r^* < x_2 < \phi^2(x_2) < \phi^4(x_2) < \phi^6(x_2) < \cdots \text{ and}$$
$$x_r^* > \phi(x_2) > \phi^3(x_2) > \phi^5(x_2) > \phi^7(x_2) > \cdots$$

Thus it is clear that the even iterates $\phi^2(x_2), \phi^4(x_2), \phi^6(x_2), \ldots$ tend to x_r and the odd iterates $\phi(x_2), \phi^3(x_2), \phi^5(x_2), \phi^7(x_2), \ldots$ move towards $(1/2)_+$. Thus at some stage of iteration, $\phi^{2r}(x_2)$ becomes infinitesimally close to x_r. This iterate will be considered as x_r itself by the computer and the sequence of iterates will be repeated. The value of r depends on the precision used in the computation. Thus we have a cycle of periodicity $n = 2r + 2$. Case (2): $x_2 < x_r^*$. Following the same procedure as for case (1), we see that the sequence of iterates have the ordering,

$$x_r^* > x_2 > \phi^2(x_2) > \phi^4(x_2) > \phi^6(x_2) > \cdots \text{ and}$$
$$x_r^* < \phi(x_2) < \phi^3(x_2) < \phi^5(x_2) < \phi^7(x_2) < \cdots$$

Here, the odd iterates of x_2 move towards x_r and the even iterates approach $(1/2)_+$. Thus at some stage of iteration, we have $\phi^{2r+1}(x_2) = x_r$, resulting in a cycle of period $n = 2r + 3$. If the map function were $\phi(x) = 4\lambda x(1 - x) + C$ throughout the interval $(0, 1)$, the role of C would be that of an additive constant applied to the logistic map [20] and the system would still have a stable 2-periodic behaviour. When the value of λ is slightly greater than λ_r, the 'virtual' 2-cycle (x_1^*, x_2^*) falls outside $[1/2, x_r]$, i.e., say, $x_1^* < 1/2$ and $x_2^* > x_r$. Once x_2^* is attained, the next iterate x_1 falls in the interval $(0, 1/2)$. The image of this point under the mapping f, falls inside the interval $[1/2, x_r]$. This point,

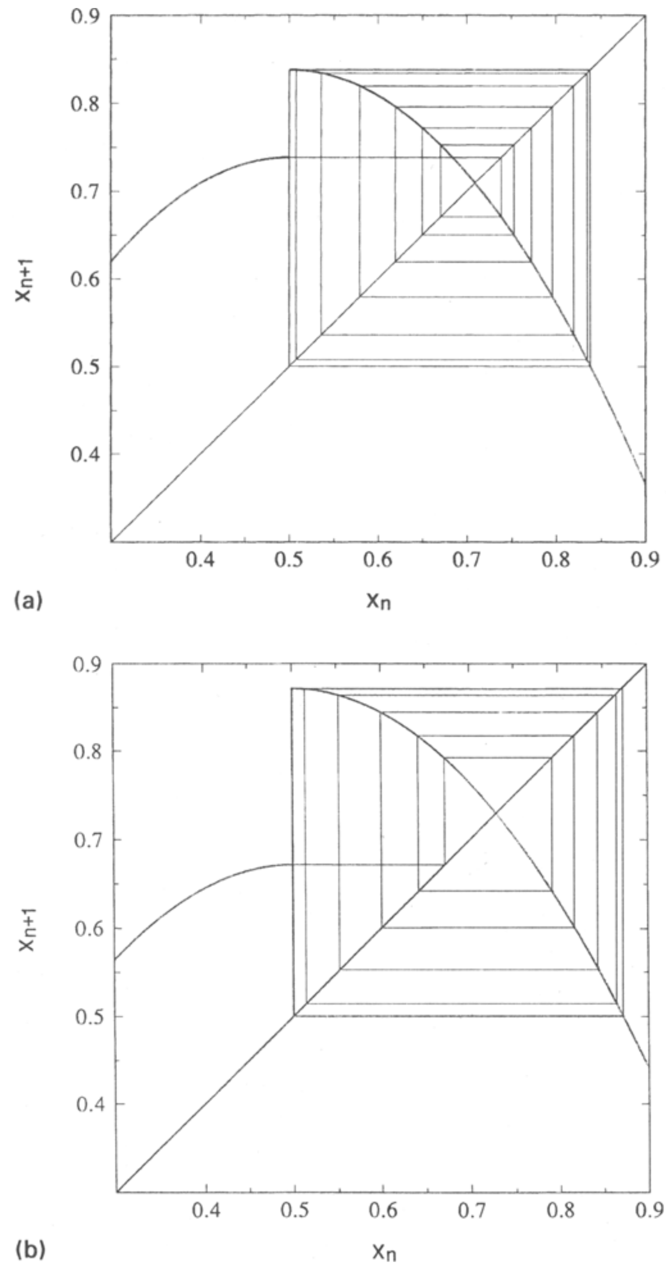
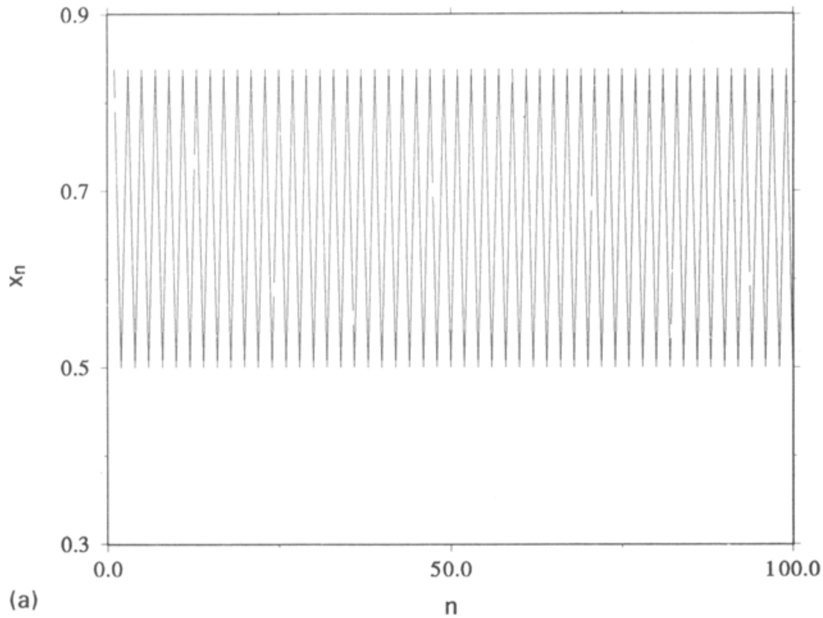


Figure 2. (**a**). Graphical representation of the outward spiralling of the iterates to an attractor of a large periodicity at the parameter value $\lambda = \lambda_r$, for $C = 0.1$. For this choice of C, the second iterate of x_r lies to the right of the unstable fixed point x_r^*. Note that the even iterates of x_2 approach x_r and the odd iterates approach $1/2$. (**b**). Same as that in (**a**) except that the value of C in this case is 0.2. Here, the second iterate of x_r is less than x_r^*. Also the odd iterates of x_2 approach x_r and the even iterates approach $1/2$. The periodicity of the cycle is odd in this case.



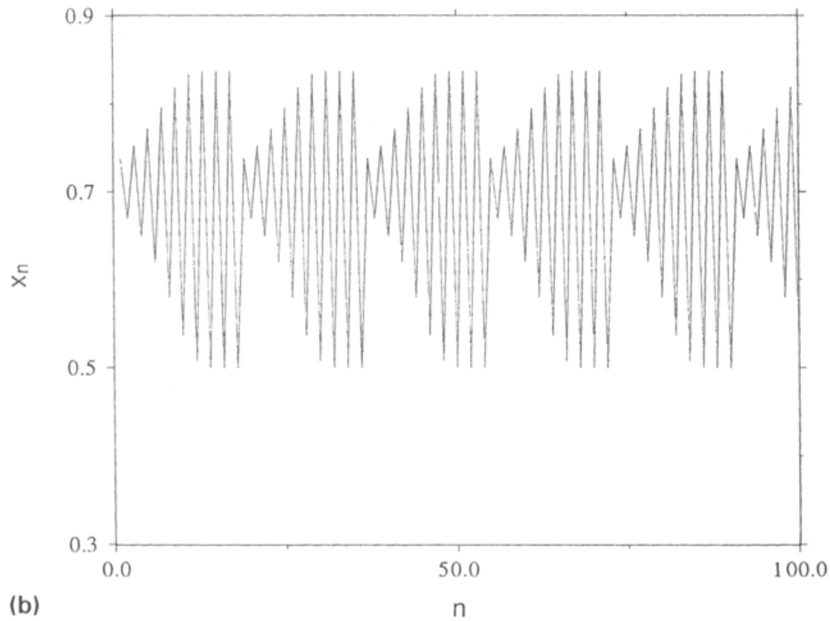
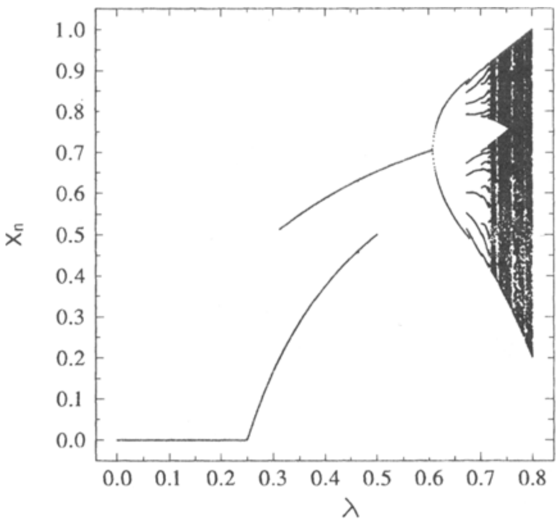


Figure 3. The time-plot of the system in eqn. (1). The value of C is taken as 0.1. Case (a) corresponds to the control parameter λ slightly less than λ_r and case (b) is for $\lambda > \lambda_r$. 10^4 iterates have been left for transients and the subsequent iterates are plotted against the iteration number. The system stabilises to a 2-cycle in the first case; in the second case, the periodicity is 18.

under repeated iterations by $\phi(x)$ approaches the virtual 2-cycle and the whole process is repeated. The outward spiralling of the iterates to an n-periodic attractor is shown in figure 2 a and b for two typical cases of $C = 0.1$ and $C = 0.2$ corresponding to even and odd periods respectively. The behaviour of the iterates of the map for values of λ immediately below and above λ_r can be easily understood from figures 3 a and b. In the period doubling process, the slope of the function at the bifurcation point is -1. But in the case of bifurcation of the 2-cycle to an n-cycle, the slope of $\phi^2(x) = 0$ at the bifurcation point. i.e., the 2-cycle bifurcates to an n-cycle at the superstable point. The elements of the n-cycle are fixed points of $T^n(x)$. The outermost element of the cycle is greater than x_r; the next element $x_1 < 1/2$. All the subsequent elements, namely, $x_2, x_3, x_4, \ldots, x_{n-1}$ fall inside the open interval $(1/2, x_r)$ and the nth iterate $x_n = x_2^*$, within the precision used. With increase of λ, the cycle elements move out until at some value of λ, the interval boundary x_r is reached by x_{n-2}. The periodicity of the system is thus lowered by 2. From the expression for x_r, it is clear that x_r increases (very slowly) with λ and that one set of alternate iterates are repelled by x_r^* towards one side and the other set of alternate iterates to the other side. With further increase of λ, the outermost element increases beyond the corresponding value of x_r and the cycle element nearest to x_r within $(1/2, x_r)$ moves towards x_r until at some stage, x_{n-4} becomes equal to x_r, resulting in a cycle of period $(n-4)$. Proceeding like this, it can be seen that as λ increases beyond λ_r, there exist different ranges of the parameter λ, for which cycles of periods decreasing by 2 exist. Based on similar arguments, it can be seen that different levels of inverse cascades in which the periods decrease in arithmetic progressions are possible; the common differences of the progressions will be even numbers, since only alternate iterates move towards one boundary of $[1/2, x_r]$ [11, 12]. Again, the bifurcations within an inverse cascade occur whenever one of the cycle elements approaches the discontinuity of $T(x)$ at $x = 1/2$ and the another element approaches x_r. [In this case, all the cycle elements approach the discontinuities of the nth iterate of the map where n is the period of the cycle]. A given cycle of period n can exhibit the usual period-doubling bifurcation, if the slope of the nth iterate becomes -1 and the cycle loses stability before its elements collide with the discontinuity. The bifurcation process continues until the iterates become aperiodic at a parameter value (λ_∞) and the system enters the chaotic region.

4. Numerical results

In this section we present the numerical studies conducted to check the validity of the conclusions in the previous section. Keeping C fixed, we have done a detailed numerical analysis of the system, by varying the control parameter λ. Figure 4 shows a bifurcation diagram for various values of x_0 in the interval $(0, 1)$ and for $C = 0.2$. For $0 < \lambda < 1/4$, the system stabilises to the zero fixed point. The attractor x_l^* exists for $1/4 < \lambda < 1/2$. For values of $\lambda > 0.3$, the fixed point x_r^* is observed. The two attractors x_l^* and x_r^* co-exist for $0.3 < \lambda < 0.5$. x_r^* undergoes period-doubling at a particular parameter value (λ_1). The two cycle behaviour continues for values of λ in the range (λ_1, λ_r) and then bifurcates into a cycle of large periodicity at $\lambda = \lambda_r$. This periodicity is found to depend on the precision used in the computation. The logistic map with a discontinuity at the centre belongs to the class of maps with precision dependent periods [21]. When λ is increased



1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0.0
x_n
0.0 0.1 0.2 0.3 0.4 0.5 0.6 0.7 0.8
λ

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