ON APPROXIMATION METHODS FOR UNBOUNDED OPERATORS

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DECLARATION

I hereby declare that the thesis entitled ***On Approximation Methods for Unbounded Operators*** is based on the original research work carried out by me under the guidance of Dr.M.N.Narayanan Namboodiri, Department of Mathematics, Cochin University of Science and Technology, Kochi – 22 and that no part of this work has previously formed the basis of the award of any degree, diploma, associateship, fellowship or any other title or recognition.

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CERTIFICATE

This is to certify that the thesis entitled ***On Approximation Methods for Unbounded Operators*** submitted to the Cochin University of Science and Technology by J. Robert Victor Edward for the award of the degree of Doctor of Philosophy in the Faculty of Science is a *bona fide* record of the work carried out by him under my supervision and that this has not been submitted previously for the award of any degree, fellowship or similar titles elsewhere.

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Chapter 1

INTRODUCTION

1.1 Summary of the Thesis

Many of the practical problems that occur in Physics, Chemistry, Engineering Sciences, Economic etc. lead to problems of solving linear operator equations or computing singular values or spectral values of linear operators. And many of the operators that arise in this way are unbounded. Functional Analysis plays a pivotal role in supplying tools for attacking these problems. An attempt is made in this thesis to concentrate on the solutions of operator equations and approximation numbers (generalizations of singular values) of linear operators. Naturally, the operators in consideration are unbounded.

Though there are effective techniques and efficient algorithms for the analysis and computation of solutions of operator equations, there are still plenty of equations whose actual solutions defy all computational endeavors. In respect of those cases, the obvious interest is to make approximations of the actual solutions.

One of the goals of Numerical Functional Analysis is to investigate the problem of approximating solutions of operator equations. Much significant work has been done in this direction by many eminent mathematicians for equations involving special types of operators such as Toeplitz operators, differential operators and integral operators. The basics of computing approximate solutions of equations involving certain kinds of bounded operators can be seen in [2], [4], [7] and [9].

The present study focuses on developing a general theory on approximate solutions of unbounded operator equations, in arbitrary Hilbert spaces. The treatment is entirely different from that of bounded operators. The usual convergence may not be expected in the case of unbounded operators. A variant of the classical notion of resolvent convergence is used for the approximation.

Yet another area the present study focuses is to discuss certain approximation numbers of unbounded operators. Approximation numbers are generalizations of the classical singular values (of compact operators). If A is a compact operator on a Hilbert space H, the A^*A is a compact self-adjoint (in fact, positive) operator. Let $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots$ be the sequence of (non-zero) eigen values of $(A*A)^{1/2}$ (counting multiplicity). $s_k(A) = \lambda_k$ is the k^{th} singular value of A. It is true that [8] $s_k(A) = \inf\{\|A - F\| / F \in \mathcal{B}(H)$, rank $F \leq k-1\}$, where $\mathcal{B}(H)$ denotes the class of bounded operators on H. The same definition works for bounded operators as well. In the general setting, $s_k(A)$ are called approximation numbers. Inspired with this, several other approximation number are introduced for bounded operators by many mathematicians.

The notion of approximation numbers was further extended to unbounded operators of M.N.N. Nammboodiri and A.V. Chithra [14]. One among them is the relative approximation numbers. And a few other similar approximation numbers have been introduced in this study.

The thesis consists of four chapters. In addition to the summary of the thesis, the basic definitions and results used in the subsequent chapters are delineated, in chapter 1.

Chapter 2 is devoted to the study of approximation methods for unbounded self-adjoint operators. The (triangular) connection between resolvent convergence, applicability and stability has been established. The role of stability is significant in the theory of approximation methods for unbounded operators as much as for bounded operators. This chapter concludes with a discussion on the finite section method.

In chapter 3, the trunction method for unbounded matrices is analyzed. Some of the results established in the previous chapter are applied here to investigate the resolvent convergence of the truncations to the original matrix. Also, an attempt is made to discuss unbounded Toeplitz matrices.

In chapter 4, a few relative approximation numbers of unbounded operators are introduced and their properties are discussed. As a final result it has been proved that the τ^* - numbers of an unbounded operator A which is bounded relative to a closed operator T can be approximated by the τ^* - numbers of its truncations.

1.2 Banach and Hilbert Spaces

Definition 1.2.1. Let X be a vector space over $\mathbb{K}$, where $\mathbb{K}$ is the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. A norm on X is a function $\|\cdot\| : X \to \mathbb{R}$ satisfying the following:

(i) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = 0$.

(ii) $\|\alpha x\| = |\alpha|\|x\|$ for all $x \in X$ and $\alpha \in \mathbb{K}$.

(iii) $||x + y|| \leq ||x|| + ||y||$ for all $x, y \in X$.

A vector space with a norm defined on it is called a normed space.

Let X be a normed space. $d(x, y) = ||x - y||$ defines a metric on X. Thus, every normed space is a metric space. If X is complete with respect to this metric, then X is called a Banach space.

Definition 1.2.2. Let X be a vector spaces over $\mathbb{K}(=\mathbb{R}$ or $\mathbb{C})$. An inner product on X is a function $\langle , \rangle : X \times X \to \mathbb{K}$ satisfying the following:

(i) $\langle x, x\rangle \geq 0$ for all $x \in X$ and $\langle x, x\rangle = 0$ if and only if $x = 0$.

(ii) $\overline{\langle x, y\rangle} = \langle y, x\rangle$ for all $x, y \in X$, where the bar denotes complex conjugation.

(iii) $\langle \alpha x, y\rangle = \alpha\langle x, y\rangle$ for all $x, y \in X, \alpha \in \mathbb{K}$.

(iv) $\langle x + y, z\rangle = \langle x, z\rangle + \langle y, z\rangle$ for all $x, y, z \in X$

A vector space with an inner product defined on it is called an inner product space.

Let X be an inner product space. For $x \in X$, define $\|x\| = \langle x, x\rangle^{1/2}$. Then $\| \quad \|$ is a norm on X. Thus, every inner product space is a normed space. If X is complete, we call it a Hilbert space. We use the letter H to denote a Hilbert space.

Definition 1.2.3. Let X_1 and X_2 be Banach spaces over $\mathbb{K}$. A function $A : X_1 \to X_2$ satisfying $A(x + y) = Ax + Ay$ and $A(\alpha x) = \alpha Ax$ for all $x, y \in X$ and $\alpha \in \mathbb{K}$ is called a linear operator. A is said to be bounded if there exist a real number $c < \infty$ such that $\|Ax\| \leq c\|x\|$ for all $x \in X$.

Let A be a bounded operator. The norm of A is defined as $\|A\| = \sup\{\|Ax\|/x \in X_1, \|x\| \leq 1\}$ and is called the operator norm.

The set of all bounded operators: $X_1 \to X_2$ is denoted by $\mathcal{B}(X_1, X_2)$. It is true that $\mathcal{B}(X_1, X_2)$ is a normed space in the operator norm and is a Banach space if X_2 is a Banach space. $\mathcal{B}(X, X)$ is denoted by $\mathcal{B}(X)$. For $A \in \mathcal{B}(X_1, X_2), R(A) = \{Ax/x \in X_1\}$ is called the range of A and $N(A) = \{x \in X_1/Ax = 0\}$ is called the null space of A.

Definition 1.2.4. Let $A \in \mathcal{B}(H_1, H_2)$, where H_1 and H_2 are Hilbert spaces. There is a unique operator $A^* \in \mathcal{B}(H_2, H_1)$ satisfying $\langle Ax, y\rangle = \langle x, A^*y\rangle$ for all $x \in H_1$ and $y \in H_2$. A^* is called the adjoint of A.

For $A, B \in \mathcal{B}(H_1, H_2), (\alpha A + \beta B)^* = \overline{\alpha}A^* + \overline{\beta}B^*$ for $\alpha, \beta \in \mathbb{K}, (A^*)^* = A, \|A^*\| = \|A\|$ and $\|A^*A\| = \|A\|^2$. If $H_1 = H_2$, $(AB)^* = B^*A^*$.

$A \in \mathcal{B}(H)$ is said to the self-adjoint if $A^* = A$.

Definition 1.2.5. $P \in \mathcal{B}(H)$ is said to be an orthogonal projection if $P^2 = P$ and $P^* = P$. By a projection we always mean a (bounded) orthogonal projection.

Definition 1.2.6. Let X be an inner product space. Let $x, y \in X$. x is said to be orthogonal to y if $\langle x, y\rangle = 0$. Two subsets E and F of X are said to be orthogonal to each other if $\langle x, y\rangle = 0$ for every $x \in E$ and $y \in F$. In this case we write $E \perp F$.

Let $S \subseteq X$. The orthogonal complement of S, denoted by $S^\perp$, is defined as $S^\perp = \{y \in X / \langle y, x\rangle = 0 \text{ for all } x \in S\}$.

$S \subseteq X$ is said to be an orthonormal set if for all $x, y \in S$,

$$\langle x, y\rangle = \begin{cases} 0 & \text{if} \quad x \neq y \\ 1 & if \quad x = y. \end{cases}$$

An orthonormal set S in a Hilbert space H is said to be an orthnormal basis for H if $\langle x, u\rangle = 0$ for all $u \in S$ implies $x = 0$. It is true that an orthonormal set S is an orthonormal basis for H if and only if *span* S is dense in H. Also, it is true that every non zero Hilbert space has an orthonormal basis.

A metric space is said to be separable if it has a countable dense subset. A Hilbert space H has a countable orthonormal basis if and only if it is separable.

Theorem 1.2.7. (Gram-Schmidt orthonormalization).

Let $\{x_1, x_2, x_3, \ldots\}$ be a linearly independent set in an inner product space X. Then there exists an orthonormal set $\{u_1, u_2, u_3, \ldots\}$ in X such that *span* $\{u_1, u_2, \ldots, u_n\} =$ *span* $\{x_1, x_2, \ldots, x_n\}$ for every n. □

Theorem 1.2.8. Let $\{u_\alpha\}$ be an orthonormal basis for H. Let $x \in H$. Then

(i) $\{u_\alpha / \langle x, u_\alpha\rangle \neq 0\}$ is countable.

(ii) $x = \sum_{n=1}^{\infty} \langle x, u_n\rangle\, u_n$, where $\{u_1, u_2, \ldots\} = \{u_\alpha / \langle x, u_\alpha\rangle \neq 0\}$.

(iii) $\|x\|^2 = \sum_{n=1}^{\infty} |\langle x, u_n\rangle|^2$, where $\{u_1, u_2, \ldots\} = \{u_\alpha / \langle x, u_\alpha\rangle \neq 0\}$.

(ii) is called the Fourier expansion of x and (iii) is the Parseval's formula. □

Finally, we state two fundamental theorems.

Theorem 1.2.9. (The Uniform Boundedness Theorem)

Let X be a Banach space and Y a normed space. Let $\{T_n\}$ be a non-empty set of bounded operators: $X \to Y$ such that $\{T_n x\}$ is a bounded set in Y for every $x \in X$. Then there exists $c < \infty$ such that $\|T_n\| \leq c$ for every n. □

Before we state the other theorem, let us define what is called the graph of an operator.

Let $T : X \to Y$. The graph of T, denoted by $\Gamma(T)$ is defined as $\Gamma(T) = \{(x, Tx)/x \in X\}$.

Theorem 1.2.10. (The Closed Graph Theorem)

Let X and Y be Banach spaces and $T : X \to Y$ be linear. If the graph of T is closed in $X \times Y$, then $T \in \mathcal{B}(X, Y)$. □

1.3 Filtrations

Definition 1.3.1. [1]

Let H be a Hilbert space. A sequence $(H_1, H_2, H_3, \ldots)$ of finite dimensional subspaces of H is said to be a filtration of H if $H_n \subseteq H_{n+1}$ for every n and $\overline{\cup_n H_n} = H$. The bar denotes closure.

It is easy to note that every Hilbert space H with a countable orthonormal basis has a filtration. If $\{u_1, u_2, u_3, \ldots\}$ is an orthonormal basis for H, we can take $H_n =$ *span* $\{u_1, u_2, \ldots, u_n\}, n = 1, 2, 3, \ldots$ The converse is also true.

Proposition 1.3.2. Every Hilbert space H with a filtration has a countable orthonormal basis.

Proof. Let $(H_1, H_2, H_3, \ldots)$ be a filtration of H so that $H_1 \subseteq H_2 \subseteq H_3 \subseteq \ldots$ and $\overline{\cup H_n} = H$, where each H_n is a finite dimensional subspace of H. Find a linearly independent set $\{x_1, x_2, x_3, \ldots\}$ in H such that $\{x_1, x_2, \ldots, x_{i_1}\}$ is a basis for H_1 $\{x_1, x_2, \ldots, x_{i_1}, x_{i_1+1}, \ldots, x_{i_2}\}$ is a basis for H_2 and so on.

By Gram-Schmidt orthonormalization process, we get an orthonormal set $\{u_1, u_2, u_3, \ldots\}$ such that *span* $\{u_1, u_2, u_3, \ldots, u_r\}$ $=$ *span* $\{x_1, x_2, x_3, \ldots, x_r\}, r = 1, 2, 3, \ldots$

Now, *span* $\{u_1, u_2, u_3, \ldots\} = \cup H_n$.

Therefore, $\overline{span\{u_1, u_2, u_3, \ldots\}} = \overline{\cup H_n} = H$.

Hence, $\{u_1, u_2, u_3, \ldots\}$ is a countable orthonormal basis for H. □

Remark 1.3.3. A Hilbert space H has a countable orthonormal basis if and only if H is separable. Hence we have the result:

A Hilbert space H has a filtration if and only if it is separable.

By a projection on H we mean an orthogonal projection.

Proposition 1.3.4. Let (H_n) be a filtration of a Hilbert space H. Let P_n be the projection on H with range H_n, for $n = 1, 2, 3, \ldots$. Then $P_n \to I$ strongly, where I is the identity operator.

Proof. For $x \in H, x = \sum_{r=1}^{\infty} \langle x, u_r \rangle u_r$, where $\{u_r\}$ is the orthonormal basis for H, as in the proof of proposition 1.3.2.

Now $P_n x = \sum_{r=1}^{i_n} \langle x, u_r \rangle u_r \to x$ as $n \to \infty$. □

1.4 Unbounded Operators

Definition 1.4.1. Let H be a Hilbert space. Consider a linear operator $A : D(A) \to H$, where $D(A)$ (called the domain of A) is a subspace of H. $R(A) = \{Ax / x \in D(A)\}$ is called the range of A and $N(A) = \{x \in D(A)/Ax = 0\}$ is called the null space of A. A is said to be bounded if there exists $c < \infty$ such that $\|Ax\| \leq c\|x\|$ for every $x \in D(A)$. If no such 'c' exists, A is said to be unbounded. If $D(A)$ is dense in H, we say that A is densely defined. We say that A is an operator on H if $D(A) = H$ and A is an operator in H if $D(A) \subseteq H$.

The graph of A, denoted by $\Gamma(A)$, is a subspace of $H \times H$ defined by $\Gamma(A) = \{(x, Ax)/x \in D(A)\}$.

$H \times H$ is a Hilbert space with the inner product $\langle x, y \rangle = \langle x_1, y_1 \rangle + \langle x_2, y_2 \rangle$, where $x = (x_1, x_2)$ and $y = (y_1, y_2)$.

If $\Gamma(A)$ is closed in $H \times H$, we say that A is a closed operator. Equivalently, A is closed if and only if $x_n \to x$ $(x_n \in D(A))$ and $Ax_n \to y$ in H imply $x \in D(A)$ and $y = Ax$.

Let A and B be linear operators in H. B is said to be an extension of A if $D(A) \subseteq D(B)$ and $Ax = Bx$ for every $x \in D(A)$. Equivalently, B is an extension of A if and only if $\Gamma(A) \subseteq \Gamma(B)$. A is said to be closable if A has a closed extension.

Let A be closable. The smallest closed extension of A is called the closure of A and is denoted by $\overline{A}$.

Now we define the adjoint of an unbounded operator. In order that the adjoint operator T^* of a linear operator T exists, T must be densely defined.

Definition 1.4.2. Let A be a densely defined linear operator in H. Let $D^* = \{y \in H /$ there exists a $z \in H$ satisfying $\langle Ax, y \rangle = \langle x, z \rangle$ for all $x \in D(A)\}$. For $y \in D^*$, define $A^*y = z$. A^* is a linear operator in H with domain D^*, and is called the adjoint of A.

A^* is a closed operator. If A^* is an extension of A (that is, if $D(A) \subseteq D(A^*)$ and $Ax = A^*x$ for all $x \in D(A)$), A is called a symmetric operator. Equivalently, A is symmetric if and only if $\langle Ax, y\rangle = \langle x, Ay\rangle$ for all $x, y \in D(A)$.

A is called self-adjoint if $A = A^*$. Thus, A is self-adjoint if and only if A is symmetric and $D(A) = D(A^*)$.

For a closable operator A, $\Gamma(\bar{A}) = \overline{\Gamma(A)}$. It is true that every self-adjoint operator is closed and every symmetric operator is closable.

A symmetric operator A is said to be essentially self-adjoint if its closure $\bar{A}$ is self-adjoint.

If a closed linear operator A is defined on all of a Hilbert space H, then it is bounded, by the closed graph Theorem. Thus, an unbounded closed operator A can not be defined on all of H; its domain $D(A)$ can only be a dense subspace of H.

Definition 1.4.3. An operator A in H is said to be invertible if A is a bijection of $D(A)$ onto H and $A^{-1} \in \mathcal{B}(H)$.

Remark 1.4.4. (i) Every bounded operator is closed. Thus $\mathcal{B}(H) \subseteq \mathcal{C}(H)$, where $\mathcal{B}(H)$ denotes the set of all bounded operators on H and $\mathcal{C}(H)$ denotes the set of all closed operators in H.

(ii) Every invertible operator is closed. For a proof, let A be invertible in H. Let (x_n) be a sequence in $D(A)$ such that $x_n \to x$ and $Ax_n \to y$ in H. Then, $A^{-1}Ax_n \to A^{-1}y$. That is, $x_n \to A^{-1}y$. Hence $x = A^{-1}y$ or $Ax = y$. Hence A is closed.

Definition 1.4.5. Let A be a closed operator in a complex Hilbert space H. The resolvent set $\rho(A)$ of A is defined as

$$\rho(A) = \{\lambda \in \mathbb{C}/\lambda I - A \text{ is invertible }\}.$$

Thus, $\lambda \in \rho(A)$ if and only if $\lambda I - A$ is a bijection of $D(A)$ onto H and $(\lambda I -$

$A)^{-1} \in \mathcal{B}(H)$. Here I denotes the identity operator on H. For $\lambda \in \rho(A)$, put $R_\lambda(A) = (\lambda I - A)^{-1}$. $R_\lambda(A)$ is called the resolvent of A at λ.

The (set theoretic) complement of the resolvent set $\rho(A)$ of A is called the spectrum of A, and is denoted by $\sigma(A)$.

Thus, $\sigma(A) = \{\lambda \in \mathbb{C}/\lambda I - A \text{ is not invertible }\}$.

λ is said to be an eigenvalue of A if there is an $x \neq 0$ in H such that $Ax = \lambda x$. The set of all eigenvalues of A, denoted by $e(A)$, is called the eigenspectrum of A. Thus, $e(A) = \{\lambda \in \mathbb{C}/\lambda I - A \text{ is not injective }\}$.

We also define $a(A)$ as follows: $\lambda \in a(A)$ if and only if there is a sequence (x_n) in H with $\|x_n\| = 1$ for all n such that $\|(\lambda I - A)x_n\| = \|\lambda x_n - Ax_n\| \to 0$ as $n \to \infty$. $a(A)$ is called the approximate eigenspectrum of A.

We have the following inclusion relation:

$$e(A) \subseteq a(A) \subseteq \sigma(A). \tag{1.1}$$

Theorem 1.4.6. [16] Let A be a closed densely-defined linear operator in a complex Hilbert space H. Then $\rho(A)$ is an open subset of $\mathbb{C}$ on which the resolvent is an analytic operator-valued function. Furthermore, $\{R_\lambda(A)/\lambda \in \rho(A)\}$ is a commuting family of bounded operators satisfying

$$R_\lambda(A) - R_\mu(A) = (\mu - \lambda)R_\mu(A)R_\lambda(A). \tag{1.2}$$

(1.2) is called the resolvent equation. □

Proposition 1.4.7. [10] Let A be self-adjoint and $\lambda \in \mathbb{C} \setminus \mathbb{R}$. Then $\lambda I - A$ has a bounded inverse and

$$\|R_\lambda(A)\| = \|(\lambda I - A)^{-1}\| \leq |Im\lambda|^{-1}. \tag{1.3}$$

Proof.

$$\begin{aligned}\lambda I - A &= (Re\lambda)I + i(Im\lambda)I - A \\ &= B + i(Im\lambda)I, \text{ where } B = (Re\lambda)I - A.\end{aligned}$$

As A is self-adjoint, B is self-adjoint.

For $x \in D(A), (\lambda I - A)x = Bx + i(Im\ \lambda)x$.

Therefore,

$$\begin{aligned}\|(\lambda I - A)x\|^2 &= \langle(\lambda I - A)x, (\lambda I - A)x\rangle \\ &= \langle Bx + i(Im\lambda)x, Bx + i(Im\lambda)x\rangle \\ &= \|Bx\|^2 + i(Im\lambda)\langle x, Bx\rangle - i(Im\lambda)\langle Bx, x\rangle + \|i(Im\lambda)x\|^2 \\ &= \|Bx\|^2 + |Im\lambda|^2\|x\|^2, \text{ since } \langle x, Bx\rangle = \langle Bx, x\rangle, \text{ as } B \text{ is self-adjoint}.\end{aligned}$$

Hence, $|Im\lambda|^2\|x\|^2 \leq \|(\lambda I - A)x\|^2$ for every $x \in D(A)$, which implies, $\|x\| \leq |Im\ \lambda|^{-1}\|(\lambda I - A)x\|$ for every $x \in D(A)$.

For $y \in R(A)$, put $x = (\lambda I - A)^{-1}y$.

Then we get, $\|(\lambda I - A)^{-1}y\| \leq |Im\lambda|^{-1}\|y\|$, which gives the desired result. □

Theorem 1.4.8. [16] Let A be a symmetric operator in a Hilbert space H. Then the following are equivalent:

(a) A is self-adjoint

(b) A is closed and $N(A^* \pm iI) = \{0\}$

(c) $R(A \pm iI) = H$.

Theorem 1.4.9. [16] Let A be a symmetric operator in a Hilbert space H. Then the following are equivalent:

(a) A is essentially self-adjoint

(b) $N(A^* \pm iI) = \{0\}$

(c) $R(A \pm iI)$ are dense in H.

1.5 Unitary Groups of Operators

Let A be a bounded self-adjoint operator on H. Then we can define $e^{itA}(t \in \mathbb{R})$, by the power series which converges in norm:

$$e^{itA} = \sum_{n=0}^{\infty} \frac{(it)^n A^n}{n!}. \tag{1.4}$$

For an unbounded self-adjoint operator A, e^{itA} can be defined using functional calculus or projection-valued measures.

First we state the multiplication operator form of the spectral theorem.

Theorem 1.5.1. [16, Theorem VIII. 4]

Let A be a self-adjoint operator in a separable Hilbert space H with domain $D(A)$. Then there is a measure space $(M,\ \mu)$ with μ a finite measure, a unitary operator $U : H \to L^2(M,\ d\mu)$ and a real-valued function f on M which is finite a.e. so that

(a) $\psi \in D(A)$ if and only if $f(\cdot)(U\psi)(\cdot)\ \in L^2(M, d\mu)$

(b) If $\varphi \in U(D(A))$, then $(UAU^{-1}\varphi)(m) = f(m)\varphi(m)$ for $m \in M$. □

Using the U and f in the above Theorem, we can define functions of a self-adjoint operator [16, p.261–262].

Let h be a bounded Borel function on $\mathbb{R}$. If A is a self-adjoint operator in H, we can define $h(A)$ by

$$h(A) = U^{-1} T_{h(f)} U, \tag{1.5}$$

where $T_{h(f)}$ is the operator on $L^2(M, d\mu)$ which acts by multiplication by the function $h(f(m))$.

Definition 1.5.2. For a measurable set $\Omega \subseteq \mathbb{R}$, let P_Ω be the operator $\chi_\Omega(A)$, where χ_Ω is the characteristic function of Ω. The family of operators $\{P_\Omega\}$ has the following properties:

(a) Each P_Ω is an orthogonal projection.

(b) $P_\phi = 0; P_{(-\infty,\infty)} = I$.

(c) If $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$ with $\Omega_n \cap \Omega_m = \phi$ for $n \neq m$, then $\sum_{n=1}^{N} P_{\Omega n}$ converges to P_Ω strongly as $N \to \infty$.

(d) $P_{\Omega_1} P_{\Omega_2} = P_{\Omega_1 \cap \Omega_2}$.

Such a family is called a projection-valued measure.

Now we state the projection-valued measure form of the spectral theorem.

Theorem 1.5.3. [16, Theorem VIII. 6] There is a one-to-one correspondence between self-adjoint operators A and projection-valued measures $\{P_\Omega\}$ on H, the correspondence being given by

$$A = \int_{-\infty}^{\infty} \lambda dP_\lambda \tag{1.6}$$

in the sense that

$$\langle x, Ay \rangle = \int_{-\infty}^{\infty} \lambda d\langle x, P_\lambda y \rangle. \tag{1.7}$$

□

If $\{P_\Omega\}$ is a projection-valued measure, then for any $x \in H, \langle x, P_\Omega x \rangle$ is a Borel measure on $\mathbb{R}$, which is denoted by $d\langle x, P_\lambda x \rangle$. The complex measure $d\langle x, P_\lambda y \rangle$ is defined by polarization:

$$\begin{aligned} 4d\langle x, P_\lambda y \rangle &= d\langle x+y, P_\lambda(x+y) \rangle - d\langle x-y, P_\lambda(x-y) \rangle \\ &\quad + id\langle x+iy, P_\lambda(x+iy) \rangle - id\langle x-iy, P_\lambda(x-iy) \rangle. \end{aligned} \tag{1.8}$$

For a bounded Borel function h, we define $h(A) = \int_{-\infty}^{\infty} h(\lambda) dP_\lambda$.

That is,

$$\langle x, h(A)y \rangle = \int_{-\infty}^{\infty} h(\lambda) d\langle x, P_\lambda(y) \rangle. \tag{1.9}$$

This definition of $h(A)$ coincides with the $h(A)$ given by (1.5). $h(A)$ will be self-adjoint if h is real-valued.

For $t \in \mathbb{R}$, take $h(\lambda) = e^{it\lambda}$. Then e^{itA} is given by $h(A)$. Thus

$$e^{itA} = \int_{-\infty}^{\infty} e^{it\lambda} dP_\lambda \tag{1.10}$$

or

$$\langle x, e^{itA} y \rangle = \int_{-\infty}^{\infty} e^{it\lambda} d\langle x, P_\lambda(y) \rangle.$$

Theorem 1.5.4. [16, Theorem VIII . 7] Let A be self-adjoint. Define $U(t) = e^{itA}$. Then $U(t)$ is a strongly continuous one-parameter unitary group, in the sense that

(a) For each $t \in \mathbb{R}, U(t)$ is a unitary operator and $U(t+s) = U(t)U(s)$ for all $t, s \in \mathbb{R}$.

(b) If $x \in H$ and $t \to t_o$, then $U(t)x \to U(t_o)x$. □

The converse of the above theorem is also true and is known as Stone's Theorem.

1.6 Resolvent Convergence

In this section we introduce the notion of resolvent convergence of unbounded self-adjoint operators. The spectrum of a self-adjoint operator A is contained in $\mathbb{R}$. So, for any $\lambda \in \mathbb{C} \setminus \mathbb{R}, R_\lambda(A) = (\lambda I - A)^{-1} \in \mathcal{B}(H)$ and $\|R_\lambda(A)\| \leq |Im\lambda|^{-1}$, by proposition 1.4.7.

Definition 1.6.1. [16] Let $A_n (n = 1, 2, 3, \ldots)$ and A be self-adjoint operators. Then we say that $A_n \to A$ in the norm resolvent sense if

$$\|R_\lambda(A_n) - R_\lambda(A)\| \to 0 \text{ for all } \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{1.11}$$

And we say that $A_n \to A$ in the strong resolvent sense if

$$R_\lambda(A_n) \to R_\lambda(A) \text{ strongly for all } \lambda \in \mathbb{C} \setminus \mathbb{R}.$$

That is, if

$$R_\lambda(A_n)y \to R_\lambda(A)y \text{ for all } \lambda \in \mathbb{C} \setminus \mathbb{R} \text{ and for all } y \in H. \tag{1.12}$$

Strong (norm) resolvent convergence is the right generalization of strong (norm) convergence of bounded operators.

To prove resolvent convergence one need not show convergence of the resolvents for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$. It is enough if one shows the convergence for some $\lambda_\circ \in \mathbb{C} \setminus \mathbb{R}$.

Theorem 1.6.2. [16, Theorem VIII .19]

Let $A_n (n = 1, 2, 3, \ldots)$ and A be self-adjoint operators and $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$.

(a) If $\|R_{\lambda_0}(A_n) - R_{\lambda_0}(A)\| \to 0$, then $A_n \to A$ in the norm resolvent sense.

(b) If $R_{\lambda_0}(A_n)y \to R_{\lambda_0}(A)y$ for all $y \in H$, then $A_n \to A$ in the strong resolvent sense. $\square$

Theorem 1.6.3. [16, Theorem VIII. 20] Let $A_n (n = 1, 2, 3, \ldots)$ and A be self-adjoint operators.

(a) If $A_n \to A$ in the norm resolvent sense and f is a continuous function on $\mathbb{R}$ vanishing at ∞, then $\|f(A_n) - f(A)\| \to 0$.

(b) If $A_n \to A$ in the strong resolvent sense and f is a bounded continuous function on $\mathbb{R}$, then $f(A_n) \to f(A)$ strongly. $\square$

The following result is due to Trotter.

Theorem 1.6.4. [16, Theorem VIII. 21] Let $A_n (n = 1, 2, 3, \ldots)$ and A be self-adjoint operators. Then $A_n \to A$ in the strong resolvent sense if and only if $e^{itA_n} \to e^{itA}$ strongly for each $t \in \mathbb{R}$. $\square$

Finally we state the Trotter-Kato Theorem.

Theorem 1.6.5. [16, Theorem VIII. 22]

Let (A_n) be a sequence of self-adjoint operators. Suppose there exist points λ_0 in the upper half-plane and μ_0 in the lower half-plane such that $R_{\lambda_0}(A_n)x$ and $R_{\mu_0}(A_n)x$ converge for each $x \in H$. Suppose further that one of the limiting operators, T_{λ_0} or T_{μ_0}, has a dense range. Then there exists a self-adjoint operator A such that $A_n \to A$ in the strong resolvent sense. □

1.7 Relative Boundedness

In this section we introduce the notion of relative boundedness of operators in Hilbert spaces. We follow [10] for the definition of relative boundedness.

Let T and A be operators in a Hilbert space H such that $D(T) \subseteq D(A)$ and $\|Ax\| \leq a\|x\| + b\|Tx\|$ for all $x \in D(T)$, where a and b are non-negative constants. Then we say that A is relatively bounded with respect to T or simply T-bounded.

The infemum b_0 of all possible 'b' is called the relative bound of A with respect to T or simply the T-bound of A.

We present two examples here.

(a) [10, Chapter IV, Example 1.6]

Let $H = L^2(a,b)$, where (a,b) is a finite interval. Define T in H by $Tu = -u''$, with domain

$$D(T) = \{u \in H/u' \text{ is absolutely continuous on } (a,b) \text{ and } u'' \in L^2(a,b)\}.$$

And define A in H by $Au = u'$, with domain

$$D(A) = \{u \in H/u \text{ is absolutely continuous on } (a,b) \text{ and } u' \in L^2(a,b)\}.$$

Then A is T-bounded with T-bound 0.

Here u' and u'' are the first and second derivatives of u.

(b) [10, Chapter IV, Example 1.10]

Let $H = L^2(a, b)$.

Let $D = \{u \in H / u', u'' \in H\}$.

Let T and A be second order ordinary differential operators, defined by

$$Tu = p_0 u'' + p_1 u' + p_2 u$$

and

$$Au = q_0 u'' + q_1 u' + q_2 u,$$

with $D(T) = D(A) = D$, where p_0, p_1, p_2, q_0, q_1 and q_2 are real-valued, p_0'', q_0'', p_1', q_1', p_0 and q_0 are continuous on $[a, b]$.

Then, A is T-bounded. □

Also we define relative compactness. Let T and A be operators in H with $D(T) \subseteq D(A)$. A is said to be T-compact if for any sequence $(x_n) \subseteq D(T)$ with both (x_n) and (Tx_n) bounded, (Ax_n) contains a convergent subsequence.

A is said to be T-degenerate if A is T-bounded and $R(A)$ is finite-dimensional.

It can be easily proved [10] that

(a) If A is bounded, then A is T-bounded.

(b) If A is T-compact, then A is T-bounded.

(c) If A is T-degenerate, then A is T-compact.

1.8 Infinite Matrices

Let (e_j) be the standard orthonormal basis for l^2. $e_j = (0, 0, \ldots 0, 1, 0, 0, \ldots)$, where 1 occurs in the j^{th} place and all other entries are 0. let A be a linear operator in l^2

such that $e_j \in D(A)$ for all j. The infinite matrix $(a_{j,k})_{j,k\in\mathbb{N}}$, where $a_{j,k} = \langle Ae_k, e_j\rangle$, is the matrix associated with A (corresponding to the orthonormal basis (e_j)). If $e_j \in D(A^*)$ for all j, then, $(\alpha_{j,k})_{j,k\in\mathbb{N}}$, where $\alpha_{j,k} = \overline{a_{k,j}}$, is the matrix associated with A^*. Here A^* denote the adjoint operator of A.

Conversely, every infinite matrix $(a_{j,k})_{j,k\in\mathbb{N}}$ represents a linear operator A in l^2. Formally, for $x = (x_j)$ in $D(A), Ax$ is given by $Ax = y$, where $y = (y_j)$ with $y_j = \sum\limits_{k=1}^{\infty} a_{j,k}x_k$.

The domain of A is given by

$$D(A) = \{x = (x_j) \in l^2/\ y_j = \sum_{k=1}^{\infty} a_{j,k}x_k \text{ is convergent for every } j \text{ and } y = (y_j) \in l^2\}.$$

A is called Hermitian if $a_{j,k} = \overline{a_{k,j}}$ for all j, k. We call A self-adjoint if A defines a self-adjoint operator in l^2. A is called symmetric if A defines a symmetric operator. If A is self-adjoint, then A, as a matrix, is Hermitian. Thus, for self-adjoint A, $a_{j,k} = \overline{a_{k,j}}$.

We say that A is bounded if A defines a bounded operator on l^2. A is said to be unbounded if A is not bounded. If A is bounded, then A (as an operator) is self-adjoint if and only if A (as a matrix) is Hermition.

Toeplitz Matrices

An infinite matrix of the form

$$A = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \cdots \\ a_1 & a_0 & a_{-1} & \cdots \\ a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \cdots \end{pmatrix}, \text{ where the } a_n \text{ are} \tag{1.13}$$

complex numbers, is called a Toeplitz matrix.

Theorem 1.8.1. [2, Theorem 1.9]

The Toeplitz matrix (1.13) generates a bounded operator on l^2 if and only if there is a function $a \in L^\infty(\mathbb{T})$ whose sequence of Fourier coefficients is the sequence $(a_n)_{n\in\mathbb{Z}}$.

Here $\mathbb{T}$ denotes the unit circle $\{z \in \mathbb{C}/|z| = 1\}$. $L^\infty(\mathbb{T})$ denotes the Banach space of essentially bounded functions on $\mathbb{T}$. The Fourier coefficients $a_n (n \in \mathbb{Z})$ are defined by

$$a_n = \frac{1}{2\pi} \int_0^{2\pi} a(e^{i\theta}) e^{-in\theta} d\theta. \tag{1.14}$$

□

1.9 Approximation Methods for Bounded Operators

Let H be a separable Hilbert space and A be a bounded operator on H. Let (H_n) be a filtration of H. Let P_n be the orthogonal projection on H with range H_n. Then $P_n \to I$ strongly, where I is the identity operator.

Definition 1.9.1. [9, Definition 1.1] A sequence (A_n) of operators with $A_n \in \mathcal{B}(H_n)$ is an approximation method for $A \in \mathcal{B}(H)$ if $A_n P_n$ converges strongly to A as $n \to \infty$.

Definition 1.9.2. [9, Definition 1.2] The approximation method (A_n) for A is applicable if for every $y \in H$, there exists n_0 such that the equations

$$A_n x_n = P_n y \ (n = 1, 2, 3, \ldots)$$

possess unique solutions x_n for every $n \geq n_0$ and if these solutions converge in norm to a solution of the equation

$$Ax = y.$$

Definition 1.9.3. [9, Definition 1.3]

A sequence (A_n) of operators with $A_n \in \mathcal{B}(H_n)$ is stable if there exists a positive integer n_0 such that the operators A_n are invertible for every $n \geq n_0$ and $\sup\limits_{n \geq n_0} \|A_n^{-1} P_n\| < \infty$.

Theorem 1.9.4. (Polski)[9, Theorem 1.4.]

Let (A_n) with $A_n \in \mathcal{B}(H_n)$ be an approximation method for the operator $A \in \mathcal{B}(H)$. This method is applicable if and only if A is invertible and the sequence (A_n) is stable. □

Remark 1.9.5. From the above theorem we remark that if A is invertible, then (A_n) is an applicable method for A if and only if (A_n) is stable.

1.10 Approximation Numbers of Bounded Operators

Let H and H' be Hilbert spaces and $A \in \mathcal{B}(H, H')$, where $\mathcal{B}(H, H')$ denotes the Banach space of all bounded operators: $H \to H'$. The k^{th} approximation number $s_k(A)$ is defined by

$$s_k(A) = \inf\{\|A - F\|/F \in \mathcal{B}(H, H'),\ \text{rank } F \leq k - 1\}. \tag{1.15}$$

It is clear from (1.15) that

$$s_1(A) = \|A\| \tag{1.16}$$

and

$$s_1(A) \geq\ s_2(A) \geq \cdots \tag{1.17}$$

Also, it can be shown that

$$\lim_{k \to \infty} s_k(A) = \|A\|_{ess.}, \tag{1.18}$$

where $\|A\|_{ess.}$ is the essential norm of A, given by

$$\|A\|_{ess.} = \inf\{\|A - K\|\}/K \in \mathcal{K}(H, H')\}. \tag{1.19}$$

$\mathcal{K}(H, H')$ denotes the space of compact operators: $H \to H'$

A proof of (1.18) for $H' = H$ is given in [8, Theorem 10.1].

Let H and H' be separable. Let $\{u_1, u_2, u_3, \ldots\}$ and $\{v_1, v_2, v_3, \ldots\}$ be orthonormal bases for H and H' respectively. For $n = 1, 2, 3, \ldots,$ let $H_n = \mathit{span}\ \{u_1, u_2, \ldots, u_n\}$ and $H'_n = \mathit{span}\ \{v_1, v_1, \ldots v_n\}$; let P_n be the orthogonal projection on H with range H_n and Q_n be the orthogonal projection on H' with range H'_n. For $A \in \mathcal{B}(H, H')$, define $A_n = Q_n A P_n / H_n$. Then $A_n \in \mathcal{B}(H_n, H'_n)$. Now

$$s_k(A_n) = \inf\{\|A_n - F_n\| / F \in \mathcal{B}(H_n, H'_n),\ \text{rank}\ F_n \leq k - 1\}. \tag{1.20}$$

Theorem 1.10.1. [3, Theorem 1.1.]

$$\lim_{n \to \infty} s_k(A_n) = s_k(A). \tag{1.21}$$

The above result is proved in [3] for the case $H = H'$. The same proof applies even when H and H' are different Hilbert spaces. □

Chapter 2

APPROXIMATION METHODS FOR UNBOUNDED SELF-ADJOINT OPERATORS

In this chapter, an attempt is made to develop a general theory on approximation methods for unbounded self-adjoint operators acting in arbitrary Hilbert spaces. We use the classical notion of resolvent convergence for the approximation.

2.1 Introduction

Let H be an infinite dimensional separable Hilbert space with inner product $\langle , \rangle$ and A be a densely defined linear operator in H. Let $D(A)$ denote the domain of A. Suppose A is invertible. That is, $A : D(A) \to H$ is a bijection and $A^{-1} \in \mathcal{B}(H)$, where $\mathcal{B}(H)$ denotes the class of bounded operators on H. Then A is uniquely solvable, in the sense that, for every $y \in H$, the equation

$$Ax = y \tag{2.1}$$

has a unique solution $x \in H$.

If A is not invertible but onto, then, for each $y \in H$ the equation (2.1) has a solution, which need not be unique.

Supplementing tools for the computation of the solution x of (2.1) is one of the fundamental problems in Functional Analysis. Computation of the exact solution x is a challenging task and, in fact, may not be possible in many cases. We will be interested in finding approximate solutions of (2.1) in those cases.

Thus, the problem is reduced to finding a sequence of 'uniquely solvable' operators A_n defined on certain (finite dimensional) subspaces D_n of $D(A)$ such that (A_n) converges to A in some sense, and a sequence (y_n) in H with $y_n \in R(A_n)$, the range of A_n, such that (y_n) converges to y in H satisfying the following:

if x_n is the unique solution of the equation

$$A_n x_n = y_n \quad (n = 1, 2, 3, \ldots), \tag{2.2}$$

then, (x_n) converges in H to the unique solution x of (2.1).

By a uniquely solvable operator, we mean an operator T such that for every y in the co-domain of T, the equation $Tx = y$ has a unique solution x in the domain of T. Thus, T is uniquely solvable if and only if T is invertible.

Here, A_n are operators on finite dimensional spaces. So, computation of the unique solutions x_n of (2.2) is not a difficult task in this age of computing.

A is unbounded in general. So, it may not be possible to define convergence in the usual sense. But there are other notions of convergence such as resolvent convergence, generalized convergence, etc.

Thus, the tasks before us are:

1. To find a sequence (A_n) of operators such that $A_n \to A$ in some sense, and
2. To see whether (x_n) converges to x in H.

Let us now define an approximation method for an unbounded operator:

Definition 2.1.1. *Let A be a densely-defined linear operator in an infinite dimensional separable Hilbert space H. A sequence (A_n), where A_n is an operator defined on certain finite dimensional subspace D_n of $D(A)$, such that $A_n \to A$ in some sense is called an approximating sequence or an approximation method for A.*

2.2 Resolvent Convergence w.r.t. Filtrations

Our main objective in this chapter is to discuss approximating sequences for unbounded self-adjoint operators. It may not be possible to define convergence in the usual sense for unbounded operators. There are other notions of convergence among unbounded operators. Resolvent convergences are the most familier ones among self-adjoint (unbounded) operators. Let us recall the definition of strong resolvent convergence.

By an operator we mean a linear operator.

Definition 2.2.1. *Let A, A_n $(n = 1, 2, 3, \ldots)$ be self-adjoint operators in a Hilbert space H. (A_n) is said to converge to A in the strong resolvent sense if $R_\lambda(A_n) \to R_\lambda(A)$ strongly for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$. That is, if*

$$R_\lambda(A_n)y \to R_\lambda(A)y \quad \text{for every } y \in H \text{ and } \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{2.3}$$

Here $R_\lambda(A)$ is the resolvent of A:

$$R_\lambda(A) = (\lambda I - A)^{-1},$$

where I is the identity operator on H.

In the above definition the operators A_n are defined on dense subspaces of H and $R_\lambda(A_n) \in \mathcal{B}(H)$, the class of bounded linear operators on H. But, in our case, the A_n are defined on certain finite dimensional subspaces of H and hence $R_\lambda(A_n)$ too are bounded operators on finite dimensional spaces.

So, we need to slightly modify the definition of resolvent convergence to satisfy our needs. First we recall the definition of a filtration:

A filtration of a separable Hilbert space H is a sequence $(H_1, H_2, \ldots)$ of finite dimensional subspaces of H with $H_n \subseteq H_{n+1}$ for every n such that $\overline{\bigcup_n H_n} = H$. The bar denotes closure.

Now we define resolvent convergence with respect to a filtration.

Definition 2.2.2. *Let A be a self-adjoint operator in H. Let (H_n) be a filtration of H. For $n = 1, 2, 3, \ldots$, let A_n be a self-adjoint operator on H_n and let P_n be the orthogonal projection on H with range H_n. We say that $A_n \to A$ in the strong resolvent sense if $R_\lambda(A_n)P_n \to R_\lambda(A)$ strongly for every $\lambda \in \mathbb{C} \setminus \mathbb{R}$. That is, if*

$$R_\lambda(A_n)P_n y \to R_\lambda(A)y \quad \text{for every } y \in H \text{ and for every } \lambda \in \mathbb{C} \setminus \mathbb{R}. \tag{2.4}$$

For each n, A_nP_n is an operator: $H \to H_n$. As H_n is a subspace of H, A_nP_n can be considered as an operator on H. Being an operator on the finite dimensional space H_n, A_n is bounded. Hence A_nP_n is bounded.

Moreover A_nP_n is self-adjoint. Self-adjointness can be verified directly.

Let $x, y \in H$.

Being a finite dimensional space, H_n is closed. Therefore, $H = H_n \oplus H_n^{\perp}$, where $H_n^{\perp}$ denotes the orthogonal complement of H_n in H.

Thus, x and y can be uniquely expressed in the form $x = x_1 + x_2$ and $y = y_1 + y_2$, where $x_1, y_1 \in H_n$ and $x_2, y_2 \in H_n^{\perp}$.

Then $P_n x = x_1$ and $P_n y = y_1$.

Therefore,

$$\langle A_n P_n x, y \rangle = \langle A_n x_1, y \rangle = \langle A_n x_1, y_1 + y_2 \rangle = \langle A_n x_1, y_1 \rangle + \langle A_n x_1, y_2 \rangle. \tag{2.5}$$

Now $A_n x_1 \in H_n$ as A_n is an operator: $H_n \to H_n$ and $y_2 \in H_n^{\perp}$.

So $\langle A_n x_1, y_2 \rangle = 0$.

Hence (2.5) implies that

$$\langle A_n P_n x, y \rangle = \langle A_n x_1, y_1 \rangle. \tag{2.6}$$

And

$$\begin{aligned} \langle x, A_n P_n y \rangle &= \langle x, A_n y_1 \rangle \\ &= \langle x_1 + x_2, A_n y_1 \rangle \\ &= \langle x_1, A_n y_1 \rangle + \langle x_2, A_n y_1 \rangle. \end{aligned} \tag{2.7}$$

Now $x_2 \in H_n^{\perp}$ and $A_n y_1 \in H_n$.

So $\langle x_2, A_n y_1 \rangle = 0$.

Hence (2.7) implies that

$$\langle x, A_n P_n y \rangle = \langle x_1, A_n y_1 \rangle. \tag{2.8}$$

But $\langle A_n x_1, y_1\rangle = \langle x_1, A_n y_1\rangle$ as A_n is self-adjoint on H_n.

Therefore, from (2.6) and (2.8) we get

$$\langle A_n P_n x, y\rangle = \langle x, A_n P_n y\rangle \quad \text{for every } x, y \in H \tag{2.9}$$

and $A_n P_n$ is self-adjoint. □

Thus, with respect to a filtration (H_n) of H and a sequence (A_n) of self-adjoint operators with $A_n \in B(H_n)$, we get a sequence $(A_n P_n)$ of self-adjoint operators on H. So we can speak of resolvent convergences of the sequence $(A_n P_n)$ in the usual sense. This resolvent convergence and the resolvent convergence we defined in Definition 2.2.3 are in fact one and the same. We prove this in the following proposition.

Proposition 2.2.3. *Let A be a (densely-defined) self-adjoint operator in H and (H_n) be a filtration of H. For $n = 1, 2, 3, \ldots,$ let A_n be a self-adjoint operator on H_n and P_n be the projection on H with range H_n. Then $A_n \to A$ in the strong resolvent sense if and only if $A_n P_n \to A$ in the strong resolvent sense.*

Proof: Let $y \in H$ and $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

Let I denote the identity operator on H and I_n denote the identity operator on H_n.

Consider

$$\begin{aligned} R_\lambda(A_n P_n) - R_\lambda(A_n) P_n &= (\lambda I - A_n P_n)^{-1} - (\lambda I_n - A_n)^{-1} P_n \\ &= (\lambda I - A_n P_n)^{-1}\{I - (\lambda I - A_n P_n)(\lambda I_n - A_n)^{-1} P_n\}. \end{aligned}$$

Therefore,

$$\{R_\lambda(A_n P_n) - R_\lambda(A_n) P_n\}(y) = (\lambda I - A_n P_n)^{-1}\{y - (\lambda I - A_n P_n)(\lambda I_n - A_n)^{-1} P_n y\}. \tag{2.10}$$

Now $P_n y \in H_n$ and so $(\lambda I_n - A_n)^{-1} P_n y \in H_n$.

For $z \in H_n$,

$$P_n z = z \quad \text{and} \quad Iz = I_n z. \tag{2.11}$$

Put $z = (\lambda I_n - A_n)^{-1} P_n y \in H_n$. (2.10) implies

$$\begin{aligned}
\{R_\lambda(A_n P_n) - R_\lambda(A_n)P_n\}(y) &= (\lambda I - A_n P_n)^{-1}\{y - (\lambda I - A_n P_n)z\} \\
&= (\lambda I - A_n P_n)^{-1}\{y - (\lambda I_n - A_n)z\}, \quad \text{by (2.11)} \\
&= (\lambda I - A_n P_n)^{-1}\{y - (\lambda I_n - A_n)(\lambda I_n - A_n)^{-1} P_n y\} \\
&= (\lambda I - A_n P_n)^{-1}\{y - P_n y\}. \qquad (2.12)
\end{aligned}$$

Hence

$$\begin{aligned}
\|R_\lambda(A_n P_n)y - R_\lambda(A_n)P_n y\| &\le \|(\lambda I - A_n P_n)^{-1}\| \|y - P_n y\| \\
&\le |Im(\lambda)|^{-1} \|\|y - P_n y\|, \quad \text{by 1.3} \\
&\to 0 \quad \text{as } n \to \infty, \text{ since } P_n y \to y.
\end{aligned}$$

Hence $R_\lambda(A_n)P_n y \to R_\lambda(A)y$ if and only if $R_\lambda(A_n P_n)y \to R_\lambda(A)y$ and the result is proved. □

Remark 2.2.4. *In order to prove strong resolvent convergence one need not show the convergence for all* $\lambda \in \mathbb{C} \setminus \mathbb{R}$. *One need to show the convergence only for some* $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$.

This is evident from proposition 2.2.3 and Theorem 1.6.2(b).

Using proposition 2.2.3, Theorem 1.6.5 can be restated as follows:

Theorem 2.2.5. *Let* (H_n) *be a filtration of* H. *For* $n = 1, 2, 3, \ldots,$ *let* A_n *be a self-adjoint operator on* H_n *and* P_n *be the (orthogonal) projection on* H *with range* H_n.

Suppose there exist points $\lambda_\circ$ in the upper half-plane and $\mu_\circ$ in the lower half-plane such that $R_{\lambda_\circ}(A_nP_n)x$ and $R_{\mu_\circ}(A_nP_n)x$ converge for each $x \in H$. Suppose further that one of the limiting operators, $T_{\lambda_\circ}$ or $T_{\mu_\circ}$, has a dense range. Then there exists a self-adjoint operator A in H such that $A_n \to A$ in the strong resolvent sense.

Combining Theorem 1.6.4 and Proposition 2.2.3 we get the following result:

Theorem 2.2.6. *Let A be a self-adjoint operator in H and A_n be a self-adjoint operator on H_n, $n = 1, 2, 3, \ldots$, where (H_n) is a filtration of H. Then $A_n \to A$ in the strong resolvent sense if and only if $e^{itA_nP_n} \to e^{itA}$ strongly for each $t \in \mathbb{R}$, where P_n is the orthogonal projection on H with range H_n, $n = 1, 2, 3, \ldots$*

Proof: (A_nP_n) is a sequence of self-adjoint operators on H. By Proposition 2.2.3, $A_n \to A$ in the strong resolvent sense if and only if $A_nP_n \to A$ in the strong resolvent sense.

But, $A_nP_n \to A$ in the strong resolvent sense if and only if $e^{itA_nP_n} \to e^{itA}$ strongly for each $t \in \mathbb{R}$, by Theorem 1.6.4. Hence the proof. □

Lemma 2.2.7. *Under the assumptions of Theorem 2.2.6., $e^{itA_nP_n}(x) \to e^{itA}(x)$ for all $x \in H$ if and only if $e^{itA_n}(x) \to e^{itA}(x)$ for all $x \in \cup H_n$.*

Proof: Assume that $e^{itA_nP_n}(x) \to e^{itA}(x)$ for all $x \in H$.
A_nP_n and A_n are bounded self-adjoint operators on H and H_n respectively.

Now

$$e^{itA_nP_n} = \sum_{k=0}^{\infty} \frac{(it)^k (A_nP_n)^k}{k!} \tag{2.13}$$

and

$$e^{itA_n} = \sum_{k=0}^{\infty} \frac{(it)^k A_n^k}{k!}. \tag{2.14}$$

For $x \in H_n$, $P_n x = x$ and $A_n P_n x = A_n x$.

Now

$$\begin{aligned}(A_nP_n)^2x &= A_nP_nA_nP_nx = A_nP_n(A_nx)\\ &= A_n(A_nx) \quad \text{ as } A_nx \in H_n\\ &= A_n^2x.\end{aligned}$$

Similarly, $(A_nP_n)^3x = A_n^3x$ and so on.

In general, $(A_nP_n)^kx = A_n^kx$, $k = 1, 2, 3, \ldots$ and is trivial for $k = 0$.

Therefore, from (2.13) and (2.14) we get,

$$e^{itA_nP_n}(x) = e^{itA_n}(x) \quad \text{for all } x \in H_n. \tag{2.15}$$

Now, let $x \in \cup H_n$.

Then $x \in H_{n_0}$ for some n_0.

So, $x \in H_n$ for all $n \geq n_0$ as $H_n \subseteq H_{n+1}$ for every n.

Hence, by (2.15), $e^{itA_nP_n}(x) = e^{itA_n}(x)$ for all $n \geq n_0$.

So,

$$\begin{aligned}\lim_{n\to\infty} e^{itA_n}(x) &= \lim_{n\to\infty} e^{itA_nP_n}(x)\\ &= e^{itA}(x), \quad \text{by our assumption.}\end{aligned}$$

Thus, $e^{itA_n}(x) \to e^{itA}(x)$ for all $x \in \cup H_n$.

Conversely, assume that $e^{itA_n}(x) \to e^{itA}(x)$ for all $x \in \cup H_n$.

Let $y \in H$ and $\varepsilon > 0$.

Since $\cup H_n$ is dense in H, there exists $x \in \cup H_n$ such that

$$\|y - x\| < \frac{\varepsilon}{3}. \tag{2.16}$$

By assumption, $e^{itA_n}(x) \to e^{itA}(x)$.

Therefore, there exists a positive integer n_1 such that

$$\|e^{itA_n}(x) - e^{itA}(x)\| < \frac{\varepsilon}{3} \quad \text{for all } n \geq n_1 . \tag{2.17}$$

As $x \in \cup H_n$, $x \in H_{n_2}$ for some n_2.

Then,

$$x \in H_n \quad \text{for all } n \geq n_2 .$$

So,

$$e^{itA_nP_n}(x) = e^{itA_n}(x) \quad \text{for all } n \geq n_2. \tag{2.18}$$

Let $n_0 = \max\{n_1, n_2\}$ and $n \geq n_0$.

Then,

$$\begin{aligned}
\|e^{itA_nP_n}(y) - e^{itA}(y)\| &\leq \|e^{itA_nP_n}(y) - e^{itA_nP_n}(x)\| \\
&\quad + \|e^{itA_nP_n}(x) - e^{itA_n}(x)\| \\
&\quad + \|e^{itA_n}(x) - e^{itA}(x)\| \\
&\quad + \|e^{itA}(x) - e^{itA}(y)\| \\
&< \|e^{itA_nP_n}\|\|y - x\| + 0 + \frac{\varepsilon}{3} \\
&\quad + \|e^{itA}\|\|x - y\|, \quad \text{by (2.18) and (2.17)} \\
&< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3}, \quad \text{from (2.17)}, \quad \text{as } \|e^{itA_n}\| = 1 = \|e^{itA}\| \\
&= \varepsilon.
\end{aligned}$$

Hence $e^{itA_nP_n}(y) \to e^{itA}(y)$. □

Using this lemma, we can restate Theorem 2.2.6 as follows:

Theorem 2.2.8. *Let A_n be a self-adjoint operator on H_n for each $n = 1, 2, 3, \ldots$, where (H_n) is a filtration of H and let A be a self-adjoint operator in H. Then, $A_n \to A$ in the strong resolvent sense if and only if $e^{itA_n}(x) \to e^{itA}(x)$ for every $x \in \cup H_n$ and for every $t \in \mathbb{R}$.* $\square$

Theorem 2.2.9. *Let H_n be a filtration of H. For $n = 1, 2, 3, \ldots$, let A_n be self-adjoint on H_n and let A be a self-adjoint operator on H. Then $A_n \to A$ in the strong resolvent sense if and only if $e^{itA_n} P_n \to e^{itA}$ strongly, for every $t \in \mathbb{R}$, where P_n is the orthogonal projection on H with range H_n.*

Proof: Because of Theorem 2.2.8, it is enough to prove that $e^{itA_n} P_n(x) \to e^{itA}(x)$ for all $x \in H$ if and only if $e^{itA_n}(x) \to e^{itA}(x)$ for all $x \in \cup H_n$ and for all $t \in \mathbb{R}$. Let $t \in \mathbb{R}$. Suppose

$$e^{itA_n} P_n(x) \to e^{itA}(x) \quad \text{for all } x \in H. \tag{2.19}$$

Let $x \in \cup H_n$. Then, $x \in H_{n_0}$ for some n_0.

So, $x \in H_n$ for all $n \geq n_0$.

Now for $n \geq n_0$,

$$\begin{aligned}
&\|e^{itA_n}(x) - e^{itA}(x)\| \\
&\quad \leq \|e^{itA_n}(x) - e^{itA_n} P_n(x)\| + \|e^{itA_n} P_n(x) - e^{itA}(x)\| \\
&\quad \leq \|e^{itA_n}\| \|x - P_n x\| + \|e^{itA_n} P_n(x) - e^{itA}(x)\| \\
&\quad = \|x - P_n x\| + \|e^{itA_n} P_n(x) - e^{itA}(x)\|, \quad \text{as } \|e^{itA_n}\| = 1 \\
&\quad \to 0, \quad \text{since } P_n x \to x \text{ and } e^{itA_n} P_n(x) \to e^{itA}(x) \text{ by our assumption.}
\end{aligned}$$

Thus, $e^{itA_n}(x) \to e^{itA}(x)$ for all $x \in \cup H_n$.

Conversely assume that

$$e^{itA_n}(x) \to e^{itA}(x) \quad \text{for all } x \in \cup H_n \,. \tag{2.20}$$

Let $x \in H$ and $\varepsilon > 0$.

Since $\cup H_n$ is dense in H, there exists $y \in \cup H_n$ such that

$$\|x - y\| < \varepsilon/4. \tag{2.21}$$

By (2.20),

$$e^{itA_n}(y) \to e^{itA}(y). \tag{2.22}$$

Consider

$$\begin{aligned}
\|e^{itA_n}P_n(x) - e^{itA}(x)\| &\le \|e^{itA_n}P_n(x) - e^{itA_n}P_n(y)\| + \|e^{itA_n}P_n(y) - e^{itA_n}(y)\| \\
&\quad + \|e^{itA_n}(y) - e^{itA}(y)\| + \|e^{itA}(y) - e^{itA}(x)\| \\
&\le \|e^{itA_n}\|\|P_n\|\|x - y\| + \|e^{itA_n}\|\|P_n y - y\| \\
&\quad + \|e^{itA_n}(y) - e^{itA}(y)\| + \|e^{itA}\|\|y - x\| \\
&\le \|x - y\| + \|P_n y - y\| + \|e^{itA_n}(y) - e^{itA}(y)\| + \|y - x\| \\
&\qquad \text{as } \|e^{itA_n}\| = 1 = \|e^{itA}\| \text{ and } \|P_n\| \le 1 \\
&\le \frac{\varepsilon}{4} + \|P_n y - y\| + \|e^{itA_n}(y) - e^{itA}(y)\| \\
&\quad + \frac{\varepsilon}{4}, \quad \text{as } \|x - y\| < \frac{\varepsilon}{4}.
\end{aligned} \tag{2.23}$$

Since $P_n y \to y$ and $e^{itA_n}(y) \to e^{itA}(y)$, there exists a positive integer n_0 such that

$$\|P_n y - y\| < \frac{\varepsilon}{4} \quad \text{and} \quad \|e^{itA_n}(y) - e^{itA}(y)\| < \frac{\varepsilon}{4} \quad \text{for all } n \ge n_0 \,.$$

Therefore, (2.23) implies

$$\|e^{itA_n}P_n(x) - e^{itA}(x)\| < \varepsilon \quad \text{for all } n \ge n_0 \,. \tag{2.24}$$

Hence

$$e^{itA_n}P_n(x) \to e^{itA}(x) \quad \text{for all } x \in H\,.$$

This completes the proof. □

2.3 Applicability and Resolvent Convergence

In this section we discuss applicable approximation methods. The relation between applicability and resolvent convergence is established. First we define applicability.

Definition 2.3.1. *Let H be a separable Hilbert space and A be a densely-defined operator in H which is onto. Let (A_n) be a sequence of operators with $A_n \in \mathcal{B}(H_n)$, where (H_n) is a filtration of H such that $H_n \subseteq D(A)$. (A_n) is said to be an applicable method for A if the following holds:*

There exists a positive integer n_o such that A_n is invertible for all $n \geq n_0$ and for every $y \in H$ the sequence (x_n) converges to x, where x_n is the unique solution of

$$A_n x_n = P_n y \quad \textit{for every } n \geq n_o \tag{2.25}$$

and x is a solution of

$$Ax = y. \tag{2.26}$$

We consider an unbounded self-adjoint operator A which is onto, so that for every $y \in H$, the equation $Ax = y$ has a solution. Suppose (A_n) is an applicable method for A. Then (A_n) will be stable (Theorem 2.3.3) and $A_n \to A$ in the strong resolvent sense (Theorem 2.3.5). We will prove in Theorem 2.4.1 that A is injective. Thus, A is invertible and x (in 2.26) is the unique solution of $Ax = y$.

The definition of applicability may be reformulated mathematically as follows: (A_n) is an applicable method for A if and only if A_n are invertible for all sufficiently large n and $A_n^{-1}P_n y \to A^{-1}y$ for every $y \in H$.

Also we define stability of approximation methods.

Definition 2.3.2. *Let H be a separable Hilbert space and (H_n) be a filtration of H. A sequence of operators (A_n) with $A_n \in \mathcal{B}(H_n)$ is said to be stable if there exists a positive integer n_0 such that A_n is invertible for every $n \geq n_0$ and*

$$\sup_{n \geq n_0} \|A_n^{-1}P_n\| < \infty. \tag{2.27}$$

Stability is a necessary condition for applicability of approximation methods. We prove this in the following proposition. The proof is the same as in the case of bounded operators.

Theorem 2.3.3. *Every applicable method is stable.*

Proof: Let A be a densely-defined operator in a separable Hilbert space H and let (A_n) be an applicable method for A, with $A_n \in \mathcal{B}(H_n)$, where (H_n) is a filtration of H.

Then, there exists a positive integer n_0 such that A_n is invertible for every $n \geq n_0$ and $A_n^{-1}P_n y \to A^{-1}y$ for every $y \in H$, where P_n is the orthogonal projection on H with range H_n.

So, $\{A_n^{-1}P_n y / n \geq n_0\}$ is a bounded set for each $y \in H$.

Hence, by the uniform boundedness theorem, $\{\|A_n^{-1}P_n\| / n \geq n_0\}$ is bounded. Thus (A_n) is stable. □

The relationship between applicability and resolvent convergence is established in the following results.

Theorem 2.3.4. *Let H be a separable Hilbert space. Let A be a self-adjoint, invertible operator in H. For $n = 1, 2, 3, \dots,$ let A_n be a self-adjoint, invertible operator on H_n, where (H_n) is a filtration of H. If $A_n \to A$ in the strong resolvent sense and (A_n) is stable, then, (A_n) is an applicable method for A.*

Proof: For $n = 1, 2, 3, \dots,$ let P_n be the orthogonal projection on H with range H_n.

As $A_n \to A$ in the strong resolvent sense, $A_n P_n \to A$ in the strong resolvent sense.

So,

$$(iI - A_n P_n)^{-1} \to (iI - A)^{-1} \text{ strongly, where } i = \sqrt{-1}. \tag{2.28}$$

Let $x \in D(A)$.

Put $x_n = (iI - A_n P_n)^{-1}(iI - A)x, n = 1, 2, 3, \dots.$

Then $x_n \to (iI - A)^{-1}(iI - A)x = x$.

And

$$\begin{aligned} A_n P_n x_n &= ix_n - (iI - A_n P_n)x_n \\ &= ix_n - (iI - A_n P_n)(iI - A_n P_n)^{-1}(iI - A)x \\ &= ix_n - (iI - A)x \\ &\to ix - (iI - A)x = Ax. \end{aligned}$$

As (A_n) is stable, A_n^{-1} exists for all $n \geq n_0$, for some positive integer n_0, and

$$\sup_{n \geq n_0} \|A_n^{-1}\| = \sup_{n \geq n_0} \|A_n^{-1} P_n\| < \infty. \tag{2.29}$$

Now for $n \geq n_0$,

$$
\begin{aligned}
[A_n^{-1}P_n - A^{-1}]Ax &= A_n^{-1}P_nAx - x \\
&= (A_n^{-1}P_nAx - P_nx_n) + (P_nx_n - P_nx) + (P_nx - x) \\
&= A_n^{-1}(P_nAx - A_nP_nx_n) + P_n(x_n - x) + (P_nx - x).
\end{aligned}
$$

This implies,

$$\|(A_n^{-1}P_n - A^{-1})Ax\| \leq \|A_n^{-1}\|\|P_nAx - A_nP_nx_n\| + \|P_n\|\|x_n - x\| + \|P_nx - x\|. \quad (2.30)$$

(2.29) implies that, there exists $M < \infty$ such that $\|A_n^{-1}\| \leq M$ for each $n \geq n_0$.

As $n \to \infty$, $A_nP_nx_n \to Ax$, $P_nAx \to Ax$, $P_nx \to x$ and $x_n \to x$. Also $\|P_n\| \leq 1$. Hence (2.30) implies,

$$(A_n^{-1}P_n - A^{-1})Ax \to 0.$$

Thus, $(A_n^{-1}P_n - A^{-1})Ax \to 0$ for every $x \in D(A)$.

That is, $(A_n^{-1}P_n - A^{-1})y \to 0$ for every $y \in R(A)$.

But $R(A) = H$ as A is invertible. Therefore, $(A_n^{-1}P_n - A^{-1})y \to 0$ for every $y \in H$.

That is, $A_n^{-1}P_ny \to A^{-1}y$ for every $y \in H$.

In other words, (A_n) is an applicable method for A. □

Theorem 2.3.5. *Let A and A_n be as in the previous theorem. If (A_n) is an applicable method for A, then, $A_n \to A$ in the strong resolvent sense.*

Proof: As (A_n) is an applicable method for A, there exists a positive integer n_0 such that A_n is invertible for all $n \geq n_0$ and $A_n^{-1}P_n \to A^{-1}$ strongly.

Let $x \in D(A)$.

For $n \geq n_0$, put $x_n = A_n^{-1} P_n A x$.

Then $x_n \in H_n$ for every n and $x_n \to A^{-1}Ax = x$.

And

$$
\begin{aligned}
A_n P_n x_n &= A_n x_n, \quad \text{since } P_n x_n = x_n, \text{ as } x_n \in H_n \\
&= A_n A_n^{-1} P_n A x \\
&= P_n A x \to Ax.
\end{aligned}
$$

Let $\lambda \in \mathbb{C} \backslash \mathbb{R}$.

Consider

$$
\begin{aligned}
&[R_\lambda(A_n P_n) - R_\lambda(A)](\lambda I - A)x \\
&\quad = [(\lambda I - A_n P_n)^{-1} - (\lambda I - A)^{-1}](\lambda I - A)x \\
&\quad = (\lambda I - A_n P_n)^{-1}(\lambda I - A)x - x \\
&\quad = [(\lambda I - A_n P_n)^{-1}(\lambda I - A)x - x_n] + [x_n - x] \\
&\quad = (\lambda I - A_n P_n)^{-1}[(\lambda I - A)x - (\lambda I - A_n P_n)x_n] + [x_n - x]. \qquad (2.31)
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\|[R_\lambda(A_n P_n) - R_\lambda(A)](\lambda I - A)x\| \\
&\qquad \leq \|(\lambda I - A_n P_n)^{-1}\| \|(\lambda I - A)x - (\lambda I - A_n P_n)x_n\| + \|x_n - x\| \\
&\qquad \leq |Im\lambda|^{-1} \|(\lambda I - A)x - (\lambda I - A_n P_n)x_n\| + \|x_n - x\|, \text{ by Proposition 1.4.7} \\
&\qquad \to 0, \quad \text{as } x_n \to x \text{ and } A_n P_n x_n \to Ax.
\end{aligned}
$$

Thus,

$$[R_\lambda(A_n P_n) - R_\lambda(A)]y \to 0 \quad \text{for every } y \in R(\lambda I - A).$$

But $\lambda I - A$ is invertible as A is self-adjoint and λ is non-real.

So, $R(\lambda I - A) = H$.

Therefore, (2.33) implies

$$R_\lambda(A_n P_n)y \to R_\lambda(A)y \quad \text{for every } y \in H.$$

That is, $A_n \to A$ in the strong resolvent sense. □

Remark 2.3.6. *We consolidate the Theorems 2.3.3, 2.3.4 and 2.3.5 as follows:*

Let H be a separable Hilbert space and (H_n) be an filtration of H. Let A be a invertible, self-adjoint operator in H. For $n = 1, 2, 3, \ldots,$ let A_n be a self-adjoint operator on H_n.

(a) If (A_n) is stable, then, (A_n) is an applicable method for A if and only if $A_n \to A$ in the strong resolvent sense.

(b) If $A_n \to A$ in the strong resolvent sense, then, (A_n) is an applicable method for A if and only if (A_n) is stable. □

Let A be an invertible self-adjoint operator in a separable Hilbert space H. Let (H_n) be a filtration of H. For $n = 1, 2, 3, \ldots,$ let A_n be an invertible self-adjoint operator on H_n.

Suppose (A_n) is an applicable method for A. Then by Theorem 2.3.5, $A_n \to A$ in the strong resolvent sense. Therefore, by Theorem 2.2.9, $e^{itA_n}P_n \to e^{itA}$ strongly, for every $t \in \mathbb{R}$.

Now e^{itA} and e^{itA_n} are bounded and invertible (in fact, unitary) operators on H and H_n respectively.

Also, (e^{itA_n}) is stable as $\|(e^{itA_n})^{-1}P_n\| = \|e^{-itA_n}P_n\| = \|e^{-itA_n}\| = 1$ for every n.

Hence by Polski's theorem (Theorem 1.9.4), (e^{itA_n}) is an applicable method for e^{itA}.

Conversely, assume that (e^{itA_n}) is an applicable method for e^{itA}.

Then $e^{-itA_n}P_n \to e^{-itA}$ strongly for every $t \in \mathbb{R}$, since $(e^{itA})^{-1} = e^{-itA}$.

So, $e^{itA_n}P_n \to e^{itA}$ strongly for every $t \in \mathbb{R}$.

By Theorem 2.2.9, $A_n \to A$ in the strong resolvent sense.

Hence by Theorem 2.3.4, (A_n) is an applicable method for A, provided (A_n) is stable.

Thus we have proved the result:

Theorem 2.3.7. *Let A be an invertible self-adjoint operator in a separable Hilbert space H. For $n = 1, 2, 3, \ldots$, let A_n be an invertible self-adjoint operator on H_n, where (H_n) is a filtration of H. If (A_n) is stable, then, (A_n) is an applicable method for A if and only if (e^{itA_n}) is an applicable method for e^{itA}.* □

Thus, from an approximation method for an unbounded self-adjoint operator A, we come to an approximation method for a bounded operator e^{itA}.

For a self-adjoint A, $e^{itA}(t \in \mathbb{R})$ is a strongly continuous one-parameter unitary group (Theorem 1.5.4).

Now, under the conditions of Theorem 2.3.7., (A_n) is an applicable method for A if and only if (e^{itA_n}) is an applicable method for e^{itA} if and only if $e^{itA_n} \to e^{itA}$ strongly for every $t \in \mathbb{R}$. In this case we shall say that the sequence of strongly continuous one-parameter unitary groups (e^{itA_n}) converges to the strongly continuous one-parameter unitary group e^{itA}.

We can restate Theorem 2.3.7. as follows:

Theorem 2.3.8. *Let A be invertible and self-adjoint in a separable Hilbert space H. For $n = 1, 2, 3, \ldots$, let A_n be invertible and self-adjoint on H_n, where (H_n) is a filtration of H. Suppose (A_n) is stable. Then, (A_n) is an applicable method for A if and only if the sequence of strongly continuous one-parameter unitary groups (e^{itA_n}) converges to the strongly continuous one-parameter unitary group e^{itA}.*

Example 2.3.9. *Consider the time-dependent Shrodinger equation*

$$\frac{du}{dt} = -iAu, \tag{2.32}$$

where $u = u(t)$ is a vector in some L^2- space H and A is a self-adjoint operator in H.

The solution of (2.32) is formally given by $u = u(t) = e^{-itA}u(0)$ [10,p.479–480]. (2.33)

Suppose (H_n) is a filtration of H and $A_n \in B(H_n)$ are self-adjoint.

Let $A_n \to A$ in the strong resolvent sense.

Then, by Theorem 2.29, $e^{-itA_n}P_n \to e^{-itA}$ strongly for every $t \in \mathbb{R}$, where for each n, P_n is the projection on H with range H_n.

So, $e^{itA_n}P_n \to e^{itA}$ strongly for every $t \in \mathbb{R}$. Hence $u_n = u_n(t) = e^{-itA_n}P_n u(0)$ approximates the solution of (2.32).

Here, the A_n are bounded. So, e^{-itA_n} has power series expansion and hence can be approximated by $I + \sum_{m=1}^{k} \frac{(-itA_n)^m}{m!}$. □

2.4 Stability

Stability plays a significant role in the theory of approximation methods. We saw in the previous section that every applicable method is stable. Also, if A is self-adjoint and (A_n) converges to A in the strong resolvent sense, then, (A_n) is an applicable method for A if and only if (A_n) is stable. These results expose the significance of stability.

The aim of this section is to analyze a few other properties of stability. The following result is an interesting one.

Theorem 2.4.1. *Let H be a separable Hilbert space and A be a self-adjoint operator in H. Let A_n be self-adjoint and invertible on H_n, for each n, where (H_n) is a filtration of H. Suppose $A_n \to A$ in the strong resolvent sense. If (A_n) is stable, then A is injective.*

Proof: Suppose $Ax = 0$, where $x \in D(A)$.

Let k be a positive integer.

Put $\lambda = k^{-1}i,$ where $i = \sqrt{-1}$.

Since A is self-adjoint, $(\lambda I - A)$ is invertible, where I is the identity operator on H.

Now

$$\begin{aligned} x &= (\lambda I - A)^{-1}(\lambda I - A)x \\ &= (\lambda I - A)^{-1}(\lambda x), \qquad \text{as } Ax = 0. \end{aligned}$$

So,

$$(\lambda I - A)^{-1}x = \lambda^{-1}x. \tag{2.34}$$

By assumption, $A_n \to A$ in the strong resolvent sense.

So, $R_\lambda(A_n)P_n x \to R_\lambda(A)x$ as $n \to \infty$, where P_n is the orthogonal projection on H with range H_n.

That is, $(\lambda I_n - A_n)^{-1} P_n x \to (\lambda I - A)^{-1} x$, where I_n is the identity operator on H_n.

Hence, by (2.34),

$$(\lambda I_n - A_n)^{-1} P_n x \to \lambda^{-1} x \quad \text{as } n \to \infty. \tag{2.35}$$

This implies,

$$\|(\lambda I_n - A_n)^{-1} P_n x\| \to \|\lambda^{-1} x\| \quad \text{as } n \to \infty.$$

That is,

$$\|(\lambda I_n - A_n)^{-1} P_n x\| \to k\|x\| \quad \text{as } n \to \infty, \tag{2.36}$$

since $\lambda = k^{-1} i$.

As (A_n) is stable, A_n is invertible on H_n for each $n \geq n_o$, for some n_o, and $\sup_{n \geq n_0} \{\|A_n^{-1} P_n\|\} < \infty$.

Put

$$\beta = \sup_{n \geq n_0} \{\|A_n^{-1} P_n\|\} = \sup_{n \geq n_0} \{\|A_n^{-1}\|\}. \tag{2.37}$$

By the resolvent equation (1.2), we have, $R_\lambda(A_n) - R_\mu(A_n) = (\mu - \lambda) R_\mu(A_n) R_\lambda(A_n)$ for any λ and μ in the resolvent set $\rho(A)$ of A.

Let $n \geq n_o$.

As A_n is invertible (on H_n), $0 \in \rho(A_n)$.

Putting $\mu = 0$ in the resolvent equation, we get,

$$R_\lambda(A_n) - R_0(A_n) = -\lambda R_0(A_n) R_\lambda(A_n).$$

That is, $(\lambda I_n - A_n)^{-1} - (-A_n)^{-1} = -\lambda(-A_n)^{-1}(\lambda I_n - A_n)^{-1}$.

That is, $(\lambda I_n - A_n)^{-1} + A_n^{-1} = \lambda A_n^{-1}(\lambda I_n - A_n)^{-1}$.

Hence,

$$\begin{aligned}\|(\lambda I_n - A_n)^{-1} + A_n^{-1}\| &\leq |\lambda|\|A_n^{-1}\|\|(\lambda I_n - A_n)^{-1}\| \\ &\leq |\lambda|\beta\|(\lambda I_n - A_n)^{-1}\|, \quad \text{from (2.37)} \\ &\leq |\lambda|\beta|Im\lambda|^{-1}, \quad \text{since } \|(\lambda I_n - A_n)^{-1}\| \leq |Im\lambda|^{-1}.\end{aligned}$$

Thus,

$$\begin{aligned}\|(\lambda I_n - A_n)^{-1} + A_n^{-1}\| &\leq \frac{1}{k}\beta k, \quad \text{as } \lambda = \frac{i}{k} \\ &= \beta.\end{aligned}$$

So,

$$\begin{aligned}\|(\lambda I_n - A_n)^{-1}\| &\leq \|(\lambda I_n - A_n)^{-1} + A_n^{-1} - A_n^{-1}\| \\ &\leq \|(\lambda I_n - A_n)^{-1} + A_n^{-1}\| + \|A_n^{-1}\| \\ &\leq \beta + \beta = 2\beta.\end{aligned}$$

Now,

$$\begin{aligned}\|(\lambda I_n - A_n)^{-1}P_n x\| &\leq \|(\lambda I_n - A_n)^{-1}\|\|P_n\|\|x\| \\ &\leq 2\beta\|x\|, \quad \text{as } \|P_n\| \leq |.\end{aligned}$$

Since this is true for every $n \geq n_0$, we get

$$\lim_{n\to\infty} \|(\lambda I_n - A_n)^{-1}P_n x\| \leq 2\beta\|x\|.$$

Therefore, by (2.36),

$$k\|x\| \leq 2\beta\|x\|. \tag{2.38}$$

(2.38) is true for every positive integer k.

Hence $\|x\|$ must be 0 and hence $x = 0$.

Thus A is injective. □

Let $A_n (n = 1, 2, 3, \ldots)$ be self-adjoint. For any $\lambda \in \mathbb{C} \backslash \mathbb{R}$, $(\lambda I_n - A_n)$ is invertible (on H_n) and $\|\lambda I_n - A_n\| \leq |Im \lambda|^{-1}$.

Thus $(\lambda I_n - A_n)$ is a stable sequence, even if (A_n) is non-stable. In otherwords, the stability of $(\lambda I_n - A_n)$ does not imply the stability of (A_n). In this regard, the following result has some significance.

Proposition 2.4.2. *Let (H_n) be a filtration of H. Let A_n be self-adjoint on H_n for each n and let A_n be invertible for all $n \geq n_0$, for some n_0. Suppose* $\sup\{\|(iI_n - A_n)^{-1}\|\} < 1$. *Then (A_n) is stable.*

Proof: Put

$$\beta = \sup \|(iI_n - A_n)^{-1}\|. \tag{2.39}$$

Then $\beta < 1$, by our assumption.

By the resolvent equation (Theorem 1.4.6), we have, $R_\lambda(A_n) - R_\mu(A_n) = (\mu - \lambda) R_\mu(A_n) R_\lambda(A_n)$ for all $\lambda, \mu \in \rho(A_n)$.

Let $n \geq n_0$.

Now, $o, i \in \rho(A_n)$ as A_n is invertible and self-adjoint.

Hence, $R_o(A_n) - R_i(A_n) = i R_i(A_n) R_o(A_n)$.

That is, $(-A_n)^{-1} - (iI_n - A_n)^{-1} = i(iI_n - A_n)^{-1}(-A_n)^{-1}$,
which implies, $A_n^{-1} + (iI_n - A_n)^{-1} = i(iI_n - A_n)^{-1} A_n^{-1}$.

Hence, using (2.39),

$$\begin{aligned}\|A_n^{-1} + (iI_n - A_n)^{-1}\| &\leq \|(iI_n - A_n)^{-1}\| \|A_n^{-1}\| \\ &\leq \beta \|A_n^{-1}\|. \end{aligned} \tag{2.40}$$

Now,

$$\begin{aligned}\|A_n^{-1}\| &= \|A_n^{-1} + (iI_n - A_n)^{-1} - (iI_n - A_n)^{-1}\| \\ &\leq \|A_n^{-1} + (iI_n - A_n)^{-1}\| + \|(iI_n - A_n)^{-1}\| \\ &\leq \beta \|A_n^{-1}\| + \beta,\end{aligned}$$

which implies $\|A_n^{-1}\| \leq \beta/1 - \beta$.

Hence $\sup_{n \geq n_0} \|A_n^{-1}\| \leq \frac{\beta}{1-\beta}$.

Thus (A_n) is stable. □

Remark 2.4.3. Following is a generalization of the above result:

Let $\lambda \in \mathbb{C} \backslash \mathbb{R}$. *If* $\sup\{\|(\lambda I_n - A_n)^{-1}\|\} < |\lambda|^{-1}$, *then* (A_n) *is stable.*

Proof:

Put $\beta = \sup \|(\lambda I_n - A_n)^{-1}\| < \frac{1}{|\lambda|}$.

By the resolvent equation, for $n \geq n_o$,

$$R_o(A_n) - R_\lambda(A_n) = (\lambda - 0) R_\lambda(A_n) R_o(A_n).$$

That is,

$$(-A_n)^{-1} - (\lambda I_n - A_n)^{-1} = \lambda(\lambda I_n - A_n)^{-1}(-A_n)^{-1}.$$

Hence,

$$\|A_n^{-1} + (\lambda I_n - A_n)^{-1}\| \leq |\lambda| \; \|(\lambda I_n - A_n)^{-1}\| \; \|A_n^{-1}\|$$
$$\leq |\lambda|\beta \; \|A_n^{-1}\|.$$

So,

$$\|A_n^{-1}\| \leq \|A_n^{-1} + (\lambda I_n - A_n)^{-1}\| + \|(\lambda I_n - A_n)^{-1}\|$$
$$\leq |\lambda|\beta\|A_n^{-1}\| + \beta.$$

From this we get,

$$\|A_n^{-1}\| \leq \frac{\beta}{1 - |\lambda|\beta} \quad \text{for each } n \geq n_o\,.$$

Hence $\sup_{n \geq n_o} \|A_n^{-1}\| \leq \frac{\beta}{1-|\lambda|\beta} < \infty$ as $\beta < \frac{1}{|\lambda|}$. □

2.5 The Finite Section Method*

Consider an unbounded operator A in an infinite dimensional separable Hilbert space H. Let $\{e_1, e_2, e_3, \ldots\}$ be an orthonormal basis for H contained in $D(A)$, the domain of A. For $n = 1, 2, 3, \ldots,$ let $H_n = span\{e_1, e_2, \ldots, e_n\}$ and P_n be the orthogonal projection on H with range H_n. Then, $A_n := P_n A P_n / H_n$ is an (bounded) operator on H_n. The sequence (A_n) is called the finite section method for A.

Now, the first question that naturally arises is whether the finite section method (A_n) an approximation method for A, or, whether (A_n) converges to A in some sense. If A is a bounded operator, the answer is yes. In fact, $A_n \to A$ strongly, if A is a bounded operator. For the unbounded case, we are able to give certain sufficient conditions for the strong resolvent convergence of (A_n) to A, for a self-adjoint A.

* The contents of this section have appeared in J. Analysis Vol. 14 (2006), 69–78.

Theorem 2.5.1. *Let A be a self-adjoint operator in a separable Hilbert space H. Let (H_n) be a filtration of H such that $H_n \subseteq D(A)$ for every n. Suppose there exists a positive integer n_0 such that $A(H_n) \subseteq H_n$ for every $n \geq n_0$. Let P_n be the orthogonal projection on H with range H_n and let $A_n := P_n A P_n / H_n$. Then $A_n \to A$ in the strong resolvent sense.*

Proof:

Let $x \in H$ and $\lambda \in \mathbb{C} \backslash \mathbb{R}$.

Consider

$$\begin{aligned}
[R_\lambda(A) - R_\lambda(A_n)P_n](x) &= [(\lambda I - A)^{-1} - (\lambda I_n - A_n)^{-1}P_n](x) \\
&= (\lambda I - A)^{-1}[I - (\lambda I - A)(\lambda I_n - A_n)^{-1}P_n](x) \\
&= (\lambda I - A)^{-1}[I - (\lambda I - A)(\lambda I_n - P_nAP_n)^{-1}P_n](x) \\
&= (\lambda I - A)^{-1}[x - (\lambda I - A)(\lambda I_n - P_nAP_n)^{-1}P_n x]. \qquad (2.41)
\end{aligned}$$

Now, $(\lambda I_n - P_nAP_n)^{-1}P_n x \in H_n$.

Let $n \geq n_0$.

For $z \in H_n$, $P_n z = z$.

So, $AP_n z = Az \in H_n$, by our assumption that $A(H_n) \subseteq H_n$.

And, $P_nAP_n z = P_n A z = Az$.

Also, $I_n z = z = Iz$.

Put $z = (\lambda I_n - P_nAP_n)^{-1}P_n x \in H_n$.

Then (2.41) becomes,

$$
\begin{aligned}
[R_\lambda(A) - R_\lambda(A_n)P_n](x) &= (\lambda I - A)^{-1}[x - (\lambda I - A)z] \\
&= (\lambda I - A)^{-1}[x - (\lambda I_n z - P_n A P_n z)] \\
&= (\lambda I - A)^{-1}[x - (\lambda I_n - P_n A P_n)z] \\
&= (\lambda I - A)^{-1}[x - (\lambda I_n - P_n A P_n)(\lambda I_n - P_n A P_n)^{-1} P_n x] \\
&= (\lambda I - A)^{-1}[x - P_n x] \qquad (2.42)
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\|[R_\lambda(A) - R_\lambda(A_n)P_n](x)\| &\leq \|(\lambda I - A)^{-1}\| \, \|x - P_n x\| \\
&\leq |Im\lambda|^{-1} \|x - P_n x\|, \quad \text{as } \|(\lambda I - A)^{-1}\| \leq |Im\lambda|^{-1} \\
&\to 0 \quad \text{as } n \to \infty, \text{ since } P_n x \to x.
\end{aligned}
$$

This implies that $R_\lambda(A_n)P_n(x) \to R_\lambda(A)(x)$ for every $x \in H$.

That is, $A_n \to A$ in the strong resolvent sense. □

Now we prove a more general result.

Theorem 2.5.2. *Let A be a self-adjoint operator in a separable Hilbert space H and $A_n := P_n A P_n / H_n$, where (H_n) is a filtration of H such that $H_n \subseteq D(A)$. Let P_n be the orthogonal projection on H with $R(P_n) = H_n$. Let $\Gamma(A) = \{(x, Ax)/x \in D(A)\}$, the graph of A, and let $G = \{(x, Ax)/x \in \cup H_n\}$. Suppose G is dense in $\Gamma(A)$. Then $A_n \to A$ in the strong resolvent sense.*

Proof:

Put $D_1 = \cup H_n$.

Let $x \in D_1$.

Then $x \in H_{n_0}$ for some n_0.

Since $H_n \subseteq H_{n+1}$ for every n, $x \in H_n$ for every $n \geq n_0$.

Therefore,

$$P_n x = x \quad \text{for every } n \geq n_0 \tag{2.43}$$

Now, for $n \geq n_0$,

$$\begin{aligned} A_n P_n x &= P_n A P_n x \\ &= P_n A x \quad \text{(by 2.43)} \\ &\to Ax \quad \text{as } n \to \infty. \end{aligned}$$

Thus,

$$A_n P_n x \to Ax \quad \text{as } n \to \infty \text{ for all } x \in D_1. \tag{2.44}$$

Let $\lambda \in \mathbb{C} \backslash \mathbb{R}$.

Let $x \in D_1 = \cup H_n$.

Put $z = \lambda x - Ax = (\lambda I - A)x$.

Consider

$$\begin{aligned} [R_\lambda(A_n P_n) - R_\lambda(A)]z &= [(\lambda I - A_n P_n)^{-1} - (\lambda I - A)^{-1}]z \\ &= [(\lambda I - A_n P_n)^{-1} - (\lambda I - A)^{-1}](\lambda I - A)x \\ &= (\lambda I - A_n P_n)^{-1}(\lambda I - A)x - x \\ &= (\lambda I - A_n P_n)^{-1}[(\lambda I - A)x - (\lambda I - A_n P_n)x] \\ &= (\lambda I - A_n P_n)^{-1}[\lambda x - Ax - \lambda x + A_n P_n x] \\ &= (\lambda I - A_n P_n)^{-1}(A_n P_n x - Ax). \end{aligned} \tag{2.45}$$

So,

$$\begin{aligned}\|[R_\lambda(A_nP_n) - R_\lambda(A)]z\| &\le \|(\lambda I - A_nP_n)^{-1}\|\,\|A_nP_nx - Ax\| \\ &\le |Im\lambda|^{-1}\|A_nP_nx - Ax\|, \quad \text{by (1.3)} \\ &\to 0 \quad \text{as } n \to \infty, \quad \text{from (2.44).}\end{aligned}$$

Hence,

$$R_\lambda(A_nP_n)z \to R_\lambda(A)z \quad \text{for every } z \in (\lambda I - A)D_1. \tag{2.46}$$

We claim that $(\lambda I - A)D_1$ is dense in $(\lambda I - A)D$ where $D = D(A)$.

Let $(\lambda I - A)x \in (\lambda I - A)D$, where $x \in D = D(A)$.

Now $(x, Ax) \in \Gamma(A)$.

By assumption, $G = \{(x, Ax)/x \in \cup H_n\}$ is dense in $\Gamma(A)$.

Therefore, there is a sequence (x_n, Ax_n) in G such that $(x_n, Ax_n) \to (x, Ax)$ in $H \times H$.

This implies that $x_n \to x$ and $Ax_n \to Ax$.

Hence, $(\lambda I - A)x_n = \lambda x_n - Ax_n \to \lambda x - Ax = (\lambda I - A)x$.

Here (x_n) is a sequence in $D_1 = \cup H_n$.

Thus $(\lambda I - A)D_1$ is dense in $(\lambda I - A)D$.

But $(\lambda I - A)D = R(\lambda I - A)$, the range of $\lambda I - A$.

Note that $R(\lambda I - A) = H,$ as $(\lambda I - A)$ is invertible, since A is self-adjoint.

Thus, $(\lambda I - A)D_1$ is dense in H.

Now we shall prove that

$$R_\lambda(A_nP_n)x \to R_\lambda(A)x \quad \text{for every } x \in H. \tag{2.47}$$

Let $x \in H$ and $\varepsilon > 0$.

As $(\lambda I - A)D_1$ is dense in H, there exists $z \in (\lambda I - A)D_1$

$$\text{such that } \|x - z\| < |Im\lambda|\frac{\varepsilon}{3}. \tag{2.48}$$

From (2.46), there exists a positive integer n_0 such that

$$\|R_\lambda(A_nP_n)z - R_\lambda(A)z\| < \frac{\epsilon}{3} \quad \text{for every } n \geq n_0. \tag{2.49}$$

Now consider, for $n \geq n_0$,

$$\begin{aligned}
&\|R_\lambda(A_nP_n)(x) - R_\lambda(A)x\| \\
&\quad \leq \|R_\lambda(A_nP_n)x - R_\lambda(A_nP_n)z\| + \|R_\lambda(A_nP_n)z - R_\lambda(A)z\| + \|R_\lambda(A)z - R_\lambda(A)x\| \\
&\quad \leq \|R_\lambda(A_nP_n)\|\,\|x - z\| + \frac{\varepsilon}{3} + \|R_\lambda(A)\|\,\|z - x\|, \quad \text{using (2.49)} \\
&\quad = \|(\lambda I - A_nP_n)^{-1}\|\,\|x - z\| + \frac{\varepsilon}{3} + \|(\lambda I - A)^{-1}\|\,\|z - x\| \\
&\quad \leq |Im\lambda|^{-1}\|x - z\| + \frac{\varepsilon}{3} + |Im\lambda|^{-1}\|z - x\| \\
&\quad < |Im\lambda|^{-1}|Im\lambda|\frac{\varepsilon}{3} + \frac{\varepsilon}{3} + |Im\lambda|^{-1}|Im\lambda|\frac{\varepsilon}{3}, \quad \text{using (2.48)} \\
&\quad = \varepsilon
\end{aligned}$$

That is, $\|R_\lambda(A_nP_n)(x) - R_\lambda(A)(x)\| < \varepsilon$ for every $n \geq n_0$.

Hence, $R_\lambda(A_nP_n)(x) \to R_\lambda(A)(x)$ as $n \to \infty$.

Thus (2.47) is proved and $A_n \to A$ in the strong resolvent sense. □

Remark 2.5.3. *In the above proof, we observe that* $A_n \to A$ *in the strong resolvent sense if* $(\lambda I - A)D_1$ *is dense in* H *for every* $\lambda \in \mathbb{C}\backslash\mathbb{R}$, *where* $D_1 = \cup H_n$.

But, $(\lambda I - A)D_1$ is dense in H for every $\lambda \in \mathbb{C}\backslash\mathbb{R}$ if and only if $(\lambda_0 I - A)D_1$ is dense in H for some $\lambda_0 \in \mathbb{C}\backslash\mathbb{R}$.

The proof of this statement is direct.

Suppose $(\lambda_0 I - A)D_1$ is dense in H for some $\lambda_0 \in \mathbb{C}\backslash\mathbb{R}$.

That is, $(\lambda_0 I - A)D_1$ is dense in $(\lambda_0 I - A)D$, where $D = D(A)$, since $(\lambda_0 I - A)D = R(\lambda_0 I - A) = H$.

Then, for any $\lambda \in \mathbb{C}\backslash\mathbb{R}$, we shall prove that $(\lambda I - A)D_1$ is dense in $(\lambda I - A)D = H$.

Let $\lambda \in \mathbb{C}\backslash\mathbb{R}$.

Let $y = (\lambda I - A)x \in (\lambda I - A)D$, where $x \in D$.

Now $(\lambda_0 I - A)x \in (\lambda_0 I - A)D$.

But $(\lambda_0 I - A)D_1$ is dense in $(\lambda_0 I - A)D$.

Therefore, there exists a sequence (x_n) in D_1 such that

$$(\lambda_0 I - A)x_n \to (\lambda_0 I - A)x \quad \text{as } n \to \infty. \tag{2.50}$$

Since $(\lambda_0 I - A)^{-1}$ exists and is a bounded operator, we get $(\lambda_0 I - A)^{-1}(\lambda_0 I - A)x_n \to (\lambda_0 I - A)^{-1}(\lambda_0 I - A)x$ as $n \to \infty$.

That is,

$$x_n \to x \quad \text{as } n \to \infty. \tag{2.51}$$

This implies,

$$\lambda_0 x_n \to \lambda_0 x \quad \text{as } n \to \infty. \tag{2.52}$$

From (2.52) and (2.50), we get,

$$\lambda_0 x_n - (\lambda_0 I - A)x_n \to \lambda_0 x - (\lambda_0 I - A)x \quad \text{as } n \to \infty.$$

$$\text{That is,} \quad Ax_n \to Ax \quad \text{as } n \to \infty. \tag{2.53}$$

Thus, from (2.51) and (2.53),

$$(\lambda I - A)x_n = \lambda x_n - Ax_n \to \lambda x - Ax = (\lambda I - A)x \quad \text{as } n \to \infty.$$

Hence, $(\lambda I - A)D_1$ is dense in $(\lambda I - A)D = H$.

Thus we have the result:

Corollary 2.5.4. *Let A be self-adjoint and $A_n := P_nAP_n/H_n$, where (H_n) is a filtration of H such that $H_n \subseteq D(A)$ and P_n be the orthogonal projection on H with range H_n. If $(\lambda I - A)D_1$ is dense in H for some $\lambda \in \mathbb{C}\backslash\mathbb{R}$, where $D_1 = \cup H_n$, then $A_n \to A$ in the strong resolvent sense. In particular, if $(iI - A)D_1$ is dense in H, then $A_n \to A$ in the strong resolvent sense.* □

Corollary 2.5.5. *$A_n \to A$ in the strong resolvent sense if $AP_nx \to Ax$ for every $x \in D(A)$.*

Proof:

Let $G = \{(x, Ax)/x \in D_1 = \cup H_n\}$.

$\Gamma(A) = \{(x, Ax)/x \in D = D(A)\}$.

It is enough to prove that G is dense in $\Gamma(A)$.

Let $(x, Ax) \in \Gamma(A)$ so that $x \in D(A)$.

Let $x_n = P_nx \in H_n \subseteq D_1$.

Then (x_n, Ax_n) is a sequence in G.

Now $x_n = P_nx \to x$.

And $Ax_n = AP_nx \to Ax$, by assumption.

Hence $(x_n, Ax_n) \to (x, Ax)$ in $H \times H$.

Thus G is dense in $\Gamma(A)$.

So, by Theorem 2.5.2, $A_n \to A$ in the strong resolvent sense. □

Corollary 2.5.6. *$A_n \to A$ in the strong resolvent sense if (AP_nx) is a cauchy sequence for every $x \in D(A)$.*

Proof: Assume that (AP_nx) is cauchy for every $x \in D(A)$.

Then (AP_nx) is convergent in H for every $x \in D(A)$, as H is complete.

Let $x \in D(A)$. Then $AP_nx \to y$ for some y in H.

Also, $P_nx \to x$ and A is closed as it is self-adjoint.

Hence $y = Ax$.

Thus, $AP_nx \to Ax$ for every $x \in D(A)$. So, by Corollary 2.5.5, $A_n \to A$ in the strong resolvent sense. □

Corollary 2.5.7. *Let A be an invertible self-adjoint operator. Then $A_n \to A$ in the strong resolvent sense if $A(D_1)$ is dense in H.*

Proof:

$D_1 = \cup H_n$, where (H_n) is a filtration of H with $H_n \subseteq D = D(A)$.

As A is invertible, $A(D) = R(A) = H$.

By assumption, $A(D_1)$ is dense in H.

That is, $A(D_1)$ is dense in $A(D)$.

Let $G = \{(x, Ax)/x \in D_1\}$.

Let $(x, Ax) \in \Gamma(A) = \{(x, Ax)/x \in D = D(A)\}$.

Now $Ax \in A(D)$.

But $A(D_1)$ is dense in $A(D)$.

Therefore, there is a sequence (x_n) in D_1 such that $A(x_n) \to Ax$ as $n \to \infty$.

Now A^{-1} exists and is bounded, since A is assumed to be invertible.

So, $A^{-1}(Ax_n) \to A^{-1}(Ax)$ as $n \to \infty$.

That is $x_n \to x$ as $n \to \infty$.

Hence, $(x_n, Ax_n) \to (x, Ax)$ in $H \times H$.

Thus the sequence (x_n, Ax_n) in G converges to (x, Ax).

This implies that G is dense in $\Gamma(A)$.

Therefore, by Theorem 2.5.2., $A_n \to A$ in the strong resolvent sense. □

In Proposition 2.5.1, we have proved that if $A(H_n) \subseteq H_n$ for every n, then $A_n \to A$ in the strong resolvent sense. In the case that A is invertible, $A_n \to A$ in the strong resolvent sense even if $A(H_n) \supseteq H_n$.

Corollary 2.5.8. *Let A be an invertible self-adjoint operator in H. If $A(H_n) \supseteq H_n$ for every n, then $A_n \to A$ in the strong resolvent sense.*

Proof:

Suppose

$$A(H_n) \supseteq H_n \quad \text{for every } n. \tag{2.54}$$

As $H_n \subseteq \cup H_n$,

$$A(H_n) \subseteq A(\cup H_n) = A(D_1) \quad \text{for every } n. \tag{2.55}$$

From (2.54) and (2.55), $H_n \subseteq A(D_1)$ for every n.

Hence, $\cup H_n \subseteq A(D_1)$.

But $\cup H_n$ is dense in H, as (H_n) is a filtration of H.

Hence, $A(D_1)$ is dense in H.

Then by Corollary 2.5.7, $A_n \to A$ in the strong resolvent sense. □

Thus, in most of the cases we can expect strong resolvent convergence of (A_n) to A. For (A_n) to be an applicable method, (A_n) must be stable (Remark 2.3.6). The stability of (A_n) depends on the nature of A.

If A is a bounded self-adjoint operator, e^{itA} is definec by $e^{itA} = \sum_{n=0}^{\infty} \frac{(it)^n A^n}{n!}$, for any real t. Here, the series converges in norm. If A is an unbounded self-adjoint operator, e^{itA} is defined using functional calculus. But for many operators, the series expansion is valid in a dense subset of the domain. We prove this in the following theorem.

Theorem 2.5.9. *Let A be a self-adjoint operator in a separable Hilbert space H. Let (H_n) be a filtration of H with $H_n \subseteq D(A)$ such that $A(H_n) \subseteq H_n$ for every n. Then $e^{itA}x = \sum_{k=0}^{\infty} \frac{(itA)^k}{k!} x$ for every $x \in \cup H_n$.*

Proof:

For $n = 1, 2, 3, \ldots$, let P_n be the orthogonal projection on H with range H_n and let $A_n := P_n A P_n / H_n$.

Since $A(H_n) \subseteq H_n$, by Theorem 2.5.1, $A_n \to A$ in the strong resolvent sense.

Now, by Theorem 2.2.6,

$$e^{itA_nP_n} \to e^{itA} \quad \text{strongly.} \tag{2.56}$$

Let $x \in \cup H_n$.

Then $x \in H_{n_0}$ for some n_0.

But $H_{n_0} \subseteq H_{n_0+1} \subseteq \ldots$.

Therefore, $x \in H_n$ for every $n \geq n_0$.

Hence, $P_n x = x$ for every $n \geq n_0$.

Also, for $n \geq n_0$,

$$\begin{aligned} A_n P_n x &= P_n A P_n P_n x \\ &= P_n A P_n x \\ &= P_n A x \\ &= Ax, \quad \text{since } Ax \in H_n \text{ as } A(H_n) \subseteq H_n. \end{aligned} \tag{2.57}$$

$$\begin{aligned} (A_n P_n)^2 x &= A_n P_n A_n P_n x \\ &= A_n P_n A x, \quad \text{by (2.57)} \\ &= A(Ax), \quad \text{again by (2.57), since } Ax \in H_n. \\ &= A^2 x \end{aligned}$$

and so on.

In general,

$$(A_n P_n)^k x = A^k x \quad \text{for } n \geq n_0, k = 1, 2, 3, \ldots. \tag{2.58}$$

$A_n P_n$ is a bounded operator.

Therefore, for $t \in \mathbb{R}$,

$$\begin{aligned} e^{itA_nP_n}x &= \left[I + \sum_{k=1}^{\infty} \frac{(itA_nP_n)^k}{k!} \right] x \\ &= \left[x + \sum_{k=1}^{\infty} \frac{(it)^k A^k x}{k!} \right], \quad \text{using (2.58)} \\ &= \left[I + \sum_{k=1}^{\infty} \frac{(itA)^k}{k!} \right] x \\ &= \left[\sum_{k=0}^{\infty} \frac{(itA)^k}{k!} \right] x. \end{aligned} \tag{2.59}$$

Now by 2.56,

$$\begin{aligned} e^{itA}(x) &= \lim_{n\to\infty} e^{itA_nP_n}(x) \\ &= \lim_{n\to\infty} \left[\sum_{k=0}^{\infty} \frac{(itA)^k}{k!} \right] x \\ &= \sum_{k=0}^{\infty} \frac{(itA)^k}{k!} x. \end{aligned}$$

Thus $e^{itA}(x) = \sum_{k=o}^{\infty} \frac{(itA)^k}{k!} x$ for every $x \in \cup H_n$. □

Example 2.5.10. *Let A be a self-adjoint operator in a separable Hilbert space H with compact resolvent.*

Then, there is an orthonormal basis $\{u_1, u_2, u_3, \ldots\}$ of H and $\{\mu_1, \mu_2, \mu_3, \ldots\} \subseteq \mathbb{C}$ such that

$$Au_n = \mu_n u_n \quad \text{for } n = 1, 2, 3, \ldots \tag{2.60}$$

and

$$Ax = \sum_n \mu_n \langle x, u_n \rangle u_n \quad \text{for every } x \in D(A). \tag{2.61}$$

Let $H_n = \text{span}\{u_1, u_2, \ldots, u_n\}$, $n = 1, 2, 3, \ldots$.

Then (H_n) is a filtration of H and $A(H_n) \subseteq H_n$ for every n.

So, by Theorem 2.5.8,

$$e^{itA}x = \sum_{k=0}^{\infty} \frac{(itA)^k}{k!} x \quad \text{for every } x \in \cup H_n.$$

There are plenty of operators, in particular, many of the differential operators, with compact resolvent. For these operators A, the series expansion for $e^{itA}x$ is valid in the dense set $\cup H_n$. □

Finite Section Method and the Gap

Let us define the gap between two subspaces of a Banach space [10]:

Let X and Y be two subspaces of a Banach space. For $X \neq \{0\}$, we set

$$\delta(X,Y) = \sup_{u \in S_X} \text{ dist } (u,Y),$$

where $S_X = \{x \in X/\|x\| = 1\}$, the unit sphere of X, and dist $(u,Y) = \inf\{\|u - y\|/y \in Y\}$, the distance from u to Y. If $X = \{0\}$, we set $\delta(0,Y) = 0$. Now define $\hat{\delta}(X,Y) = \max\{\delta(X,Y), \delta(Y,X)\} \cdot \hat{\delta}(X,Y)$ is called the gap between X and Y.

Lemma 2.5.11. *Let X and Y be subspaces of a Banach space such that $X \subseteq Y$. Then $\hat{\delta}(X,Y) = 0$ if and only if X is dense in Y.*

Proof. Assume that X is dense in Y. As $X \subseteq Y$, $\delta(X,Y) = 0$.

Now let $u \in S_Y$.

That is, $u \in Y$ and $\|u\| = 1$.

Since X is dense in Y, there exists a sequence $\{x_n\}$ in X such that $\{x_n\}$ converges to u.

$$\text{That is, } \|u - x_n\| \to 0.$$

Hence, $\text{dist}(u,X) = \inf\{\|u - x\|/x \in X\} = 0$.

Thus, for every $u \in S_Y$, $\text{dist}(u, X) = 0$.

$$\text{So,} \quad \sup_{u \in S_Y} \text{dist}(u, X) = 0.$$

That is,

$$\delta(Y, X) = 0.$$

Now

$$\begin{aligned} \hat{\delta}(X, Y) &= \max\{\delta(X, Y), \delta(Y, X)\} \\ &= 0. \end{aligned}$$

Conversely assume that $\hat{\delta}(X, Y) = 0$.

That is, $\max\{\delta(X, Y), \delta(Y, X)\} = 0$.

Then, $\delta(X, Y) = 0$ and $\delta(Y, X) = 0$.

Take $\delta(X, Y) = 0$.

That is, $\sup_{u \in S_Y} \text{dist}(u, X) = 0$,

which implies that $\text{dist}\,(u, X) = 0$ for every $u \in S_Y$.

That is, $\inf\{\|u - x\| / x \in X\} = 0$ for every $u \in S_Y$. So, for every $u \in S_Y$, there is a sequence $\{x_n\}$ in X such that $\|u - x_n\| \to 0$.

Hence X is dense in $S_Y = \{y \in Y / \|y\| = 1\}$. By linearity, X is dense in Y. □

Theorem 2.5.12. *Suppose A is a self-adjoint operator in H and $A_n := P_n A P_n / H_n$, where (H_n) is a filtration of H such that $H_n \subseteq D(A)$ and P_n is the orthogonal projection on H with range H_n. Put $G = \{(x, Ax)/x \in \cup H_n\}$. If $\widehat{\delta}(G, \Gamma(A)) = 0$, where $\Gamma(A)$ is the graph of A, then $A_n \to A$ in the strong resolvent sense.*

Proof. As $\hat{\delta}(G, \Gamma(A)) = 0$, by Lemma 2.5.11, G is dense in $\Gamma(A)$.

So, by Theorem 2.5.2., $A_n \to A$ in the strong resolvent sense. □

Chapter 3

UNBOUNDED MATRICES AND THE TRUNCATION METHOD

Suppose we want to solve the matrix equation $Ax = y$, where A is an infinite matrix and y is a given element of l^2, the space of all square summable scalar sequences. Once the computation of the actual solution is not possible, one will be interested in finding approximate solutions. In this chapter we give an overview of how and when the truncations of the given matrix can be used to approximate the solution.

3.1 Introduction

Let $A = (a_{j,k})_{j,k \in \mathbb{N}}$, where $\mathbb{N} = \{1, 2, 3, \ldots\}$, be an infinite matrix of scalars. Then A represents a linear operator from a subspace $D(A)$ of l^2 into l^2, where l^2 is the space of square-summable scalar sequences:

$$l^2 = \{x = (x_1, x_2, \ldots)/x_j \in \mathbb{C}, j = 1, 2, 3, \ldots, \sum_{j=1}^{\infty} |x_j|^2 < \infty\}.$$

l^2 is a Hilbert space with the inner product $\langle .,. \rangle$ given by

$$\langle x, y\rangle = \sum_{j=1}^{\infty} x_j, \bar{y}_j \text{ for all } x = (x_j), y = (y_j) \in l^2. \tag{3.1}$$

The norm induced by the inner product is given by

$$\|x\| = \langle x, x\rangle^{1/2} = \left(\sum_{j=1}^{\infty} |x_j|^2\right)^{1/2} \tag{3.2}$$

and the corresponding metric is

$$d(x, y) = \|x - y\| = \left(\sum_{j=1}^{\infty} |x_j - y_j|^2\right)^{1/2}. \tag{3.3}$$

$D(A)$ is called the domain of A and is explicitly given by $D(A) = \{x \in l^2 / Ax \in l^2\}$. Here $Ax = y = (y_1, y_2, \ldots)$ with $y_j = \sum\limits_{k=1}^{\infty} a_{jk} x_k$.

A generates a bounded operator on l^2 if $Ax \in l^2$ for every $x \in l^2$ (that is, $D(A) = l^2$) and there exists a constant $M < \infty$ such that

$$\|Ax\| \leq M\|x\| \text{ for every } x \in l^2. \tag{3.4}$$

If no such M exists, A is said to be unbounded.

For $j = 1, 2, 3 \ldots,$ let e_j be the element in l^2 whose j^{th} co-ordinate is 1 and all other co-ordinates are 0. Then $(e_1, e_2, e_3, \ldots)$ will constitute an orthonormal basis for l^2. We shall denote l^2 by H. Put $H_n =$ span $\{e_1, e_2, \ldots e_n\}$, $n = 1, 2, 3, \ldots$ Let P_n be the orthogonal projection on l^2 with range H_{n}. Infact, P_{n} is the operator on l^2 defined by

$$P_n : (x_1, x_2, x_3, \ldots) \mapsto (x_1, x_2, x_3, \ldots, x_n, 0, 0, 0, \ldots). \tag{3.5}$$

Let A_n be the truncated matrix

$$A_n = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}. \tag{3.6}$$

For $n = 1, 2, 3, \ldots, A_n$ can be identified with the operator

$$A_n(: H_n \to H_n) = P_n A P_n / H_n, \tag{3.7}$$

provided $H_n \subseteq D(A)$.

Suppose we want to solve the matrix equation

$$Ax = y \tag{3.8}$$

for a given $y \in l^2$. Computation of the actual solution x of (3.8) is not possible in many cases. In those cases, we will be interested in finding approximate solutions of (3.8), by means of computational methods. In this chapter we discuss how and when truncations can be used to approximate the solutions x of (3.8).

We restrict ourselves to invertible A's, so that, for every $y \in l^2$, (3.8) has a unique solution. (3.8) is the infinite system

$$\begin{pmatrix} a_{11} & a_{12} & a_{13} & \ldots \\ a_{21} & a_{22} & a_{23} & \ldots \\ a_{31} & a_{32} & a_{33} & \ldots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \end{pmatrix}. \tag{3.9}$$

Let us consider the truncated systems

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} x_1^{(n)} \\ x_2^{(n)} \\ \vdots \\ x_n^{(n)} \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}. \tag{3.10}$$

(3.10) can be written as

$$A_n x^{(n)} = P_n y, \tag{3.11}$$

where $x^{(n)} \in H_n$.

When A_n is invertible, (3.11) has a unique solution $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, \ldots, x_n^{(n)})$. Computation of the solution $x^{(n)}$ of the truncated system is not difficult in this age of computing.

The fundamental question before us is that when do the unique solutions $x^{(n)}$ of (3.11) approximate the unique solution x of (3.8) (or, when does the sequence $(x^{(n)})$ converge to x in l^2?)

Let us recall the definitions of applicability and stability.

Definition 3.1.1. Let A be invertible. The sequence of truncations (A_n) of A is an applicable method or an applicable sequence for A if the following hold:

There exists a positive integer n_0 such that

i) A_n is invertible (as an operator on H_n) for every $n \geq n_o$ and

ii) $A_n^{-1} P_n y$ converges to $A^{-1} y$ for every $y \in H$.

Also we recall the definition of stability, which has close connection with applicability.

The norm of a matrix A is defined as $\|A\| = \sup\{\|Ax\|/x \in D(A), \|x\| \leq 1\} \cdot$ $\|A\|$ is finite if and only if A is bounded.

Definition 3.1.2. (A_n) is a stable method or a stable sequence if there exists a positive integer n_0 such that A_n is invertible (as an operator on H_n) for $n \geq n_0$ and

$$\sup\{\|A_n^{-1}\|/n \geq n_0\} < \infty. \tag{3.12}$$

3.2 Unbounded Matrices

Let us recall that the infinite matrix $A = (a_{j,k})_{j,k \in \mathbb{N}}$ is said to be unbounded if A is not bounded as an operator on l^2 . In that case, the domain $D(A)$ of A can not be all of l^2 , in general.

Now, the basic question is: when is A densely-defined and $D(A)$ contains the standard orthonormal basis? Following proposition answers this question.

Proposition 3.2.1. $A = (a_{j,k})_{j,k \in \mathbb{N}}$ defines a densely-defined operator in l^2 whose domain contains the standard orthonormal basis $\{e_1, e_2, e_3, \ldots\}$ if and only if each of the columns of A lies in l^2 .

Proof. First we remark that if $D(A)$ contains $\{e_1, e_2, e_3, \ldots\}$, then A is densely-defined. For a proof, suppose $\{e_1, e_2, e_3, \ldots\} \subseteq D(A)$. Then, *span* $\{e_1, e_2, e_3, \ldots\} \subseteq D(A)$, since $D(A)$ is a subspace (of l^2). But *span* $\{e_1, e_2, e_3, \ldots\}$ is dense in l^2 . So, $D(A)$ is dense in l^2 and A is densely-defined. Hence, it is enough to show that $D(A)$ contains $\{e_1, e_2, e_3, \ldots\}$ if and only if every column of A lies in l^2 . $e_k = (0, 0, \ldots, 0, 1, 0, 0, \ldots.)$, where 1 occurs in the k^{th} place and all other co-ordinates are

0. Fix $k \in \mathbb{N}$.

$$Ae_k = \begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{pmatrix} = \begin{pmatrix} a_{1k} \\ a_{2k} \\ a_{3k} \\ \cdot \\ \cdot \\ \cdot \end{pmatrix}, \tag{3.13}$$

the k^{th} column of A. Now, $e_k \in D(A)$ if and only if $Ae_k \in l^2$. That is, if and only if the k^{th} column of A lies in l^2. So, $\{e_1, e_2, e_3, \ldots\} \subseteq D(A)$ if and only if every column of A lies in l^2. Hence the proof. □

Definition 3.2.2. Let $A = (a_{j,k})_{j,k\in\mathbb{N}}$ be an infinite matrix. The adjoint of A is the matrix $A^* = (\alpha_{j,k})_{j,k\in\mathbb{N}}$, where

$$\alpha_{j,k} = \overline{a_{k,j}} \quad \text{for all} \quad j, k \in \mathbb{N}. \tag{3.14}$$

A is said to be Hermitian if $A = A^*$.

Thus, A is Hermitian if and only if

$$a_{j,k} = \overline{a_{k,j}} \quad \text{for all} \quad j, k \in \mathbb{N}. \tag{3.15}$$

It is true that the matrix A is Hermitian if A, considered as an operator in l^2, is self-adjoint. We call an infinite matrix A self-adjoint if A, considered as an operator in l^2, is self-adjoint. Thus, for self-adjoint A, $a_{j,k} = \overline{a_{k,j}}$ for all j, k.

For the domain of a self-adjoint A to contain the standard orthonormal basis $\{e_1, e_2, e_3, \ldots\}$, it is necessary and sufficient that each of the rows and columns lies in l^2.

Proposition 3.2.3. Let $A = (a_{j,k})_{j,k\in\mathbb{N}}$ be self-adjoint. Then, the following are equivalent:

(i) The domain $D(A)$ of A contains the standard orthonormal basis $\{e_1, e_2, e_3, \ldots\}$.

(ii) Every column of A lies in l^2.

(iii) Every row of A lies in l^2.

Proof. (i) $\Longleftrightarrow$ (ii).

By Proposition 3.2.1, $D(A)$ contains $\{e_1, e_2, e_3, \ldots\}$ if and only if every column of A lies in l^2. Thus, (i) and (ii) are equivalent.

(ii) $\Longleftrightarrow$ (iii).

The k^{th} column of A lies in l^2 if and only if

$$\sum_{j=1}^{\infty} |a_{j,k}|^2 < \infty. \tag{3.16}$$

The k^{th} row of A lies in l^2 if and only if

$$\sum_{j=1}^{\infty} |a_{k,j}|^2 < \infty. \tag{3.17}$$

But, $a_{k,j} = \overline{a_{j,k}}$ for every j, k, as A is self-adjoint. So, $|a_{k,j}| = |\overline{a_{j,k}}| = |a_{j,k}|$. Thus,

$$\sum_{j=1}^{\infty} |a_{j,k}|^2 < \infty \text{ if and only if } \sum_{j=1}^{\infty} |a_{k,j}|^2 < \infty.$$

That is, the k^{th} column of A lies in l^2 if and only if the k^{th} row of A lies in l^2. Hence (ii) and (iii) are equivalent. $\square$

Example 3.2.4.

(a) Let $(k_1, k_2, k_3, \ldots)$ be a sequence of non-zero numbers such that $|k_n| \to \infty$ as $n \to \infty$. Consider the diagonal matrix

$$A = \begin{pmatrix} k_1 & & & 0 \\ & k_2 & & \\ & & k_3 & \\ 0 & & & \ddots \end{pmatrix}. \tag{3.18}$$

Then A is unbounded. A is self-adjoint if and only if $k_1, k_2, k_3, \ldots$ are all real. Now $Ae_j = (0, 0, \ldots, 0, k_j, 0, \ldots)$ (written as a row vector), where k_j occurs in the j^{th} place and all other co-ordinates are 0.

So, $\|Ae_j\| = |k_j| < \infty, \quad j = 1, 2, 3, \ldots$. Hence $e_j \in D(A)$ for $j = 1, 2, 3, \ldots$ and $D(A)$ is dense in l^2. Thus, every diagonal matrix defines a densely-defined operator in l^2.

(b) Let c_{00} be the space of ultimately zero sequences. That is, $c_{00} = \{x = (x_j) /$all but finitely many x_j's are $0\}$. Note that, if $x \in c_{00}$, then $x \in l^2$. Let $s_1, s_2, s_3, \ldots \in c_{00}$ and A be the infinite matrix $A = (s_1, s_2, s_3, \ldots)$ with s_j as the j^{th} column of $A \cdot s_j \in l^2$ as it belongs to c_{00}.

Hence, by proposition 3.2.1., A defines a densely-defined operator in l^2. Thus, if each column of a matrix A contains only a finite number of non-zero terms, then A defines a densely-defined operator in l^2.

(c) Consider the matrix

$$A = \begin{pmatrix} 1 & 1/2 & 1/4 & 1/8 & 1/16 & \cdots \\ 1/2 & 2 & 1/2 & 1/4 & 1/8 & \cdots \\ 1/4 & 1/2 & 4 & 1/2 & 1/4 & \cdots \\ 1/8 & 1/4 & 1/2 & 8 & 1/2 & \cdots \\ 1/16 & 1/8 & 1/4 & 1/2 & 16 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \end{pmatrix}.$$

Ae_j is the j^{th} column of A and the j^{th} column contains the entry 2^{j-1}. So,

$$\|Ae_j\| \geq 2^{j-1}, \text{ for } j = 1, 2, 3, \ldots \tag{3.19}$$

Thus $\|Ae_j\| \to \infty$ as $j \to \infty$, whereas $\|e_j\| = 1$ for every j. Hence A is unbounded. Now, each column of A is square-summable and hence belongs to l^2. So, by proposition 3.2.1., $D(A)$ contains (e_j) and A is densely-defined. □

3.3 Truncations and the Resolvent Convergence

Let $A = (a_{j,k})_{j,k \in \mathbb{N}}$ define a self-adjoint operator and $A_n = P_n A P_n / H_n$, where $H_n = span\{e_1, e_2, \ldots, e_n\}$ and P_n is the (orthogonal) projection on $H = l^2$ with range H_n. Here $e_j = (0, 0, \ldots, 0,\ 1, 0, 0, \ldots)$, the element in l^2 whose j^{th} co-ordinate is 1 and all other co-ordinates are 0. A_n is, in fact, the $n \times n$ truncation ($n \times n$ principal section) of A, given in (3.6).

Since A is self-adjoint, $a_{j,k} = \overline{a_{k,j}}$ for all j, k. Hence A_n is self-adjoint, as A_n is bounded for all n.

Suppose A is invertible and (A_n) converges to A in the strong resolvent sense. Then, by Remark 2.3.6 (b), (A_n) is an applicable method for A if and only if (A_n)

is stable. So, the primary question is: when does the sequence of truncations (A_n) converge to A in the strong resolvent sense? (A_n) is said to converge to A in the strong resolvent sense if $R_\lambda(A_n)P_n$ converges to $R_\lambda(A)$ strongly for every (consequently for some) $\lambda \in \mathbb{C}\setminus\mathbb{R}$, where $R_\lambda(A) = (\lambda I - A)^{-1}$. I being the identity matrix (operator).

We have established certain sufficient conditions in Section 2.5 of Chapter 2 for the resolvent convergence of (A_n) to A. We list them here.

(a) (**Theorem 2.5.1**) If $A(H_n) \subseteq H_n$ for every $n \geq n_o$, for some n_o, then $A_n \to A$ in the strong resolvent sense.

(b) (**Theorem 2.5.2**) If $G = \{(x, Ax)/x \in \cup H_n\}$ is dense in the graph $\Gamma(A)$ of A, then $A_n \to A$ in the strong resolvent sense.

(c) (**Corollary 2.5.5**) If $AP_n x \to Ax$ for every $x \in D(A)$, then $(A_n) \to A$ in the strong resolvent sense.

For invertible A, we have

(d) (**Corollary 2.5.6**) If $A(\cup H_n)$ is dense in $H(= l^2)$, then $A_n \to A$ in the strong resolvent sense.

Example 3.3.1. Let A be the diagonal matrix

$$A = \begin{pmatrix} k_1 & & & 0 \\ & k_2 & & \\ & & k_3 & \\ & & & \\ & 0 & & \ddots \end{pmatrix},$$

where $k_1, k_2, k_3 \ldots$ are all real, so that A is self-adjoint.

Let $x = (x_1, x_2, \ldots, x_n, 0, 0, \ldots) \in H_n$. Then

$$Ax = \begin{pmatrix} k_1 & 0 & 0 & 0 & \cdots \\ 0 & k_2 & 0 & 0 & \cdots \\ 0 & 0 & k_3 & 0 & \cdots \\ 0 & 0 & 0 & k_4 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ 0 \\ \vdots \end{pmatrix}$$

$$= (k_1x_1, k_2x_2, \ldots k_nx_n, 0, 0, \ldots) \in H_n.$$

Thus $A(H_n) \subseteq H_n$ for every n. Hence by Theorem 2.5.1, $A_n \to A$ in the strong resolvent sense. □

Lemma 3.3.2. Let $A = (a_{j,k})_{j,k\in\mathbb{N}}$ be self-adjoint and invertible. Let $c_k = (a_{1k}, a_{2k}, a_{3k}, \ldots)$. That is, c_k is the k^{th} column of A. Suppose *span* $\{c_1, c_2, c_3, \ldots\}$ is dense in l^2. Then, $A_n \to A$ in the strong resolvent sense.

Proof. Let (e_n) be the standard orthonormal basis for l^2. Then, for $k = 1, 2, 3, \ldots$, $Ae_k = c_k$, the k^{th} column of A. For $n = 1, 2, 3, \ldots, H_n = \textit{span}\,\{e_1, e_2, \ldots, e_n\}$. Therefore, $\cup H_n = \textit{span}\,\{e_1, e_2, e_3, \ldots\}$. Hence, $A(\cup H_n) = \textit{span}\,\{Ae_1, Ae_2, Ae_3, \ldots\}$, by the linearity of A. Thus, $A(\cup H_n) = \textit{span}\,\{c_1, c_2, c_3, \ldots\}$, as $Ae_k = c_k, k = 1, 2, 3, \ldots$.

By assumption, span $\{c_1, c_2, c_3, \ldots\}$ is dense l^2.

So, $A(\cup H_n)$ is dense in l^2. Therefore, by Corollary 2.5.6, $A_n \to A$ in the strong resolvent sense. □

Let $A = (a_{j,k})_{j,k\in\mathbb{N}}$ be self-adjoint. Let c_j be the j^{th} column of A, so that $A = (c_1, c_2, c_3, \ldots)$. Let $\Delta(A)$ be the subset of $D(A)$ defined by $\Delta(A) = \{x =$

$(x_1, x_2, x_3, \ldots) \in D(A)/Ax = c_1x_1 + c_2x_2 + \cdots\}$. Then, $\Delta(A)$ is a dense subspace of $D(A)$. That $\Delta(A)$ is a subspace can be directly verified. To show the denseness of $\Delta(A)$ in $D(A)$, we will show that $\Delta(A)$ contains $\cup H_n$. For, let $x \in \cup H_n$. Then, $x \in H_m$ for some m. But $H_m = \text{span}\{e_1, e_2, \ldots, e_m\}$, where for each j, e_j is the vector whose j^{th} co-ordinate is 1 and all other co-ordinates are 0.

So, $x = (x_1, x_2, \ldots, x_m, 0, 0, 0, \ldots)$, where $x_1, x_2, \ldots, x_m$ are scalars, or, $x = x_1e_1 + x_2e_2 + \cdots + x_me_m$. Since A is linear, $Ax = x_1Ae_1 + x_2Ae_2 + \cdots + x_m Ae_m = x_1c_1 + x_2c_2 + \cdots + x_mc_m$, as $Ae_j = c_j$. Thus, $Ax = c_1x_1 + c_2x_2 + \cdots + c_mx_m + c_{m+1}0 + \cdots$ So $x \in \Delta(A)$. Hence, $\cup H_n \subseteq \Delta(A) \subseteq D(A)$. Now, $\cup H_n$ is dense in $D(A)$. Therefore, $\Delta(A)$ is dense in $D(A)$.

Lemma 3.3.3.

$$\Delta(A) = \{x \in D(A)/AP_nx \to Ax\}.$$

Proof. Let $x = (x_1, x_2, x_3, \ldots) \in D(A)$.

Then,

$$\begin{aligned} P_nx &= (x_1, x_2, \ldots, x_n, 0, 0, \ldots) \\ &= x_1e_1 + x_2e_2 + \cdots + x_ne_n. \end{aligned}$$

So,

$$\begin{aligned} AP_nx &= A(x_1e_1 + x_2e_2 + \cdots + x_ne_n) \\ &= x_1Ae_1 + x_2Ae_2 + \cdots + x_nAe_n \\ &= x_1c_1 + x_2c_2 + \cdots + x_nc_n, \end{aligned}$$

since $Ae_j = c_j$, for each j.

By definition, $\Delta(A) = \{x \in D(A) / Ax = c_1x_1 + c_2x_2 + \cdots\}$.

So, $x \in \Delta(A)$ if and only if $Ax = c_1x_1 + c_2x_2 + \cdots$ if and only if $Ax = \lim_{n\to\infty}(c_1x_1 + c_2x_2 + \cdots + c_nx_n)$ if and only if $Ax = \lim_{n\to\infty} AP_nx$ if and only if $AP_nx \to Ax$.

Thus, $\Delta(A) = \{x \in D(A) / AP_nx \to Ax\}$. □

Theorem 3.3.4. Let $A = (a_{j,k})_{j,k\in\mathbb{N}}$ be self-adjoint. Put $A_\Delta = A|_{\Delta(A)}$. Suppose A_Δ is essentially self-adjoint. Then $A_n \to A$ in the strong resolvent sense.

Proof. $D(A_\Delta) = \Delta(A)$. Let $x, y \in D(A_\Delta)$. Then $\langle A_\Delta x, y\rangle = \langle Ax, y\rangle = \langle x, Ay\rangle$, as A is self-adjoint.

And $\langle x, A_\Delta y\rangle = \langle x, Ay\rangle$.

Thus, $\langle A_\Delta x, y\rangle = \langle x, A_\Delta y\rangle$ for all $x, y \in D(A_\Delta)$.

So, A_Δ is symmetric.

Let $x \in \Delta(A)$.

By Lemma 3.3.3, $AP_nx \to Ax$. Hence, $A_nP_nx = P_nAP_nx \to Ax$, since $P_n \to I$ strongly.

Take $\lambda = i$. Put $z = (\lambda I - A)x$. Then, as in the proof of Theorem 2.5.2 (for proving (2.46)), we can prove that

$$R_\lambda(A_nP_n)z \to R_\lambda(A)z.$$

That is, $R_\lambda(A_nP_n)z \to R_\lambda(A)z$ for all $z \in (\lambda I - A)(\Delta(A))$.

Or,

$$R_\lambda(A_nP_n)z \to R_\lambda(A)z \text{ for all } z \in R(\lambda I - A_\Delta). \tag{3.20}$$

By assumption, A_Δ is essentially self-adjoint. So by Theorem 1.4.9, $R(A_\Delta - iI) = R(A_\Delta - \lambda I)$ is dense in l^2.

Hence, $R(\lambda I - A_\Delta)$ is dense in l^2, since $R(\lambda I - A_\Delta) = R(A_\Delta - \lambda I)$. Let $y \in l^2$ and $\varepsilon > 0$.

There exists $z \in R(\lambda I - A_\Delta)$ such that

$$\|y - z\| < \varepsilon/3. \tag{3.21}$$

(3.20) implies that there is a positive integer n_0 such that

$$\|R_\lambda(A_nP_n)z - R_\lambda(A)z\| < \frac{\varepsilon}{3} \text{ for all } n \geq n_0. \tag{3.22}$$

Now consider,

$$\begin{aligned}\|R_\lambda(A_nP_n)y - R_\lambda(A)y\| &\leq \|R_\lambda(A_nP_n)y - R_\lambda(A_nP_n)z\| \\ &\quad + \|R_\lambda(A_nP_n)z - R_\lambda(A)z\| + \|R_\lambda(A)z - R_\lambda(A)y\| \\ &\leq \|R_\lambda(A_nP_n)\| \; \|y - z\| + \varepsilon/3 + \|R_\lambda(A)\| \; \|z - y\|\end{aligned}$$

for all $n \geq n_0$, from(3.22). Now, $\|R_\lambda(A_nP_n)\| \leq |Im\lambda|^{-1} = 1$, as $\lambda = i$, using Proposition 1.4.7 and similarly $\|R_\lambda(A)\| \leq 1$.

Hence, $\|R_\lambda(A_nP_n)y - R_\lambda(A)y\| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3$ for all $n \geq n_0$, using (3.21).

That is, $\|R_\lambda(A_nP_n)y - R_\lambda(A)y\| < \varepsilon$ for all $n \geq n_0$. Thus, $R_\lambda(A_nP_n)y \to R_\lambda(A)y$. Since $y \in l^2$ is arbitrary, $A_nP_n \to A$ is the strong resolvent sense.

Equivalently, $A_n \to A$ in the strong resolvent sense. □

3.4 Unbounded Toeplitz Matrices

Let $(a_n)_{n=-\infty}^{\infty}$ be a sequence of-complex numbers. The Toeplitz matrix defined by $(a_n)_{n=-\infty}^{\infty}$ is the infinite matrix

$$A = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \cdots \\ a_1 & a_0 & a_{-1} & \cdots \\ a_2 & a_1 & a_0 & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{pmatrix} \tag{3.23}$$

It is well known that (Theorem 1.8.1) A generates a bounded operator on l^2 if and only if (a_n) is the sequence of Fourier coefficients of a function $a \in L^\infty(\mathbb{T})$, where $\mathbb{T}$ denotes the unit circle in the complex plane:

$$\mathbb{T} = \{z \in \mathbb{C}/|z| = 1\}.$$

This celebrated result was established by Toeplitz (1911). We may denote the matrix A in (3.23) by $T(a)$. Thus, if $a \in L^1(\mathbb{T})\backslash L^\infty(\mathbb{T})$, the operator generated by $T(a)$ is not bounded on l^2.

In this section we deal with unbounded self-adjoint Toeplitz matrices $T(a)$, where $a \in L^2(\mathbb{T})$. The reason for taking $a \in L^2(\mathbb{T})$ instead of $L^1(\mathbb{T})$ is justified in the following proposition.

Proposition 3.4.1. Let $T(a)$ be self-adjoint. Then the domain of $T(a)$ contains the standard orthonormal basis $\{e_1, e_2, e_3, \cdots\}$ if and only if $a \in L^2(\mathbb{T})$.

Proof. By proposition 3.2.3., the domain of $T(a)$ contains the standard orthonormal basis $\{e_1, e_2, e_3, \cdots\}$ if and only if each of the columns (rows) of $T(a)$ lies in l^2. Hence, it is enough to prove that the columns (rows) of $T(a)$ lie in l^2 if and only if $a \in L^2(\mathbb{T})$.

$$T(a) = \begin{pmatrix} a_0 & a_{-1} & a_{-2} & \cdots \\ a_1 & a_0 & a_{-1} & \cdots \\ a_2 & a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots \end{pmatrix}$$

Here, a_n is the n^{th} Fouries coefficient of a $(n \in \mathbb{Z})$. But $a \in L^2(\mathbb{T})$ if and only if $(a_n) \in l^2(\mathbb{Z})$. Thus, it suffices to show that the columns (rows) of $T(a)$ lie in l^2 if and only if $(a_n)_{n\in\mathbb{Z}} \in l^2(\mathbb{Z})$. Suppose $(a_n) \in l^2(\mathbb{Z})$. That is,

$$\sum_{n=-\infty}^{\infty} |a_n|^2 < \infty. \tag{3.24}$$

Then, obviously

$$\sum_{n=N}^{\infty} |a_n|^2 < \infty \text{ for } N = 0, -1, -2, \ldots \tag{3.25}$$

Thus, every column of $T(a)$ lies in l^2.

Conversely, assume that each column of $T(a)$ lies in l^2. In particular, the first column of $T(a)$ lies in l^2. So,

$$\sum_{n=0}^{\infty} |a_n|^2 < \infty. \tag{3.26}$$

Now, by Proposition 3.2.3, each row of $T(a)$ lies in l^2.

In particular, the first row lies in l^2. So,

$$\sum_{n=0}^{\infty} |a_{-n}|^2 < \infty. \tag{3.27}$$

From (3.26) and (3.27),

$$\sum_{n=0}^{\infty} |a_n|^2 + \sum_{n=0}^{\infty} |a_{-n}|^2 < \infty.$$

That is,

$$|a_0|^2 + \sum_{n=-\infty}^{\infty} |a_n|^2 < \infty,$$

which implies $\sum_{n=-\infty}^{\infty} |a_n|^2 < \infty$, or $(a_n) \in l^2(\mathbb{Z})$. This completes the proof. $\square$

Theorem 3.4.2. Let $A = T(a)$ be self-adjoint with $a \in L^2(\mathbb{T})$. Let $D' = D(A) \cap l^1$. Let $G' = \{(x, Ax)/x \in D'\}$. Suppose G' is dense in $\Gamma(A)$. Then the sequence (A_n) of truncations of A converges to A in the strong resolvent sense.

Proof. $D' = D(A) \cap l^1$. Then, D' is dense in $D(A)$. Let $x \in D'$.

Let $(a_n)_{n\in\mathbb{Z}}$ be the sequence of Fourier coefficients of a.

Since $a \in L^2(\mathbb{T}), \sum_{n\in\mathbb{Z}} |a_n|^2 < \infty$.

Let $\sum_{n\in\mathbb{Z}} |a_n|^2 = \alpha^2$.

Let c_j be the j^{th} column of $A = T(a)$.

Then $\|c_j\|^2 \leq \sum_{n\in\mathbb{Z}} |a_n|^2 \leq \alpha^2$, so that

$$\|c_j\|^2 \leq \alpha \text{ for each } j. \tag{3.28}$$

Here $\| \ \|$ denotes the l^2-norm.

Suppose $x = (x_1, x_2, x_3, \ldots)$.

Then for each n,

$$\begin{aligned} P_n x &= (x_1, x_2, \ldots, x_n, 0, 0, \ldots) \\ &= x_1 e_1 + x_2 e_2 + \cdots + x_n e_n. \end{aligned}$$

Therefore,

$$\begin{aligned} AP_n x &= x_1 Ae_1 + x_2 Ae_2 + \cdots + x_n Ae_n, \text{as } A \text{ is linear} \\ &= x_1 c_1 + x_2 c_2 + \cdots + x_n c_n. \end{aligned}$$

Thus, for $n > m$,

$$AP_n x - AP_m x = x_{m+1}c_{m+1} + \cdots + x_n c_n.$$

Hence,

$$\begin{aligned}\|AP_n x - AP_m x\| &\leq |x_{m+1}|\|c_{m+1}\| + \cdots + |x_n|\|c_n\| \\ &\leq \alpha(|x_{m+1}| + \cdots + |x_n|) \qquad (3.29)\end{aligned}$$

using(3.28).

Now $\sum_{n=1}^{\infty} |x_n| < \infty$, as $x = (x_n) \in D' \subseteq l^1$. So, for given $\varepsilon > 0$, there exists n_0 such that $|x_{m+1}| + \cdots + |x_n| < \varepsilon$ for all $m, n, \geq n_0$. Hence, (3.29) implies that $(AP_n x)$ is a Cauchy sequence in $H = l^2$. Since l^2 is complete, there exists $y \in l^2$ such that $(AP_n x)$ converges to y in l^2. But $(P_n x)$ converges to x and A is closed.

So,

$$y = Ax.$$

Thus, $AP_n x \to Ax$ for all $x \in D'$.

Hence, $(P_n x, AP_n x) \to (x, Ax)$ in $l^2 \times l^2$ for all $x \in D'$.

Let

$$G = \{(x, Ax)/x \in \cup H_m\}.$$

For $x \in D'$, $P_n x \in H_n \subseteq \cup H_m$. So, $(P_n x, AP_n x)$ is a sequence in G. Thus, for every $(x, Ax) \in G'$, there is a sequence $(P_n x, AP_n x) \subseteq G$ such that $(P_n x, AP_n x) \to (x, Ax)$. So, G is dense in G'.

By assumption, G' is dense in $\Gamma(A)$. Hence G is dense in $\Gamma(A)$.

So, by Theorem 2.5.2, $A_n \to A$ in the strong resolvent sense. □

Chapter 4

APPROXIMATION NUMBERS OF UNBOUNDED OPERATORS

Approximation numbers are generalizations of the classical singular values. In this chapter, we discuss a few approximation numbers of unbounded operators.

4.1 Relative Approximation Numbers

Relative approximation numbers of relatively bounded operators are introduced by M.N.N. Namboodiri and A.V. Chithra [14]. Let us prove a limiting property of relative approximation numbers in this section. Relative bounds are used to define relative approximation numbers.

Definition 4.1.1. *Let T be a densely-defined linear operator in H. A linear operator A in H with $D(A) \supseteq D(T)$ is said to be relatively bounded with respect to T, or simply T-bounded, if there are nonnegative constants a and b such that*

$$\|Ax\| \leq a\|x\| + b\|Tx\| \quad \textit{for all } x \in D(T). \tag{4.1}$$

The infemum of all possible b satisfying (4.1) is called the relative bound of A with respect to T, or simply the T-bound of A. We may denote the T-bound of A by $b_T(A)$.

Suppose A is bounded on H and T is any densely-defined operator in H.

Then $\|Ax\| \leq \|A\|\|x\| = \|A\| \; \|x\| + 0\|Tx\|$ for all $x \in D(T)$.

Thus, every bounded operator A on H is T-bounded for any T and $b_T(A) = 0$.

We write $A \in \mathcal{B}_T(H)$ to mean that A is T-bounded. It is clear that $\mathcal{B}(H) \subseteq \mathcal{B}_T(H)$ for any T, where $\mathcal{B}(H)$ denotes the class of bounded operators on H.

Remark 4.1.2. Let $A \in \mathcal{B}_T(H)$ and $L \in \mathcal{B}(H)$. Then, $b_T(A+L) = b_T(A)$.

Proof. We have, for $x \in D(T)$,

$$\begin{aligned}
\|(A+L)x\| &\leq \|Ax\| + \|Lx\| \\
&\leq a\|x\| + b\|Tx\| + \|Lx\|, \quad \text{by (4.1)} \\
&\leq a\|x\| + b\|Tx\| + \|L\|\|x\| \\
&= (a + \|L\|)\|x\| + b\|Tx\| \qquad (4.2)
\end{aligned}$$

Hence, $A+L \in B_T(H)$ and $b_T(A+L) \leq b$, for every b satisfying (4.1).

This implies, $b_T(A+L) \leq \inf\{b\}$, where the infemum is taken over all 'b' satisfying (4.1).

The infemum of all such 'b' is $b_T(A)$.

Hence,

$$b_T(A+L) \leq b_T(A). \qquad (4.3)$$

Replacing A by $A+L$ and L by $-L$ in (4.3), we get,

$$b_T(A) \leq b_T(A+L). \qquad (4.4)$$

From (4.3) and (4.4), we get,

$$b_T(A) = b_T(A+L). \tag{4.5}$$

□

Proposition 4.1.3. *Suppose T is densely-defined and A and B are T-bounded in H. Then $b_T(A+B) \leq b_T(A) + b_T(B)$.*

Proof. As A and B are T-bounded, $D(T) \subseteq D(A)$ and $D(T) \subseteq D(B)$.

Therefore, $D(T) \subseteq D(A) \cap D(B)$. Thus, $A+B$ is a densely-defined operator in H with domain $D(A+B) = D(A) \cap D(B) \supseteq D(T)$.

Let $\varepsilon > 0$ be given.

Then, by the definition of $b_T(A)$ and $b_T(B)$, there are constants b_1 and b_2 with $b_1 < b_T(A) + \varepsilon$ and $b_2 < b_T(B) + \varepsilon$ and constants a_1 and a_2 such that

$$\|Ax\| \leq a_1\|x\| + b_1\|Tx\| \quad \text{for all } x \in D(T) \text{ and}$$

$$\|Bx\| \leq a_2\|x\| + b_2\|Tx\| \quad \text{for all } x \in D(T).$$

Now,

$$\begin{aligned} \|(A+B)x\| &\leq \|Ax\| + \|Bx\| \\ &\leq (a_1+a_2)\|x\| + (b_1+b_2)\|Tx\| \quad \text{for all } x \in D(T). \end{aligned}$$

Hence

$$\begin{aligned} b_T(A+B) &\leq b_1 + b_2 \\ &< b_T(A) + \varepsilon + b_T(B) + \varepsilon = b_T(A) + b_T(B) + 2\varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ is arbitrary,

$$b_T(A+B) \leq b_T(A) + b_T(B). \tag{4.6}$$

□

Remark 4.1.4. If b is chosen very close to $b_T(A)$ in (1), the other constant a will have to be chosen very large, in general. Thus, there may not exist a finite constant a, satisfying

$$\|Ax\| \leq a\|x\| + b_T(A)\|Tx\| \quad \text{for all } x \in D(T). \tag{4.7}$$

But for many cases, finite 'a' satisfying (4.7) exists. In those cases, we may say that A is strongly T-bounded. We denote the class of strongly T-bounded operators by $\mathcal{B}'_T(H)$.

Definition 4.1.5. *Suppose $A \in \mathcal{B}'_T(H)$. Let a_0 be the infemum of all 'a' satisfying (4.7). a_0 is called the relative co-bound of A with respect to T or simply the co T-bound of A, and is denoted by $b'_T(A)$.*

If A is T-bounded, but not strongly, we set $b'_T(A) = \infty$.

From (4.7), it is explicit that, for $A \in \mathcal{B}'_T(H)$,

$$\|Ax\| \leq b'_T(A)\|x\| + b_T(A)\|Tx\| \quad \textit{for all } x \in D(T). \tag{4.8}$$

Note. If $A \in \mathcal{B}(H)$, then $A \in \mathcal{B}'_T(H)$ with $b_T(A) = 0$ and $b'_T(A) = \|A\|$. Also we note that if $A \in \mathcal{B}'_T(H)$, then $A + L \in \mathcal{B}'_T(H)$ for every bounded operator L and

$$b'_T(A+L) \leq b'_T(A) + \|L\|, \tag{4.9}$$

$$\text{since} \quad b_T(A+L) = b_T(A).$$

Now let us define relative approximation numbers [14].

Definition 4.1.6. *Let T be a closed operator in H and A be T-bounded. For $k = 1, 2, 3, \ldots,$ the k^{th} relative approximation number $s'_{k,T}(A)$ is defined as*

$$s'_{k,T}(A) = \inf\{b'_T(A - F)/F \in \mathcal{B}(H), \quad \text{rank } F \leq k - 1\}. \tag{4.10}$$

Also we define

$$s'_{\infty,T}(A) = \inf\{b'_T(A - K)/K \in \mathcal{K}(H)\}, \tag{4.11}$$

where $\mathcal{K}(H)$ denotes the class of compact operators on H.

We note that $s'_{k,T}(A)$ can be ∞.

It is clear from the definitions that

$$b'_T(A) = s'_{1,T}(A) \geq s'_{2,T}(A) \geq \cdots \geq 0 \tag{4.12}$$

and

$$s'_{k,T}(A) \geq s'_{\infty,T}(A) \quad \text{for } k = 1, 2, 3, \ldots. \tag{4.13}$$

Theorem 4.1.7. *Let $A \in \mathcal{B}'_T(H)$. Then*

$$\lim_{k\to\infty} s'_{k,T}(A) = \inf_{k=1,2,3,\ldots} s'_{k,T}(A) = s'_{\infty,T}(A).$$

Proof. Let

$$m = \inf_{k=1,2,3,\ldots} s'_{k,T}(A).$$

Then (4.13) implies that

$$m \geq s'_{\infty,T}(A). \tag{4.14}$$

Now let $\varepsilon > 0$ and $K \in \mathcal{K}(H)$.

We can find a finite rank operator F such that

$$\|K - F\| < \varepsilon. \tag{4.15}$$

For $x \in D(T)$, consider

$$\begin{aligned} \|(A-F)x\| &= \|(A-K)x + (K-F)x\| \\ &\leq \|(A-K)x\| + \|K-F\|\|x\| \\ &\leq b'_T(A-K)\|x\| + b_T(A-K)\|Tx\| + \varepsilon\|x\|, \end{aligned} \tag{4.16}$$

by (4.15).

But $b_T(A-K) = b_T(A) = b_T(A-F)$, from (4.5).

Therefore, (4.16) becomes

$$\|(A-F)x\| \leq [b'_T(A-K)+\varepsilon]\|x\| + b_T(A-F)\|Tx\| \quad \text{for every } x \in D(T). \tag{4.17}$$

Hence

$$b'_T(A-F) \leq b'_T(A-K) + \varepsilon. \tag{4.18}$$

Let rank $F = k-1$.

Then (4.18) implies

$$s'_{k,T}(A) \leq b'_T(A-K) + \varepsilon, \quad \text{as } s'_{k,T}(A) \leq b'_T(A-F),$$

$$\text{which implies } m \leq b'_T(A-K) + \varepsilon, \tag{4.19}$$

from the definition of m.

As $\varepsilon > 0$ is arbitrary, we get, $m \leq b'_T(A-K)$.

Since this is true for every $K \in \mathcal{K}(H)$,

$$m \leq \inf\{b'_T(A-K)/K \in \mathcal{K}(H)\}.$$

$$\text{That is,} \quad m \leq s'_{\infty,T}(A). \tag{4.20}$$

From (4.14) and (4.20) we get, $m = s'_{\infty,T}(A)$.

That is, $\inf\{s'_{k,T}(A)/k=1,2,3,\dots\}=s'_{\infty,T}(A)$.

Again, since $s'_{1,T}(A)\geq s'_{2,T}(A)\geq\cdots$, $\inf_{k=1,2,3,\dots} s'_{k,T}(A)=\lim_{k\to\infty} s'_{k,T}(A)$.

Hence the proof. $\square$

Proposition 4.1.8. *Let A be T-bounded. If $s'_{k,T}(A)$ is finite for some k, then $s'_{k,T}(A)$ is finite for every k.*

Proof. Let $k\geq 1$.

It is enough to prove that $s'_{k,T}(A)$ is finite if and only if $s'_{1,T}(A)$ is finite.

Suppose $s'_{1,T}(A)$ is finite.

Then, obviously $s'_{k,T}(A)$ is finite, since $s'_{1,T}(A)\geq s'_{k,T}(A)\geq 0$.

Conversely, assume that $s'_{k,T}(A)$ is finite.

That is,

$$\inf\{b'_T(A-F)/F\in\mathcal{B}(H),\quad \text{rank } F\leq k-1\}=s'_{k,T}(A)<\infty.$$

Hence, there is $F\in\mathcal{B}(H)$ with rank $F\leq k-1$ such that

$$b'_T(A-F)<s'_{k,T}(A)+1<\infty.$$

Now

$$\begin{aligned} b'_T(A) &= b'_T(A-F+F)\\ &\leq b'_T(A-F)+\|F\|,\quad \text{by (4.9)}\\ &<\infty. \end{aligned}$$

That is, $s'_{1,T}(A)<\infty$. $\square$

Remark 4.1.9.

(a) From the above result we observe the following:

$s'_{k,T}(A)$ is finite for some k if and only if $s'_{k,T}(A)$ is finite for every k if and only if $s'_{1,T}(A) = b'_T(A)$ is finite. Thus, for $A \in \mathcal{B}'_T(A), s'_{k,T}(A)$ is finite for $k = 1, 2, 3, \ldots$

(b) In a similar manner we can prove that $s'_{\infty,T}(A)$ is finite if and only if $s'_{1,T}(A) = b'_T(A)$ is finite. That is, if and only if A is strongly T-bounded. If

$$s'_{\infty,T}(A) = \inf\{b'_T(A - K)/K \in \mathcal{K}(H)\} < \infty,$$

then, there is $K \in \mathcal{K}(H)$ such that $b'_T(A - K) < \infty$.

Now,

$$\begin{aligned} b'_T(A) &= b'_T(A - K + K) \\ &\leq b'_T(A - K) + \|K\| < \infty. \end{aligned}$$

The other part is obvious, since $b'_T(A) = s'_{1,T}(A) \geq s'_{\infty,T}(A) \geq 0$.

(c) We note that A is strongly T-bounded if and only if $b'_T(A) = s'_{1,T}(A)$ is finite. Also, $s'_{1,T}(A)$ is finite if and only if $s'_{k,T}(A)$ is finite for any k and $s'_{\infty,T}(A)$ is finite. So, if A is T-bounded but not strongly, then $s'_{k,T}(A) = \infty$ for $k = 1, 2, 3, \ldots$ and $s'_{\infty,T}(A) = \infty$. Thus, Theorem 4.1.7 is still valid even if A is T-bounded but not strongly. However, this is an uninteresting case as all $s'_{k,T}(A)$ and $s'_{\infty,T}(A)$ are ∞. □

Example 4.1.10. *Let H be a separable Hilbert space and let $\{e_1, e_2, e_3, \ldots\}$ be an orthonormal basis for H. Then $x = \sum_{k=1}^{\infty} \langle x, e_k \rangle e_k$ for every $x \in H$.*

Define T and A by

$$Tx = \sum_{k=1}^{\infty} k\langle x, e_k\rangle e_k \quad \textit{and}$$
$$Ax = \sum_{k=1}^{\infty} k^{1/2}\langle x, e_k\rangle e_k.$$

Then T and A are densely-defined operators in H with domains

$$D(T) = \left\{ x \in H \Big/ \sum_{k=1}^{\infty} k^2 |\langle x, e_k\rangle|^2 < \infty \right\} \quad \textit{and}$$
$$D(A) = \left\{ x \in H \Big/ \sum_{k=1}^{\infty} k |\langle x, e_k\rangle|^2 < \infty \right\} \quad \textit{respectively.}$$

In fact,

$$D(T) \supseteq \textit{ span } \{e_1, e_2, e_3, \dots\} \textit{ and}$$

$$D(A) \supseteq \textit{ span } \{e_1, e_2, e_3, \dots\}.$$

Claim: $D(A) \supseteq D(T)$.

Let $x \in D(T)$. Then

$$\sum_{k=1}^{\infty} k^2 |\langle x, e_k\rangle|^2 < \infty.$$

Hence, $\sum_{k=1}^{\infty} k|\langle x, e_k\rangle|^2 < \infty$, by comparision test, since $k \leq k^2$, for $k = 1, 2, 3, \dots$

So, $x \in D(A)$.

Thus, $D(A) \supseteq D(T)$.

Now, for $x \in D(T)$,

$$\|Tx\|^2 = \sum_{k=1}^{\infty} k^2 |\langle x, e_k\rangle|^2 \quad \textit{and}$$
$$\|Ax\|^2 = \sum_{k=1}^{\infty} k |\langle x, e_k\rangle|^2.$$

Also,

$$\|x\|^2 = \sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2.$$

It is clear that $\|Ax\|^2 \leq \|Tx\|^2$ *for every* $x \in D(T)$ *, as* $k \leq k^2, k = 1, 2, 3, \dots$.

Hence $\|Ax\| \leq \|Tx\| = 0\,\|x\| + \|Tx\|$ *for every* $x \in D(T)$ *.*

Thus A *is* T*-bounded.*

It is proved in [14] that $s'_{1,T}(A) = \infty$ *.*

Let us give a proof in a slightly different manner:

Let $\varepsilon > 0$ *be given. Choose* $a \geq \frac{1}{2\varepsilon}$ *.*

Then $4a^2\varepsilon^2 \geq 1$ *or* $1 - 4a^2\varepsilon^2 \leq 0$ *.*

This implies, $\varepsilon^2 t^2 - t + a^2 \geq 0$ *for all real* t *.*

In particular, $\varepsilon^2 k^2 - k + a^2 \geq 0$ *for* $k = 1, 2, 3, \dots$

So, $\varepsilon^2 k^2 + a^2 \geq k$ *for* $k = 1, 2, 3, \dots$

Hence, $\varepsilon^2 \sum_{k=1}^{\infty} k^2 |\langle x, e_k\rangle|^2 + a^2 \sum_{k=1}^{\infty} |\langle x, e_k\rangle|^2 \geq \sum_{k=1}^{\infty} k |\langle x, e_k\rangle|^2$ *, for* $x \in D(T)$ *.*

That is,

$$\varepsilon^2 \|Tx\|^2 + a^2 \|x\|^2 \geq \|Ax\|^2 \quad \textit{for all } x \in D(T)\,.$$

Thus,

$$\begin{aligned} \|Ax\|^2 &\leq a^2\|x\|^2 + \varepsilon^2\|Tx\|^2 \\ &\leq a^2\|x\|^2 + \varepsilon^2\|Tx\|^2 + 2a\varepsilon\|x\|\|Tx\| \\ &= (a\|x\| + \varepsilon\|Tx\|)^2 \quad \textit{for every } x \in D(T). \end{aligned}$$

This implies, $\|Ax\| \leq a\|x\| + \varepsilon\|Tx\|$ *for every* $x \in D(T)$ *.*

Hence $b_T(A) \leq \varepsilon$ *.*

Since $\varepsilon > 0$ *is arbitrary,* $b_T(A) = 0$ *.*

Then, $b'_T(A)$ *must be* ∞ *. For, if* $b'_T(A) < \infty$ *, then,*

$$\|Ax\| \leq b'_T(A)\|x\| + b_T(A)\|Tx\| = b'_T(A)\|x\|, \quad \text{as } b_T(A) = 0\,.$$

This is not possible, since A *is unbounded.*

So, $b'_T(A) = \infty$ *. That is,* $s'_{1,T}(A) = \infty$ *. Now by Proposition 4.1.8 and the subsequent remark,* $s'_{k,T}(A) = \infty$ *for every* k *and* $s'_{\infty,T}(A) = \infty$ *.*

4.2 Approximation Number Sets

The notion of approximation number sets was introduced and few properties were discussed in [14]. Our objective is to see the relationship between the eigenspectrum and the first approximation number set of a closed (unbounded) operator. First let us recall the definition of approximation number sets. For a Hilbert space $H, \mathcal{C}(H)$ denotes the class of closed operators in H .

Definition 4.2.1. *For* $A \in \mathcal{C}(H)$ *, let* $\omega_A(H) = \{T \in \mathcal{C}(H)/A$ *is strongly* T*-bounded*$\}$ *. Let* k *be a positive integer. The* k^{th} *approximation number set* $\tilde{s}_k(A)$ *is defined as*

$$\tilde{s}_k(A) = \{s'_{k,T}(A)/T \in \omega_A(H)\}. \tag{4.21}$$

In the following theorem we will establish the partial connection between the approximation number set $\tilde{s}_1(A)$ of a closed operator A and its approximate eigenspectrum.

A scalar λ is said to be an approximate eigenvalue of A if there exists a sequence (x_n) in H with $\|x_n\| = 1$ for all n such that

$$\|Ax_n - \lambda x_n\| \to 0.$$

The set of all approximate eigenvalues of A is called the approximate eigenspectrum of A and is denoted by $a(A)$. It is clear that $e(A) \subseteq a(A)$, where $e(A)$ is the eigenspectrum of A.

A is said to be bounded below if there is a constant $c > 0$ such that $c\|x\| \leq \|Ax\|$ for every $x \in D(A)$. Equivalently, A is bounded below if and only if $c\|x\| \leq \|Ax\|$ for every $x \in D(A)$ with $\|x\| = 1$, provided $H \neq \{0\}$.

Let $H \neq \{0\}$.

Then, $\lambda \in a(A)$ if and only if there exists a sequence (x_n) in H with $\|x_n\| = 1$ for all n such that $\|(A - \lambda I)x_n\| \to 0$ if and only if there does not exists a $c > 0$ such that $c\|x\| \leq \|(A - \lambda I)x\|$ for all $x \in D(A)$ with $\|x\| = 1$ if and only if $A - \lambda I$ is not bounded below.

Thus,

$$a(A) = \{\lambda / A - \lambda I \text{ is not bounded below}\}.$$

Theorem 4.2.2. *Let A be an unbounded, closed operator in a Hilbert space* $H \neq \{0\}$. *Then,* $|a(A)| \subseteq \tilde{s}_1(A)$.

$$|a(A)| \textit{ denotes the set } \{|k| / k \in a(A)\}.$$

Proof. If $a(A)$ is empty, the result is obvious. So, assume that $a(A)$ is not empty.

$$\text{Let} \quad \lambda \in a(A).$$

$$\text{Put} \quad T = A - \lambda I.$$

Then $D(T) = D(A)$ and for $x \in D(T)$,

$$\|Ax\| = \|(A - \lambda I)x + \lambda x\|$$
$$\leq \|(A - \lambda I)x\| + |\lambda|\|x\|.$$
$$\text{That is,} \quad \|Ax\| \leq |\lambda|\|x\| + \|Tx\|. \tag{4.22}$$

Thus A is T-bounded, with T-bound ≤ 1.

We claim that the T-bound is actually equal to 1.

Suppose

$$\|Ax\| \leq a\|x\| + t\|(A - \lambda I)x\| \tag{4.23}$$

for every $x \in D(T)$, for some $t < 1$.

Then, for every $x \in D(T)$, we have,

$$\|(A - \lambda I)x\| \leq \|Ax\| + |\lambda|\|x\|$$
$$\leq a\|x\| + t\|(A - \lambda I)x\| + |\lambda|\|x\|, \quad \text{using (4.23)}.$$

This implies, $(1 - t)\|(A - \lambda I)x\| \leq (\alpha + |\lambda|)\|x\|$ for all $x \in D(T)$.

$$\text{So, } \|(A - \lambda I)x\| \leq \left(\frac{a + |\lambda|}{1 - t}\right)\|x\| \quad \text{for all } x \in D(T) = D(A). \tag{4.24}$$

(4.24) implies that $A - \lambda I$ is bounded, which in turn implies that A is bounded. But A is unbounded. Therefore, (4.23) cannot be true for any $t < 1$. Hence, the T-bound of $A, b_T(A)$ equals 1.

Thus (4.22) becomes,

$$\|Ax\| \leq |\lambda|\|x\| + b_T(A)\|Tx\| \text{ for all } x \in D(T).$$

So, A is strictly T-bounded and

$$b'_T(A) \leq |\lambda|. \tag{4.25}$$

Now we claim that $b'_T(A) = |\lambda|$.

If possible, let

$$\|Ax\| \leq \beta\|x\| + \|(A - \lambda I)x\| \quad \text{for every } x \in D(T)\,, \text{ for some } \beta < |\lambda|\,. \tag{4.26}$$

Then, for $x \in D(T)$,

$$\begin{aligned} |\lambda|\|x\| &= \|\lambda x\| \\ &= \|(\lambda I - A)x + Ax\| \\ &\leq \|(A - \lambda I)x\| + \|Ax\| \\ &\leq \|(A - \lambda I)x\| + \beta\|x\| + \|(A - \lambda I)x\|, \quad \text{using (4.26)} \\ &= 2\|(A - \lambda I)x\| + \beta\|x\|. \end{aligned}$$

This implies that

$$\|(A - \lambda I)x\| \geq \frac{1}{2}(|\lambda| - \beta)\|x\| \quad \text{for every } x \in D(T) = D(A)\,.$$

Thus, $A - \lambda I$ is bounded below.

So, $\lambda \notin a(A)$, which is not the case.

Hence $b'_T(A) = |\lambda|$.

Thus, $s'_{1,T}(A) = b'_T(A) = |\lambda|$.

So, $|\lambda| \in \tilde{s}_1(A)$.

Hence, $|a(A)| \subseteq \tilde{s_1}(A)$. □

Corollary 4.2.3. $|e(A)| \subseteq \tilde{s}_1(A)$ *, where* $e(A)$ *is the eigenspectrum of* A *.*

Proof. we have, $e(A) \subseteq a(A)$. So, $|e(A)| \subseteq |a(A)|$.

Hence, by Theorem 4.2.2, we get, $|e(A)| \subseteq \tilde{s_1}(A)$. □

Example 4.2.4. *Let H be an infinite dimensional separable Hilbert space and let $\{e_1, e_2, \dots\}$ be an orthonormal basis for H. Define*

$$Ax = \sum_{n=1}^{\infty} k_n \langle x, e_n \rangle,$$

where (k_n) is a sequence of positive numbers such that $(k_n) \to \infty$ as $n \to \infty$.

Then A is a closed operator. In fact, A is a self-adjoint operator.

For $j = 1, 2, \dots, Ae_j = k_j e_j$, so that $k_j \in e(A)$.

Hence $\tilde{s}_1(A)$ is unbounded, by Corollary 4.2.3. □

Remark 4.2.5. *If A is an unbounded closed operator whose eigenspectrum is unbounded, then $\tilde{s}_1(A)$ is an unbounded set. (This is evident from Corollary 4.2.3). In particular, $\tilde{s}_1(A)$ is unbounded for every closed operator A with compact resolvent.*

Caution: Theorem 4.2.2 is not true for bounded operators.

Let A be a bounded operator on H.

$$\begin{aligned}\tilde{s}_1(A) &= \{s'_{1,T}(A)/T \in \omega_A(H)\} \\ &= \{b'_T(A)/T \in \omega_A(H)\}.\end{aligned}$$

But $b'_T(A) = \|A\|$ for any T if A is a bounded operator.

Thus $\tilde{s}_1(A)$ is the singleton set $\{\|A\|\}$.

Take $H = l^2$ and suppose $A : l^2 \to l^2$ is given by $A(x_1, x_2, \dots) = (x_1, \frac{x_2}{2}, \frac{x_3}{3}, \dots)$.

Then,

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\} \subseteq e(A) \subseteq a(A).$$

$|a(A)|$, being an infinite set, cannot be contained in the singleton set $\tilde{s}_1(A)$.

Remark 4.2.6. $s'_{k,T}(A_n)$ need not converge to $s'_{k,T}(A)$ even when $A_n \to A$ in the resolvent sense.

For example, let $A_n : l^2 \to l^2$ be defined by

$$A_n(x_1, x_2, x_3, \ldots) = (x_1, 2x_2, 3x_3, \ldots, nx_n, 0, 0, 0, \ldots), \quad n = 1, 2, 3, \ldots$$

and

$$A(x_1, x_2, x_3, \ldots) = (x_1, 2x_2, 3x_3, \ldots)$$
$$\text{with } D(A) = \left\{ x = (x_1, x_2, \ldots) \in l^2 \Big/ \sum_{n=1}^{\infty} n^2 |x_n|^2 < \infty \right\}.$$

Then $A_n \to A$ in the resolvent sense.

Let k be any positive integer. Then $k \in e(A) \subseteq a(A)$.

$$\text{For } T = A - kI, \quad s'_{1,T}(A) = b'_T(A) = |k| = k.$$

But

$$\begin{aligned} s'_{1,T}(A_n) &= b'_T(A_n) \\ &= \|A_n\|, \quad \text{as } A_n \text{ is bounded} \\ &= n. \end{aligned}$$

Thus, $s'_{1,T}(A_n) \to \infty$ as $n \to \infty$, whereas $s'_{1,T}(A) = k$.

Note: In the above example, $s'_{1,T}(A_n)$ are independent of T whereas $s'_{1,T}(A)$ is dependent on T.

4.3 Generalized Relative Approximation Numbers

In this section we introduce another kind of relative approximation numbers, namely, generalized relative approximation numbers, of unbounded operators which are bounded relative to a given unbounded operator. These numbers satisfy several properties satisfied by the numbers $s_{k,T}$, including the limiting property. There is an advantage with these numbers. For every operator A, bounded relative to T, the generalized relative approximation numbers exist and are finite, even if A is not strongly T-bounded.

Generalized relative approximation numbers are defined using generalized relative bounds. We introduce generalized relative bounds first.

Let us recall the definition of relative boundedness:

Let H be a Hilbert space and T be a densely-defined linear operator in H. A linear operator A in H with $D(A) \supseteq D(T)$ is T-bounded if there are non-negative (finite) constants a and b such that

$$\|Ax\| \leq a\|x\| + b\|Tx\| \quad \text{for all } x \in D(T)\,.$$

We remark that A is T-bounded if and only if there exists $\alpha < \infty$ such that

$$\|Ax\| \leq \alpha(\|x\| + \|Tx\|) \quad \text{for all } x \in D(T)\,. \tag{4.27}$$

If A is T-bounded, (4.27) is satisfied with $\alpha = \max\{a, b\}$.

Conversely, if (4.27) holds for some α, then, (4.1) is satisfied with $a = b = \alpha$ in (4.1) and A is T-bounded.

Definition 4.3.1. *Let A be T-bounded in H. The infemum of all 'α' satisfying (4.27) is called the generalized relative bound of A with respect to T, or simply the generalized T-bound of A, and is denoted by $g_T(A)$.*

Remark 4.3.2.

(i) (4.27) implies,

$$\|Ax\| \leq g_T(A)(\|x\| + \|Tx\|) \quad \text{for all } x \in D(T)\,. \tag{4.28}$$

(ii) If A is a bounded operator, we have,

$$\|Ax\| \leq \|A\|\|x\| \leq \|A\|(\|x\| + \|Tx\|) \quad \text{for all } x \in D(T)\,. \tag{4.29}$$

Hence

$$g_T(A) \leq \|A\| \quad \text{for } A \in \mathcal{B}(H)\,. \tag{4.30}$$

(iii) Suppose A and B are T-bounded.

Then $D(A) \supseteq D(T)$ and $D(B) \supseteq D(T)$ so that $D(A) \cap D(B) \supseteq D(T)$.

Thus, $A + B$ is a densely-defined operator with $D(A+B) = D(A) \cap D(B)$.

Also,

$$\|Ax\| \leq g_T(A)(\|x\| + \|Tx\|) \quad \text{for all } x \in D(T) \tag{4.31}$$

and,

$$\|Bx\| \leq g_T(B)(\|x\| + \|Tx\|) \quad \text{for all } x \in D(T)\,. \tag{4.32}$$

Now, for $x \in D(T)$,

$$\begin{aligned} \|(A+B)x\| &\leq \|Ax\| + \|Bx\| \\ &\leq [g_T(A) + g_T(B)](\|x\| + \|Tx\|) \quad \text{from (4.31) and (4.32).} \end{aligned}$$

Hence

$$g_T(A+B) \leq g_T(A) + g_T(B). \tag{4.33}$$

(iv) Suppose A is T-bounded and L is bounded on H.

Then, for $x \in D(T)$,

$$\begin{aligned} \|(A+L)x\| &\leq \|Ax\| + \|Lx\| \\ &\leq g_T(A)(\|x\| + \|Tx\|) + \|L\|\|x\| \\ &\leq (g_T(A) + \|L\|)(\|x\| + \|Tx\|). \end{aligned} \tag{4.34}$$

Hence $A + L$ is T-bounded and

$$g_T(A + L) \leq g_T(A) + \|L\|. \tag{4.35}$$

□

Now we define the generalized relative approximation numbers $\theta_{k,T}(A)$.

Definition 4.3.3. *Let A be T-bounded. For $k = 1, 2, 3, \ldots$, define*

$$\theta_{k,T}(A) = \inf\{g_T(A - F)/F \in \mathcal{B}(H),\ \text{rank } F \leq k - 1\}. \tag{4.36}$$

$\theta_{k,T}(A)$ is called the k^{th} generalized approximation number of A relative to T.

Also we define

$$\theta_{\infty,T}(A) = \inf\{g_T(A - K)/K \in \mathcal{K}(H)\}, \tag{4.37}$$

where $\mathcal{K}(H)$ denotes the class of compact operators on H.

Remark 4.3.4.

$$\textit{For } A \in B(H), \theta_{k,T}(A) \leq s_k(A), \tag{4.38}$$

where $s_k(A)$ are the classical approximation numbers defined by

$$s_k(A) = \inf\{\|A - F\|/F \in \mathcal{B}(\mathcal{H}),\ \text{rank } F \leq k - 1\}.$$

(4.38) is clear from the definitions, since

$$g_T(A - F) \leq \|A - F\|, \quad \textit{from (4.30).}$$

□

Proposition 4.3.5. *Let A be T-bounded and L be bounded. Then,*

$$\theta_{k,T}(A + L) \leq \theta_{k,T}(A) + \|L\|. \tag{4.39}$$

Proof. Let $F \in B(H)$ and rank $F \leq k-1$.

Now

$$\begin{aligned}\theta_{k,T}(A+L) &= \inf\{g_T(A+L-F_1)/F_1 \in \mathcal{B}(H),\ \text{rank } F_1 \leq k-1\} \\ &\leq g_T(A+L-F) \\ &\leq g_T(A-F) + \|L\|, \end{aligned} \tag{4.40}$$

from (4.35).

(4.40) holds good for every $F \in \mathcal{B}(H)$ with rank $F \leq k-1$.

Hence

$$\theta_{k,T}(A+L) \leq \inf\{g_T(A-F)/F \in \mathcal{B}(H),\ \text{rank } F \leq k-1\} + \|L\|.$$

That is,

$$\theta_{k,T}(A+L) \leq \theta_{k,T}(A) + \|L\|.$$

□

Theorem 4.3.6. *For* $A \in \mathcal{B}_T(H)$,

$$\lim_{k\to\infty} \theta_{k,T}(A) = \inf_{k=1,2,3,\ldots} \theta_{k,T}(A) = \theta_{\infty,T}(A). \tag{4.41}$$

Proof. Let

$$m = \inf\{\theta_{k,T}(A)/k = 1,2,3,\ldots\}.$$

As every finite-rank operator is compact, it is clear from (4.36) and (4.37) that

$$\theta_{k,T}(A) \geq \theta_{\infty,T}(A) \quad \text{for every } k. \tag{4.42}$$

(4.42) implies that

$$m \geq \theta_{\infty,T}(A). \tag{4.43}$$

Let $\varepsilon > 0$ be given and $K \in \mathcal{K}(H)$.

We can find a finite-rank operator F_1 such that

$$\|K - F_1\| < \varepsilon. \tag{4.44}$$

Let $k-1$ be the rank of F_1.

Now, for $x \in D(T)$,

$$\|(A-K)x\| \leq g_T(A-K)[\|x\| + \|Tx\|] \tag{4.45}$$

and

$$\begin{aligned}
\|(A-F_1)x\| &= \|(A-K)x + (K-F_1)x\| \\
&\leq \|(A-K)x\| + \|K-F_1\|\|x\| \\
&= g_T(A-K)[\|x\| + \|Tx\|] + \varepsilon\|x\|, \quad \text{by (4.45) and (4.44)} \\
&\leq g_T(A-K)[\|x\| + \|Tx\|] + \varepsilon[\|x\| + \|Tx\|].
\end{aligned}$$

Thus,

$$\|(A-F_1)x\| \leq (g_T(A-K) + \varepsilon)[\|x\| + \|Tx\|]. \tag{4.46}$$

Hence,

$$g_T(A-F_1) \leq g_T(A-K) + \varepsilon. \tag{4.47}$$

Now, $\theta_{k,T}(A) \leq g_T(A-F_1)$, as $\theta_{k,T}(A) = \inf\{g_T(A-F)/F \in \mathcal{B}(H),\ \text{rank } F \leq k-1\}$ and rank $F_1 = k-1$.

Hence,

$$\theta_{k,T}(A) \leq g_T(A-K) + \varepsilon.$$

So,

$$m = \inf\{\theta_{k,T}(A)/k = 1, 2, 3, \ldots\} \leq g_T(A-K) + \varepsilon. \tag{4.48}$$

As $\varepsilon > 0$ is arbitrary, we get

$$m \leq g_T(A - K). \tag{4.49}$$

(2.49) is true for every $K \in \mathcal{K}(H)$.

Hence

$$m \leq \inf\{g_T(A - K)/K \in \mathcal{K}(H)\}.$$

That is

$$m \leq \theta_{\infty,T}(A). \tag{4.50}$$

From (4.43) and (4.50),

$$m = \theta_{\infty,T}(A).$$

That is,

$$\inf\{\theta_{k,T}(A)/k = 1, 2, 3, \dots\} = \theta_{\infty,T}(A). \tag{4.51}$$

It is direct from the definition of $\theta_{k,T}$ (Definition 4.3.3) that

$$\theta_{1,T}(A) \geq \theta_{2,T}(A) \geq \cdots \geq 0.$$

So, $\lim_{k\to\infty} \theta_{k,T}(A)$ exists and is equal to $\inf\{\theta_{k,T}(A)/k = 1, 2, 3, \dots\}$.

Hence

$$\lim_{k\to\infty} \theta_{k,T}(A) = \inf\{\theta_{k,T}(A)/k = 1, 2, 3, \dots\} = \theta_{\infty,T}(A).$$

□

Theorem 4.3.7. *Suppose A and B are T-bounded. Then we have*

$$\theta_{k_1+k_2-1,T}(A + B) \leq \theta_{k_1,T}(A) + \theta_{k_2,T}(B). \tag{4.52}$$

Proof. Let $F_1, F_2 \in \mathcal{B}(H)$ with rank $F_1 \leq k_1 - 1$ and rank $F_2 \leq k_2 - 1$.

Put $F = F_1 + F_2$.

Then, rank $F \le k_1 + k_2 - 1 - 1$.

Now

$$\begin{aligned} g_T(A+B-F) &= g_T(A+B-F_1-F_2) \\ &\le g_T(A-F_1) + g_T(B-F_2), \quad \text{by (4.33).} \end{aligned}$$

But,

$$\theta_{k_1+k_2-1,T}(A+B) \le g_T(A+B-F), \quad \text{as rank } F \le k_1+k_2-1-1.$$

Hence

$$\theta_{k_1+k_2-1,T}(A+B) \le g_T(A-F_1) + g_T(B-F_2). \tag{4.53}$$

Since (4.53) is true for every $F_1 \in \mathcal{B}(H)$ with rank $F_1 \le k_1 - 1$ and for every $F_2 \in \mathcal{B}(H)$ with rank $F_2 \le k_2 - 1$, we get,

$$\begin{aligned} \theta_{k_1+k_2-1,T}(A) &\le \inf\{g_T(A-F_1)/F_1 \in \mathcal{B}(H),\ \text{rank } F_1 \le k_1 - 1\} \\ &\quad + \inf\{g_T(B-F_2)/F_2 \in \mathcal{B}(H),\ \text{rank } F_2 \le k_2 - 1\}. \end{aligned}$$

That is,

$$\theta_{k_1+k_2-1,T}(A+B) \le \theta_{k_1,T}(A) + \theta_{k_2,T}(B).$$

□

Corollary 4.3.8. *If A and B are T-bounded, then,*

$$\text{(i)} \qquad \theta_{k,T}(A+B) \le \theta_{k,T}(A) + g_T(B) \tag{4.54}$$

and

$$\text{(ii)} \qquad \theta_{k,T}(A+B) \le \theta_{k,T}(B) + g_T(A). \tag{4.55}$$

Proof. Put $k_1 = k$ and $k_2 = 1$ in (4.52).

We get, $\theta_{k,T}(A+B) \le \theta_{k,T}(A) + \theta_{1,T}(B)$.

But

$$\begin{aligned}\theta_{1,T}(B) &= \inf\{g_T(B-F)/F \in \mathcal{B}(H),\ \text{rank } F \le 1-1=0\} \\ &= \inf\{g_T(B-0)\} \\ &= g_T(B).\end{aligned}$$

The proof of the other part is similar. □

Proposition 4.3.9. *Let A and B be T-bounded. Then*

$$|\theta_{k,T}(A) - \theta_{k,T}(B)| \le g_T(A-B). \tag{4.56}$$

Proof. Let $F \in \mathcal{B}(H)$ be such that rank $F \le k-1$.

Then $\theta_{k,T}(A) \le g_T(A-F)$

$$= g_T(A-B+B-F) \le g_T(A-B) + g_T(B-F).$$

$$\text{So,} \quad \theta_{k,T}(A) - g_T(A-B) \le g_T(B-F). \tag{4.57}$$

Since (4.57) is true for every $F \in \mathcal{B}(H)$ with rank $F \le k-1$, we get,

$$\begin{aligned}\theta_{k,T}(A) - g_T(A-B) &\le \inf\{g_T(B-F)/F \in \mathcal{B}(H),\ \text{rank } F \le k-1\} \\ &= \theta_{k,T}(B).\end{aligned}$$

This implies,

$$\theta_{k,T}(A) - \theta_{k,T}(B) \leq g_T(A - B). \tag{4.58}$$

Interchanging A and B, we get,

$$\theta_{k,T}(B) - \theta_{k,T}(A) \leq g_T(B - A).$$

But

$$g_T(B - A) = g_T(A - B) \quad \text{as } g_T(-A) = g_T(A).$$

So,

$$\theta_{k,T}(B) - \theta_{k,T}(A) \leq g_T(A - B). \tag{4.59}$$

From (4.58) and (4.59), we get

$$|\theta_{k,T}(A) - \theta_{k,T}(B)| \leq g_T(A - B).$$

□

Corollary 4.3.10. *If T is densely defined in H and A and B are bounded on H, then,*

$$|\theta_{k,T}(T + A) - \theta_{k,T}(T + B)| \leq \|A - B\|. \tag{4.60}$$

Proof. From (4.56) we get,

$$\begin{aligned} |\theta_{k,T}(T + A) - \theta_{k,T}(T + B)| &\leq g_T((T + A) - (T + B)) \\ &= g_T(A - B). \end{aligned}$$

But $g_T(A - B) \leq \|A - B\|$, from (4.30), as $A - B \in \mathcal{B}(H)$.

Hence

$$|\theta_{k,T}(T + A) - \theta_{k,T}(T + B)| \leq \|A - B\|.$$

□

4.4 Relative Square Bounds

In this section we introduce relative square bounds of relatively bounded operators. Our aim, in fact, is to discuss about square approximation numbers, that we will do in the next section.

There are two kinds of square bounds, namely, relative square bounds and generalized relative square bounds. We will discuss certain properties of these bounds here.

As usual, H will denote a Hilbert space. Let T be a densely-defined closed operator in H. Its domain $D(T)$ will become a Hilbert space with an induced inner product and every T-bounded operator will become a bounded operator in the new Hilbert space. The closedness of T is essential here.

Once again we recall the definition of relative boundedness:

Let T be densely-defined and closed in H. A linear operator A with $D(A) \supseteq D(T)$ is T-bounded if there exist non-negative real numbers a and b such that

$$\|Ax\| \leq a\|x\| + b\|Tx\| \quad \text{for all } x \in D(T). \tag{4.61}$$

Theorem 4.4.1. *A is T-bounded if and only if there are real constants c and d such that*

$$\|Ax\|^2 \leq c^2\|x\|^2 + d^2\|Tx\|^2 \quad \textit{for all } x \in D(T). \tag{4.62}$$

Proof. Assume that (4.62) holds.

That is, $\|Ax\|^2 \leq c^2\|x\|^2 + d^2\|Tx\|^2$ for all $x \in D(T)$, for some real numbers c and d.

Then

$$\|Ax\|^2 \le c^2\|x\|^2 + d^2\|Tx\|^2 + 2cd\|x\|\|Tx\|$$

$$= (c\|x\| + d\|Tx\|)^2 \quad \text{for all } x \in D(T).$$

Thus,

$$\|Ax\| \le c\|x\| + d\|Tx\| \quad \text{for all } x \in D(T)$$

and A is T-bounded.

Now assume that A is T-bounded.

Then,

$$\|Ax\| \le a\|x\| + b\|Tx\| \quad \text{for all } x \in D(T),$$

for some non-negative real constants a and b.

This implies,

$$\|Ax\|^2 \le a^2\|x\|^2 + b^2\|Tx\|^2 + 2ab\|x\|\ \|Tx\| \quad \text{for all } x \in D(T). \tag{4.63}$$

Let

$$X = \{x \in D(T)/\|x\| \le \|Tx\|\}$$

and

$$Y = \{x \in D(T)/\|x\| > \|Tx\|\}.$$

Then $D(T) = X \cup Y$.

For $x \in X$, (4.63) implies,

$$\|Ax\|^2 \le a^2\|x\|^2 + b^2\|Tx\|^2 + 2ab\|Tx\|^2$$

$$= a^2\|x\|^2 + (b^2 + 2ab)\|Tx\|^2$$

$$\le (a^2 + 2ab)\|x\|^2 + (b^2 + 2ab)\|Tx\|^2. \tag{4.64}$$

For $x \in Y$, (4.63) implies,

$$\begin{aligned}\|Ax\|^2 &\leq a^2\|x\|^2 + b^2\|Tx\|^2 + 2ab\|x\|^2 \\ &= (a^2 + 2ab)\|x\|^2 + b^2\|Tx\|^2 \\ &\leq (a^2 + 2ab)\|x\|^2 + (b^2 + 2ab)\|Tx\|^2. \qquad (4.65)\end{aligned}$$

From (4.64) and (4.65),

$$\|Ax\|^2 \leq (a^2 + 2ab)\|x\|^2 + (b^2 + 2ab)\|Tx\|^2 \quad \text{for all } x \in D(T).$$

Thus,

$$\|Ax\|^2 \leq c^2\|x\|^2 + d^2\|Tx\|^2 \quad \text{for all } x \in D(T),$$

where $c^2 = a^2 + 2ab$ and $d^2 = b^2 + 2ab$.

This completes the proof. □

Definition 4.4.2. *Let T be a densely-defined closed operator in H and A be T-bounded. The infemum of all possible d satisfying (4.62) is called the relative square bound of A with respect to T, or simply, the $T(2)$-bound of A and is denoted by $\phi_T(A)$.*

Suppose there is a real number c such that

$$\|Ax\|^2 \leq c^2\|x\|^2 + \phi_T(A)^2\|Tx\|^2 \quad \textit{for all } x \in D(T). \qquad (4.66)$$

The infemum of all possible c satisfying (4.66) is called the $T(2)$-co bound of A, and is denoted by $\phi'_T(A)$.

In this case, from (4.66) we get,

$$\|Ax\|^2 \leq (\phi'_T(A))^2\|x\|^2 + (\phi_T(A))^2\|Tx\|^2 \quad \textit{for all } x \in D(T). \qquad (4.67)$$

If no 'c' satisfying (4.66) exists, we set $\phi'_T(A) = \infty$.

Example 4.4.3. *Let H be any separable Hilbert space and let $\{e_1, e_2, e_3, \dots\}$ be an orthonormal basis for H.*

Define T and A by

$$Tx = \sum_{k=1}^{\infty} k\langle x, e_k\rangle e_k$$

and

$$Ax = \sum_{k=1}^{\infty} k^{1/2}\langle x, e_k\rangle e_k.$$

We have proved in Section 4.1 (Example 4.1.10) that A is T-bounded and

$$\|Ax\|^2 \leq \|Tx\|^2 \quad \text{for all } x \in D(T).$$

Thus,

$$\|Ax\|^2 \leq 0\|x\|^2 + \|Tx\|^2 \quad \text{for all } x \in D(T).$$

Also, it is proved that for given $\varepsilon > 0$ and for $a \geq 1/2\varepsilon$,

$$\|Ax\|^2 \leq a^2\|x\|^2 + \varepsilon^2\|Tx\|^2 \quad \text{for every } x \in D(T).$$

So, $\phi_T(A) \leq \varepsilon$.

As $\varepsilon > 0$ is arbitrary, $\phi_T(A) = 0$.

Hence, since A is unbounded, there exists no c such that

$$\|Ax\|^2 \leq c^2\|x\|^2 + \phi_T(A)^2\|Tx\|^2 = c^2\|x\|^2 \quad \text{for all } x \in D(T).$$

Thus, $\phi'_T(A) = \infty$. □

Remark 4.4.4. For a bounded operator A, $\phi_T(A) = 0$ and $\phi'_T(A) = \|A\|$, for any T.

Proof. Let $A \in \mathcal{B}(H)$ and T be any densely defined operator in H.

Then, for $x \in D(T)$,

$$\|Ax\| \leq \|A\|\|x\|.$$

So,

$$\|Ax\|^2 \leq \|A\|^2\|x\|^2 = \|A\|^2\|x\|^2 + 0\|Tx\|^2$$

Hence,

$$\phi_T(A) = 0.$$

Now,

$$\begin{aligned}
\phi'_T(A) &= \inf\{c \geq 0 / \|Ax\|^2 \leq c^2\|x\|^2 + \phi_T(A)^2\|Tx\|^2 \text{ for all } x \in D(T)\} \\
&= \inf\{c \geq 0 / \|Ax\|^2 \leq c^2\|x\|^2 \text{ for all } x \in D(T)\} \\
&= \inf\{c \geq 0 / \|Ax\| \leq c\|x\| \text{ for all } x \in D(T)\} \\
&= \inf\{c \geq 0 / \|Ax\| \leq c\|x\| \text{for all } x \in H\}, \quad \text{as } A \text{ is bounded and } D(T) \text{ is dense in } H \\
&= \|A\|.
\end{aligned}$$

□

Now let us define generalized square bounds of relatively bounded operators. For that we need the following simple result:

Proposition 4.4.5. *Let T be a closed operator in H and A be an operator such that $D(A) \supseteq D(T)$. Then, A is T-bounded if and only if*

$$\|Ax\|^2 \leq \alpha^2(\|x\|^2 + \|Tx\|^2) \quad \textit{for all } x \in D(T), \tag{4.68}$$

for some finite α.

Proof. By Theorem 4.4.1, A is T-bounded if and only if $\|Ax\|^2 \leq c^2\|x\|^2 + d^2\|Tx\|^2$ for every $x \in D(T)$, for some real c and d.

Suppose there exist c and d such that

$$\|Ax\|^2 \leq c^2\|x\|^2 + d^2\|Tx\|^2 \quad \text{for every } x \in D(T).$$

Then

$$\|Ax\|^2 \leq \alpha^2(\|x\|^2 + \|Tx\|^2), \quad \text{where } \alpha^2 = \max\{c^2, d^2\}.$$

Conversely, if $\|Ax\|^2 \leq \alpha^2(\|x\|^2 + \|Tx\|^2)$, it is obvious that $\|Ax^2\| \leq c^2\|x\|^2 + b^2\|Tx\|^2$, with $c^2 = d^2 = \alpha^2$.

Hence the proof. □

Definition 4.4.6. *Let T be a densely defined operator in H and A be T-bounded in H. The infemum of all α satisfying (4.68) is called the generalized $T(2)$-bound of A, and is denoted by $G_T(A)$.*

In this case (4.68) implies,

$$\|Ax\|^2 \leq G_T(A)^2(\|x\|^2 + \|Tx\|^2) \quad \textit{for all } x \in D(T). \tag{4.69}$$

Remark 4.4.7.

$$\textit{For} \quad A \in \mathcal{B}(H), G_T(A) \leq \|A\| \quad \textit{for any } T. \tag{4.70}$$

Proof. Let $A \in \mathcal{B}(H)$.

Then, for any $x \in D(T)$,

$$\begin{aligned} \|Ax\|^2 &\leq \|A\|^2\|x\|^2 \\ &\leq \|A\|^2(\|x\|^2 + \|Tx\|^2). \end{aligned}$$

Hence $G_T(A) \leq \|A\|$. □

Suppose A and B are T-bounded, where T is a densely-defined operator in H. Then, since $D(A) \supseteq D(T)$ and $D(B) \supseteq D(T), D(A) \cap D(B) \supseteq D(T)$. Hence $A+B$ is a densely-defined operator in H with domain $D(A+B) = D(A) \cap D(B)$.

Then, $A+B$ is also T-bounded.

Proposition 4.4.8. *Suppose A and B are T-bounded. Then,*

$$G_T(A+B) \leq G_T(A) + G_T(B). \tag{4.71}$$

Proof. Let $x \in D(T)$.

We have $\|Ax\|^2 \leq G_T(A)^2[\|x\|^2 + \|Tx\|^2]$, which implies,

$$\|Ax\| \leq G_T(A)[\|x\|^2 + \|Tx\|^2]^{1/2}. \tag{4.72}$$

Similarly,

$$\|Bx\| \leq G_T(B)[\|x\|^2 + \|Tx\|^2]^{1/2}. \tag{4.73}$$

Now,

$$\begin{aligned} \|(A+B)x\| &\leq \|Ax\| + \|Bx\| \\ &\leq [G_T(A) + G_T(B)][\|x\|^2 + \|Tx\|^2]^{1/2} \end{aligned}$$

Hence,

$$\|(A+B)x\|^2 \leq (G_T(A) + G_T(B))^2(\|x\|^2 + \|Tx\|^2),$$

which implies,

$$G_T(A+B) \leq G_T(A) + G_T(B).$$

□

4.5 Square Approximation Numbers*

The objective in this section is to define the square approximation numbers $\tau_{k,T}$ and $\tau^*_{k,T}$ of T-bounded operators. We define these numbers using generalized relative square bounds, discussed in the previous section.

First we define the τ-numbers.

Definition 4.5.1. *Let T be a closed operator in H and A be T-bounded. For $k = 1, 2, 3, \ldots$, we define*

$$\tau_{k,T}(A) = \inf\{G_T(A - F)/F \in \mathcal{B}(H), \quad \text{rank } F \leq k - 1\} \tag{4.74}$$

and

$$\tau_{\infty,T}(A) = \inf\{G_T(A - K)/K \in \mathcal{K}(H)\}, \tag{4.75}$$

where $\mathcal{B}(H)$ and $\mathcal{K}(H)$ respectively denote the classes of bounded and compact operators on H.

Remark 4.5.2. It is clear from the definitions that

$$\tau_{1,T}(A) \geq \tau_{2,T}(A) \geq \cdots \geq \tau_{\infty,T}(A) \geq 0. \tag{4.76}$$

Now let us discuss a few properties of the τ-numbers.

Theorem 4.5.3.

$$\lim_{k\to\infty} \tau_{k,T}(A) = \inf\{\tau_{k,T}(A)/k = 1, 2, 3, \ldots\} = \tau_{\infty,T}(A). \tag{4.77}$$

Proof. Let

$$m = \inf\{\tau_{k,T}(A)/k = 1, 2, 3 \ldots\}. \tag{4.78}$$

* The contents of this section have appeared in J. Analysis Vol. 12 (2004), 125–134.

(4.76) implies that $m \geq \tau_{\infty,T}(A)$.

To prove the reverse inequality, let $K \in \mathcal{K}(H)$ and $\varepsilon > 0$ be given. Since the class of finite rank operators on H is dense in $\mathcal{K}(H)$, there is a finite rank operator F on H such that

$$\|K - F\| < \varepsilon. \tag{4.79}$$

Let $x \in D(T)$.

Then,

$$\|(A-K)x\|^2 \leq G_T(A-K)^2[\|x\|^2 + \|Tx\|^2]$$

or,

$$\|(A-K)x\| \leq G_T(A-K)[\|x\|^2 + \|Tx\|^2]^{1/2}. \tag{4.80}$$

Consider,

$$\begin{aligned}
\|(A-F)x\| &\leq \|(A-K+K-F)x\| \\
&\leq \|(A-K)x\| + \|(K-F)x\| \\
&< G_T(A-K)[\|x\|^2 + \|Tx\|^2]^{1/2} + \varepsilon\|x\|, \quad \text{using (4.80) and (4.79)} \\
&\leq G_T(A-K)[\|x\|^2 + \|Tx\|^2]^{1/2} + \varepsilon[\|x\|^2 + \|Tx\|^2]^{1/2} \\
&= (G_T(A-K) + \varepsilon)[\|x\|^2 + \|Tx\|^2]^{1/2}.
\end{aligned}$$

Thus,

$$\|(A-F)x\|^2 \leq (G_T(A-K) + \varepsilon)^2[\|x\|^2 + \|Tx\|^2]$$

for every $x \in D(T)$.

Hence,

$$G_T(A-F) \leq G_T(A-K) + \varepsilon. \tag{4.81}$$

But,

$$\tau_{k,T}(A) \leq G_T(A-F), \quad \text{if rank } F = k-1\,.$$

Thus, from (4.81) we get,

$$\tau_{k,T}(A) \leq G_T(A-K) + \varepsilon,$$

which implies, $m \leq G_T(A-K) + \varepsilon$, as $m = \inf\{\tau_{k,T}(A)/k = 1,2,3,\dots\}$.

Since $\varepsilon > 0$ is arbitrary, we get,

$$m \leq G_T(A-K). \tag{4.82}$$

(4.82) is true for every $K \in \mathcal{K}(H)$.

Hence,

$$m \leq \inf\{G_T(A-K)/K \in \mathcal{K}(H)\}.$$

That is,

$$m \leq \tau_{\infty,T}(A). \tag{4.83}$$

From (4.78) and (4.83),

$$m = \tau_{\infty,T}(A).$$

That is,

$$\inf\{\tau_{k,T}(A)/k = 1,2,3,\dots\} = \tau_{\infty,T}(A).$$

Again, since $\{\tau_{k,T}(A)\}_{k=1}^{\infty}$ is monotonic decreasing and bounded below,

$$\lim_{k\to\infty} \tau_{k,T}(A) = \inf\{\tau_{k,T}(A)/k = 1,2,3,\dots\} = \tau_{\infty,T}(A).$$

□

Definition 4.5.4. *An operator T is said to be semi-expanding if*

$$\|Tx\| \geq \|x\| \quad \textit{for every } x \in D(T) \textit{ with } Tx \neq 0\,, \tag{4.84}$$

That is, T is semi-expanding if

$$\|Tx\| \geq \|x\| \quad \textit{for every } x \notin N(T)\,,$$

where $N(T)$ is the null space of T.

T is said to be expanding if

$$\|Tx\| \geq \|x\| \quad \textit{for every } x \in D(T)\,. \tag{4.85}$$

We note that every expanding operator is semi-expanding.

Example 4.5.5. *Let H be a separable Hilbert space. Let $\{e_n/n = 1, 2, 3, \dots\}$ be an orthonormal basis for H. We have $x = \sum_{n=1}^{\infty}\langle x, e_n\rangle e_n$ for every $x \in H$. Let (k_n) be a sequence of scalars such that $|k_n| \geq 1$ for every n. Define T by*

$$T(x) = \sum_{n=1}^{\infty} k_n \langle x, e_n\rangle e_n.$$

Then T is a densely-defined operator in H with domain

$$D(T) = \{x \in H / \sum_{n=1}^{\infty} |k_n|^2 |\langle x, e_n\rangle|^2 < \infty\}.$$

Now, for $x \in D(T)$,

$$\begin{aligned}\|x\|^2 &= \sum_{n=1}^{\infty} |\langle x, e_n\rangle|^2 \\ &\leq \sum_{n=1}^{\infty} |k_n|^2 |\langle x, e_n\rangle|^2, \quad \textit{as } |k_n| \geq 1 \\ &= \|Tx\|^2.\end{aligned}$$

Thus, $\|Tx\| \geq \|x\|$ *for every* $x \in D(T)$ *and* T *is expanding.* $\square$

If the sequence (k_n) *in the above example is bounded, then* T *will be a bounded operator. If we choose* (k_n) *such that* $|k_n| \to \infty$ *as* $n \to \infty$, *then* T *will be unbounded. In fact, many of the unbounded operators are expanding.*

It is clear from (4.85) that every expanding linear operator is $1{-}1$. *So, an unbounded operator* T *which is not* $1-1$ *cannot be expanding. But* T *can be semi-expanding, even if it is not* $1-1$.

If T is expanding or semi-expanding, we have the result:

Theorem 4.5.6. *Let* A *be* T*-bounded and* L *be bounded on* H. *Then for* $k = 1, 2, 3, \ldots$,

$$\tau_{k,T}(A + L) \leq \tau_{k,T}(A) + \|L\|, \tag{4.86}$$

provided T *is expanding or semi-expanding.*

Proof. It is enough to prove the result for the case that T is semi-expanding. Let $k \geq 1$ and $F \in \mathcal{B}(H)$ be with rank $F \leq k - 1$.

For any $x \in D(T)$, we have,

$$\|(A - F)x\|^2 \leq (G_T(A - F))^2 \ (\|x\|^2 + \|Tx\|^2). \tag{4.87}$$

Since $\|x\|^2 + \|Tx\|^2 \leq (\|x\| + \|Tx\|)^2$, (4.87) implies,

$$\|(A - F)x\|^2 \leq (G_T(A - F))^2(\|x\| + \|Tx\|)^2 .$$

$$\text{So,} \quad \|(A - F)x\| \leq G_T(A - F)(\|x\| + \|Tx\|). \tag{4.88}$$

Now consider,

$$\begin{aligned}
\|(A+L-F)x\|^2 &\leq (\|(A-F)x\| + \|Lx\|)^2 \\
&\leq (\|(A-F)x\| + \|L\|\|x\|)^2 \\
&= \|(A-F)x\|^2 + \|L\|^2\|x\|^2 + 2\|(A-F)x\| \ \|L\| \ \|x\| \\
&\leq (G_T(A-F))^2(\|x\|^2 + Tx\|^2) + \|L\|^2\|x\|^2 \\
&\quad + 2G_T(A-F)(\|x\| + \|Tx\|)\|L\| \ \|x\|, \quad \text{using (4.87) and (4.88)} \\
&= \{(G_T(A-F))^2 + \|L\|^2 + 2G_T(A-F)\|L\|\}\|x\|^2 \\
&\quad + (G_T(A-F))^2\|Tx\|^2 + 2G_T(A-F)\|L\| \ \|x\| \ \|Tx\|.
\end{aligned}$$

Thus,

$$\begin{aligned}
\|(A+L-F)x\|^2 &\leq (G_T(A-F) + \|L\|)^2\|x\|^2 + (G_T(A-F))^2\|Tx\|^2 \\
&\quad + 2G_T(A-F)\|L\| \ \|x\| \ \|Tx\| \quad \text{for all } x \in D(T). \qquad (4.89)
\end{aligned}$$

As T is semi-expanding, $\|x\| \leq \|Tx\|$ for every $x \in D(T)$ with $Tx \neq 0$.

Using this, for $x \in D(T)$ with $Tx \neq 0$, (4.89) implies,

$$\begin{aligned}
\|(A+L-F)x\|^2 &\leq (G_T(A-F) + \|L\|)^2\|x\|^2 + (G_T(A-F))^2\|Tx\|^2 \\
&\quad + 2G_T(A-F)\|L\| \ \|Tx\|^2 \\
&= (G_T(A-F) + \|L\|)^2\|x\|^2 \\
&\quad + \{(G_T(A-F))^2 + 2G_T(A-F)\|L\|\}\|Tx\|^2 \\
&\leq (G_T(A-F) + \|L\|)^2\|x\|^2 + (G_T(A-F) + \|L\|)^2\|Tx\|^2 \\
&= (G_T(A-F) + \|L\|)^2(\|x\|^2 + \|Tx\|^2). \qquad (4.90)
\end{aligned}$$

By comparing with (4.89), we see that (4.90) is still valid even if $Tx = 0$.

Thus (4.90) is true for all $x \in D(T)$.

Hence

$$G_T(A + L - F) \leq G_T(A - F) + \|L\|. \tag{4.91}$$

This implies,

$$\tau_{k,T}(A + L) \leq G_T(A - F) + \|L\|, \tag{4.92}$$

since $\tau_{k,T}(A + L) = \inf\{G_T(A + L - F),\ F \in \mathcal{B}(H),\ \text{rank}\ F \leq k - 1\}$.

(4.92) is true for every $F \in \mathcal{B}(H)$ with rank $F \leq k - 1$.

Hence

$$\tau_{k,T}(A + L) \leq \inf\{G_T(A - F)/F \in \mathcal{B}(H),\ \text{rank}\ F \leq k - 1\} + \|L\|.$$

That is,

$$\tau_{k,T}(A + L) \leq \tau_{k,T}(A) + \|L\|.$$

□

Remark 4.5.7. The above result is also true for $k = \infty$.

That is,

$$\tau_{\infty,T}(A + L) \leq \tau_{\infty,T}(A) + \|L\|, \tag{4.93}$$

if T is semi-expanding, A is T-bounded and L is bounded.

The proof is similar. □

If A and B are T-bounded, then, $G_T(A + B) \leq G_T(A) + G_T(B)$, by Proposition 4.4.8. Using this, we are able to prove the following result:

Theorem 4.5.8. *Let A and B be T-bounded. Then,*

$$\tau_{k_1+k_2-1,T}(A + B) \leq \tau_{k_1,T}(A) + \tau_{k_2,T}(B). \tag{4.94}$$

Proof. The proof is the same as that of Theorem 4.3.7 □

τ^*-numbers

Now let us introduce the τ^*-numbers. The following definitions are needed in the sequel.

Definition 4.5.9. [10]. *Assume that* $D(A) \supseteq D(T)$. A *is said to be relatively compact with respect to* T, *or simply,* T*-compact if for any sequence* (x_n) *in* $D(T)$ *with both* (x_n) *and* (Tx_n) *bounded,* $(A(x_n))$ *contains a convergent subsequence.*

A is said to be T-degenerate if A is T-bounded and $R(A)$ is finite-dimensional, where $R(A)$ denotes the range of A.

Definition 4.5.10. *Let* T *be a closed operator in* H *and* A *be* T*-bounded. For* $k = 1, 2, 3, \ldots,$ *we define*

$$\tau^*_{k,T}(A) = \inf\{G_T(A - F)/F \textit{ is } T\textit{-degenerate and } \dim R(F) \leq k - 1\}. \qquad (4.95)$$

Here, $\dim R(F)$ *means the dimension of* $R(F)$.

Also we define

$$\tau^*_{\infty,T}(A) = \inf\{G_T(A - K)/K \textit{is } T\textit{-compact}\}. \qquad (4.96)$$

The τ^*-numbers of A can be regarded as the classical approximation numbers of an induced (bounded) operator A_T between two Hilbert spaces.

Let us develop the framework needed first.

Let H be a Hilbert space with inner product $\langle , \rangle$ and T be a densely-defined closed operator in H. Define $\langle , \rangle_T$ on $D(T)$ by

$$\langle x, y\rangle_T = \langle x, y\rangle + \langle Tx, Ty\rangle. \tag{4.97}$$

It can be easily verified that $\langle , \rangle_T$ is an inner product on $D(T)$. Let $\| \|_T$ be the norm induced by the inner product $\langle , \rangle_T$.

That is

$$\begin{aligned} \|x\|_T &= \langle x, x\rangle_T^{1/2} = (\langle x, x\rangle + \langle Tx, Tx\rangle)^{1/2} \\ &= (\|x\|^2 + \|Tx\|^2)^{1/2} \quad \text{for all } x \in D(T). \end{aligned} \tag{4.98}$$

Here $\| \|$ is the norm on H induced by $\langle , \rangle$:

$$\|x\| = \langle x, x\rangle^{1/2}.$$

Denote the inner product space $D(T)$ with the inner product $\langle , \rangle_T$ by H_T.

Theorem 4.5.11. *H_T is a Hilbert space.*

Proof. Let (x_n) be a Cauchy sequence in H_T.

For $n, m = 1, 2, 3, \dots,$

$$\|x_n - x_m\|_T^2 = \|x_n - x_m\|^2 + \|Tx_n - Tx_m\|^2. \tag{4.99}$$

Hence, both (x_n) and (Tx_n) are Cauchy sequences in H.

Let $(x_n) \to x$ and $(Tx_n) \to y$. Since T is closed, $x \in D(T) = H_T$ and $Tx = y$.

Hence,

$$\begin{aligned} \|x_n - x\|_T^2 &= \|x_n - x\|^2 + \|Tx_n - Tx\|^2 \\ &= \|x_n - x\|^2 + \|Tx_n - y\|^2 \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Thus $(x_n) \to x$ in H_T and H_T is complete. □

Theorem 4.5.12. *If H is separable, then H_T is separable.*

Proof. Let H be separable. Then $H \times H$ is separable.

Being a subspace of $H \times H, \Gamma(T)$ is separable, where $\Gamma(T)$ denotes the graph of T.

Let E be a countable dense subset of $\Gamma(T)$.

Let $x \in H_T = D(T)$. Then $(x, Tx) \in \Gamma(T)$.

Since E is dense in $\Gamma(T)$, there is a sequence (x_n, Tx_n) contained in E such that $(x_n, Tx_n) \to (x, Tx)$ in $H \times H$, as $n \to \infty$.

This implies $\|x_n - x\| \to 0$ and $\|Tx_n - Tx\| \to 0$ in H as $n \to \infty$.

Hence $\|x_n - x\|^2 + \|Tx_n - Tx\|^2 \to 0$ as $n \to \infty$. That is $\|x_n - x\|_T^2 \to 0$ as $n \to \infty$. Thus, the set $\{x/(x, Tx) \in E\}$ is a countable dense set in H_T. So H_T is separable. □

Suppose $D(A) \supseteq D(T)$. Let A_T be the restriction of A to $D(T)$. Then A_T can be considered as an operator: $H_T \to H$. (Note that $H_T = D(T)$ as sets.)

Proposition 4.5.13. *A is T bounded if and only if $A_T : H_T \to H$ is bounded. In this case*

$$\|A_T\| = G_T(A). \tag{4.100}$$

Proof. Assume that A is T-bounded.

Then, by Proposition 4.4.5, there is $\alpha < \infty$ such that

$$\begin{aligned}\|Ax\|^2 &\le \alpha^2(\|x\|^2 + \|Tx\|^2) \\ &= \alpha^2\|x\|_T^2 \quad \text{for all } x \in D(T) = H_T.\end{aligned}$$

So,

$$\|Ax\| \leq \alpha\|x\|_T \quad \text{for all } x \in D(T) = H_T. \tag{4.101}$$

That is, $\|A_T(x)\| \leq \alpha\|x\|_T$ for all $x \in H_T$.

Hence $A_T \in \mathcal{B}(H_T, H)$.

Now

$$\begin{aligned}
\|A_T\| &= \inf\{\alpha / \|A_T(x)\| \leq \alpha\|x\|_T \quad \text{for all } x \in H_T\} \\
&= \inf\{\alpha / \|A_T(x)\|^2 \leq \alpha^2\|x\|_T^2 \quad \text{for all } x \in H_T\} \\
&= \inf\{\alpha / \|Ax\|^2 \leq \alpha^2(\|x\|^2 + \|Tx\|^2) \quad \text{for all } x \in D(T)\} \\
&= G_T(A), \quad \text{from the definition 4.4.6.}
\end{aligned}$$

Conversely assume that $A_T \in \mathcal{B}(H_T, H)$.

Then

$$\|A_T(x)\| \leq \|A_T\|\|x\|_T \quad \text{for all } x \in H_T.$$

So,

$$\|A_T x\|^2 \leq \|A_T\|^2\|x\|_T^2 \quad \text{for all } x \in H_T.$$

That is,

$$\|Ax\|^2 \leq \|A_T\|^2(\|x\|^2 + \|Tx\|^2) \quad \text{for all } x \in D(T). \tag{4.102}$$

Then by Proposition 4.4.5, A is T-bounded. □

Proposition 4.5.14. *Let T be a (densely-defined) closed operator in H. The following properties can easily be verified:*

(i) *Every compact operator is T-compact.*

(ii) *Every (bounded) finite rank operator is T-degenerate.*

(iii) *Every T-compact operator is T-bounded.*

(iv) *Every T-degenerate operator is T-compact.*

Proof.

(i) If $A : D(A) \to H$ is compact, then, for every bounded sequence (x_n) in $D(A)$, $(T(x_n))$ has a convergent subsequence and $D(T) \subseteq D(A)$. Hence from the definition of relative compactness (Definition 4.5.9) A is T-compact.

(ii) Let A be a bounded finite-rank operator. Then, A is T-bounded and $R(A)$ is finite dimensional. So A is T-degenerate.

(iii) can be proved directly using the definitions. However, we give a proof with the help of the Hilbert space H_T.

First we prove the following result:

A is T-compact if and only if A_T is compact. Assume that A is T-compact. To prove A_T is compact, take any bounded sequence (x_n) in H_T.

Then, there exists $c < \infty$ such that $\|x_n\|_T \leq c$ for every n.

That is, $\|x_n\|^2 + \|Tx_n\|^2 \leq c^2$ for every n.

This implies that both (x_n) and (Tx_n) are bounded sequences.

Since A is T-compact (Definition 4.5.9), (Ax_n) contains a convergent subsequence.

That is, $(A_T(x_n))$ contains a convergent subsequence. Hence A_T is compact.

Conversely, assume that A_T is compact. To prove that A is T-compact, take a sequence (x_n) in $D(T)$ such that both (x_n) and (Tx_n) are bounded.

Since $\|x_n\|_T^2 = \|x_n\|^2 + \|Tx_n\|^2, (\|x_n\|_T)$ is a bounded sequence in H_T.

Now the compactness of A_T implies that $(A_T(x_n)) = (Ax_n)$ contains a convergent subsequence. Thus, from the definition of T-compactness, A is T-compact.

Now we prove (iii).

Suppose A is T-compact. Then A_T is compact. So, A_T is bounded. Hence by Proposition 4.5.13, A is T-bounded.

(iv) Assume that A is T-degenerate. Then A is T-bounded and $R(A)$ is finite dimensional. Hence, A_T is bounded (by Proposition 4.5.13) and $R(A_T)$ is finite-dimensional. Thus, being a finite rank (bounded) operator, A_T is compact. Hence A is T-compact. □

Note: Now the following relations are obvious:

(i)

$$\tau^*_{1,T}(A) \geq \tau^*_{2,T}(A) \geq \cdots \geq \tau^*_{\infty,T}(A) \geq 0. \tag{4.103}$$

(ii)

$$\tau_{k,T}(A) \geq \tau^*_{k,T}(A) \quad \text{for any } k\,. \tag{4.104}$$

Proposition 4.5.15. *Let T be a closed operator in H and A be T-bounded. Then,*

(i)

$$\tau^*_{k,T}(A) = s_k(A_T), k = 1, 2, 3, \ldots \tag{4.105}$$

(ii)

$$\tau^*_{\infty,T}(A) = \|A_T\|_{ess.}, \tag{4.106}$$

the essential norm of A_T.

(iii)

$$\lim_{k\to\infty} \tau^*_{k,T}(A) = \tau^*_{\infty,T}(A). \tag{4.107}$$

Proof. (i) Note that F is T-degenerate with $\dim R(F) \leq k-1$ if and only if $F_T \in \mathcal{B}(H_T, H)$ with rank $F_T \leq k-1$.

Also $(A-B)_T = A_T - B_T$ and $\|(A-B)_T\| = G_T(A-B)$, from (4.100).

Hence for $k = 1, 2, 3, \ldots$

$$\begin{aligned}
\tau^*_{k,T}(A) &= \inf\{G_T(A-F)/F \text{ is } T\text{-degenerate, } \dim R(F) \leq k-1\} \\
&= \inf\{\|(A-F)_T\|/F \text{ is } T\text{-degenerate, } \dim R(F) \leq k-1\} \\
&= \inf\{\|A_T - F_T\|/F_T \in \mathcal{B}(H_T, H),\ \text{rank } F_T \leq k-1\} \\
&= s_k(A_T).
\end{aligned}$$

(ii)

$$\begin{aligned}
\tau^*_{\infty,T}(A) &= \inf\{G_T(A-K)/K \text{ is } T\text{-compact}\} \\
&= \inf\{\|A_T - K_T\|/K \text{ is } T\text{-compact}\}.
\end{aligned}$$

But K is T-compact if and only if K_T is compact.

Therefore,

$$\begin{aligned}
\tau^*_{\infty,T}(A) &= \inf\{\|A_T - K_T\|/K_T \in \mathcal{K}(H_T, H)\} \\
&= \|A_T\|_{\text{ess.}}, \quad \text{the essential norm of } A_T.
\end{aligned}$$

(iii) For any bounded operator B we have,

$$\lim_{k\to\infty} s_k(B) = \|B\|_{\text{ess}}.$$

Hence, since A_T is bounded, we have,

$$\lim_{k\to\infty} s_k(A_T) = \|A_T\|_{\text{ess}}.$$

That is,

$$\lim_{k\to\infty} \tau^*_{k,T}(A) = \tau^*_{\infty,T}(A),$$

from (i) and (ii). □

Approximation of τ^*-numbers by truncation

Let H be an infinite-dimensional separable Hilbert space with inner product $<,>$ and T be a densely-defined closed operator in H. Then H_T is also a separable Hilbert space with inner product $<,>_T$ defined by (4.97) (Theorems 4.5.11 and 4.5.12).

Let $\{u_1, u_2, u_3, \ldots\}$ be an orthonormal basis for H, contained in $D(T)$. Then $\{u_1, u_2, u_3, \ldots\}$ is a linearly independent set in $D(T)$. As $H_T = D(T), \{u_1, u_2, u_3, \ldots\}$ is a linearly independent set in H_T. By Gram-Schmidt orthonormalization process (Theorem 1.2.7), there is an orthonormal sequence $\{v_1, v_2, v_3, \ldots\}$ in H_T such that *span* $\{u_1, u_2, \ldots, u_n\}$ = *span* $\{v_1, v_2, \ldots, v_n\}, n = 1, 2, \ldots$. Put H_n = *span* $\{u_1, u_2, \ldots, u_n\}$ = *span* $\{v_1, v_2, \ldots, v_n\}$. Let P_n be the orthogonal projection on H with range H_n and Q_n be the orthogonal projection on H_T with range H_n.

Suppose S is T-bounded. Put

$$(A_T)_n = P_n A_T Q_n|_{H_n} \tag{4.108}$$

and

$$A_n = P_n A P_n|_{H_n}, \tag{4.109}$$

where $A_T : H_T \to H$ is the restriction of A to $H_T = D(T)$.

Now we prove the desired result:

Theorem 4.5.16. *Let T be a densely-defined, closed operator in a separable Hilbert space H and let A be T-bounded. Then, for $k = 1, 2, 3, \ldots$,*

$$\lim_{n\to\infty} \tau^*_{k,T}(A_n) = \tau^*_{k,T}(A), \tag{4.110}$$

where A_n is given by (4.109).

Proof. A_T is a bounded operator: $H_T \to H$, by proposition 4.5.13.

Now, by Theorem 1.10.1,

$$\lim_{n\to\infty} s_k((A_T)_n) = s_k(A_T). \tag{4.111}$$

$$\text{We claim that } (A_T)_n = (A_n)_T, \tag{4.112}$$

where $(A_n)_T$ is the operator A_n considered as an operator: $H_n \subseteq H_T \to H_n$.

$(A_T)_n$ is also an operator: $H_n \subseteq H_T \to H_n$.

Let $x \in H_n$.

Then

$$\begin{aligned}(A_T)_n x &= P_n A_T Q_n x \\ &= P_n A_T x, \text{ since } Q_n x = x \text{ as } x \in H_n = R(Q_n). \\ &= P_n A x, \text{ as } A_T = A/_{D(T)}.\end{aligned}$$

Now,

$$\begin{aligned}A_n x &= P_n A P_n x \\ &= P_n A x, \text{ since } P_n x = x \text{ as } x \in H_n = R(P_n).\end{aligned}$$

Therefore,

$$(A_n)_T x = P_n A x,$$

since $(A_n)_T = A_n$ algebraically.

Thus,

$$(A_T)_n = (A_n)_T.$$

So, (4.111) becomes,

$$\lim_{n\to\infty} s_k((A_n)_T) = s_k(A_T). \tag{4.113}$$

But

$$s_k(A_T) = \tau^*_{k,T}(A),$$

by proposition 4.5.15(i).

Hence, from (4.113) we get,

$$\lim_{n\to\infty} \tau^*_{k,T}(A_n) = \tau^*_{k,T}(A).$$

□

Illustration 4.5.17.

A linear operator T in a Hilbert space H is said to have a compact normal resolvent if there is a $\lambda_\circ \in \rho(T)$, the resolvent set of T, such that $(\lambda_\circ I - T)^{-1}$ is compact and normal.

Suppose T has a compact normal resolvent. Then there exists an orthonormal basis $\{u_1, u_2, u_3, \ldots\}$ in H and $\{\mu_1, \mu_2, \mu_3, \ldots\} \subseteq \mathbb{C}$ such that

(a) $Tu_n = \mu_n u_n, n = 1, 2, 3, \ldots.$ and

(b) $Tx = \sum_n \mu_n \langle x, u_n \rangle u_n$ for every $x \in D(T)$ [15, p. 487]

Now, for $i \neq j$,

$\langle u_i, u_j \rangle_T = \langle u_i, u_j \rangle + \langle Tu_i, Tu_j \rangle$, where $<, >$ is the inner product in H.

Since $Tu_i = \mu_i u_i$, we have,

$$\begin{aligned}\langle u_i, u_j \rangle_T &= \langle u_i, u_j \rangle + \mu_i \overline{\mu_j} \langle u_i, u_j \rangle \\ &= 0, \text{ as } \{u_1, u_2, u_3, \ldots\} \text{ is orthonormal in } H.\end{aligned}$$

Thus, $\{u_1, u_2, u_3, \ldots\}$ is orthogonal in H_T as well; but not orthonormal.

$$\begin{aligned}\langle u_i, u_i \rangle_T &= \langle u_i, u_i \rangle + \langle Tu_i, Tu_i \rangle \\ &= \langle u_i, u_i \rangle + \mid \mu_i \mid^2 \langle u_i, u_i \rangle \\ &= (1 + \mid \mu_i \mid^2), \text{ since } \langle u_i, u_i \rangle = 1.\end{aligned}$$

Take

$$\omega_i = \frac{u_i}{\sqrt{1 + \mid \mu_i \mid^2}}.$$

Then $\{\omega_1, \omega_2, \omega_3, \ldots\}$ is orthonormal in H.

It is clear that *span* $\{u_1, u_2, \ldots, u_n\} =$ *span* $\{\omega_1, \omega_2, \ldots, \omega_n\}$ for every n.

By Theorem 4.5.16,

$$\lim_{n \to \infty} \tau^*_{k,T}(A_n) = \tau^*_{k,T}(A),$$

where $A_n = P_n A P_n \mid_{H_n}$ and P_n is the projection on H with range $H_n =$ *span* $\{u_1, u_2, \ldots, u_n\}$. □

Conclusions

An attempt has been made to develop a general theory on approximation methods for unbounded self-adjoint operators acting in arbitrary Hilbert spaces. Resolvent convergence is used for the approximation. The connection between resolvent convergence, applicability and stability is established. The significance of stability in the theory of approximation methods has been highlighted. Also, the resolvent convergence of the finite section method has been investigated.

The resolvent convergence of the truncations of an unbounded matrix is necessary for the sequence of truncation to be an applicable method for the matrix. A few sufficient conditions for this convergence have been highlighted. An attempt is also made to discuss unbounded Toeplitz matrices.

Further, a few relative approximation numbers of unbounded operators are introduced and their properties are discussed. The partial connection between the eigenspectrum of an unbounded closed operator and its first approximation number set is established. Finally, it is proved that the τ^* - numbers of an unbounded operator A which is bounded relative to a closed operator T can be approximated by the τ^* - numbers of its truncations.

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