ON SZEGÖ'S TYPE THEOREMS

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CERTIFICATE

*This is to certify that the thesis entitled **"ON SZEGŐ'S TYPE THEOREMS"** submitted to the Cochin University of Science and Technology by Remadevi.S. for the award of degree of Doctor of Philosophy in the Faculty of Science is a bonafide record of studies done by her under my supervision. This report has not been submitted previously for considering the award of any degree, fellowship or similar titles elsewhere.*

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CHAPTER I
INTRODUCTION

Gabour Szegö's (1895-1985) most important work was in the area of extremal problems and Toeplitz matrices. He proved a number of limit theorems, now known as Szegö's limit theorem, the strong Szegö's limit theorem [2] and Szegö's orthogonal polynomials.

In early twenties G. Szegö studied in detail the distribution of eigenvalues of the section of Toeplitz forms associated with a function defined in $[-\pi,\pi]$.

The basic idea used by Szegö is the so called equidistribution of sequences introduced by H. Weyl.

Equidistribution of Sequence [13]

Let (u_k) $(k \geq 1)$ be a sequence of real numbers contained in an interval I of length $|I|$. For any subinterval J of I, of length $|J|$, let $J(n)$ denote the number of points among $u_1, u_2, \ldots, u_n$ that lie in J. The sequence is said to be equidistributed or uniformly distributed on I if for each J contained in I,

$$\lim_{n} \frac{J(n)}{n} = \frac{|J|}{|I|} .$$

(Intervals may be open or half-open.)

The following measure theoretic version can also found in [13].

Theorem

The sequence (u_k) contained in $[0,2\pi)$ is uniformly distributed on that interval if and only if

$$\lim_{n} \frac{\sum_{k=1}^{n} f(u_k)}{n} = \int_0^{2\pi} f \, d\sigma$$

For every function f that is continuous and periodic with period 2π.

Toeplitz has studied the distribution of eigenvalues of an infinite matrix $(C_{\upsilon-\mu})$ where the indices υ and μ range from $-\infty$ to ∞. The asymptotic distribution of the eigenvalues of Toeplitz forms can be expressed in the terminology of theory of equal distribution due to H. Weyl. The well known Szegö's theorem throws light into the asymptotic distribution of eigenvalues of truncations.

Szegö's Theorem [12]

The Szegö's theorem on Toeplitz matrices states that if $\lambda_1(A)_N, \lambda_2(A)_N, \ldots, \lambda_N(A)_N$ are the eigenvalues of the $N \times N$ truncations $(A)_N$ of the matrix $A = (a_{i-j})$, where

$$a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx$$

is the k^{th} Fourier coefficient of the multiplier f in $L^\infty(-\pi,\pi)$, and F is any continuous function on R, then

$$\lim_{N\to\infty}\frac{\sum_{k=1}^{N} F(\lambda_k(A)_N)}{N} = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(f(x))dx \qquad \ldots \qquad (1)$$

The above theorem is well known for its applications to trigonometric moment problems, stochastic process [12] and to problems in edge detection [14].

The classical Szegö's theorem is based on Fourier system $\{e_n : n \in Z\}$ where $e_n(x) = e^{inx}$. In this thesis we study similar results in the context of Haar System.

1.1 Summary of the Thesis

The problem considered is the validity of conclusion of Szegö under the following changes in the hypothesis.

(i) The Fourier basis is reordered

(ii) The Fourier system is replaced by other systems like Haar wavelet system with various ordering.

The thesis is divided into five chapters including introductory Chapter I.

In Chapter II we look into the effect of change in the ordering of the Fourier system on Szegö's classical observations of asymptotic distribution of eigenvalues of finite Toeplitz forms. This is done by checking proofs and Szegö's properties in the new set up. It is observed that there is no change in the conclusion of Szegö. The first section deals with minimum property of Toeplitz forms and its limits in the changed system. The second one deals with asymptotic distribution of eigenvalues of finite Toeplitz forms in the new system. This is an imitation of the method adopted by Szegö in the original case.

In Chapter III we consider the multiplication operators under Haar system in $L^2(0,1)$. To be more precise the corner $N \times N$ truncations and the associated asymptotic distribution of eigenvalues are analyzed, analogous to Szegö's theorem classical version. This chapter is divided into two sections. In section one, $L^2(0,1)$ with Haar system under lexicographic ordering is considered. The main theorem of this chapter [3.1.3] says that the conclusion of classical theorem does not remain valid in the changed setup. It is also observed that when the same operator is considered with respect to another ordering, the distribution of eigenvalues converges. In section two we consider spectral approximations of multiplication operators under Haar system in $L^2(0,1)$. This work is quite similar to the work of Kent E. Morrison.[17].

In chapter IV analogous to classical Szegö's theorem we define Szegö's Type theorem for operators in $L^2(R_+)$ and in $L^2(R)$ and checks its validity for certain multiplication operators with respect to a chosen ordering of the Haar basis. It is observed that for certain multiplication operators T_f with multiplier $f = h_{io}$, $i \geq 0$, the distribution of eigenvalues converges but not to the "Szegö limit" and for multiplication operators T_f with $f = h_{ij}, i \geq 0, j > 0$, the distribution of eigenvalues exists and Szegö's Type theorem is valid. This can be considered the main result of this chapter. The theorem 4.11 provides a partial $L^2(R)$ version of the above result.

In the fifth and final chapter, we discuss classes of orderings of Haar System in $L^2(R_+)$ and in $L^2(R)$ in which Szegö's Type Theorem is valid for certain multiplication operators. This chapter is divided into two sections. In the first section, we give an ordering to Haar system in $L^2(R_+)$ and prove that with respect to this ordering, Szegö's Type Theorem holds for general class of multiplication operators T_f with multiplier $f \in L^2(R_+)$, subject to some conditions on f. This is given in 5.1.13, which is the main result of this chapter. Finally in second section more general classes of orderings of Haar system in $L^2(R_+)$ and in $L^2(R)$ are identified in such a way that for certain classes of multiplication operators the asymptotic distribution of eigenvalues exists. Some illustrative examples are also given.

Apart from these five chapters a result on spectral approximation and a proposal for future investigation to higher dimensional $L^2(R^n)$ is given in the appendix.

1.2 Basic Definitions and Theorems

Some basic definitions and theorems which are quoted in the subsequent chapters are given here.

1.2.1 Toeplitz's Forms [12]

Let $f(x)$ be a real-valued function of class L and

$$f(x) \sim \sum_{-\infty}^{\infty} c_n e^{inx}$$

its Fourier Series, where

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx\ ,\ c_{-n} = \overline{c_n}\ .$$

Then the Hermitian form $T_n = \sum c_{\upsilon-\mu} u_\mu \overline{u_\upsilon}, \nu, \mu = 0,1,\ldots,n$ is called the Toeplitz form associated with the function $f(x)$ and the matrix $(c_{\upsilon-\mu})$ is called Toeplitz matrix. We have in this case

$$T_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| u_0 + u_1 e^{ix} + \ldots + u_n e^{inx} \right|^2 f(x)\, dx \, .$$

1.2.2. Equal distribution of numbers [12]

For each n we consider a set of $n+1$ real numbers $a_1^{(n)}, a_2^{(n)}, \ldots, a_{n+1}^{(n)}$ and another set of the same kind $b_1^{(n)}, b_2^{(n)}, \ldots, b_{n+1}^{(n)}$. We assume that for each ν and n

$$\left|a_\nu^{(n)}\right| < K, \quad \left|b_\nu^{(n)}\right| < K$$

where K is independent of ν and n. We say that $\{a_\nu^{(n)}\}$ and $\{b_\nu^{(n)}\}$, $n \to \infty$, are equally distributed in the interval $[-K, K]$ if the following holds. Let $F(t)$ be an arbitrary continuous function in the interval $[-K, K]$; we have then

$$\lim_{n\to\infty} \frac{\sum_{\nu=1}^{n+1} [F(a_\nu^{(n)}) - F(b_\nu^{(n)})]}{n+1} = 0 \, .$$

1.2.3 Multiplication Operator [17]

Suppose $I \subseteq R$ is an interval and $f: I \to C$ is a bounded measurable function. Define the multiplication operator

$$T_f : L^2[I] \to L^2[I] : g \to fg, \; g \in L^2[I].$$

Let $\{e_1, e_2, \ldots\}$ be an orthonormal basis of $L^2[I]$. We define the $N \times N$ matrix $(T_f)_N = (a_{ij}), \; 1 \le i, j \le N$, where

$$a_{ij} = \int f(x)\, e_j(x)\, \overline{e_i}(x)\, dx \, .$$

The infinite matrix $(T_f) = (a_{ij})$ $0 \le i, j$, represents the operator T_f. T_f is the bounded linear operator and we use the operator norm

$$\|T_f\| = \sup_{\|\xi\|=1} \| T_f(\xi) \|.$$

Let P_N denote the orthogonal projection of H onto the span $\{e_1, e_2, \ldots e_n\}$ and put $T_{fN} = P_N T_f P_N$. As it is done in [1], we freely consider T_{fN} as $N \times N$ corner truncation of the matrix (T_f). We can regard $(T_f)_N$ as a matrix approximation of T_f.

1.2.4 Hausdorff metric [15,17]

Let $H(c)$ denote the set of compact subsets of C. Define the Hausdorff metric h on $H(C)$ by

$$h(M,N) = \max[h^*(M,N), h^*(N,M)]$$

(The housdorff distance between M & N) where

$$h^*(M,N) = \sup_{m \in M} \inf_{n \in N} |m - n|.$$

1.2.5 Essential Range [6]

Let E be a measurable subset of R and $f \in L^\infty(E)$. The set

$$\{k \in R : m\{t \in E : |f(t) - k| < \varepsilon\} > 0 \text{ for every } \varepsilon > 0\}$$

is called the essential range of f and is denoted by $R(f)$.

1.2.6 Haar Wavelet Theory [3,5]

Wavelets are mathematical functions that cut up data into different frequency components and then study each component with a resolution matched to its scale. They have advantages over traditional Fourier methods in analyzing physical situations where the signal contains discontinuities and sharp spikes. A comparison of Fourier transform and Wavelet transform is given in [3].

The first mention of wavelet appeared in an appendix to the thesis of A.Haar. The theory of wavelets lies in the boundaries between (i)Mathematics (ii) Scientific Calculations (iii) Signal Processing (iv) Image

Processing. The main branch of mathematics leading to wavelets began with Joseph Fourier who introduced Fourier Synthesis.

In 1910 Haar constructed an orthonormal basis for $L^2(0,1)$ now known as Haar system which provides a local analysis.

For $m,n \in Z$, let I_{mn} be the closed interval

$$I_{mn} = \left[\frac{n}{2^m}, \frac{n+1}{2^m}\right] \subseteq R.$$

Such intervals are called dyadic intervals. The collection $\{I_{mn} : m,n \in Z\}$ of all dyadic intervals has the nesting property: if the interiors of I_{mn} and I_{pq} have nonempty intersection, then either $I_{mn} \subseteq I_{pq}$ or $I_{pq} \subseteq I_{mn}$. The Haar function $\{h_{mn} : m,n \in Z\}$ on R are defined as

$$h_{mn}(x) = \begin{cases} 2^{m/2} & \frac{n}{2^m} \le x < \frac{n+1/2}{2^m} \\ -2^{m/2} & \frac{n+1/2}{2^m} \le x < \frac{n+1}{2^m} \\ 0 & \text{otherwise} \end{cases}$$

Each h_{mn} is nonzero on I_{mn} and $\{h_{mn} : m,n \in Z\}$ is an orthonormal set. $\{h_{mn} : m,n \in Z\}$ is complete in $L^2(R)$,so we have the identity

$$f = \sum_{m,n \in Z} \langle f, h_{mn} \rangle \, h_{mn} \quad \text{in } L^2(R).$$

This expansion is local in the sense that if $f = 0$ on h_{mn}, then $\langle f, h_{mn} \rangle = 0$.

Let

$$h = h_{00} = \begin{cases} 1 & 0 \le x < 1/2 \\ -1 & 1/2 \le x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Then for each $m,n \in Z$

$$h_{mn}(x) = 2^{m/2} \, h(2^m x - n).$$

Hence all basis elements are obtained by certain translations and dilations of one element. This is the characteristic structure of wavelet basis. That one element is called the wavelet. Thus Haar system is regarded as a simplest example of a wavelet basis. 'h' is known as Haar Wavelet.

One of the properties of the Haar wavelet is that it has compact support. It is the only known simple symmetrical wavelet with compact support. Also the simplest wavelet basis suitable for edge detection problems is the Haar basis [11,24]. Unfortunately Haar wavelet is not continuously differentiable which somewhat limits its applications like problems in differential equation.

Let ϕ be the characteristic function of the unit interval [0,1] and for each $j \in Z$, $\phi_j(x) = \phi(x-j)$. Then the collection $\{\phi_j(x)\}$ is an orthonormal set in $L^2(R)$. Let V_0 be the closed linear span of $\phi_j(x)$. Let

$$h_{ij}(x) = 2^{i/2}\ h(2^i x - j) \quad i, j \in Z .$$

For each $i \geq 0$, let W_i be the closed linear span of $\{h_{ij}(x)\ j \in Z\ ,\ i \geq 0\}$. Then it is known that

$$L^2(R) = V_0 \oplus \left\{ \bigoplus_{i=0}^{\infty} W_i \right\}.$$

Hence the collection $\{\phi_j, h_{ij},\ j \in Z\ \ i \geq 0\}$ is an orthonormal basis in $L^2(R)$. The analysis carried in $L^2(R)$ is using the above orthonormal basis. In the case of $L^2(R_+)$ and in $L^2(0,1)$, the restriction of these functions are considered.

1.2.7 Weyl's Theorem[8]

Let A and B be the Hermitian matrices. Then

$$\max_j \left| \lambda_j^{\downarrow}(A) - \lambda_j^{\downarrow}(B) \right| \leq \|A - B\|$$

where $\lambda_j^{\downarrow}(A)$ and $\lambda_j^{\downarrow}(B)$ be the eigenvalues of A and B arranged in decreasing order.

1.2.8 Iterated Limit Theorem [7]

Let (a_{mn}) be the double sequence. Suppose that the single limits $y_m = \lim_n(a_{mn})$, $z_n = \lim_m(a_{mn})$ exist for all $m, n \in N$, and that the convergence of one of these collections is uniform. Then both iterated limits and the double limit exist and all three are equal.

We conclude this chapter by giving some of Kent E. Morrison's work on Szegö's Type theorem based on Walsh system. A brief sketch of Morrison's work [17] is as follows:

In his paper he considered how well the eigenvalues of the matrices approximate the spectrum of the multiplication operator, which is the essential range of the multiplier. The choice of the orthonormal basis strongly affects the convergence. He considered the spectral convergence of multiplication operators acting on the L^2 functions on an interval with respect to Fourier basis, Legendre basis and Walsh basis in the following sense.

(i) $\Lambda_n(f) \to R(f)$ in $H(C)$

(ii) $\mu_n(f) \to \phi*(m)$ weakly.

Where

$\Lambda_n(f)$ - The set of eigenvalues of $(T_f)_N$

$R(f)$ - Essential range of f

$H(C)$ - The Hausdorff space of C

$\mu_n(f)$ - The measure on C such that $\mu_n(f) = \dfrac{\sum_{\lambda\in\Lambda_n}\delta_\lambda}{n}$ where δ_λ is the Dirac delta measure concentrated at λ .

$\phi*(m)$ - The measure defined in C such that

$$\phi*(m)\,[F] = \frac{1}{b-a}\int_a^b F[f(x)]\,dx$$ for any continuous function F on C.

In the case of Legendre basis, Szegö proved the following theorem and another version of this is given in Morrison's paper.

1.2.9 Theorem [12]

Let f be a real valued L^{∞} function on [-1, 1]. Then the sequence of spectral measures $\mu_n(f)$ converges weakly to the measure μ defined by

$$\mu(a,b) = \frac{1}{\pi}\left[\phi\left(\cos^{-1} b\right) - \phi\left(\cos^{-1} a\right)\right]$$

In the case of Walsh basis, Morrison proved the following theorem.

1.2.10 Theorem

Let $f(x) = \sum_{i=0}^{k} c_i \psi_i(x)$ with k less than 2^m where ψ_i is the Walsh functions for $i \geq 0$. Then

(i) $\mu_n(f)$ converges weakly to $\phi * (dx)$.

(ii) For $n = 2^m$ and m sufficiently large, $\Lambda_n(f) = R(f)$.

In this thesis, theorems 3.2.1 and 3.2.2 are analogous to the above mentioned theorem, with Haar system as the underlined basis.

1.3 Notations that are frequently used

T_f	- Multiplication Operator with multiplier f.
(T)	- The matrix of a bounded linear operator on a Hilbert space with respect to a chosen base.
$(T)_N$	- The $N \times N$ corner truncation of (T).
P_N	- Orthogonal projection of L^2 space to span of first n basis elements .
T_N	- $P_N T P_N$ ♦

CHAPTER II

CLASSICAL FOURIER THEOREY

Fourier system forms a basis for the Hilbert space $L^2[-\pi,\pi]$. The classical Szegö's theorem [4,12] is based on Fourier System, $\{ e_n : n \in Z \}$, where $e_n(x) = e^{inx}$. In this chapter we look into the effect of change in the ordering of the Fourier System on Szegö's classical observations of asymptotic distribution of eigenvalues of finite Toeplitz forms. This is done by checking proofs and Szegö's propositions in the new set up. Since the Fourier system is unconditional [19], any arbitrary ordering of the Fourier system forms a basis for the Hilbert space $L^2[-\pi,\pi]$.

This chapter comprises of two sections of which the first one, deals with extremum (minimum) property of the Toeplitz forms and its limits in the changed system. Second one, deals with asymptotic distribution of eigenvalues of finite Toeplitz forms in the new system and the validity of Szegö's theorem.

2.1 Extremum properties of Toeplitz Forms (minimum)

In this section we define a system of orthogonal polynomials with respect to arbitrary ordered Fourier system and the associated Toeplitz forms and find its extremum properties as in the original work of Szegö [12] . Let the arbitrary ordered Fourier system be denoted by $\left\{ e^{is_n x},\ n = 0,1,\ldots \quad s_0 = 0 \right\}$ where 's' is the permutation on N, the set of Natural numbers.

2.1.1 Definition

Let $\alpha(x)$ be a distribution function of the infinite type, $-\pi \le x \le \pi$ and

$$c_{s_n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-is_n x}\, d\alpha(x), \qquad n = 0, 1, 2 \;\ldots$$

be its Fourier-Stieltjes coefficients in the new system. Using the orthogonalization procedure, we form a system of polynomials $\hat{\varphi}_0(x), \hat{\varphi}_1(x), \hat{\varphi}_2(x), \ldots \hat{\varphi}_n(x) \ldots$ of the complex variable z which are orthogonal on the unit circle $|z| = 1$ with the weight $\frac{d\alpha(x)}{2\pi}$.

The system $\{\hat{\varphi}_n(z)\}$ is uniquely determined by the conditions

(i) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{\varphi}_n(z)\hat{\varphi}_m(z)\, d\alpha(x) = \delta_{nm}$$

(ii) $\hat{\varphi}_n(z)$ is a polynomial in which coefficient of z^{s_n} is real and positive.

Let

$$f_n(x) = e^{is_n x} \qquad n = 0, 1, \ldots$$

$$\langle f_\mu, f_\nu \rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{is_\mu x} e^{-is_\nu x}\, d\alpha(x)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i(s_\nu - s_\mu)x}\, d\alpha(x)$$

$$= c_{s_\nu - s_\mu}$$

Define

$$\hat{D}_n = \det(c_{s_\nu - s_\nu})_{\nu,\mu=0}^{n}$$

$$= \begin{vmatrix} c_0 & c_{-s_1} & \cdots & \cdots & c_{-s_{n-1}} & c_{-s_n} \\ c_{s_1} & c_0 & \cdots & \cdots & c_{s_1 - s_{n-1}} & c_{s_1 - s_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{s_{n-1}} & c_{s_{n-1} - s_1} & \cdots & \cdots & c_0 & c_{s_{n-1} - s_n} \\ c_{s_n} & c_{s_n - s_1} & \cdots & \cdots & c_{s_n - s_{n-1}} & c_0 \end{vmatrix}$$

$$\hat{\varphi}_n(x) = \left(\hat{D}_{n-1}\hat{D}_n\right)^{-1/2} \begin{vmatrix} c_0 & c_{-s_1} & \cdots & \cdots & c_{-s_{n-1}} & c_{-s_n} \\ c_{s_1} & c_0 & \cdots & \cdots & c_{s_1-s_{n-1}} & c_{s_1-s_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{s_{n-1}} & c_{s_{n-1}-s_1} & \cdots & \cdots & c_0 & c_{s_{n-1}-s_n} \\ 1 & z^{s_1} & \cdots & \cdots & z^{s_{n-1}} & z^{s_n} \end{vmatrix}$$

where $z = e^{ix}$.

The coefficient of z^{s_n} in $\hat{\varphi}_n(z)$ is denoted by the special notation

$$\hat{k}_n = \left(\frac{\hat{D}_{n-1}}{\hat{D}_n}\right)^{1/2} .$$

2.1.2 Definition

The Toeplitz forms [1, 25] with respect to the new system is defined as

$$\hat{T}_n = \sum_{\mu,\nu=0\text{--}n} c_{s_\nu - s_\mu} u_\mu \overline{u_\nu} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|u_0 + u_1 z^{s_1} + \ldots + u_n z^{s_n}\right|^2 d\alpha(x) \qquad \ldots \qquad (1)$$

Then $\hat{D}_n = \det\left(c_{s_\nu - s_\mu}\right)$ is the determinant of the Toeplitz form. They are called Toeplitz determinants associated with $\alpha(x)$ in the new system. Since (1) is positive definite, we have $\hat{D}_n > 0 \quad \forall n$.

The next theorem gives the extremum property (minimum) of the Toeplitz forms, in the new system.

2.1.3 Theorem

The polynomial $\hat{k}_n^{-1}\hat{\varphi}_n(z)$ minimizes the integral $\frac{1}{2\pi}\int_{-\pi}^{\pi}|g(z)|^2 d\alpha(x)$, $z = e^{ix}$ where $g(z) = z^{s_n} + a_1 z^{s_{n-1}} + \ldots + a_n$ is an arbitrary polynomial generated by $z^0, z^{s_1}, \ldots, z^{s_n}$ in which coefficient of $z^{s_n} = 1$. The minimum itself is $\hat{k}_n^{-2} = \dfrac{\hat{D}_n}{\hat{D}_{n-1}}$.

Proof:

This follows by representing $g(z)$ in the form

$$g(z) = v_0\hat{\varphi}_0(z) + v_1\hat{\varphi}_1(z) + \ldots + v_n\hat{\varphi}_n(z)$$

where $v_0, v_1, \ldots, v_n$ are complex variables and v_n is subjected to the condition $v_n\hat{k}_n = 1$. $\therefore v_n = \hat{k}_n^{-1}$

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}|g(z)|^2\, d\alpha(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}|v_0\hat{\varphi}_0(z) + v_1\hat{\varphi}_1(z) + \ldots + v_n\hat{\varphi}_n(z)|^2\, d\alpha(x) \\ &= |v_0|^2 + \ldots + |v_n|^2 \geq |v_n|^2 \\ &\geq \hat{k}_n^{-2} = \frac{\hat{D}_n}{\hat{D}_{n-1}} \qquad \ldots (1)\end{aligned}$$

When $g(z) = \hat{k}_n^{-1}\hat{\varphi}_n(z)$, then

coefficient of z^{s_n} in $g(z) = \hat{k}_n^{-1}\hat{k}_n = 1$ and

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|g(z)|^2\, d\alpha(x) = \hat{k}_n^{-2} \qquad \ldots (2)$$

Hence from (1) and (2), we get

$$\min \frac{1}{2\pi}\int_{-\pi}^{\pi}|g(z)|^2\, d\alpha(x) = \hat{k}_n^{-2} = \frac{\hat{D}_n}{\hat{D}_{n-1}}$$

Therefore when $g(z) = \hat{k}_n^{-1}\hat{\varphi}_n(z)$,

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}|g(z)|^2\, d\alpha(x) &= \frac{1}{2\pi}\int_{-\pi}^{\pi}|\hat{k}_n^{-1}\hat{\varphi}_n(z)|^2\, d\alpha(x) \\ &= \hat{k}_n^{-2} = \text{minimum value.}\end{aligned}$$

Hence the polynomial $\hat{k}_n^{-1}\hat{\varphi}_n(z)$ minimizes the integral $\frac{1}{2\pi}\int_{-\pi}^{\pi}|g(z)|^2\, d\alpha(x)$.

Now we find the limit of the minimum of the Toeplitz forms under the side condition $u_0 = 1$, which is given in the following limit theorem.

2.1.4 Theorem

Let $\alpha(x)$ be a distribution function of the infinite type. We consider the Toeplitz forms $\hat{T}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|u_0 + u_1 z^{s_1} + \ldots + u_n z^{s_n}\right|^2 d\alpha(x)$ with the side condition $u_0 = 1$. Let $\hat{\mu}_n$ denote the minimum. Then

$$\lim_{n\to\infty}\hat{\mu}_n = \hat{\mu} = G(\omega) = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\log(\omega(x))dx\right\}$$

where $\omega(x)$ is the almost every where existing derivative of $\alpha(x)$.

Proof:

$$\hat{T}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|u_0 + u_1 z^{s_1} + \ldots + u_n z^{s_n}\right|^2 d\alpha(x) \ .$$

The minima $\hat{\mu}_n$ are non increasing as n increases. Hence $\lim_{n\to\infty}\hat{\mu}_n = \hat{\mu}$ exists.

$$\hat{\mu}_n \;=\; \min \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|u_0 + u_1 z^{s_1} + \ldots + u_n z^{s_n}\right|^2 d\alpha(x), \qquad u_0 = 1$$

$$\geq \min\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|u_0 + u_1 z^1 + u_2 z^2 + \ldots + u_{s_n} z^{s_n}\right|^2 d\alpha(x), \;\; u_0 = 1$$

$$= \min\left(T_{s_n}\right)_{u_0=1} = \mu_{s_n} \qquad .$$

Taking limit we get,

$$\hat{\mu} \geq \mu = G(\omega(x)) \qquad [12, \text{Chapter } 3] \qquad \ldots \qquad (1)$$

In order to prove the reverse inequality, first we show that it is always possible to find a large enough m such that

$$\{0, 1, 2, \ldots, n\} \subset \{s_0, s_1, \ldots, s_m\}.$$

There exist positive integers

$\alpha_0 = 0, \alpha_1, \ldots, \alpha_n$ such that $s_{\alpha_0} = 0, s_{\alpha_1} = 1, \ldots, s_{\alpha_n} = n$.

Choose $m \geq \max\{\alpha_0, \alpha_1, \ldots, \alpha_n\}$.

Then

$$\{\alpha_o, \alpha_1, \ldots, \alpha_n\} \quad \subset \quad \{0, 1, 2, \ldots, m\}$$

Hence

$$\{s_{\alpha_0}, s_{\alpha_1}, \ldots, s_{\alpha_n}\} \quad \subset \quad \{s_0, s_1, \ldots, s_m\}$$

ie. $$\{0, 1, \ldots, n\} \quad \subset \quad \{s_0, s_1, \ldots, s_m\}$$

Let

$$T_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|u_0 + u_1 z^1 + \ldots + u_n z^n\right|^2 d\alpha(x)$$

Then

$$\text{Min } T_n = \min\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|u_0 + u_1 z^1 + \ldots + u_n z^n\right|^2 d\alpha(x), \quad u_0 = 1$$

$$\geq \min\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|u_0 + u_1 z^{s_1} + \ldots + u_m z^{s_m}\right|^2 d\alpha(x), \quad u_0 = 1$$

Taking limit, the above inequality reduces to

$$\lim_{n\to\infty}\mu_n \geq \lim_{m\to\infty}\hat{\mu}_m$$

ie. $$\mu = G(\omega(x)) \geq \hat{\mu} \qquad \ldots \quad (2)$$

From (1) and (2), we get

$$\hat{\mu} = G(\omega) = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\log(\omega(x))dx\right\}$$

2.1.5 Theorem

Consider the Toeplitz form

$$\hat{T}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|u_0 + u_1 z^{s_1} + \ldots + u_n z^{s_n}\right|^2 d\alpha(x), \; z = e^{ix}\,.$$

Let $(\hat{\mu}_n)_{u_0=1}$ and $(\hat{\mu}_n)_{u_n=1}$ denote the minimum of $\hat{T}_n$ under the side condition $u_0 = 1$ and $u_n = 1$ respectively. Then

$$\lim_{n\to\infty}(\hat{\mu}_n)_{u_n=1} = \lim_{n\to\infty}\frac{\hat{D}_n}{\hat{D}_{n-1}} = G(\omega)$$

$$= \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi}\log(\omega(x))dx\right\}$$

Proof

We show that $(\hat{\mu}_n)_{u_0=1} = (\hat{\mu}_n)_{u_n=1}$.

Then the theorem is evident from theorems 2.1.3 and 2.1.4.

Case :1 u_n is the leading coefficient

$$\hat{T}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|u_0 + u_1 z^{s_1} + \ldots + u_n z^{s_n}\right|^2 d\alpha(x) \ , \ u_0 = 1 \qquad \ldots \quad (1)$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|u_0 z^{-s_n} + u_1 z^{-(s_n - s_1)} + \ldots + u_n\right|^2 d\alpha(x), \quad u_0 = 1$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left| z^{s_n} + u_1 z^{s_n - s_1} + \ldots + u_{n-1} z^{s_n - s_{n-1}} + u_n\right|^2 d\alpha(x) \qquad \ldots \quad (2)$$

Therefore

$$\min \hat{T}_n = \min \frac{1}{2\pi}\int_{-\pi}^{\pi}\left| z^{s_n} + u_1 z^{s_n - s_1} + \ldots + u_{n-1} z^{s_n - s_{n-1}} + u_n\right|^2 d\alpha(x)$$

Hence from theorem 2.1.3, we get

$$(\hat{\mu}_n)_{u_0=1} = \frac{\widetilde{D}_n}{\widetilde{D}_{n-1}} \qquad \ldots \quad (3)$$

where $\widetilde{D}_n$ is the determinant of the Toeplitz form (2).

The Toeplitz forms (1) and (2) are same. Therefore their determinants are also same. This can be proved in the following way.

Evaluation of $\widetilde{D}_n$

Let $h(z) = z^{s_n} + u_1 z^{s_n - s_1} + \ldots + u_n$. Then $h(z)$ is an arbitrary polynomial generated by $z^0, z^{s_n - s_{n-1}}, z^{s_n - s_{n-2}}, \ldots, z^{s_n - s_1}, z^{s_n}$ such that coefficient of $z^{s_n} = 1$.

The determinant of the Toeplitz from (1) is

$$\widetilde{D}_n = \begin{vmatrix} c_{s_0} & c_{s_{n-1}-s_n} & c_{s_{n-2}-s_n} & \cdots & c_{s_1 - s_n} & c_{-s_n} \\ c_{s_n - s_{n-1}} & c_0 & c_{s_{n-2}-s_{n-1}} & \cdots & c_{s_1 - s_{n-1}} & c_{-s_{n-1}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{s_n - s_1} & c_{s_{n-1}-s_1} & c_{s_{n-2}-s_2} & \cdots & c_0 & c_{-s_1} \\ c_{s_n} & c_{s_{n-1}} & c_{s_{ns-2}} & \cdots & c_{s_1} & c_0 \end{vmatrix}$$

Interchanging the rows R_i, and R_{n-i}, then the columns C_i and C_{n-i} for $i = 0, 1, 2, \ldots\ n/2$ when n is even and for $i = 0, 1, \ldots, (n-1)/2$ when n is odd and then taking transpose we get,

$$\widetilde{D}_n = \begin{vmatrix} c_0 & c_{-s_1} & \cdots & \cdots & c_{-s_{n-1}} & c_{-s_n} \\ c_{s_1} & c_0 & \cdots & \cdots & c_{s_1-s_{n-1}} & c_{s_1-s_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{s_{n-1}-s_0} & c_{s_{n-1}-s_1} & \cdots & \cdots & c_0 & c_{s_{n-1}-s_n} \\ c_{s_n} & c_{s_n-s_1} & \cdots & \cdots & c_{s_n-s_{n-1}} & c_0 \end{vmatrix} = \hat{D}_n$$

Hence equation (3) reduces to

$$(\hat{\mu}_n)_{u_0=1} = \frac{\widetilde{D}_n}{\widetilde{D}_{n-1}} = \frac{\hat{D}_n}{\hat{D}_{n-1}}$$
$$= \min\left(\hat{\mathrm{T}}_n\right)_{u_n=1} = (\hat{\mu}_n)_{u_n=1}$$

Taking limit, then from theorem 2.1.4 we get,

$$\lim_{n\to\infty} (\hat{\mu}_n)_{u_n=1} = \lim_{n\to\infty} \frac{\hat{D}_n}{\hat{D}_{n-1}} = G(\omega)$$

$$= \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \log(\omega(x))dx\right\}$$

Case 2: when u_n is not the leading coefficient.

Let u_k be the leading coefficient. Then divide the polynomial by z^{s_k}. The rest of the proof can be carried out in the same way as in case 1.

Following are some observations obtained by comparing the results in the standard Fourier System and in the new System.

2.1.6 Remarks

It is observed that

(i) In the standard Fourier system, minimum T_n under the side condition $u_0 = 1$ is equal to the minimum of the same Toeplitz form T_n under the side condition $u_n = 1$. But in the new system, $\min\left(\hat{T}_n\right)_{u_0=1}$ is equal to the minimum of another Toeplitz form under the side condition $u_n = 1$.

(ii) The trace of the matrix $\left(c_{\nu-\mu}\right)_{\nu,\mu=0}^{n}$ of the Toeplitz Form

$$T_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|u_0 + u_1 z^1 + \ldots + u_n z^n\right|^2 d\alpha(x)$$

in the standard Fourier system is same as the trace of the matrix $\left(c_{s_\nu - s_\mu}\right)_{s_\nu, s_\mu=0}^{n}$ of the Toeplitz form

$$\hat{T}_n = \frac{1}{2\pi}\int_{-\pi}^{\pi}\left|u_0 + u_1 z^{s_1} + \ldots + u_n z^{s_n}\right|^2 d\alpha(x)$$

in the arbitrary ordered Fourier System. That is, in any arbitrary ordering of the Fourier basis the trace of the Toeplitz matrix remains the same.

2.2 Asymptotic distribution of eigenvalues

In this section the validity of Szegö's Theorem is established. We do this by checking various stages of the proof of Szegö in the new set up. Toeplitz has studied the distribution of eigenvalues of an infinite matrix $\left(c_{\nu-\mu}\right)$, where the indices ν & μ range from $-\infty$ to ∞ under the standard Fourier system. A value λ is called an eigenvalue of the matrix T if the matrix $T - \lambda I$ has no bounded inverse, I denote the unit matrix.

Now we recall the definition of equal distribution of numbers.

2.2.1 **Definition** [1.2.2]

For each n we consider a set of $n+1$ real numbers $a_1^{(n)}, a_2^{(n)}, \ldots, a_{n+1}^{(n)}$ and another set of the same kind $b_1^{(n)}, b_2^{(n)}, \ldots, b_{n+1}^{(n)}$. We assume that for each ν and n

$$\left|a_\nu^{(n)}\right| < K, \quad \left|b_\nu^{(n)}\right| < K,$$

where K is independent of ν and n. We say that $\left\{a_\nu^{(n)}\right\}$ and $\left\{b_\nu^{(n)}\right\}$, $n \to \infty$, are equally distributed in the interval $[-K, K]$ if the following holds. Let $F(t)$ be an arbitrary continuous function in the interval $[-K, K]$; we have then

$$\lim_{n\to\infty} \frac{\sum_{\nu=1}^{n+1}\left[F\left(a_\nu^{(n)}\right) - F\left(b_\nu^{(n)}\right)\right]}{n+1} = 0.$$

Let $f(x)$ be a real valued function of the class L and let

$$c_{s_n} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-is_n x} f(x)dx \qquad n = 0, \pm 1, \pm 2 \ldots.$$

We consider the finite Toeplitz forms

$$\hat{T}_n(f) = \sum_{\mu,\nu=1}^{n} c_{s_\nu - s_\mu} u_\mu \bar{u}_\nu$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} \left|u_0 + u_1 z^{s_1} + \ldots + u_n z^{s_n}\right|^2 f(x)dx \qquad \ldots \qquad (1)$$

. The eigenvalues of $\hat{T}_n(f)$ are defined as the root of the characteristic equation $\det\left(\hat{T}_n(f - \lambda)\right) = 0$. Hence the eigenvalues of $\hat{T}_n(f)$ are the eigenvalues of the matrix

$$\left(c_{s_\nu - s_\mu}\right)_{\nu,\mu=0}^{n} = \begin{bmatrix} c_0 & c_{-s_1} & c_{-s_2} & \cdots & \cdots & c_{-s_n} \\ c_{s_1} & c_0 & c_{s_1 - s_2} & \cdots & \cdots & c_{s_1 - s_n} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{s_{n-1}} & c_{s_{n-1} - s_1} & c_{s_{n-2} - s_2} & \cdots & \cdots & c_{s_{n-1} - s_n} \\ c_{s_n} & c_{s_n - s_1} & c_{s_n - s_2} & \cdots & \cdots & c_0 \end{bmatrix}$$

We denote them by $\beta_1, \beta_2, \ldots, \beta_{n+1}$. Also if $m \le f(x) \le M$ for all real x then from (1) we have $m \le \hat{T}_n(f) \le M$. Also we have

$$m \le \beta_\nu \le M \quad \nu = 1,2, \ \ldots, n+1 .$$

The main result of this chapter is the following theorem and it is the well known Szegö's Theorem in the new arbitrary ordered Fourier System.

2.2.2 Theorem

Let $f(x)$ be a real-valued function of the class L. We denote by m and M the 'essential' lower and upper bound of $f(x)$ respectively and assume that m and M are finite. If $F(\beta)$ is any continuous function defined in the finite interval $m \le \beta \le M$ we have

$$\lim_{n\to\infty} \frac{F(\beta_1)+F(\beta_2)+ \ldots + F(\beta_{n+1})}{n+1} = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(f(x))dx \quad \ldots \quad (2)$$

Proof

Using the definition of the equal distribution the above limit relation can be expressed as follows. The sets $\{\beta_\nu\}$ and $\left\{f\left(-\pi + \frac{2\nu\pi}{n+2}\right)\right\}$, $n \to \infty$ are equally distributed.

It is well known that the limit relation will be proved for all continuous functions $F(t)$ if it holds for certain special sets of continuous functions $F(t) = t^s$ $s = 0,1,2,\ldots.$ and $F(t) = \log t$.

We show that the limit relation is true for $F(t) = \log t$. Then the result follows for t^s also [12, Chapter V]. This will yield the required result (2).

Let $\hat{D}_n$ be the determinant of the Toeplitz form (1), then from theorem 2.1.5, we have

$$\lim_{n\to\infty} \frac{\hat{D}_n}{\hat{D}_{n-1}} = \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log(f(x))dx \right\}$$

ie. $$\lim_{n\to\infty} \left[\hat{D}_n(f)\right]^{1/n+1} = \exp\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} \log(f(x))dx \right\}$$

Therefore

$$\lim_{n\to\infty} \log\left(\hat{D}_n\right)^{1/n+1} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log(f(x))dx$$

Substituting $\hat{D}_n = \beta_1 \beta_2, \ldots \beta_{n+1}$, we get

$$\lim_{n\to\infty} \log(\beta_1 \beta_2 \ldots \beta_{n+1})^{1/n+1} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log(f(x))dx$$

ie. $$\lim_{n\to\infty} \frac{\log\beta_1 + \log\beta_2 + \ldots + \log\beta_{n+1}}{n+1} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \log(f(x))dx$$

Hence the result (2) is true for $F(\beta) = \log\beta$, which completes the proof.

♦

CHAPTER III

MULTIPLICATION OPERATORS AND HAAR WAVELETS IN $L^2(0,1)$

In this chapter, we deal with asymptotic distribution of eigenvalues of multiplication operators under Haar wavelet basis in $L^2(0,1)$ [1.2.6]. This chapter is divided into two sections. In section one, it is shown that the conclusion of the classical Szegö's theorem on asymptotic distribution of eigenvalues of finite sections of multiplication operators, does not remain valid when the trigonometric basis is replaced by the Haar basis. It is also observed that when the same operator is considered with respect to Haar system under a different ordering, the distribution of eigenvalues converges.

In section two, we consider the spectral approximations of multiplication operators under the Haar basis. This work is quite similar to the work of Kent E. Morrison. [17]

3.1 Non existence of '**Szegö** limit'

First of all we recall the statement of Szegö's Theorem

3.1.1 Szegö's Theorem [Chapter I]

The Szegö's theorem on Toeplitz matrices states that if $\lambda_1(A)_N, \lambda_2(A)_N, \ldots, \lambda_N(A)_N$ are the eigenvalues of the $N \times N$ truncations $(A)_N$ of the matrix $A = (a_{i-j})$, where

$$a_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx$$

is the k^{th} Fourier coefficient of the multiplier f in $L^\infty(-\pi,\pi)$, and F is any continuous function on R, then

$$\lim_{N\to\infty} \frac{\sum_{k=1}^{N} F(\lambda_k(A)_N)}{N} = \frac{1}{2\pi}\int_{-\pi}^{\pi} F(f(x))dx \qquad \ldots \qquad (1)$$

3.1.2 Lexicographic ordering of Haar basis [18]

The lexicographically ordered Haar basis can be represented as a sequence $\{\psi_0, \psi_1, \psi_2, \ldots, \psi_n, \ldots\}$ of functions where

$$\psi_0(x) \equiv 1$$

$$\begin{aligned}\psi_n(x) &= h_{rp}(x)\\ &= 2^{r/2}, \quad \frac{p}{2^r} \le x < \frac{(p+1/2)}{2^r}\\ &= -2^{r/2} \quad \frac{(p+1/2)}{2^r} \le x < \frac{(p+1)}{2^r}\\ &= 0 \quad \text{otherwise}\end{aligned}$$

where $n = 2^r + p, \quad 0 \le p < 2^r \; (r \ge 0)$.

The main result in this section is given in the following theorem.

3.1.3 Theorem [18]

Let T_f be the multiplication operator on $L^2(0,1)$ with $f = h_{00}$. Then the asymptotic formula (1) is not satisfied when the trigonometric basis is replaced by lexicographically ordered Haar basis.

Proof

Let $(T_f) = (a_{ij})$ where

$$\begin{aligned}a_{ij} &= \int_0^1 h_{00}(x)\, \psi_i(x)\, \psi_j(x)\, dx\\ &= 1 \quad i = j = n = 2^r + p, \quad 0 \le p < 2^{r-1}, i \ne j = 0,1\\ &= -1 \quad i = j = n, \quad 2^{r-1} \le p < 2^r\\ &= 0 \quad i \ne j \ne 0,1, \quad i = j = 0,1\end{aligned}$$

Therefore the matrix (T_f) is given by

$$(T_f) = \begin{bmatrix} 0 & 1 & 0 & 0 & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & 0 & \dots & \dots & \dots & \dots \\ 0 & 0 & 1 & 0 & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & -1 & 0 & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 1 & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

Now consider the following truncations $(T_f)_N$ of (T_f) when

(*i*) $N = 2^r$, $r = 0, 1, 2, \dots$

(*ii*) $N = 2^r + 2^{r-1}$, $r = 0, 1, 2, \dots$

In both truncations the upper left 2×2 matrix is same and the eigenvalues obtained from this matrix is 1 and -1. Hence in both cases the only eigenvalues are 1 and -1. We compute the multiplicities of eigenvalues in each case. Let N_1 and N_{-1} denote the multiplicities of 1 and -1 respectively.

Evaluation of N_1:

$$T_{fN}[\psi_k(x)] = \psi_k(x) \quad \text{if } x \in \left[0, \frac{1}{2}\right]$$

where $T_{fN} = P_N T_f P_N$.

Hence the eigenvectors corresponding to the eigenvalue 1 are those $\psi_k(x)$ whose support lies in $\left[0, \frac{1}{2}\right]$. In this case we will have

$$\left[\frac{j}{2^i}, \frac{j+1}{2^i}\right] \subset \left[0, \frac{1}{2}\right].$$

Thus multiplicity, namely the number of j s satisfying the above relation equals 2^{i-1} for each i.

In case (i), $N_1 = \sum_{i=0}^{r-1} 2^{i-1}+1 = 2^{r-1}$ and

in case (ii) $N_1 = \sum_{i=0}^{r-2} 2^{i-1} + 2^{r-1} + 1 = 2^r$.

Similarly we can calculate N_{-1} by counting ψ_k s whose support lies in $\left[\frac{1}{2},1\right]$ and which is given in case (i) and (ii) by

$$N_{-1} = 2^{r-1}$$

Therefore in (i) the eigenvalues 1 and -1 each have multiplicity $\frac{N}{2}$. On the other hand +1 has multiplicity $\frac{2N}{3}$ and -1 has multiplicity $\frac{N}{3}$ for sections of type (ii). Let F be any continuous function on R then,

$$\lim_{N\to\infty} \frac{\sum_{k=1}^{N} F(\lambda_k (T_f)_N)}{N} = \frac{F(1)+F(-1)}{2} \quad \text{for type (i)}$$

$$= \frac{2F(1)+F(-1)}{3} \quad \text{for type (ii)}$$

Hence the limit depends upon the truncation of the matrix and therefore the Szegö's theorem fails to hold in this case.

3.1.4 Remarks

Following are some observations obtained from the analysis of the limits.

(i) When $N = 2^r + p, p = 0$ or $p =$ a fixed constant independent of r, then

$$\lim_{N\to\infty} \frac{\sum_{k=1}^{N} F(\lambda_k (T_f)_N)}{N} = \frac{F(1)+F(-1)}{2} = \int_0^1 F(h_{00}(x))dx$$

(ii) For those subsequences for which the growth rate of p is slower than 2^r, then also

$$\lim_{N\to\infty}\frac{\sum_{k=1}^{N}F(\lambda_k(T_f)_N)}{N} = \frac{F(1)+F(-1)}{2} = \int_0^1 F(h_{00}(x))dx .$$

(iii) For any p fixed, the limit becomes $\lambda_p F(1)+(1-\lambda_p)F(-1)$, where $0\le\lambda_p\le 1$. The maximum value of λ_p is 2/3 and the minimum value of λ_p is 1/2.

It is a matter of curiosity to know the outcome when the multiplier is $f=\psi_k$ for $k>1$. Let $\psi_k = h_{mn}$. It is not surprising to see that the conclusions are the same.

In this case the matrix $(\mathrm{T_f})=(a_{ij})$ is given by

$$(T_f) = \begin{pmatrix} \begin{matrix} a_{11}\ a_{12}\ \dots a_{1K} \\ a_{21}\ a_{22}\ \dots a_{2K} \\ \cdots \\ a_{K1}\ a_{K2}\ \dots a_{KK} \end{matrix} & \bigcirc & & \\ & a_{K+1K+1} & & \\ \bigcirc & & a_{K+2K+2} & \\ & & & \ddots \end{pmatrix}$$

where

$$\text{some } a_{ij}\ne 0 \ \ \mathrm{i}\ne\mathrm{j} \ \ \& \ \ 1\le i,\ j\le K,$$

K is the position of h_{mn} in the basis 3.1.2 and

$$a_{ii} = 2^{m/2}, -2^{m/2}, 0 \ \text{ for } \ i\ge K .$$

In $(\mathrm{T}_f)_N$, the upper left $K\times K$ corner entries remains the same for all values of $N \geq K$. Therefore it will enough to consider the diagonal part, while calculating the multiplicities of $2^{m/2}, -2^{m/2}$ and 0. More explicitly the contributions of the above $K\times K$ block will be zero in the limiting case.

Now we calculate the multiplicities of the eigenvalues. The following propositions give the multiplicity of eigenvalues $2^{m/2}$, $-2^{m/2}$ and 0.

3.1.5 Proposition

The multiplicity of eigenvalue $2^{m/2}$ in $(\mathrm{T}_f)_N$ where $N = 2^r + p,\ p < 2^r,\quad r = 0,1,\ldots$ is

$$N_{2^{m/2}} = \begin{cases} 2^{r-m-1} - 1 & \text{if} \quad p < n2^{r-m} \\ 2^{r-m-1} + p - n2^{r-m} & \text{if} \quad n2^{r-m} \leq p < (2n+1)2^{r-m-1} \\ 2^{r-m} - 1 & \text{if} \quad (2n+1)2^{r-m-1} \leq p < 2^r \end{cases}$$

Proof:

Let $N_{2^{m/2}}$ denote the multiplicity of the eigenvalue $2^{m/2}$. Then

$$T_{f_N}[\psi_i(x)] = 2^{m/2}\psi_i(x) \qquad \text{if } x \in \left[\frac{n}{2^m}, \frac{(n+1/2)}{2^m}\right]$$

where $T_{fN} = P_N T_f P_N$.

Hence the eigenvectors corresponding to $2^{m/2}$ be those h_{ij}s whose support lies in $\left[\frac{n}{2^m}, \frac{(n+1/2)}{2^m}\right]$. Then we will have

$$\left[\frac{j}{2^i}, \frac{j+1}{2^i}\right] \subset \left[\frac{n}{2^m}, \frac{(n+1/2)}{2^m}\right].$$

$$\Leftrightarrow n2^{i-m} \leq j < (2n+1)2^{i-m-1},\ i = m+1, m+2, \ldots$$

Thus multiplicity, namely the number of j s which satisfies the above inequality is equal to

$$N_{2^{m/2}} = \begin{cases} \sum_{i=m+1}^{r-1} 2^{i-(m+1)} & \text{if} \quad p < n2^{r-m} \\ \sum_{i=m+1}^{r-1} 2^{i-m-1} + p + 1 - n2^{r-m} & \text{if} \quad n2^{r-m} \le p < (2n+1)2^{r-m-1} \\ \sum_{i=m+1}^{r} 2^{i-m-1} & \text{if} \quad p \ge (2n+1)2^{r-m-1} \end{cases}$$

Therefore

$$N_{2^{m/2}} = \begin{cases} 2^{r-m-1} - 1 & \text{if} \quad p < n2^{r-m} \\ 2^{r-m-1} + p - n2^{r-m} & \text{if} \quad n2^{r-m} \le p < (2n+1)2^{r-m-1} \\ 2^{r-m} - 1 & \text{if} \quad (2n+1)2^{r-m-1} \le p < 2^r \end{cases}$$

3.1.6 Proposition

The multiplicity of eigenvalue $-2^{m/2}$ in $(T_f)_N$ where $N = 2^r + p, p < 2^r$ $, r = 0, 1, \ldots$ is

$$N_{-2^{m/2}} = \begin{cases} 2^{r-m-1} - 1 & \text{if} \quad p < (2n+1)2^{r-m-1} \\ p - n2^{r-m} & \text{if} \quad (2n+1)2^{r-m-1} \le p < (n+1)2^{r-m} \\ 2^{r-m} - 1 & \text{if} \quad (n+1)2^{r-m} \le p < 2^r \end{cases}$$

Proof:

Let $\psi_i(x) = h_{ij}$. We calculate $N_{-2^{m/2}}$ by counting the h_{ij} s whose support satisfies the condition

$$\left[\frac{j}{2^i}, \frac{j+1}{2^i}\right] \subseteq \left[\frac{(n+1/2)}{2^m}, \frac{n+1}{2^m}\right].$$

The rest of the proof of this proposition can be carried out in the same way as in proposition 3.1.5.

3.1.7 Proposition

The multiplicity of eigenvalue zero in $(T_f)_N$ where $N = 2^r + p, p < 2^r, \quad r = 0, 1, \ldots$ is

$$N_0 = \begin{cases} (2^m - 1)(2^{r-m} - 1) + p + 1 & \text{if } \ p < n2^{r-m} \\ (2^m - 1)(2^{r-m} - 1) + n2^{r-m} & \text{if } \ n2^{r-m} \le p < (n+1)2^{r-m} \\ (2^m - 2)(2^{r-m} - 1) + p & \text{if } \ (n+1)2^{r-m} \le p < 2^r \end{cases}$$

Proof:

Let $\psi_i(x) = h_{ij}$. We calculate N_0 by counting the h_{ij} s whose support

$$\left[\frac{j}{2^i}, \frac{j+1}{2^i}\right] \text{ is disjoint from } \left[\frac{n}{2^m}, \frac{n+1}{2^m}\right] \text{ and } \frac{j+1}{2^i} \le 1.$$

The rest of the proof of this proposition can be carried out in the same way as in proposition 3.1.5.

In table 3.1.8 the multiplicities of eigenvalues and the limit of distribution of eigenvalues of various truncations obtained by assigning various values for N are given.

$N = 2^r + p$, $p < 2^r$ Value of p	Multiplicity of eigenvalue $2^{m/2}$	Multiplicity of eigenvalue $-2^{m/2}$	Multiplicity of eigenvalue 0	$\lim_{N\to\infty} \frac{\sum_{k=1}^{N} F(\lambda_k(T_f)_N)}{N}$
0	$2^{r-m-1}-1$	$2^{r-m-1}-1$	$(2^{r-m}-1)(2^m-1)+1$	$\frac{F(2^{m/2})+F(-2^{m/2})}{2^{m+1}}+\frac{2^m-1}{2^m}F(0)$
n	$2^{r-m-1}-1$	$2^{r-m-1}-1$	$(2^{r-m}-1)(2^m-1)+n+1$	$\frac{F(2^{m/2})+F(-2^{m/2})}{2^{m+1}}+\frac{2^m-1}{2^m}F(0)$
$(n-5)2^{r-m}$	$2^{r-m-1}-1$	$2^{r-m-1}-1$	$(2^{r-m}-1)(2^m-1)+$ $(n-5)2^{r-m}+1$	$\frac{F(2^{m/2})+F(-2^{m/2})}{2^{m+1}+(n-5)2}+\frac{2^m+n-6}{2^m+n-5}F(0)$
$n2^{r-m}-1$	$2^{r-m-1}-1$	$2^{r-m-1}-1$	$(2^{r-m}-1)(2^m-1)+n2^{r-m}$	$\frac{F(2^{m/2})+F(-2^{m/2})}{2^{m+1}+2n}+\frac{2^m+n-1}{2^m+n}F(0)$
$n2^{r-m}$	$2^{r-m-1}-1$	$2^{r-m-1}-1$	$(2^{r-m}-1)(2^m-1)+n2^{r-m}$	$\frac{F(2^{m/2})+F(-2^{m/2})}{2^{m+1}+2n}+\frac{2^m+n-1}{2^m+n}F(0)$
$n2^{r-m}+2^{r-m-3}$	$2^{r-m-1}+2^{r-m-3}$	$2^{r-m-1}-1$	$(2^{r-m}-1)(2^m-1)+n2^{r-m}$	$\frac{5F(2^{m/2})+4F(-2^{m/2})}{2^{m+3}+8n+1}+\frac{(2^m-1)8+8n}{2^{m+3}+8n+1}F(0)$
$(n+1)2^{r-m}$	$2^{r-m}-1$	$2^{r-m}-1$	$(2^{r-m}-1)(2^m-2)+(n+1)2^{r-}$	$\frac{F(2^{m/2})+F(-2^{m/2})}{2^m+n+1}+\frac{2^m+n-1}{2^m+n+1}F(0)$
$(2n+1)2^{r-m-1}$	$2^{r-m}-1$	2^{r-m-1}	$(2^{r-m}-1)(2^m-1)+n2^{r-m}$	$\frac{2F(2^{m/2})+F(-2^{m/2})}{2^{m+1}+2n+1}+\frac{2^{m+1}+2n-2}{2^{m+1}+2n+1}F(0)$
2^r-1	$2^{r-m}-1$	$2^{r-m}-1$	$(2^{r-m}-1)(2^m-2)+2^r-1$	$\frac{F(2^{m/2})+F(-2^{m/2})}{2^{m+1}}+\frac{2^m-1}{2^m}F(0)$

3.1.8 Table

One can see that for different subsequences, their limits need not be the same and hence Szegö's theorem fails in this case.

In the following theorem, we show that there is an ordering for the Haar system such that the averages in the asymptotic formula (1) of various sections of T_f converge.

3.1.9 Theorem [18]

Let H be the Haar system in $L^2(0,1)$ ordered as

$\{\varphi_0, h_{00}, h_{10}, \ldots, h_{n0}, h_{n-1,1}, \ldots, h_{n-k,k}, \ldots\}$, $\frac{k+1}{2^{n-k}} \le 1$. Let (T_f) be the matrix of the multiplication operator T_f in $L^2(0,1)$ with respect to this basis. If $\lambda_1(T_f)_N, \lambda_2(T_f)_N, \ldots, \lambda_N(T_f)_N$ are the eigenvalues of $(T_f)_N$, and F any continuous function on R, then

$$\lim_{N\to\infty} \frac{\sum_{k=1}^{N} F(\lambda_k(T_f)_N)}{N} = F(1)$$

Proof

Let N be a positive integer and let $h_{m-k,k}$ be the N^{th} basis element. We show that $N = O(m^2)$. For positive integers n and k_n, $0 \le n \le m$, we have

$$\frac{k_n+1}{2^{n-k_n}} \le 1$$

$$\Leftrightarrow \log(k_n+1) + k_n \le n \qquad \ldots \quad (1)$$

Let $n' = \left[\left[\frac{n-1}{2}\right]\right]$, where [[ ]] denote the integral part of it. Then

$$n' \le \frac{n-1}{2} \le n'+1 \qquad \ldots \quad (2)$$

Let $k_n = n' + 1$ where $n' = \left[\left[\frac{n-1}{2}\right]\right]$

One can see that it satisfies the inequality (1). Hence we have

$$\log(n'+2) + n' + 1 \le n$$

Therefore the total number of h_{n-k_m, k_n} for which support is contained in $(0,1)$ is at least equal to

$$n' + 1 > \frac{n-1}{2}$$

$$\Rightarrow k_n \ge \frac{n-1}{2}$$

$$\therefore N \ge \sum_{n=0}^{m} \frac{n-1}{2} = O(m^2)$$

Also, the only eigenvalues of $(T_f)_N$ are $+1$ and -1. Let N_1 and N_{-1} be the multiplicities of the eigenvalues 1 and -1. We show that

$$\frac{N_{-1}}{N} \to 0 \text{ as } \mathrm{N} \to \infty.$$

It is clear that

$N_{-1} =$ Number of $h_{n-k,k}$ such that its support $\subseteq \left[\frac{1}{2}, 1\right]$ & $k < n$ where $n \le m$.

$$\text{Support of } h_{n-k,k} \subseteq \left[\frac{1}{2}, 1\right] \Leftrightarrow \frac{k}{2^{n-k}} \ge 1/2$$

$$\Leftrightarrow k2^{k+1} \ge 2^n$$

$$\Leftrightarrow k + 1 + \log k \ge n$$

$$\therefore k \ge n - \log n - 1 \quad \text{and} \quad k < n$$

Hence the number of such k s is at most equal to $\log n + 1$ for each n.

$$\therefore N_{-1} \le \sum_{n=1}^{m} (\log n + 1) \le m(\log m + 1).$$

ie.

$$N_{-1} = O(m \log m)$$

Hence

$$\frac{N_{-1}}{N} \to 0 \text{ as } N \to \infty .$$

This completes the proof.

3.2 Spectral Approximations of Multiplication Operators under Haar Wavelet Basis

The spectral convergence of multiplication operators and their eigenvalue distributions are important areas of research. Multiplication operators on L^2 spaces are not compact, and therefore the approximations with finite matrices in Hilbert space cannot converge in norm topology on the space of operators. The choice of orthogonal basis of the Hilbert space affects the convergence. The spectral convergence of multiplication operators under Fourier, Legendre and Walsh basis has been done in detail by Szegö [12] and Kent E. Morrison[17]. In this section we consider the spectral convergence of it under Haar wavelet basis. It is quite similar to the work of Kent E. Morrison.

Let $H(C)$ denote the set of compact subsets of C. Suppose $I \subset R$ is an interval and f be a bounded measurable function. Let T_f be the multiplication operator with f as multiplier. Let $H(C)$ denote the set of compact subsets of C and h denote the Hausdorf metric [1.2.4] on $H(C)$. Let Λ_N be the set of eigenvalues of $(T_f)_N$ and consider Λ_N as an element of $H(C)$. Let $R(f)$ be the essential range [1.2.5] of the multiplier f. $R(f)$ is also an element of $H(C)$. One of the convergences we considered is the convergence of Λ_N to $R(f)$ in $H(C)$.

Another convergence considered is the weak* convergence of measures. Let δ_x denote the Dirac delta measure concentrated at x. Let

$$\mu_N = \frac{\sum_{i=1}^{N} \delta_{\lambda_i}}{N} \text{ where } \lambda_i = \lambda_i (T_f)_N$$

is the measure defined for each N. Define another measure μ on C such that for every continuous function F on R,

$$\mu(F) = \int_0^1 F(f(x))dx$$

In this section we will discuss the spectral convergence in the following sense.

(i) $\Lambda_N \to R(f)$ in $H(C)$

(ii) $\mu_N \to \mu$ weakly.

i.e., $\int F d\mu_N \to \int F d\mu$ as $N \to \infty$, where F is defined as above.

3.2.1 Theorem

Let T_f be the multiplication operator on $L^2(0,1)$ with $f = h_{00}$.

Then, with respect to the lexicographically ordered Haar system,

(i) $\Lambda_N \to R(h_{00})$ in $H(C)$

(ii) For $N = 2^{r+1}$ (r is any positive integer), $\mu_N \to \mu$ weakly.

Proof

Recall that

$$\psi_1(x) = h_{00}(x) = \begin{cases} 1 & \text{if} \quad 0 \le x < \frac{1}{2} \\ -1 & \text{if} \quad \frac{1}{2} \le x < 1 \end{cases}$$

To prove (i) :

From theorem 3.1.3, we have

$$\Lambda_N = \{1, -1\}$$

Now we show that $R(h_{00}) = \{1,-1\}$

$$R(h_{00}) = \{ k \in R / m[x \in R,\ |\psi_1(x) - k| < \varepsilon] > 0\ \forall \varepsilon > 0 \}$$

Let ε be any number >0.

When $k = 1$, we have

$$|\psi_1(x) - k| = 0 \quad \text{if} \quad x \in \left[0, \frac{1}{2}\right)$$
$$= 2 \quad \text{if} \quad x \in \left[\frac{1}{2}, 1\right)$$

$$\therefore \left[0, \frac{1}{2}\right) \subseteq \{x \in R / |\psi_1(x) - 1| < \varepsilon\}$$

$$\therefore m[x \in R / |\psi_1(x) - 1| < \varepsilon] \geq m\left[0, \frac{1}{2}\right] = \frac{1}{2} > 0$$

$$\therefore k = 1 \in R(h_{00})$$

Similarly when $k = -1$

$$|\psi_1(x) + 1| = 0 \quad \text{if} \quad x \in \left[\frac{1}{2}, 1\right)$$
$$= 2 \quad \text{if} \quad x \in \left[0, \frac{1}{2}\right)$$

$$\therefore \left[\frac{1}{2}, 1\right) \subseteq \{x \in R / |\psi_1(x) + 1| < \varepsilon\}$$

$$\therefore m[x \in R / |\psi_1(x) + 1| < \varepsilon] \geq m\left[\frac{1}{2}, 1\right] = \frac{1}{2} > 0$$

$$\therefore k = -1 \in R(h_{00})$$

when $k \neq \pm 1$, choose $0 < \varepsilon < |1 - k|$. Then no x exists such that

$$|\psi_1(x) - k| < |1 - k|$$

Therefore

$$\{x \in R / |\psi_1(x) - k| < |1 - k|\} = \phi$$

$$m[x \in R / |\psi_1(x) - k| < \varepsilon] = 0. \text{ Therefore } k \notin R(h_{00})$$

Hence for all N,

$$\Lambda_N = R(h_{00}) = \{1, -1\}$$

Therefore

$$h(\Lambda_N, R(h_{00})) = \max[d(\Lambda_N, R(h_{00})), d(R(h_{00}), \Lambda_N)] = 0.$$

To prove (ii):

When $N = 2^{r+1}$, from theorem 3.1.3,

$$N_1 = 2^r \text{ and } N_{-1} = 2^r.$$

Hence

$$\lim_{N \to \infty} \int F d\mu_N = \lim_{N \to \infty} \frac{F(\lambda_1) + F(\lambda_2) + \ldots + F(\lambda_N)}{N}$$

$$= \frac{F(1) + F(-1)}{2} = \int_0^1 F(h_{00}(x)) dx$$

Now consider the finite sum $f(x) = \sum_{i=0}^{k} \alpha_i \psi_i$ with $k = 2^{m+1} - 1$.

The functions $\psi_0, \psi_1, \ldots \psi_k$ are Haar functions in $L^2(0,1)$ taken as in the lexicographic ordering 3.1.2. Then we have the following theorem.

3.2.2 Theorem

Let T_f be the multiplication operator on $L^2(0,1)$ with

$f = \sum_{i=0}^{k} \alpha_i \psi_i$, $k = 2^{m+1} - 1$. Then

(i) $\Lambda_N \to R(f)$ in $H(C)$

(ii) For $N = 2^{r+1}$ (r is any positive integer), $\mu_N \to \mu$ weakly.

Proof:

Given $f = \sum_{i=0}^{k} \alpha_i \psi_i$, $k = 2^{m+1} - 1$. Here $\psi_k = h_{m,2^m-1}$.

Let $(T_f) = (a_{ij})$ where

$$a_{ij} = \int_0^1 f(x)\, \psi_i(x)\, \psi_j(x)\, dx$$

Consider a $N \times N$ truncation $(T_f)_N$ where $N = 2^{r+1}$. Then, from [12, Chapter7] the eigenvalues of $(T_f)_N$ are $f(x_0), f(x_1), \ldots, f(x_{N-1})$ where x_s is the mid-point of the interval $\left[\dfrac{s}{2^{r+1}}, \dfrac{s+1}{2^{r+1}}\right]$.

Let these eigenvalues be denoted by $\lambda_1(A)_N, \lambda_2(A)_N, \ldots, \lambda_N(A)_N$ repeated according to multiplicity.

Since $f = \sum_{i=0}^{k} \alpha_i \psi_i$, $k = 2^{m+1} - 1$, $f(x_0), f(x_1), \ldots, f(x_{N-1})$ takes only 2^{m+1} distinct values. Let it be denoted by $\beta_1, \beta_2, \ldots \beta_{2^{m+1}}$. Its values are

$$f(x_s) = \beta_1 \quad \text{if} \quad x_s \in \left[0, \frac{1}{2^{m+1}}\right]$$

$$f(x_s) = \beta_2 \quad \text{if} \quad x_s \in \left[\frac{1}{2^{m+1}}, \frac{2}{2^{m+1}}\right]$$

$$f(x_s) = \beta_3 \quad \text{if} \quad x_s \in \left[\frac{2}{2^{m+1}}, \frac{3}{2^{m+1}}\right]$$

$$\ldots$$

$$f(x_s) = \beta_{2^m} \quad \text{if} \quad x_s \in \left[\frac{2^{m+1}-2}{2^{m+1}}, \frac{2^{m+1}-1}{2^{m+1}}\right]$$

$$f(x_s) = \beta_{2^m+1} \quad \text{if} \quad x_s \in \left[\frac{2^{m+1}-1}{2^{m+1}}, 1\right].$$

To prove (i) :

We have $\quad \Lambda_N = \{\beta_1, \beta_2, \ldots, \beta_{2^{m+1}}\}$

Hence it is enough to show that

$$R(f)=\{\beta_1,\beta_2,\ldots,\beta_{2^{m+1}}\}.$$

For any $\varepsilon>0$ be given.

$$|f(x)-\beta_k|=0 \quad \text{if} \quad x\in\left[\frac{k-1}{2^{m+1}},\frac{k}{2^{m+1}}\right]$$
$$\neq 0 \quad \text{otherwise}$$

$$\therefore \left[\frac{k-1}{2^{m+1}},\frac{k}{2^{m+1}}\right]\subset\{x\in R/|f(x)-\beta_k|<\varepsilon\}$$

$$\therefore m[x\in R/|f(x)-\beta_k|<\varepsilon]\geq m\left[\frac{k-1}{2^{m+1}},\frac{k}{2^{m+1}}\right]=\frac{1}{2^{m+1}}>0$$

$$\therefore \beta_k\in R(f), \qquad k=1,2,\ \ldots,2^{m+1}$$

When $f(x)\neq\beta_k$ for any k, choose $0<\varepsilon<\min_k|\beta_k-\lambda|$

Let $x\in(0,1)$, then $x\in\left[\frac{k-1}{2^{m+1}},\frac{k}{2^{m+1}}\right]$ for exactly one value of k. Therefore

$$\{x/\ |f(x)-\lambda\ |<\min_k|\beta_k-\lambda\ |\ \}=\phi$$
$$ie, \quad \{x/|f(x)-\lambda\ |<\varepsilon\}=\phi$$
$$\therefore m[x\in R/|f(x)-\lambda\ |<\varepsilon]=0$$

$\therefore \lambda\notin R(f)$. Hence $\Lambda_N\to R(f)$ in $H(C)$

Proof of (ii):

First we calculate the multiplicities of the eigenvalues β_k. In general the eigenvalue $f(x_s)$ is β_k, if the point x_s satisfies the condition

$$x_s=\frac{2s+1}{2^{r+2}}\in\left[\frac{k-1}{2^{m+1}},\frac{k}{2^{m+1}}\right],\ k=1,2,\ \ldots,2^{m+1}\ \&\ s=0,1,\ldots,N-1$$

$$(k-1)2^{r-m+1} \leq 2s+1 \leq k2^{r-m+1}$$

$$(k-1)2^{r-m} < s < k2^{r-m}$$

Then, for each k, number of s satisfying the above inequality equals 2^{r-m}.

Hence multiplicity of

$$\beta_k = 2^{r-m} \ , \ k = 1,2, \ . \ . \ . \ ,2^{m+1} \ .$$

$$\text{i.e.,} \quad N_{\beta_k} = 2^{r-m}, k = 1,2, \ . \ . \ . \ ,2^{m+1}$$

$$\therefore \lim_{N\to\infty} \int F d\mu_N = \lim_{N\to\infty} \frac{F(\lambda_1)+F(\lambda_2)+ \ . \ . \ . \ +F(\lambda_N)}{N}$$

$$= \lim_{N\to\infty} \frac{2^{r-m}\left(F(\beta_1)+F(\beta_2)+ \ . \ . \ . \ +F\left(\beta_{2^{m+1}}\right)\right)}{2^{r+1}}$$

$$= \frac{F(\beta_1)+F(\beta_2)+ \ . \ . \ . \ +F\left(\beta_{2^{m+1}}\right)}{2^{m+1}}$$

$$= \int_0^1 F(f(x))\,\mathrm{dx}$$

Hence

$$\lim_{N\to\infty} \mu_N(F) = \mu(F) \ .$$

i.e., $\mu_N \to \mu$ weakly when $N = 2^{r+1}$, $r = 0, 1, 2, \ . \ . \ . \ .$

3.2.3 Remarks:

It follows immediately from 3.1.3 that in the above proposition when $N \neq 2^{r+1}$, for certain sequences of truncations, $\lim_{N\to\infty} \frac{\sum_{k=1}^{N} F(\lambda_k (T_f)_N)}{N}$ will not exists at all.

♦

CHAPTER IV

HAAR SYSTEM IN $L^2(R_+)$ AND SZEGÖ'S TYPE THEOREMS

In this chapter analogous to classical Szegö's theorem we define, Szegö's type theorem for operators in $L^2(R_+)$ and check its validity for certain multiplication operators. Since the trigonometric basis is not available in $L^2(R_+)$ or in $L^2(R)$, we consider Szegö's type theorems with respect to a chosen ordering of the Haar basis in these spaces.

4.1 Definition: Szegö's Type Theorem[21]

Let f be a real function in $L^\infty(R_+)$, and let T_f be the multiplication operator defined on $L^2(R_+)$ (respectively $L^2(R)$). With respect to a given orthonormal basis, let $\{\lambda_1(T_f)_N, \lambda_2(T_f)_N, \ldots, \lambda_N(T_f)_N\}$ be the eigenvalues (repeated according to multiplicity) of the associated corner truncations $(T_f)_N$. Then T_f is said to satisfy Szegö's type theorem if

$$\lim_{N\to\infty} \frac{F(\lambda_1(T_f)_N) + F(\lambda_2(T_f)_N) + \ldots + F(\lambda_N(T_f)_N)}{N} = \lim_{M\to\infty} \frac{1}{M}\int_0^M F[f(x)]\,dx$$

$$= \lim_{M\to\infty} \frac{1}{2M}\int_{-M}^{M} F(f(x))\,dx$$

(respectively)

where F is any continuous function on R.

To carry out further analysis, we consider the following ordered Haar System in $L^2(R_+)$.

4.2 An ordering of the Haar wavelet basis for $L^2(R_+)$

Recall that $\{\phi_j(x), h_{ij}(x),\ i,j \in Z_+ \cup \{0\}\}$ is the Haar system for $L^2(R_+)$ where

$$h_{ij}(x) = 2^{i/2}, \quad \frac{j}{2^i} \le x < \frac{j+1/2}{2^i}$$

$$= -2^{i/2}, \quad \frac{j+1/2}{2^i} \le x < \frac{j+1}{2^i}$$

$$= 0 \quad \text{otherwise}$$

$\phi_j(x) = \phi(x-j)$ where ϕ is the characteristic function of $[\,0,1\,]$.

Consider the following ordering,

$$\{h_{00}, \phi_0, h_{10}, h_{01}, \phi_1, \ldots, h_{r0}, h_{r-1,1}, \ldots, h_{0r}, \ldots\}.$$

Let us denote this ordered basis by $\{\psi_k : k = 1, 2, \ldots\}$

4.3 Position of h_{ij} and ϕ_{i+j} in this ordering

To determine the position of each ψ_k, we write the above basis in the triangular form

$$h_{00}, \phi_0$$

$$h_{10}, h_{01}, \phi_1$$

$$h_{20}, h_{11}, h_{02}, \phi_2$$

$$h_{30}, h_{21}, h_{12}, h_{03}, \phi_3$$

$$\ldots$$

$$h_{i+j,0}, h_{i+j-1,1}, h_{i+j-2,2} \ldots h_{1,i+j-1}, h_{0,i+j}, \phi_{i+j}$$

$$\ldots$$

Then h_{ij} occupies the k th position where $k = \frac{(i+j)(i+j+3)}{2} + j + 1$ and the position of ϕ_{i+j} is $\frac{(i+j+1)(i+j+4)}{2}$.

Hence the ordered basis is given by $\{\psi_k : k = 0, 1, 2, \ldots\}$ where

$$\psi_k = h_{ij} \quad \text{if} \quad k = \frac{(i+j)(i+j+3)}{2} + j + 1$$
$$= \phi_{i+j} \quad \text{if} \quad k = \frac{(i+j+1)(i+j+4)}{2}$$

4.4 Proposition

The m^{th} basic element is h_{ij} where $j = m - \left\{1 + \frac{n(n+3)}{2}\right\}$, $i = n - j$ and $n = \left[\left[\frac{-3 + \left[\sqrt{9 + 8(m-1)}\right]}{2}\right]\right]$. If $j = n+1$ then the m^{th} basis element is ϕ_n, where [[ ]] denotes the integral part of it.

Proof

From the above ordering, there exists unique i and j such that the m^{th} basic element is h_{ij} or ϕ_{i+j}.

Case: 1 Assume m^{th} basis element is h_{ij}

Then the position of h_{ij} in the above ordering is $\frac{(i+j)(i+j+3)}{2} + j + 1$.

Hence we determine the values of i and j for which

$$\frac{(i+j)(i+j+3)}{2} + j + 1 = m$$

ie $$(i+j)^2 + 3(i+j) + 2(j+1-m) = 0$$

ie. $$i + j = \frac{-3 + \left(\sqrt{9 - 8(j+1-m)}\right)}{2} \qquad \ldots \qquad (1)$$

From this equation we find out the values of i and j using the following conditions.

1. $i+j \geq 0$
2. $i+j$ cannot be a fraction
3. $j \geq 0$

Now considering the first condition,

$$9-8(j+1-m) \geq 0$$

$$j+1 < \frac{9}{8}+m$$

$$j \leq m$$

when $j = m$, $i+j$ is negative which is not true by condition one.

$$\therefore j = 0,1, \ldots, m-1$$

The second condition is satisfied only when $9-8(j+1-m)$ is a perfect square. This is possible only when

$$j = m - \left[1+\frac{n(n+3)}{2}\right], \quad i = n-j$$

where n is some positive integer.

Then from equation (1)

$$i+j = n \qquad \ldots \qquad (2)$$

Next we find the value of n using the third condition.

Since $j \geq 0$

$$m - \left[1+\frac{n(n+3)}{2}\right] \geq 0$$

$$ie, n^2+3n-2(m-1) \leq 0$$

Let $$n' = \left[\left[\frac{-3+\left[\sqrt{9+8(m-1)}\right]}{2}\right]\right]$$

where [[ ]] denote the integral part.

We prove that $n = n'$.

Let α_1, α_2 be the roots of the equation $n^2 + 3n - 2(m-1) = 0$. Then

$$\alpha_1 = \frac{-3 + \sqrt{9 + 8(m-1)}}{2}$$

$$\alpha_2 = \frac{-3 - \sqrt{9 + 8(m-1)}}{2}$$

Therefore

$$n^2 + 3n - 2(m-1) = (n - \alpha_1)(n - \alpha_2)$$

Since $(n' - \alpha_2) > 0$, and $n' \le \alpha_1$, we have

$$(n' - \alpha_1)(n' - \alpha_2) \le 0$$

$\therefore n = n'$ and from equation (2) we get

$$i = n - j \quad \text{if } j \le n.$$

Hence the m^{th} basis element is h_{ij} where

$$j = m - \left[1 + \frac{n(n+3)}{2}\right] \; \& \, i = n - j \quad (\text{if } j \le n) \text{ where}$$

$$n = \left[\left[\frac{-3 + \left[\sqrt{9 + 8(m-1)}\right]}{2}\right]\right].$$

Case: 2

When j exceeds n, i becomes negative and no h_{ij} occur in the above basis other than ϕ_n which corresponds to $j = n+1$ (and only $n+1$). This is quite clear from the triangular form. It is also evident from the above calculations that the value of m is $\frac{(n+1)(n+4)}{2}$ when $j = n+1$. This is nothing but the position of ϕ_n in the above arrangement.

The main result in this chapter is the following theorem, which gives the asymptotic distribution of eigenvalues of certain multiplication operator with respect to the ordering 4.2. It is an improved version of the corresponding result in [21, pp 120-122]

4.5 Theorem

Let T_f be is the multiplication operator on $L^2(R_+)$ with $f = h_{00}$ and $\{\lambda_1(T_f)_N, \lambda_2(T_f)_N, \ldots, \lambda_N(T_f)_N\}$ be the eigenvalues of $(T_f)_N$ with respect to the basis 4.2, then for any continuous function F on R,

$$\lim_{N\to\infty} \frac{\sum_{k=1}^{N} F(\lambda_k(T_f)_N)}{N} = F(1)$$

Proof:

Let $(T_f) = (a_{ij})$, where

$$a_{ij} = \langle T_{\psi_1}(\psi_i(x)), \psi_j(x)\rangle$$

$$= \int_0^1 h_{00}(x).\psi_i(x)\psi_j(x)dx$$

$$= 0 \quad \text{when } i \neq j \text{ and } i, j > 2$$

The matrix (T_f) is given by

$$(T_f) = \begin{bmatrix} 0 & 1 & 0 & \ldots & \ldots & \ldots. \\ 1 & 0 & 0 & 0 & \ldots & \ldots \\ 0 & 0 & 0 & \ldots & \ldots & \ldots \\ \ldots & \ldots & \ldots & 1 & \ldots & \ldots \\ \ldots & \ldots & \ldots & \ldots & -1 & \ldots \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \end{bmatrix}$$

Let N be a positive integer and let the N^{th} basis element be h_{mn} or ϕ_{m+n}. Then $N = \frac{(m+n)(m+n+3)}{2} + n + 1$ or $\frac{(m+n+1)(m+n+4)}{2}$. Let $m+n=s$, then $N = O(s^2)$ in both cases.

Consider the N^{th} stage truncation $(T_f)_N$ of T_f where N is as above. In $(T_f)_N$, the multiplicity contributed by the upper left 2×2 matrix is same for all values of $N \geq 2$. Since we consider only the limits of averages, it is enough to consider the multiplicity of eigenvalues in the diagonal part of the truncated matrix after neglecting the above 2×2 matrix. The eigenvalues are 1, -1 and 0. Let N_1, N_{-1}, N_0 denote the multiplicities of the eigenvalues 1, -1 and 0 respectively. We show that

$$\frac{N_{-1}}{N} \to 0, \frac{N_0}{N} \to 0 \quad \text{as} \quad N \to \infty .$$

Estimation of N_{-1}:

$$T_{fN}(\psi_k(x)) = -\psi_k(x) \quad \text{where } T_{fN} = P_N T_f P_N$$

$$\Leftrightarrow \psi_k = h_{ij} \quad \text{such that its support} \subset \left[\frac{1}{2}, 1\right] \quad \& \ i + j \leq s$$

$$\Leftrightarrow \left[\frac{j}{2^i}, \frac{j+1}{2^i}\right] \subset \left[\frac{1}{2}, 1\right] \& \ i + j \leq s$$

$$\Leftrightarrow 2^{i-1} \leq j < 2^i - 1 \ \ \& \ i + j \leq s \qquad \ldots \quad (1)$$

Since $2^{i-1} \leq j$ we have,

$$2^{i-1} + i \leq i + j \leq s \Rightarrow 2^{i-1} < s$$

$$\text{ie, } \ i - 1 \leq \log_2 s \Rightarrow i \leq \log_2 s + 1 \qquad \ldots \quad (2)$$

From equation (1) the number of j s corresponding to each i is 2^{i-1}.

$\therefore$ Total number of h_{ij} s $\leq \sum_{i=1}^{\log s+1} 2^{i-1} \leq (\log s + 1)s$

That is, $\qquad N_{-1} \leq s(\log s + 1)$

Therefore we have $N_{-1} = O(s \log s)$.

Hence

$$\lim_{N\to\infty} \frac{N_{-1}}{N} = 0 \quad \text{(since } N = O(s^2)\text{.)}$$

Now we show that $\frac{N_0}{N} \to 0$ as $N \to \infty$.

This can be seen as follows.

$T_{fN}(\psi_k) = 0 \Leftrightarrow (i) \quad \psi_k = h_{ij}$ whose support is disjoint from $[0, 1]$ & $i + j \le s$

$(ii) \ \psi_k = \phi_{i+j}$,& $\ i + j \le s$

Considering (i), we get

$$T_{fN}(h_{ij}) = 0 \Leftrightarrow \frac{j}{2^i} \ge 1 \;\&\; i + j \le s$$

$$\Leftrightarrow 2^i \le j \le s - i \qquad \ldots \qquad (3)$$

ie, $\qquad 2^i + i \le i + j \Rightarrow 2^i < i + j < s \Rightarrow i < \log_2 s$

From equation (3), the number of j corresponding to each i is equal to $s + 1 - (2^i + i)$. Therefore the number of j s corresponding to each i is at most equal to $s - 2$.

Hence considering (i) and (ii)

$$N_0 \le (s-2)\log s + s$$

$$\therefore \lim_{N\to\infty} \frac{N_0}{N} = 0 \quad .$$

and therefore $\qquad \frac{N_1}{N} \to 1$ as $N \to \infty$.

$$\lim_{N\to\infty} \frac{\sum_{k=1}^{N} F(\lambda_k (T_f)_N)}{N} = F(1)$$

4.6 Remarks

(i) For each x, let δ_x denote the Dirac delta measure concentrated at x. For simplicity put $\lambda_i = \lambda_i(T_f)_N$. Let $\mu_N = \dfrac{\delta_{\lambda_1} + \delta_{\lambda_2} + \ldots + \delta_{\lambda_N}}{N}$ be the measure defined for each N and let $\mu = \delta_1$. Then the above theorem implies that for all continuous functions F on R,

$$\int_0^\infty F d\mu_N \to \int_0^\infty F d\mu$$

ie, $\mu_N \to \mu$ weakly as $N \to \infty$.

(ii) Since $F(0) = \lim_{M \to \infty} \frac{1}{M} \int_0^M F(h_{00}(x))dx$, we can interpret the above result as 'failure' of Szegö's Type Theorem in general. Of course, one should admit that this type of interpretation is not fair, since the asymptotic limit exists.

(iii) When we consider T_f where $f = h_{u0}$ (u any positive integer) with respect to the same basis 4.2, then also the above remarks holds.

Now we consider the case of all multiplication operators T_f where $f = h_{uv}$ ($v \neq 0$). Then it is surprising to see that, T_f satisfies Szegö's Type Theorems. This result is given in the following theorem.

4.7 Theorem

Let T_f be the multiplication operator on $L^2(R_+)$ with $f(x) = h_{uv}(x)$ ($v \neq 0$). Let $\lambda_1(T_f)_N, \lambda_2(T_f)_N, \ldots, \lambda_N(T_f)_N$ be the eigenvalues of the truncated matrix $(T_f)_N$ with respect to the basis 4.2, then for any continuous function F on R, the following asymptotic formula holds.

$$\lim_{N\to\infty}\frac{\sum_{k=1}^{N}F(\lambda_k(T_f)_N)}{N} = \lim_{M\to\infty}\frac{1}{M}\int_0^M F(h_{uv}(x))dx$$

Proof:

In the ordering 4.2, the position of h_{uv} is K where

$$K=\frac{(u+v)(u+v+3)}{2}+v+1.$$

Let $\quad (T_f) = (a_{ij}) \quad$ where

$$a_{ij} = \int_0^\infty h_{uv}(x)\psi_i(x)\psi_j(x)dx$$

$$= 0, \text{ when } i\neq j \text{ and } i \text{ or } j>K$$

$$= 2^{u/2}, -2^{u/2}, 0 \text{ when } i=j.$$

Let N be a positive integer and let the N^{th} basis element be h_{mn} or ϕ_{m+n}. Then $N=\frac{(m+n)(m+n+3)}{2}+n+1$ or $N=\frac{(m+n+1)(m+n+4)}{2}$. Put $m+n=s$. Then in both cases $N=O(s^2)$.

Consider the truncations $(T_f)_N = (a_{ij}),\ 1\le i,j\le N$ of (T_f) where N is as above. Since the upper left $K\times K$ block is same for all values of N, as in previous case, it is enough to consider the diagonal block $(T_f)_N$ for each $N\ge K$. Therefore eigenvalues of $(T_f)_N$ are $2^{u/2}, -2^{u/2}$ and 0.

Now we estimate the multiplicity of these eigenvalues.

Claim : $N_{2^{u/2}} \le \frac{s\log s}{2}$

The proof is as follows.

$$T_f(\psi_k(x)) = 2^{u/2}\psi_k(x) \;\Leftrightarrow\; x \in \left[\frac{v}{2^u}, \frac{v+\frac{1}{2}}{2^u}\right].$$

This is possible only when $\psi_k(x) = h_{ij}(x)$. In this case we will have,

$$\left[\frac{j}{2^i}, \frac{j+1}{2^i}\right] \subseteq \left[\frac{v}{2^u}, \frac{v+\frac{1}{2}}{2^u}\right] \qquad \ldots \quad (1)$$

From inequality (1)

$$j \geq v2^{i-u} \quad \text{and} \quad \left(v+\frac{1}{2}\right)2^{i-u} \geq j+1$$

$$\Rightarrow j + \frac{2^{i-u}}{2} \geq j+1 \Rightarrow i > u$$

Let $i = u+k$ for some $k = 1, 2, \ldots, s-u$. (since $i+j \leq s$)

From (1) we get

$$\frac{v}{2^u} \leq \frac{j}{2^{u+k}} \Rightarrow v \leq \frac{j}{2^k} \qquad \ldots \quad (2)$$

Also we have

$$\frac{j+1}{2^{u+k}} \leq \frac{v+\frac{1}{2}}{2^u} \Rightarrow j \leq \left(v+\frac{1}{2}\right)2^k - 1 \qquad \ldots \quad (3)$$

Hence from (2) & (3), $\quad v2^k \leq j \leq v2^k + 2^{k-1} - 1$.

Therefore the number of j s is at most equal to 2^{k-1} for each i.

But $i + j \geq u + k + v2^k$ and also $i + j \leq s$.

$$\therefore u + k + v2^k \leq s \Rightarrow v2^k \leq s$$

That is, $$2^k < s \Rightarrow k < \log_2(s).$$

Therefore the total number of h_{ij} s which satisfies(1) is at most equal to $k2^{k-1}$.

$$\therefore N_{2^{u/2}} \leq k2^{k-1} \leq \frac{s(\log_2 s)}{2}$$

Similarly we can also estimate $N_{-2^{s/2}}$ and which is given by

$$N_{-2^{s/2}} \leq \frac{s(\log_2 s)}{2}$$

$$\Rightarrow \frac{N_{2^{s/2}}}{N} \text{ and } \frac{N_{-2^{s/2}}}{N} \to 0 \text{ as } N \to \infty. \text{ (since } N = O(s^2)\text{)}$$

$$\Rightarrow \frac{N_0}{N} \to 1 \text{ as } N \to \infty.$$

Therefore

$$\lim_{N\to\infty} \frac{\sum_{k=1}^{N} F(\lambda_k (T_f)_N)}{N} = F(0)$$

$$= \lim_{M\to\infty} \frac{1}{M} \int_0^M F(h_{uv}(x))dx.$$

Szegö Limits in $L^2(R)$

We consider the Hilbert space $L^2(R)$ together with the following ordered Haar system in it.

4.8 An ordering of the Haar wavelet basis for $L^2(R)$ [20]

The Haar system $\{\phi_j(x), h_{ij}(x), i = 0,1,2, \ldots \text{ and } j \in Z\}$ form an orthonormal basis for $L^2(R)$. A particular ordering of the basis is given by

$$\{h_{0,0}, \phi_0, h_{10}, h_{01}, \phi_1, \phi_{-1}, h_{0,-1} \ldots\}.$$

Let us denote this by $\{\omega_k : k = 0,1,2,\ldots\}$.

4.9 Position of h_{ij} and ϕ_{i+j} in this ordering

To determine the position of each h_{ij} and ϕ_{i+j}, we write the above basis in the triangular form as,

$$h_{00}, \phi_0$$

$$h_{10}, h_{01}, \phi_1, \phi_{-1}, h_{0,-1}$$

$$h_{20}, h_{11}, h_{02}, \phi_2, \phi_{-2}, h_{1,-1}, h_{0,-2}$$

$$\cdots$$

$$h_{i+j,0}, h_{i+j-1,1}, h_{i+j-2,2} \ \ldots h_{1,i+j-1}, h_{0,i+j}, \phi_{i+j}, \phi_{-(i+j)} \cdots \ h_{0,-(i+j)}$$

$$\cdots$$

4.10 Proposition

In the above ordering ω_k s are as follows

$$\begin{array}{ll} \omega_k = \phi_{i+j} & \text{if} \quad k = (i+j)^2 + 3(i+j) + 1 \\ \omega_k = \phi_{-(i+j)} & \text{if} \quad k = (i+j)^2 + 3(i+j) + 2 \\ \omega_k = h_{ij} & \text{if} \quad k = (i+j)^2 + 2(i+j) + j \\ \omega_k = h_{i,-j} & \text{if} \quad k = (i+j)^2 + 3(i+j) + j + 2 \end{array}$$

Proof :

Case: 1 When $\omega_k = \phi_{i+j}$

In the above triangular form, ϕ_{i+j} lies in $(i+j+2)^{th}$ position of $(i+j+1)^{th}$ row.

$$\begin{aligned} \therefore k &= 2[2+3+ \ldots + (i+j)] + i + j + 1 + (i+j+2) \\ &= 2\left[\frac{(i+j)(i+j+1)-1}{2}\right] + 2(i+j) + 3 \\ &= (i+j)^2 + 3(i+j) + 1 \end{aligned}$$

Case: 2 When $\omega_k = \phi_{-(i+j)}$

Its position is $(i+j+3)$ in $(i+j+1)^{th}$ row.

$$\therefore k = (i+j)^2 + 3(i+j) + 2 .$$

Case: 3 When $\omega_k = h_{ij}$

In the above triangular form, h_{ij} lies in $(j+1)^{th}$ position of $(i+j+1)^{th}$ row.

$$\therefore k = 2[2 + \ldots + (i+j)] + (i+j+1) + j + 1$$
$$= (i+j)^2 + 2(i+j) + j$$

Case : 4 When $\omega_k = h_{i,-j}$

It lies in $(i+j+1)^{th}$ row in $(i+2j+3)$ position.

$$\therefore k = 2[2 + \ldots + (i+j)] + (i+j+1) + (i+2j+3)$$
$$= (i+j)^2 + 3(i+j) + j + 2$$

In the next theorem we observe the behavior of the distribution of eigenvalues for certain multiplication operators in $L^2(R)$ with respect to the above ordering. Also it is the $L^2(R)$ version of theorem 4.5.

4.11 Theorem

Let T_f be the multiplication operator on $L^2(R)$ with $f = h_{00}$ and consider $(T_f)_N$ with respect to the above basis. Then for any continuous function F on R,

$$\lim_{N\to\infty} \frac{\sum_{k=1}^{N} F(\lambda_k (T_f)_N)}{N} = \frac{F(0)+F(1)}{2}$$

where $\lambda_k (T_f)_N$, $k = 1,2,\ldots N$ are the eigenvalues of $(T_f)_N$.

Proof

The method used here is essentially as in theorem 4.5. The only change is in the multiplicity of eigenvalue 0.

Let

$$(T_f) = (a_{ij}) \qquad \text{where}$$
$$(a_{ij}) = \int_{-\infty}^{\infty} h_{00}(x)\omega_i(x)\omega_j(x)\,dx$$

Consider the N^{th} stage truncation $(T_f)_N$ of (T_f) where the N^{th} basis element is $h_{mn}, \phi_{m+n}, \phi_{-(m+n)}$ or $h_{m,-n}$. Then for all values of N, from the above proposition $N = O(s^2)$ where $s = m + n$. As in theorem 4.5, the only eigenvalues are 1,-1 and 0.

To prove the theorem it is enough to show that

$$\frac{N_{-1}}{N} \to 0 \text{ and } \frac{N_0}{N} \to 1/2 \text{ as } N \to \infty$$

Claim: $N_{-1} \le s(\log s + 1)$

$$T_{fN}(\omega_k) = -\omega_k \Leftrightarrow \omega_k = h_{ij}$$

if its support is contained in $\left[\frac{1}{2}, 1\right]$ and $i + j \le s$

$$\Leftrightarrow \left[\frac{j}{2^i}, \frac{j+1}{2^i}\right] \subset \left[\frac{1}{2}, 1\right] \text{ and } i + j \le s$$

$$\Leftrightarrow \frac{j}{2^i} \ge \frac{1}{2} \text{ and } \frac{j+1}{2^i} \le 1$$

$$\Leftrightarrow 2^{i-1} \le j < 2^i - 1 \qquad \ldots \qquad (1)$$

and $\qquad i + j \le s$

adding i we get ,

$$2^{i-1} + i \le j + i < 2^i - 1 + i$$

That is,

$$2^{i-1} + i \le s \Rightarrow 2^{i-1} < s \Rightarrow i \le \log s + 1$$

From equation (1), the number of j s corresponding to each i is 2^{i-1}.

Therefore

$$N_{-1} \le \sum_{i=1}^{\log s} 2^{i-1} < (\log s + 1)s .$$

$$\Rightarrow \lim_{N \to \infty} \frac{N_{-1}}{N} = \lim_{N \to \infty} \frac{O(s \log s)}{O(s^2)} = 0$$

Now we show that

$$N_0 = \frac{s(s+1)}{2} + 2s + O(s \log s).$$

This can be seen as follows.

$T_{fN}(\omega_k)=0 \Leftrightarrow (i)\omega_k = h_{ij}$ if its support is disjoint from $(0,1)$ and $i+j \le s$.

(ii) $\omega_k = h_{i,-j}$ where $i+j \le s$.

(iii) $\omega_k = \phi_{i+j}$ where $i+j \le s$

(iv) $\omega_k = \phi_{-(i+j)}$ where $i+j \le s$.

Considering (i), then

$$T_f(h_{ij})=0 \Leftrightarrow \left[\frac{j}{2^i},\frac{j+1}{2^i}\right] \text{ is disjoint from } [0,\ 1] \text{ and } i+j \le s.$$

$$\Leftrightarrow \frac{j}{2^i} \ge 1,\ i+j \le s$$

$$\Leftrightarrow j \ge 2^i, j \le s-i \qquad \dots \quad (2)$$

Adding i we get,

$$2^i + i \le i + j \le s$$
$$2^i < s \Rightarrow i < \log_2 s$$

From equation (2), the number of j corresponding to each i is at most equal to $s-2$. Hence number of h_{ij} satisfying condition (i) is less than or equal to $(s-2)\log s = O(s \log s)$.

From the position of $h_{i,-j}$ in the ordering 4.9, it is clear that the number of $h_{i,-j}$ satisfying condition (ii) $= s+(s-1)+ \dots +2+1$

$$= \frac{s(s+1)}{2}$$

Similarly the number of ϕ_{i+j}s and $\phi_{-(i+j)}$s satisfying condition (iii) and (iv) respectively is equal to s.

Adding all these we get

$$N_0 = \frac{s(s+1)}{2} + 2s + O(s \log s).$$

Therefore,

$$\frac{N_0}{N} \to 1/2 \text{ as } N \to \infty.$$

$$\Rightarrow \frac{N_1}{N} \to 1/2 \text{ as } N \to \infty.$$

Hence

$$\lim_{N\to\infty} \frac{\sum_{k=1}^{N} F\left(\lambda_k\left(T_f\right)_N\right)}{N} = \frac{F(0)+F(1)}{2} .$$

We conclude this chapter with the following remarks.

4.12 Remarks

For each x, let δ_x denote the Dirac delta measure concentrated at x. For simplicity put $\lambda_i = \lambda_i\left(T_f\right)_N$. Let $\mu_N = \frac{\delta_{\lambda_1} + \delta_{\lambda_2} + \ldots \delta_{\lambda_N}}{N}$ be the measure defined for each N and let $\mu = \frac{\delta_0 + \delta_1}{2}$. Then the above theorem implies that $\int_{-\infty}^{\infty} F d\mu_N \to \int_{-\infty}^{\infty} F d\mu$ for all continuous functions F on R.

ie, $\mu_N \to \mu$ weakly as $N \to \infty$. ♦

CHAPTER V

GENERAL ORDERINGS OF

HAAR SYSTEM IN $L^2(R_+)$ AND IN $L^2(R)$

In this chapter we discuss some classes of orderings of Haar system in $L^2(R_+)$ and in $L^2(R)$ in which Szegö's Type Theorem is valid for certain multiplication operators. This Chapter is divided into two sections.

In the first section we have given an ordering different from 4.2 to Haar System in $L^2(R_+)$ and we consider Szegö's type theorems for Multiplication operators with respect to this new ordered Haar system. Then it is interesting to see that the Szegö's Type theorems hold in this case. We also prove that Szegö's Type Theorems holds for general class of multiplication operators T_f with multiplier $f \in L^2(R_+)$ subject to some conditions on f.

Finally in the second section more general classes of orderings of Haar System in $L^2(R_+)$ and in $L^2(R)$ are identified in such a way that for certain classes of multiplication operators the asymptotic distribution of eigenvalues exists. Some illustrative examples are also considered other than the previous once.

5.1 Szegö's Type Theorems

In this section we consider an ordering of the Haar system and with respect to this new ordered system, Szegö's type theorems for a class of multiplication operators are analyzed.

5.1.1 An ordering of the Haar wavelet basis for $L^2(R_+)$

Recall that $\{\phi_j(x), h_{ij}(x),\ i, j \in Z_+ \cup \{0\}\}$ forms an orthonormal basis for $L^2(R_+)$ where $\phi_r(x) = \phi(x-r)$, where ϕ is the characteristic function in $[0,1]$ and h_{rp} is the Haar function defined by

$$h_{r,p}(x) = 2^{r/2} \qquad \frac{p}{2^r} \le x < \frac{p+1/2}{2^r}$$
$$= -2^{r/2} \qquad \frac{p+1/2}{2^r} \le x < \frac{p+1}{2^r}$$
$$= 0 \qquad \text{otherwise.}$$

where r & p are non negative integers.

Now consider an ordering of the Haar system which is given by the filling arrangement

$$\{\phi_0, h_{00}, h_{01}, \phi_1, h_{10}, \ldots, \phi_r, h_{r0}, \ldots\}.$$

Let this be denoted by $\{\psi_k : k = 1, 2, \ldots\}$.

5.1.2 Position of h_{ij} & ϕ_{i+j} in the above mentioned ordering

We can easily determine the position of each h_{ij} & ϕ_{i+j} by arranging the basis in the triangular form as given below.

$$\phi_0,\ h_{00},\ h_{01}$$
$$\phi_1,\ h_{10},\ h_{11},\ h_{12},\ h_{13},\ h_{02},\ h_{03}$$
$$\cdots$$
$$\phi_{r-1},\ h_{r-1,0}, \ldots, h_{r-i,2^r-1}, \ \cdots\ , h_{0,2^{r-1}}, \ldots, h_{0,2^r-1}$$
$$\phi_r,\ h_{r,0},\ \ldots, h_{r,2^{r+1}-1}, \ldots, h_{02^r},\ \cdots\ , h_{0,2^{r+1}-1}$$
$$\phi_{r+s}, h_{r+s,0}, \ldots, h_{r+s,2^{r+s+1}-1}, \ldots, h_{r,2^{r+s}}, \ldots, h_{r+s,2^{r+s+1}-1}, \ldots, h_{0,2^{r+s}}, \ldots, h_{0,2^{r+s+1}-1}$$
$$\cdots$$

5.1.3 Proposition

In the above ordering, the ψ_k s are as follows

$$\psi_k = \varphi_r \qquad\qquad \text{if} \quad k = r(2^r + 1) + 1$$
$$= h_{rp},\ p < 2^{r+1} \qquad \text{if} \quad k = r(2^r + 1) + p + 2$$
$$= h_{rp},\ p \ge 2^{r+s}\ s \in \mathrm{N}, \quad \text{if} \quad k = (r + 2s)(2^{r+s} + 1) + p - s + 2$$

Proof:

Case 1 : When $p < 2^{r+1}$

Then in the above ordering h_{rp} lies in the $(r+1)^{th}$ row in $(p+2)^{th}$ position.

Therefore position of h_{rp} $= 2.2^0 + 3.2^1 + \ldots + (r+1)2^{r-1} + p + 1 + r + 1$

$= r(2^r + 1) + p + 2$

Case 2: When $p \geq 2^{r+s}$, $s = 1, 2, \ldots$

Then h_{rp} lies in $(r+s+1)^{th}$ row and in $(s2^{r+s} + p + 2)^{th}$ position.

Therefore

Position of h_{rp} $= 2.2^0 + 3.2^1 + \ldots + (r+s+1)2^{r+s-1} + s2^{r+s} + p + 2 + r + s$

$= (r+2s)(2^{r+s} + 1) + p - s + 2$

Position of ϕ_r $= r(2^r + 1) + 1$

Now we analyze the behavior of certain multiplication operators with respect to the above ordering of the Haar system.

5.1.4 Theorem [20]

Let T_f be the multiplication operator on $L^2(R_+)$ with $f = h_{mn}$, $n < 2^{m+1}$. Let (T_f) be the matrix of T_f with respect to the above basis and let $\lambda_1(T_f)_N, \lambda_2(T_f)_N, \ldots, \lambda_N(T_f)_N$ be the eigenvalues (repeated according to multiplicity) of the truncated matrix $(T_f)_N$ then the following asymptotic formula holds:

$$\lim_{N\to\infty} \frac{\sum_{k=1}^{N} F(\lambda_k (T_f)_N)}{N} = \lim_{M\to\infty} \frac{1}{M} \int_0^M F[f(x)]$$

where F is any continuous function on R .

Proof

Let

$(T_f) = (a_{ij})$ where

$$a_{ij} = \int_0^\infty h_{mn}(x)\psi_i(x)\psi_j(x)dx$$

$= 0$ when $i \neq j$ and $i, j > K$,

where K is the position of h_{mn} in the above basis and when $i = j$

$$a_{ii} = \begin{cases} 2^{m/2} \\ -2^{m/2} \\ 0 \end{cases}$$

Let N be a positive integer and let the N^{th} basis element be h_{rp} where $p < 2^{r+1}$, h_{rp} where $p \geq 2^{r+s}$ $s \in N$ or ϕ_r. Then $N = r(2^r + 1) + p + 2$, $N = (r + 2s)(2^{r+s} + 1) + p - s + 2$ or $N = r(2^r + 1) + 1$. Therefore in all cases $N = O(r2^r)$. Consider the truncation $(T_f)_N$ of (T_f) where N is as above. The multiplicities contributed by the upper left $K \times K$ block is same for all values of $N \geq K$. Therefore it is enough to consider the multiplicity of eigenvalues in the diagonal part of the truncated matrix after neglecting the $K \times K$ block as before. Then the eigenvalues considered are $2^{m/2}, -2^{m/2}, 0$.

Let $N_{2^{m/2}}, N_{-2^{m/2}}, N_0$ denote the multiplicities of the eigenvalues $2^{m/2}, -2^{m/2}, 0$ of $(T_f)_N$ respectively. First we calculate these multiplicities.

$N_{2^{m/2}}$ can be calculated as follows.

Case1: N^{th} basis element is ϕ_r

By ordering 5.1.2, it lies in the $(r+1)^{th}$ row and $N = r(2^r + 1) + 1$.

$$T_{fN}[\psi_k(x)] = 2^{m/2}\psi_k(x) \quad \text{if} \quad x \in \left[\frac{n}{2^m}, \frac{n + 1/2}{2^m}\right]$$

where $T_{fN} = P_N T_f P_N$ and P_N is the orthogonal projection of $L^2(R_+)$ to span $\{\psi_1, \psi_2, \ldots, \psi_N\}$.

Hence the eigenvectors corresponding to $2^{m/2}$ are those $\psi_k(x)$ whose support lies in $\left[\frac{n}{2^m}, \frac{n + 1/2}{2^m}\right]$. For any value of k, $\phi_k(x)$ is not an eigenvector of $2^{m/2}$ and

. $h_{ij}(x)$ is an eigenvector $\Leftrightarrow$ $\left[\frac{j}{2^i},\frac{j+1}{2^i}\right]\subset\left[\frac{n}{2^m},\frac{n+\frac{1}{2}}{2^m}\right]$.

$$\Leftrightarrow \quad n2^{i-m} \le j < (2n+1)2^{i-(m+1)} .$$

Thus multiplicity, namely the number of j s satisfying the above relation equals $2^{i-(m+1)}$ for each i.

Hence, $$N_{2^{m/2}} = \sum_{i=m+1}^{r-1} 2^{i-(m+1)} = 2^{r-m-1} - 1$$

Case2: N^{th} basis element is h_{rp}, $p<2^{r+1}$

It is the $(p+2)^{th}$ element of $(r+1)^{th}$ row in the ordering and $N = r(2^r+1) + p + 2$. Hence as before we can calculate $N_{2^{m/2}}$ and it is given by

$$N_{2^{m/2}} = \begin{cases} 2^{r-m-1} - 1 & \text{if} \quad p < n2^{r-m} \\ 2^{r-m-1} + p - n2^{r-m} & \text{if} \quad n2^{r-m} \le p < (2n+1)2^{r-m-1} \\ 2^{r-m} - 1 & \text{if} \quad (2n+1)2^{r-m-1} \le p < 2^{r+1} \end{cases}$$

Case 3: N^{th} basis element is h_{rp}, $p \ge 2^{r+s}$, $s = 1, 2, \ldots$

Then in the ordering 5.1.2, h_{rp} lies in $(r+s+1)^{th}$ row and $N = (r+2s)(2^{r+s}+1) + p - s + 2$.

$$N_{2^{m/2}} = \sum_{i=m+1}^{r+s} 2^{i-(m+1)} = 2^{r+s-m} - 1.$$

Similarly we can calculate $N_{-2^{m/2}}$ and N_0 by counting the ψ_k s whose support lies in $\left[\frac{n+\frac{1}{2}}{2^m},\frac{n+1}{2^m}\right]$ and whose support lies outside $\left[\frac{n}{2^m},\frac{n+1}{2^m}\right]$ respectively. These results are summarized in table 5.1.5. Now $N_{2^{m/2}}, N_{-2^{m/2}}$ are $O(2^r)$ and therefore $\frac{N_{2^{m/2}}}{N} \to 0$ and $\frac{N_{-2^{m/2}}}{N} \to 0$ as $N \to \infty$. Hence $\frac{N_0}{N} \to 1$ as $\mathrm{N} \to \infty$, which completes the proof.

Value of N	Value of p	$N_{2^{m/2}}$	$N_{-2^{m/2}}$	N_0
$r(2^r+1)+1$	$p<(2n+1)2^{r-m-1}$	$2^{r-m-1}-1$	$2^{r-m-1}-1$	$r(2^r+1)-m(2^m+1)-2^{r-m}-n-2$
$r(2^r+1)+p+2$	$p<n2^{r-m}$	$2^{r-m-1}-1$	$2^{r-m-1}-1$	$r(2^r+1)-m(2^m+1)-2^{r-m}-n+3+p$
$r(2^r+1)+p+2$	$n2^{r-m}\le p<(2n+1)2^{r-m-1}$	$2^{r-m-1}+p$	$2^{r-m-1}-1$	$r(2^r+1)-m(2^m+1)-(n-1)2^{r-m}-n-2$
$r(2^r+1)+p+2$	$(2n+1)2^{r-m-1}\le p<(n+1)2^{r-m}$	$2^{r-m}-1$	$p-n2^{r-m}$	$r(2^r+1)-m(2^m+1)+(n-1)2^{r-m}-n+2$
$r(2^r+1)+p+2$	$(n+1)2^{r-m}\le p<2^{r+1}$	$2^{r-m}-1$	$2^{r-m}-1$	$r(2^r+1)-m(2^m+1)+(n-1)2^{r-m}-n+2$
$(r+2s)(2^{r+s}+1)+p+2-s$	$p\ge 2^{r+s}$, $s=1,2,\dots$	$2^{r+s-m}-1$	$2^{r+s-m}-1$	$(r+2s)(2^{r+s}+1)-m(2^m+1)-2^{r+s+1-m}+3+p-s$

5.1.5 Table

Now we consider the multiplication operators under the same basis with more general multipliers by taking the linear combinations of $\psi_k s$. Then also the conclusion is same. This is given in the next theorem.

5.1.6 Theorem [20]

Let T_f be the multiplication operator on $L^2(R_+)$ with $f=\sum_{k=1}^{n}\alpha_k\psi_k$. α_ks are real. With respect to the ordered basis 5.1.1, let $\lambda_1(T_f)_N, \lambda_2(T_f)_N, \ldots, \lambda_N(T_f)_N$ be the eigenvalues (repeated according to multiplicity) of the associated truncated matrix $(T_f)_N$. Then the following asymptotic formula holds for any continuous function F on R .

$$\lim_{N\to\infty}\frac{\sum_{k=1}^{N}F(\lambda_k(T_f)_N)}{N} = \lim_{M\to\infty}\frac{1}{M}\int_0^M F[f(x)]$$

Proof

First we show that the theorem is true for the operators T_f with multiplier $f=\sum_{k=1}^{n}\alpha_k\psi_k, n=(m+1)(2^{m+1}+1)$ where $\psi_1,\psi_2,\ldots,\psi_n$ are taken as in the ordering 5.1.1. From this case we deduce the result for operators T_f where $f=\sum_{k=1}^{n}\alpha_k\psi_k$,

Let (T_f) be matrix of transformation of the operator T_f. The behavior of (T_f) is same as in the above case. Here the upper left $K\times K$ block is nonzero, where $K=b(2^b+1)+1, b=2^{m+1}-1$ and the remaining nonzero elements exists only in the main diagonal. Consider the N th truncation $(T_f)_N$ where $N=O(r2^r)$. Since $\psi_k(x)=h_{0,2^{m+1}-1}$, there exist $(m+2)2^{m+1}+1$ diagonal elements in $(T_f)_N$. The only eigenvalues considered are the diagonal elements. Let these values be $\beta_0,\beta_1,\ldots,\beta_{(m+2)2^{m+1}}$. For finding the multiplicities of these eigenvalues, we arrange them as $\beta_0 = 0$, β_q where $q = k2^{m+1} + c$, $0 \le k \le m+1$ and $0 < c \le 2^{m+1}$. Let N_q denote the maximum multiplicity of β_q . The values of N_q for different values of q and for all possible values of N are given in table 5.1.8. Here also N_q is of $O(2^r)$. Therefore

$$\frac{N_q}{N} \to 0 \quad \text{as } N \to \infty .$$

This completes the proof in the first case.

Now if $f = \sum_{k=1}^{n} \alpha_k \psi_k,\ n \ne (m+1)(2^{m+1}+1)$, without loss of generality we may assume that by taking coefficients α_ks to be zero for sufficiently many values of k, this will affect only add to possibly an increase in the multiplicity in the eigenvalue zero. Hence the result.

5.1.7 Remarks

We calculate the eigenvalues β_qs using the following computations .

(i) $T_{fN}(\psi_i(x)) = \beta_q(\psi_i(x)) \Leftrightarrow \psi_i(x) = h_{ij}(x)$ where $T_{fN} = P_N T_f P_N$, P_N is the orthogonal projection.

$$\Leftrightarrow \left[\frac{j}{2^i}, \frac{j+1}{2^i}\right] \subseteq \left[\frac{q-1}{2^{m+1}}, \frac{q}{2^{m+1}}\right],$$

where $q = k2^{m+1} + c, k < 2, 0 < c \le 2^{m+1}$.

(ii) $T_{fN}(\psi_i(x)) = \beta_q(\psi_i(x)) \Leftrightarrow \psi_i(x) = h_{ij}(x)$

$$\Leftrightarrow \left[\frac{j}{2^i}, \frac{j+1}{2^i}\right] \subseteq \left[\frac{2^{m+k} + (c-1)2^{k-1}}{2^{m+1}}, \frac{2^{m+k} + c2^{k-1}}{2^{m+1}}\right]$$

where $q = k\,2^{m+1} + c,\ 2 \le k \le m+1,\ 0 < c \le 2^{m+1}$.

(iii) $T_{fN}(\psi_i(x)) = 0 \Leftrightarrow (a)\ \psi_i(x) = \phi_i(x) \quad \forall\ i \ge 2^{m+1}$

$$(b)\ \psi_i(x) = h_{ij}(x) \Leftrightarrow \left[\frac{j}{2^i}, \frac{j+1}{2^i}\right] \not\subset \left[0, 2^{m+1}\right]$$

Eigenvalue β_q, $q = k2^{m+1} + c$ $0 \le k \le m+1$, $0 \le c \le 2^{m+1}$	Multiplicity	Value of p	Value of N
β_q, $q = k2^{m+1} + c$ $k < 2$, $0 < c \le 2^{m+1}$	$2^{r-m-1} - 1$ $(2-q)2^{r-m-1} + p$ $2^{r-m} - 1$ $2^{r+s-m} - 1$	$p < (q-1)2^{r-m-1}$ $(q-1)2^{r-m-1} \le p <$ $q2^{r-m-1} - 1$ $q2^{r-m-1} - 1 \le p < 2^{r+1}$ $p \ge 2^{r+s}, s = 1,2,--$	$r(2^r+1)+1$ or $r(2^r+1)+p+2$ $r(2^r+1)+p+2$ $r(2^r+1)+p+2$ $(r+2s)(2^{r+s}+1)+p-s+2$
β_q, $q = 22^{m+1} + c$ $0 < c \le 2^{m+1}$	$2(2^{r-m-2} - 1)$ $2(2^{r+s-m-1} - 1)$ $2(2^{r+s-m-2} - 1)$	$p < 2^{r+1}$ $p \ge 2^{r+s}, s > 1$ $p \ge 2^{r+s}, s \le 1$	$r(2^r+1)+1$ or $r(2^r+1)+p+2$ $(r+2s)(2^{r+s}+1)+p-s+2$ $(r+2s)(2^{r+s}+1)+p-s+2$
β_q, $q = 32^{m+1} + c$ $0 < c \le 2^{m+1}$	$2^2(2^{r-m-3} - 1)$ $2^2(2^{r+s-m-2} - 1)$ $2^2(2^{r+s-m-3} - 1)$	$p < 2^{r+1}$ $p \ge 2^{r+s}, s > 2$ $p \ge 2^{r+s}, s \le 2$	$r(2^r+1)+1$ or $r(2^r+1)+p+2$ $(r+2s)(2^{r+s}+1)+p-s+2$ $(r+2s)(2^{r+s}+1)+p-s+2$

5.1.8 Table

Eigenvalue β_q, $q = k2^{m+1} + c$ $0 \le k \le m+1$, $0 \le c \le 2^{m+1}$	Multiplicity	Value of p	Value of N
β_q, $q = 42^{m+1} + c$ $0 < c \le 2^{m+1}$	$2^3(2^{r-m-4} - 1)$ $2^3(2^{r+s-m-3} - 1)$ $2^3(2^{r+s-m-4} - 1)$	$p < 2^{r+1}$ $p \ge 2^{r+s}, s > 3$ $p \ge 2^{r+s}, s \le 3$	$r(2^r + 1) + 1$ or $r(2^r + 1) + p + 2$ $(r+2s)(2^{r+s} + 1) + p - s + 2$ $(r+2s)(2^{r+s} + 1) + p - s + 2$
β_q, $q = 52^{m+1} + c$ $0 < c \le 2^{m+1}$	$2^4(2^{r-m-5} - 1)$ $2^4(2^{r+s-m-4} - 1)$ $2^4(2^{r+s-m-5} - 1)$	$p < 2^{r+1}$ $p \ge 2^{r+s}, s > 4$ $p \ge 2^{r+s}, s \le 4$	$r(2^r + 1) + 1$ or $r(2^r + 1) + p + 2$ $(r+2s)(2^{r+s} + 1) + p - s + 2$ $(r+2s)(2^{r+s} + 1) + p - s + 2$
β_q, $q = m2^{m+1} + c$ $0 < c \le 2^{m+1}$	$2^{m=1}(2^{r-m-2} - 1)$ $2^{m-1}(2^{r+s-m-1} - 1)$ $2^{m-1}(2^{r+s-m-2} - 1)$	$p < 2^{r+1}$ $p \ge 2^{r+s}, s > m$ $p \ge 2^{r+s}, s \le m$	$r(2^r + 1) + 1$ or $r(2^r + 1) + p + 2$ $(r+2s)(2^{r+s} + 1) + p - s + 2$ $(r+2s)(2^{r+s} + 1) + p - s + 2$
β_q, $q = (m+1)2^{m+1} + c$ $0 < c \le 2^{m+1}$	$2^m(2^{r-m-2} - 1)$ $2^m(2^{r+s-m-1} - 1)$ $2^m(2^{r+s-m-2} - 1)$	$p < 2^{r+1}$ $p \ge 2^{r+s}, s > (m+1)$ $p \ge 2^{r+s}, s \le (m+1)$	$r(2^r + 1) + 1$ or $r(2^r + 1) + p + 2$ $(r+2s)(2^{r+s} + 1) + p - s + 2$ $(r+2s)(2^{r+s} + 1) + p - s + 2$

5.1.8 Table (Contd.)

Next we consider the asymptotic distribution of eigenvalues of multiplication operators on $L^2(R_+)$ with multiplier $f = \sum_{k=1}^{\infty} \alpha_k \psi_k, \ \sum |\alpha_k|^2 < \infty$ in $L^2(R_+)$.

Given $f = \sum_{k=1}^{\infty} \alpha_k \psi_k, \ \sum_{k=1}^{\infty} |\alpha_k|^2 < \infty$ and α_ks are real and $f_n = \sum_{k=1}^{n} \alpha_k \psi_k$. Assume that the sequence (f_n) converge uniformly to f on R. That is, for every $\varepsilon > 0$, there exists a positive integer $\widetilde{N}$ (depending only on ε) such that $\forall$ n $\geq \widetilde{N}$ implies

$$|f_n(x) - f(x)| < \varepsilon \qquad \forall x \qquad \ldots \qquad (1)$$

Let (T_f) and (T_{f_n}) be the matrix of transformations of T_f and (T_{f_n}) and $\{\lambda_1(T_f)_N, \lambda_2(T_f)_N, \ldots, \lambda_N(T_f)_N\}$, $\{\lambda_1(T_{f_n})_N, \lambda_2(T_{f_n})_N, \ldots, \lambda_N(T_{f_n})_N\}$ be the eigenvalues of $(T_f)_N$ and $(T_{f_n})_N$ respectively. For simplicity we denote these eigenvalues by $\{\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_N\}$ and $\{\lambda_1^{(n)}, \lambda_2^{(n)}, \ldots, \lambda_N^{(n)}\}$ respectively.

If $f_n \to f$ uniformly on R, then the operators T_f, T_{f_n}, T_{fN} and $T_{f_n N}$ where $T_{fN} = P_N T_f P_N$ and $T_{f_n \mathrm{N}} = P_N T_{f_n} P_N$ as before, has the following property.

5.1.9 Proposition

If $f_n \to f$ uniformly on R, then $\| T_{f_n} - T_f \| \to 0$ and $\| T_{f_n \mathrm{N}} - T_{fN} \| \to 0$ for all N.

Proof

First we prove that $T_{f_n} \xrightarrow{n} T_f$ uniformly (in the operator norm).

Consider

$$\left\|T_{f_n} - T_f\right\|^2 = \sup\left\{\left\|\left(T_{f_n} - T_f\right)(\xi)\right\|^2 : \|\xi\| = 1\right\} \qquad \ldots \quad (1)$$

$$\begin{aligned}
\left\|\left(T_{f_n} - T_f\right)(\xi)\right\|^2 &= \| f_n\xi - f\xi \|^2 \\
&= \int_0^\infty |(f_n - f)(x)|^2 |\xi(x)|^2 dx \\
&\le \sup|(f_n - f)|^2 \|\xi\|^2 \\
&\le \sup|(f_n - f)|^2 \int_0^\infty |\xi(x)|^2 dx
\end{aligned}$$

Therefore Equation (1) becomes

$$\left\|T_{f_n} - T_f\right\|^2 \le \sup|(f_n - f)|^2$$

Since $\| f_n - f \| \to 0$, the result follows.

Now

$$\begin{aligned}
\left\| T_{f_n N} - T_{f N} \right\| &= \| P_N T_{f_n} P_N - P_N T_f P_N \| \\
&\le \| P_N \|^2 \left\| T_{f_n} - T_f \right\| \le \left\| T_{f_n} - T_f \right\|
\end{aligned}$$

Hence the result.

Since f_n and f are real, the matrices $\left(T_{f_n}\right)_N$ and $\left(T_f\right)_N$ are self adjoint. Let $\lambda_1^{(n)} \ge \lambda_2^{(n)} \ge \ldots \ge \lambda_N^{(n)}$ and $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_N$ be the eigenvalues of these matrices arranged in non increasing order. Then using Weyl's perturbation theorem [1.2.7], we can relate these eigenvalues. It is given in the following proposition.

5.1.10 Proposition

Let $\lambda_1^{(n)} \ge \lambda_2^{(n)} \ge \ldots \ge \lambda_N^{(n)}$ and $\lambda_1 \ge \lambda_2 \ge \ldots \ge \lambda_N$ be the eigenvalues, arranged in non increasing order of the matrices $\left(T_{f_n}\right)_N$ and

$(T_f)_N$ respectively, then, $(\lambda_k^{(n)}) \xrightarrow{n} \lambda_k$ uniformly for all values of $k = 1, 2, \ldots, N$.

Proof :

The matrices $(T_{f_n})_N$ and $(T_f)_N$ are self adjoint and therefore using Weyl's perturbation theorem [1.2.7] we have

$$\max_k \left| \lambda_k^{(n)} - \lambda_k \right| \leq \left\| T_{f_n N} - T_{f N} \right\|, \ k = 1, 2, \ldots, N$$

Hence the result follows from the proposition 5.1.9.

Since $T_{f_n N}$ and $T_{f N}$ are self adjoint, using the upper semi continuity and lower semi continuity [16] of the eigenvalues, we have the following remarks.

5.1.11 Remarks

(i) Let

$$A = \Lambda_N(f_n) = \{\lambda_1^{(n)}, \lambda_2^{(n)}, \ldots, \lambda_N^{(n)}\}$$

$$B = \Lambda_N(f) = \{\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_N\}$$

Then the Hausdorff distance between A and B tends to zero as $n \to \infty$.

That is

$$h(\Lambda_N(f_n), \Lambda_N(f_n)) \to 0.$$

The proof is as follows:

Since $f_n \to f$ uniformly, $T_{f_n N} \to T_{f N}$ uniformly as $n \to \infty$.

Therefore using upper semi continuity [16, page 70],

$$\max_{\lambda_i^{(n)} \in A} \{dis(\lambda_i^{(n)}, B)\} \to 0 \quad \text{as } n \to \infty.$$

Also lower semi continuity holds at λ_j [16]. Therefore

$$dis(\lambda_j, A) \to 0 \text{ as } n \to \infty$$

where $\lambda_j \in B$. Since $(T_{f_n})_N$ are self adjoint, the lower semi continuity holds at every eigenvalue λ_j of B [16, Chapter 2]. Therefore

$$\max_{\lambda_j \in B} \{dis(\lambda_j, A)\} \to 0 \quad \text{as} \quad n \to \infty \quad .$$

Hence the Hausdorf distance between A and B

$$h(A, B) = \max \left\{ \max_{\lambda_j^{(n)} \in A} \{dis(\lambda_i^{(n)}, B)\}, \max_{\lambda_j \in B} \{dis(\lambda_j, A)\} \right\} \to 0 \ .$$

(ii) Since $f_n \to f$ uniformly on R, $T_{f_n N} \to T_{f N}$ uniformly , and therefore we have $tr(T_{f_n}) \to tr(T_f)$ uniformly where 'tr' denote the trace of the matrix.

The main result of this section is given in the following theorems and it gives the asymptotic distribution of eigenvalues of the multiplication operators T_f.

5.1.12 Theorem

Let T_f be the multiplication operator with $f = \sum_{k=1}^{\infty} \alpha_k \psi_k$, $\sum |\alpha_k|^2 < \infty$, α_ks are real. Assume that $f_n \to f$ uniformly on R where $f_n = \sum_{k=1}^{n} \alpha_k \psi_k$, and let $\{\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_N\}$ be the eigenvalues of $(T_f)_N$ repeated according to multiplicity, then the following asymptotic formula holds for any continuous function F on R.

$$\lim_{N \to \infty} \frac{\sum_{k=1}^{N} F(\lambda_k)}{N} = \lim_{M \to \infty} \frac{1}{M} \int_0^M F[f(x)]$$

Proof

Consider the double sequence $\{a_{N,n}\}$ in N and n , where

$$a_{N,n} = \frac{F(\lambda_1^{(n)}) + F(\lambda_2^{(n)}) + \ldots + F(\lambda_N^{(n)})}{N}, N, n = 1, 2, \ldots$$

and $\{\lambda_1^{(n)}, \lambda_2^{(n)}, \ldots, \lambda_N^{(n)}\}$ be the eigenvalues of $(T_{f_n})_N$.

We show that the limit (double limit) [7, 10] of the double sequence $\{a_{N,n}\}$ exists as $N, n \to \infty$ and it is equal to $F(0)$

Let $\{Y_N^n\}$, $N = 1, 2, \ldots$ denote the row sequences of $\{a_{N,n}\}$, where

$$Y_N^n = \frac{F(\lambda_1^{(n)}) + F(\lambda_2^{(n)}) + \ldots + F(\lambda_N^{(n)})}{N},$$

and y_N denote its limit as $n \to \infty$.

Let $\{Z_n^N\}$, $n = 1, 2, \ldots$ denote the column sequences of $\{a_{N,n}\}$,where

$$Z_n^N = \frac{F(\lambda_1^{(n)}) + F(\lambda_2^{(n)}) + \ldots + F(\lambda_N^{(n)})}{N},$$

and z_n denote its limit $N \to \infty$.

Now we prove the following results,

(i) For every N and n the limits y_N and z_n exists.

(ii) The collection of row sequences $\{Y_N : N = 1, 2, \ldots\}$ converges uniformly.

Then the theorem follows immediately from Iterated limit theorem [1.2.8].

To prove (i)

Using proposition 5.1.9, for each value of $k = 1, 2, \ldots,$

$$\lim_{n \to \infty} \lambda_k^{(n)} = \lambda_k \text{ (Uniformly in } n).$$

Since F is continuous,

$$F(\lambda_k^{(n)}) \to F(\lambda_k) \quad k = 1, 2, \ldots \quad .$$

Therefore for all values of N,

$$\sum_{k=1}^{N} \frac{F(\lambda_k^{(n)})}{N} \to \sum_{k=1}^{N} \frac{F(\lambda_k)}{N} = y_N \text{ as } n \to \infty \quad , \quad N = 1, 2, \ldots$$

Now by theorem 5.1.6, Szegö's limit exists for T_{f_n} s for each $n = 1, 2, \ldots$ and hence z_n exists for each n.

To prove (ii)

Let $\varepsilon > 0$ be given. Consider,

$$|a_{Nn} - y_N| = \left| \frac{\sum_{k=1}^{N} F(\lambda_k^{(n)})}{N} - \frac{\sum_{k=1}^{N} F(\lambda_k)}{N} \right|$$

$$= \frac{1}{N} \left| \sum_{k=1}^{N} F(\lambda_k^{(n)}) - \sum_{k=1}^{N} F(\lambda_k) \right|$$

$$= \frac{1}{N} \left| \sum_{k=1}^{N} \left[F(\lambda_k^{(n)}) - F(\lambda_k) \right] \right|$$

$$\leq \frac{1}{N} \sum_{k=1}^{N} \left| F(\lambda_k^{(n)}) - F(\lambda_k) \right| \qquad \ldots \quad (1)$$

From proposition 5.1.11, there exists $\widetilde{N}$ such that

$$\left| F(\lambda_k^{(n)}) - F(\lambda_k) \right| < \varepsilon \qquad \forall\ n \geq \widetilde{N}, \&\ \forall k$$

Therefore Equation (1) reduces to

$$|a_{Nn} - y_N| < \varepsilon \qquad \forall\ n \geq \widetilde{N}, N = 1, 2, \ldots$$

which completes the proof.

Next we have taken the weaker condition, where the sequences $f_n = \sum_{k=1}^{n} \alpha_k \psi_k$, converges uniformly to $f = \sum_{k=1}^{\infty} \alpha_k \psi_k$, $\sum |\alpha_k|^2 < \infty$ (α_k s are real), on compact subsets E of R and the basis is taken as in 5.1.1.

Let $F_n = f - f_n = \sum_{k=n+1}^{\infty} \alpha_k \psi_k$. Now we choose α_k s such that $|F_n(x)|$ is uniformly bounded. Under these conditions, the asymptotic distribution of eigenvalues of T_f converges and Szegö's Type Theorem is valid. This is given in the following theorem.

5.1.13 Theorem

Let T_f be the multiplication operator with $f = \sum_{k=1}^{\infty} \alpha_k \psi_k$, $\sum |\alpha_k|^2 < \infty$, α_ks are real. Assume that $f_n \to f$ uniformly on compact subsets E of R where $f_n = \sum_{k=1}^{n} \alpha_k \psi_k$. Choose α_ks such that $|F_n| = |f_n - f|$ is uniformly bounded. Let $\{\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_N\}$ be the eigenvalues of $(T_f)_N$ repeated according to multiplicity, then the following asymptotic formula holds.

$$\lim_{N\to\infty} \frac{\sum_{k=1}^{N} F(\lambda_k)}{N} = \lim_{M\to\infty} \frac{1}{M} \int_0^M F[f(x)]$$

where F is any continuous function on R.

Proof

The proof follows immediately from the theorem 5.1.12 if we establish the following results.

(i) $T_{f_n} \to T_f$ pointwise.

(ii) $T_{f_n N} \xrightarrow{n} T_{f N}$ uniformly for all N

Proof of (i)

Since $|F_n(x)| = |f_n - f|$ is uniformly bounded, we have $|F_n(x)| \le \beta \quad \forall x$

and since $f_n(x)$ converges uniformly to f in the set E ,

$$|F_n(x)| \to 0 \quad \forall\ x \in E\ .$$

Now we show that $T_{F_N} \to 0$ ($T_{f_n} \to T_f$) point wise.

Consider $T_{f_n} - T_f = T_{f_n - f} = T_{F_n}$. Then,

$$\| T_{F_n} \| = \sup_{\|\xi\|=1} \left\{ \| T_{F_n}(\xi) \| \right\}$$

$$\| T_{F_n}(\xi) \|^2 = \int_0^\infty | F_n(x)\xi(x) |^2 \, dx$$

Then for any $0 < N_0 < \infty$ we have,

$$\| T_{F_n}(\xi) \|^2 = \int_0^{N_0} | F_n(x)\xi(x) |^2 \, dx + \int_{N_0}^\infty | F_n(x)\xi(x) |^2 \, dx$$

$$= \int_0^{N_0} | F_n(x) |^2 \, |\xi(x)|^2 \, dx + \int_{N_0}^\infty | F_n(x) |^2 \, |\xi(x)|^2 \, dx \quad \ldots \ (1)$$

Let $\varepsilon > 0$ be given. Since $\xi(x) \in L^2(R_+)$, N_0 can be chosen such that

$$\int_{N_0}^\infty |\xi(x)|^2 \, dx < \frac{\varepsilon}{2\beta^2} .$$

Now

$$\int_{N_0}^\infty |F_n(x)\xi(x)|^2 \, dx = \int_{N_0}^\infty | F_n(x) |^2 \, |\xi(x)|^2 \, dx$$

$$< \int_{N_0}^\infty \beta^2 |\xi(x)|^2 \, dx \quad (\because | F_n(x) | < \beta \,)$$

$$< \beta^2 \int_{N_0}^\infty |\xi(x)|^2 \, dx < \frac{\varepsilon}{2} \qquad \forall \ n \qquad \ldots \ (2)$$

Let $E = [0, N_0]$ be the compact set.

Since $F_n \to 0$ uniformly on E, we have for every $\varepsilon > 0$ there exists N_1 such that

$$|F_n(x)| < \frac{\varepsilon}{2} \quad \forall \ n \geq N_1 \ \ \& \ \forall \ \ x \in E$$

$$\therefore \int_0^{N_0} | F_n(x) |^2 \, |\xi(x)|^2 \, dx < \frac{\varepsilon}{2} \qquad \forall n \geq N_1 \qquad \ldots \ (3)$$

Then using equations (2) and (3), equation (1) reduces to

$$\int_0^{\infty} |F_n(x)|^2 |\xi(x)|^2 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \quad \forall n \geq N_1 .$$

Hence $T_{F_N} \to 0$. ie. $T_{f_n} \to T_f$ point wise on R.

Proof of (ii):

Since $T_{F_N} \to 0$ point wise on R and P_N is compact, $T_{f_n} P_N \to T_f P_N$ uniformly on R.

Now consider,

$$\|T_{f_n N} - T_{f N}\| = \|P_N T_{f_n} P_N - P_N T_f P_N\|$$

$$= \|P_N (T_{f_n} - T_f) P_N\|$$

$$\leq \|P_N\| \, \|(T_{f_n} - T_f) P_N\|$$

$$\leq \| \, (T_{f_n} P_N - T_f P_N) \| \to 0 .$$

which completes the proof of (ii) and hence the theorem.

5.1.14 Remarks

(i) For each x, let δ_x denote the Dirac delta measure concentrated at x.

Let $\mu_N = \dfrac{\delta_{\lambda_1} + \delta_{\lambda_2} + \ldots + \delta_{\lambda_N}}{N}$ be the measure defined for each N and let $\mu = \delta_0$. Then the above theorems 5.1.12 and 5.1.13 implies that for all continuous functions F on R.

$$\int_0^{\infty} F d\mu_N \to \int_0^{\infty} F d\mu$$

ie, $\mu_N \to \mu$ weakly .

(ii) Theorem 5.1.12 follows from 5.1.13.

(iii) The coefficients α_ks in the theorem 5.1.13 can be chosen as follows.

$\alpha_k \le \frac{1}{k}$ $k = 1, 2, 3, \ldots,$ then $|F_n(x)|$ is uniformly bounded.

The proof is as follows.

$f = \sum_{k=1}^{\infty} \alpha_k \psi_k,$ where $\psi_k = h_{rp}$ or ϕ_r. From the ordering 5.1.1, $k = O(r2^r)$.

Then, for $x \in R$,

$$|f(x)| \le \sum_{k=1}^{\infty} |\alpha_k \psi_k(x)| \qquad \ldots \qquad (1)$$

Case: 1 When r is fixed and p varies

Then supports of h_{ij} is disjoint for each i and j. Then $f(x)$ contains only one term for a given $x \in R$. Hence (1) reduces to

$$|f(x)| < |\alpha_k \psi_k(x)| \text{ , where } x \text{ belongs to the support of } \psi_k$$

$$= |\alpha_k| |h_{rp}(x)| \le \frac{|2^{r/2}|}{k}$$

which is finite, where $k = O(r2^r)$.

Case: 2 when r varies,

then (1) reduces to

$$|f(x)| < \sum_{k=1}^{\infty} |\alpha_k| \, |\psi_k(x)| < \sum_{r=1}^{\infty} \frac{2^{r/2}}{O(r2^r)}$$

Since $k > r2^r$

$$|f(x)| < \sum_{r=1}^{\infty} \frac{2^{r/2}}{r2^r} < \beta \quad \text{for all } x$$

Therefore,

$$|F_n(x)| = \left| \sum_{k=n+1}^{\infty} \alpha_k \psi_k \right| < \left| \sum_{k=1}^{\infty} \alpha_k \psi_k \right| < \beta \text{ for all } x .$$

Hence $|F_n(x)|$ is uniformly bounded on R.

5.2 Generalization of orderings of Haar System

In this section we identify different classes of orderings of Haar system in $L^2(R_+)$ and in $L^2(R)$ so that for certain multiplication operators the asymptotic distribution of eigenvalues converges to a fixed limit and Szegö's type theorem is valid. Also we have given examples for orderings other than the orderings mentioned earlier. Throughout $H = \{\phi_j(x), h_{ij}(x), i, j \in Z_+ \cup \{0\}\}$ will denote the Haar system in $L^2(R_+)$.

First of all we have the simple result.

5.2.1 Proposition

Let $\{\psi_k : k \in N\}$ be an ordering of the Haar System where Szegö's Type Theorem holds for an operator T. Then with respect an ordering obtained by changing the positions of a finite number of elements, the Szegö's theorem will remain valid for the same operator T.

Now we construct a class of orderings of Haar system for the case of multiplication operators T_f in $L^2(R_+)$ with $f = h_{00}$ so that asymptotic distribution of eigenvalues converges. This result is given in the following theorem.

5.2.2 Theorem

Let $\lambda_1(T_f)_N = 0, \lambda_2(T_f)_N = 1$, and $\lambda_3(T_f)_N = -1$ be the distinct eigenvalues of $(T_f)_N$ where $f = h_{00}$. Let M_j be the eigenspace associated with $\lambda_j(T_f)_N$, $j = 1, 2, 3$ and $H_j = H \cap M_j$. For a sequence (j_n) of positive integers, let $A_1, A_2 \dots$, $B_1, B_2, \dots$ be partitions of H_j and $H \cap M_j^C$ respectively such that $|A_n| = n\, j_n$ and $|B_n| = j_n$. Consider an ordering of it whose entries are arranged as $\{B_1, A_1, B_2, A_2, \ldots\}$. Then this is a basis for $L^2(R_+)$ for which

$$\lim_{N\to\infty} \frac{\sum_{k=1}^{N} F\left(\lambda_k\left(T_f\right)_N\right)}{N} = F\left(\lambda_j\left(T_f\right)_N\right)$$

where F is any continuous function on R.

Proof:

To prove the theorem it is enough to show that $\frac{N_j}{N} \to 1$ as $N \to \infty$, where N_j is the multiplicity of eigenvalue $\lambda_j\left(T_f\right)_N$ of $\left(T_f\right)_N$. Let N be a positive integer. Then for some n depending on N,

$$N = \sum_{k=1}^{n-1}((k+1)j_k) + K, \quad K \le j_n + K_1 \text{ where } K_1 < nj_n \quad .$$

Then

$$N_j = \sum_{k=1}^{n-1} kj_k + K_1 \qquad \text{where } K_1 \text{ is defined as above.}$$

$$\therefore \lim_{n\to\infty} \frac{N_j}{N} = \lim_{n\to\infty} \frac{\sum_{k=1}^{n-1} kj_k + K_1}{\sum_{k=1}^{n-1} (k+1)j_k + K} = \lim_{n\to\infty} \frac{\left(1 + \frac{K_1}{\sum_{k=1}^{n-1} kj_k}\right)}{1 + \left(\frac{\sum_{k=1}^{n-1} j_k + K}{\sum_{k=1}^{n-1} kj_k}\right)} = 1$$

Therefore

$$\lim_{N\to\infty} \frac{N_k}{N} = 0, \quad k \ne j.$$

Hence the result.

5.2.3 Remarks:

(i) When $\lambda_j(T_f)_N = \lambda_1(T_f)_N = 0$, then

$$\lim_{N\to\infty}\frac{\sum_{k=1}^{N}F(\lambda_k(T_f)_N)}{N} = F(0) = \lim_{M\to\infty}\left[\frac{1}{M}\int_0^M F(f(x))dx\right]$$

Hence, Szegö's type Theorem is valid in this case.

(ii) When $\lambda_j(T_f)_N = \lambda_2(T_f)_N = 1$, then the asymptotic distribution of eigenvalues converges to $F(1)$ and when $\lambda_j(T_f)_N = \lambda_3(T_f)_N = -1$, then it converges to $F(-1)$.

The above theorem gives three classes of orderings for multiplication operator T_f where $f = h_{00}$, so that asymptotic distribution of eigenvalues of $(T_f)_N$ converges to $F(\lambda_j(T_f)_N)$, where $\lambda_j(T_f)_N = 1, -1, 0$. Now we give an example for the case when $\lambda_j(T_f)_N = 0$.

5.2.4 Example

Let $f = h_{00}$. Therefore support of $f = [0,1]$. Recall that $H = \{\phi_j(x), h_{ij}(x), i, j \in Z_+ \cup \{0\}\}$. Let $M_0 = \{h_{ij}(x), \phi_j(x) / \text{its support} \not\subset [0,1]\}$. For a positive integer t, let $j_n = 2^{n-t}$, and let $A_t, A_{t+1}, \ldots$ be a partition of M_0 such that $|A_n| = (n-t+1)2^{n-t}+1$, $n = t, t+1, \ldots$. Define

$$A = \{\phi_0, \phi_1, \ldots, \phi_{t-1}, h_{t-1,0}, h_{t-2,0}, \ldots, h_{00}, \phi_t\}$$

$$A_n = \{h_{n-t,2^{n-t}}, \ldots, h_{n-t,2^{n-t+1}-1}, h_{n-t-1\,2^{n-t}}, \ldots, h_{0,2^{n-t}}, \ldots, h_{0,2^{n-t+1}-1}, \phi_{n+1}\}, n > t$$

Let $B_t, B_{t+1}, \ldots$ be a partition of $H \cap {M_0}^C$ such that $|B_t| = (t+1)2^{n-t}$ and for any $n > t$ define $B_n = \{h_{n0}, \ldots, h_{n,2^{n-t+1}-1}, h_{n-1,2^{n-t}}, \ldots, h_{n-t+1,2^{n-t+1}-1}\}$. Then

$H = \{A, B_t, A_t, B_{t+1}, A_{t+1}, \ldots\}$ is a basis for $L^2(R_+)$, for which Szegö's Type Theorem is valid.

Let us denote this basis by $\{\psi_k : k = 1,2,--\}$. Here also we have calculated the position of h_{ij} and ϕ_j in the above ordering and it is given in the following proposition.

5.2.5 Proposition

In the above example ψ_k s are as follows.

$$\begin{array}{lll} \psi_k = \phi_{k-1} & \text{if} \quad k = 1,2,\ldots,t & \\ \psi_k = h_{r0} & \text{if} \quad k = 2t - r, & r = 0,1,\ldots,t-1, \\ \psi_k = \phi_r & \text{if} \quad k = r(2^{r-t}+1)+1\ , & r = t, t+1, \ldots \\ \psi_k = h_{rp}\ \& p < 2^{r-t+1} & \text{if} \quad k = r(2^{r-t}+1)+p+2\ , & r = t, t+1, \ldots \\ \psi_k = h_{rp}\ \& p \geq 2^{r-t+s} & \text{if} \quad k = (r+2s)(2^{r-t+s}+1) & \\ & \qquad + p - s + 2\ , & s = 1,2,\ldots \end{array}$$

Proof

We arrange the above basis elements as

$$\phi_0, \phi_1, \phi_2, \ldots, \phi_{t-1}, h_{t-1,0}, h_{t-2,0}, \ldots, h_{1,0}, h_{0,0}, \phi_t \qquad (A)$$

and the remaining basis elements are arranged in the triangular form

$$h_{t,0}, h_{t,1}, h_{t-1,1}, \ldots, h_{1,1}, h_{0,1}, \phi_{t+1} \qquad (B_t\ , A_t)$$

$$h_{t+1,0}, h_{t+1,1}, h_{t+1,2}, \ldots, h_{0,2}, h_{0,3}, \phi_{t+2} \qquad (B_{t+1}, A_{t+1})$$

$$\ldots$$

$$\phi_r, h_{r,0}, \ldots, h_{r,(2^{r-t+1}-1)}, h_{(r-1),2^{r-t}}, \ldots, h_{r-1,(2^{r-t+1}-1)}, \ldots, h_{0,2^{r-t}}, \ldots, h_{0,(2^{r-t+1}-1)}, \phi_{r+1} \quad (B_r\ , A_r)$$

$$\ldots$$

From the arrangement it is clear that

$$\psi_k = \phi_{k-1} \quad \text{if} \quad k = 1,2,\ldots,t$$
$$\psi_k = h_{r0} \quad \text{if} \quad k = 2t - r, r = 0,1,\ldots,t-1$$

Now we find the position of ϕ_r for $r = t, t+1, \ldots$.

ϕ_r lies in the $(r-t+1)^{\text{th}}$ row.

Therefore position of

$$\phi_r = 2t + r - t + 1 + (t+2)2^0 + (t+3)2^1 + \ldots + (r+1)2^{r-t+1}$$
$$= 2t + r - t + 1 + r2^{r-t} - t$$
$$= r(2^{r-t}+1)+1, \quad r = t, t+1, \ldots$$

Now we find the position of h_{rp}.

Case 1: *when* $p < 2^{r-t+1}$ and $r = t, t+1, \ldots$

Then h_{rp} lies in the $(r-t+1)^{\text{th}}$ row in the above ordering of basis.

Therefore position of h_{rp}

$$= 2t + r - t + 1 + p + 1 + (t+2)2^0 + (t+3)2^1 + \ldots + (r+1)2^{r-t+1}$$
$$= r(2^{r-t}+1) + p + 2$$

Case 2: *when* $p \geq 2^{r=t+s}, s = 1,2,\ldots \& r = 0,1,2,\ldots$

Then h_{rp} lies $(r-t+s+1)^{th}$ row and the position of h_{rp} in this row is $s2^{r-t+s} + p + 2$. Therefore the position of h_{rp}

$$= 2t + (t+2)2^0 + (t+3)2^1 + \ldots + (r+s+1)2^{r-t+s-1} + p + 2 + r - t + s$$
$$= (r+2s)(2^{r-t+s}+1) + p - s + 2$$

5.2.6 Remarks

In the above example for $t = 0,1,\ldots$, we get a collection of orderings for which Szegö s Type Theorem is valid. In particular when $t = 0$ the ordering reduces to the ordering in 5.1.1, where $j_n = 2^n$, $|B_n| = 2^n$ $|A_n| = (n+1)2^n + 1$, such that, $A = \{\phi_0\}$,

$A_n = \{h_{n,2^n}, \ldots, h_{n,2^{n+1}-1}, h_{n-1,2^n}, \ldots h_{n-1,2^{n+1}-1}, \ldots h_{0,2^n}, \ldots h_{0,2^{n+1}-1}, \phi_{n+1}\}$ and

$B_n = \{h_{n0}, h_{n,1}, \ldots h_{n,2^n-1}\}$ and the ordering taken is $\{A, B_0, A_0, B_1, A_1, \ldots\}$.

5.2.7 Theorem

Let $\lambda_1(T_f)_N = 0$, $\lambda_2(T_f)_N = 1$ and $\lambda_3(T_f)_N = -1$ be the eigenvalues of $(T_f)_N$ where $f = h_{00}$. Let M_0 M_1, and M_{-1} be the subsets of H such that $M_0 = \{h_{ij}, \phi_i \,/\, \text{its support} \not\subset [0,1]\}$, $M_1 = \left\{h_{ij} \,/\, \text{support of } h_{ij} \subset \left[0, \frac{1}{2}\right]\right\}$ and $M_{-1} = \left\{h_{ij} \,/\, \text{support of } h_{ij} \subset \left[\frac{1}{2}, 1\right]\right\}$. For a sequence of positive integers (j_k), let $A_1, A_2, \ldots,$ $B_1, B_2, \ldots,$ $C_1, C_2, \ldots$ be partitions of M_0, M_1 *and* M_{-1} respectively such that $|A_k| = a_1 j_k$, $|B_k| = a_2 j_k$ and $|C_k| = a_3 j_k$ where a_1, a_2, a_3 are constants. Consider an ordering of H whose entries are arranged as $A_1, B_1, C_1, A_2, B_2, C_2, \ldots$. Then, this is a basis for $L^2(R_+)$ for which

$$\lim_{N\to\infty} \frac{\sum_{k=1}^{N} F\left(\lambda_k (T_f)_N\right)}{N} = \gamma_1 F(0) + \gamma_2 F(1) + \gamma_3 F(-1)$$

where $\gamma_1 + \gamma_2 + \gamma_3 = 1$ and F is any continuous function on R.

Proof

From theorem 5.1.4 (when $m, n = 0$), M_0, M_1 and M_{-1} are the eigenspaces of the eigenvalues 0,1,-1 respectively. Therefore to prove the theorem it is enough to show that

$$\lim_{N\to\infty} \frac{N_{\lambda_j}}{N} = \gamma_j$$

where N_{λ_j} is the multiplicity of eigenvalue λ_j of $(T_f)_N$.

Let N be a positive integer. Then for some n,

$$N = \sum_{k=1}^{n-1} (a_1 + a_2 + a_3) j_k + K, \qquad K \le (a_1 + a_2 + a_3) j_n$$

and then

$$N_{\lambda_j} = \sum_{k=1}^{n-1} (a_j j_k) + K_j, \qquad K_j \le a_j j_n \;, \; j = 1, 2, 3.$$

$$\lim_{N\to\infty}\frac{N_{\lambda_j}}{N} = \frac{a_j\left(\sum_{k=1}^{n-1} j_k\right)+K_j}{\sum_{k=1}^{n-1}(a_1+a_2+a_3)j_k+K}$$

$$= \lim_{N\to\infty}\frac{a_j+\frac{K_j}{\sum_{k=1}^{n-1} j_k}}{(a_1+a_2+a_3)+\frac{K}{\sum_{k=1}^{n-1} j_k}} = \gamma_j.$$

Therefore

$$\lim_{N\to\infty}\frac{\sum_{k=1}^{N} F(\lambda_k(T_f)_N)}{N} = \gamma_1 F(0)+\gamma_2 F(1)+\gamma_3 F(-1)$$

where $\gamma_1+\gamma_2+\gamma_3=1$.

5.2.8 Remarks:

(i) For each x, let δ_x denote the Dirac delta measure concentrated at x. For simplicity let $\lambda_j(T_f)_N = \lambda_j$ where $\lambda_j(T_f)_N$ =1, -1, 0. Let $\mu_N = \frac{\delta_{\lambda_1}+\delta_{\lambda_2}+\ldots+\delta_{\lambda_N}}{N}$ be the measure defined for each N and let $\mu = \gamma_1\delta_0+\gamma_2\delta_1+\gamma_3\delta_{-1}$, where γ_j is defined as above. Then the above theorems implies that $\int_0^{\infty} F d\mu_N \to \int_0^{\infty} F d\mu$ for all continuous functions F on R.

Now we identify a class of orderings for the case of multiplication operators with multiplier f having compact support. This is given in the following theorem.

5.2.9 Theorem

Let T_f be the multiplication operator with $f = \sum_{k=1}^{n} \alpha_k \psi_k$ where $\psi_k(x) = h_{ij}(x)$, or $\phi_i(x)$ and assume that the support of $f = [0,\ 2^t]$ for a non-negative integer t. Let $M_0 = \{h_{ij}(x), \phi_i(x)$ / whose support $\not\subset [0,\ 2^t]\ \}$. For a sequence of positive integers (j_n), let $A_1, A_2, \ldots,\ B_1, B_2, \ldots$ be partitions of M_0 and $H \cap M_0^{\ C}$ such that $|A_n| = n\, j_n$ and $|B_n| = j_n$. Then $H = \{A_1, B_1, A_2, B_2, \ldots\}$ is a basis for $L^2(R_+)$ and for which

$$\lim_{N\to\infty} \frac{F(\lambda_1) + F(\lambda_2) + \ldots + F(\lambda_N)}{N} = \lim_{M\to\infty} \frac{1}{M} \int_0^M F[f(x)]\,dx$$

where F is any continuous function on R and $\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ are the eigenvalues of $(T_f)_N$.

Proof :

From theorem 5.1.5,

$$T_{fN}(\psi_i(x)) = 0 \Leftrightarrow (i)\ \ \psi_i(x) = \phi_i(x) \quad \forall i\ \geq 2^t \text{ where } T_{fN} = P_N T_f P_N$$

$$(ii)\ \ \psi_i(x) = h_{ij}(x) \Leftrightarrow \left[\frac{j}{2^i}, \frac{j+1}{2^i}\right] \not\subset [0,\ 2^t]$$

Therefore M_0 is the eigenspace corresponding to the eigenvalue zero and the rest of the proof is similar to the proof of the theorem 5.2.2 .

5.2.10 Corollary :

The above theorem indicates that there are variety of orderings for which Szegö's Type Theorem is valid. For example let H, f and M_0 be defined as in the above theorem and $j_n = 2^{n+t}$, where t is a fixed non-negative integer. Define $A_n = \{h_{n,2^{n+t}}, \ldots, h_{n,2^{n+t+1}-1}, h_{n-1,2^{n+t}}, \ldots, h_{0,2^{n+t+1}-1}\}$

$$B_n = \{\phi_n, h_{n,0}, \ldots, h_{n,2^{n+t}-1}\}\quad \phi_n \in A_{n-1} \text{ if } n \geq 2^t \text{ such that}$$

$$\begin{aligned} |A_n| &= (n+1)2^{n+t} && \text{if } n < 2^t - 1 \\ &= (n+1)2^{n+t} + 1 && \text{if } n \geq 2^t - 1 \end{aligned}$$

$$\begin{aligned} |B_n| &= 2^{n+t}+1 \qquad \text{if} \quad n<2^t \\ &= 2^{n+t} \qquad \text{if} \quad n \geq 2^t \end{aligned}$$

respectively. Then the ordering $\{B_0, A_0, B_1, A_1, \ldots\}$ is a basis for $L^2(R_+)$ for which Szegö's Type Theorem is true.

For curiosity we have found the position of $h_{ij}(x)$ and $\phi_j(x)$ in the above ordering and it is given in the following proposition.

5.2.11. Proposition:

The ordered Haar basis can be represented as a sequence $\{\psi_k : k = 1,2,..\}$ where

$$\psi_k = \begin{cases} \phi_r & \text{if} \quad k = r(2^{r+t}+1)+1 \\ h_{rp}, \; p < 2^{r+t+1} & \text{if} \quad k = r(2^{r+t}+1)+p+2 \\ h_{rp}, \; p \geq 2^{r+t+1} & \text{if} \quad k = (r+2s)(2^{r+t+s}+1)+p-s+2, \; s = 1,2, \ldots \end{cases}$$

Proof :

The proof is obvious by arranging the basis elements in the triangular form as given below.

$$\phi_0,\ h_{0,0},\ h_{0,1},\ h_{0,2}, \ldots, h_{0,2^{t+1}-1} \qquad (B_0\ ,\ A_0\ )$$

$$\phi_1,\ h_{1,0},\ \ldots, h_{1,2^{t+2}-1},\ h_{0,2^{t+1}}, \ldots, h_{0,2},\ h_{0,2^{t+2}-1} \qquad (B_1\ ,\ A_1\ )$$

$$\ldots$$

$$\phi_{r-1},\ h_{r-1,0}, \ldots, h_{r-1,2^{r+t}-1},\ h_{r-2,2^{r+t-1}}, \ldots, h_{r-2,2^{r+t}-1}, \ldots, h_{0,2^{r+t-1}}, \ldots, h_{0,2^{r+t}-1} \qquad (B_{r-1}\ ,\ A_{r-1}\ )$$

$$\phi_r,\ h_{r,0}, \ldots, h_{r,2^{r+t+1}-1},\ h_{r-1,2^{r+t}}, \ldots, h_{r-1,2^{r+t+1}-1}, \ldots, h_{0,2^{r+t}}, \ldots, h_{0,2^{r+t+1}-1} \qquad (B_r\ ,\ A_r\ )$$

$$\ldots$$

Then ϕ_r lies in the $(r+1)^{th}$ row. Therefore,

$$\begin{aligned} \text{Position of } \phi_r &= 2.2^t + 3.2^{t+1} + \ldots + (r+1)2^{r+t-1} + r + 1 \\ &= r(2^{r+t}+1)+1 \end{aligned}$$

Now we find the position of h_{rp}

Case 1 : When $p < 2^{r+t+1}$

Then h_{rp} lies in the $(r+1)^{th}$ row in the above ordering of the basis.

$$\text{Position of } h_{rp} = 2.2^t + 3.2^{t+1} + \ldots + (r+1)2^{r+t-1} + r + 1 + p + 1$$
$$= r(2^{r+t} + 1) + p + 2$$

Case 2: When $p \geq 2^{r+t+s}, s = 1,2,\ldots$

If $p \geq 2^{r+t+s}, s = 1,2,\ldots$, then h_{rp} lies (r +s +1) th row and the position of h_{rp} in this row is $s2^{r+t+s} + p + 2$. Therefore Position of h_{rp}

$$= 2.2^t + 3.2^{t+1} + \ldots + (r+s+1)2^{r+t+s-1} + s2^{r+t+s} + p + 2 + r + t + s$$
$$= (r+2s)(2^{r+t+s} + 1) + p - s + 2$$

5.2.12 Remarks :

In the above ordering when $t = 0$, then the ordering reduces to the ordering 5.1.1 for which $j_n = 2^n$, $|A_n| = (n+1)2^n + 1$, $\forall \; n \geq 0$ and $|B_n| = 2^n \;\; \forall \; n \geq 1$ where $B_0 = \{\phi_0, h_{00}\}$, $A_0 = \{h_{01}, \phi_1\}$,

$A_n = \{h_{n,2^n}, \ldots, h_{n,2^{n+1}-1}, h_{n-1,2^n}, \ldots, h_{n-1,2^{n+1}-1}, \ldots, h_{0,2^{n}}, \ldots, h_{0,2^{n+1}-1}, \phi_{n+1}\}$ and

$B_n = \{h_{n0}, h_{n,1}, \ldots, h_{n,2^n-1}\} \;\; \forall \; n \geq 1$.

5.2.13 Haar System in $L^2(R_-)$

Now we transform the problem in $L^2(R_-)$ to that of $L^2(R_+)$ with out changing the spectra as well as eigenvalues of truncations as follows.

For $f \in L^2(R_-), \tilde{f}$ is defined as $\tilde{f}(t) = f(-t), \;\; \forall \;\; t \in R_+$. Then $\tilde{f} \in L^2(R_+)$, and vice versa. Let T be the multiplication operator in $L^2(R_-)$. Define $\tilde{T} \in L^2(R_+)$ such that $\tilde{T}(\tilde{f}) = \widetilde{T(f)}$. Therefore T and $\tilde{T}$ have the same spectrum.

$$\therefore \sigma[(T)_N] = \sigma[(\tilde{T})_N] \; .$$

Hence results considered for $L^2(R_+)$ can be easily carried over to the context of $L^2(R_-)$.

Now we consider the case of multiplication operators in the case in $L^2(R)$. The following theorem gives a class of orderings in $L^2(R)$ for which Szegö's Type Theorem holds.

5.2.14 Theorem :

Let $\{\psi_k : \quad k = 1,2,\ldots\}$, $\{\eta_k : \quad k = 1,2,\ldots\}$ be any ordered Haar system in $L^2(R_+)$ and in $L^2(R_-)$ respectively for which Szegö's Type Theorem is valid for certain multiplication operators. Then with respect to the ordering $\{\psi_1, \psi_2, \ldots, \psi_{n_1}, \eta_1, \eta_2, \ldots, \eta_{n_2} \ldots\}$ in $L^2(R)$, Szegö's Type Theorem is valid for certain multiplication operators .

Proof:

$$L^2(R) = L^2(R_+) \oplus L^2(R_-).$$

Let $g \in L^2(R)$,

$$\therefore g = g\psi_{R_+} \oplus g\psi_{R_-} = g_1 \oplus g_2 \quad , g_1 \in L^2(R_+),\ g_2 \in L^2(R_-),$$

where ψ_{R_+}, ψ_{R_-} are the characteristic function of $L^2(R_+)$ and $L^2(R_-)$ respectively. Let T_f be the multiplication operator in $L^2(R)$ with multiplier $f \in L^2(R)$. Therefore

$$f = f_1 \oplus f_2 \ ,\ f_1 \in L^2(R_+),\ f_2 \in L^2(R_-).$$

$$\begin{aligned} T_f &: g \to fg \\ T_f(g) &= fg = (f_1 \oplus f_2)(g_1 \oplus g_2) = f_1 g_1 + f_2 g_2 \\ &= \left(T_{f_1} \oplus T_{f_2}\right)(g_1 \oplus g_2) = \left(T_{f_1} \oplus T_{f_2}\right)g \\ \Rightarrow T_f &= T_{f_1} \oplus T_{f_2}. \end{aligned}$$

$\{\psi_1, \psi_2, \ldots, \psi_{n_1}, \eta_1, \eta_2, \ldots, \eta_{n_2} \ldots\}$ is the Haar System in $L^2(R)$ and P_N be the orthogonal projection onto first N elements. Let these N elements be $\{\psi_1, \psi_2, \text{-----}, \psi_{n_1}, \eta_1, \eta_2, \text{---}, \eta_{n_2}\}$ such that $n_1 + n_2 = N$, any positive integer.

$$P_N = P_{n_1} \oplus P_{n_2},\ P_{n_1} \in L^2(R_+),\ P_{n_2} \in L^2(R_-)\ .$$

Consider an $N \times N$ corner truncations of T_f which is given by

$$(T_f)_N = P_N T_f P_N = (P_{n_1} \oplus P_{n_2})(T_{f_1} \oplus T_{f_2})(P_{n_1} \oplus P_{n_2})$$

$$= (P_{n_1} T_{f_1} P_{n_1}) \oplus (P_{n_2} T_{f_2} P_{n_2})$$

Let $\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$, $\{\beta_1, \beta_2, \ldots, \beta_{n_1}\}$ and $\{\gamma_1, \gamma_2, \ldots, \gamma_{n_2}\}$ be the eigenvalues of $(T_f)_N$, $(T_{f_1})_N$ and $(T_{f_2})_N$ respectively. Since Szegö's Type Theorem is valid for the spaces $L^2(R_+)$ and for $L^2(R_-)$ with respect to the basis $\{\psi_k : \quad k = 1,2,\ldots\}$, and $\{\eta_k : \quad k = 1,2,\ldots\}$ respectively, we have

$$\lim_{n_1 \to \infty} \frac{F(\beta_1) + \ldots + F(\beta_{n_1})}{n_1} = F(0) = \lim_{M \to \infty} \frac{1}{M} \int_0^M F[f(x)]dx \qquad \ldots \qquad (1)$$

$$\lim_{n_2 \to \infty} \frac{F(\gamma_1) + \ldots + F(\gamma_{n_2})}{n_2} = F(0) \quad = \lim_{M \to \infty} \frac{1}{M} \int_{-M}^0 F[f(x)]dx \qquad \ldots \qquad (2)$$

Now consider

$$\lim_{N \to \infty} \frac{F(\lambda_1) + \ldots + F(\lambda_N)}{N} = \lim_{N \to \infty} \left[\frac{F(\beta_1) + \ldots + F(\beta_{n_1})}{N} + \frac{F(\gamma_1) + \ldots + F(\gamma_{n_2})}{N} \right]$$

$$= \lim_{N \to \infty} \frac{\left[\dfrac{F(\beta_1) + \ldots + F(\beta_{n_1})}{n_1} \right]}{\left(\dfrac{n_1 + n_2}{n_1} \right)} + \frac{\left[\dfrac{F(\gamma_1) + \ldots + F(\gamma_{n_2})}{n_2} \right]}{\left(\dfrac{n_1 + n_2}{n_2} \right)}$$

$$(\because\ n_1 + n_2 = N)$$

$$\lim_{N\to\infty}\left[\frac{F(0)n_1}{n_1+n_2}+\frac{F(0)n_2}{n_1+n_2}\right] = F(0)\left[\underset{N\to\infty}{Lt}\left[\frac{n_1}{n_1+n_2}+\frac{n_2}{n_1+n_2}\right]\right]$$

$$= F(0)$$

Therefore from equations (1) and (2) we get,

$$\lim_{N\to\infty}\frac{F(\lambda_1)+\ldots+F(\lambda_N)}{N} = F(0)$$

$$= \lim_{M\to\infty}\left[\frac{1}{2M}\left[\int_0^M F(f(x))dx+\int_{-M}^0 F(f(x))dx\right]\right]$$

$$= \lim_{M\to\infty}\left[\frac{1}{2M}\left[\int_{-M}^M F(f(x))dx\right]\right]$$

which completes the proof.

For example we give below an ordering of the Haar System for the space $L^2(R)$ in which case Szegö's Type Theorem holds for certain multiplication operators.

5 2.15 An ordering of the Haar wavelet basis in $L^2(R)$

The ordered Haar wavelet basis for $L^2(R)$ is given by the filling arrangement $\{\phi_0, h_{00}, h_{01}, h_{0,-1}, \phi_1, \ldots, h_{r0}, \ldots, h_{0,-(2^r-1)}, \ldots\}$. This can be written in the triangular form as

$$\phi_0,\ h_{00},\ h_{011},\ h_{0,-1}$$

$$\phi_1,\ h_{1,0},\ h_{1,1},h_{1,2}\ h_{13},\ h_{02},\ h_{03},\ \phi_{-1},\ h_{1,-1},\ldots,h_{0,-2},\ h_{0,-1}$$

$$\phi_2,\ h_{20},\ldots,h_{27},\ h_{14},\ldots,h_{17},h_{0,4},\ldots,h_{0,7},\phi_{-1},h_{1,-1},\ldots,h_{0,-1}$$

$$\ldots$$

$$\phi_r,\ h_{r0},\ldots,h_{r,2^{r+1}-1},\ h_{r-12^r},\ldots,h_{0,2^{r+1}-1},\phi_{-r},h_{r,-1},\ldots,h_{r,-(2^{r+1}-1)},\ldots,h_{0,-(2^{r+1}-1)}$$

$$\ldots$$

Let us denote this basis by $\{\omega_k : k \in N\}$. Then from the above triangular form the ψ_k s are as follows.

$$\omega_k = \begin{cases} \varphi_r & \text{if} \quad k = r(2^{r+1}+1)+1 \\ h_{rp} \quad , p < 2^{r+1}-1 & \text{if} \quad k = r(2^{r+1}+1)+p+1, \\ h_{rp} \quad , \mathrm{p} \geq 2^{\mathrm{r+s+1}} \text{-} 1\, \mathrm{s} \in \mathrm{N}, & \text{if} \quad k = (2r+3s)(2^{r+s}+1)+r+s+p+1 \\ \varphi_{-r} & \text{if} \quad k = (3r+2)2^r+r+1 \\ h_{r,-p}, \quad p < 2^{r+1}-1 & \text{if} \quad k = (3r+2)2^r+r+p+1, \\ h_{r,-p}, \quad p \geq 2^{r+s+1}-1 & \text{if} \quad k = (3r+4s+2)2^{r+s}+r+s+p+1 \end{cases}$$

where $\varphi_r(x)$ and h_{rp} are defined as before.

Now we consider the case of multiplication operators in $L^2(R)$ under the above ordering. In the next theorem we observe the behavior of the distribution of eigenvalues for certain multiplication operators in $L^2(R)$.

5.2.16 Theorem

Let $(T_f)_N$ be the N^{th} stage truncations of the operator T_f in $L^2(R)$ with the multiplier $f(x) = h_{00}(x)$ and let $\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$ be the eigenvalues of $(T_f)_N$ repeated according to multiplicity, F any continuous function on R then

$$\lim_{N\to\infty} \frac{F(\lambda_1)+\ldots+F(\lambda_N)}{N} = F(0)$$

Proof

We have $L^2(R) = L^2(R_+) \oplus L^2(R_-)$.

For simplicity let $T_f = T$.

$$\therefore\ T_f = T \in L^2(R) \Rightarrow T = T_1 \oplus T_2$$

where $T_1 \in L^2(R_+)$ and $T_2 \in L^2(R_-)$ such that

$$T_1 : \psi \to f\psi \text{ where } \psi \in L^2(R_+)$$
$$T_2 : \eta \to f\eta \text{ where } \eta \in L^2(R_-)$$

From theorem 5.1.5, we have the theorem for operator T_1 in $L^2(R_+)$ and similarly for the operator T_2 in $L^2(R_-)$.

Let N_0 , N_{10} and N_{20} be the multiplicities of the eigenvalue zero of $(T)_N$,$(T_1)_N$, $(T_2)_N$ respectively. We show that the sum of N_{10} and N_{20} is less than N_0. Then the multiplicity of the eigenvalue zero increases and hence the theorem follows for the operator $T_f = T$ in $L^2(R)$.

Claim: $N_{10} + N_{20} \le N_0$.

The proof is as follows.

Let E , E_1 and E_2 be the eigenspaces of the eigenvalue zero of the matrices $(T)_N$,$(T_1)_N$, $(T_2)_N$ respectively.

$$E = \{g \in L^2(R): \quad T(g) = 0\}$$
$$E_1 = \{g_1 \in L^2(R_+): \quad T_1(g_1) = 0\}$$
$$E_2 = \{g_2 \in L^2(R_-): \quad T_2(g_2) = 0\} .$$

Let $f_1 \in E_1$ then,

$$\begin{aligned} T(f_1) &= (T_1 \oplus T_2) f_1 \\ &= T_1(f_1) \oplus T_2(f_1) = 0 \end{aligned}$$

Hence $E_1 \subset E$ and similarly we have $E_2 \subset E$, which completes the proof..

Finally we generalize the theorem 5.2.7 and it is given in the following theorem.

5.2.17 Theorem

Let $\lambda_1(T_f)_N = 0, \lambda_2(T_f)_N = 1$, and $\lambda_3(T_f)_N = -1$ be the eigenalues of $(T_f)_N$ where $f = h_{00}$. Let M_0 M_1, and M_{-1} be the subsets of H defined by

$M_0 = \{h_{ij}, \phi_i \,/\, \text{its support} \not\subset [0,1]\}$, $M_1 = \left\{h_{ij} \,/\, \text{support of } h_{ij} \subset \left[0, \frac{1}{2}\right]\right\}$ and $M_{-1} = \left\{h_{ij} \,/\, \text{support of } h_{ij} \subset \left[\frac{1}{2}, 1\right]\right\}$. Let $A_1, A_2, \ldots,\ B_1, B_2, \ldots,\ C_1, C_2, \ldots$ be partitions of M_0, M_1 and M_{-1} respectively such that $|A_k| = a_1 j_k^{(1)}$, $|B_k| = a_2 j_k^{(2)}$ & $|C_k| = a_3 j_k^{(3)}$ where a_1, a_2, a_3 are constants and $(j_k^{(1)})$, $(j_k^{(2)})$, $(j_k^{(3)})$ be any sequences of positive integers such that

$$\lim_n \frac{\sum_{k=1}^{n} j_k^{(2)}}{\sum_{k=1}^{n} j_k^{(1)}}, \quad \lim_n \frac{\sum_{k=1}^{n} j_k^{(3)}}{\sum_{k=1}^{n} j_k^{(1)}} \text{ exists and}$$

$$\lim_n \frac{\sum_{k=1}^{n} j_k^{(3)}}{\sum_{k=1}^{n} j_k^{(2)}} \text{ exists or diverges.}$$

Consider an ordering of H whose entries are arranged as $\{A_1, B_1, C_1, A_2, B_2, C_2, \ldots\}$. Then with respect to this ordering, for all continuous functions F on R the limit

$$\lim_{N \to \infty} \frac{\sum_{k=1}^{N} F(\lambda_k (T_f)_N)}{N} \text{ exists.}$$

Proof:

Let N be any positive integer. Then for some n,

$$N = a_1 \sum_{k=1}^{n-1} j_k^{(1)} + a_2 \sum_{k=1}^{n-1} j_k^{(2)} + a_3 \sum_{k=1}^{n-1} j_k^{(3)} + K,$$

where $K = K_1 + K_2 + K_3$ where $K_1 \le a_1 j_n^{(1)}$, $K_2 \le a_2 j_n^{(2)}$, $K_3 \le a_3 j_n^{(3)}$.

Consider the N^{th} truncation of (T_f) where N is as above. Let N_j be the multiplicity of $\lambda_j (T_f)_N$, $j = 1,2,3$.

$$N_1 = a_1 \sum_{k=1}^{n-1} j_k^{(1)} + K_1$$

$$N_2 = a_2 \sum_{k=1}^{n-1} j_k^{(2)} + K_2 \quad \text{and}$$

$$N_3 = a_3 \sum_{k=1}^{n-1} j_k^{(3)} + K_3 \,.$$

Hence

$$\lim_{N \to \infty} \frac{\sum_{k=1}^{N} F\left(\lambda_k (T_f)_N\right)}{N} = \lim_{N \to \infty} \frac{N_1 F(0) + N_2 F(1) + N_3 F(-1)}{N} \quad \ldots (1)$$

Case :1 when $\lim_{n} \frac{\sum_{k=1}^{n} j_k^{(3)}}{\sum_{k=1}^{n} j_k^{(2)}}$ *exists.*

Then

$$\lim_{N \to \infty} \frac{N_j}{N} = \lim_{N \to \infty} \frac{a_j \left(\sum_{k=1}^{n-1} j_k{}^{(j)} \right) + K_j}{a_1 \sum_{k=1}^{n-1} j_k{}^{(1)} + a_2 \sum_{k=1}^{n-1} j_k{}^{(2)} + a_3 \sum_{k=1}^{n-1} j_k{}^{(3)} + K} \quad \ldots (2)$$

where $j = 1, 2, 3$

In the statement of the theorem, it is given that

$$\lim_{n} \frac{\sum_{k=1}^{n} j_k^{(2)}}{\sum_{k=1}^{n} j_k^{(1)}}, \quad \lim_{n} \frac{\sum_{k=1}^{n} j_k^{(3)}}{\sum_{k=1}^{n} j_k^{(1)}} \quad \text{and} \quad \lim_{n} \frac{\sum_{k=1}^{n} j_k^{(3)}}{\sum_{k=1}^{n} j_k^{(2)}} \text{ exists} \quad \ldots (3)$$

Dividing equation (2) by $\sum_{k=1}^{n-1} j_k{}^{(j)}$ and using condition (3), we can easily show that

$$\lim_{N \to \infty} \frac{N_j}{N} \quad \text{exists for each } j = 1, 2, 3$$

Hence the limit in equation (1) exists.

Case :2 *When* $\lim_{n} \frac{\sum_{k=1}^{n} j_k^{(3)}}{\sum_{k=1}^{n} j_k^{(2)}}$ *diverges.*

$$\lim_{N\to\infty} \frac{N_2}{N} = \lim_{N\to\infty} \frac{a_2\left(\sum_{k=1}^{n-1} j_k^{(2)}\right) + K_2}{a_1\sum_{k=1}^{n-1} j_k^{(1)} + a_2\sum_{k=1}^{n-1} j_k^{(2)} + a_3\sum_{k=1}^{n-1} j_k^{(3)} + K} \qquad \ldots (4)$$

Since $\lim_{n} \frac{\sum_{k=1}^{n} j_k^{(3)}}{\sum_{k=1}^{n} j_k^{(2)}}$ diverges, we have $\lim_{n} \frac{\sum_{k=1}^{n} j_k^{(3)}}{\sum_{k=1}^{n} j_k^{(2)}} \to \infty$.

Therefore dividing equation (4) by $\sum_{k=1}^{n-1} j_k^{(2)}$, we get

$$\lim_{N\to\infty} \frac{N_2}{N} \to 0$$

For $j = 1,3$, as in case 1, we can easily prove that

$$\lim_{N\to\infty} \frac{N_j}{N} \text{ exists.}$$

Therefore the limit in equation (1) exists in this case also.

We conclude this chapter with the following remarks.

5. 2. 18 Remarks:

(i) When $\left(j_k^{(1)}\right) = \left(j_k^{(2)}\right) = \left(j_k^{(1)}\right) = \left(j_k\right)$ then the above theorem reduces to theorem 5.2.7.

(ii) An outline of a proposal for further investigation has been given in the appendix of the thesis. This is based on the theory of modified Haar functions in $L^2\left(R^n\right) \otimes C_{(n)}$ where $C_{(n)}$ is 2^n - dimensional Clifford algebra.

♦

APPENDIX

Here we have given a result on spectral approximation of certain multiplication operators in $L^2(R_+)$ with respect to Haar basis and a proposal for future investigation to higher dimensional $L^2(R^n)$.

A .1 A Result on Spectral Approximation

First we discuss some definitions and results which are used in this appendix.

A.1.1 Filtration [4]

A filtration for Hilbert Space H is a sequence $F = \{H_1, H_2, \ldots\}$ of finite dimensional subspaces of H such that $H_n \subseteq H_{n+1}$ and $\bigcup_n H_n$ is dense in H.

A.1.2 Degree of a Bounded Linear Operator [4]

Let $F = \{H_n\}$ be a filtration of H and P_n be the orthogonal projection onto H_n. The degree of an operator $A \in B(H)$ is defined by

$$\deg(A) = \sup_{n \geq 1} rank(P_n A - AP_n).$$

A.1.3 Arvesons Class [4]

Let M denote the class of all A in $B(H)$ such that $A = \sum_{1}^{\infty} A_k$ where $A_k \in B(H)$ and $\deg(A_k) < \infty$ such that norm of A

$$\|A\| = \inf \sum_{k=1}^{\infty} \left[1 + \deg(A_k)^{1/2}\right] \|A_k\| < \infty$$

Then M is called Arvesons Class.

A.1.4 Arvesons Criteria for an Operator to belong M (Arvesons Class) [4]

Let $\{e_n : n \in Z\}$ be a bilateral orthonormal basis for a Hilbert space H and let $\{H_n : n = 1,2,...\}$ be the filtration $H_n = [e_{-n}, e_{-n+1}, ..., e_n]$. Let (a_{ij}) be the matrix of an operator $A \in B(H)$ relative to $\{e_n\}$, and for every $k \in Z$, let

$$d_k = \sup_{i \in Z} |a_{i+k,i}| \ .$$

Then A will belong to Arvesons class M whenever the series $\sum_k |k|^{1/2} d_k$ converges.

A.1.5 Band Operator [23]

We call $A \in L(H)$ a band operator with respect to H if

$$\sup_n tr(P_n A - AP_n) < \infty$$

Here we consider,

A	- Multiplication operator T_f
H	- The Hilbert space $L^2(R_+)$
$\{e_n : n \in Z\}$	- The Haar system $\{\psi_n : n = 1,2,...\}$ordered as in 5.1.1
H_n	- $\{\psi_1, \psi_2, ..., \psi_n\}$
P_n	- Orthogonal projection onto $\{\psi_1, \psi_2, ..., \psi_n\}$

In this section, using the above criteria we show that the certain multiplication operators in $L^2(R_+)$ belongs to Arvesons class with respect to the Haar basis and it is given in the following theorem.

A.1.6 Theorem

Let $\{\psi_n : n = 1,2,...\}$ be the ordered Haar basis in 5.1.1 and T_f be the multiplication operator on $L^2(R_+)$ with $f = \sum_{k=1}^{\infty} \alpha_k \psi_k, \ \alpha_k \le \frac{1}{2^{2k}}$. Then T_f belongs to the Arvesons Class.

Proof:

Let

$$(T_f) = (a_{ij}) \text{ where } a_{ij} = \int_0^{\infty} f \psi_i(x) \psi_j(x) dx.$$

$$d_k = \sup_{i \in Z_+} |a_{i+k,i}|$$

$$= \sup_{i \in Z_+} | \langle T\psi_i(x), \psi_{i+k}(x) \rangle |$$

$$= \sup_{i \in Z_+} \left| \int_0^{\infty} f(x) \psi_i(x) \psi_{i+k}(x) \, dx \right|$$

$$= \sup_{i \in Z} \left\{ 0, \ \left| \int_{j/2^i}^{j+1/2^i} f(x) h_{ij}(x) dx \right|, \ \left| 2^{m/2} \int_{j/2^i}^{j+1/2^i} f(x) h_{ij}(x) dx \right| \right\}$$

(if support of $h_{ij}(x) \subset$ in the support of $h_{mn}(x)$ or $\phi_m(x)$)

$$= \sup_{i \in Z_+} \left\{ 0, \ |\alpha_{i+k}|, \ \left| 2^{m/2} \alpha_{i+k} \right| \right\}$$

If $(\alpha_k) = \left(1/2^{2k}\right)$ or $\alpha_k \le 1/2^{2k}$ then,

$$d_k = \sup_{i \in Z_+} \left\{ 0, \ |2^{-2(i+k)}|, \ \left| 2^{m/2} 2^{-2(i+k)} \right| \right\}$$

$$< \frac{2^{m/2}}{2^{2k}} < \frac{2^{k/2}}{2^{2k}} < \frac{1}{2^k}$$

Since T_f is symmetric $d_{-k} = d_k$. Then

$$\sum |k|^{1/2} d_k = 2\left[\sqrt{1}d_1 + \sqrt{2}d_2 + \ldots\right]$$

$$< 2\left[\frac{\sqrt{1}}{2} + \frac{\sqrt{2}}{2^2} + \ldots\right] < \infty$$

Hence the theorem follows from A.4.

We conclude this section of the appendix with the following remarks.

A.1.7 Remarks:

(i) Since T_f belongs to Arvesons class we have the following estimate of commutator 2-norms

$$\sup_n \left\| T_f P_n - P_n T_f \right\|_2 < \infty \qquad \ldots \quad (1)$$

where Hilbert – Schmidt norm of operator B is $\|B\|_2 = (trace(B * B))^{1/2}$.

Therefore T_f is a band operator.

(ii) As a consequence of Arvesons Theorem [4, Theorem3.8] $\sigma_e(T_f)$ can be fully recovered by the eigenvalues of the truncations $(T_f)_n$.

A.2 A Proposal for Future Investigation

We give an outline of a proposal for further investigation. This is based on the theory of modified Haar functions in $L^2(R^n) \otimes C_{(n)}$, where $C_{(n)}$ is 2^n -dimensional Clifford algebra.

The construction of the modified Haar functions can be found in the article "**The Cauchy singular integral operator and Clifford wavelets**" **by Lars Andersson, Bjorn Jawerth, and Marius Mitrea** [5, Chapter15].

First we give the definition of Clifford algebras over R. Fix a non-negative integer n and let $e_0, e_1, \ldots, e_n$ be the standard basis in R^{n+1} (or C^{n+1}).

A.2.1 Clifford Algebra [5]

The 2^n- dimensional Clifford algebra $R_{(n)}$ (or $C_{(n)}$) is the algebra over R (or C respectively) freely generated by $e_0, e_1, \ldots, e_n$ subject to the relations:

1. e_0 is the multiplicative identity
2. for $1 \le j,k \le n$, $e_j e_k + e_k e_j = -2\delta_{jk} e_0 = \begin{cases} 0, & \text{if } j \ne k \\ -2e_0, & \text{if } j = k \end{cases}$

In particular the Clifford algebras $R_{(0)}, R_{(1)}$, and $R_{(2)}$ are the real numbers, complex numbers, and quaternions, respectively. We embed R^{n+1} in $R_{(n)}$ (or $C_{(n)}$) by

$$x \in R^{n+1}, x = (x_0, x_1, \ldots, x_n) \mapsto \sum_{j=0}^{n} x_j e_j \in R_{(n)} \subset C_{(n)}$$

The image of R^{n+1} under this embedding is called the set of Clifford numbers in $R_{(n)}$ (or $C_{(n)}$) .

For $A \subseteq \{1, 2, \ldots, n\}$, $A = \{i_1 < i_2 <, \ldots < i_k\}$, we set $e_A = e_{i_1} e_{i_2} \cdots e_{i_k}$. We also write $e_0 = 1$. Then $\{e_A\}_{A \subseteq \{1,2,\ldots,n\}}$ is a basis for $R_{(n)}$ and $C_{(n)}$. Hence

$$x \in R_{(n)} \Rightarrow x = \sum x_A e_A \, .$$

We shall write $x_0 e_0 = x_0$, and refer to x_0 as the scalar part of x, i.e., $x_0 =: \operatorname{Re} x$. The above basis is orthonormal with respect to the inner product

$$\langle x, y \rangle = \sum x_A \bar{y}_A$$

where $x, y \in R_{(n)}$ and the Euclidian norm is defined as

$$|x| = \left(\sum_A |x_A|^2 \right)^{1/2} .$$

With this norm $R_{(n)}$ and $C_{(n)}$ become normed algebras.

Now consider the Clifford algebra $L^2(R^n) \otimes C_{(n)}$, the Clifford algebra version of $L^2(R^n)$. In this space modified Haar system behaves like an (bi) orthogonal set of functions with respect to the form

$$\langle f_1, f_2 \rangle = \int f_1(x) N(x) f_2(x) dx \qquad \ldots \quad (1)$$

where f_1, f_2 are $C_{(n)}$ valued functions, Σ - the graph of a Lipschitz function $g : R^n \to R$ and $N(x)$ is the Clifford number defined by $(1, -\nabla g(x))$. Therefore, $\operatorname{Re} N(x) = 1$ and $|N(x)| \sim 1$.

A.2.2 Modified Haar system in $L^2(R^n) \otimes C_{(n)}$

Let F denote the collection of all dyadic cubes of R^n .

$Q = Q_{k,v} = \left\{ x \in R^n : \frac{v_i}{2^k} \le x_i \le \frac{v_i + 1}{2^k} \quad i = 1, 2, \ldots, n \right\}$ for $k \in Z, v \in R^n$.

Each dyadic cube has 2^n subcubes.

$$\{Q^j\}_{j=1}^{2^n} = \left\{ Q^j \in F l(Q^j) = \frac{1}{2} l(Q), \ Q^j \subset Q \right\}$$

where $l(Q)$ is the side length of Q.

Let $F_k = \left\{ Q \in F; l(Q) = \frac{1}{2^k} \right\}$. Let $|Q|$ denote the volume of Q . Define

$$m(Q) = |Q|^{-1} \int_Q N(x) dx$$

where $N(x)$ is defined as above. Then $\operatorname{Re} m(Q) = 1$ and $|m(Q)| \sim 1$.

For $Q \in F$ and $i = 1, 2, \ldots, 2^n - 1$, define the $C_{(n)}$- valued functions $\{\beta^L{}_{Q,i}\}_{Q,i}$ and $\{\beta^R{}_{Q,i}\}_{Q,i}$ by

$$\beta^{L}{}_{Q,i} = 2^{n/2}|Q|^{-1/2} M(Q,i)\left\{ i^{-1} m\left(\bigcup_{\nu=1}^{i} Q^{\nu}\right)^{-1} \left(\sum_{\nu=1}^{i} \chi_{Q^{i}}\right) - m\left(Q^{i+1}\right)^{-1} \chi_{Q^{i+1}} \right\}$$

and

$$\beta^{R}{}_{Q,i} = 2^{n/2}|Q|^{-1/2} \left\{ i^{-1} m\left(\bigcup_{\nu=1}^{i} Q^{\nu}\right)^{-1} \left(\sum_{\nu=1}^{i} \chi_{Q^{i}}\right) - m\left(Q^{i+1}\right)^{-1} \chi_{Q^{i+1}} \right\} M(Q,i)$$

where $M(Q,i) := \left(\left\{ i^{-1} m\left(\bigcup_{\nu=1}^{i} Q^{\nu}\right)^{-1} + m\left(Q^{i+1}\right)^{-1} \right\}^{-1} \right)^{1/2}$.

The next two results shows that β s behave much like an orthonormal basis for $L^2(R^n)_{(n)}$.

A.2.3 Corollary [5]

For each $f \in L^2(R^n)_{(n)}$,

$$f = \sum_{Q\in F} \sum_{j=1}^{2^n-1} \langle f \; , \; \beta^{R}_{Q,i} \rangle_{\Sigma} \beta^{L}_{Q,i}$$

and

$$f = \sum_{Q\in F} \sum_{j=1}^{2^n-1} \beta^{R}_{Q,i} \langle f \; , \; \beta^{L}_{Q,i} \rangle_{\Sigma}$$

A.2.4 Theorem [5]

If $f \in L^2(R^n)_{(n)}$ then

$$\|f\|_2^2 \sim \sum_{Q\in F} \sum_{i=1}^{2^n-1} \left| \langle f , \beta^{R}_{Q,i} \rangle_{\Sigma} \right|^2 \sim \sum_{Q\in F} \sum_{i=1}^{2^n-1} \left| \langle \beta^{L}_{Q,i} , f \rangle_{\Sigma} \right|^2$$

A.2.5 Modified Haar System in $L^2(R^n)$

From the above basis, an orthonormal basis for $L^2(R^n)$ can be derived by taking $\Sigma = R^n$, then $g(x) = 0$. Therefore, $N(x) = (1, 0, 0, \ldots, 0)$. Then the above β s reduces to

$$h_Q^i := 2^{n/2}|Q|^{-1/2}\left(\frac{i}{i+1}\right)^{1/2} \left\{ \frac{1}{i}\sum_{\nu=1}^{i} \chi_{Q^{\nu}} - \chi_{Q^{i+1}} \right\}, \; i = 1, 2, \ldots, 2^n - 1, \; Q \in F$$

The family $\{h_Q^i\}_{Q,i}$ forms an orthonormal basis for $L^2(R^n)$ with respect to the standard inner product

$$\langle f_1, f_2 \rangle_{R^n} = \int_{R^n} f_1(x) f_2(x) dx, \; x \in R^n \; (\because N(x) = (1, 0, \ldots, 0) \;)$$

This is the modified Haar system in $L^2(R^n)$. In this case for $f \in L^2(R^n)$,

$$f = \sum_{Q \in F_k} \sum_{i=1}^{2^n - 1} \langle f, h_Q^i \rangle h_Q^i$$

Thus one can formulate the problems investigated in this thesis in the above set up and carry out investigations. ♦

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