STOCHASTIC MODELLING

ANALYSIS OF SOME SINGLE AND TWO COMMODITY INVENTORY PROBLEMS

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CERTIFICATE

Certified that the thesis entitled **"ANALYSIS OF SOME SINGLE AND TWO COMMODITY INVENTORY PROBLEMS"** is a bonafide record of work done by Smt. Merlymole Joseph K. under my guidance in the Department of Mathematics, Cochin University of Science and Technology, and that no part of it has been included any where previously for the award of any Degree or any other similar title.

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CHAPTER I

INTRODUCTION

1.1. Introduction.

In many disciplines of the social and natural sciences dynamic systems are encountered that are made up of a large number of separate but interacting units. Due to complexity, inherent random effects or incompleteness of information about the dynamic structure, a stochastic model is appropriate for many of these systems.

This thesis is devoted to the study of some stochastic models in inventories. An inventory system is a facility at which items of materials are stocked. In order to promote smooth and efficient running of business, and to provide adequate service to the customers, an inventory of materials is essential for any enterprise. When uncertainty is present, inventories are used as a protection against risk of stock out. It is advantageous to procure the item before it is needed at a lower marginal cost. Again, by bulk purchasing, the advantage of price discounts can be availed. All these contribute to the formation of inventory.

Maintaining inventories is a major expenditure for any organization. For each inventory, the fundamental question is how much new stock should be ordered and when should the orders be placed. If large quantities are ordered, the organization has to pay excessive storage cost. On the other hand, very small order quantities result in very high procurement cost. Hence, a trade off between the two is called for. Management of any such inventory involves monitoring the input and withdrawals of inventoried items, as well as making decisions as to the best means of replenishing the inventory.

In the present study, we have considered several models for single and two commodity stochastic inventory problems. By model building, we mean providing a model that will provide a good fit to a set of data and that will give good estimates of parameters and good prediction of future values for given values of the independent variables.

1.2. Historical Background.

The first quantitative analysis in inventory studies started with the work of Harris in 1915. He formulated mathematically a simple inventory situation and obtained its solution. Wilson rediscovered the same formula in 1918. After the second world war, several researchers like Pierre Masse (1946), Arrow, Harris and Marschack (1951) Dvoretsky, Kiefer and Wolfowitz (1952) and Whitin (1953) have discussed the stochastic nature of inventory problems.

A systematic analysis of (s, S) inventory model based on renewal theory is first provided by Arrow Karlin and Scarf (1958). The book by Hadley and Whitin (1963), provides an excellent account of applications. A computational approach for finding optimal (s, S) inventory policies is given by Veinott and Wagner (1965). An excellent review by Veinott (1966), summarizes the status of mathematical theory of inventory until the early sixties. He focuses his attention on the determination of optimal policies of multi - item and / or for multi echelon inventory systems with certain and uncertain demands. The cost analysis of different inventory systems along with several other characteristics is given in Naddor (1966). Gross and Harris (1971) develop continuous review (s, S) inventory models with state dependent lead times. Sivazlian (1974) considered a continuous review (S, s) inventory system with arbitrary inter arrival time distribution between demands, where each arrival demands exactly one unit. He obtains the transient and steady state distribution for the position inventory and shows that the limiting distribution of the position inventory is uniform and is independent of the inter arrival time distribution under many sharp assumptions. The same result for the case with arbitrarily distributed demand quantity has been obtained by Richards (1975). An indepth study of (s, S) inventory policy with arbitrarily distributed lead time is available in Srinivasan (1979). Here he assumes the demand process as a renewal process where as Sahin (1979) considers an inventory problem with the item being continuously measured; inter arrival times form a renewal process. However, she assumes the lead time to be a degenerate random variable. This was further extended by Manoharan, Krishnamoorthy and Madhusoodhanan (1987) to the case of non-identically distributed inter arrival demand times and random lead times, which however is restricted to demand quantity being exactly equal to one unit.

An (s, S) inventory system with demand for items dependent on an external environment is studied by Feldmann (1975). Ramaswami (1981) obtains algorithms for an (s, S) inventory model where the demand is according to a versatile Markovian point process. The binomial moments of the time dependent and limiting distributions of the deficit in the case of a continuous review (s, S) policy with random lead time and demand process following a compound renewal process have been obtained by Sahin (1983).

Thangaraj and Ramanarayanan (1983) discuss an inventory system with two reordering levels and random lead time. Ramanarayanan and Jacob (1986) analyze the same problem with relaxation that the lead time is random and several reordering levels. Krishnamoorthy and Manoharan (1991) discuss the same problem in which they have obtained the time dependent probability distribution of the inventory level and the correlation between the number of demands during a lead time and the length of the next inventory dry period. Krishnamoorthy and Manoharan (1990) consider an (s,S) inventory problem with state dependent demand quantities. They obtain the system state probabilities.

The review by Nahmias (1982) provides the state of art on perishable inventory models until the beginning of the eighties. Kalpakom and Arivarignan introduce perishability of exhibiting item(s) and provide several characterization of the underlying inventory process. They (1985a) consider the case of an inventory system with arbitrary inter arrival time between demands in which one item is put into operation as an exhibiting item whose lifetime has the exponential distribution. Non exhibited items do not deteriorate. The transient and steady state distributions for position inventory are derived under assumption that quantity demanded at a demand epoch depends the time elapsed since the previous arrival. Again the same system having one exhibiting item subject to random failures with failure times following exponential distribution and unit demand is dealt with by the same authors (1985b) and the expression for the limiting distribution of the position inventory is derived by applying the techniques of semi-regenerative process. Manoharan and Krishnamoorthy (1989) consider an inventory problem with all items subject to decay and derive the limiting probability distribution. They assume that quantities demanded by arrivals are independently and identically distributed random variables and inter arrival times follow an arbitrary distribution. Kalpakom and Arivarignan (1989) analyze a perishable inventory model in which the inventoried items have life times with negative exponential distribution with demands forming a Poisson process which is extended by Krishnamoorthy and Varghese (1995) to one, subject to disasters.

Ramanarayanan and Jacob (1987) analyze an inventory system with random lead time and bulk demands. They use the matrix of transition time densities and its convolutions to arrive at the expression for the probability distribution of the inventory level. Inventory systems with random lead times and server vacations when the inventory becomes dry is introduced by Daniel and Ramanarayanan (1987, 1988).

Sivazlian and Stanfel (1975) discuss a two commodity single period inventory problem. Krishnamoorthy, Basha and Lakshmi (1994) consider a two commodity inventory system with demand quantities exactly one unit of either or both type at each demand epoch. They investigate the stationary distribution of the system state. Some optimization problems associated with this model are also examined. Also

Krishnamoorthy, Lakshmi and Basha (1997) generalize the above set up by analyzing a two commodity inventory problem with Markov shift in demand of either type of commodity, and derive the stationary distribution of the system state. They provide a characterization for the system state distribution to be uniform.

Berg, Posner and Zhao (1994) consider production inventory system with unreliable machines. Dhandra and Prasad (1995) analyze a two commodity inventory model for one-way substitutable item.

N Policy is introduced into inventory problem by Krishnamoorthy and Raju (1998a, b) wherein local purchase is resorted to when the backlog reaches a threshold N. Three types of local purchases are discussed by them-local purchase to bring the level to S cancelling outstanding order, local purchase to bring the level to s and the local purchase to meet the backlog alone without cancelling the outstanding orders.They examine the N value that minimizes the total expected cost.

1.3. An Outline of the Present Work :

The thesis is divided into six chapters, including this introductory chapter. Chapters two and three are about single commodity inventory problems and the last three derived on two commodity problems. We have analyzed the models to get the inventory level probabilities at any instant of time and determined the cost functions. Most of the models are illustrated with numerical examples.

Chapter two deals with single commodity, continuous review, (s, S) inventory system with disasters. In most of the analysis of inventory systems the decay and disaster factors are ignored. But in several practical situations, these factors play an important role in decision making. Examples are electronic equipment stored and exhibited on a sales counter where there is possibility of damage to the equipment due to lightning, crops subject to natural calamity etc.

We have examined two models. In Model I, inventory level depletes due to both disasters and demands. Shortages are not allowed and lead time is zero. The inter arrival times of disasters have arbitrary distribution G(.) and the quantity destructed depends on the time elapsed between disasters. Demands form a Compound Poisson process. The assumptions of Model II are similar to Model I except that the time elapsed between two consecutive demand points are independently and identically distributed with common distribution function G(.) and demand magnitude depends only on the time elapsed since the previous demand points. The probability distribution of stock level at arbitrary time points and also the steady state inventory level distribution are obtained for both the models. Cost functions associated with the models are also studied.

In chapter III, we have introduced correlation in (s, S) inventory problems in two different ways. Model I discusses analysis of correlated order quantity. Model II studies correlation between order quantity and replenishment quantity. The inventory level at arbitrary time point and its limiting distribution are computed. Some optimization problems are also examined for both the models.

Chapter IV deals with linearly correlated bulk demand two commodity inventory problem, where each arrival demands a random number of items of each commodity C_1 and C_2, the maximum quantity demanded being $a\ (< s_1)$ and $b\ (< s_2)$ respectively. The particular case of linearly correlated demand is also discussed. Numerical illustrations are also provided.

Chapter V deals with two models. First model describes a bulk demand two commodity inventory problem. We follow $(s_k,\ S_k)$ policy for the commodity C_k ($k = 1, 2$). The probability that a demand occurs for commodity C_k alone is p_k and a demand for both C_1 and C_2 together is assumed not to occur. Thus $p_1 + p_2 = 1$. Lead time is assumed to be zero.

In Model II, all assumptions are similar to Model I except that the probability for a demand of both commodities together is allowed. Lead time is exponentially distributed for first commodity and sales of C_1 restricted to those customers, that demand second commodity C_2 also until C_1 is replenished. The limiting probabilities and optimization problems are examined for both models. Some numerical illustrations are also provided.

In the last chapter, we analyze a two commodity inventory problem with lead time under N policy. Local purchase by shopkeepers are very common. Situations of this sort arise in practice in shops when certain goods run out of stock and on reaching a threshold (negative level), the owner goes for local purchase. Though this results in higher cost to the system, it ensures goodwill of customers.

In this model, all assumptions are similar to Model II described in Chapter V except that we introduce the N policy for local purchase of the first commodity. Three variants of the problem are investigated. The limiting probabilities of the system size are derived. An optimization problem is examined. Numerical illustrations are also provided.

The notations used in this thesis are explained in each chapter. The thesis ends with a list of references.

CHAPTER II

SINGLE COMMODITY INVENTORY PROBLEMS WITH DISASTERS

2.1. INTRODUCTION

In this chapter, we discuss a continuous review inventory system in which inventory level depletes due to disasters and demands. Two models are discussed. First we examine the case in which the time elapsed between two consecutive demand points are independent and identically distributed with common distribution function F(.) with mean μ (assumed finite) and in which demand magnitude depends only on the time elapsed since the previous demand epoch. The time between disasters has an exponential distribution with parameter λ. This is Model I.

In Model II, the inter arrival time of disasters have general distribution (F.) with mean λ ($< \infty$) and the quantity destructed depends on the time elapsed between disasters. Demands form a compound Poisson process with inter arrival times of demands having mean $1/\mu$.

The review by Nahmias (1982) discusses several perishable inventory models. Kalpakom and Arivarignan (1985) introduced perishability of exhibiting item and provide several characterisation of the underlying inventory process. Further the same authors (1988) analyse a perishable inventory model in which the life time of inventoried items is negative exponential with demands forming a Poisson process. Krishnamoorthy and Varghese have extended the above to one, subject to disasters . In this chapter the dependence structure is introduced to the (s, S) inventory models with disasters in two different ways. In Model I, the successive quantities demanded are dependent - dependence being on the time elapsed since the previous demand points. In Model II, the quantity destructed depends on the time elapsed between disasters. Both models deal with zero lead time. The assumption of zero lead time may restrict the application of the model yet we find several applications of the models in our day to day life. One such is the case of certain electrical and electronic equipments damaged due to lightning. The replacement can be done within no time, due to the abundance of such items in the market.

Section 2.2 provides the description of Model I. System size probability distribution at arbitrary time point in finite time and steady state behaviour are obtained. and a suitable cost function is also examined in the same section.

In Section 2.3, the description and analysis of Model II are given. System size probabilities and the limiting distribution are obtained. An optimal decision rule is also discussed.

The following notations are used in this chapter.

S - maximum inventory level

s - reordering level

M - $S - s$

E - $\{s + 1, \dots, S\}$

X (t) - Inventory level at time t ($t \geq o$)

X_n - $X(T_n+)$, $n \in \{1,2,3,\ldots\}$

$*$ - convolution. For example $(F*G)t = \int_{-\infty}^{\infty} F(t)dG(t-u)$

$f^{*n}(.)$ - n fold convolution of f (.) with itself.

$H_{i,k}(u)$ - probability that starting with i units the inventory level reaches

k at time u, as a consequence of one disaster in (u, u+du).

$$= \begin{cases} \binom{i}{k} p^k (1-p)^{i-k} \lambda \bar{e}^{\lambda u}, & s+1 \le k \le i \\ \sum_{j=0}^{s} \binom{i}{j} p^j (1-p)^{i-j} \lambda e^{-\lambda u} & for\ k = S \end{cases}$$

$$H_{i,k}^{(m)}(u) = H_{i,k}^{*m}(u) \text{ and define } H_{i,k}^{(0)}(u) = e^{-\lambda u}$$

$g_r(u)$ - Probability of r units demanded at a demand epoch when u time units elapsed from the last demand occurrence point.

2.2. MODEL I

An (s, S) inventory model with the maximum capacity of the ware house being fixed at S is considered. The stock is brought to S whenever the inventory level falls to s or below s, due to disasters and or demands for the first time after the previous replenishment. Lead time is assumed to be zero. Shortages are not allowed. The basic assumption of our model is that the time elapsed between two consecutive demand points are independent and identically distributed with common distribution function F(.) having mean μ (assumed finite). The quantity demanded by each arrival depends only on the time elapsed since the previous demand points. The time between disasters is exponentially distributed with parameter λ. Due to a disaster a random number of units are destroyed. Each unit in the inventory survives a disaster with probability p and succumbs to it with probability 1-p.

2.2.1. Analysis of the Model :

Suppose $0 = T_0 < T_1 < \ldots < T_n < \ldots$ are the times at which demand occurs and X_0, $X_1, \ldots, X_n \ldots$ be the corresponding inventory levels, $X(T_n+) = X_n$, $n \in \{1,2,3,\ldots\}$. Then we have

Theorem:- $(X,T) = \{ (X_n, T_n), n = 0,1,2\ldots\}$ forms a Markov renewal process (MRP) with semi – Markov kernel,

$$Q(i, j, t) = P[X_{n+1} = j , T_{n+1} - T_n \le t / X_n = i]\ i, j \in E , t \ge 0 .$$

Proof follows easily from the definition of MRP.

Q(i , j, t) represents the transition probability from i to j in time less than or equal to t. We have

$$Q(i,j,t) = \int_{u=0}^{t} \sum_{\substack{k\in E \\ k\ge j}} \sum_{m=0}^{\infty} H_{i,k}^{(m)}(u)\, g_{k-j}(u)\,dF(u)$$

........(1)

$$Q(i,S,t) = \int_{u=0}^{t} \sum_{k\in E} \sum_{m=0}^{\infty} H_{i,k}^{(m)}(u)\, g_{(k-s)}(u)\,dF(u)$$

The right hand side of (1) is arrived at as follows. From the level i, the inventory position reaches k at time u, as a consequence of m disasters until time u, and k-j units are demanded at the next demand epoch when u time units elapse from the last demand occurrence point, which has probability $H_{i,k}^{(m)}(u)\, g_{k-j}(u)$

Second part of equation (1) is obtained as – from the level i reaches k at time u, as a consequence of m disasters until time u, atleast k-s units are demanded at the next demand epoch when u time units elapse from the last demand occurrence point so that inventory level reaches S which has probability $H_{i,k}^{(m)}(u)\, g_{(k-s)}(u)$

The next step is to obtain an expression for the Markov renewal function. To this end we proceed as follows.

As soon as the stock level falls to s or below s, for the first time after the previous replenishment an order for replenishment is placed, so as to bring the inventory level back to S. Looking at the successive epochs $0 = T_0^1, T_1^1, \ldots$ at which the inventory level is brought to S (these can be either disaster or demand epochs). Let F (S, S, t) be the probability distribution of time between two consecutive S to S transition. S to S transition can occur in two mutually exclusive ways with each one again having two possibilities.

Initially due to a demand the inventory level drops to the ordering set. Consequently an order is placed and replenishment occurs at instant of commencement of inventory. Then next passage to S can be due to either

(i) k demands and . $n_1 + \ldots + n_{k+1}$ disasters take away atmost M-1 units and due to the next demand the inventory level drops to the ordering set. Or

(ii) k demands and $n_1 + \ldots + n_{k+1}$ disasters take away atmost M-1 units and due to the next disaster the level drops to the ordering set. The distribution function of this time duration is represented by F_1 (S, S, t).

Again, initially due to a disaster, the inventory level drops to the ordering set and an order is placed and replenishment occurs at instant of commencement of inventory. Here also for S to S transition two possibilities are there. Either

(i) k demands and $n_1 + \ldots + n_{k+1}$ disasters take away atmost M-1 units and due to the next demand the inventory level drops to the ordering set and triggering in an order placement. Or

(ii) k demands and $(n_1 + \ldots + n_{k+1})$ disasters take away atmost (M-1) units and due to the next disaster inventory level drops to the ordering set. Replenishment occurs due to instant order placement. Here we obtain F_2 [S, S, t].

Hence F [S, S, t] = F_1 [S, S, t] + F_2[S, S, t].

where

$$F_1[S,S,t] = \sum_{i_1,\ldots,i_{k+1} > 0} \sum_{j_1,\ldots,j_{k+1} \geq 0} \sum_{n_1,\ldots,n_{k+1} \geq 0} \sum_{\substack{i_1+\ldots+i_k+j_1+\ldots+j_{k+1} < M \\ i_1+\ldots+i_k+j_1+\ldots+j_{k+1}+i_{k+1} \geq M}}$$

$$\int_{u_1=0}^{t} \int_{u_2=u_1}^{t} \cdots \int_{u_k=u_{k-1}}^{t} \int_{w=u_k}^{t} H_{S,S-j_1}^{(n_1)}(u_1) g_{i_1}(u_1)$$

$$H^{(n_2)}_{S-(i_1+j_1),S-(i_1+j_1+j_2)}(u_2-u_1) \quad g_{i_2}(u_2-u_1) \quad \ldots$$

$$H^{(n_k)}_{S-(i_1+\ldots+i_{k-1}+j_1+\ldots+j_{k-1}),S-(i_1+\ldots+i_{k-1}+j_1+\ldots+j_k)}(u_k-u_{k-1})$$

$$g_{i_k}(u_k-u_{k-1})H^{(n_{k+1})}_{S-(i_1+\ldots+i_k+j_1+\ldots+j_k),S-(i_1+\ldots+i_k+j_1+\ldots+j_{k+1})}(w-u_k)$$

$$g_{i_{k+1}}(w-u_k) \quad [1-F(t-w)] \quad e^{-\lambda(t-w)} \quad dwdu_k du_{k-1}\ldots du_2 du_1$$

$$+$$

$$\sum_{i_1,\ldots,i_k>0}\ \sum_{j_1,\ldots,j_{k+2}\geq 0}\ \sum_{n_1,\ldots,n_{k+1}\geq 0}\ \sum_{\substack{i_1+\ldots+i_k+j_1+\ldots+j_{k+1}<M \\ i_1+\ldots+i_k+j_1+\ldots+j_{k+1}+j_{k+2}\geq M}}$$

$$\int_{u_1=0}^{t}\ \int_{u_2=u_1}^{t}\cdots\int_{u_k=u_{k-1}}^{t}\ \int_{v=u_k}^{t}\ \int_{x=v}^{t}\ H^{(n_1)}_{S,S-j_1}(u_1)g_{i_1}(u_1)\ldots\ldots$$

$$H^{(n_k)}_{S-(i_1+\ldots+i_{k-1}+j_1+\ldots+j_{k-1}),S-(i_1+\ldots+i_{k-1}+j_1+\ldots+j_k)}(u_k-u_{k-1})g_{i_k}(u_k-u_{k-1})$$

$$H^{(n_{k+1})}_{S-(i_1+\ldots+i_k+j_1+\ldots+j_k),S-(i_1+\ldots+i_k+j_1+\ldots+j_{k+1})}(v-u_k)$$

$$\lambda e^{-\lambda(x-v)}\binom{S-(i_1+\ldots+i_k+j_1+\ldots+j_{k+1}}{j_{k+2}}p^{S-(i_1+\ldots+i_k+j_1+\ldots+j_{k+2})}$$

$$(1-p)^{j_{k+2}}\ [1-F(t-u_k)] \quad e^{-\lambda(t-x)} \quad dxdvdu_k du_{k-1}\ldots du_2 du_1 \qquad \ldots(2)$$

and

$$F_2[S,S,t] = \sum_{i_1,\dots,i_{k+1}>0}\ \sum_{j_1,\dots,j_{k+1}\geq 0}\ \sum_{n_1,\dots,n_{k+1}\geq 0}\ \sum_{\substack{i_1+\dots+i_k+j_1+\dots+j_{k+1}<M \\ i_1+\dots+i_{k+1}+j_1+\dots+j_{k+1}\geq M}}$$

$$\int_{u=0}^{\infty}\ \int_{u_1=u}^{t}\cdots\int_{u_k=u_{k-1}}^{t}\ \int_{w=u_k}^{t} H_{S,S-j_1}^{(n_1)}(u_1)\frac{g_{i_1}(u+u_1)}{1-F(u)}$$

$$H_{S-(i_1+j_1),S-(i_1+j_1+j_2)}^{(n_2)}(u_2-u_1) \quad g_{i_2}(u_2-u_1)\dots$$

$$H_{S-(i_1+\dots+i_{k-1}+j_1+\dots+j_{k-1}),S-(i_1+\dots+i_{k-1}+j_1+\dots+j_k)}^{(n_k)}(u_k-u_{k-1})$$

$$g_{i_k}(u_k-u_{k-1})H_{S-(i_1+\dots+i_k+j_1+\dots+j_k),S-(i_1+\dots+i_k+j_1+\dots+j_{k+1})}^{(n_{k+1})}(w-u_k)$$

$$g_{i_{k+1}}(w-u_k)\ [1-F(t-w)]\ e^{-\lambda(t-w)} \quad dwdu_kdu_{k-1}\dots du_2du_1d_u$$

$$+$$

$$\sum_{i_1,\dots,i_k>0}\ \sum_{j_1,\dots,j_{k+2}\geq 0}\ \sum_{n_1,\dots,n_{k+1}\geq 0}\ \sum_{\substack{i_1+\dots+i_k+j_1+\dots+j_{k+1}<M \\ i_1+\dots+i_k+j_1+\dots+j_{k+2}\geq M}}$$

$$\int_{u=0}^{\infty}\ \int_{u_1=u}^{t}\cdots\int_{u_k=u_{k-1}}^{t}\ \int_{v=u_k}^{t}\ \int_{x=v}^{t} H_{S,S-j_1}^{(n_1)}(u_1)\frac{g_{i_1}(u+u_1)}{1-F(u)} \quad \dots$$

$$H_{S-(i_1+\dots+i_{k-1}+j_1+\dots+j_{k-1}),S-(i_1+\dots+i_{k-1}+j_1+\dots+j_k)}^{(n_k)}(u_k-u_{k-1})\ g_{i_k}(u_k-u_{k-1})$$

$$H_{S-(i_1+\dots+i_k+j_1+\dots+j_k),S-(i_1+\dots+i_k+j_1+\dots+j_{k+1})}^{(n_{k+1})}(v-u_k)$$

$$\lambda e^{-\lambda(x-v)} \binom{S-(i_1+\ldots+i_k+j_1+\ldots+j_{k+1}}{j_{k+2}} p^{S-(i_1+\ldots+i_k+j_1+\ldots+j_{k+2})}(1-p)^{j_{k+2}}$$

$$[1-F(t-u_k)] \quad e^{-\lambda(t-x)} \quad dx dv du_k du_{k-1} \ldots du_2 du_1 d_u \text{ ------(3)}$$

The right hand side of equation (2) is arrived at as follows. Initially the inventory level is S. We take this as the time origin. Then n_1 disasters take place until time u_1 (first demand epoch) which altogether destroy j_1 units, the demand that takes place at time u_1 take away i_1 units and the inventory position at time u_1 just after meeting the demands and disasters in between is $S-(i_1+j_1)$ (>s). Again n_2 disasters take place until time u_2, destroy j_2 units, the demand at u_2 takes i_2 units and inventory level at u_2 is $S-(i_1 + i_2 + j_1 + j_2)$ (>s). Proceeding in this way, a total of k demands and $n_1+\ldots+n_{k+1}$ disasters take away atmost (M-1) units until demand epoch w (the demand at w takes i_{k+1} units) at which the inventory level drops to the ordering set. Hence the first part of equation (2).

For getting the second part of equation (2) proceed in the same way as mentioned above. Total of k demands and $n_1+\ldots+n_{k+1}$ disasters until time v take away atmost (M-1) units and due to a disaster during (v, x) inventory level drops to the ordering set.

The only difference in arriving at equation (3) is that initially due to a disaster inventory level drops to the ordering set. Identify this epoch as the initial time and an order is placed. At this, u time units has elapsed since the last demand epoch. Total of n_1 disasters takes place until time u_1 which together take away j_1 units. The first demand after the replenishment takes place at u_1 due to which the inventory level is down by i_1 units. Proceed assigning like this to arrive at (3).

Now we define

$$R[S,S,t] = \sum_{n=0}^{\infty} F^{*n}[S,S,t]$$ which is the expected number of visits to S in (0, t] starting initially at S.

2.2.2. Time Dependent System State Probabilities :

Defining P(i, j, t) = P [X (t) = j /X (0+) = i] with i , j ∈ E. We see that

that P (i, j, t) satisfies the Markov renewal equations (Cinlar 1975). Thus

$$P[S, j, t] = \Pr[X(t) = j, T_1 > t / X(o+) = S] +$$

$$\Pr[X(t) = j, T_1, \leq t / X(0+) = S]$$

$$= L(S,j,t) + \int_0^t F(S,S,du) P(S,j,t-u)$$

$$where \quad L(S,j,t) = \int_0^t \sum_{m=0}^{\infty} H_{S,j}^{(m)}(u)(1-F(u))du \;, \; j = s+1,\ldots,S$$

and the solution is given by

$$P(S,j,t) = \int_0^t R(S,S,du)\, L(S,j,t-u) \qquad for\; j = s+1,\ldots,S$$

----------(4)

2.2.3. Steady State Analysis

In order to obtain the limiting distribution of the stock level, consider the Markov chain $\{X_n,\ n\in(1,2,3,\ldots)\}$ associated with Markov renewal process (X,T). The transition probability matrix $P = ((p(i,j)))$ of order M, where p(i,j) is given by

$$p(i,j) = \int_{u=0}^{\infty} \sum_{k\in E} \sum_{m=0}^{\infty} H_{i,k}^{(m)}(u)\ g_{k-j}(u)\ dF(u) \qquad ---(5)$$

The following lemma gives a necessary and sufficient condition for the chain to be irreducible.

Lemma:

The necessary and sufficient condition for the chain $\{X_n\ ,n\in\ (1,2,3,\ldots)\}$ to be irreducible is that $g_1(u) \neq 0$ for some interval in $[0, \infty]$.

Proof :

If $g_1(u) = 0$ almost everywhere, then column of transition probability matrix corresponding to state S-1 becomes a null vector, as such the state S-1 is inaccessible from any other state. Thus, the Markov chain becomes reducible which proves the necessary part of the lemma.

To prove sufficiency, we assume $g_1(u) \neq 0$ for some interval in $[0,\infty]$. Then we have $p(i, j) > 0$. Thus every state is accessible from all states. Hence Markov chain is irreducible and pocesses a unique stationary distribution $\bar{\pi} = (\pi_{s+1},\ldots,\pi_S)$ which satisfies $\bar{\pi}\mathbf{P}=\bar{\pi}$ and $\bar{\pi}\underline{\mathbf{e}}=\mathbf{1}$.

Let $q_j = \lim_{t\to\infty} P(i,j,t)$ be the limiting distribution of the stock level.

Theorem:

. If $g_1(u) \neq 0$ for some interval in $[0,\infty]$ and F(t) is absolutely continuous with E(X)< ∞. Then

$$q_n = \frac{\sum_{\substack{j\in E \\ j\geq n}} \pi_j \int_0^\infty L(j,n,t)dt}{\sum_{j\in E} \pi_j m_j}$$ where m_j is the mean sojourn time in state j.

Proof :

We have $g_1(u) \neq 0$, it follows that the Markov chain $\{X_n, n\in (1,2,3,\ldots)\}$ is irreducible and recurrent. Hence the Markov renewal process (X ,T) becomes irreducible and recurrent. It is aperiodic also. Thus from Cinlar (1975)

$$q_n = \frac{\sum_{\substack{j\in E \\ j\geq n}} \pi_j \int_0^\infty L(j,n,t)dt}{\sum_{j\in E} \pi_j m_j},$$ where m_j is the mean sojourn time in state j.

Special Case : No disaster occurs.

We have $\lambda \to 0$, So

$p(i,j) = \int_{u=0}^{t} g_{i-j}(u)dF(u)$. Then transition probability matrix is

$$P = \begin{bmatrix} 0 & 0 & 0 & \beta_1 \\ \beta_1 & 0 & 0 & \beta_2 \\ . & . & 0 & \\ \beta_{M-2} & \beta_{M-3} & 0 & \beta_{M-1} \\ \beta_{M-1} & \beta_{M-2} & \beta_1 & \beta_M \end{bmatrix} \text{where } \beta_i = \int_{u=0}^{t} g_i(u)\,dF(u)$$

The stationary distribution π can be obtained by normalizing **W**=($W_{s+1},\ldots, W_{s+M}$) where **W** is determined by solving **WP=W** ----(a). The last column of **P** can be deleted as it is reduntant in computing W s. Taking W_{s+M} =1 the system of equations (a) can be rewritten as

$$\begin{bmatrix} 1 & -\beta_1 & -\beta_2 & -\beta_{M-2} \\ 0 & 1 & -\beta_1 & -\beta_{M-3} \\ 0 & 0 & 1 & -\beta_{M-4} \\ . & . & & \\ 0 & 0 & . & . \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} W_{s+1} \\ W_{s+2} \\ W_{s+3} \\ \\ \\ W_{s+M-1} \end{bmatrix} = \begin{bmatrix} \beta_{M-1} \\ \beta_{M-2} \\ \beta_{M-3} \\ \\ . \\ \beta_1 \end{bmatrix} \quad ---(b)$$

Let γ_j be the discrete analogue of the sequence $\{\beta_j, j \geq 1\}$. Then we have (Feller)

$\gamma_j = \sum_{k=1}^{\infty} \beta_j^{(k)}$ where $\beta_j^{(k)}$ is the k fold convolution of β_j with itself and also

$$\gamma_j = \beta_j + \sum_{k=1}^{j-1} \gamma_k \beta_{j-k}, \quad j = 2,3,\ldots.$$

$$\gamma_1 = \beta_1 \qquad ------(c)$$

The set of equations (c) imply that

$$\beta_j = \gamma_j - \sum_{k=1}^{j-1} \gamma_k \beta_{j-k} \qquad , j = 2,3,\ldots$$

$$\beta_1 = \gamma_1$$

which can be written for j = 1, 2, … , M as

$$\begin{bmatrix} 1 & -\beta_1 & -\beta_2 & & -\beta_{M-2} \\ 0 & 1 & -\beta_1 & & -\beta_{M-3} \\ 0 & 0 & 1 & & -\beta_{M-4} \\ . & . & & & \\ 0 & 0 & . & & . \\ 0 & 0 & 0 & & 1 \end{bmatrix} \begin{bmatrix} \gamma_{M-1} \\ \gamma_{M-2} \\ \gamma_{M-3} \\ \\ \\ \gamma_1 \end{bmatrix} = \begin{bmatrix} \beta_{M-1} \\ \beta_{M-2} \\ \beta_{M-3} \\ \\ . \\ \beta_1 \end{bmatrix} \qquad ---(d)$$

Hence the system of equations (b) has the following solution :

$$W_{s+j} = \gamma_{M-j} \ , \qquad j = 1,2,\ldots,M-1 \, and \, W_{s+M} = 1.$$

Therefore, we get

$$\pi_{s+j} = W_{s+j} \Big/ \left(\sum_{k=1}^{M} W_{s+k} \right) \qquad , j = 1,2,\ldots,M-1 \quad and$$

$$\pi_{s+M} = 1 \Big/ \left(\sum_{k=1}^{M} W_{s+k} \right)$$

We have

$$\sum_{k=1}^{M} W_{s+k} = 1 + \sum_{k=1}^{M-1} \gamma_{M+k}$$
$$= 1 + \sum_{k=1}^{M-1} \gamma_k$$
$$= 1 + R_{M-1}$$

where $R_j = \sum_{k=1}^{j} \gamma_k$ is the discrete analogue of the renewal function of the sequence $\{\beta_n, n \geq 1\}$. Hence we have

$$\pi_{s+j} = W_{s+j} \Big/ (1 + R_{M-1})$$

$$= \gamma_{M-j} \Big/ (1 + R_{M-1}) \quad , j = 1,2,\ldots,M-1 \qquad \textit{and}$$

$$\pi_{s+M} = 1 \Big/ (1 + R_{M-1})$$

The limiting distribution of the stock level is given by

$$q_n = \frac{\sum_{\substack{j \in E \\ j \geq n}} \pi_j \int_0^\infty g_{j-n}(t)\, dF(t)}{\sum_{j \in E} \pi_j m_j} \quad , \qquad n = s+1,\ldots,S$$

2.2.4. Optimization Problem

For any inventory model, the decision variables are to be so chosen that the objective function associated with the model attains the minimum value at these values of the decision variables. Here the objective function is the total expected cost per unit time in the steady state. The decision variables s and S should be so chosen that the objective function is minimum for those values of s and S.

Let T be the time duration between two consecutive S to S transition. F [S, S, t] denotes the distribution of the time duration T between two consecutive S to S transitions. Using this we can calculate the expected length of a cycle E (T). Hence the expected number of orders placed per unit time is 1/E (T).

Let Z be the fixed ordering cost for the commodity. The expected cost of ordering for the commodity per unit time is Z/ E (T). The holding cost of the commodity per unit time is $h\left(\sum_{j=s+1}^{S} q_j\right)$ where h is the holding cost per unit per unit time. For calculating procurement cost, we consider the probability of inventory level dropping to j from i ($j \in E$) due to a demand

$$= \int_0^\infty \frac{1/\mu}{1/\mu + \lambda}\; g_{i-j}(u)\, dF(u)$$

$$= \int_0^\infty \frac{1}{1+\lambda\mu}\; g_{i-j}(u)\, dF(u)$$

and probability of inventory level dropping to j from i due to a disaster

$$= \int_0^\infty \frac{\lambda}{\lambda + 1/\mu} \binom{i}{j} p^{i-j} (1-p)^j \ \lambda e^{-\lambda u} du$$

Total procurement cost per unit time is

$$r\sum_{j=0}^{s} \left((s-j)+M\right) \int_0^\infty \frac{1}{1+\lambda\mu} g_{i-j}(u) dF(u) +$$

$$r\sum_{j=0}^{\infty} \left((s-j)+M\right) \int_0^\infty \frac{\lambda}{\lambda + 1/\mu} \binom{i}{j} p^{i-j} (1-p)^j \lambda e^{-\lambda u} du$$

Where r is the unit procurement cost of the item. The total expected cost for the system is

$$= \frac{Z}{E(T)} + h\left(\sum_{j=s+1}^{S} q_j\right) + \ r\sum_{j=0}^{s} \left((s-j)+M\right) \int_0^\infty \frac{1}{1+\lambda\mu} g_{i-j}(u) dF(u) +$$

$$r\sum_{j=0}^{\infty} \left((s-j)+M\right) \int_0^\infty \frac{\lambda}{\lambda + 1/\mu} \binom{i}{j} p^{i-j} (1-p)^j \lambda e^{-\lambda u} du \qquad ------(6)$$

2.3. MODEL II

Following notations are used in this model.

$G_{i,k}(u)$ = Probability that starting with i units inventory level reaches k (with or without replenishment in between) at time u, as a consequence of one demand in (u, u+du))

$$= \begin{cases} g_{i-k} \mu e^{-u\mu} & for \quad i > k > s \\ \\ g_{<i-s>} \mu e^{-u\mu} & for\ k = S \end{cases}$$

$$G_{i,k}^{(m)}(u) = G_{i,k}^{*m}(u)$$

$$G_{i,k}^{(m)}(u) = \sum_{m=0}^{\infty} \sum_{r=0}^{\infty} \sum_{\substack{l_1+\ldots+l_m \le rM+i-k \\ l_1,\ldots,l_m \ge 0}} g_{l_1} \ldots g_{l_m} \frac{e^{-\lambda u}(\lambda u)^m}{m!}$$

$$with\ G_{i,k}^{(0)}(u) = e^{-\lambda u}$$

p_u = probability of a unit being destroyed due to a disaster when time elapsed since the previous disaster is u

g_l = probability that l units are demanded by an arrival

$g_{<l>}$ = probability that atleast l units are demanded by an arrival

$h_r(u)$ = r units destructed at a disaster epoch when u time units elapsed since the occurrence of last disaster

In this model, we assume the inter arrival times of disasters to have general distribution F (.) with mean λ (assumed finite) and the quantity destructed depends on the time elapsed between disasters. We assume probability of a unit being destroyed due to a disaster when u time units elapsed since the previous disaster as p_u. Demands form a compound Poisson process with inter arrival times of demands having mean 1/μ

2.3.1. Analysis of the Model

Let $0 = T_0 < T_1 < \ldots$ be the times at which disasters occur and $Y_0, Y_1, \ldots$ be the corresponding inventory levels, immediately after the initial, first, … disasters. i.e .

$Y(T_n+) = Y_n$, n = 0, 1, 2, 3,……Then

Theorem :

$(Y, T) = \{(Y_n, T_n), n = 0, 1, 2, \ldots\}$ forms a Markov renewal process (MRP) with semi-Markov Kernal,

$$Q(i, j, t) = P[\, Y_{n+1} = j,\ T_{n+1} - T_n \le t \ /\ Y_n = i],\ i, j \in E,\ t \ge 0,$$

Q(i, j, t) represents the transition probability from i to j in time less than or equal to t. We have

$$Q(i,j,t) = \int_{u=0}^{t} \sum_{\substack{k\in E \\ k\geq j}} \sum_{m=0}^{\infty} G_{i,k}^{(m)}(u) \binom{k}{j} p_u^{\,k-j}(1-p_u)^j f(u)du$$

and

$$Q(i,S,t) = \int_{u=0}^{t} \sum_{k\in E} \sum_{m=0}^{\infty} G_{i,k}^{(m)}(u) \sum_{j=0}^{s} \binom{k}{j} p_u^{\,k-j}(1-p_u)^j f(u)du$$

---------(7)

The right hand side of equation. (7) is obtained as follows:

The inventory level immediately after a disaster is i.It moves to k at time u, as a consequence of m demands in (0, u), k-j units are destroyed due to a disaster when time elapsed since the previous disaster is u.

Next we obtain the expression for the probability distribution F(S, S, t) of the time between two consecutive S to S transitions. The S to S transition can occur in two mutually exclusive ways. Consider the epoch at which inventory level is brought to S due to a disaster. Then the next passage to S can be due to either

(i) k disasters and intermediate $m_1 + \dots + m_{k+1}$ demands that take away atmost (M-1) units and due to the next disaster the inventory level drops to the ordering set.Or

(ii) k disasters and $m_1 + \ldots + m_{k+1}$ demands take away atmost (M-1) units and due to the next demand the inventory level drops to the ordering set. We denote the distribution of the duration of this time by $F_1(S, S, t)$.

Again, due to a demand the inventory level drops to the ordering set and an order is placed and replenishment occurs at instant of commencement of inventory. Then next passage to S can be due to either :

(i) k disasters and intermediate $m_1 + \ldots + m_{k+1}$ demands that take away atmost M-1 units and due to the next disaster that take the inventory level to the ordering set. Or

(ii) k disasters and intermediate $m_1 + \ldots + m_{k+1}$ demands that take away atmost M-1 units and due to the next demand the inventory level drops to the ordering set. The distribution function in this case is represented by $F_2(S, S, t)$.

Hence $F(S, S, t) = F_1(S, S, t) + F_2(S, S, t)$ Where

$$F_1(S,S,t) = \int_{t_1=0}^{t} \int_{t_2=t_1}^{t} \cdots \int_{w=t_k}^{t} \sum_{l_1,\ldots l_{k+1} \geq 0} \sum_{r_1,\ldots,r_{k+1} \geq 0} \sum_{m_1,\ldots,m_{k+1} \geq 0} \sum_{\substack{r_1+\ldots+r_k+l_1+\ldots+l_{k+1} < M \\ r_1+\ldots+r_{k+1}+l_1+\ldots+l_{k+1} \geq M}}$$

$$G_{S,S-l_1}^{(m_1)}(t_1)\, h_{r_1}(t_1)\; G_{S-(r_1+l_1),S-(r_1+l_1+l_2)}^{(m_2)}(t_2-t_1)\, h_{r_2}(t_2) \cdots$$

$$G_{S-(r_1+\ldots+r_{k-1}+l_1+\ldots+l_{k-1}),S-(r_1+\ldots+r_{k-1}+l_1+\ldots+l_k)}^{(m_k)}(t_k-t_{k-1})$$

$$h_{r_k}(t_k)\; G^{(m_{k+1})}_{S-(r_1+\dots+r_k+l_1+\dots+l_k),\,S-(r_1+\dots+r_k+l_1+\dots+l_{k+1})}(w-t_k)$$

$$h_{r_{k+1}}(w)\,(1-F(t-w))e^{-\lambda(t-w)}dw\,dt_k\,dt_{k-1}\dots dt_2\,dt_1$$

$$+$$

$$\int_{t_1=0}^{t}\int_{t_2=t_1}^{t}\cdots\int_{v=t_k}^{t}\int_{x=v}^{t}\;\sum_{l_1,\dots l_{k+2}\ge 0}\;\sum_{r_1,\dots,r_k\ge 0}\;\sum_{m_1,\dots,m_{k+1}\ge 0}\;\sum_{\substack{r_1+\dots+r_k+l_1+\dots+l_{k+1}<M\\ r_1+\dots+r_k+l_1+\dots+l_{k+2}\ge M}}$$

$$G^{(m_1)}_{S,S-l_1}(t_1)\,h_{r_1}(t_1)\;G^{(m_2)}_{S-(r_1+l_1),\,S-(r_1+l_1+l_2)}(t_2-t_1)\,h_{r_2}(t_2)\;\cdots$$

$$G^{(m_k)}_{S-(r_1+\dots+r_{k-1}+l_1+\dots+l_{k-1}),\,S-(r_1+\dots+r_{k-1}+l_1+\dots+l_k)}(t_k-t_{k-1})$$

$$h_{r_k}(t_k)\,G^{m_{k+1}}_{S-(l_1+\dots+l_k+r_1+\dots+r_k),\,S-(l_1+\dots+l_{k+1}+r_1+\dots+r_k)}(v-t_k)$$

$$\frac{e^{-\lambda(x-v)}(\lambda(x-v))^{l_{k+2}}}{l_{k+2}!}\,g_{l_{k+2}}(1-F(t-t_k))e^{-\lambda(t-x)}dx\,dv\,dt_k\dots dt_2\,dt_1$$

-----(8)

$$F_2(S,S,t) = \sum_{l_1,\dots l_{k+1}\geq 0} \sum_{r_1,\dots,r_{k+1}\geq 0} \sum_{m_1,\dots,m_{k+1}\geq 0}$$

$$\sum_{\substack{l_1+\dots+l_{k+1}+r_1+\dots+r_k<M \\ l_1+\dots+l_{k+1}+r_1+\dots+r_{k+1}\geq M}} \int_{y=0}^{\infty} \int_{t_1=y}^{t} \int_{t_2=t_1}^{t} \cdots \int_{w=t_k}^{t} \cdot$$

$$G_{S,S-l_1}^{(m_1)}(t_1)\frac{h_{r_1}(y+t_1)}{1-F(y)}\; G_{S-(r_1+l_1),S-(r_1+l_1+l_2)}^{(m_2)}(t_2-t_1)\, h_{r_2}(t_2)\cdots$$

$$G_{S-(r_1+\dots+r_k+l_1+\dots+l_k),S-(r_1+\dots+r_k+l_1+\dots+l_{k+1})}^{(m_{k+1})}(w-t_k)$$

$$h_{r_{k+1}}(w)1-F(t-w)e^{-\lambda(t-w)}dwdt_k\dots dt_2dt_1dy$$

$$+$$

$$\sum_{l_1,\dots l_{k+2}\geq 0} \sum_{r_1,\dots,r_k\geq 0} \sum_{m_1,\dots,m_{k+1}\geq 0}$$

$$\sum_{\substack{l_1+\dots+l_{k+1}+r_1+\dots+r_k<M \\ l_1+\dots+l_{k+2}+r_1+\dots+r_k>M}} \int_{y=0}^{\infty} \int_{t_1=y}^{t} \int_{t_2=t_1}^{t} \cdots \int_{v=t_k}^{t} \int_{x=v}^{t}$$

$$G_{S,S-l_1}^{(m_1)}(t_1) \qquad \frac{h_{r_1}(y+t_1)}{1-F(y)}\;\dots\dots\dots$$

$$G_{S-(r_1+\dots+r_{k-1}+l_1+\dots+l_{k-1}),S-(r_1+\dots+r_{k-1}+l_1+\dots+l_k)}^{(m_k)}(t_k-t_{k-1})$$

$$h_{r_k}(t_k)\;\; G_{S-(r_1+\dots+r_k+l_1+\dots+l_k),S-(r_1+\dots+r_k+l_1+\dots+l_{k+1})}^{(m_{k+1})}(v-t_k)$$

$$\frac{e^{-\lambda(x-v)}(\lambda(x-v))^{l_{k+2}}}{l_{k+2}!} g_{l_{k+2}}(1-F(t-t_k))e^{-\lambda(t-x)} dx dv dt_k \ldots dt_2 dt_1 dy$$

---------(9)

The right hand side of Equation. (8) is arrived at as follows:

Initially, the inventory level is S. We take this as the time origin. Then m_1 demands take place until time t_1 (first disaster epoch) which altogether take away l_1 units and the disaster that takes place at t_1 destroys r_1 units. The inventory position at t_1, just after meeting the demands and removing the destroyed items due to disaster is $S-(r_1+l_1)$. Proceeding in this way, total of k disasters and $m_1+\ldots+m_{k+1}$demands take away atmost M-1 units prior to the last (in that cycle) disaster epoch W. Due to the disaster at W (which destroys r_{k+1} units) inventory level drops to the ordering set. Hence the first part of equation (8).

For the second part of equation (8) proceed in the same way as mentioned above. Total of k disasters and $m_1+\ldots\ldots+m_{k+1}$ demands until time V take away atmost M-1 units and due to the demand during (v, x) inventory level drops to the ordering set. Similarly we get the other two parts of the equation. (9).

Now, we define $R(S,S,t)=\sum_{n=0}^{\infty} F^{*n}(S,S,t)$

which is the expected number of visits to S in (0 , t] starting initially at S.

2.3.2. Time Dependent System State Probabilities:

Defining P[i, j, t] = P [Y (t) = j / Y (0+) = i] with i, j $\in$ E. The system state probabilities at time t satisfy the Markov renewal equation. (Cinlar 1975). Thus,

$$P[S, j, t] = P(Y(t) = j, T_1 > t / Y(0+) = S) + P(Y(t) = j, T_1 \leq t / Y(0+) = S)$$

$$= m(S,j,t) + \int_0^t F(S,S,du)P(S,j,t-u)$$

where $m(S,j,t) = \int_0^t \sum_{m=0}^{\infty} G_{S,j}^{(m)}(u)(1-F(u))du, \; j = s+1,\ldots,S$

And the solution is given by

$$P(S,j,t) = \int_0^t R(S,S,du)m(S,j,t-u) \quad for\; j = s+1,\ldots,S$$

2.3.3. Steady State Analysis

To get the limiting distribution of the inventory level probabilities, consider the Markov chain [Y_n, n $\in$ (1,2,3,.....)] associated with the MRP (Y,T). The transition probability matrix of order M is given by

v = ((v (i, j))) where v (i, j) is given by

$$v(i,j) = \int_{u=0}^{\infty} \sum_{k\in E} \sum_{m=0}^{\infty} G_{i,k}^{(m)}(u)h_{k-j}(u)du \qquad and$$

$$h_{k-j}(u) = \int_0^{\infty} \binom{k}{j} p_u^{\,k-j}(1-p_u)^j f(u)du \qquad ----(10)$$

Lemma:

The necessary and sufficient condition for the Markov chain $[Y_n,\ n \in (1, 2, 3, \ldots)]$ to be irreducible is that $h_1(u) \neq 0$ for some interval in $[0, \infty)$.

Proof:

If $h_1(u) = 0$, then column of transition probability matrix corresponding to the state S-1 becomes a null vector so that Markov chain becomes reducible, which is the necessity part. To prove sufficiency we assume $h_1(u) \neq 0$, then $v(i, j) > 0$. Thus every state is accessible from all other states. So Markov chain is irreducible and possesses a unique distribution $\bar{\pi} = (\pi_{s+1}, \ldots, \pi_S)$ *which satisfies* $\bar{\pi} v = \bar{\pi}$ & $\bar{\pi} \underline{e} = 1$

Let $y_j = \lim_{t \to \infty} v(i, j, t)$ is the limiting distribution of the stock level.

Theorem.

If $h_1(u) \neq 0$ for some interval in $[0,\infty)$, and F(t) is absolutely continuous with $E(X) < \infty$, Then

$$y_n = \frac{\sum_{\substack{j \in E \\ j \geq n}} \pi_j \int_0^\infty m(j,n,t)\,dt}{\sum_{j \in E} \pi_j m_j}$$

where m_j is the mean sojourn time in state j .Proof follows easily from Cinlar (1975)

2.3.4. Optimization problem

Let T be the time duration between two consecutive S to S transition . F (S, S, t) denotes the distribution of time duration T . Using expression (8) we can calculate the expected length of the cycle. The expected number of orders placed per unit time is

1/ E (T). Let Z be the ordering cost for the commodity. The expected cost of ordering per unit time is Z / E(T). The holding cost per unit time is $h\sum_{j\in E} y_j$, Where h is the holding cost per unit per unit time. For calculating procurement cost , the probability of inventory level dropping from i to j due to a demand $=\int_0^\infty \frac{\mu}{\mu+1/\lambda} g_{i-j}\mu e^{-u\mu}du$ and probability of inventory level dropping to j from i due to a disaster

$$=\int_0^\infty \frac{1/\lambda}{1/\lambda+\mu}\binom{i}{j} p_u^{\,i-j}(1-p_u)^j dF(u)$$

$$=\int_0^\infty \frac{1}{1+\lambda\mu}\binom{i}{j} p_u^{\,i-j}(1-p_u)^j dF(u)$$

Total procurement cost per unit time is =

$$r\left\{\sum_{j=0}^{s}((s-j)+M)\int_0^\infty \frac{\mu}{\mu+1/\lambda} g_{i-j}\mu e^{-u\mu}du\right\}+$$

$$r\left\{\sum_{j=0}^{\infty}((s-j)+M)\int_0^\infty \frac{1}{1+\lambda\mu}\binom{i}{j} p_u^{\,i-j}(1-p_u)^j dF(u)\right\}$$

The total expected cost for the system is $=\frac{Z}{E(T)}+h\left(\sum_{j=s+1}^{S} y_j\right)+$

$$r\left\{\sum_{j=0}^{s}((s-j)+M)\int_0^\infty \frac{\mu}{\mu+1/\lambda} g_{i-j}\mu e^{-u\mu}du\right\}+$$

$$r\left\{\sum_{j=0}^{\infty}((s-j)+M)\int_0^\infty \frac{1}{1+\lambda\mu}\binom{i}{j} p_u^{\,i-j}(1-p_u)^j dF(u)\right\}$$

CHAPTER III

SOME CORRELATED INVENTORY MODELS WITH LEAD TIME

3.1. INTRODUCTION :

In this chapter, we have introduced correlation in (s, S) inventory problems in two different ways. In Model I, we analyze correlated order quantity. In Model II, the effect of correlation between order quantity and replenishment quantity is studied. Some details concerning correlated inventory problems can be found in Thangarag and Ramanarayanan (1983). They discuss an inventory system with random lead time with two reordering levels. Ramanarayanan and Jacob (1986) consider the same problem with zero lead time and varying reordering levels. Inventory system with varying reordering levels and random lead time is discussed Krishnamoorthy and Manoharan (1991). They obtained the time dependent probability distribution of the inventory level and the correlation between the number of demands during the lead time and the length of the next inventory dry period.

In the first model, we consider a continuous review, single commodity inventory problem under (s, S) policy with the modification that any two consecutive order quantities are correlated. The demand forms renewal process with distribution function G(.) with mean μ (assumed finite). Due to a demand at time zero, the inventory level falls to s and an order is placed for M units. It is assumed that whatever is ordered, gets replenished. Order quantities belong to the set {M-a, ..., M } for some positive integer a with M-a > s. Lead time is exponentially distributed with parameter λ.

The results of this Chapter have been presented in the International Conference on Stochastic Processes held at Cochin (1996).

In the second model, replenishment quantity need not be equal to the quantity ordered for, but they are correlated. Arrival of demands form a Poisson process with parameter λ. Whenever the inventory levels falls to s, for the first time, after the previous replenishment, an order is placed. Lead time follows an arbitrary distribution function F (.).

In Section 3. 2, we obtain the system state probabilities, limiting distribution and cost analysis of Model I. Analysis, system state probabilities, limiting distribution and cost analysis of Model II are provided in Section 3.3. The following notations are used in this chapter :

S	=	Maximum inventory level.
s	=	Reordering Point.
M	=	S - s
*	=	Convolution.
E	=	$\{M\text{-}a,\dots,M\}$, $M > a > 0$ and $M - a > s$.
E_1	=	$\{ 0, 1, \dots , s, \dots , S \}$.
δ_{ij}	=	$\begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}$
π_k	=	Probability that the order quantity in the steady state is k.
π_{ij}	=	Probability that the order quantity and replenishment quantity in the steady state are i and j respectively. $i, j \in E$

3.2. MODEL I

3.2.1. Analysis of the Model :

Let $0 = T_0, T_1, \dots$ be the epochs at which the initial, first, ... orders are placed for replenishment. $Y_0, Y_1, \dots$ be the quantity ordered at these epochs. $I_0, I_1, \dots$ be the inventory level at these epochs.

Let $p_{ij} = P(Y_n = i, Y_{n+1} = j)$ $i, j \in E$

We obtain the expression for the probability distribution of time between two consecutive s to s transition. This event can occur in two mutually exclusive ways.

(1) during the transition from s to s, in time t, no dry period (inventory level does not drop to zero) due to demands during lead time.

(2) during the transition from s to s, in time t, inventory level drops to zero due to demands during lead time and so there is a dry period.

Hence,

$F((s, M), (s, j), t) = F_1((s, M), (s, j), t) + F_2((s, M), (s, j), t)$ where $F_1((s, M), (s, j), t)$ and $F_2((s, M), (s, j), t)$ correspond respectively to a transition from s to s in time t, when number of demands during lead time is less than s or greater than or equal to s.

We have
$$F_1(s,M), (s,j), t) = \int_{u=0}^{t} \int_{v=u}^{t} \int_{w=v}^{t} \sum_{i \in E} \sum_{k=0}^{s-1} g^{*k}(u) p_{ij} \lambda e^{-\lambda v} g^{*(i-k)}(w-u)/(1-G(v-u))\, dw\, dv\, du \qquad \text{---(1)}$$

$$F_2((s,M), (s,j), t) = \int_{u=0}^{t} \int_{v=u}^{t} \int_{w=v}^{t} \sum_{i \in E} \sum_{k \geq s} g^{*k}(u) p_{ij} \lambda e^{-\lambda v} g^{*(i-s)}(w-u)/(1-G(v-u))\, dw\, dv\, du \qquad \text{---(2)}$$

The right hand side of (1) is arrived at as follows. Due to a demand inventory level drops to s. A replenishment order is placed for j units at or prior to the elapse of t units of time, since the previous order placement with the order quantity i at the previous epoch. Then, k demands takes until time u (the k th being at u; where k is less than or equal to s-1). Then the replenishment of i units takes place in (v, v+dv) but no demand during this time. Now, the inventory level is s + i - k. Exactly i - k demands takes place in (v, w) which brings the inventory level to s. A similar argument yields the right hand side of (2), except that in this case, there is dry period.

Now define $R((s,M),(s,j),t) = \sum_{m=0}^{\infty} F^{*m}((s,M),(s,j),t)$ which is the Markov renewal equation.

3.2.2. Time Dependent System State Probabilities :

Without loss of generality, we may assume that at time $T_0 = 0$, the state of the system is $(I_0, Y_0) = (s, M)$ (assumed fixed). Consider the two dimensional process $Z(t)=\{I(t), Y(t)\}$. Then the process $\{Z(t), t \geq 0\}$ is a semi- Markov process with the state space $E_1 \times E$.

Defining

$P((s, M), (n, j), t) = P\{Z(t) = (n, j) / Z(0) = (s, M)\}$, we see that $P((s, M), (n, j), t)$ satisfies the Markov renewal equations (Cinlar 1975).

(1) For $n = 1, 2, \ldots, s$

$$P((s, M), (n, j), t) = P\{Z(t) = (n, j), T_1 > t / Z(0) = (s, M)\}$$
$$P\{Z(t) = (n, j), T_1 \leq t / Z(0) = (s, M)\}$$

$$= H^{(1)}((s,M),(n,j),t) + \int_0^t R((s,M),(s,j),du)$$
$$\left(G^{*(s-n)}(t-u) - G^{*(s-n+1)}(t-u)\right)e^{-\lambda(t-u)}du$$

where

$$H^{(1)}((s,M),(n,j),t) = \int_0^t g^{*(s-n)}(u)e^{-\lambda u}du$$

(2) For n = 0

$$P((s,M),(n,j),t) = H^{(2)}((s,M),(n,j),t) + \int_0^t R((s,M),(s,j),du) \sum_{l \geq s} \left(G^{*(l)}(t-u) - G^{*(l+1)}(t-u)\right) e^{-\lambda(t-u)} du$$

Where

$$H^{(2)}((s,M),(n,j),t) = \sum_{l \geq s} e^{-\lambda t} G^{*l}(t)$$

(3) For n = S - q, q = 0, 1, … , a

$$P((s,M),(n,j),t) = \delta_{Mj} H^{(3)}((s,M),(n,j),t) + \int_{u=0}^{t} \int_{v=u}^{t} \sum_{i \in E} R((s,M),(s,j),du) p_{ij} \lambda e^{-\lambda v} \left(G^{*(j+q-M)}(t-u) - G^{*(j+q-M+1)}(t-u)\right) dvdu$$

where

$$\delta_{Mj} H^{(3)}((s,M),(n,j),t) = \int_{u=0}^{t} \int_{v=u}^{t} \sum_{q=0}^{a} g^{*q}(u) \lambda e^{-\lambda v} /(1-G(v-u)) p_{Mj} dvdu$$

where we define $g^{*0}(u)$ as identically equal to one.

(4) For n = s+1, … , M

$$P(s,M),(n,j),t) = \delta_{Mj} H^{(4)}(s,M),(n,j),t) + \int_{u=0}^{t} \int_{v=u}^{t} \int_{w=v}^{t} \sum_{i \in E} R(s,M),(s,j),du) p_{ij} \sum_{l=0}^{s-1} g^{*l}(v-u) \lambda e^{-\lambda w} \left[G^{*(s+j-l-n)}(t-v) - G^{*(s+j-l-n+1)}(t-v)\right] dwdvdu$$

$$+\int_{u=0}^{t}\int_{v=u}^{t}\int_{w=v}^{t}\sum_{i\in E}R((s,M),(s,j),du)p_{ij}$$

$$\sum_{l\geq s}g^{*l}(v-u)\lambda e^{-\lambda w}\left[\frac{G^{*(j-n)}(t-w)-G^{*(j-n+1)}(t-w)}{1-G(w-v)}\right]dwdvdu$$

$$where\ \delta_{Mj}H^{(4)}((s,M),(n,j),t)=\int_{u=0}^{t}\int_{v=u}^{t}\sum_{l\leq s}g^{*l}(u)\left[\frac{\lambda e^{-\lambda v}}{1-G(v-u)}\right]p_{Mj}$$

$$\left[G^{*(S-l-n)}(t-v)-G^{*(S-l-n+1)}(t-v)\right]dvdu$$

(5) For n=M+1,…,S-a-1

$$P\{(s,M),(n,j),t\}=\ \delta_{Mj}H^{(5)}\{(s,M),(n,j),t\}+$$

$$\int_{u=0}^{t}\int_{v=u}^{t}\int_{w=v}^{t}\sum_{i\in E}R\{(s,M),(s,j),du\}p_{ij}$$

$$\sum_{l=0}^{s-1}g^{*l}(v-u)\lambda e^{-\lambda w}\left[\frac{G^{*(s+j-l-n)}(t-w)-G^{*(s+j-l-n+1)}(t-w)}{1-G(w-v)}\right]\ dwdvdu$$

$$where\ \delta_{Mj}H^{(5)}\{(s,M),(n,j),t\}=\int_{u=0}^{t}\int_{v=u}^{t}\sum_{l=0}^{s-1}g^{*l}(u)\lambda e^{-\lambda v}$$

$$\left[\frac{G^{*(S-l-n)}(t-v)-G^{*(S-l-n+1)}(t-v)}{1-G(v-u)}\right]dvdu$$

Hence the solution is given by

$$P\{(s,M),(n,j),t\} = \int_0^t R\{(s,M),(s,j),du\}\ H^{(1)}\{(s,M),(n,j),t-u\}$$

$$for\ n = 1,2,\ldots,s.$$

$$P\{(s,M),(n,j),t\} = \int_0^t R\{(s,M),(s,j),du\}\ H^{(2)}\{(s,M),(n,j),t-u\}$$

$$for\ n = 0.$$

$$P\{(s,M),(n,j),t\} = \int_0^t R\{(s,M),(s,j),du\}\delta_{Mj} H^{(3)}\{(s,M),(n,j),t-u\}$$

$$for\ n = S-q, q = 0,1,\ldots,a.$$

$$P\{(s,M),(n,j),t\} = \int_0^t R\{(s,M),(s,j),du\}\delta_{Mj} H^{(4)}\{(s,M),(n,j),t-u\}$$

$$for\ n = s+1,\ldots,M.$$

$$P\{(s,M),(n,j),t\} = \int_0^t R\{(s,M),(s,j),du\}\delta_{Mj} H^{(5)}\{(s,M),(n,j),t-u\}$$

$$for\ n = M+1,\ldots,S-a-1$$

3.2.3. Limiting Distribution :

Stationary distribution π can be computed using **πP=π** and **π$\underline{e}$=1** where π is a row vector of $(a+1)^2$ elements. $\underline{e} = (1, \ldots, 1)^T$ and **P** is the transition probability matrix of the Markov chain under consideration. The mean time to return to s starting from s is

$m_k = pm_k^{(1)} + (1-p)m_k^{(2)}$. Where $m_k^{(1)}$ is the mean time to return to s starting from s when order quantity is k with no dry period during lead time and $m_k^{(2)}$ is the mean time to return to s starting from s when the order quantity is k and there is dry period during lead time. Thus

$$m_k^{(1)} = \int_{t=0}^{\infty}\int_{u=0}^{t}\int_{v=u}^{t} t\sum_{l=0}^{s-1} g^{*l}(u)\lambda e^{-\lambda v}\left\{\frac{G^{*(k-l)}(t-v)-G^{*(k-l+1)}(t-v)}{1-G(v-u)}\right\}dvdudt$$

$$m_k^{(2)} = \int_{t=0}^{\infty}\int_{u=0}^{t}\int_{v=u}^{t} t\sum_{l\geq s} g^{*l}(u)\lambda e^{-\lambda v}\left\{\frac{G^{*(k-s)}(t-v)-G^{*(k-s+1)}(t-v)}{1-G(v-u)}\right\}dvdudt$$

and p = P (no dry period) = Probability that replenishment takes place at or prior to the (s- 1) th demand = $\int_0^{\infty}\sum_{k=0}^{s-1}(1-e^{-\lambda u})g^{*k}(u)du$

Let $q_n = \lim_{t\to\infty} P\{(s,M),(n,j),t\}$

Then, following Cinlar (1975) the limiting probabilities are given by

(1) For n = 0

$$q_n = \frac{\pi_k \int_0^{\infty} g^{*s}(t)e^{-\lambda t}dt}{\sum_{k=M-a}^{M}\pi_k m_k}$$

(2) For n = 1, 2, ... , s

$$q_n = \frac{\pi_k \int_0^{\infty} g^{*(s-n)}(t)e^{-\lambda t}dt}{\sum_{k=M-a}^{M}\pi_k m_k}$$

(3) For n = S - q, q = 0, 1, … , a.

$$q_n = \frac{\pi_k \int_0^\infty \left(G^{*(k-M+q)}(t) - G^{*(k-M+q+1)}(t)\right)\lambda e^{-\lambda t} dt}{\sum_{k=M-a}^{M} \pi_k m_k}$$

(4) For n = s + 1, … , M

$$q_n = \frac{\pi_k \int_0^\infty g^{*(s+k-n)}(t)\lambda e^{-\lambda t} dt}{\sum_{k=M-a}^{M} \pi_k m_k}$$

(5) For n = M+1, …, S– a -1

$$q_n = \frac{\pi_k \int_0^\infty g^{*(s+k-n)}(t)\lambda e^{-\lambda t} dt}{\sum_{k=M-a}^{M} \pi_k m_k}$$

3.2.4. Cost Analysis :

Let T be the time duration between two consecutive ordering points of the commodity. Then probability distribution of time between two consecutive ordering points of the commodity is given by F((s, M), (s, j), t). Then expected length of the cycle is

$$E_i(T) = \sum_{j\in E}\left\{\sum_{k=0}^{s-1}\left(\frac{k}{\mu} + \frac{i-k}{\mu}\right)\right\}p_{ij} + \sum_{j\in E}\left\{\sum_{k\geq s}\left(\frac{k}{\mu} + \frac{i-s}{\mu}\right)\right\}p_{ij}$$

Hence the expected number of orders placed per unit time is $1/E_i(T)$. Let k_1 be the fixed ordering cost for the commodity. The expected cost of ordering for the commodity per unit time is $k_1/E_i(t)$. Let h_1 be the holding cost of the commodity per unit time. The holding cost of the commodity per unit time is $h_1\left(\sum_{n=1}^{S} nq_n\right)$. Expected procurement cost is given by $r_1 \sum_{k=M-a}^{M} k\pi_k$, where r_1 is the unit procurement cost of the item. The total expected cost for the system per unit time is

$$k_1/E_i(t) + h_1\left(\sum_{n=1}^{S} nq_n\right) + r_1 \sum_{k=M-a}^{M} k\pi_k \qquad \text{-----------------(3)}$$

3.3 MODEL II :

In this model, we consider an (s, S) inventory policy in which ordering quantities and replenishment quantities are not the same but correlated. We assume that at time 0, due to a demand the inventory level fall to s, so that an order for replenishment by a quantity M is placed. Initial replenishment takes place for M units. Whenever the inventory level fall to s for the first time after the previous replenishment, an order is placed for j units, $j \in E$. The replenishment quantity need not be equal to the quantity ordered for, but they are correlated. Arrival of demands form a Poisson process with parameter λ. Lead time follows an arbitrary distribution F(.).

3.3.1. Analysis of the Model :

Let $0=T_0, T_1,\ldots,T_n,\ldots$ be the ordering epochs. $X_0, X_1,\ldots,X_n,\ldots$ be the ordering quntities at these epochs. $(X_i \in E,\ i=1, 2, \ldots, n, \ldots)$ and $Y_0, Y_1,\ldots,Y_n,\ldots$ be the replenishment quantities ($y_j \in E$, $j = 1, 2, \ldots, n, \ldots$) and $I_0, I_1, \ldots, I_n, \ldots$ be the inventory levels at these epochs ($I_l \in E_1$, $l = 1, 2, \ldots, n, \ldots$)

Let $p_{ij} = P(X_n = i, \; Y_n = j) \qquad i, j \in E$

We obtain the expression for the probability distribution of time between two consecutive s to s transition. This event can occur in two mutually exclusive ways as in Model I. Hence

$F((s, M, M), (s, i, j), t) = F_1((s, M, M), (s, i, j), t) + F_2((s, M, M), (s, i, j), t)$ where

$F_1((s, M, M), (s, i, j), t)$ and $F_2((s, M, M), (s, i, j), t)$ correspond respectively to a transition from s to s in time t, when number of demands during lead time is less than s, or greater than or equal to s. where

$$F_1((s,M,M),(s,i,j),t) = \int_{v=0}^{t} \int_{w=v}^{t} \sum_{k=0}^{s-1} p_{ij} \frac{e^{-\lambda v}(\lambda v)^k}{k!} f(v) \frac{e^{-\lambda(w-v)}(\lambda(w-v))^{j-k}}{(j-k)!} \, dw dv$$

and

$$F_2((s,M,M),(s,i,j),t) = \int_{v=0}^{t} \int_{w=v}^{t} \sum_{k \geq s} p_{ij} \frac{e^{-\lambda v}(\lambda v)^k}{k!} f(v) \frac{e^{-\lambda(w-v)}(\lambda(w-v))^{j-s}}{(j-s)!} \, dw dv$$

The right hand side of $F_1((s, M, M), (s, i, j), t)$ is arrived at as follows. Due to a demand inventory level drops to s. An order is placed for i units. Then k demands takes place in (0, v) and replenishment of j units occurs in (v, v + dv). Exactly (j-k) demands take place in (v, w) which brings the inventory level to s. A similar argument yields the right hand side of $F_2((s, M, M), (s, i, j), t)$ except that in this case,there is dry period during lead time. Now we define

$$R((s,M,M),(s,i,j),t) = \sum_{m=0}^{\infty} F^{*m}(s,M,M),(s,i,j),t)$$

which is the Markov renewal function.

3.3.2.Time Dependent System State Probabilities :

Initially at time T_0, we assume that state of the system $(I_0,X_0,Y_0) = (s, M, M)$. Consider the three dimensional process $Z(t) = \{I(t), X(t), Y(t)\}$. Then the process $\{Z(t), t\geq 0\}$ is a semi-Markov process with the state space $E_1 \times E \times E$.

Defining

$$P(s, M, M), (n, i, j), t) = P\ \{Z(t) = (n, i, j) / Z(0) = (s, M, M)\}$$

We see that P(s, M, M),(n ,i ,j),t) satisfies the Markov renewal equations.

(1) For n = 1, 2, ... , s

$$P(s,M,M),(n,i,j),t) = P((Z(t) = (n,i,j),T_1 > t / Z(0) = (s,M,M)) +$$
$$P((Z(t) = (n,i,j),T_1 \leq t / Z(0) = (s,M,M))$$

$$= H^{(1)}((s,M,M),(n,i,j),t) + \int_{u=0}^{t} R((s,M,M),(s,i,j),du)$$

$$(1-F(t-u)) \quad \frac{e^{-\lambda(t-u)}(\lambda(t-u))^{s-n}}{(s-n)!} du$$

where

$$H^{(1)}((s,M,M),(n,i,j),t) = \frac{e^{-\lambda t}(\lambda t)^{s-n}}{(s-n)!}(1-F(t))$$

(2) For n = 0

$$P((s,M,M),(n,i,j),t) = \quad H^{(2)}(s,M,M),(n,i,j),t) + \int_{u=0}^{t} R((s,M,M),(s,i,j),du)$$

$$\sum_{l \geq s} \frac{e^{-\lambda}(t-u)(\lambda(t-u)^{l}}{l!}(1-F(t-u))du$$

where

$$H^{(2)}((s,M,M),(n,i,j),t) = \sum_{l \geq s} \frac{e^{-\lambda t}(\lambda t)^{l}}{l!}(1-F(t))$$

(3) For n = S-q , q = 0, 1, … , a

$$P((s,M,M),(n,i,j),t) = \delta_{M\,j} H^{(3)}((s,M,M),(n,i,j),t) + \int_{u=0}^{t} \int_{v=u}^{t} R((s,M,M),(s,i,j),du) p_{ij} f(v)$$

$$\frac{e^{-\lambda(t-v)}(\lambda(t-v)^{j+q-M}}{(j+q-M)!} e^{-\lambda(v-u)} dv du$$

where

$$\delta_{Mj} H^{(3)}((s,M,M),(n,i,j),t) = \int_{u=0}^{t} \int_{v=u}^{t} \sum_{q=0}^{a} \frac{e^{-\lambda u}(\lambda u)^{q}}{q!}$$

$$p_{ij} \; f(v) e^{-\lambda(v-u)} dvdu$$

(4) For n = s+1, … , M

$$P((s,M,M),(n,i,j),t) = \delta_{Mj} \; H^{(4)}((s,M,M),(n,i,j),t) + \int_{u=0}^{t} \int_{v=u}^{t} R((s,M,M),(s,i,j),du)$$

$$\sum_{l=0}^{s-1} \frac{e^{-\lambda(v-u)}(\lambda(v-u))^{l}}{l!} p_{ij} f(v-u) \frac{e^{-\lambda(t-v)}(\lambda(t-v))^{s+j-l-n}}{(s+j-l-n)!} dvdu$$

$$+$$

$$\int_{u=0}^{t} \int_{v=u}^{t} R((s,M,M),(s,i,j),du) \sum_{l \geq s} \frac{e^{-\lambda(v-u)}(\lambda(v-u))^{l}}{l!} \; p_{ij} \; f(v-u)$$

$$\frac{e^{-\lambda(t-v)}(\lambda(t-v))^{j-n}}{(j-n)!} dvdu$$

where

$$\delta_{Mj} H^{(4)}(s,M,M),(n,i,j),t) = \int_{u=0}^{t} \sum_{l \leq s} \frac{e^{-\lambda u}(\lambda u)^{l}}{l!} \; f(u)$$

$$p_{Mj\,j} \; \frac{e^{-\lambda(t-u)}(\lambda(t-u))^{S-l-n}}{(S-l-n)!} du$$

(5) For n = M+1, … , S – a – 1

$$P((s,M,M),(n,i,j),t) = \delta_{Mj}\ H^{(5)}((s,M,M),(n,i,j),t) + \int_{u=0}^{t}\int_{v=u}^{t} R((s,M,M),(s,i,j),du)$$

$$\sum_{l=0}^{s-1} \frac{e^{-\lambda(v-u)}(\lambda(v-u))^{l}}{l!} p_{ij} f(v-u) \frac{e^{-\lambda(t-v)}(\lambda(t-v))^{s+j-l-n}}{(s+j-l-n)!} dvdu$$

where

$$\delta_{Mj} H^{(5)}(s,M,M),(n,i,j),t) = \int_{u=0}^{t} \sum_{l=0}^{S-1} \frac{e^{-\lambda u}(\lambda u)^{l}}{l!} f(u)\ p_{Mj}\ \frac{e^{-\lambda(t-u)}(\lambda(t-u))^{S-l-n}}{(S-l-n)!} du$$

Hence the solution is given by

$$P((s,M,M),(n,i,j),t) = \int_{0}^{t} R((s,M,M),(s,i,j),du) H^{(1)}((s,M,M),(n,i,j),t-u)$$

$$for\ n = 1,2,\ldots,s$$

$$P((s,M,M),(n,i,j),t) = \int_{0}^{t} R((s,M,M),(s,i,j),du) H^{(2)}((s,M,M),(n,i,j),t-u)$$

$$for\ n = 0$$

$$P((s,M,M),(n,i,j),t) = \int_{0}^{t} R((s,M,M),(s,i,j),du) \delta_{Mj}\ H^{(3)}((s,M,M),(n,i,j),t-u)$$

$$for\ n = S-q,\ q = 0,1,\ldots,a$$

$$P((s,M,M),(n,i,j),t) = \int_0^t R((s,M,M),(s,i,j),du)\delta_{Mj}\ H^{(4)}((s,M,M),(n,i,j),t-u)$$

$$\textit{for } n = s+1,\dots,M$$

$$P((s,M,M),(n,i,j),t) = \int_0^t R((s,M,M),(s,i,j),du)\delta_{Mj}\ H^{(5)}((s,M,M),(n,i,j),t-u)$$

$$\textit{for } n = M+1,\dots,S-a-1$$

3.3.3. Limiting Distribution :

Stationary distribution π_{ij} / (i,j) $\in$E) can be computed using $\pi\mathbf{P} = \pi$ and $\pi\,\underline{\mathbf{e}}=1$ where $\underline{e}= (1,\dots,1)^T$ and π is a row vector of $(a+1)^2$ elements. The mean time to return to s starting from s is $m_{ij} = q\,m_{ij}^{(1)} + (1-q)\,m_{ij}^{(2)}$ where $m_{ij}^{(1)}$ is the mean time to return to s starting from s when the order quantity and the replenishment quantity are i and j respectively and there is no dry period during lead time. $m_{ij}^{(2)}$ is the mean time to return to s starting from s and there is dry period during lead time and is given by

$$m_{ij}^{(1)} = t \int_{t=0}^{\infty}\int_{v=0}^{t}\int_{w=v}^{t}\sum_{k=0}^{s-1} p_{ij}\ \frac{e^{-\lambda v}(\lambda v)^k}{k!} f(v)\ \frac{e^{-\lambda(w-v)}(\lambda(w-v))^{j-k}}{(j-k)!}\ dwdvdt$$

$$m_{ij}^{(2)} = t \int_{t=0}^{\infty}\int_{v=0}^{t}\int_{w=v}^{t}\sum_{k\geq s} p_{ij}\ \frac{e^{-\lambda v}(\lambda v)^k}{k!} f(v)\ \frac{e^{-\lambda(w-v)}(\lambda(w-v))^{j-s}}{(j-s)!}\ dwdvdt$$

and q = P (no dry period) = Probability that replenishment takes place at or prior to the (s-1) th demand.

$$= \int_0^{\infty}\sum_{k=0}^{s-1}\frac{e^{-\lambda u}(\lambda u)^k}{k!}(1-F(u))du$$

Let $q_n = \lim_{t \to \infty} p\{(s,M,M),(n,i,j),t\}$

Then, following Cinlar (1975), the limiting probabilities are obtained as given below:

(1) For n = 0

$$q_n = \frac{\pi_{ij} \int_0^\infty \sum_{k \geq s} \frac{e^{-\lambda t}(\lambda t)^k}{k!}(1 - F(t))dt}{\sum_{i,j \in E} \pi_{ij} m_{ij}}$$

(2) For n = 1, 2, ... , s

$$q_n = \frac{\pi_{ij} \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^{s-n}}{(s-n)!}(1 - F(t))dt}{\sum_{i,j \in E} \pi_{ij} m_{ij}}$$

(3) For n = s + 1, ... , M

$$q_n = \frac{\pi_{ij} \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^{s+j-n}}{(s+j-n)!} f(t)dt}{\sum_{i,j \in E} \pi_{ij} m_{ij}}$$

(4) For n = M+1, ... , S – a - 1

$$q_n = \frac{\pi_{ij} \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^{s+j-n}}{(s+j-n)!} f(t)dt}{\sum_{i,j \in E} \pi_{ij} m_{ij}}$$

(5) For n = S - q, q = 0, 1, …, a

$$q_n = \frac{\pi_{ij} \int_0^{\infty} \frac{e^{-\lambda t} (\lambda t)^{s+j-n}}{(s+j-n)!} f(t)dt}{\sum_{i,j \in E} \pi_{ij} m_{ij}}$$

3.3.4. Cost Analysis :

The objective function corresponding to this model is the total expected cost per unit time under steady state. Let T_1 be the time duration between two consecutive order placement epochs. Distribution of time between two consecutive ordering points of the commodity is given by F((s, M, M), (s ,i, j), t). Then expected length of the cycle is given by

$$E_{i,j}(T_1) = \left(\left(\sum_{k=0}^{s-1} \frac{k}{\lambda} \right) + \left(\frac{j-k}{\lambda} \right) \right) p_{ij} + \left(\left(\sum_{k \geq s} \frac{k}{\lambda} \right) + \left(\frac{j-s}{\lambda} \right) \right) p_{ij}$$

Expected number of orders placed per unit time is 1 / $E_{i,j}$ (T_1). Then expected cost of ordering for the commodity per unit time is k_1/ $E_{i,j}$ (T_1) where k_1 is the fixed ordering cost for the commodity. Let h_1 be the holding cost per unit time. The holding cost of the commodity per unit time is $h_1 \sum_{n=1}^{S} n q_n$. Total procurement cost is given by $r(\sum_{j=M-a}^{M} j \sum_{i=M-a}^{M} \pi_{ij})$. Thus, the total expected cost per unit time under steady state is

$$k_1 / E_{i,j}(T_1) + h_1 \sum_{n=1}^{S} n q_n + r(\sum_{j=M-a}^{M} j \sum_{i=M-a}^{M} \pi_{ij}).$$

CHAPTER-IV

ANALYSIS OF GENERAL CORRELATED BULK DEMAND TWO COMMODITY INVENTORY PROBLEM

4.1 INTRODUCTION

Inventory systems of (s, S) type for single commodity have been studied quite extensively in the past. The details of the initial developments in this field can be found in Arrow, Karlin and Scarf (1958), Hadley and Whitin (1963), Veinott (1966), Srinivasan (1979), Sahin (1983), and Ramanarayan and Jacob (1987) consider single commodity inventory problem with random lead time under (s, S) policy and obtain second measures of effectiveness. Sahin (1979) examines the (s, S) policy for a continuous measurement item under constant lead time. Krishnamoorthy and Lakshmi [1991] deal with a single commodity inventory problem with Markov modulated demand quantities and obtain the long run system state distribution. They analyze a Markov decision process.

Sivazlian [1971] considers the stationary characteristics of a multi-commodity inventory problem. Krishnamoorthy, Lakshmi and Basha [1994] have dealt with a two commodity inventory problem with unit demand with no dependence, whatever between the commodities demanded, and provide a characterisation to the system state. They [1996] have also considered a two commodity inventory problem with Markov shift in the type of commodity demanded and derive the stationary distribution of the system state. They provide a characterization for the system state distribution to be uniform.

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Krishnamoorthy, Merlymole and Ravindranathan [1998] generalize this result to a bulk demand two commodity inventory problem.

In this chapter, we consider correlated bulk demand two commodity inventory problem with the commodities represented by C_1 and C_2 respectively. The (s_k, S_k) policy is followed for the commodity C_k (k = 1, 2). The probability that an arrival demands i units of C_1 and j units of C_2 is $P_{i,j}$ (i = 1, 2,..., a; j = 1, 2,..., b). The inter arrival times of demands are independent and identically distributed random variables following the distribution function G (.) with mean ($\mu < \infty$). No shortage is permitted. Two types of ordering policies are considered-individual ordering and joint ordering. In the former as soon as the inventory level of any commodity falls to or below its reordering level for the first time after the previous replenishment, an order for replenishment is placed for that commodity alone. In the joint ordering policy, whenever the inventory level of any commodity falls to or below its reordering level for the first time after the previous replenishment, an order for replenishment is placed for both so as to bring their levels S_1 and S_2 respectively. In both cases lead time is assumed to be zero.

Section 4.2 deals with the analysis of the models. Limiting distribution of the inventory level is computed in section 4.3. An optimisation problem is discussed in section 4.4. Numerical illustrations are provided in section 4.5. Section 4. 6 deals with the particular case of linear correlation.

Notations:

$X(t)$	=	Inventory level of C_1 at time t
$Y(t)$	=	Inventory level of C_2 at time t
T_n	=	nth demand epoch n = 0,1,2,... with $T_0 = 0$
X_n	=	$X(T_n+)$
Y_n	=	$Y(T_n+)$
M_k	=	$S_k - s_k$ (k = 1,2)

$I(t) = \{X(t), Y(t)\}$

$*$ = Convolution

$E_k = \{s_k + 1, \ldots, S_k\} \quad k = 1, 2.$

$E = E_1 \times E_2$

N = Set of non-negative integers.

$P_{i,j}$ = Probability that an arrival demands i units of C_1 and j units of C_2;

$i = 1, 2, \ldots, a; \quad j = 1, 2, \ldots, b.$

$$P_{i.} = \sum_{j=1}^{b} P_{i,j}$$

$$P_{.j} = \sum_{i=1}^{a} P_{i,j}$$

$$\delta_{[x]} = \begin{cases} 1 & \text{if } x \text{ is not an integer} \\ 0 & \text{otherwise} \end{cases}$$

$[k]$ = Largest integer in k.

$F_1(.,.,.)$ = The distribution of the time between two consecutive replenishments of C_1 in the individual ordering policy.

$F_2(.,.,.)$ = The distribution of the time between two consecutive replenishments of C_2 in the individual ordering policy.

$F_{12}(.,.,.)$ = The distribution of the time between two consecutive (S_1, S_2) to (S_1, S_2) transition in the joint ordering policy.

4.2 ANALYSIS OF THE MODELS :

We analyze two types of policies separately.

4.2.1. Individual Ordering Policy:

Here the replenishment is such that whenever the inventory level of anyone of the commodities fall to the level (s_k) or below due to demands after the previous replenishment, an order is placed and an instantaneous replenishment occurs for that commodity alone so as to bring the level back to S_k ($k = 1,2$). Suppose exactly r demands results in the replenishment of C_1. Thus (r-1) demands take away atmost ($S_1 - s_1$-1) units of C_1. Probability distribution of the time between two consecutive replenishments of C_1 is

$$F_1[(S_1,S_2),(S_1,\eta),t] = \sum_{r=[M_1/a]+\delta_{[M_1/a]}}^{M_1} \sum_{\ell=0}^{\infty} \sum_{\substack{j_1,\dots,j_r=1,\dots,b \\ j_1+\dots+j_r=\ell M_2+S_2-\eta}} \sum_{\substack{i_1,\dots,i_r=1,\dots,a \\ i_1+\dots+i_{r-1}<M_1 \\ i_1+\dots+i_r\geq M_1}} P_{i_1,j_1}\dots P_{i_r,j_r} G^{*r}(t)$$

Similarly the probability distribution of the time between two consecutive replenishments of C_2 is

$$F_2[(S_1,S_2),(\gamma,S_2),t] = \sum_{r=[M_2/b]+\delta_{[M_2/b]}}^{M_2} \sum_{m=0}^{\infty} \sum_{\substack{i_1,\dots,i_r=1,\dots,a \\ i_1+\dots+i_r=mM_1+S_1-\gamma}} \sum_{\substack{j_1,\dots,j_r=1,\dots,b \\ j_1+\dots+j_{r-1}<M_2 \\ j_1+\dots+j_r\geq M_2}} P_{i_1,j_1}\dots P_{i_r,j_r} G^{*r}(t)$$

For computing the time dependent system state probabilities, let $I(t) = \{X(t),Y(t)\}$ be the system state at time t. Suppose T_n, $n = 0,1,\dots,$ is the nth demand epoch with $T_0 = 0$. After the demand at T_0, suppose the inventory levels of C_1 and C_2 are brought back to S_1 and S_2 respectively. We have $I(t) = \{X(T_n+),Y(T_n+)\}$, for $T_n \leq t < T_{n+1}$. It is easily seen that $\{I(t),\ t \geq 0\}$ is a semi-Markov process on E and $\{X_n,Y_n\}$ $n \in N = \{X(T_n+), (T_n+)\}$ $n\in N$ is the embedded Markov renewal process on E. The system state probabilities at time t satisfy the equation, (In what follows we write $P\{(S_1,S_2), (\ell,q), t\}$for $P\{X(t),Y(t)) = (\ell,q)/ (X(0),Y(0)) = (S_1,S_2)\}$ with appropriate suffix for F to indicate whether the replenishment policy is individual or joint).

$$P\{(S_1,S_2),(\ell,q),t\} = H\{(S_1,S_2),(\ell,q),t\} + \int_0^t \sum_{\eta\in E_2} F_1\{(S_1,S_2),(S_1,\eta),du\}$$

$$P\{(S_1,\eta),(\ell,q),t-u\},(\ell,q),(S_1,S_2)\in E \qquad \text{------} \qquad (1)$$

where $H\{(S_1,S_2),(\ell,q),t\} = P\{(X(t),Y(t)) = (\ell,q)/(X(0),Y(0)=(S_1,S_2),$

$$l\neq S_1, X(u)\neq S_1 \, for \, 0 < u \leq t\}$$

Hence the time dependent system state probabilities are given by

$$P\{(S_1,S_2),(\ell,q),t\} = \int_0^t \sum_{\eta\in E_2} F_1\{(S_1,S_2),(S_1,\eta),du\}$$

$$P\{(S_1,\eta),(\ell,q),t-u\};\ (S_1,\eta),(\ell,q)\in E$$

Similarly

$$P\{(S_1,S_2),(\alpha,\beta),t\} = \int_0^t \sum_{\gamma\in E_1} F_2\{(S_1,S_2),(\gamma,S_2),du\}$$

$$P\{(\gamma,S_2),(\alpha,\beta),t-u\};\ (\gamma,S_2),(\alpha,\beta)\in E$$

4.2.2. Joint Ordering Policy:

Suppose exactly r demands result in a replenishment. Thus (r-1) demands take away at most (S_1-s_1-1) units of C_1 and (S_2-s_2-1) units of C_2. Then the probability distribution of the time between two consecutive transitions to (S_1, S_2) is

$$F_{12}\{(S_1,S_2),(S_1,S_2),t\} = \sum_{r=\min([M_1/a]+\delta[M_1/a],[M_2/b]+\delta[M_2/b])}^{\min(M_1,M_2\}}$$

$$\sum_{\substack{i_1,\ldots,i_r=1,\ldots,a;\ j_1,\ldots,j_r=1,\ldots,b \\ i_1+\ldots+i_{r-1}<M_1;\ j_1+\ldots+j_{r-1}<M_2 \\ either\ i_1+\ldots+i_r\geq M_1\ or\ j_1+\ldots+j_r\geq M_2}} P_{i_1,j_1,\ldots}P_{i_r,j_r} G^{*r}(t)$$

The system state probabilities at time t satisfies the equation

$$P\{(S_1,S_2),(\ell,q),t\} = H\{(S_1,S_2),(\ell,q),t\}+\int_0^t F_{12}\{(S_1,S_2),(S_1,S_2),du\}$$

$$P\{(S_1,S_2),(\ell,q),t-u\};\ (\ell,q),(S_1,S_2)\in E \text{ -----(2)}$$

where $H\{(S_1,S_2),(\ell,q),t\} = P\{(X(t),Y(t)) = (\ell,q)/(X(0),Y(0)=(S_1,S_2),$

$$\ell\neq S_1, q\neq S_2, X(u)\neq S_1.\ X(v)\neq S_2 \text{ for } 0<u\leq t,\ 0<v\leq t\}$$

Hence the time dependent system state probabilities are given by

$$P\{(S_1,S_2),(\ell,q),t\} = \int_0^t F_{12}\{(S_1,S_2),(S_1,S_2),du\}$$

$$P\{(S_1,S_2),(\ell,q),t-u\};\ (\ell,q),(S_1,S_2)\in E$$

4.3. LIMITING DISTRIBUTIONS

4.3.1. Individual Ordering Policy

Let $\lim_{t\to\infty} P\{(S_1,S_2),(\ell,q),t\} = A(\ell,q),(\ell,q)\in E.$ Note that these probabilities are independent of the initial state since in a finite state space, irreducible, aperiodic Markov chain, this characteristics holds and the Markov chain under study satisfies these conditions. In order to compute the limiting probabilities, the transition probability matrix P corresponding to the two dimensional Markov chain $\{X_n, Y_n\}$ $n\in N$ is to be obtained. Since the Markov chain $\{X_n, Y_n\}$ $n\in N$ is irreducible and aperiodic its stationary distribution, $\theta=\{\theta(\ell,q),(\ell,q)\in E\}$ can be computed using $\theta P = \theta$ and $\theta \underline{e} = 1$

where $\underline{e} = (1,1,\ldots,1)^T$ and θ is a row vector of M_1M_2 elements. The mean sojourn time in any state (ℓ,q) is

$$m(\ell,q)=\int_0^\infty \{1-G(t)\}dt=\mu(<\infty)$$

Thus.

$$A(\ell,q)=\frac{\theta(\ell,q)\int_0^\infty P[I(t)=(\ell,q),T_1>t|I(0)=(\ell,q)]dt}{\sum_{(\ell,q)\in E}\theta(\ell,q)m(\ell,q)}$$

$$=\theta(\ell,q)$$

Hence from the above expression,

$$\lim_{t\to\infty} P\{(S_1,S_2),(\ell,q),t\} = A(\ell,q)=\theta(\ell,q)$$

and are independent of the initial state as is expected from the theory of finite state irreducible aperiodic Markov chains.

4.3.2 Joint Ordering Policy:

Let $\lim_{t\to\infty} P\{(S_1,S_2),(\ell,q),t\} = Q(\ell,q),(\ell,q)\in E$. From the transition probability matrix P_1 of the Markov chain $\{X_n, Y_n\}$, its stationary distribution $\prod=\{\prod(\ell,q)|(\ell,q)\in E\}$ can be computed using $\Pi P_1 = \Pi$ and $\Pi\underline{e} = 1$ where $\underline{e} = (1,1,\ldots,1)^T$ and Π is a row vector of

M_1M_2 elements. We can easily see that the limiting probabilities of the system state are given by

$$Q(\ell,q) = \lim_{t\to\infty} P\{X(t),Y(t)\} = (\ell,q)\} = \prod (\ell,q), (\ell,q) \in E$$

4.4. OPTIMIZATION PROBLEM

4.4.1. Individual Ordering Policy:

The decision variables should be chosen so that the objective function associated with the model attains an optimal value at these chosen values of the decision variables. The objective function corresponding to this model is the total expected cost per unit time under steady state and the decision variables are S_1,s_1, S_2,s_2. Let U_1 be the time duration between two consecutive replenishments of C_1. Then

$$E(u_1) = \sum_{r=[M_1/a]+\delta_{[M_1/a]}}^{M_1} \sum_{\ell=0}^{\infty} \sum_{\substack{j_1,\ldots,j_r=1,\ldots,b \\ j_1+\ldots+j_r=\ell M_2+S_2-\eta}} \sum_{\substack{i_1,\ldots,i_r=1,\ldots,a \\ i_1+\ldots+i_{r-1}<M_1 \\ i_1+\ldots+i_r\geq M_1}} P_{i_1,j_1}\ldots P_{i_r,j_r} r\mu$$

Similarly, if U_2 is the time duration between two consecutive replenishments of C_2, then

$$E(u_2) = \sum_{r=[M_2/b]+\delta_{[M_2/b]}}^{M_2} \sum_{m=0}^{\infty} \sum_{\substack{i_1,\ldots,i_r=1,\ldots,a \\ i_1+\ldots+i_r=mM_1+S_1-\gamma}} \sum_{\substack{j_1,\ldots,j_r=1,\ldots,b \\ j_1+\ldots+j_{r-1}<M_2 \\ j_1+\ldots+j_r\geq M_2}} P_{i_1,j_1}\ldots P_{i_r,j_r} r\mu$$

The expected number of orders placed per unit time for C_1 is $1/E(U_1)$ and that for C_2 is $1/E(U_2)$. Expected quantity demanded of C_1 is $\sum_{i=1}^{a} ip_i$ per demand and that of C_2 is

$\sum_{j=1}^{b} jp_{.j}$. Hence the expected demand for C_1 per unit time is $\frac{1}{\mu}\sum_{i=1}^{a} ip_{i.}$ and that of C_2 is $\frac{1}{\mu}\sum_{j=1}^{b} jp_{.j}$. Let k_1 and k_2 be the fixed ordering costs of C_1 and C_2 respectively for individual ordering and V_1 and V_2 be the holding cost of one unit of C_1 and C_2 per unit time. Then the total average holding cost of C_1 and C_2 per unit time is

$$V(S_1,s_1,S_2,s_2)=V_1\sum_{\ell=s_1+1}^{S_1}\ell\sum_{q=s_2+1}^{S_2}\theta(\ell,q)+V_2\sum_{q=s_2+1}^{S_2}q\sum_{\ell=s_1+1}^{S_1}\theta(\ell,q)$$

Thus the total expected cost per unit time under steady state is

$$Z(S_1,s_1,S_2,s_2)=V(S_1,s_1,S_2,s_2)+\frac{k_1}{E(U_1)}+\frac{k_2}{E(U_2)}+r_1\left[\frac{1}{\mu}\sum_{i=1}^{a} ip_{i.}\right]+r_2\left[\frac{1}{\mu}\sum_{j=1}^{b} jp_{.j}\right]$$

where r_k is the unit procurement cost of item C_k, $k = 1,2$.

4.4.2 Joint Ordering Policy :

Let U_3 be the time duration between two consecutive replenishments. Then

$$E(u_3)=\sum_{r=\min\{[M_1/a]+\delta_{[M_1/a]},[M_2/b]+\delta_{[M_2/b]}\}}^{\min\{M_1,M_2\}}\ \sum_{\substack{i_1,\dots,i_r=1,\dots,a;\ j_1,\dots,j_r=1,\dots,b\\ i_1+\dots+i_{r-1}<M_1;\ j_1+\dots+j_{r-1}<M_2\\ \text{either } i_1+\dots+i_r\ge M_1 \text{ or } j_1+\dots+j_r\ge M_2}} P_{i_1,j_1}\dots P_{i_r,j_r}\,r\mu$$

Expected quantity demanded of C_1 is $\sum_{i=1}^{a} ip_{i.}$ per demand and that of C_2 is $\sum_{j=1}^{b} jp_{.j}$. Thus in a cycle the expected demand for C_1 is $\frac{E(U_3)}{\mu}\sum_{i=1}^{a} ip_{i.}$, and per unit time $\frac{1}{\mu}\sum_{i=1}^{a} ip_{i.}$ units of

C_1 on the average are demanded. Also the expected demand for C_2 per unit time is $\frac{1}{\mu}\sum_{j=1}^{b} jp_{.j}$. Let k be the fixed ordering cost. Suppose V_1 and V_2 are the holding costs of one unit of C_1 and C_2 respectively per unit time. Then the total average holding cost of

C_1 and C_2 per unit time is

$$V^1(S_1,s_1,S_2,s_2)=V_1\sum_{\ell=s_1+1}^{S_1}\ell\sum_{q=s_2+1}^{S_2}\pi(\ell,q)+V_2\sum_{q=s_2+1}^{S_2}q\sum_{\ell=s_1+1}^{S_1}\pi(\ell,q)$$

Thus the total expected cost per unit time under steady state is

$$Z^1(S_1,s_1,S_2,s_2)=V^1(S_1,s_1,S_2,s_2)+\frac{k}{E(U_3)}+r_1\left[\frac{1}{\mu}\sum_{i=1}^{a} ip_{i.}\right]+r_2\left[\frac{1}{\mu}\sum_{j=1}^{b} jp_{.j}\right]$$

where r_k is the unit procurement cost of item C_k, $k = 1,2$.

4.5.NUMERICAL ILLUSTRATION :

Consider the inventory system with $k = 10$, $k_1 = 10$, $k_2 = 12$, $r_1 = 6$, $r_2 = 8$, $v_1 = 2$, $v_2 = 1.5$, $a = 2$, $b = 3$ and mean of the distribution of the inter-arrival time of demands $\mu = 4$. For three sets of values of $P_{i,j}$ s ($i = 1, 2$; $j = 1, 2, 3$), expected value of the time duration between two consecutive replenishments and the average costs for individual ordering policy and joint ordering policy are computed and is given in Table – I. From the table we see that joint ordering policy is preferable.

Table – I

	S_1 s_1 S_2 s_2	P_{11} P_{12} P_{13} P_{21} P_{22} P_{23}	$E(U_1)$& $E(U_2)$ (for I.O) resp.	$E(U_3)_{J.O}$	(Average Cost)$_{I.O}$	(Average Cost)$_{J.O}$
I						
A	5 1 6 2	.2 .2 .2 .2 .1 .1	3.04960 2.59200	9.31200	28.14506	22.24794
B	5 1 6 2	.1 .2 .1 .1 .2 .3	2.62720 1.27600	8.6040	34.17717	23.27071
C	5 1 6 2	.3 .1 .1 .1 .3 .1	2.31490 1.49600	9.7480	32.56906	21.86949
II						
A	8 2 7 2	.2 .2 .2 .2 .1 .1	3.40720 2.7280	11.70400	32.39254	27.75051
B	8 2 7 2	.1 .2 .1 .1 .2 .3	3.56624 2.5360	10.2780	30.93859	19.86483
C	8 2 7 2	.3 .1 .1 .1 .3 .1	2.67632 4.03620	12.25960	31.69489	27.31797
III						
A	9 2 9 3	.2 .2 .2 .2 .1 .1	2.44704 2.69600	13.77005	36.86572	31.14606
B	9 2 9 3	.1 .2 .1 .1 .2 .3	2.42768 2.75060	12.19883	37.84931	32.44686
C	9 2 9 3	.3 .1 .1 .1 .3 .1	1.88784 4.2576	14.39026	37.3777	30.56229

4.6. LINEAR CORRELATION :

In this section we consider linearly correlated demand quantities of the two commodities. U_n and V_n be the demand quantities for C_1 and C_2 respectively at the nth demand epoch. Let $U_n = i$ then due to the linear correlation between the demand quantities of C_1 and C_2 we may write $V_n = m + id$; $-1 \leq d \leq 1$, $m > 0$, and $m+id$ is a positive integer which is not larger than b. For individual ordering policy, probability distribution of time between two consecutive replenishments of C_1 is

$$F_1\big((S_1,S_2),(S_1,k),t\big) = \sum_{r=[M_1/a]+\delta_{[M_1/a]}}^{M_1} \sum_{\alpha=0}^{\infty} \sum_{\substack{j_1,\ldots,j_r=2,\ldots,b \\ j_\ell = m+i_\ell d(\ell=1,2,\ldots,r);m,d>0 \\ (r-1)m+d[i_1+\ldots+i_{r-1}]<M_2 \\ rm+d[[i_1+\ldots+i_r]=\alpha M_2+S_2-k}} \sum_{\substack{i_1,\ldots,i_r=1,\ldots,a \\ i_1+\ldots+i_{r-1}<M_1 \\ i_1+\ldots+i_r \geq M_1}} P_{i_1,j_1,\ldots}P_{i_r,j_r}G^{*r}(t)$$

Similarly probability distribution of the time between two consecutive replenishments of C_2 is

$$F_2\big((S_1,S_2),(\beta,S_2),t\big) = \sum_{r=[M_2/b]+\delta_{[M_2/b]}}^{M_2} \sum_{\gamma=0}^{\infty} \sum_{\substack{i_1,\ldots,i_r=2,\ldots,a \\ i_\ell = m+j_\ell d(\ell=1,2,\ldots,r);m,d>0 \\ (r-1)m+d[j_1+\ldots+j_{r-1}]<M_1 \\ rm+d[[j_1+\ldots+j_r]=\gamma M_1+S_1-\beta}} \sum_{\substack{j_1,\ldots,j_r=1,\ldots,b \\ j_1+\ldots+j_{r-1}<M_2 \\ j_1+\ldots+j_r \geq M_2}} P_{i_1,j_1,\ldots}P_{i_r,j_r}G^{*r}(t)$$

For joint ordering policy probability distribution of the time between two consecutive replenishments is

$$F_{12}\{(S_1,S_2),(S_1,S_2),t\} = \sum_{r=\min([M_1/a]+\delta[M_1/a],[M_2/b]+\delta[M_2/b])}^{\min(M_1,M_2)}$$

$$\sum_{\substack{i_1,\ldots,i_r=1,\ldots,a;\,j_1,\ldots,j_r=1,\ldots,b \\ j_\ell = m+i_\ell d(\ell=1,2,\ldots,r);m,d>0 \\ i_1+\ldots+i_{r-1}<M_1;(r-1)m+d[i_1+\ldots+i_{r-1}]<M_2 \\ \text{either } i_1+\ldots+i_r \geq M_1 \text{ or } rm+d[i_1+\ldots+i_r] \geq M_2}} P_{i_1,j_1,\ldots}P_{i_r,j_r}G^{*r}(t)$$

In individual ordering policy, as in the general case, we get the time dependent system state probabilities foe commodity C_1 is

$$P\{(S_1,S_2),(\ell,q),t\} = \int_0^t \sum_{k\in E_2} F_1\{(S_1,S_2),(S_1,k),du\}$$

$$P\{(S_1,k),(\ell,q),t-u\};\ (S_1,k),(\ell,q)\in E$$

Similarly for commodity C_2

$$P\{(S_1,S_2),(\ell,q),t\} = \int_0^t \sum_{\beta\in E_1} F_2\{(S_1,S_2),(\beta,S_2),du\}$$

$$P\{(\beta,S_2),(\ell,q),t-u\};\ (\beta,S_2),(\ell,q)\in E$$

In joint ordering policy we get the time dependent system state probability as

$$P\{(S_1,S_2),(\ell,q),t\} = \int_0^t F_{12}\{(S_1,S_2),(S_1,S_2),du\}$$

$$P\{(S_1,S_2),(\ell,q),t-u\};\ (\ell,q),(S_1,S_2)\in E$$

4.6.1 Optimzation Problem :

For individual ordering policy let B_1 be the time duration between two consecutive replenishments of C_1. Then

$$E(B_1) = \sum_{r=[M_1/a]+\delta_{[M_1/a]}}^{M_1} \sum_{\alpha=0}^{\infty} \sum_{\substack{j_1,\dots,j_r=2,\dots,b \\ j_\ell=m+i_\ell d(\ell=1,2,\dots,r);m,d>0 \\ (r-1)m+d[i_1+\dots+i_{r-1}]<M_2 \\ rm+d[[i_1+\dots+i_r]=\alpha M_2+S_2-k}} \sum_{\substack{i_1,\dots,i_r=1,\dots,a \\ i_1+\dots+i_{r-1}<M_1 \\ i_1+\dots+i_r\ge M_1}} P_{i_1,j_1,\dots}P_{i_r,j_r} r\mu$$

Similarly B_2 be the time duration between two consecutive replenishments of C_2. Then

$$E(B_2) = \sum_{r=[M_2/b]+\delta_{[M_2/b]}}^{M_2} \sum_{\gamma=0}^{\infty} \sum_{\substack{i_1,\dots,i_r=2,\dots,a \\ i_\ell=m+j_\ell d(\ell=1,2,\dots,r);m,d>0 \\ (r-1)m+d[j_1+\dots+j_{r-1}]<M_1 \\ rm+d[[j_1+\dots+j_r]=\gamma M_1+S_1-\beta}} \sum_{\substack{j_1,\dots,j_r=1,\dots,b \\ j_1+\dots+j_{r-1}<M_2 \\ j_1+\dots+j_r\ge M_2}} P_{i_1,j_1,\dots}P_{i_r,j_r} r\mu$$

Expected quantity demand of C_1 per unit time is $\frac{1}{\mu}\sum_{i=1}^{a} ip_{i.}$ and that of C_2 is $\frac{1}{\mu}\sum_{j=1}^{b} jp_{.j}$. k_1 and k_2 be fixed ordering costs of C_1 and C_2 respectpively. Then the total expected cost of ordering per unit time is $\frac{k_1}{E(B_1)}+\frac{k_2}{E(B_2)}$. V_1 and V_2 be the holding costs of one unit of C_1 and C_2 respectively per unit time. Then the total average holding cost of C_1 and C_2 per unit time is

$$V(S_1,s_1,S_2,s_2)=V_1\sum_{\ell=s_1+1}^{S_1}\ell\sum_{q=s_2+1}^{S_2}\theta(\ell,q)+V_2\sum_{q=s_2+1}^{S_2}q\sum_{\ell=s_1+1}^{S_1}\theta(\ell,q)$$

where $\theta(\ell,q)=\lim_{t\to\infty}P\{(S_1,S_2),(\ell,q),t\}$. Thus the total expected cost per unit time under steady state is

$$Z(S_1,s_1,S_2,s_2)=V(S_1,s_1,S_2,s_2)+\frac{k_1}{E(B_1)}+\frac{k_2}{E(B_2)}+r_1\left[\frac{1}{\mu}\sum_{i=1}^{a}ip_{i.}\right]+r_2\left[\frac{1}{\mu}\sum_{j=1}^{b}jp_{.j}\right]$$

where r_k is the unit procurement cost of item C_k, $k = 1, 2$. For joint ordering policy

$$E(B_3)=\sum_{r=\min\{[M_1/a]+\delta_{[M_1/a]},[M_2/b]+\delta_{[M_2/b]}\}}^{\min\{M_1,M_2\}}\ \sum_{\substack{i_1,\dots,i_r=1,\dots,a;\ j_1,\dots,j_r=2,\dots,b\\ j_\ell=m+i_\ell d;(l=1,2,\dots,r);m,d>0\\ i_1+\dots+i_{r-1}<M_1;(r-1)m+d[i_1+\dots+i_{r-1}]<M_2\\ \text{either } i_1+\dots+i_r\ge M_1 \text{ or } rm+d[i_1+\dots+i_r]\ge M_2}} P_{i_1,j_1}\dots P_{i_r,j_r}\, r\mu$$

The total expected cost per unit time under steady state is

$$Z^1(S_1,s_1,S_2,s_2)=V^1(S_1,s_1,S_2,s_2)+\frac{k}{E(B_3)}+r_1\left[\frac{1}{\mu}\sum_{i=1}^{a}ip_{i.}\right]+r_2\left[\frac{1}{\mu}\sum_{j=1}^{b}jp_{.j}\right]$$ where

$$V^1(S_1,s_1,S_2,s_2)=V_1\sum_{\ell=s_1+1}^{S_1}\ell\sum_{q=s_2+1}^{S_2}\prod(\ell,q)+V_2\sum_{q=s_2+1}^{S_1}q\sum_{\ell=s_1+1}^{S_2}\prod(\ell,q)$$ and

$$\prod(\ell,q)=\lim_{t\to\infty}P\{(S_1,S_2),(\ell,q),t\}$$

4.6.2 Numerical Illustration :

Consider an inventory system with $k = 10$, $k_1 = 10$, $r_1 = 6$, $r_2 = 8$, $V_1 = 2$, $V_2 = 1.5$, $a=2$, $b = 3$ and $\mu = 4$. For two sets of fixed values of $P_{i,j}$ s ($i=1,2$; $j=1,2,3$), expected value of the time duration between two consecutive replenishments and average cost for individual ordering policy and joint ordering policy are given in Table-II. Here also we see that joint ordering is preferable to individual ordering policy as is expected (from the general case considered in section 4. 5).

Table-II

	S_1 s_1 S_2 s_2	P_{11} P_{12} P_{13} P_{21} P_{22} P_{23}	$E(B_1)$& $E(B_2)$ (for I.O) resp.	$E(B_3)_{J.O}$	(Average Cost)$_{I.O}$	(Average Cost)$_{J.O}$
I						
A	5 1 6 2	.2 .2 .2 .2 .1 .1	0.0800 0.0800	0.0800	295.2363	146.1746
B	5 1 6 2	.1 .2 .1 .1 .2 .3	0.3200 0.3200	0.3200	90.03145	45.4086
C	5 1 6 2	.3 .1 .1 .1 .3 .1	0.7200 0.7200	0.7200	50.7805	34.7329
II						
A	8 2 7 2	.2 .2 .2 .2 .1 .1	0.0120 0.0120	0.0120	1857.5844	860.2297
B	8 2 7 2	.1 .2 .1 .1 .2 .3	0.0960 0.0960	0.0960	252.6986	132.3494
C	8 2 7 2	.3 .1 .1 .1 .3 .1	0.3240 0.3240	0.3240	95.85229	57.3673

CHAPTER V

SOME BULK DEMAND TWO COMMODITY INVENTORY MODELS

5.1. INTRODUCTION

In this chapter, we consider two models. In Model-I, we analyze a bulk demand two commodity inventory problem which generalize the results of Krishnamoorthy, Lakshmi and Basha (1994). They have considered a two commodity inventory problem with unit demand with no dependants, whatever between the type of commodities demanded. They provide a characterization of the system state probabilities. In our model we consider a bulk demand two commodity inventory problem with the commodities represented by C_1 and C_2 respectively. The (s_k, S_k) policy is adopted for commodity C_k ($k = 1, 2$). A demand for both $C_1$and C_2 together is assumed not to occur. No shortage is permitted. Replenishment is such that whenever the inventory level of C_k falls to s_k ($k = 1, 2$) or below that due to a demand after the previous replenishment, an order is placed and instantaneous replenishment of that occurs so as to bring the inventory level back to S_k.

In Model-II, the probability p_{12} for a demand of both commodities together is assumed positive i.e. $(p_1 + p_2 + p_{12} = 1)$. On the inventory level of C_1 reaching the level s_1, unit demand for it is only entertained for the first commodity, and sales of C_1 is restricted to those demands which demand the second commodity also until replenishment of C_1 occurs. Due to a bulk demand, if the inventory level of C_1 falls below s_1, such a demand will be satisfied by units sufficient enough to maintain the reordering level. We assume $M_1 > 2s_1$ and $a < s_1$.

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Lead time is exponentially distributed for the first commodity and it is zero for the second commodity. Unmet demands are not backlogged.

Section 5.2 deals with description and the stochastic formulation of Model-I. Limiting probabilities of the system state is obtained in Section 5.3. An optimization problem is discussed in Section 5.4. An application of the model along with a numerical illustration is given in the same section.

Section 5.5 gives the description and analysis of Model-II. Transient state probabilities are obtained in Section 5.6. Section 5.7 deals with the limiting probabilities and cost analysis is discussed in Section 5.8.

5.2. DESCRIPTION OF MODEL I :

In this model, we consider a bulk demand two–commodity inventory problem with the commodities represented by C_1 and C_2 respectively. The (s_k, S_k) policy is adopted for commodity C_k ($k = 1, 2$). Given that a demand occured, the probability that it is for commodity C_k is p_k ($k = 1, 2$), ($p_1 + p_2 = 1$), conditioned on a demand taking place for C_1 (C_2) the probability that it is for i (j) units of C_1 (C_2) is g_i (h_j) $i = 1, 2, \dots,$ a ($j = 1, 2, \dots, b$). A demand for both C_1 and C_2 together is assumed not to occur. The inter arrival times of demands are independently and identically distributed random variables following distribution function G(.) with mean ($\mu < \infty$). The demand quantities are independent of the type of the commodity demanded. No shortage is permitted. Replenishment is such that whenever the inventory level of C_k falls to s_k ($k = 1, 2$) or below that due to a demand after the previous replenishment, an order is placed and instantaneous replenishment of that occurs so as to bring the inventory level back to S_k. The following notations are used in this model.

X(t) = Inventory level of C_1 at time t

$Y(t)$ = Inventory level of C_2 at time t

$I(t)$ = $X(t), Y(t))$

M_k = $S_k - s_k$; $k = 1, 2.$

$*$ = Convolution.

E_k = $(s_k+1,\ldots,S_k)$ $k = 1,2.$

E = $E_1 \times E_2$

g_i = Probability that i units of C_1 are demanded at a demand epoch $i = 1,2,\ldots,a$

h_j = Probability that j units of C_2 are demanded at a demand epoch $j = 1,2,\ldots,b$

$$\phi_1(z) = \sum_{i=1}^{a} g_i\, z^i$$

$$\phi_2(z) = \sum_{j=1}^{b} h_j\, z^j$$

$$(\phi_k(z))^{*l} = (\phi_k(z))^{l-1}(\phi_k(z)) \qquad l = 2, 3, \ldots;\ k = 1, 2.$$

$g_i(l)$ = Probability of l demands for C_1 consuming i units of it. This is the coefficient of z^i in $(\phi_1(Z))^{*l}$

$h_j(q)$ = Probability of q demands for C_2 consuming j units of it. This is the coefficient of z^j in $((\phi_2(Z))^{*q}$

g_{i_m} = Probability that i_m units of C_1 are demanded at the m th demand epoch,

$i_m = 1, 2, \ldots, a$; $m = 1, 2, \ldots, l$ where l denotes the number of demand epochs for C_1 between its two consecutive replenishments. i.e. l th demand epoch leads to next replenishment after the previous replenishment.

$l = [M_1/a] + \delta_{[M_1/a]}, \ldots, M_1$ where $\delta_{\left[\frac{M_1}{a}\right]} = \begin{cases} 1 \text{ if } M_1/a \text{ is not an integer.} \\ 0 \quad \text{Otherwise.} \end{cases}$

h_{j_n} = Probability that j_n units of C_2 are demanded at the nth demand epoch,

$j_n = 1, 2, \ldots, b$; $n = 1, 2, \ldots, q$ where q denotes the number of demand epochs for C_2 between its two consecutive replenishments, i.e. q th demand epoch leads to next replenishment after the previous replenishment.

$q = [M_2/b] + \delta_{[M_2/b]}, \ldots, M_2$ where $\delta_{\left[\frac{M_2}{b}\right]} = \begin{cases} 1 \text{ if } M_2/b \text{ is not an integer.} \\ 0 \quad \text{Otherwise.} \end{cases}$

Analysis :

Let $0 = T_o < T_1 < \ldots < T_n < \ldots$ be the successive demand epochs and X_o, $X_1, \ldots, X_n, \ldots$ and $Y_o, Y_1, \ldots, Y_n, \ldots$ be the inventory levels of C_1 and C_2 respectively, immediately after demands at these epochs. We may denote the inventory level process at time t by $(X(t), Y(t))$ $t \geq 0$ with $X_n = X(T_n +)$ and $Y_n = Y(T_n +)$.

Some Distribution Functions of Interest :

Result 5.2.1.

Let T_1 be the time elapsed between two consecutive S_1 to S_1 transition of C_1 and $F_1((S_1,i),(S_1,j),t)$ be its distribution function. Then we have

$$F_1((S_1,i),(S_1,j),t) = \sum_{l=\left[M_1/a\right]+\delta_{\left[M_1/a\right]}}^{M_1} \sum_{\substack{i_1+\ldots+i_{l-1}<M_1 \\ i_1+\ldots+i_l \geq M_1}} \sum_{r_1,\ldots,r_l\geq 0} p_2^{r_1} p_1 p_2^{r_2} p_1 \cdots p_2^{r_l} p_1 \quad g_{i_1}\cdots g_{i_l}$$

$$q_{ij}^{(r_1+\ldots+r_l)} \quad G^{*(l+r_1+\ldots+r_l)}(t)$$

Where $q_{ij}^{(r)}$ is the probability of a transition from i to j of C_2 due to r demands for that commodity, $i, j \in E_2$, $r = 0, 1, 2, \ldots$ with

$$q_{ij}^{(0)} = \begin{cases} 1 \textit{ if } i = j \\ 0 \textit{ Otherwise} \end{cases}$$

$$q_{ij}^{(r)} = \begin{cases} h_{i-j} & \textit{if } i > j > s_2 \quad (i-j \leq b) \\ \sum_{k=i-s_2}^{b} h_k & \textit{if } j = S_2 \quad (i-s_2 \leq b) \end{cases}$$

Proof :-

To derive the expression for $F_1((S_1, i), (S_1, j), t)$, set time to zero when an order for C_1 is placed. Instantaneous replenishment of that occurs, so that inventory level of C_1 reaches S_1. Then a total of l demands occurs for the first commodity, resulting in its replenishment. Thus $l - 1$ demands take away atmost $S_1 - s_1 - 1$ units of C_1. In between there can be a number of demands for C_2. Suppose that r_{v+1} (≥ 0) demands for C_2 occur in the interval containing v th and (v+1) th demand epochs of C_1 (v = 0, 1, 2,..., l - 1).

Result 5.2.2:

Let T_2 be the time elapsed between two consecutive S_2 to S_2 transition of C_2 and $F_2((i,, S_2), (j, S_2), t)$ its distribution function. Then we have

$$F_2((i,S_2),(j,S_2),t) = \sum_{q=\left[M_2/b\right]+\delta_{\left[M_2/b\right]}}^{M_2} \sum_{\substack{j_1+\ldots+j_{q-1}<M_2 \\ j_1+\ldots+j_q \geq M_2}} \sum_{l_1,\ldots l_q \geq 0} p_1^{l_1} p_2 p_1^{l_2} p_2 \ldots p_1^{l_q} p_2 \quad h_{j_1} \ldots h_{j_q}$$

$$y_{ij}^{(l_1+\ldots l_q)} \quad G^{*(q+l_1+\ldots+l_q)}(t)$$

where

$Y_{ij}^{(l)}$ is the probability of a transition from i to j of C_1 due to l demands, l = 0, 1, ...; i, j $\in E_1$ with

$$y_{ij}^{(0)} = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{Otherwise} \end{cases}$$

and

$$y_{ij}^{(l)} = \begin{cases} g_{i-j} & if\, i>j>s_1 \quad (i-j \leq a) \\ \sum_{k=i-s_1}^{a} g_k & if\ j=S_1 \quad (i-s_1 \leq a) \end{cases}$$

5.3. TIME DEPENDENT SYSTEM STATE PROBABILITIES :

Define $R_1[(S_1,i), (S_1,j), t] = \sum_{n=0}^{\infty} F_1^{*n}[(S_1,i),(S_1,j),t]\ for\ (S_1,i),(S_1,j) \in E$

Next we compute the time dependent system state probabilities. Let $I(t) = (X(t), (Y(t))$ be the system state at time t and $I(t) = (X_n, Y_n)$, $T_n \le t < T_{n+1}$. Then $\{ I(t), t \ge 0\}$ is a semi- Markov process on E. The system state probabilities at time t satisfy the equation (Cinlar 1975)

$$P((S_1, S_2), (i, j), t) = H((S_1, S_2), (i, j), t) + \int_0^t \sum_{k \in E_2} F_1((S_1,S_2), (S_1,k), du)$$

$$P((S_1, k),(i, j), t-u) \qquad (i, j) \in E$$

where $H((S_1, S_2), (i, j), t)$ is the probability of transition from (S_1, S_2) to (i, j) with the state S_1 of C_1 not revisited in $(0, t)$ if atleast one demand for C_1 occurs.

$$= \begin{cases} \displaystyle\sum_{n=l}^{\infty} \sum_{l=\left[\frac{S_1-i}{a}\right]+\delta_{\left[\frac{S_1-i}{a}\right]}}^{S_1-i} y_{S_1,i}^{(l)} \; q_{S_2,j}^{(n-l)} \left(G^{*(n)}(t) - G^{*(n+1)}(t)\right) & if i \ne S_1 \\ \displaystyle\sum_{n=0}^{\infty} q_{S_2,j}^{(n)} \left(G^{*(n)}(t) - G^{*(n+1)}(t)\right) & if\ i = S_1 \end{cases}$$

Hence the time dependent system state probabilities satisfy the integral equation.

$$P((S_1, S_2),(i, j), t) = H((S_1, S_2), (i, j), t) + \int_0^t \sum_{k \in E_2} F_1((S_1,S_2),(S_1,k),du) \; P((S_1,k),(i, j),t-u)$$

and probability is given by

$$P((S_1,S_2),(i,j),t) = \int_0^t \sum_{k \in E_2} R_1((S_1,S_2,(S_1,k),du)\, H((S_1,k),(i,j),t-u)$$

$$(S_1,S_2),(S_1,k),(i,j) \in E$$

Similar derivation leads to expression for $P((S_1, S_2), (i, j), t)$ looking at regeneration points of C_2. However we note that, either can be used to compute the limiting probabilities of the system state.

5.4. LIMITING DISTRIBUTIONS :

Let $\lim_{t \to \infty} P((S_1, S_2), (i, j), t) = p(i, j), (i, j) \in E.$

From the transition probability matrix $\mathbf{P}$ of Markov chain (X_n, Y_n), its stationary

distribution $\pi = (\pi(i, j)), (i, j) \in E$ can be computed using $\pi \mathbf{P} = \pi$ and $\pi \underline{e} = 1$ where

$\underline{e} = (1,\ldots,1)^T$ and π is a row vector having $M_1 M_2$ elements.

5.4.1. Theorem :

The limiting probabilities of the system state are given by $p(i,j) = \pi(i, j)$; $(i, j) \in E$.

Proof:

The mean sojourn time in any state (i, j) is $m(i,j) = \int_0^\infty (1-G(t))dt = \mu$

(assumed finite). Hence the expected sojourn time is same for every state $(i, j), (i, j) \in E$

Thus

$$p(i,j) = \frac{\pi(i,j)\int_0^\infty p\,(I(t)=(i,j);T_1\rangle t/I(o)=(i,j)dt}{\sum_{(i,j)\in E}\pi(i,j)m(i,j)}$$

$$=\pi(i,j)$$

From the above expression

$$\lim_{t\to\infty} p((S_1,S_2),(i,j),t) = p(i,j) = \pi(i,j)$$

and independent of the initial state, as is expected from the theory of finite state irreducible Markov chains.

5.4.2. Theorem:

If $p_1 = p_2 = p$ (= 1/2) then the inventory level follows the discrete uniform distribution, given by

$$\pi(i,j) = \frac{1}{M_1M_2} \text{ for every } (i,j)\in E.$$

Proof:

From $\pi\mathbf{P} = \pi$ and $\pi\underline{\mathbf{e}} = \mathbf{1}$.

We see that the equation $\pi(i, j+1)p + \pi(i+1, j)p = \pi(i, j)$ for $i = s_1+1, \ldots, S_1$ and $j = s_2+1, \ldots, S_2$ has a solution given by

$$\pi(i,j) = \frac{1}{M_1M_2} \text{ for} (i,j)\in E$$

However this solution is unique since the Markov chain has a finite state space. If we assume $p_2 = 0$ so that $p_1 = 1$ or $p_1 = 0$ so that $p_2 = 1$, we have a single commodity inventory problem.

5.5. OPTIMIZATION PROBLEM :

The objective function corresponding to this model is the total expected cost per unit time under steady state. Here the decision variables are S_1, s_1, S_2, s_2. Let T_1 be the time duration between two consecutive replenishment epochs of C_1. Then, we define T_1 as the length of a cycle. So the expected length of a cycle is $E(T_1)$. Distribution of time between two successive visits to S_1 is

$$= \sum_{l=[M_1/a]+\delta_{[M_1/a]}}^{M_1} \sum_{k=0}^{\infty} p_1^l p_2^k \sum_{r=1}^{a} \sum_{j=0}^{a-r} g_{M_1-r}(l-1) g_{r+j}(1) G^{*(l+k)}(t)$$

where $g_k(r)$ is defined in page 69. Then

$$E(T_1) = \sum_{l=[M_1/a]+\delta_{[M_1/a]}}^{M_1} \sum_{k=0}^{\infty} (l+k) E(\text{int}\, erarrivaltime) p_1^l p_2^k$$

$$\sum_{r=1}^{a} \sum_{j=0}^{a-r} g_{M_1-r}(l-1) g_{r+j}(1)$$

$$= \sum_{l=[M_1/a]+\delta_{[M_1/a]}}^{M_1} \sum_{k=0}^{\infty} (l+k) \mu p_1^l p_2^k \sum_{r=1}^{a} \sum_{j=0}^{a-r} g_{M_1-r}(l-1) g_{r+j}(1)$$

Hence the expected number of orders placed per unit time for C_1 is $1/E(T_1)$. The expected number of demands for C_2 in time $E(T_1)$ is $[E(T_1)/\mu - M_1^1]^+$ where

r_1 and r_2 being the unit procurement cost of item C_1 and C_2 respectively. The optimal values M_1 and M_2 are calculated for the values k_1, k_2, v_1, v_2, r_1, r_2, p_1, p_2, g_i s h_j s, μ, S_1, s_1, S_2, s_2, a and b.

5.6. AN APPLICATION:

Suppose the system has S_1 identical components of type-I and S_2 identical components of type-II. The system is considered operating if atleast s_1+1 of type-I and s_2+1 of type-II of the components function. Otherwise, the system is in the failed state. We assume that the lifetime of all components of type-I follows exponential distribution with mean μ_1 and that of type-II follows exponential distribution with mean μ_2. At time origin, all components are operating. Let T be the random variable denoting the time to failure of the system starting with S_1 of type-I and S_2 of type-II components at time zero. The system reliability in (0, t] is given by

$$P(T>t)=\sum_{l=0}^{M_1-1}\sum_{k=0}^{M_2-1}\binom{S_1}{l}(1-e^{-\mu_1 t})^l(e^{-\mu_1 t})^{S_1-l}\binom{S_2}{k}(1-e^{-\mu_2 t})^k(e^{-\mu_2 t})^{S_2-k}$$

$P_0(t)$ denotes the probability that the system is in failed state at time t where

$$P_0(t)=\sum_{l=0}^{M_1-1}\sum_{k=0}^{M_2-1}\binom{S_1}{l}(1-e^{-\mu_1 t})^l(e^{-\mu_1 t})^{S_1-l}\binom{S_2}{k}(1-e^{-\mu_2 t})^k(e^{-\mu_2 t})^{S_2-k}$$

Failed components are replaced by new identical components as soon as the system fails. Let Y be the random variable denoting the time elapsed between two successive replacements. We assume that $\mu_1 = \mu_2$

Then

$$E(Y)=\int_0^\infty P(Y>t)dt$$

$$=\int_0^\infty \sum_{l=0}^{M_1-1}\sum_{k=0}^{M_2-1}\binom{S_1}{l}(1-e^{-v})^l(e^{-v})^{S_1-l}\binom{S_2}{k}(1-e^{-v})^k(e^{-v})^{S_2-k}\,dv/\mu_1$$

$$=\sum_{l=0}^{M_1-1}\sum_{k=0}^{M_2-1}\binom{S_1}{l}\binom{S_2}{k}1/\mu_1 B[S_1+S_2-(l+k),l+k+1]$$

Particular Case:

When there is only one type of component the above reduces to the problem of multiple satellite launch discussed by Sivazlian and Stanfel (1975).

5.7 NUMERICAL ILLUSTRATION :

Consider a two commodity inventory system with $k_1=10$, $k_2=12$, $r_1=5$, $r_2=7.5$, $v_1=1.00$, $v_2=1.50$, $a=5$, $b=4$ and mean of the distribution of the inter arrival time of demands $\mu=4$. For four sets of fixed values of p_1, p_2, g_i s and h_j s, $i=1,\ldots 5$; $j=1,\ldots,4$. $E(T_1)$ and the average cost are computed and tabulated. Then the optimal values of M_1 and M_2 are obtained.

Sl No.	S_1	s_1	S_2	s_2	a	b	p_1	p_2	g_i	h_j	$E(T_1)$	A.C
1	20	1	10	8	5	4	.4	.6	.2	.4	.02	2546.30
2							.5	.5	.2	.2	.04	1130.84
3							.6	.4	.3	.2	.10	523.74
4							.7	.3	.1	.2	.22	242.17
5							.8	.2	.2	.3	.56	110.85
1	20	2	10	5	5	4	.4	.6	.2	.4	.08	606.71
2							.5	.5	.2	.2	.18	279.27
3							.6	.4	.3	.2	.40	138.89
4							.7	.3	.1	.2	.91	73.89
5							.8	.2	.2	.3	2.28	43.66
1	20	3	10	6	5	4	.4	.6	.2	.4	.22	227.65
2							.5	.5	.2	.2	.50	113.74
3							.6	.4	.3	.2	1.10	64.93
4							.7	.3	.1	.2	2.51	42.38
5							.8	.2	.2	.3	6.18	31.92
1	20	4	10	7	5	4	.4	.6	.2	.4	.54	105.03
2							.5	.5	.2	.2	1.23	60.67
3							.6	.4	.3	.2	2.72	41.67
4							.7	.3	.1	.2	6.15	32.92
5							.8	.2	.2	.3	14.85	28.87

From the table we see that the optimal pair is $M_1 = 16$ and $M_2 = 3$.

5.8. MODEL II :

In this model, all assumptions of Model-I hold. Further we permit the possibility of the occurrence of a demand for both C_1 and C_2, the probability for which is p_{12} $(p_1 + p_2 + p_{12} = 1)$. Conditioned on a demand taking place for C_1 and C_2 together the probability that i units of C_1 and j units of C_2 are demanded is $q_{i,j}$ $(i = 1, 2, \ldots, a;\ j = 1, 2, \ldots, b)$. Whenever the inventory level of C_1 reaches the level s_1, then only unit demand is entertained for the first commodity. Further, sales of C_1 is restricted, to those demands which require the second commodity also. Due to a bulk demand if the inventory level for C_1 falls below s_1, such a demand will be satisfied by units sufficient enough to maintain its reordering level. We assume $M_1 > 2s_1$ and $a < s_1$. Lead time is exponentially distributed for the first commodity and it is zero for the second commodity. Unmet demands are not backlogged. The following additional notations are used for this model :

$$E_1 = [\,0, \ldots, s_1, \ldots, S_1]$$

$$E_2 = [\,s_2+1, \ldots, S_2\,]$$

This result has been presented in the National Conference on Applied Statistics and Operations research held at Nagpur (1998)

$E = E_1 \times E_2$

$q_{i,j}$ = Probability that i units of C_1 and j units of C_2 are demended at a demand epoch

(i=1,2,…,a ;j=1,2,…,b)

m_1^1 = Total number of units demanded by m_1 demands of C_1

R_+ = Set of non negative real numbers

k_2^1 = Total number of units demanded by k_2 demands of C_2

$W_{ij}^{(r)}$ = Probability of a transition from i to j of C_2 due to r demands

$r = 0, 1, 2, \ldots, i, \; j \in E_2$.

Result 5.8.1 :

$\{((X_n,Y_n),T_n), n = 0, 1, 2,\ldots\}$ is a Markov renewal process on the state space $E \times R_+$ with semi-Markov kernel $\{Q[(l_1,m_1), (l_2,m_2), t], (l_1,m_1), (l_2,m_2) \in E$ and $t \in R_+\}$ where

$Q((l_1,m_1),(l_2,m_2),t) = P((X_{n+1},Y_{n+1}) = (l_2,m_2), T_{n+1} - T_n \le t / (X_n,Y_n) = (l_1,m_1))$ and are as given below :

$$Q((l_1,m_1),(l_2,m_2),t)=\begin{cases} p_1 g_{i_1} G(t) & \text{for } l_2 = l_1 - i_1 \text{ if } l_1 - i_1 \geq s_1 \text{ and} \\ & = s_1 \quad \text{if } l_1 - i_1 < s_1 \\ & \text{where } s_1 + 1 \leq l_1 \leq S_1 \text{ with } m_1 = m_2 \\ & s_2 + 1 \leq m_1 \leq S_2,\ i_1 = 1,2,\ldots,a \\ p_2 h_{j_1} G(t) & \text{for } m_2 = m_1 - j_1 \text{ if } m_1 - j_1 > s_2 \text{ and} \\ & = S_2 \quad \text{if } m_1 - j_1 \leq s_2 \\ & \text{with } l_1 = l_2 \text{ for } s_1 + 1 \leq l_1 \leq S_1 \\ & s_2 + 1 \leq m_1 \leq S_2,\ j_1 = 1,2,\ldots,b \\ p_{12} q_{i_1,j_1} G(t) & \text{for } l_2 = l_1 - i_1 \text{ if } l_1 - i_1 \geq s_1 \text{ and} \\ & = s_1 \quad \text{if } l_1 - i_1 < s_1 \\ & \text{with } m_2 = m_1 - j_1 \text{ if } m_1 - j_1 > s_2 \text{ and} \\ & = S_2 \quad \text{if } m_1 - j_1 \leq s_2 \\ & \text{where } s_1 + 1 \leq l_1 \leq S_1 \ \&\ s_2 + 1 \leq m_1 \leq S_2 \\ & i_1 = 1,2,\ldots,a,\ j_1 = 1,2,\ldots,b \end{cases}$$

$$
= \begin{cases}
p_{12} q_{i_1, j_1} \int_0^t e^{-\lambda u} g(u) du & \text{for } l_2 = l_1 - 1 \quad \text{if } 0 \le l_1 \le s_1 \\
& \text{with } m_2 = m_1 - j_1 \text{ if } m_1 - j_1 > s_2 \\
& \quad = S_2 \quad \text{if } m_1 - j_1 \le s_2 \\
& s_2 + 1 \le m_1 \le S_2 \ ; i_1 = 1,2,\ldots,a; j_1 = 1,2,\ldots,b \\
p_1 \int_0^t e^{-\lambda u} g(u) du & \text{for } l_1 = l_2 \ , m_1 = m_2 \quad \text{if } 0 \le l_1 \le s_1 \\
& s_2 + 1 \le m_1 \le S_2 \\
p_2 h_{j_1} \int_0^t e^{-\lambda u} g(u) du & \text{for } m_2 = m_1 - j_1 \text{ if } m_1 - j_1 > s_2 \\
& \quad = S_2 \quad \text{if } m_1 - j_1 \le s_2 \\
& \text{with } l_1 = l_2 \quad \text{for } 0 \le l_1 \le s_1 \\
& s_2 + 1 \le m_1 \le S_2 \ ; j_1 = 1,2,\ldots,b \\
h_{j_1} (p_2 + p_{12}) \int_0^t e^{-\lambda u} g(u) du & \text{for } m_2 = m_1 - j_1 \text{ if } m_1 - j_1 > s_2 \\
& \quad = S_2 \quad \text{if } m_1 - j_1 \le s_2 \\
& \text{with } l_1 = l_2 \quad , l_1 = 0 \\
& s_2 + 1 \le m_1 \le S_2 \ ; j_1 = 1,2,\ldots,b \\
p_1 g_{i_1} \int_0^t (1 - e^{-\lambda u}) g(u) du & \text{for } \ l_2 = l_1 + M_1 - i_1 \text{ if } 0 \le l_1 \le s_1 \\
& \text{with } m_1 = m_2 \quad \text{for } s_2 + 1 \le m_1 \le S_2 \\
& i_1 = 1,2,\ldots,a \\
p_2 h_{j_1} \int_0^t (1 - e^{-\lambda u}) g(u) du & \text{for } \ l_2 = l_1 + M_1 \text{ if } 0 \le l_1 \le s_1 \\
& \text{with } m_2 = m_1 - j_1 \text{ if } m_1 - j_1 > s_2 \\
& \quad = S_2 \text{ if } m_1 - j_1 \le s_2, \ j_1 = 1,\ldots,b \\
& s_2 + 1 \le m_1 \le S_2
\end{cases}
$$

$$
=\begin{cases} p_{12} q_{i_1, j_1} \int_0^t (1-e^{-\lambda u}) g(u) du & for\ l_2 = l_1 + M_1 - i_1\ if\ 0 \le l_1 \le s_1 \\ & with\ m_2 = m_1 - j_1\ if\ m_1 - j_1 > s_2 \\ & \quad = S_2 \quad if\ m_1 - j_1 \le s_2 \\ & s_2 + 1 \le m_1 \le S_2,\ i_1 = 1,2,\ldots,a;\ j_1 = 1,2,\ldots,b \\ 0 & otherwise \end{cases}
$$

Some Distribution Functions of Interest :

Let $T_{(1)}$ be the time elapsed between two consecutive order placment epochs of C_1 and $F_{(1)}[(s_1, s_2), (s_1, k), t]$ its distribution function. Then we have

Result 5.8.2 :

$$F_{(1)}[(s_1,s_2),(s_1,k),t] = F_{(1)}^1[(s_1,s_2),(s_1,k),t] + F_{(1)}^2[(s_1,s_2),(s_1,k),t]$$

where

$$
F_{(1)}^1((s_1,S_2),(s_1,k),t) = \int_{u=0}^{t} \int_{v=u}^{t} \int_{w=v}^{t} \sum_{l_1=0}^{\infty} \sum_{l_2=0}^{\infty} \sum_{l_{12}<s_1} \sum_{m_1=\left[\frac{M_1-l_{12}}{a}\right]+\delta_{\left[\frac{M1-l12}{a}\right]}}^{M_1-l_{12}} \sum_{m_1^1=m_1}^{\min(am_1, M_1-l_{12})}
$$

$$
\sum_{m_2=0}^{\infty} \sum_{\substack{m_{12}=\left[\frac{M_1-l_{12}-m_1^1}{a}\right]+\delta_{\left[\frac{M1-l12-m_1^1}{a}\right]} \\ i_1+\ldots+i_{m_1}+c_1+\ldots+c_{m_{12}}-1<M_1-l_{12} \\ i_1+\ldots+i_{m_1}+c_1+\ldots+c_{m_{12}} \ge M_1-l_{12} \quad and(a)}}^{M_1-l_{12}-m_1^1} p_1^{\ l_1} p_2^{\ l_2} p_{12}^{\ l_{12}} h_{j_1} \ldots h_{j_{l_2}} \cdot
$$

$$
q_{i_1, j_1} \ldots q_{i_{l_{12}}, j_{l_{12}}} \quad g^{*(l_1+l_2+l_{12})}(u) \frac{\lambda e^{-\lambda v}}{1-G(v-u)} p_1^{\ m_1} p_2^{\ m_2} p_{12}^{\ m_{12}} g_{i_1} \ldots g_{i_{m_1}}
$$

$$h_{j_1}\cdots h_{j_{l_2}}\, q_{c_1,d_1}\cdots q_{c_{m_{12}},d_{m_{12}}}\, W_{S_2 k}^{(l_2+l_{12}+m_2+m_{12})}\, g^{*(m_1+m_2+m_{12})}(w-u)\big((1-G(t-w)\big)dw\,dv\,du$$

$$F_{(1)}^{2}((s_1,S_2),(s_1,k),t)=\int_{u=0}^{t}\int_{v=u}^{t}\int_{w=v}^{t}\sum_{l_1=0}^{\infty}\sum_{l_2=0}^{\infty}\sum_{l_{12}\geq s_1}\sum_{m_1=\left[\frac{M_1-s_1}{a}\right]+\delta_{\left[\frac{M_1-s_1}{a}\right]}}^{M_1-s_1}$$

$$\sum_{m_1{}^1=m_1}^{\min(am_1,M_1-s_1)}\sum_{m_2=0}^{\infty}\sum_{\substack{m_{12}=\left[\frac{M_1-s_1-m_1{}^1}{a}\right]+\delta_{\left[\frac{M_1-s_1-m_1{}^1}{a}\right]}\\ i_1+\ldots+i_{m_1}+c_1+\ldots+c_{m_{12}}-1<M_1-s_1\\ i_1+\ldots+i_{m_1}+c_1+\ldots+c_{m_{12}}\geq M_1-s_1 \quad and \quad (b)}}^{M_1-s_1-m_1{}^1}$$

$$p_1^{\,l_1}p_2^{\,l_2}p_{12}^{\,l_{12}}h_{j_1}\cdots h_{j_{l_2}}\,q_{i_1,j_1}\cdots q_{i_{l_{12}},j_{l_{12}}}\,g^{*(l_1+l_2+l_{12})}(u)\frac{\lambda e^{-\lambda v}}{1-G(v-u)}$$

$$p_1^{\,m_1}p_2^{\,m_2}p_{12}^{\,m_{12}}\quad g_{i_1}\cdots g_{i_{m_1}}\,h_{j_1}\cdots h_{j_{m_2}}\,q_{c_1,d_1}\cdots q_{c_{m_{12}},d_{m_{12}}}$$

$$W_{S_2 k}^{(l_2+l_{12}+m_2+m_{12})}\,g^{*(m_1+m_2+m_{12})}(w-u)\big((1-G(t-w)\big)dw\,dv\,du$$

where

$$(a)=i_1+\ldots+i_{m_1-1}+c_1+\ldots+c_{m_{12}}<M_1 \quad or$$

$$i_1+\ldots+i_{m_1}+c_1+\ldots+c_{m_{12}-1}<M_1$$

$$with\ i_1+\ldots+i_{m_1}+c_1+\ldots+c_{m_{12}}\geq M_1$$

depending on whether the last demand is for C_1 alone or for C_1 and C_2 together.

$$(b) = i_1 + \ldots + i_{m_1 - 1} + c_1 + \ldots + c_{m_{12}} < M_1 - s_1 \quad or$$

$$i_1 + \ldots + i_{m_1} + c_1 + \ldots + c_{m_{12} - 1} < M_1 - s_1$$

$$with \;\; i_1 + \ldots + i_{m_1} + c_1 + \ldots + c_{m_{12}} \geq M_1 - s_1$$

depending on whether the last demand is for C_1 alone or for C_1 and C_2 together.

Proof :

Clearly, the order placement epochs coincide with some demand epochs. Between any two consecutive order placement epochs of C_1, there is exactly one replenishment of C_1. The term $F_{(1)}^1$ correspond to the case of no dry period during lead time, $F_{(1)}^2$ correspond to the case of dry period during lead time of C_1.

To derive the expression for $F_{(1)}^1[(s_1, S_2), (s_1, k), t]$. Set time to zero, when an order for C_1 is placed. During (0, u) there may be number of demands which will be for C_1 alone or C_2 alone or for both C_1 and C_2. Let those be l_1, l_2, l_{12} respectively. But at this time only demands for C_2 alone or for C_1 and C_2 together will be saisfied. Note that $l_{12} < s_1$. Due to replenishment in (v, v + dv) the inventory level of C_1 rises to S_1-l_{12}. Due to m_1 demands for C_1 alone , m_{12} demands for C_1 and C_2 together with none, one or more demands for C_2 alone in (u, w) conditioned on no demand in (u, v), the level of C_1 drops to s_1. Hence an order for replenishment is placed. (Here $m_1+m_{12}-1$ demands take away less than M_1-l_{12} units of C_1. m_1+m_{12} demands take away atleast M_1-l_{12} units of C_1.) The expression for $F_{(1)}^2$ can be obtained in a similar way. Only difference is that $l_{12} \geq s_1$, so that dry period is there during lead time of C_1.

Result 5.8.3.

Let $T_{(2)}$ be the time between two consecutive order placement epochs of C_2. Probability distribution of the time between two consecutive visits to s_2 of C_2 is given by

$$F_2((.,S_2),(.,S_2),t) = \int_0^t \sum_{k_1=0}^{\infty} \sum_{k_2=\left[\frac{M_2}{b}\right]+\delta_{\left[\frac{M_2}{b}\right]}}^{M_2} \sum_{k_2^1=k_2}^{\min(bk_2,M_2)} \sum_{\substack{k_{12}=\left[\frac{M_2-k_2^1}{b}\right]+\delta_{\left[\frac{M_2-k_2^1}{b}\right]} \\ j_1+\ldots+j_{k_2}+d_1+\ldots+d_{k_{12}}-1<M_2 \\ j_1+\ldots+j_{k_2}+d_1+\ldots+d_{k_{12}}\geq M_2 \text{ and}(c)}}^{M_2-k_2^1} p_1^{k_1} p_2^{k_2} p_{12}^{k_{12}} h_{j_1}\ldots h_{j_{k_2}}$$

$$g_{i_1}\cdots g_{i_{k_1}} q_{c_1,d_1}\cdots q_{c_{k_{12}},d_{k_{12}}} g^{*(k_1+k_2+k_{12})}(u)(1-G(t-u))du$$

where

$$\begin{aligned} (c) \quad &= j_1+\ldots+j_{k_2-1}+d_1+\ldots+d_{k_{12}} < M_2 \\ \textit{or} \quad & j_1+\ldots+j_{k_2}+d_1+\ldots+d_{k_{12-1}} < M_2 \\ \textit{with} \quad & j_1+\ldots+j_{k_2}+d_1+\ldots+d_{k_{12}} \geq M_2 \end{aligned}$$

depending on whether the last demand is for C_2 alone or for C_1 and C_2 together.

Proof :

Shift the time origin to the epoch of placing an order for C_2. Lead time of C_2 being zero, there are S_2 units of C_2 now available in the system. Due to $k_2 + k_{12}$ demands during (0, u) the inventory level drops down to s_2 for the first time and due to instantaneous replenishment, the level of C_2 is brought to S_2. During this time, there can be number of demands for C_1 alone. This provides the expression for $F_2(.)$

5.9. TRANSIENT STATE PROBABILITIES :

Let I(t) be the system state at time t and I(t) = (X_n, Y_n), $T_n \le t \le T_{n+1}$. Then [I(t), t≥0] is a semi-Markov process on E. The system state probabilities at time t stisfies the equation (Cinlar 1975)

$$P[(s_1,S_2),(i,j),t] = k[(s_1,S_2),(i,j),t] + \int_0^t \sum_{k\in E_2} F_{(1)}[(s_1,S_2),(s_1,k),du]$$
$$P[(s_1,k),(i,j),t-u]$$

where $k[(s_1,S_2),(i,j),t] = \Pr[I(t)=(i,j), T_1>t \,/\, I(0)=(s_1,S_2)]$

$$=1-G(t)$$

Hence the time dependent system size probability is given by

$$P[(s_1,S_2),(i,j),t] = \int_0^t \sum_{k\in E_2} R_1[(s_1,S_2),(s_1,k),du]$$
$$k[(s_1,k),(i,j),t-u],(s_1,k),(i,j),(s_1,S_2)\in E$$

$$\text{where } R_1[(s_1,S_2),(s_1,k),t] = \sum_{n=0}^{\infty} F_{(1)}^{*n}[(s_1,S_2),(s_1,k),t]\, for(s_1,S_2),(s_1,k)\in E$$

5.10. LIMITING PROBABILITIES :

Let $\pi(i,j) = \lim_{n\to\infty} P(X_n = i, Y_n = j)\, for(i,j)\in E$. Then π (i,j)s can be uniquely obtained as the solution of **πp= π** and π **e=1** where

$$P = \lim_{t\to\infty} Q[(l_1,m_1),(l_2,m_2),t],(l_1,m_1),(l_2,m_2)\in E$$

Define

$$H(i,j)=\lim_{t\to\infty} P[(s_1,S_2),(i,j),t];(i,j)\in E$$

Then

$$H(i,j)=\frac{\pi(i,j)\int_0^\infty p(I(t)=(i,j);\ T_1\rangle t/I(o)=(i,j)]dt}{\sum_{(i,j)\in E}\pi(i,j)m(i,j)}$$

$=\pi$ (i,j) , for $i>s_1+1$;$s_2+1 \le j \le S_2$

$$H(S_1,j)=\frac{\pi(i,j)\int_0^\infty p[(I(t)=(i,j);T_1\rangle t/I(o)=(i,j)]\lambda e^{-\lambda t}(1-G(t))\,dt}{\sum_{(i,j)\in E}\pi(i,j)m(i,j)}$$

$$i\le s_1,\quad s_2+1\le j\le S_2$$

where m (i, j) is the mean sojourn time of the Markov renewal process $\{(X_n,Y_n)\,T_n,\ n=0,1,2,\ldots\}$ in the state (i, j) and is given by

$$m(i,j)=\int_0^\infty[1-G(t)]dt\ for(i,j)\in E$$

5.11. COST ANALYSIS :

Let $T_{(1)}$ be the time duration between two consecutive reordering epochs of C_1. We define $T_{(1)}$ as the length of the cycle. Probability distribution of time for s_1 to s_1 transition is given by $F_{(1)}$ [(s_1, s_2), (s_1, k), t]. Then

$$E(T_{(1)}) = \sum_{l_1=0}^{\infty}\sum_{l_2=0}^{\infty}\sum_{l_{12}<s_1} p_1^{\,l_1} p_2^{\,l_2} p_{12}^{\,l_{12}} \left(e^{-\lambda\mu(l_1+l_2+l_{12})} - e^{-\lambda\mu(l_1+l_2+l_{12}+1)}\right)$$

$$\sum_{m_1=\left\lceil\frac{M_1-l_{12}}{a}\right\rceil+\delta_{\left\lceil\frac{M_1-l_{12}}{a}\right\rceil}}^{M_1-l_{12}} \quad \sum_{m_1^1=m_1}^{\min(am_1,M-l_{12})} \quad \sum_{m_2=0}^{\infty}$$

$$\sum_{m_{12}=\left\lceil\frac{M_1-l_{12}-m_1^1}{a}\right\rceil+\delta_{\left\lceil\frac{M_1-l_{12}-m_1^1}{a}\right\rceil}}^{M_1-l_{12}-m_1^1} p_1^{\,m_1} p_2^{\,m_2} p_{12}^{\,m_{12}} (l_1+l_2+l_{12}+m_1+m_2+m_{12})\mu$$

$$+$$

$$\sum_{l_1=0}^{\infty} \sum_{l_2=0}^{\infty} \sum_{l_{12}\geq s_1} p_1^{\,l_1} p_2^{\,l_2} p_{12}^{\,l_{12}} \left(e^{-\lambda\mu(l_1+l_2+l_{12})}\right)$$

$$\sum_{m_1=\left\lceil\frac{M_1-s_1}{a}\right\rceil+\delta_{\left\lceil\frac{M_1-s_1}{a}\right\rceil}}^{M_1-s_1} \quad \sum_{m_1^1=m_1}^{\min(am_1,M_1-s_1)} \quad \sum_{m_{12}=\left\lceil\frac{M_1-s_1-m_1^1}{a}\right\rceil+\delta_{\left\lceil\frac{M_1-s_1-m_1^1}{a}\right\rceil}}^{M_1-s_1-m_1^1} p_1^{\,m_1} p_2^{\,m_2} p_{12}^{\,m_{12}}$$

$$(l_1+l_2+l_{12}+m_1+m_2+m_{12})\mu$$

Similarly $T_{(2)}$ be the time duration between two consecutive order placement epochs of C_2. Probability distribution of the time between two consecutive visits to s_2 of C_2 is given by $F_2\{(., S_2), (., S_2), t\}$. Then

$$E(T_{(2)}) = \sum_{k_1=0}^{\infty} \quad \sum_{k_2=\left\lceil\frac{M_2}{b}\right\rceil+\delta_{\left\lceil\frac{M_2}{b}\right\rceil}}^{M_2} \quad \sum_{k_2^1=k_2}^{\min(bk_2,M_2)} \quad \sum_{k_{12}=\left\lceil\frac{M_2-k_2^1}{b}\right\rceil+\delta_{\left\lceil\frac{M_2-k_2^1}{b}\right\rceil}}^{M_2-k_2^1}$$

$$p_1^{\,k_1} p_2^{\,k_2} p_{12}^{\,k_{12}} (k_1+k_2+k_{12})\mu$$

The expected number of orders placed per unit time for C_1 is $1/E(T_{(1)})$ and that for C_2 is $1/E(T_{(2)})$. Let k_1 and k_2 be the fixed ordering cost for C_1 and C_2 respectively. Then the total expected cost of ordering for C_1 and C_2 per unit time is $(k_1/E(T_{(1)}))+(k_2/E(T_{(2)}))$. The total average holding cost of C_1 and C_2 per unit time is $= V_1(\sum_{i=1}^{S_1} i \sum_{j=s_2+1}^{S_2} \pi(i,j)) + V_2(\sum_{j=s_2+1}^{S_2} j \sum_{i=1}^{S_1} \pi(i,j))$, v_1 and v_2 being the holding cost of C_1 and C_2 per unit time. Thus, the total expected cost per unit time under steady state is

$$Z(S_1,s_1,S_2,s_2) = V_1(\sum_{i=1}^{S_1} i \sum_{j=s_2+1}^{S_2} \pi(i,j)) + V_2(\sum_{j=s_2+1}^{S_2} j \sum_{i=1}^{S_1} \pi(i,j))$$

$$+(k_1/E(T_{(1)}))+(k_2/E(T_{(2)}))+r_1M_1+r_2((M_2-1)+\sum_{i=1}^{a}\sum_{j=1}^{b} j(h_j+q_{i,j}))$$

r_1 and r_2 being the unit procurement cost of C_1 and C_2 respectively.

CHAPTER VI

ANALYSIS OF A TWO COMMODITY INVENTORY PROBLEM WITH LEAD TIME UNDER N-POLICY

6.1. INTRODUCTION

Some results about multi-commodity, continuous review inventory system can be found in Sivazlian (1971), Krishnamoorthy, Basha and Lakshmi (1994) Krishnamoorthy, and Varghese (1995) Krishnamoorthy, Merlymol and Ravindranathan (1999) , N policy for local purchase is introduced by Krishnamoorthy and Raju (1998). However, the literature on multicommodity item is much less than this on single commodity inventory.

Local purchase by shop-keepers are very common. Situations of this sort arise in practice in shops when certain goods run out of stock and reaches a threshold (negative level), the owner goes for local purchase. This involves higher cost to the system. Even this will ensure goodwill of customers to a great extent. Here a two commodity inventory problem under N – policy for local purchase with lead time is considered. In this continuous review inventory system the two commodities are represented by C_1 and C_2 respectively. The (s_k, S_k) policy is followed for the commodity C_k $(k=1,2)$. The inter-arrival times of demands are independently and identically distributed random variables following distribution function G(.) with mean μ $(< \infty)$. Each arrival demands either one unit of first commodity C_1 alone with probability p_1 or one unit of second commodity C_2 alone with probability p_2 or one unit of C_1 and C_2 with probability p_{12} such that $p_1+p_2+p_{12} = 1$. Lead time is exponentially distributed with parametre λ for C_1 whereas for C_2 it is zero. Whenever the inventory level of C_1 reaches s_1 , its sales is restricted to those customers who demand one unit of C_2 also. Whenever the inventory level of C_k falls to s_k , an order is placed for M_k units of that commodity $k = 1, 2$. At the epoch at which the backlog of C_1 reaches N ,due to a demand during a lead time, we take one of the three decisions regarding the replenishment

(i) Cancel the existing order placed for C_1 and make a local purchase to bring its level to S_1, that is buy $N + S_1$ units of C_1 locally. The outstanding order is cancelled to avoid the possibility of exceeding the inventory level of C_1 beyond S_1, due to both local purchase and the replenishment of the order. Or

(ii) A local purchase is made to raise the inventory level to s_1,without cancelling the order placed. Or

(iii) A local purchase to clear the backlogs alone is made without cancelling the replenishment order.

Several transactions in real life takes place as described above. Consider a shop selling tube and tyre. The demand can be for exactly one of the items or for both together with certain probabilities. Whenever the inventory level of tube reaches the level s_1 , then the sales of tube is restricted to those who take one unit of tyre also. At the epoch at which due to a demand, the backlog of tube reaches N, during a lead time of tube we take one of the three decisions regarding the replenishment. We compute the limiting distribution of the inventory level for all the three cases and examined associated cost functions.

This chapter is organized as follows. In Section 3, Model 1 (local purchase upto S_1 cancelling replenishment order) is formulated and analyzed. The time dependent and stationary probabilities are obtained and the cost analysis is carried out. Model 2 (local purchase upto s_1, not cancelling replenishment order) and Model 3 (local purchase to meet all outstanding demands, retaining the replenishment order) are discussed in Sections 4 and 5 respectively.

The following notations are used in this chapter.

E_1 = $\{-N+1,\ldots, s_1,\ldots,S_1\}$.Thus $\{-N\}$ is an instantaneous state.

E_2 = $\{s_2+1, \ldots , S_2\}$.

E = $E_1 \times E_2$.

$X(t)$ = Inventory level of C_1 at time t.

T_0 = 0, $T_1,\ldots$ are the successive epochs at which demands takes place.

Then

X_n = $X(T_n+)$, inventory level of C_1 immediately after the n th demand.

$Y(t)$ = inventory level of C_2 at time t

Y_n = $Y(T_n+)$, inventory level of C_2 immediately after the n th demand.

$I(t)$ = The system state at time t; $I(t)= (X(t), Y(t))$, $I(t) = (X_n,Y_n)$, $T_n \leq t < T_{n+1}$

M_k = $S_k - s_k$; $k = 1, 2$.

$R+$ = Set of non negative real numbers.

6.2. ANALYSIS OF THE MODELS :

Let $0=T_0<T_1<\ldots<T_n<\ldots$ be the successive demand epochs such that $\{T_n, n\geq 0\}$ constitutes a renewal process. Let $X_0, X_1,\ldots,X_n,\ldots$ and $Y_0, Y_1\ldots Y_n,\ldots$ be the inventory levels of C_1 and C_2, just after meeting the demands at these epochs. Because of the N-policy, for local purchase during lead time of C_1, X_n assumes values in the set $E_1 =\{-N+1, \ldots,s_1,\ldots,S_1\}$ and Y_n takes values in the set $E_2 =\{s_2+1,\ldots, S_2\}$. The process $\{(X_n, Y_n),n=0,1,\ldots\}$ forms a Markov chain on the state space E with initial probability

$$P[(X_0,Y_0)=(s_1,S_2)]=\delta_{(i,j),(s_1,S_2)}=\begin{cases}1 & if\ i=s_1, j=S_2\\ 0 & otherwise\end{cases}$$

We assume that $S_1-2s_1+1>N$ to avoid perpetual ordering of C_1.

6.3. MODEL 1 :

In this model we assume that a local purchase is made to raise the inventory level of C_1 to S_1, even at a much higher cost, after cancelling the order already pending, if during lead time the backlog accumulates to N.

Result 6.1

$\{((X_n,Y_n),T_n), n=0,1,2,\ldots\}$ is a Markov renewal process on the state space $E \times R_+$ with semi – Markov kernel ,

$Q_{(1)} = \{Q_{(1)}[(i_1, j_1),(i_2, j_2), t]; (i_1, j_1),(i_2, j_2) \in E \text{ and } t \in R_+\}$

Where $Q_{(1)}$ [(i_1,j_1), (i_2,j_2) ,t] = P [$(X_{n+1},Y_{n+1}) = (i_2,j_2)$, $T_{n+1}-T_n \le t$ / $(X_n, Y_n) = (i_1,j_1)$]
and are as given below.

$$
Q_{(1)}((i_1,j_1),(i_2,j_2),t)= \begin{cases}
p_1 G(t) & for\, i_2 = i_1 - 1 \quad if \quad s_1 + 1 \le i_1 \le S_1 \\
 & with\; j_2 = j_1 \quad for\, s_2 + 1 \le j_1 \le S_2 \\
p_2 G(t) & for\; j_2 = S_2 \quad if\; j_1 = s_2 + 1 and \\
 & j_2 = j_1 - 1\; if\, s_2 + 2 \le j_1 \le S_2 \\
 & with \quad i_2 = i_1, \quad s_1 + 1 \le i_1 \le S_1 \\
p_{12} G(t) & for\, i_2 = i_1 - 1 \quad if \quad s_1 + 1 \le i_1 \le S_1 \\
 & with \;\; j_2 = S_2 \quad if\; j_1 = s_2 + 1 and \\
 & j_2 = j_1 - 1\; if\, s_2 + 2 \le j_1 \le S_2 \\
\int_0^t p_1 e^{-\lambda u} g(u) du & for\, i_2 = i_1, \;\; j_2 = j_1 \quad if \\
 & -N + 1 \le i_1 \le s_1, s_2 + 1 \le j_1 \le S_2 \\
\int_0^t p_2 e^{-\lambda u} g(u) du & for\; j_2 = S_2 \quad if\; j_1 = s_2 + 1 and \\
 & j_2 = j_1 - 1\; if\, s_2 + 2 \le j_1 \le S_2 \\
 & with\, i_2 = i_1, \;\; for - N + 1 \le i_1 \le s_1
\end{cases}
$$

$$
= \begin{cases}
\int_0^t p_{12} e^{-\lambda u} g(u)du & \text{for } j_2 = S_2 \quad \text{if } j_1 = s_2 + 1 \text{ and} \\
& j_2 = j_1 - 1 \quad \text{if } s_2 + 2 \le j_1 \le S_2 \\
& \text{with } i_2 = i_1 - 1 \quad \text{for} -N + 2 \le i_1 \le s_1 \\
& \text{and } i_2 = S_1 \quad \text{for } i_1 = -N + 1 \\
\int_0^t p_1 (1 - e^{-\lambda u}) g(u)du & \text{for } i_2 = i_1 + M_1 \quad \text{if } -N + 1 \le i_1 \le s_1 \\
& \text{with } j_2 = j_1 \quad \text{for } s_2 + 1 \le j_1 \le S_2 \\
\int_0^t p_2 (1 - e^{-\lambda u}) g(u)du & \text{for } i_2 = i_1 + M_1 \quad \text{if } -N + 1 \le i_1 \le s_1 \\
& \text{with } j_2 = j_1 - 1 \quad \text{for } s_2 + 2 \le j_1 \le S_2 \\
& \text{and } j_2 = S_2 \quad \text{if } j_1 = s_2 + 1 \\
\int_0^t p_{12} (1 - e^{-\lambda u}) g(u)du & \text{for } i_2 = i_1 + M_1 - 1 \quad \text{if } -N + 2 \le i_1 \le s_1 \\
& \text{with } j_2 = j_1 - 1 \quad \text{if } s_2 + 2 \le j_1 \le S_2 \\
& \text{and } j_2 = S_2 \quad \text{if } j_1 = s_2 + 1 \\
0 & \text{otherwise}
\end{cases}
$$

6.3.1. Some Distribution Functions of Interest :

Let T_1 be the time elapsed between two consecutive order placement epochs of C_1.and $F_{(1)}\{(s_1,.),(s_1,.),t\}$ its distribution function. Then we have

Result 6.2 :

$F_{(1)}\{(s_1,.),(s_1,.),t\} = F_{(1)}^1\{(s_1,.),(s_1,.),t\} + F_{(1)}^2\{(s_1,.),(s_1,.),t\}$

where

$$F_{(1)}^1\{(s_1,.),(s_1,.),t\} = \int_{u_1=0}^{t}\int_{u_2=u_1}^{t}\int_{u_3=u_2}^{t}\int_{u_4=u_3}^{t}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_{12}=0}^{s_1+N-1}\frac{(k_1+k_2+k_{12})!}{k_1!k_2!k_{12}!}$$

$$p_1^{k_1}\ p_2^{k_2}\ p_{12}^{k_{12}}\quad g^{*(k_1+k_2+k_{12})}(u_1)\quad \lambda e^{-\lambda u_2}$$

$$\sum_{l_1=0}^{M_1-k_{12}-1}\ \sum_{l_2=0}^{\infty}\ \sum_{\substack{l_{12}=0\\ l_1+l_{12}=M_1-k_{12}-1}}^{M_1-k_{12}-l_1-1}\frac{(l_1+l_2+l_{12})!}{l_1!l_2!l_{12}!}$$

$$p_1^{l_1}p_2^{l_2}p_{12}^{l_{12}}\ \frac{g^{*(l_1+l_2+l_{12})}(u_3-u_1)}{1-G(u_2-u_1)}(p_1+p_{12})g(u_4-u_3)$$

$$[1-G(t-u_4)]du_4du_3du_2du_1$$

and

$$F_{(1)}^2\{(s_1,.),(s_1,.),t\} = \int_{u_1=0}^{t}\int_{u_2=u_1}^{t}\int_{u_3=u_2}^{t}\int_{u_4=u_3}^{t}\sum_{k_1=0}^{\infty}\cdot\sum_{k_2=0}^{\infty}\sum_{\substack{k_{12}\\ k_{12}=s_1+N-1}}\frac{(k_1+k_2+k_{12})!}{k_1!k_2!k_{12}!}$$

$$p_1^{k_1}p_2^{k_2}p_{12}^{k_{12}}\ \ g^{*(k_1+k_2+k_{12})}(u_1)\ \ (p_1+p_{12})\ \ g(u_2-u_1)\ e^{-\lambda u_2}$$

$$\sum_{l_1=0}^{M_1-1}\ \sum_{l_2=0}^{\infty}\ \sum_{\substack{l_{12}=0\\ l_1+l_{12}=M_1-1}}^{M_1-l_1-1}(\ \frac{(l_1+l_2+l_{12})!}{l_1!l_2!l_{12}!}$$

$$p_1^{l_1}p_2^{l_2}p_{12}^{l_{12}}g^{*(l_1+l_2+l_{12})}(u_3-u_2)(p_1+p_{12})g(u_4-u_3)$$

$$[1-G(t-u_4)]du_4du_3du_2du_1$$

Proof :

Clearly the order placement epochs coincide with some demand epochs .Between any two consecutive order placement epochs of C_1, there is exactly one replenishment of C_1 and this may be against the order placed or due to a local purchase. The terms $F_{(1)}{}^1$ and $F_{(1)}{}^2$ correspond to the cases of natural replenishment and local purchase respectively.

To derive the expression for $F_{(1)}{}^1\{(s_1,.),(s_1,.),t\}$: Set time to zero when an order for C_1 is placed. There may be a number of demands which will be for C_1 alone or C_2 alone or for both C_1 and C_2 during $(0, u_1)$. Let these be k_1, k_2, and k_{12} respectively. But during this time interval only demands for C_2 alone or for C_1 and C_2 together will be satisfied. Note that $k_{12} \leq s_1+N-1$. Due to the replenishment against the order placed in (u_2, u_2+du_2), the inventory level of C_1 rises to S_1-k_{12}. Then during (u_2, u_3) l_1 demands for C_1 alone, l_2 demands for C_2 alone, l_{12} demands for C_1 and C_2 together occur ,conditioned on no demand in (u_1, u_2) .Due to l_1 and l_{12} demands the inventory level of C_1 comes down to s_1+1. Finally due to a demand in $(u_4, u_4+d\,u_4)$ for C_1 or for both C_1 and C_2 together, the level of C_1 comes down to the reorder level s_1. Hence an order for replenishment by a quantity S_1-s_1 is placed. It may be noted that there can be any number of demands for C_2 in this duration. The expression for $F_{(1)}{}^2$ can be obtained in a similar fashion. Here we note that at the epoch at which due to a demand for C_1 or for C_1 and C_2 together the backlog of C_1 reaches N, a local purchase is made to raise the inventory level of C_1 to S_1.

Result 6.3 :

Let T_2 be the time elapsed between two consecutive order placement epochs of C_2.Probability distribution of the time between two consecutive visits to s_2 of C_2 is given by

$$F_2\{(.,S_2),(.,S_2),t\} = \int_{u=0}^{t} \int_{w=u}^{t} \sum_{k_1=0}^{\infty} \sum_{\substack{k_2,k_{12} \\ k_2+k_{12}=M_2-1}} \frac{(k_1+k_2+k_{12})!}{k_1!k_2!k_{12}!}$$

$$p_1^{k_1} p_2^{k_2} p_{12}^{k_{12}} \quad g^{*(k_1+k_2+k_{12})}(u)(p_2+p_{12})g(w-u)dwdu$$

Proof :

Shift the time origin to the epoch of placing an order for C_2. Led time of C_2 being zero, there are S_2 units of C_2 now available in the system. At some point in (0,u) the level of C_2 drops down to s_2+1 for the first time. Next demand takes place in (w, w+dw) and this is for C_2 alone or C_1 and C_2 together, there by bringing down the level of C_2 to s_2 and due to the instantaneous replenishment, the level of C_2 is brought to S_2.This provides the expression for F_2 (.).

6.3.2. Transient State Probabilities :

Define P [(s_1,S_2) ,(i, j), t]=P[(X(t),Y(t))=(i, j)/(X(0),Y(0))=(s_1,S_2)].

Then it can be seen that

$$P_{1(i,j)}(t) = P[(s_1,S_2),(i\,,j),t] = \int_0^t \sum_{(i_1,j_1)\in A} Q_{(1)}[(s_1,S_2)(i_1,j_1),du]\; p[(i_1,j_1),(i,j),t-u]$$

where A={ $(s_1\text{-}1, S_2\text{-}1)$, $(s_1, S_2\text{-}1)$, $(S_1\text{-}1, S_2\text{-}1)$, $(S_1\text{-}1, S_2)$, $(S_1, S_2\text{-}1)$ }.

Clearly from the state (s_1, S_2), the system can move to state (i_1, j_1) in A defined above, in the transition which takes place in (u, u+du) after the one at time zero and the transition from (i_1, j_1) to (i, j) in time (t-u) is governed by the P function.

6.3.3.Stationary Distribution :

Let $\pi_1(i,j) = \lim_{n\to\infty} P(X_n = i, Y_n = j)\, for (i,j) \in E$

Then $\pi_1(i,j)$ s can be uniquely obtained as the solution of $\pi_1 P_1 = \pi_1 \, and \, \pi_1 \underline{e} = 1$

Where $P_1 = \lim_{t\to\infty} Q_{(1)}\{(i_1,j_1),(i_2,j_2),t\},(i_1,j_1),(i_2,j_2) \in E$

Define

$$H_1(i,j) = \lim_{t\to\infty} P_{1_{(i,j)}}(t)$$

$$H_1(S_1,j) = \frac{\pi_1(S_1,j)\int_0^\infty (1-G(t)dt + \pi_1(s_1,j)\int_0^\infty (1-e^{-\lambda t})(1-G(t))dt}{\sum_{(i,j)\in E} \pi_1(i,j)m(i,j)}$$

$$for\, s_2 \le j \le S_2$$

$$H_1(i,j) = \begin{cases} \dfrac{\pi_1(i,j)\int_0^\infty (1-G(t))dt + \pi_1(i-(S_1-s_1),j)\int_0^\infty (1-e^{-\lambda t})(1-G(t))dt}{\sum_{(i,j)\in E} \pi_1(i,j)m(i,j)} \\ \qquad for\; S_1 - s_1 - N + 1 \le i \le S_1 - 1, s_2 \le j \le S_2 \\ \dfrac{\pi_1(i,j)\int_0^\infty e^{-\lambda t}(1-G(t))dt}{\sum_{(i,j)\in E} \pi_1(i,j)m(i,j)} \quad for\; -N+1 \le i \le s_1,\;\; s_2 \le j \le S_2 \end{cases}$$

where m (i, j) is the mean sojourn time of the Markov renewal process$\{[(X_n,Y_n),T_n], n=0,1,2,\ldots\}$ in the state (i, j) and is given by

$$m(i,j)=\int_0^{\infty}(1-G(t))dt \qquad for(i,j)\in E$$

6.3.4. Cost Analysis :

The objective function corresponding to this model is the total expected cost per unit time under steady state .Let T_1 be the time duration between two consecutive reordering epochs of C_1. Then

$$E(T_1)=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_{12}=0}^{s_1+N-1} p_1^{k_1}p_2^{k_2}p_{12}^{k_{12}}\left(e^{-\lambda\mu(k_1+k_2+k_{12})}-e^{-\lambda\mu(k+k_2+k_{12}+1)}\right)$$

$$\sum_{l_1=0}^{M_1-k_{12}}\sum_{l_2=0}^{\infty}\sum_{l_{12}=0}^{M_1-k_{12}-l_1} p_1^{l_1}p_2^{l_2}p_{12}^{l_{12}}(k_1+k_2+k_{12}+l_1+l_2+l_{12})\mu$$

$$+$$

$$\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_{12}=s_1+N} p_1^{k_1}p_2^{k_2}p_{12}^{k_{12}}\ \frac{1}{\lambda}\ \left(e^{-\lambda\mu(s_1+N+k_1+k_2)}\right)$$

$$\sum_{l_1=0}^{M_1}\sum_{l_2=0}^{\infty}\sum_{l_{12}=0}^{M_1-l_1} p_1^{l_1}p_2^{l_2}p_{12}^{l_{12}}(k_1+k_2+k_{12}+l_1+l_2+l_{12})\mu$$

Hence the expected number of orders placed per unit time for C_1 is $1/E(T_1)$. The expected number of orders placed for C_2 in $E(T_1)$=$E(T_1)$/expected time duration between two consecutive replenishments of C_2.

$$= E(T_1)/\left(\frac{\mu M_2}{p_2 + p_{12}}\right) = E(T_1)(p_2 + p_{12})/(\mu M_2)$$

Hence, the expected number of orders placed per unit time for C_2 is $(p_2+p_{12}) / (\mu M_2)$. Let k_1 and k_2 be the fixed ordering costs for C_1 and C_2 respectively. Then the total fixed expected cost of ordering for C_1 and C_2 per unit time is

$$=(k_1 / E(T_1)) + k_2 ((p_2 + p_{12}) / (\mu M_2)).$$

The total average holding cost of C_1 and C_2 per unit time is

$h_1\left(\sum_{i=1}^{S_1} i \sum_{j=s_2+1}^{S_2} \pi_1(i,j)\right) + h_2\left(\sum_{j=s_2+1}^{S_2} j \sum_{i=1}^{S_1} \pi_1(i,j)\right)$ where h_1 and h_2 being the holding costs of C_1 and C_2 per unit time. Purchase cost per unit of C_1 (C_2) be v_1 (v_2); $v_1^1 (>v_1)$ be the local purchase cost per unit of C_1. Then the total procurement cost per unit time is.

$$v_1\left(\frac{M_1}{E(T_1)}\right) + v_2\left(\frac{p_2 + p_{12}}{\mu}\right) + v_1^1\left(\frac{S_1 + N}{E(T_1)}\right)$$

Thus the total expected cost per unit time under steady state is

$$= k_1/E(T_1)) + k_2((p_2+p_{12})/(\mu M_2) + h_1\left(\sum_{i=1}^{S_1} i \sum_{j=s_2+1}^{S_2} \pi_1(i,j)\right) + h_2\left(\sum_{j=s_2+1}^{S_2} j \sum_{i=1}^{S_1} \pi_1(i,j)\right)$$

$$v_1\left(\frac{M_1}{E(T_1)}\right) + v_2\left(\frac{p_2 + p_{12}}{\mu}\right) + v_1^1\left(\frac{S_1 + N}{E(T_1)}\right) \qquad + k$$

where k is the order cancellation cost.

6.4. Model II

In this model, a local purchase is made to bring the inventory level of C_1 to s_1, without cancelling the order placed. Semi-Markov kernel in this case is given by

$$Q_{(2)}[(-N+1, j_1),(s_1, j_2),t] = p_{12}\int_0^t e^{-\lambda u} g(u)du; \; for\; j_2 = j_1 - 1,$$
$$s_2 + 2 \le j_1 \le S_2$$
$$or\; j_2 = S_2 \; when\; j_1 = s_2 + 1$$

For all other combinations, they are same as in Model-I.

Result 6.4.

Let T_3 be the time elapsed between two consecutive order placement epochs of C_1. Its distribution function $F_{(3)}\{(s_1,.),(s_1,.),t\}$ is given by

$$F_{(3)}\{(s_1,.),(s_1,.),t\} = F_{(3)}^{\ 1}\{(s_1,.),(s_1,.),t\} + F_{(3)}^{\ 2}\{(s_1,.),(s_1,.),t\}$$

where

$$F_{(3)}^{\ 1} = F_{(1)}^{\ 1} \text{ of Model-I}$$

and

$$F_{(3)}^{2}\{(s_1,.),(s_1,.),t\} = \int_{u=_1 0}^{t} \int_{u_2=u_1}^{t} \int_{u_3=u_2}^{t} \int_{u_4=u_3}^{t} \int_{u_5=u_4}^{t} \int_{u_6=u_5}^{t} \int_{u_7=u_6}^{t} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_{12}=s_1+N-1} \frac{(k_1+k_2+k_{12})!}{k_1!k_2!k_{12}!}$$

$$p_1^{k_1} p_2^{k_2} p_{12}^{k_{12}} \quad g^{*(k_1+k_2+k_{12})}(u_1) \quad (p_1+p_{12})g(u_2-u_1)$$

$$\sum_{l=0}^{\infty} b_1^{*l}[(s_1,.),(s_1,.),u_3-u_2] \sum_{l_1=0}^{\infty} \sum_{l_2=0}^{\infty} \sum_{l_{12} \prec N+s_1} \frac{(l_1+l_2+l_{12})!}{l_1!l_2!l_{12}!} p_1^{l_1} p_2^{l_2} p_{12}^{l_{12}}$$

$$g^{*(l+l_2+l_{12})}(u_4-u_3)\lambda e^{-\lambda u_5} \sum_{n_1=0}^{M_1-l_{12}-1} \sum_{n_2=0}^{\infty} \sum_{n_{12}=0}^{M_1-l_{12}-n_1-1} n \frac{(n_1+n_2+n_{12})!}{n_1!n_2!n_{12}!}$$

$$p_1^{n_1} p_2^{n_2} p_{12}^{n_{12}} \frac{g^{*(n_1+n_2+n_{12})}(u_6-u_4)}{1-G(u_5-u_4)} (p_1+p_{12})g(u_7-u_6)(1-G(t-u_7))$$

$$du_7 du_6 du_5 du_4 du_3 du_2 du_1$$

where $b_1[(s_1,.),(s_1,.),t] = \frac{d}{dt} B_1[(s_1,.),(s_1,.),t]$ and

$$B_1[(s_1,.),(s_1,.),t] = \int_{u=0}^{t} \int_{v=u}^{t} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_{12}=s_1+N-1} \frac{(k_1+k_2+k_{12})!}{k_1!k_2!k_{12}!} p_1^{k_1} p_2^{k_2} p_{12}^{k_{12}}$$

$$g^{*(k_1+k_2+k_{12})}(u) \ (p_1+p_{12}) \ g(v-u)\,[1-G(t-v)]\,dv\,du$$

Proof:

Set time equal to zero at the time of placement of order. In (0, u_1) there are exactly ($N+s_1-1$) demands for C_1 and C_2 together (with none one or more demands for C_2 alone, demands that are exclusively for C_1 are not satisfied). In (u_2, u_2+du_2) there is a demand for either C_1 or for both C_1 and C_2 so that the inventory level of C_1 drops down to -N resulting in a local purchase which bring the inventory level to s_1. This is repeated l times (l=1,2,3,...). $B_l[(s_1, .),(s_1, .), t]$ represents the distribution function of time between two consecutive local purchase epochs. The last local purchase occurs in (u_3, u_3+du_3). After the last local purchase at u_3, l_{12} demands occur for C_1 and C_2 together where $l_{12} < N+s_1$, so that the level of C_1 after the natural replenishment at u_5 is S_1-l_{12}. Then due to $M_1-l_{12}-1$ demands for C_1 (either for C_1 alone or along with that of C_2) during (u_5, u_6), the level of C_1 drops to s_1+1. Finally due to a demand in (u_7, u_7+du_7) for C_1 or for both C_1 and C_2 together the level of C_1 comes down to reorder level s_1, and the next order for replenishment is placed. Considering these facts we get the expression for $F_{(3)}{}^2$.

6.4.1. Limiting distribution

Let

$\pi_2(i,j) = \lim_{n\to\infty} p(X_n = i, Y_n = j) \quad for\,(i,j) \in E.$ Then $\pi_2(i,j)$ s can be obtained as the solution of $\pi_2 P_2 = \pi_2$ and $\pi_2 \underline{e} = 1$ where

$P_2 = \lim_{t\to\infty} Q_2\{(i_1,j_1),(i_2,j_2),t\}, (i_1,j_1),(i_2,j_2) \in E$ and $\underline{e}$ is an $(N+S_1)(S_2-s_2)$ component row vector of ones. The probability distribution of the system state at arbitrary epoch

are given by

$H_2(i,j) = \lim_{t\to\infty} P_{2_{(i,j)}}(t)$ where

$$P_{2(i,j)}(t) = \int_0^t \sum_{(i_1,j_1)\in A} Q_{(2)}((s_1,S_2),(i_1,j_1),du) P((i_1,j_1),(i,j),t-u)$$

Then

$$H_2(i,j) = \begin{cases} \dfrac{\pi_2(i,j)\int_0^\infty (1-G(t)dt + \pi_2[i-S_1+s_1,j]\int_0^\infty (1-e^{-\lambda t})(1-G(t))dt}{\sum_{(i,j)\in E} \pi_2(i,j)m(i,j)} & \text{for } S_1-s_1-N+1 \le i \le S_1 \\ & s_2+1 \le j \le S_2 \\ \dfrac{\pi_2(i,j)\int_0^\infty (1-G(t))dt}{\sum_{(i,j)\in E} \pi_2(i,j)m(i,j)} & \text{for } s_1+1 \le i \le S_1-s_1-N \\ & s_2+1 \le j \le S_2 \\ \dfrac{\pi_2(i,j)\int_0^\infty e^{-\lambda t}(1-G(t))dt}{\sum_{(i,j)\in E} \pi_2(i,j)m(i,j)} & \text{for } -N+1 \le i \le s_1 \\ & s_2+1 \le j \le S_2 \end{cases}$$

where $\mu = m(i,j) = \int_0^\infty (1-G(t))dt \quad for(i,j)\in E$

6.4.2. Cost Analysis

Let T_3 be the time elapsed between two consecutive order placement epochs of C_1. Then

$$E(T_3) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_{12}=0}^{s_1+N-1} p_1^{k_1} p_2^{k_2} p_{12}^{k_{12}} \left(e^{-\lambda\mu(k_1+k_{21}+k_{12})} - e^{-\lambda\mu(k_1+k_2+k_{12}+1)}\right)$$

$$\sum_{l_1=0}^{M_1-k_{12}} \sum_{l_2=0}^{\infty} \sum_{l_{12}=0}^{M_1-k_{12}-l_1} p_1^{l_1} p_2^{l_2} p_{12}^{l_{12}} (k_1 + k + k_{12} + l_1 + l_2 + l_{12})\mu$$

$$+$$

$$\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_{12}=s_1+N} \sum_{r=1}^{\infty} \sum_{l<s_1+N} p_1^{k_1} p_2^{k_2} p_{12}^{k_{12}} \left(e^{-\lambda\mu(k_1+k_2+r(s_1+N))} - e^{-\lambda\mu(k_1+k_2+r(s_1+N)+l)}\right)$$

$$\sum_{l_1=0}^{M_1-l} \sum_{l_2=0}^{\infty} \sum_{l_{12}=0}^{M_1-l-l_1} p_1^{l_1} p_2^{l_2} p_{12}^{l_{12}} (k_1 + k_2 + k_{12} + l_1 + l_2 + l_{12})\mu$$

In the case of cost analysis of this model , only change is that, the quantity purchased in a local purchase is only s_1+N and there is no cancellation of the orders placed. Then the total expected cost per unit time under steady state

$$= \left(k_1 / E(T_3)\right) + k_2\left((p_2 + p_{12}) / \mu M_2\right) + h_1\left(\sum_{i=1}^{S_1} i \sum_{j=s_2+1}^{S_2} \pi_2(i,j)\right) + h_2\left(\sum_{j=s_2+1}^{S_2} j \sum_{i=1}^{S_1} \pi_2(i,j)\right)$$
$$+ v_1\left(M_1 / E(T_3)\right) + v_2\left((p_2 + p_{12}) / \mu\right) + v_1^1\left((s_1 + N) / E(T_3)\right)$$

6.5.Model III

In this model local purchase to clear only the backlogs, each time the inventory level drops to-N. Semi-Markov kernel in this case is given by

$$Q_3((-N+1,j_1),(0,j_2),t)=\begin{cases} p_{12}\int_0^t e^{-\lambda u}g(u)du & for\ j_2=j_1-1, \\ & s_2+2\le j_1\le S_2 \\ & and\ j_2=S_2\quad when\ j_1=s_2+1 \end{cases}$$

For all other combinations of elements of the state space, they are same as in Model I

Result 6.5.

Let T_4 be the time elapsed between two consecutive order placement epochs of C_1. Then its distribution function $F_{(4)}\{(s_1,.),(s_1,.),t\}$ is given by
$F_{(4)}\{(s_1,.),(s_1,.),t\}= F_{(4))}^1\{(s_1,.),(s_1,.),t\}+ F_{(4)}^2\{(s_1,.),(s_1,.),t\}$ where

$F_{(4)}^1= F_{(1)}^1$ of Model I and

$$F^2_{(4)}\{(s_1,.),(s_1,.),t\}=\int_{u_1=0}^{t}\int_{u_2=u_1}^{t}\int_{u_3=u_2}^{t}\int_{u_4=u_3}^{t}\int_{u_5=u_4}^{t}\int_{u_6=u_5}^{t}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_{12}=N+s_1-1}\frac{(k_1+k_2+k_{12})!}{k_1!k_2!k_{12}!}$$

$$p_1^{k_1}p_2^{k_2}p_{12}^{k_{12}}g^{*(k_1+k_2+k_{12})}(u_1)(p_1+p_{12})g(u_2-u_1)\sum_{l=0}^{\infty}b_2^{*l}((0,.),(0,.),u_3-u_2)$$

$$\lambda e^{-\lambda u_4}\sum_{l_1=0}^{M_1-s_1-1}\sum_{l_2=0}^{\infty}\sum_{l_{12}=0}^{M_1-s_1-l_1-1}\frac{(l_1+l_2+l_{12})!}{l_1!l_2!l_{12}!}\qquad p_1^{l_1}p_2^{l_2}p_{12}^{l_{12}}\frac{g^{*(l_1+l_2+l_{12})}(u_5-u_3)}{1-G(u_4-u_3)}$$

$$(p_1+p_{12})g(u_6-u_5)(1-G(t-u_6)du_6du_5du_4du_3du_2du_1$$

where

$$b_2((0,.),(0,.),t))=\frac{d}{dt}B_2((0,.),(0,.),t)and$$

$$B_2((0,.),(0,.),t)=\int_{u=0}^{t}\int_{v=u}^{t}\sum_{k_2=0}^{\infty}\sum_{k_{12}=N-1}\frac{(k_1+k_2+k_{12})!}{k_1!k_2!k_{12}!}p_1^{k_1}p_2^{k_2}p_{12}^{k_{12}}g^{*(k_1+k_2+k_{12})}(u)(p_1+p_{12})$$

$$g(v-u)(1-G(t-v))dvdu$$

Proof:

Set time equal to zero at the time of placement of orders. In $(0,U_1)$ there are exactly $(N+s_1-1)$ demands for C_1 and C_2 together (with none , one or more demands for C_2 alone ,demands that are exclusively for C_1 is not met due to restricted sales.) In (u_2, u_2+du_2) there is a demand for either C_1 or for both C_1 and C_2 so that the inventory level of C_1 drops to $-N$ resulting in a local purchase just to clear the backlogs alone. This is repeated l times (l=1,2,3,...). $B_2((0, .),(0, .), t)$ represents the distribution function of time between two consecutive local purchase epochs. The last local purchase occurs in (u_3, u_3+du_3). After the last local purchase at u_3 the natural replenishment occurs at $(u_4, u_4+d\,u_4)$. Due to l_1 demands for C_1 alone , l_{12} demands for C_1 and C_2 together, l_2 demands for C_2 alone in (u_3, u_5) , conditioned on no demand in (u_3,u_4) for C_1 or for both C_1 and C_2 together ,the level of C_1 drops to s_1+1. Finally a demand in (u_6 , u_6+du_6) for C_1 or for both C_1 and C_2, the level of C_1 becomes the reorder level s_1. Hence an order for replenishment by a quantity S_1-s_1 is placed.

6.5.1. Limiting Distribution.

The limiting probabilities immediately after a demand epoch ie. $\lim_{n\to\infty} P((X_n,Y_n)=(i,j))$ are obtained by solving

$$\pi_3 P_3=\pi_3 \text{ and } \pi_3\underline{e}=1$$

where

$$P_3 = \lim_{t\to\infty} Q_{(3)}\{(i_1, j_1), (i_2, j_2), t\}, (i_1, j_1), (i_2, j_2) \in E$$ and **e** is an (N+S₁)(S₂-s₂)

component row vector of ones. The probability distribution of the system state at arbitrary epochs are given by

$$H_3(i,j) = \lim_{t\to\infty} P_{3_{(i,j)}}(t) \text{ where}$$

$$P_{3(i,j)}(t) = \int_0^t \sum_{(i_1,j_1)\in A} Q_{(3)}((s_1, S_2), (i_1, j_1), du) P((i_1, j_1), (i, j), t-u)$$

Then

$$H_3(i,j) = \begin{cases} \pi_3(i,j) + \dfrac{\pi_3(i - S_1 + s_1, j)}{\mu} \displaystyle\int_0^\infty (1 - e^{-\lambda t})(1 - G(t))dt & for S_1 - s_1 - N + 1 \le i \le S_1 \\ & s_2 + 1 \le j \le S_2 \\ \pi_3(i,j) & for s_1 + 1 \le i \le S_1 - s_1 - N \\ & s_2 + 1 \le j \le S_2 \\ \dfrac{\pi_3(i,j)}{\mu} \displaystyle\int_0^\infty e^{-\lambda t}(1 - G(t))dt & for -N + 1 \le i \le s_1 \\ & s_2 + 1 \le j \le S_2 \end{cases}$$

$$where\ \mu = m(i,j) = \int_0^\infty (1 - G(t))dt$$

6.5.2. Cost Analysis

Let T_4 be the time elapsed between two consecutive order placement epochs of C_1. Then

$$E(T_4) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_{12}=0}^{s_1+N-1} p_1^{k_1} p_2^{k_2} p_{12}^{k_{12}} \left(e^{-\lambda\mu(k_1+k_2+k_{12})} - e^{-\lambda\mu(k_1+k_2+k_{12}+1)}\right)$$

$$\sum_{l_1=0}^{M_1-k_{12}} \sum_{l_2=0}^{\infty} \sum_{l_{12}=0}^{M_1-k_{12}-l_1} p_1^{l_1} p_2^{l_2} p_{12}^{l_{12}} (k_1+k_2+k_{12}+l_1+l_2+l_{12})\mu$$

$$+$$

$$\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_{12}=s_1+N} \sum_{r=1}^{\infty} \sum_{l<N} p_1^{k_1} p_2^{k_2} p_{12}^{k_{12}} \left(e^{-\lambda\mu(k_1+k_2+rN)} - e^{-\lambda\mu(k_1+k_2+rN+l)}\right)$$

$$\sum_{l_1=0}^{M_1-s_1-l} \sum_{l_2=0}^{\infty} \sum_{l_{12}=0}^{M_1-s_1-l-l_1} p_1^{l_1} p_2^{l_2} p_{12}^{l_{12}} (k_1+k_2+k_{12}+l_1+l_2+l_{12})\mu$$

In the case of cost analysis of this model, the quantity purchased in a local purchase is N units and no order cancellation in this also. Then the total expected cost per unit time under steady state is

$$\left(k_1/E(T_4)\right) + k_2\left((p_2+p_{12})/\mu M_2\right) + h_1\left(\sum_{i=1}^{S_1} i \sum_{j=s_2+1}^{S_2} \pi_3(i,j)\right)$$

$$+ h_2\left(\sum_{j=s_2+1}^{S_2} j \sum_{i=1}^{S_1} \pi_3(i,j)\right) + v_1\left(M_1/E(T_4)\right) + v_2\left((p_2+p_{12})/\mu\right)$$

$$+ v_1^1\left((N)/E(T_4)\right)$$

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