Stochastic Modelling: Analysis and Applications

ANALYSIS OF QUEUEING MODELS WITH WORKING VACATIONS, WORKING INTERRUPTIONS AND ON QUEUEING MODELS WITH PROCESSING OF ITEMS FOR SERVICE

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ANALYSIS OF QUEUEING MODELS WITH WORKING VACATIONS, WORKING INTERRUPTIONS AND ON QUEUEING MODELS WITH PROCESSING OF ITEMS FOR SERVICE

Ph.D. thesis in the field of Stochastic Modelling: Analysis & Applications

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To
My family
and
Teachers

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Certificate

Certified that the work presented in this thesis entitled "**Analysis of Queueing Models with Working Vacations, Working Interruptions and on Queueing Models with Processing of Items for Service**" is based on the authentic record of research carried out by Ms. Divya V. under our guidance in the Department of Mathematics, Cochin University of Science and Technology, Kochi- 682 022 and has not been included in any other thesis submitted for the award of any degree. Also certified that all the relevant corrections and modifications suggested by the audience during the Pre-synopsis seminar and recommended by the Doctoral Committee of the candidate has been incorporated in the thesis and the work done is adequate and complete for the award of Ph.D. Degree.

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Declaration

I, Divya V, hereby declare that the work presented in this thesis entitled "**Analysis of Queueing Models with Working Vacations, Working Interruptions and on Queueing Models with Processing of Items for Service**" is based on the original research work carried out by me under the supervision and guidance of Dr. A. Krishnamoorthy, formerly Professor, Department of Mathematics, Cochin University of Science and Technology, Kochi- 682 022 and has not been included in any other thesis submitted previously for the award of any degree.

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ANALYSIS OF QUEUEING MODELS WITH WORKING VACATIONS, WORKING INTERRUPTIONS AND ON QUEUEING MODELS WITH PROCESSING OF ITEMS FOR SERVICE

Contents

Notations and Abbreviations

$\mathbf{e}(a)$	:	Column vector of $1'$s of order a
$\boldsymbol{e}'_a$	:	Transpose of $\boldsymbol{e}_a$
$\mathbf{e}_a(b)$	:	column vector of order b with 1 in the ath position and the remaining entries zero
$\boldsymbol{e}'_a(b)$	:	Transpose of $\boldsymbol{e}_a(b)$
$\mathbf{e}$	:	Column vector of $1'$s of appropriate order
I_a	:	identity matrix of order a
I	:	identity matrix of appropriate dimension
LST	:	Laplace-Steiltges Transform
PH	:	Phase type
$CTMC$	:	Continuous Time Markov Chain
QBD	:	Quasi-birth-and-death
$LIQBD$	:	Level Independent Quasi-Birth-and-Death process
$LDQBD$	:	Level Dependent Quasi-Birth-and-Death process
MAP	:	Markovian Arrival Process
WV	:	Working Vacation
WI	:	Working Interruption
$\otimes$	:	Kronecker product
$\oplus$	:	Kronecker sum
$d_{ij}^{(k)}$	:	entries of D_k, $k = 0 \,\text{or}\, 1$
δ_l	:	l^{th} row sum of D_1

Chapter 1

Introduction

Stochastic Modelling is the art of modelling natural phenomena, taking into consideration the randomness involved. It combines the possibility of theoretical beauty with a real world meaning of its key concepts. Application fields such as telecommunication or insurance bring methods and results of stochastic modelling to the attention of theoreticians and practitioners.

One of the most important domains in stochastic modelling is the field of queueing theory. We can see queues in almost all walks of life. For instance, in banks, super market check-out counters, airport check-in systems, doctor's clinic, manufacturing systems, communication systems. The queues may be visible or not. Apparently, nobody wants to be in queue for a long time. Thus analyzing these congestion situations using appropriate queueing models has a great significance in this modern world.

Queueing theory is the probabilistic study of waiting lines and it is very useful for analyzing the procedure of queueing of daily life of human being. It deals with techniques for analyzing congestion situations. Many real systems can be reduced to components which can be modelled by the concept of a socalled queue. The formation of queue is a common phenomenon which occurs whenever the current demand for a service exceeds the current capacity to provide that service. The pioneer investigator was the well-known Danish Mathematician A.K.Erlang, who in 1909 published 'The Theory of Probabilities and Telephone Conversations' in which he studied the problem of telephone traffic congestion. A queue consists of a system into which there comes a stream of users who demand some capacity of the system over a certain time interval before they leave the system. Users are served in the system by one or many servers. The former describe the input into a queue, while the latter represents the function of the inner mechanisms of a queueing system.

Until middle of 1970's queueing theorists were heavily depending on complex analytic tools for solving queueing models. Motivated by this fact, in 1975, Marcel F. Neuts developed Phase type distributions and Matrix analaytic methods. The representation of system elements by phase-type distributions and their analysis by matrix-analytic method has significantly expanded the scope of queueing systems for which many useful results can be derived.

1.1 Phase Type distribution (Continuous time)

The continuous PH distributions are introduced as a natural generalization of the exponential and Erlang distributions. A PH-distribution is obtained as the distribution of the time until absorption in a Markov chain having a finite state space and an absorbing state. Phase-type distributions have matrix representations that are not unique. Furthermore, any probability distribution defined on the nonnegative real line can be approximated arbitrarily closely by a phase-type distribution. This means that the class of PH distributions is dense in the family of continuous distributions of random variables on the non-negative half of the real line.

Consider a Markov process $\chi = \{X(t) : t \geq 0\}$ having finite state space $\{1, 2, \ldots, m+1\}$ and the infinitesimal generator matrix

$$Q = \begin{pmatrix} T & \mathbf{T}^0 \\ \mathbf{0} & 0 \end{pmatrix}$$

where T is a square matrix of order m, $\boldsymbol{T}^0$, a column vector and 0, the zero row vector of the same dimension. The initial distribution of χ is given by the row vector $\bar{\boldsymbol{\alpha}} = (\boldsymbol{\alpha}, \alpha_{m+1})$, with $\boldsymbol{\alpha}$ a row vector of dimension m. The states $\{1, ..., m\}$ are transient, while the state m + 1 is absorbing. Let $Y := \inf\{t \geq 0 : X(t) = m + 1\}$ denote the random variable of the time until absorption in state $m + 1$. The distribution of Y is called phase–type distribution (or shortly PH distribution) with parameters $(\boldsymbol{\alpha}, T)$. We write $Y \sim PH(\boldsymbol{\alpha}, T)$. The dimension m of T is called the order of the distribution PH$(\boldsymbol{\alpha}, T)$. The states $(1, ..., m)$ are also called phases, which gives rise to the name phasetype distribution. Let $\boldsymbol{e}$ denote the column vector of dimension m with all entries equal to one. Also, we have $\boldsymbol{T}^0 = -T\boldsymbol{e}$ and $\alpha_{m+1} = 1 - \boldsymbol{\alpha e}$. These follow immediately from the properties that the row sums of a generator are zero and the sum of a probability vector is one. The vector $\boldsymbol{T}^0$ is called the exit vector of the PH distribution.
The distribution function of Y is given by

$$F(t) := P(Y \leq t) = 1 - \boldsymbol{\alpha} e^{Tt}\boldsymbol{e}, \text{ for all } t \geq 0$$

and its density function is

$$f(t) = \boldsymbol{\alpha} e^{Tt}\boldsymbol{T}^0, \text{ for all } t > 0.$$

Here, the function $e^{Tt} = exp(Tt) = \sum_{n=0}^{\infty} \frac{t^n}{n!} T^n$ denotes a matrix exponential function.
The Laplace-Stieltjes transform of $F(t)$ is given by

$$\phi(s) = \int_0^{\infty} e^{-st} dF(t) = \alpha_{m+1} + \boldsymbol{\alpha}(sI - T)^{-1}\boldsymbol{T^0}$$

for all $s \in C$ with $Re(s) \geq 0$.
The moments of Y are given by

$$E(Y^n) = (-1)^n n! \boldsymbol{\alpha} T^{-n} \boldsymbol{e}$$

for all $n \in N$.

Theorem 1.1.1 (see Theorem 9.3 of [5]). *Let F denote a PH($\boldsymbol{\alpha}, T$) distribution function. F is non defective, i.e. $F(\infty) = 1$ for all $\boldsymbol{\alpha}$, if and only if T is invertible. In this case, $(-T^{-1})_{ij}$ is the expected total time spent in state j given that the process χ started in state i.*

For further information about the PH distribution, see, *Neuts*, [40], *Breuer and Baum*, [5], *Latouche and Ramaswami* , [33] and *Qi-Ming He*, [42]. Usefulness of PH distribution as service time distribution in telecommunication networks is elaborated, e.g., in *Pattavina and Parini* [41] and *Riska, Diev and Smirni* [43].

1.2 Markovian Arrival Process

Markovian Arrival Processes (MAP) introduced in *Neuts* [40] is a rich class of point processes that includes many well-known processes such as Poisson, PH-renewal processes and Markov-modulated Poisson process. A significant feature of the MAP is the underlying Markovian structure that fits ideally in the context of matrix-analytic solutions to stochastic models. MAP is a generalization of the Poisson process, which keeps many useful properties of the Poisson process. For example, the memoryless property of the Poisson process is partially preserved by the MAP by conditioning on the phase of the underlying Markov chain. Any stochastic counting process can be approximated arbitrarily closely by a sequence of Markovian arrival processes. MAP is a convenient tool to model both renewal and non-renewal arrivals. In [6], *Chakravarthy* provides an extensive survey of the Batch Markovian Arrival Process (BMAP) in which arrivals are in batches where as it is in singles in MAP.

A continuous time Markov chain $\{(N(t), I(t)), t \geq 0\}$ with state space $\{(i, j) : i \geq 0, 1 \leq j \leq m\}$ and infinitesimal generator

$$\mathcal{Q} = \begin{bmatrix} D_0 & D_1 & & \\ & D_0 & D_1 & \\ & & D_0 & D_1 \\ & & \ddots & \ddots \end{bmatrix}.$$

is called a MAP with matrix representation (D_0, D_1). Here D_0 and D_1 are square matrices of order m, where m is a positive integer. The diagonal elements of D_0 are negative and its off-diagonal elements are nonnegative, D_1 has all its elements nonnegative and $D_0 + D_1$ is an infinitesimal generator. Let $D_0 = (d_{ij}^{(0)})$ and $D_1 = (d_{ij}^{(1)})$, then $d_{ij}^{(0)}$ is the rate of transitions from phase i to j without an arrival, for $i \neq j$; $d_{ij}^{(1)}$ is the rate of transitions from phase i to j with an arrival and $-d_{ii}^{(0)}$ is the total rate of events in phase i. Let $N(t)$ denote the number of arrivals in $(0, t)$ and $I(t)$ the phase of the Markov chain at time t. Let $\boldsymbol{\pi}^*$ be the stationary probability vector of D. Then the constant $\beta^* = \boldsymbol{\pi}^* D_1 \boldsymbol{e}$, referred to as *fundemental rate*, gives the expected number of arrivals per unit of time in the stationary version of the MAP.

1.3 Quasi-birth-death processes

Consider a Markov process with $\{X(t),\ t \in \mathbf{R}^+\}$ on the bivariate state space $\Omega = \bigcup_{n \geq 0} \{(n, j) : 1 \leq j \leq m\}$. The first coordinate n represents the level, and j the phase of the n^{th} level. The number of phases in each level may be either finite or infinite. The Markov process is called a QBD process if one-step transitions from a state are restricted to the same level or to the two adjacent levels. In other words,

$$(n-1, j') \rightleftharpoons (n, j) \rightleftharpoons (n+1, j'') \quad \text{for } n \geq 1.$$

If the transition rates are level independent, the resulting QBD process is called level independent quasi-birth-death process (LIQBD); else it is called level dependent quasi-birth-death process (LDQBD). Arranging the elements of Ω in lexicographic order, the infinitesimal generator of a LIQBD process is block tridiagonal and has the following form:

$$\mathcal{Q} = \begin{pmatrix} B_1 & A_0 & & & \\ B_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \end{pmatrix} \tag{1.1}$$

where the matrices A_0, A_1, A_2 are square and have the same dimension; matrix B_1 is also square and need not have the same size as A_1. Also, the matrices B_2, A_2 and A_0 are nonnegative and the matrices B_1 and A_1 have nonnegative off-diagonal elements and strictly negative diagonals. The row sums of Q are equal to zero, so that we have $B_1\boldsymbol{e} + A_0\boldsymbol{e} = B_2\boldsymbol{e} + A_1\boldsymbol{e} + A_0\boldsymbol{e} = (A_0 + A_1 + A_2)\boldsymbol{e} = \mathbf{0}$.

Among the various tools that we used in this thesis Matrix geometric method plays an important role. A brief description of this is given below.

1.4 Matrix Geometric Method

Marcel F. Neuts pioneered matrix-geometric methods in the study of queueing models in the 1970s. The transform techniques used in solving QBD processes are replaced largely by the matrix geometric approach with the advent of high speed computers and efficient algorithms. In matrix geometric method the distribution of a random variable is defined through a matrix; its density function, moments, etc. are expressed with this matrix. The modelling tools such as Phase type distributions, Markovian Arrival Processes, Batch Markovian Arrival Processes, Markovian Service Processes etc. are well suited for Matrix Geometric Methods. The power and popularity of matrix-geometric methods come from their flexibility in stochastic modelling, ability for analytic exploration, natural algorithmic thinking, and tractability in numerical computation.

Theorem 1.4.1 (see Theorem 3.1.1. of *Neuts* [40]). *The process $\boldsymbol{Q}$ in (1.1) is positive recurrent if and only if the minimal non-negative solution R to the matrix-quadratic equation*

$$R^2 A_2 + RA_1 + A_0 = 0 \tag{1.2}$$

has all its eigenvalues inside the unit disk and the finite system of equations

$$\begin{aligned} \boldsymbol{x}_0 (B_1 + RB_2) &= \boldsymbol{0} \\ \boldsymbol{x}_0 (I - R)^{-1} \boldsymbol{e} &= 1 \end{aligned} \tag{1.3}$$

has a unique positive solution $\boldsymbol{x}_0$.

If the matrix $A = A_0 + A_1 + A_2$ is irreducible, then $sp(R) < 1$ if and only if

$$\boldsymbol{\pi} A_0 \boldsymbol{e} < \boldsymbol{\pi} A_2 \boldsymbol{e} \tag{1.4}$$

where $\boldsymbol{\pi}$ is the stationary probability vector of A.

The stationary probability vector $\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \ldots)$ of $\boldsymbol{Q}$ is given by

$$\boldsymbol{x}_i = \boldsymbol{x}_0 R^i \quad \text{for } i \geq 1. \tag{1.5}$$

Once R, the rate matrix, is obtained, the vector $\boldsymbol{x}$ can be computed. We can use an iterative procedure or logarithmic reduction algorithm (see *Latouche and Ramaswami* [33]) or the cyclic reduction algorithm (see *Bini and Meini* [4]) for computing R.

1.5 Computation of R matrix

There are many algorithms for finding rate matrix R. Here we describe one of them.

Iterative algorithm

From (1.2), we can evaluate R in a recursive procedure as follows.

Step 0: $R(0) = 0$.

Step 1:

$$R(n+1) = A_0(-A_1)^{-1} + R^2(n)A_2(-A_1)^{-1}, \quad n = 0, 1, \dots$$

Continue **Step 1** until $R(n+1)$ is close to $R(n)$.

That is, $||R(n+1) - R(n)||_\infty < \epsilon$.

1.6 Review of related work

In classical queueing systems, servers are always available. But in vacation queueing systems, the server may not be available for a certain duration of time since he has to attend some supplementary jobs or is to undergo maintenance work or by its failure resulting in interruption of current service or simply to take a break. Levy and Yechiali [35] introduced the concept of server vacation.They considered both single vacation and multiple vacation queueing models. Under a single vacation policy, after taking a vacation at the end of a busy period, the server either serves the waiting customers, if any, else stays idle. Under multiple vacation policy, the server takes vacations until it finds at least one customer waiting in the system at a vacation completion instant.

Considerable number of work in this area upto 1986 were surveyed by Doshi in [9]. More studies on vacation models could be found in Takagi [47] and in Tian and Zhang [49]. Servi and Finn [46] introduced the concept of a working vacation in which the server offers services at a lower rate during vacation if customers are available.They computed explicit formulae for the mean, variance and distribution of the number of customers and time spent by a customer in the system. Kim et al. [25] considered the M/G/1 queue with working vacations and obtained the steady-state queue length distribution. Wu and Takagi [55] considered M/G/1 queue with multiple working vacations and obtained the distribution of the queue size and the time in the system for an arbitrary customer in the steady-state. The concept of vacation interruption was introduced by Li and Tian [36]. They studied the M/M/1 queue with working vacations and vacation interruptions. Under the vacation interruption policy, the server can come back from the vacation without completing the vacation. By employing the matrix-geometric method, they obtained the distributions and the stochastic decomposition for the number of customers and the waiting time. Li et al. [37] analyzed a single server vacation queue with a general arrival process with working vacation and vacation interruption. By matrix manipulations they obtained various performance measures such as mean queue length and waiting time.

In classical queueing models N-policy is used as a control mechanism to start service when the number of customers present in the system hits N, starting from the epoch the server becomes idle due to the system becoming empty. Yadin and Naor [51] introduced the concept of N-policy for M/M/1 queueing system without start-up time. Lee et al. [34] considered an $M^X/G/1$ queueing system with N-policy and multiple vacations. They obtained the system size distribution and showed that the system size could be decomposed into three random variables one of which is the system size of ordinary $M^X/G/1$ queue.They also derived the waiting time distribution, some performance measures and also a condition under which the optimal stationary operating policy is achieved under a linear cost structure. Kasahara et al. [22] considered MAP/G/1 queueing systems with and without vacations. For both the cases, they analyzed the stationary queue length and the waiting time distributions, and derived recursive formulas to compute the moments of those distributions. Also they provided a numerical algorithm to obtain the mass function of the stationary queue length.

Zhang and Hou [56] considered the MAP/G/1 queue with working vacations and vacation interruption and obtained the queue length distributions. Cosmika and Selavaraju [14] analyzed a working vacation queueing model with priority customers where the service time of customers follows phase-type distributions. They assumed that after serving a customer in working vacation, if the server finds any customer waiting in the queue, the vacation is interrupted and the server switches to normal service mode. They derived distributions of duration of a busy period, busy cycle, queue length and waiting time for the two types of customers.

Sreenivasan et al. [45] studied a MAP/PH/1 queueing model with working vacations, vacation interruptions and N-policy. The server takes vacation and offers service at a lower rate during those times. The server returns to normal state whenever a random clock expires or the queue length hits a specific threshold value whichever occurs first. They analyzed the model in steady state using matrix analytic methods.

Queues with interruption play an important role in day to day life. We encounter different kinds of interruptions in various activities like internet browsing, banking, medical check ups, in supermarkets etc. The works so far reported in the literature discuss about interruptions such as server induced, customer induced, enviornment dependent service interruptions, server vacations, vacation interruptions and interruption due to arrival of a priority customer. The first reported work on queues with service interruption is by White and Christie[54] in which they considered a two-priority single server system with the low priority customer in service pre-empted on arrival of a high priority customer. Even in the case of single class customer system, the customer in service has to wait whenever a system breakdown occurs. The interrupted service starts from the very beginning (repeat) or from where it got interrupted (resumption) on completion of interruption. These two cases are separately considered in Keilson [24], Gaver [13] and by several other researchers. Fiems et al. [12] introduced probability measures for repeat/resumption on completion of interruption without assigning any rule. Krishnamoorthy et al. [30] are the first to give a specific rule for resumption/repetition of service. We refer the review paper by Krishnamoorthy et al. [29] for details on queueing models with system induced service interruption (priority queues not included).

Varghese et al.[50] introduced a new type of interruption called customer induced interruption in which a customer interrupts own service. They considered an infinite capacity queueing system with a single server in which customers arrive according to a Poisson process with the service time following an exponential distribution. The interruptions occur according to a Poisson process and the duration of each interruption follows an exponential distribution. The self-interrupted customers enter into a finite buffer of size K. Any interrupted customer, finding the buffer full, is considered lost. Those interrupted customers who complete their interruptions move into another buffer of same size and are given a nonpreemptive priority over new customers. They evaluated several performance measures. Numerical illustrations of the system behavior are provided and also discussed an optimization problem through an illustrative example. Krishnamoorthy et al. [31] extended this to a multi-server queueing system. They investigated the behavior of the queueing system, several performance measures are evaluated and provided numerical illustrations of the system behaviour. Also an optimization problem to maximize the revenue with respect to number of servers and optimal buffer size for the self-interrupted customers are discussed through two illustrative examples. Dudin et al. [10] extended these to MMAP/PH(PH)/c queue with negative arrivals.

Varghese and Krishnamoorthy [32] considered a single-server retrial queue with infinite capacity of the primary buffer and finite capacity of the orbit to which customers arrive according to a Poisson process, and the service time follows phase-type distribution. The customer-induced interruption occurs according to a Poisson process. The self-interrupted customers enter into the orbit. Any interrupted customer, finding the orbit full, is considered lost. The interrupted customers retry for service after the interruption is completed. Several performance measures are evaluated and some numerical illustrations of the system behavior provided.

In most of the work reported in queueing theory it is implicitly assumed that if the server is ready to serve and customers are available to receive service then the service process proceeds. Either availability of "additional" items required to provide service is not taken into consideration/ignored or its abundance is taken for granted. In the latter case the holding cost incurred is completely ignored. Sometimes the item(s) required for service may not be available. In such cases service cannot be provided even when server(s) is/are readily available and customer(s) are waiting.

Thus in several cases availability of both customers and servers alone cannot guarantee service. This naturally leads to the investigation of availability of additional item(s) required to provide service. Then some control problems also arise– how much of additional item(s) to be held, time required to procure such items and so on. This leads to the consideration of holding cost, shortage cost and associated revenue loss. Kazimirisky [23] seems to be the first to introduce 'additional items needed for service'. He considered a BMAP/G/1 queue, with the server engaged in producing additional items whenever customers are not waiting. In most of the work on queues with 'additional items' for service exactly one processed item is assumed to be required for each customer. Customer service time distribution depends on whether processed item is available or not. Thus there are two distinct service time distributions.

Baek et al. [3] considered MMAP of customers of two types– type I(high priority) and type II (low priority). Both type of customers require a certain minimum number of additional items to start their service. Type I customers do not have space to wait. If a type I customer is in service while another type I customer arrives, the latter leaves the system. On the other hand if a type II customer is in service, the former is pushed out of the system by the type I arrival, provided the number of additional items available is atleast equal to the minimum number required to start its service. Else, it leaves the system without changing the status. Type II customers have an infinite capacity waiting space. Additional items arrive to the system according to MAP. They invetigate system stability and analyze its performance. Dhanya et al.[7] extend the above to retrial queueing set up.

Hanukov et al. [17] analyze a single server queueing system where again additional items is needed for service of a customer (one item for each customer). The arrival process is Poisson and service time is exponenetially distributed. The service consists of two independent stages. The first stage can be performed even in the absence of customers, whereas the second stage requires the customer to be present. When the system is devoid of customers, the server produces an inventory of first stage called 'preliminary ' services, which is used to reduce customer's overall sojourn times. Hence in this model customer will not have to wait for the entire service to be carried out from the beginning, provided processed item is available at the time the customer is taken for service. Such customers have a shorter service time in comparison to those who encounter the system with no processed item when taken for service. Divya et al. [8] considered single server queue in which customers arrive according to MAP with representaion (D_0, D_1) of order n. The details are given in chapter 4 of this thesis.

In real life, people become impatient while waiting for service. Hence to model reality, we should take into consideration customer's impatience. To characterize customers' impatient behaviour, some terminologies like balking, reneging and retrials are employed in queueing system. Balking customers decide not to join the queue if it is too long and reneging customers leave the queue if they have waited too long for service. Retrial queues study systems where customers do not wait in a line (provided there is no buffer to wait) when server is found to be busy; instead they keep repeating their attempts to access the server at random time points (see Falin and Templeton [11], Artalejo and Gomaz-Corral [1]). Wang et al [52] has presented a review on queuing systems with impatient customers.

Wang and Zhang [53] consider a single-server service-inventory system where customers arrive according to a Poisson process and the service times are independent and exponentially distributed. A customer takes exactly one item from the inventory upon service completion. A continuous review policy is adopted to replenish the inventory. With two different information levels, i.e. the fully unobservable case and the partially observable case, arriving customers decide whether to join or to balk the system. They investigated customers' individually optimal and socially optimal strategies, and further consider the optimal pricing issue that maximises the servers revenue. Some numerical experiments are carried out to show that the individually optimal joining probability (or threshold) is not always greater than that of socially optimal one. It was observed that, to maximise the servers revenue, concealing some system information from customers may be more profitable. Conversely, to maximise the social welfare, the customers need more system information. Finally, numerical results in the fully unobservable case illustrate a reasonable phenomenon that the revenue maximum is equal to social optimum in most cases.

1.7 Summary of the thesis

In this thesis we discuss a few queueing models with working vacation, working interruption and with processing of items for service by identifying continuous time Markov chains. The modelling tools like Poisson process, Markovian

Arrival Process (MAP) and Phase type distributions (PH-distributions) are used. The resulting QBD process are analyzed algorithmically using matrix geometric method. Numerical examples are done using MATLAB Program.

Now we turn to the content of the thesis. This thesis entitled **'Analysis of Queueing Models with Working Vacations, Working Interruptions and on Queueing Models with Processing of Items for Service'**, is divided into 6 chapters including the present introductory chapter(chapter 1).

In chapter 2, we study two single server queueing models with non-preemptive priority and working vacation under two distinct N-policies. High priority(type I) customers are served even in vacation mode whereas low priority(type II) customers are served only when the server comes to normal mode of service. Type I customers have only a limited waiting space L whereas type II customers have unlimited capacity. The two distinct N-policies are as described below: In model I, while service of type I customers are in progress in vacation mode (working vacation), if the number of such customers present in the system hits N $(\leq L)$ or the vacation timer(clock) expires, whichever occurs first, the server is switched to normal mode. In model II, switching the server to normal mode from vacation mode occurs as soon as the accumulated number(those served out plus those present in the system) of type I customers during that working vacation hits N or the vacation timer expires, whichever occurs first. Type I customers arrive according to a Poisson process whereas type II customer's arrival is governed by Markovian Arrival Process(MAP). Service time of type I and type II customers follow distinct phase type distributions. At a service completion epoch, finding the system empty, the server takes an exponentially distributed working vacation. During working vacation, type I customers are served at a reduced rate. On vacation expiration, the service of the type I customer, already in service, will start from the beginning in the normal mode of service. We analyze these models in steady state to compute the distribution of the duration of service time continuously in slow mode, expected number of returns to 0 type I customer state, starting from 0 type I customer state during vacation mode of service before the arrival of a type II customer, the distribution of a p-cycle in normal mode, LSTs of busy cycle, busy period of type I customers generated during the service time of a type II customer and LSTs of waiting time distributions of type I and type II customers. We compare these models in steady state by numerical experiments to identify the superior model.

In chapter 3, we study a (M,MAP)/(PH,PH)/1 queue with nonpreemptive priority, working interruption and protection from interruption. Two types of priority classes of customers, where type I customers arrive according to a Poisson process and type II customers arrive according to Markovian Arrival Process are considered. Service time of both type I and type II customers follow mutually independent phase type distributions. The number of type I customers in the system is restricted to a maximum of L. Also type I customers are assumed to have a non-preemptive priority over type II customers. Customer services are subject to interruption by a self–induced mechanism. The interruptions occur according to a Poisson process. Instead of stopping service completely, the service continues at a slower rate during interruption. Also we assume that an interruption occuring while customer is already under interruption will not affect the customer.The server continues to serve at this lower rate until interruption is fixed. The duration of interruption is assumed to be exponentially distributed. A protection mechanism to reduce the effect of interruptions on type I customers service is arranged.The protection for the service of type I customers is provided at the epoch of realization of the clock which starts ticking at the moment a type I customer is taken for service. Type II customers are not provided protection against interruption during their service. Also we assume that type I customers get service at a faster rate starting from the epoch of providing service protection. We analyse the distribution of service time duration of both type I and type II customers and the distribution of a p-cycle. Also we provide LSTs of busy cycle, busy period of type I customers generated during the service time of a type II customer and LSTs of waiting time distributions of type I and type II customers. Also we compute the expected number of interruptions during a type I and a type II service. We perform numerical computations to evaluate important system characteristics and also optimal system cost using a cost function .

In chapter 4, we study a MAP/(PH,PH)/1 queue with processing of service items under Vacation and N-policy. We assume that customers arrive at a single server queueing system according to Markovian Arrival process. When the system is empty, the server goes for vacation and produces inventory for future use during this period. The maximum number of inventory at a stretch is L. The inventory processing time follows phase type distribution. These are required for the service of customers-one for each customer. The server returns from vacation when there are N customers in the system. The service time follows two distinct phase type distributions depending on whether there is processed item or no processed item available at service commencement epoch. We analyse the distribution of time till the number of customers hit N or the inventory level reaches L, that of idle time, the distribution of time until the number of customers hit N and also the distribution of the number of inventory processed before the arrival of the first customer in a cycle. Also we provide the distribution of a busy cycle, LSTs of busy cycles in which no item is left in the inventory and that of at least one item left in the inventory. We perform some numerical experiments to evaluate the expected idle time, standard deviation and coefficient of varaiation of idle time of the server .

In chapter 5, we extend the queueing model considered in the previous chapter to the case where the customers are impatient. Arriving customers join the queue with probability p or balk with probability $1-p$. Also the customers waiting for service become impatient and renege after a random time period which is exponentially distributed. Thus the system is level dependent. We find the distribution of time until the number of customers hit N. Several system performance characteristics are computed. Also we compute LST of the waiting time distribution for the case of no reneging. For the special case of no reneging, some numerical experiments for computing individual optimal strategy, maximum revenue to the server and social optimal strategy are also discussed.

In last chapter, we study a two-server queueing system in which the customers arrive according to Markovian Arrival Process. Each customer is to be provided with a processed item at the end of his service. Server 1 provides service only, whereas Server 2 provides service and also processes the item required to serve customers. The maximum inventory level permitted is L. The inventory processing time follows phase type distribution. After processing L items, server 2 starts serving customers, if any waiting; else stays idle. Server 1 is dedicated to service only. Service is rendered only if there are processed items. Also, if at the time of arrival of a customer both servers are idle, server 1 provides him service and server 2 continues to remain idle even if it has completed the processing of L items. The duration of service time given by both servers follow phase type distributions of same order, but server 1 provides service at a slower rate than server 2. If the inventory level drops to a predetermined level s at a customer departure epoch due to a service completion by server 2, then he starts processing items. If the inventory level drops to level s due to a service completion by server 1, then the customer served by server 2 is shifted to server 1 to provide him the residual service and server 2 starts processing items. The arrival process is independent of the inventory processing and service process. The long run behaviour of the system is analyzed under condition for stability. We derive some important distributions associated with the model. Numerical investigation of the optimal values of L and s is provided.

Finally a section "concluding remarks and suggestions for future study", is included.

Chapter 2

(M, MAP)/(PH, PH)/1 queue with Non-preemptive priority and working vacation under N-policy

In this chapter we analyze two single server queueing models with two priority classes of customers where type I customers are assumed to have a non-preemptive priority over type II. The server goes on working vacation whenever the system becomes empty. Further the working vacation ends as soon as N customers accumulate. A working vacation queueing system provides relief to customers since the server is always available for service, though at a reduced rate, at the beginning of a cycle. Thus customer impatience gets reduced

1. Presented in the International Conference on Stochastic Modelling Analysis and Applications organised by the Centre for Research, Department of Mathematics, CMS College, Kottayam held on 10 and 11 January 2018.
2. Some results of this chapter are included in the following paper.
A. Krishnamoorthy, Divya V.: **(M, MAP)/(PH, PH)/1 queue with Non-preemptive priority and working vacation under N-policy** (communicated).

through introduction of working vacation in the place of vacation (without service). We have introduced a two-priority system where high priority customers alone are served during working vacation. This is a realistic situation since the system has to take care of impatience of such customers more than that of low priority customers. Further we imposed finite capacity for the High priority queue; this is to ensure "not too large waiting time" for such customers. In the N-policy introduced by Yadin and Naor [51], the server waits (or server is not activated) until the number of customers present in the system becomes N to start service in every new cycle. A customer arriving during this time will have to wait until the server is activated. The customers could become impatient while no service is provided. The purpose is to extend the duration of a busy period and thus reduce per unit time cost to the system. In a working vacation queueing model the above definition of N-policy needs modification. In Sreenivasan et al.[45] the N-policy is introduced as follows: The server goes on vacation when, at the end of a service, no customer is left in the system. However, he starts giving service at a slower mode with the arrival of the first customer to the system. This is called working vacation since the server serves even during vacation. New customers may arrive during that service time. The service continues to be on vacation mode until either the number of customers in the system reaches N or the vacation timer expires, whichever occurs first. In the absence of occurence of these events, the server goes for another vacation when the system becoming empty again. If the vacation timer has large mean value and arrival rate is much slower than even service rate during working vacation, it will take a long time for N customers to be present at any given time. In fact, quite often the system becomes empty more often than the service hits normal mode.

We introduce another type of N-policy, in connection with working vacation. The server on vacation serves in working vacation mode customers who arrive after the just concluded busy period. This continues until the vacation timer expires or the number of customers present in the system plus number of customers already served (accumulated number) during the current vacation hits N, whichever occurs first; else the server goes for another vacation since the system is found to be empty immediately after completion of a service. We provide a comparison between the two models to check which is superior under given conditions. During working vacation type I customers alone recieve service. This assumption can be justified; type I customers are more impatient than type II, though we have not brought in this paper the customer impatience factor.

In model I, we use N-policy as a control mechanism to end a working vacation, as described: During a working vacation, either N type I customers should be present in the system at a given epoch or the vacation clock should expire, whichever occurs first, inorder to switch to normal mode of service. In model II also we use N-policy as a control mechanism to terminate a working vacation: During a working vacation, either the number of type I customers present in the system plus number of type I customers already served during that vacation hits N or the vacation clock expires, whichever occurs first inorder to switch to normal mode of service. Type I customers alone are served during working vacation. Thus the idle time of the server in the discussed N–policy is better utilized in working vacation under N–policy. This also helps in reducing impatience of high priority customers. Further since the normal mode is realized in model II at a higher rate than in model I, we expect the former to perform better, which is seen to be true through numerical experiments.

2.1 Model Description and Mathematical formulation of model I

We consider a single server queue with two priority classes of customers where type I customers arrive according to a Poisson process with rate λ and type II customer arrival follows a Markovian Arrival Process with representation

(D_0, D_1) of order n. Service time of type I customer is assumed to be of phase type distributed with representation $(\boldsymbol{\alpha},\, T)$ of order m and of a type II customer is assumed to be of phase type distributed with representation $(\boldsymbol{\alpha'},\, T')$ of order m'. The maximum number of type I customers in the system is restricted to L. They are assumed to have a non-preemptive priority over type II customers. At a service completion epoch, finding the system empty, server takes a WV. The duration of vacation is assumed to be exponentially distributed with parameter η. Type I customers arriving during vacation are served at a lower rate(WV): Phase Type distribution with representation $(\boldsymbol{\alpha}, \theta T)$, $0 < \theta < 1$. Thus the expected service rate in normal mode is $\mu = [\boldsymbol{\alpha}(-T)^{-1}\boldsymbol{e}]^{-1}$ and $\theta\mu$ is the rate of the vacation mode of service. If on completion of service of a type I customer during WV, no type I is waiting, then the server continues in vacation, even if type II customers are available in the system. The server turns to normal working mode during a WV either when the vacation clock expires or when the number of type I customers in the system hits level N, $1 \le N \le L$ whichever occurs first. Type II customers are considered for service only when on completion of vacation, no type I customer is present in the system or on service completion of a type I customer in normal mode none of type I customer is left in the system. The expected service rate of a type II customer is $\mu' = [\boldsymbol{\alpha'}(-T')^{-1}\boldsymbol{e}]^{-1}$. Also on vacation expiration, the service of the type I customer already in service, starts from the beginning in the normal mode of service.

Let $Q^* = D_0 + D_1$ be the generator matrix of the type II arrival process and $\boldsymbol{\pi^*}$ be its stationary probability vector. Hence $\boldsymbol{\pi^*}$ is the unique (positive) probability vector satisfying

$$\boldsymbol{\pi^*} Q^* = 0,\ \boldsymbol{\pi^*}\boldsymbol{e} = 1$$

The constant $\beta^* = \boldsymbol{\pi^*} D_1 \boldsymbol{e}$, referred to as *fundemental rate*, gives the expected number of type II arrivals per unit of time in the stationary version of the

MAP. It is assumed that the two arrival processes are independent of each other and are also independent of the service processes.

2.1.1 The QBD process

The model described above can be studied as a LIQBD process. First we introduce the following notations:
At time t:
$N_1(t)$: the number of type II customers in the system,
$N_2(t)$: the number of type I customers in the system,

$$S(t) = \begin{cases} 0, & \textit{if the server is on vacation/on WV} \\ 1, & \textit{if type I customer in service and service in normal mode} \\ 2, & \textit{if type II customer in service} \end{cases}$$

$J(t)$: the phase of the service process when the server is busy
$M(t)$: the phase of arrival of the type II customer.

It is easy to verify that $\{(N_1(t), N_2(t), S(t), J(t), M(t)) : t \geq 0\}$ is a LIQBD with state space

$$\Omega = \cup_{i=0}^{\infty} l(i)$$

where $l(0) = \{(0,0,k) : 1 \leq k \leq n\} \cup \{(0, i_2, j_1, j_2, k) : 1 \leq i_2 \leq N-1; j_1 = 0 \, or \, 1; 1 \leq j_2 \leq m; 1 \leq k \leq n\} \cup \{(0, i_2, 1, j_2, k) : N \leq i_2 \leq L; 1 \leq j_2 \leq m; 1 \leq k \leq n\}$ and for $i_1 \geq 1$,
$l(i_1) = \{(i_1, 0, 0, k) : 1 \leq k \leq n\} \cup \{(i_1, 0, 2, j_2, k) : 1 \leq j_2 \leq m'; 1 \leq k \leq n\} \cup \{(i_1, i_2, 0, j_2, k) : 1 \leq i_2 \leq N-1; 1 \leq j_2 \leq m; 1 \leq k \leq n\} \cup \{(i_1, i_2, 1, j_2, k) : 1 \leq i_2 \leq L; 1 \leq j_2 \leq m; 1 \leq k \leq n\} \cup \{(i_1, i_2, 2, j_2, k) : 1 \leq i_2 \leq L; 1 \leq j_2 \leq m'; 1 \leq k \leq n\}$

Note that when $N_1(t) = N_2(t) = 0$, server will be on vacation and so $S(t)$ and $J(t)$ need not be considered. Also when $N_2(t) = 0$ and $S(t) = 0$, then

$J(t)$ need not be considered.The only other component in the state vector in both cases would be $M(t)$.

The infinitesimal generator of this CTMC is

$$\mathcal{Q}_1 = \begin{bmatrix} B_0 & C_0 & & & \\ B_1 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}.$$

where B_0 contains transitions within the level 0; C_0 represents transitions from level 0 to level 1; B_1 represents transitions from level 1 to level 0; A_0 represents transitions from level h to level $h{+}1$ for $h \geq 1$, A_1 represents transitions within the level h for $h \geq 1$ and A_2 represents transitions from level h to $h-1$ for $h \geq 2$. The boundary blocks B_0, C_0, B_1 are of orders $n(1+m(L+N-1)) \times n(1+m(L+N-1))$, $n(1+m(L+N-1)) \times n(1+mN+(L-1)m+(L+1)m')$, $n(1+mN+(L-1)m+(L+1)m') \times n(1+m(L+N-1))$ respectively. A_0, A_1, A_2 are square matrices of order $n(1+mN+(L-1)m+(L+1)m')$.

Define the entries of $B_{0_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)}$, $C_{0_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)}$ and $B_{1_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)}$ as transition submatrices which contains transitions of the form $(0, i_1, j_1, k_1, l_1) \rightarrow (0, i_2, j_2, k_2, l_2)$, $(0, i_1, j_1, k_1, l_1) \rightarrow (1, i_2, j_2, k_2, l_2)$ and $(1, i_1, j_1, k_1, l_1) \rightarrow (0, i_2, j_2, k_2, l_2)$ respectively. Define the entries of $A_{0_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)}$, $A_{1_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)}$ and $A_{2_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)}$ as transition submatrices which contains transitions of the form $(h, i_1, j_1, k_1, l_1) \rightarrow (h{+}1, i_2, j_2, k_2, l_2)$, where $h \geq 1$; $(h, i_1, j_1, k_1, l_1) \rightarrow (h, i_2, j_2, k_2, l_2)$, where $h \geq 1$ and $(h, i_1, j_1, k_1, l_1) \rightarrow (h-1, i_2, j_2, k_2, l_2)$, where $h > 1$ respectively. Since none or one event alone could take place in a short interval of time with positive probability, in general, a transition such as $(i_1, i_2, j, k, l) \rightarrow (i_1', i_2', j', k', l')$ has positive rate only for exactly one of i_1', i_2', j', k', l' different from i_1, i_2, j, k, l.

$$B_{0_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)} = \begin{cases} \lambda(\boldsymbol{\alpha} \otimes I_n) & i_1 = 0, i_2 = 1; j_1 = j_2 = 0; 1 \le k_2 \le m, \\ & 1 \le l_1, l_2 \le n \\ \lambda I_{mn} & 1 \le i_1 \le N-2, i_2 = i_1 + 1; j_1 = j_2 = 0; \\ & 1 \le k_1, k_2 \le m; 1 \le l_1, l_2 \le n \\ \lambda \boldsymbol{e}(m) \otimes (\boldsymbol{\alpha} \otimes I_n) & i_1 = N-1, i_2 = N; j_1 = 0, j_2 = 1; 1 \le k_1, k_2 \le m; \\ & 1 \le l_1, l_2 \le n \\ \lambda I_{mn} & 1 \le i_1 \le L-1, i_2 = i_1 + 1; j_1 = j_2 = 1; \\ & 1 \le k_1, k_2 \le m; 1 \le l_1, l_2 \le n \\ \theta \boldsymbol{T}^0 \otimes I_n & i_1 = 1, i_2 = 0; j_1 = 0, j_2 = 0; 1 \le k_1 \le m; \\ & 1 \le l_1, l_2 \le n \\ \boldsymbol{T}^0 \otimes I_n & i_1 = 1, i_2 = 0; j_1 = 1, j_2 = 0; 1 \le k_1 \le m; \\ & 1 \le l_1, l_2 \le n \\ \theta \boldsymbol{T}^0 \boldsymbol{\alpha} \otimes I_n & 2 \le i_1 \le N-1, i_2 = i_1 - 1; j_1 = 0, j_2 = 0; \\ & 1 \le k_1, k_2 \le m; 1 \le l_1, l_2 \le n \\ \boldsymbol{T}^0 \boldsymbol{\alpha} \otimes I_n & 2 \le i_1 \le L, i_2 = i_1 - 1; j_1 = j_2 = 1; 1 \le k_1, k_2 \le m; \\ & 1 \le l_1, l_2 \le n \\ \eta \boldsymbol{e}(m) \otimes (\boldsymbol{\alpha} \otimes I_n) & 1 \le i_1 \le N-1, i_2 = i_1; j_1 = 0, j_2 = 1; \\ & 1 \le k_1, k_2 \le m; 1 \le l_1, l_2 \le n \\ D_0 - \lambda I_n & i_1 = i_2 = 0; j_1 = j_2 = 0; 1 \le l_1, l_2 \le n \\ \theta T \oplus D_0 - (\lambda + \eta) I_{mn} & 1 \le i_1 \le N-1, i_2 = i_1; j_1 = j_2 = 0; 1 \le k_1, k_2 \le m; \\ & 1 \le l_1, l_2 \le n \\ T \oplus D_0 - \lambda I_{mn} & 1 \le i_1 \le L-1, i_2 = i_1; j_1 = j_2 = 1; 1 \le k_1, k_2 \le m; \\ & 1 \le l_1, l_2 \le n \\ T \oplus D_0 & i_1 = i_2 = L; j_1 = j_2 = 1; 1 \le k_1, k_2 \le m; \\ & 1 \le l_1, l_2 \le n \end{cases}$$

$$C_{0_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)} = \begin{cases} D_1 & i_1 = 0, i_2 = 0; j_1 = j_2 = 0; 1 \le l_1, l_2 \le n \\ I_m \otimes D_1 & 1 \le i_1 \le N-1, i_2 = i_1; j_1 = j_2 = 0; 1 \le k_1, k_2 \le m; 1 \le l_1, l_2 \le n \\ I_m \otimes D_1 & 1 \le i_1 \le L, i_2 = i_1; j_1 = j_2 = 1; 1 \le k_1, k_2 \le m; 1 \le l_1, l_2 \le n \end{cases}$$

$$B_{1_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)} = \begin{cases} \boldsymbol{T'}^0 \otimes I_n & i_1 = i_2 = 0; j_1 = 2, j_2 = 0; 1 \le k_1 \le m'; 1 \le l_1, l_2 \le n \\ \boldsymbol{T'}^0 \alpha \otimes I_n & 1 \le i_1 \le L, i_2 = i_1; j_1 = 2, j_2 = 1; 1 \le k_1 \le m', 1 \le k_2 \le m; \\ & 1 \le l_1, l_2 \le n \end{cases}$$

$$A_{2_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)} = \begin{cases} \boldsymbol{T'}^0\boldsymbol{\alpha'} \otimes I_n & i_1 = i_2 = 0; j_1 = j_2 = 2; 1 \le k_1, k_2 \le m'; 1 \le l_1, l_2 \le n \\ \boldsymbol{T'}^0\boldsymbol{\alpha} \otimes I_n & 1 \le i_1 \le L, i_2 = i_1; j_1 = 2, j_2 = 1; 1 \le k_1 \le m', 1 \le k_2 \le m; \\ & 1 \le l_1, l_2 \le n \end{cases}$$

$$A_{1_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)} = \begin{cases} \lambda(\boldsymbol{\alpha} \ \otimes I_n) & i_1 = 0, i_2 = 1; j_1 = j_2 = 0; 1 \le k_2 \le m; \ 1 \le l_1, l_2 \le n \\ \lambda I_{mn} & i_1 = 0, i_2 = 1; j_1 = j_2 = 2; 1 \le k_1, k_2 \le m'; \\ & 1 \le l_1, l_2 \le n \\ \lambda I_{mn}, & 1 \le i_1 \le N-2; i_2 = i_1 + 1; j_1 = j_2 = 0; 1 \le k_1, k_2 \le m; \\ & 1 \le l_1, l_2 \le n \\ \lambda \boldsymbol{e}(m) \otimes (\boldsymbol{\alpha} \otimes I_n) & i_1 = N-1, i_2 = N; j_1 = 0, j_2 = 1; 1 \le k_1, k_2 \le m; \\ & 1 \le l_1, l_2 \le n \\ \lambda I_{mn} & 1 \le i_1 \le L-1, i_2 = i_1 + 1; j_1 = j_2 = 1; 1 \le k_1, k_2 \le m; \\ & 1 \le l_1, l_2 \le n \\ \lambda I_{mn} & 1 \le i_1 \le L-1, i_2 = i_1 + 1; j_1 = j_2 = 2; 1 \le k_1, k_2 \le m'; \\ & 1 \le k_1, l_2 \le n \\ \theta \boldsymbol{T}^0 \otimes I_n & i_1 = 1, i_2 = 0; j_1 = j_2 = 0; 1 \le k_1 \le m; \\ & 1 \le l_1, l_2 \le n \\ \boldsymbol{T}^0 \boldsymbol{\alpha'} \otimes I_n & i_1 = 1, i_2 = 0; j_1 = 1, j_2 = 2; 1 \le k_1 \le m, 1 \le k_2 \le m'; \\ & 1 \le l_1, l_2 \le n \\ \theta \boldsymbol{T}^0 \boldsymbol{\alpha} \otimes I_n & 2 \le i_1 \le N-1, i_2 = i_1 - 1; j_1 = j_2 = 0; 1 \le k_1, k_2 \le m; \\ & 1 \le l_1, l_2 \le n \\ \boldsymbol{T}^0 \boldsymbol{\alpha} \otimes I_n & 2 \le i_1 \le L, i_2 = i_1 - 1; j_1 = j_2 = 1; 1 \le k_1, k_2 \le m; \\ & 1 \le l_1, l_2 \le n \\ \eta(\boldsymbol{\alpha'} \otimes I_n) & i_1 = i_2 = 0; j_1 = 0, j_2 = 2; 1 \le k_2 \le m'; 1 \le l_1, l_2 \le n \\ \eta \boldsymbol{e}(m) \otimes (\boldsymbol{\alpha} \otimes I_n) & 1 \le i_1 \le N-1, i_2 = i_1; j_1 = 0, j_2 = 1; 1 \le k_1, k_2 \le m; \\ & 1 \le l_1, l_2 \le n \\ D_0 - (\lambda + \eta) I_n & i_1 = i_2 = 0; j_1 = j_2 = 0; 1 \le l_1, l_2 \le n \\ T \oplus D_0 - \lambda I_{mn} & i_1 = i_2 = 0; j_1 = j_2 = 2; 1 \le k_1, k_2 \le m'; 1 \le l_1, l_2 \le n \\ \theta T \oplus D_0 - (\lambda + \eta) I_{mn} & 1 \le i_1 \le N-1, i_2 = i_1; j_1 = j_2 = 0; 1 \le k_1, k_2 \le m; \\ & 1 \le l_1, l_2 \le n \\ T \oplus D_0 - \lambda I_{mn} & 1 \le i_1 \le L-1, i_2 = i_1; j_1 = j_2 = 1; 1 \le k_1, k_2 \le m; \\ & 1 \le l_1, l_2 \le n \\ T \oplus D_0 - \lambda I_{mn} & 1 \le i_1 \le L-1, i_2 = i_1; j_1 = j_2 = 2; 1 \le k_1, k_2 \le m'; \\ & 1 \le l - 1, l_2 \le n \\ T \oplus D_0 & i_1 = i_2 = L; j_1 = j_2 = 1; 1 \le k_1, k_2 \le m; 1 \le l_1, l_2 \le n \\ T \oplus D_0 & i_1 = i_2 = L; j_1 = j_2 = 2; 1 \le k_1, k_2 \le m'; 1 \le l_1, l_2 \le n \end{cases}$$

$$A_{0_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)} = \begin{cases} D_1 & i_1 = i_2 = 0; j_1 = j_2 = 0; 1 \le l_1, l_2 \le n \\ I_m \otimes D_1 & i_1 = i_2 = 0; j_1 = j_2 = 2; 1 \le k_1, k_2 \le m'; 1 \le l_1, l_2 \le n \\ I_m \otimes D_1 & 1 \le i_1 \le N-1, i_2 = i_1; j_1 = j_2 = 0; 1 \le k_1, k_2 \le m; \\ & 1 \le l_1, l_2 \le n \\ I_m \otimes D_1 & 1 \le i_1 \le L, i_2 = i_1; j_1 = j_2 = 1; 1 \le k_1, k_2 \le m; \\ & 1 \le l_1, l_2 \le n \\ I_m \otimes D_1 & 1 \le i_1 \le L, i_2 = i_1; j_1 = j_2 = 2; 1 \le k_1, k_2 \le m'; \\ & 1 \le l_1, l_2 \le n \end{cases}$$

2.2 Steady State Analysis

First we find the condition for stability of the system under study.

2.2.1 Stability condition

Let $\boldsymbol{\pi} = (\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \ldots, \boldsymbol{\pi}_L)$ denote the steady state probability vector of the generator

$$A = A_0 + A_1 + A_2 = \begin{bmatrix} F_0 & F_1 & & & & & & & & & & \\ F_2 & F_3 & \lambda I & & & & & & & & & \\ & F_4 & F_3 & \lambda I & & & & & & & & \\ & & \ddots & \ddots & \ddots & & & & & & & \\ & & & F_4 & F_3 & \lambda I & & & & & & \\ & & & & F_4 & F_3 & F_5 & & & & & \\ & & & & & F_6 & F_7 & \lambda I & & & & \\ & & & & & & F_8 & F_7 & \lambda I & & & \\ & & & & & & & \ddots & \ddots & \ddots & & \\ & & & & & & & & F_8 & F_7 & \lambda I \\ & & & & & & & & & F_8 & F_9 \end{bmatrix}$$

where

$$F_0(k,l)=\begin{cases} D_0+D_1-(\lambda+\eta)I_n & k=1,l=1\\ \eta(\boldsymbol{\alpha'}\otimes I_n) & k=1,l=2\\ 0 & k=2,l=1\\ \boldsymbol{T'}^0\boldsymbol{\alpha'}\otimes I_n+T'\oplus D_0-\lambda I_{m'n}+I_{m'}\otimes D_1 & k=2,l=2 \end{cases}$$

$$F_1(k,l)=\begin{cases} \lambda(\boldsymbol{\alpha}\otimes I_n) & k=1,l=1\\ \lambda I_{m'n} & k=2,l=3\\ 0 & \text{otherwise} \end{cases},\ F_2(k,l)=\begin{cases} \theta\boldsymbol{T}^0\otimes I_n & k=1,l=1\\ \boldsymbol{T}^0\boldsymbol{\alpha'}\otimes I_n & k=2,l=2\\ 0 & \text{otherwise} \end{cases}$$

$$F_3(k,l)=\begin{cases} \theta T\oplus D_0-(\lambda+\eta)I_{mn}+I_m\otimes D_1 & k=1,l=1\\ \boldsymbol{e}(m)\otimes\eta(\boldsymbol{\alpha}\otimes I_n) & k=1,l=2\\ T\oplus D_0-\lambda I_{mn}+I_m\otimes D_1 & k=l=2\\ T'\oplus D_0-\lambda I_{m'n}+I_{m'}\otimes D_1 & k=l=3\\ \boldsymbol{T'}^0\boldsymbol{\alpha}\otimes I_n & k=3,l=2\\ 0 & \text{otherwise} \end{cases}$$

$$F_4(k,l)=\begin{cases} \theta\boldsymbol{T}^0\boldsymbol{\alpha}\otimes I_n & k=1,l=1\\ \boldsymbol{T}^0\boldsymbol{\alpha}\otimes I_n & k=2,l=2\\ 0 & \text{otherwise} \end{cases},\ F_5(k,l)=\begin{cases} \lambda\boldsymbol{e}(m)\otimes(\boldsymbol{\alpha}\otimes I_n) & k=l=1\\ \lambda I_{mn} & k=2,l=1\\ \lambda I_{m'n} & k=3,l=2\\ 0 & \text{otherwise} \end{cases}$$

$$F_6(k,l)=\begin{cases} \boldsymbol{T}^0\boldsymbol{\alpha}\otimes I_n & k=1,l=2\\ 0 & \text{otherwise} \end{cases}$$

$$F_7(k,l) = \begin{cases} T \oplus D_0 - \lambda I_{mn} + I_m \otimes D_1 & k = l = 1 \\ T' \oplus D_0 - \lambda I_{m'n} + I_{m'} \otimes D_1 & k = l = 2 \\ 0 & k = 1, l = 2 \\ \boldsymbol{T'}^0 \alpha \otimes I_n & k = 2, l = 1 \end{cases}$$

$$F_8(k,l) = \begin{cases} \boldsymbol{T}^0 \boldsymbol{\alpha} \otimes I_n & k = l = 1 \\ 0 & \text{otherwise} \end{cases}, F_9(k,l) = \begin{cases} T \oplus D_0 + I_m \otimes D_1 & k = l = 1 \\ T' \oplus D_0 + I_{m'} \otimes D_1 & k = l = 2 \\ \boldsymbol{T'}^0 \boldsymbol{\alpha} \otimes I_n & k = 2, l = 1 \end{cases}$$

with dimension of F_0, F_1, F_2 be $n(1+m') \times n(1+m'), n(1+m') \times (2m+m')n, (2m+m')n \times n(1+m')$ respectively. F_3, F_4 are square matrices of order $(2m+m')n$, F_5 is of order $(2m+m')n \times (m+m')n$, F_6 is of order $(m+m')n \times (2m+m')n$, F_7, F_8, F_9 are square matrices of order $(m+m')n$. ie,

$$\boldsymbol{\pi} A = 0, \boldsymbol{\pi e} = 1 \tag{2.1}$$

The LIQBD description of the model indicates that the queueing system is stable (see Neuts [40]) if and only if the left drift exceeds that of right drift. That is,

$$\boldsymbol{\pi} A_0 \mathbf{e} < \boldsymbol{\pi} A_2 \mathbf{e} \tag{2.2}$$

The vector $\boldsymbol{\pi}$ cannot be obtained directly in terms of the parametres of the model. From (2.1) we get

$$\boldsymbol{\pi}_i = \boldsymbol{\pi}_{i-1} \mathcal{U}_{i-1}, 1 \leq i \leq L \tag{2.3}$$

where

$$\mathcal{U}_0 = -F_1(F_3 + \mathcal{U}_1 F_4)^{-1}$$

$$\mathcal{U}_i = \begin{cases} -\lambda(F_3 + U_{i+1}F_4)^{-1} & for\ 1 \leq i \leq N-3 \\ -\lambda(F_3 + \mathcal{U}_{N-1}F_6)^{-1} & for\ i = N-2 \\ -F_5(F_7 + \mathcal{U}_N F_8)^{-1}, & for\ i = N-1 \\ -\lambda(F_7 + \mathcal{U}_{i+1}F_8)^{-1} & for\ N \leq i \leq L-2 \\ -\lambda F_9^{-1} & for\ i = L-1 \end{cases}$$

From the normalizing condition $\pi e = 1$ we have

$$\boldsymbol{\pi}_0 \left(\sum_{j=0}^{L-1} \prod_{i=0}^{j} \mathcal{U}_i + I \right) \boldsymbol{e} = 1 \tag{2.4}$$

The inequality (2.2) gives the stability condition as

$$\boldsymbol{\pi}_0 \left[(I_{(1+m')} \otimes D_1)\boldsymbol{e} + \sum_{i=0}^{N-2} \prod_{j=0}^{i} \mathcal{U}_j (I_{(2m+m')} \otimes D_1)\boldsymbol{e} + \sum_{i=N-1}^{L-1} \prod_{j=0}^{i} \mathcal{U}_j (I_{(m+m')} \otimes D_1)\boldsymbol{e} \right]$$
$$< \boldsymbol{\pi_0} \left[A_{20} + \sum_{i=0}^{N-2} \prod_{j=0}^{i} \mathcal{U}_j A_{21} + \sum_{i=N+1}^{L-1} \prod_{j=0}^{i} \mathcal{U}_j A_{22} \right] \tag{2.5}$$

where, $\mathcal{A}_{20} = \begin{bmatrix} 0 \\ (\boldsymbol{T'}^0 \boldsymbol{\alpha'} \otimes I)\boldsymbol{e} \end{bmatrix}$, $\mathcal{A}_{21} = \mathcal{A}_{22} = \begin{bmatrix} 0 \\ (\boldsymbol{T'}^0 \boldsymbol{\alpha} \otimes I)\boldsymbol{e} \end{bmatrix}$, with 0 a zero column vector of order n, $2mn$ and mn for A_{20}, A_{21} and A_{22} respectively.

2.2.2 Steady-state probability vector

Assuming that the condition (2.5) is satisfied we proceed to find the steady-state probability of the system state.

Let $\boldsymbol{x}$ be the steady state probability vector of Q. We partition this vector as

$$\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2 \ldots),$$

where $\boldsymbol{x}_0$ is of dimension $n(1 + m(L + N - 1))$, $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots$ are of dimension

$n(1 + mN + (L-1)m + (L+1)m')$. Under the stability condition, we have

$$\boldsymbol{x}_i = \boldsymbol{x}_1 R^{i-1}, i \geq 2$$

where the matrix R is the minimal nonnegative solution to the matrix quadratic equation

$$R^2 A_2 + RA_1 + A_0 = 0$$

and the vectors $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$ are obtained by solving the equations

$$\boldsymbol{x}_0 B_0 + \boldsymbol{x}_1 B_1 = 0 \qquad (2.6)$$

$$\boldsymbol{x}_0 C_0 + \boldsymbol{x}_1 (A_1 + RA_2) = 0 \qquad (2.7)$$

subject to the normalizing condition

$$\boldsymbol{x}_0 \boldsymbol{e} + \boldsymbol{x}_1 (I - R)^{-1} \boldsymbol{e} = 1 \qquad (2.8)$$

For evaluating the performance of the system we have to compute certain distributions. We proceed to such computations.

2.2.3 Distribution of duration of slow service mode

The duration T_{slow}, of a slow service mode is defined as the time the server stays in slow service mode (through initiating a WV) until either switching to normal mode through the vacation clock realization or with the number of type I customers in the system hitting the threshold value N or the number of type I customers hitting 0 before expiration of vacation, whichever occurs first. We consider the Markov process $T_{slow}(t) = \{(N(t), J(t)) : t \geq 0\}$ where $N(t)$ is the number of type I customers in the system at time t, $J(t)$ the service phase at time t. Thus the state space of the process is $\{(i, j) : 1 \leq i \leq N-1; 1 \leq j \leq m\} \cup \{0\} \cup \{*_1\} \cup \{*_2\}$ where 0 denotes the absorbing state indicating that there is no type I customer in the system and $*_1$ denotes the absorbing state indicating the vacation expiration by vacation clock realization and $*_2$ denotes the absorbing state indicating the vacation expiration by the number of type customers in the system hitting N. The initial probability vector is given by

$$\boldsymbol{\beta}_1 = \frac{1}{d_1}(w_1, w_2, \cdots, w_m, \mathbf{0})$$

where, for, $1 \leq j \leq m$, $w_j = \sum_{k=1}^{n} \frac{\lambda\alpha_j}{\lambda+\eta-d_{kk}^{(0)}} x_{0,0,k} + \sum_{i=1}^{\infty}\sum_{k=1}^{n} \frac{\lambda\alpha_j}{\lambda+\eta-d_{kk}^{(0)}} x_{i,0,0,k}$, with

$$d_1 = \sum_{k=1}^{n} \frac{\lambda}{\lambda+\eta-d_{kk}^{(0)}} x_{0,0,k} + \sum_{i=1}^{\infty}\sum_{k=1}^{n} \frac{\lambda}{\lambda+\eta-d_{kk}^{(0)}} x_{i,0,0,k}$$

and $\mathbf{0}$ is a zero matrix of order $1 \times (N-2)m$.

The infinitesimal generator $\mathcal{S}_1$ of $T_{slow}(t)$ has the form

$$\mathcal{S}_1 = \begin{bmatrix} S_1 & \boldsymbol{S}_1^{(0)} & \boldsymbol{S}_1^{(1)} & \boldsymbol{S}_1^{(2)} \\ \mathbf{0} & 0 & 0 & 0 \end{bmatrix}$$

where

$$S_1 = \begin{bmatrix} \theta T - (\lambda+\eta)I & \lambda I & & & \\ \theta\boldsymbol{T}^0\boldsymbol{\alpha} & \theta T - (\lambda+\eta)I & \lambda I & & \\ \ddots & \ddots & \ddots & & \\ & \theta\boldsymbol{T}^0\boldsymbol{\alpha} & \theta T - (\lambda+\eta)I & \lambda I \\ & & \theta\boldsymbol{T}^0\boldsymbol{\alpha} & \theta T - (\lambda+\eta)I \end{bmatrix},$$

$$\boldsymbol{S}_1^{(0)} = \begin{bmatrix} \theta\boldsymbol{T}^0 \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}, \boldsymbol{S}_1^{(1)} = \begin{bmatrix} \eta\boldsymbol{e}(m) \\ \vdots \\ \eta\boldsymbol{e}(m) \\ \eta\boldsymbol{e}(m) \end{bmatrix}, \boldsymbol{S}_1^{(2)} = \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \lambda\boldsymbol{e}(m) \end{bmatrix}.$$

Thus we have the following Lemma.

Lemma 2.2.1. The expected duration of time the server stays continuously in WV until the number of type I customers in the system reach 0 is given by $\boldsymbol{\beta}_1(-S_1)^{-2}\boldsymbol{S_1}^{(0)}$.

Our objective is to compute the expected number of hits to zero type I customer state until the server returns to normal mode of service before the arrival of a type II customer. Define the random variable M_1 as number of returns to 0 type I customer state starting from 0 type I customer state during vacation mode of service before the arrival of a type II customer.

2.2.4 Expected value of M_1

Lemma 2.2.2 provides the expected duration of the time starting from the beginning of a vacation until the start of the next vacation, without going to normal mode of service in between, before the arrival of a type II customer. As a first step for computing expected number of such hits, we compute the following disribution. Let T_s denote the duration of slow service until the arrival of a type II customer.

Distribution of T_s

We consider the Markov process $T_s(t) = \{(N(t), J(t), M(t)) : t \geq 0\}$ where $N(t)$ is the number of type I customers in the system at time t, $J(t)$ the service phase and $M(t)$ the arrival phase of type II customer at that instant. Thus the state space of the process is $\{(i, j, k) : 1 \leq i \leq N-1; 1 \leq j \leq m; 1 \leq k \leq n\} \cup \{0\} \cup \{*_1\} \cup \{*_2\}$ where 0 denotes the absorbing state indicating that there is no type I customer in the system, $*_1$,$*_2$ denote the absorbing states indicating the vacation expiration and arrival of a type II customer respectively. The initial probability vector is given by

$$\boldsymbol{\beta}_2 = (1/d_2)(w_{1,1}, \cdots, w_{1,n}, \cdots, w_{m,1}, \cdots, w_{m,n}, \mathbf{0})$$

where, for, $1 \le j \le m$, $1 \le k \le n$,

$$w_{j,k} = \frac{\lambda \alpha_j}{\lambda + \eta - d_{kk}^{(0)}} x_{0,0,k},$$

$d_2 = \sum_{k=1}^{n} \frac{\lambda}{\lambda+\eta-d_{kk}^{(0)}} x_{0,0,k}$ and $\mathbf{0}$ is a zero matrix of order $1 \times (N-2)mn$. The infinitesimal generator $\mathcal{S}_2$ of $T_s(t)$ has the form

$$\mathcal{S}_2 = \begin{bmatrix} S_2 & \boldsymbol{S}_2^{(0)} & \boldsymbol{S}_2^{(1)} & \boldsymbol{S}_2^{(2)} \\ \mathbf{0} & 0 & 0 & 0 \end{bmatrix}$$

where

$$S_2 = \begin{bmatrix} \theta T \oplus D_0 - (\lambda+\eta)I & \lambda I & & & \\ \theta \boldsymbol{T}^0 \boldsymbol{\alpha} \otimes I & \theta T \oplus D_0 - (\lambda+\eta)I & \lambda I & & \\ \ddots & \ddots & \ddots & & \\ & \theta \boldsymbol{T}^0 \boldsymbol{\alpha} \otimes I & \theta T \oplus D_0 - (\lambda+\eta)I & \lambda I \\ & & \theta \boldsymbol{T}^0 \boldsymbol{\alpha} \otimes I & \theta T \oplus D_0 - (\lambda+\eta)I \end{bmatrix}.$$

$$\boldsymbol{S}_2^{(0)} = \begin{bmatrix} \theta \boldsymbol{T}^0 \otimes \boldsymbol{e}(n) \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}, \boldsymbol{S}_2^{(1)} = \begin{bmatrix} \delta \boldsymbol{e}(m) \\ \vdots \\ \delta \boldsymbol{e}(m) \end{bmatrix}, \boldsymbol{S}_2^{(2)} = \begin{bmatrix} \eta \boldsymbol{e}(mn) \\ \vdots \\ \eta \boldsymbol{e}(mn) \\ (\lambda+\eta)\boldsymbol{e}(mn) \end{bmatrix}$$

where $\mathbf{0}$ is a zero matrix of order $mn \times 1$ and

$$\delta = \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_n \end{bmatrix}, \tag{c1}$$

with δ_i representing the ith rowsum of D_1.
Thus we have the following Lemma.

Lemma 2.2.2. The expected duration of time the server remains continuously in WV until the number of type I customers reach 0 and before the arrival of a type II customer is given by $\boldsymbol{\beta}_2(-S_2)^{-2}\boldsymbol{S}_2^{(0)}$.

Next we compute the following distribution. Let T_s' denote the duration of time the server, starting in slow service mode until either he gets back to normal mode through vacation expiration or the arrival of a type II customer, whichever occurs first.

Distribution of T_s'

The distribution of T_s' can be studied as the time until absorption in a continuous time Markov chain with state space $\{(0,k) : 1 \leq k \leq n\} \cup \{(i,j,k) : 1 \leq i \leq N-1; 1 \leq j \leq m; 1 \leq k \leq n\} \cup \{*_1\} \cup \{*_2\}$, i denote the number of type I customers in the system, j the service phase, k, the arrival phase of type II customer, $*_1$, the absorbing state indicating the vacation expiration and $*_2$, the absorbing state indicating the arrival of a type II customer.

The initial probability vector is given by

$$\boldsymbol{\beta}_3 = (1/d_2)(\mathbf{0}, w_{1,1}, \cdots, w_{1,n}, \cdots, w_{m,1}, \cdots, w_{m,n}, \mathbf{0})$$

where, for, $1 \leq j \leq m$, $1 \leq k \leq n$, $w_{j,k}$ and d_2 are defined above, the first $\mathbf{0}$ is a zero matrix of order n and the second $\mathbf{0}$ is a zero matrix of order $1 \times (N-2)mn$.

The infinitesimal generator $\mathcal{S}_3$ of $T_s'(t)$ has the form

$$\mathcal{S}_3 = \begin{bmatrix} S_3 & \boldsymbol{S_3}^{(0)} & \boldsymbol{S_3}^{(1)} \\ \mathbf{0} & 0 & 0 \end{bmatrix}$$

where

$$S_3 = \begin{bmatrix} D_0 - \lambda I & \lambda(\boldsymbol{\alpha} \otimes I) & & & & \\ \theta\boldsymbol{T}^0 \otimes I & \theta T \oplus D_0 - (\lambda+\eta)I & \lambda I & & & \\ & \theta\boldsymbol{T}^0\boldsymbol{\alpha} \otimes I & \theta T \oplus D_0 - (\lambda+\eta)I & \lambda I & & \\ & \ddots & \ddots & \ddots & & \\ & & \theta\boldsymbol{T}^0\boldsymbol{\alpha} \otimes I & \theta T \oplus D_0 - (\lambda+\eta)I & \lambda I \\ & & & \theta\boldsymbol{T}^0\alpha \otimes I & \theta T \oplus D_0 - (\lambda+\eta)I \end{bmatrix},$$

$\boldsymbol{S}_3^{(0)} = \begin{bmatrix} \mathbf{0} \\ \eta\boldsymbol{e}(mn) \\ \vdots \\ \eta\boldsymbol{e}(mn) \\ (\lambda+\eta)\boldsymbol{e}(mn) \end{bmatrix}$, and $\boldsymbol{S}_3^{(1)} = \begin{bmatrix} \delta \\ \boldsymbol{e}(m)\otimes\delta \\ \vdots \\ \boldsymbol{e}(m)\otimes\delta \end{bmatrix}$ where $\mathbf{0}$ is a zero matrix of order n and δ is given by (c1).

Thus we have the following Lemma.

Lemma 2.2.3. The expected duration of time the server remains in WV with or without hitting zero state of type I customer until the arrival of a type II customer is given by $\boldsymbol{\beta}_3(-S_3)^{-2}\boldsymbol{S}_3^{(1)}$.

The Theorem below provides the expected number of visits to the state "no type I customer", starting from that state, before the arrival of a type II customer.

Theorem 2.2.1. *The expected number of returns to 0 type I customer state during the vacation mode of sevice starting from that state before the arrival of a type II customer is given by*

$$\left(\frac{1}{\lambda}+\boldsymbol{\beta}_3(-S_3)^{-2}\boldsymbol{S}_3^{(1)}\right) / \left(\frac{1}{\lambda}+\boldsymbol{\beta}_2(-S_2)^{-2}\boldsymbol{S}_2^{(0)}\right).$$

2.3 Waiting Time Analysis

2.3.1 Type I customer

To find the waiting time of a type I customer who arrives at time x, we have to consider different possibilities depending on the status of server at that time. The server may be on vacation, WV, normal mode 1 or in normal mode 2. Let Z_1 be the random variable representing the waiting time of a type I customer in the queue. Define $W_1(x) = \text{Prob}(Z_1 \le x)$ and $W_1^*(s)$ be the corresponding LST.

Case I

The tagged customer arrives to the system when the server is on vacation. Suppose E_1 denote the event that the system is in the state $(0, 1, 0, u, v)$, $1 \leq u \leq m$; $1 \leq v \leq n$ or in the state $(n_1, 1, 0, u, v)$, $n_1 \geq 1$; $1 \leq u \leq m$; $1 \leq v \leq n$ immediately after arrival of the tagged customer. Let $W_1^*(s/E_1)$ denote the corresponding LST. Then

$$W_1^*(s/E_1) = 1.$$

Case II

The tagged type I customer arrives to the system when the server is on WV. Suppose that $a+1$ is the position of the tagged customer when he arrives the system. For $1 \leq a \leq N-2$, let E_2 denote the event the system be in the state $(n_1, a+1, 0, u, v)$, $n_1 \geq 0$; $1 \leq u \leq m$; $1 \leq v \leq n$ immediately after arrival of the tagged customer. Let $W_1^*(s/E_2)$ denote the corresponding LST.

Case (i)

Let E denote the event that the server switches to normal mode due to random clock (vacation clock) realization during the slow service. Then $E = \cup_{i=1}^{i=a+1}(E \cap H_i)$ where H_1 denotes the event the random clock expires during the residual service time of the customer in service and for $2 \leq i \leq a$, H_i denotes the event the random clock expire during the ith service. In these cases, the waiting time of an arbitrary type I customer is the sum of time duration, starting from his arrival epoch till random clock expiration, service time of the customer in service at the time of random clock expiration from the beginning in normal mode of service and service time of the remaining customers. Let H_{a+1} denotes the event the random clock expires after the ath service. In this case, the waiting time of an arbitrary customer is the sum of the residual service time of the customer in service when the tagged customer arrives and service time of remaining $a-1$ type I customers in slow mode.

Now,

$$P(E/E_2) = \left(\int_{t=0}^{\infty} (\boldsymbol{e'}_{a+1}(N-1) \otimes \boldsymbol{e'}_u(m)) exp(S_1 t) \boldsymbol{S}_1^{(1)} dt\right)$$

where S_1, $\boldsymbol{S}_1^{(1)}$ are as defined in section 2.2.3.

Let $p_{a,u} = \left((\boldsymbol{e'}_{a+1}(N-1) \otimes \boldsymbol{e'}_u(m))(-S_1)^{-2}\boldsymbol{S_1}^{(1)}\right)^{-1}$ be the rate of absorption to $\{*_1\}$ from S_1 and $\mu^{(i)}$ denote the expected rate of sum of i service time distributions, each following PH$(\boldsymbol{\alpha}, T)$(except $\mu(^{(1)})$ (see Breuer and Baum [5])from the arrival epoch of the tagged customer. Here, $\mu^{(1)} = \theta\mu_u$ which is the rate of residual service time when the server is providing slow service in phase u. Now,

$$P(H_1/E, E_2) = \frac{p_{a,u}}{p_{a,u} + \mu^{(1)}},$$

$$P(H_i/E, E_2) = \frac{p_{a,u}}{p_{a,u} + \mu^{(i)}} - \frac{p_{a,u}}{p_{a,u} + \mu^{(i-1)}} \text{ for } 2 \leq i \leq a,$$

$$P(H_{a+1}/E, E_2) = \frac{\mu^{(a)}}{p_{a,u} + \mu^{(a)}}$$

Then the conditional LSTs are given by

$$W_1^*(s/E_2, E, H_1) = \left(\frac{\eta}{s+\eta}\right)(\boldsymbol{\alpha}(sI - T)^{-1}\boldsymbol{T}^0)^a,$$

$$W_1^*(s/E_2, E, H_i) = \left(\frac{\eta}{s+\eta}\right)(\boldsymbol{\alpha}(sI - T)^{-1}\boldsymbol{T}^0)^{a-i+1}, \text{for } 2 \leq i \leq a$$

and

$$W_1^*(s/E_2, E, H_{a+1}) = (\boldsymbol{e}'_u(sI - \theta T)^{-1}\theta T^0)(\boldsymbol{\alpha}(sI - \theta T)^{-1}\theta\boldsymbol{T}^0)^{a-1}.$$

Thus conditional LST

$$W_1^*(s/E_2, E) = \sum_{i=1}^{a+1} W_1^*(s/E_2, E, H_i) P(H_i/E_2, E).$$

Case (ii)

Let F denote the event "the server switches to normal mode when the number of type I customers in the system hit N" during the slow service. Then $F = \cup_{i=1}^{i=a+1}(F \cap J_i)$ where J_1 denotes the event: the number of type I customers in the system reaches N during the residual service time. For $2 \leq i \leq a$, J_i denotes the event: the number of type I customers in the system reaches N during the ith customer's service time. In these cases, the waiting time of an arbitrary type I customer is the sum of time duration starting from his arrival epoch till the number of type I customers hit N, service time of the customer in service at the time of switching to normal mode from the beginning in normal mode of service and service time of remaining customers. Let J_{a+1} denote the event "the number of type I customers in the system reaches N after the ath customer's service". In this case, the waiting time of an arbitrary customer is the sum of the residual service time of the customer in service when the tagged customer arrives and service time of remaining $a-1$ type I customers in slow mode.

Now,

$$P(F/E_2) = \left(\int_{t=0}^{\infty} (\boldsymbol{e'}_{a+1}(N-1) \otimes \boldsymbol{e'}_u(m)) exp(S_1 t) \boldsymbol{S}_1^{(2)} dt \right)$$

where S_1, $\boldsymbol{S}_1^{(2)}$ are as defined in section 2.2.3.
Let $q_{a,u} = \left((\boldsymbol{e'}_{a+1}(N-1) \otimes \boldsymbol{e'}_u(m))(-S_1)^{-2} \boldsymbol{S_1}^{(2)} \right)^{-1}$ be the rate of absorption to $\{*_2\}$ from S_1

$$P(J_1/F, E_2) = \frac{q_{a,u}}{q_{a,u} + \mu^{(1)}},$$

$$P(J_i/F,E_2) = \frac{q_{a,u}}{q_{a,u}+\mu^{(i)}} - \frac{q_{a,u}}{q_{a,u}+\mu^{(i-1)}}, \text{for } 2 \leq i \leq a$$

and

$$P(J_{a+1}/F,E_2) = \frac{\mu^{(a)}}{q_{a,u}+\mu^{(a)}}$$

The conditional LSTs are given by

$$W_1^*(s/E_2,F,J_1) = \left(\frac{\lambda}{s+\lambda}\right)^{N-a-1} (\boldsymbol{\alpha}(sI-T)^{-1}\boldsymbol{T}^0)^a,$$

$$W_1^*(s/E_2,F,J_i) = \left(\frac{\lambda}{s+\lambda}\right)^{N-a+i-2} (\boldsymbol{\alpha}(sI-T)^{-1}\boldsymbol{T}^0)^{a-i+1}, \text{ for } 2 \leq i \leq a$$

and

$$W_1^*(s/E_2,F,J_{a+1}) = (\boldsymbol{e}'_u(sI-\theta T)^{-1}\theta\boldsymbol{T}^0)(\boldsymbol{\alpha}(sI-\theta T)^{-1}\theta\boldsymbol{T}^0)^{a-1}.$$

Thus the conditional LST,

$$W_1^*(s/E_2,F) = \sum_{i=1}^{a+1} W_1^*(s/E_2,F,J_i)P(J_i/E_2,F)$$

Case (iii)

Let G denote the event that the system becomes empty before vacation expiration.

$$P(G/E_2) = \left(\int_{t=0}^{\infty} (\boldsymbol{e'}_{a+1}(N-1)\otimes \boldsymbol{e'}_u(m))exp(S_1t)\boldsymbol{S}_1^{(0)}dt\right)$$

where S_1, $\boldsymbol{S}_1^{(0)}$ are as defined in section 2.2.3.

In this case the conditional LST,

$$W_1^*(s/E_2, G) = (\boldsymbol{e}'_u(sI - \theta T)^{-1}\theta \boldsymbol{T}^0)(\boldsymbol{\alpha}(sI - \theta T)^{-1}\theta \boldsymbol{T}^0)^{a-1}.$$

Thus the conditional LST,

$$W_1^*(s/E_2) = W_1^*(s/E_2, E)P(E/E_2) + W_1^*(s/E_2, F)P(F/E_2) + W_1^*(s/E_2, G)P(G/E_2).$$

Case III

The customer arrives to the system when the server is in normal mode 1 of service. Suppose that $a + 1$ is the position of the tagged customer when he arrives the system. Let E_3 denote the event the system is in the state $(n_1, a + 1, 1, u, v)$, $n_1 \geq 0$; $1 \leq a \leq L - 1$; $1 \leq u \leq m$; $1 \leq v \leq n$ immediately after arrival of the tagged customer. In this case the waiting time is the sum of residual normal service of the type I customer in service and $a - 1$ remaining normal service time of type I customers. Let $W_1^*(s/E_3)$ denote the corresponding conditional LST.
Then conditional LST,

$$W_1^*(s/E_3) = (\boldsymbol{e}'_u(sI - T)^{-1}\boldsymbol{T}^0)(\boldsymbol{\alpha}(sI - T)^{-1}\boldsymbol{T}^0)^{a-1}.$$

Case IV

The customer arrives to the system when the server is in normal mode 2 of service. Suppose that $a + 1$ be the position of the tagged customer when he arrives the system. Let E_4 denote the event the system is in the state $(n_1, a + 1, 2, u, v)$, $n_1 \geq 1$; $0 \leq a \leq L - 1$; $1 \leq u \leq m'$; $1 \leq v \leq n$ immediately after arrival of the tagged customer. In this case the waiting time is the sum of residual service time of the type II customer in service and a remaining normal service time of type I customers. Let $W_1^*(s/E_4)$ denote the corresponding LST.
Then the conditional LST,

$$W_1^*(s/E_4) = (\boldsymbol{e}'_u(sI - T')^{-1}\boldsymbol{T'}^0)(\boldsymbol{\alpha}(sI - T)^{-1}\boldsymbol{T}^0)^{a}.$$

Let $w_{i_1,i_2,j,k,l}$ denote the probabilty that the system is in the state (i_1, i_2, j, k, l) immediately after arrival of the tagged customer. Then,

$$
\begin{aligned}
w_{0,1,0,u,v} &= \tfrac{\lambda\alpha_u}{\lambda+\eta-d^{(0)}_{vv}} x_{0,0,v}, \text{ for, } 1 \le u \le m,\ 1 \le v \le n \\
w_{n_1,1,0,u,v} &= \tfrac{\lambda\alpha_u}{\lambda+\eta-d^{(0)}_{vv}} x_{n_1,0,0,v}, \text{ for, } n_1 \ge 1,\ 1 \le u \le m,\ 1 \le v \le n \\
w_{n_1,a+1,0,u,v} &= \tfrac{\lambda}{\lambda+\eta-\theta T_{uu}-d^{(0)}_{vv}} x_{n_1,a,0,u,v}, \text{ for, } n_1 \ge 0,\ 1 \le a \le N-2, \\
&\quad 1 \le u \le m,\ 1 \le v \le n \\
w_{n_1,N,1,u,v} &= \textstyle\sum_{u'=1}^{m} \tfrac{\lambda\alpha_u}{\lambda+\eta-\theta T_{u'u'}-d^{(0)}_{vv}} x_{n_1,N-1,0,u',v} + \tfrac{\lambda}{\lambda-T_{uu}-d^{(0)}_{vv}} x_{n_1,N-1,1,u,v}, \\
&\quad \text{for, } n_1 \ge 0, 1 \le u, u' \le m,\ 1 \le v \le n \\
w_{n_1,a+1,1,u,v} &= \tfrac{\lambda}{\lambda-T_{uu}-d^{(0)}_{vv}} x_{n_1,a,1,u,v}, \text{ for, } n_1 \ge 0,\ 1 \le a \le N-2 \text{ or} \\
&\quad N \le a \le L-1, 1 \le u \le m,\ 1 \le v \le n \\
w_{n_1,a+1,2,u,v} &= \tfrac{\lambda}{\lambda-T_{uu}-d^{(0)}_{vv}} x_{n_1,a,2,u,v}, \text{ for, } n_1 \ge 1,\ 0 \le a \le L-1, \\
&\quad 1 \le u \le m',\ 1 \le v \le n
\end{aligned}
$$

Thus we have the following Theorem.

Theorem 2.3.1. *The LST of the waiting time of a type I customer is given by*

$$
W_1^*(s) = \frac{1}{d}\Big[\sum_{n_1=0}^{\infty}\sum_{v=1}^{n} w_{n_1,1,0,u,v} + \sum_{n_1=0}^{\infty}\sum_{a=1}^{N-2}\sum_{u=1}^{m}\sum_{v=1}^{n} W_1^*(s/E_2)(w_{n_1,a+1,0,u,v}) + \sum_{n_1=0}^{\infty}\sum_{a=1}^{L-1}\sum_{u=1}^{m}\sum_{v=1}^{n} W_1^*(s/E_3)(w_{n_1,a+1,1,u,v}) + \sum_{n_1=1}^{\infty}\sum_{a=0}^{L-1}\sum_{u=1}^{m'}\sum_{v=1}^{n} W_1^*(s/E_4)(w_{n_1,a+1,2,u,v})\Big] \tag{2.9}
$$

where

$$d = \sum_{n_1=0}^{\infty}\sum_{v=1}^{n} w_{n_1,1,0,u,v} + \sum_{n_1=0}^{\infty}\sum_{a=1}^{N-2}\sum_{u=1}^{m}\sum_{v=1}^{n} w_{n_1,a+1,0,u,v} + \sum_{n_1=0}^{\infty}\sum_{a=1}^{L-1}\sum_{u=1}^{m}\sum_{v=1}^{n} w_{n_1,a+1,1,u,v} + \sum_{n_1=1}^{\infty}\sum_{a=0}^{L-1}\sum_{u=1}^{m'}\sum_{v=1}^{n} w_{n_1,a+1,2,u,v}. \quad (2.10)$$

2.3.2 Type II Customer

To find the LST of the waiting time distribution of a type II customer, we have to compute certain distributions. We proceed to such computations.

Definition 2.3.1. Duration of time with p type I customers in the system at a service commencement epoch of type I customers until the number of type I customers become zero for the first time is defined as a *p-cycle* denoted by B_p.

Distribution of a p-cycle in normal mode

This can be studied as a phase type distribution with representation $(\boldsymbol{\gamma}_p, T_1)$ where the underlying markov chain has state space $\{(i,j) : 1 \leq i \leq L; 1 \leq j \leq m\} \cup \{0\}$ where i denotes the number of type I customers in the system, j the service phase and 0 the absorbing state indicating that the number of type I customers become zero. The infinitesimal generator $\mathcal{T}_1$ of $B_p(t)$ has the form

$$\mathcal{T}_1 = \begin{bmatrix} T_1 & \boldsymbol{T}_1^0 \\ \mathbf{0} & 0 \end{bmatrix}$$

where

$$T_1 = \begin{bmatrix} T-\lambda I & \lambda I & & & \\ T^0\alpha & T-\lambda I & \lambda I & & \\ \ddots & \ddots & \ddots & & \\ & T^0\alpha & T-\lambda I & \lambda I & \\ & & T^0\alpha & T & \end{bmatrix} \quad \text{(c2)}$$

$$\boldsymbol{T}_1^0 = \begin{bmatrix} T^0 \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \quad \text{(c3)}$$

and the initial probabilty vector is

$$\boldsymbol{\gamma}_p = \begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} & \boldsymbol{\alpha} & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}, 1 \leq p \leq L \quad \text{(c4)}$$

where $\boldsymbol{\alpha}$ is in the pth position and $\mathbf{0}$ is a zero matrix of order m. Thus we have the following Theorem.

Theorem 2.3.2. *The LST of the length of a p-cycle is given by*

$$\boldsymbol{\gamma}_p(sI - T_1)^{-1}\boldsymbol{T}_1^0.$$

LST of the busy cycle generated by type I customers arriving during the service time of a type II customer

Theorem 2.3.3. *The LST of the busy cycle generated by type I customers arriving during the service time of a type II customer is given by*

$$\hat{B_{c_L}}(s) = \boldsymbol{\alpha'}[(s+\lambda)I-T']^{-1}\boldsymbol{T'}^0 + \sum_{p=1}^{L-1} \boldsymbol{\gamma}_p(sI-T_1)^{-1}\boldsymbol{T}_1^0\lambda^p\boldsymbol{\alpha'}[(s+\lambda)I-T']^{-(p+1)}\boldsymbol{T'}^0 +$$
$$\boldsymbol{\gamma}_L(sI-T_1)^{-1}\boldsymbol{T}_1^0\boldsymbol{\alpha'}[\lambda^{-1}((s+\lambda)I-T')]^{-L}[I-\lambda[(s+\lambda)I-T']^{-1}]^{-1}$$
$$[(s+\lambda)I-T']^{-1}\boldsymbol{T'}^0. \quad (2.11)$$

Proof. Let B_{c_L} denote the length of the busy cycle generated by type I customers arriving during the service time of a type II customer , $\hat{B_{c_L}}(s)$ the LST of the length of the busy cycle and l the number of type I customers that arrive during service time of type II customer.
Then $B_{c_L} = X + B_L^1 + \cdots B_L^l$ where X denote the service time of the type II customer in service, B_L^j the busy period generated by jth type I customers that arrive during X, where $1 \le j \le l$.

$$\begin{aligned}
\hat{B_{c_L}}(s) &= E(e^{-sB_{c_L}}) \\
&= \int_{x=0}^{\infty} E(e^{-sB_{c_L}}/X=x)P(x \le X < x+dx) \\
&= \int_{x=0}^{\infty}\sum_{p=0}^{\infty} E(e^{-sB_{c_L}}/X=x, l=p)P(l=p/X=x)P(x \le X < x+dx) \\
&= \int_{x=0}^{\infty}\sum_{p=0}^{\infty} E(e^{-sB_{c_L}}/X=x, l=p)\frac{e^{-\lambda x}(\lambda x)^p}{p!}\boldsymbol{\alpha'}e^{T'x}\boldsymbol{T'}^0dx \\
&= \int_{x=0}^{\infty} e^{-(s+\lambda)x}\boldsymbol{\alpha'}e^{T'x}T'^0dx + \int_{x=0}^{\infty}\sum_{p=1}^{L-1} e^{-sx}\boldsymbol{\gamma}_p(sI-T_1)^{-1}T_1^0\frac{e^{-\lambda x}(\lambda x)^p}{p!} \\
&\quad \boldsymbol{\alpha'}e^{T'x}\boldsymbol{T'}^0dx + \int_{x=0}^{\infty}\sum_{p=L}^{\infty} e^{-sx}\boldsymbol{\gamma}_L(sI-T_1)^{-1}\boldsymbol{T}_1^0\frac{e^{-\lambda x}(\lambda x)^p}{p!}\boldsymbol{\alpha'}e^{T'x}\boldsymbol{T'}^0dx \\
&= \boldsymbol{\alpha'}[(s+\lambda)I-T']^{-1}\boldsymbol{T'}^0 + \sum_{p=1}^{L-1}\boldsymbol{\gamma}_p(sI-T_1)^{-1}\boldsymbol{T}_1^0\frac{\lambda^p\boldsymbol{\alpha'}}{p!}\int_{x=0}^{\infty} x^pe^{-[(s+\lambda)I-T']x}\boldsymbol{T'}^0dx \\
&\quad + \sum_{p=L}^{\infty}\boldsymbol{\gamma}_L(sI-T_1)^{-1}\boldsymbol{T}_1^0\frac{\lambda^p}{p!}\boldsymbol{\alpha'}\int_{x=0}^{\infty} x^pe^{[-(s+\lambda)I-T'])x}\boldsymbol{T'}^0dx
\end{aligned} \quad (2.12)$$

We have,

$$\int_{x=0}^{\infty} x^pe^{-[(s+\lambda)I-T']x}dx = \frac{p!}{[(s+\lambda)I-T']^{p+1}} \quad (2.13)$$

Substituting (2.28) in (2.26) , its third term

$$\begin{aligned}
&= \sum_{p=L}^{\infty} \boldsymbol{\gamma}_L (sI-T_1)^{-1}\boldsymbol{T_1}^0 \lambda^p \boldsymbol{\alpha'}[(s+\lambda)I-T']^{-(p+1)}\boldsymbol{T'}^0\\
&= \boldsymbol{\gamma}_L (sI-T_1)^{-1}\boldsymbol{T_1}^0 \boldsymbol{\alpha'} \sum_{p=L}^{\infty}\left[\lambda^{-1}[(s+\lambda)I-T']\right]^{-p}[(s+\lambda)I-T']^{-1}\boldsymbol{T'}^0\\
&= \boldsymbol{\gamma}_L (sI-T_1)^{-1}\boldsymbol{T}_1^0 \boldsymbol{\alpha'}\left[\lambda^{-1}[(s+\lambda)I-T']\right]^{-L} \sum_{q=0}^{\infty}\left[\lambda^{-1}[(s+\lambda)I-T']\right]^{-q}[(s+\lambda)I-T']^{-1}\boldsymbol{T'}^0\\
&= \boldsymbol{\gamma}_L (sI-T_1)^{-1}\boldsymbol{T}_1^0 \boldsymbol{\alpha'}\left[\lambda^{-1}[(s+\lambda)I-T']\right]^{-L}[I-\lambda[(s+\lambda)I-T']^{-1}]^{-1}[(s+\lambda)I-T']^{-1}\boldsymbol{T'}^0
\end{aligned} \tag{2.14}$$

Substituting (2.14) in (2.26) gives

$$\hat{B_{c_L}}(s) = \boldsymbol{\alpha'}[(s+\lambda)I-T']^{-1}\boldsymbol{T'}^0 + \sum_{p=1}^{L-1}\boldsymbol{\gamma}_p(sI-T_1)^{-1}\boldsymbol{T}_1^0\lambda^p\boldsymbol{\alpha'}[(s+\lambda)I-T']^{-(p+1)}\boldsymbol{T'}^0 + \boldsymbol{\gamma}_L(sI-T_1)^{-1}\boldsymbol{T}_1^0$$
$$\boldsymbol{\alpha'}[\lambda^{-1}[(s+\lambda)I-T']^{-L}\left[I-\lambda[(s+\lambda)I-T']^{-1}\right]^{-1}[(s+\lambda)I-T']^{-1}\boldsymbol{T'}^0 \tag{2.15}$$

□

LST of the busy period of type I customers generated during the service time of a type II customer

Theorem 2.3.4. *The LST of the busy period of type I customers generated during the service time of a type II customer is given by*

$$\bar{B}_L(s) = \boldsymbol{\alpha'}[\lambda I - T']^{-1}\boldsymbol{T'}^0 + \sum_{p=1}^{L-1}\boldsymbol{\gamma}_p(sI-T_1)^{-1}\boldsymbol{T}_1^0\lambda^p\boldsymbol{\alpha'}[\lambda I-T']^{-(p+1)}\boldsymbol{T'}^0 +$$
$$\boldsymbol{\gamma}_L(sI-T_1)^{-1}\boldsymbol{T}_1^0\boldsymbol{\alpha'}[\lambda^{-1}(\lambda I-T')]^{-L}[I-\lambda[\lambda I-T']^{-1}]^{-1}[\lambda I-T']^{-1}\boldsymbol{T'}^0. \tag{2.16}$$

Proof. Let B_L denote the length of the busy period generated by type I customers arriving during the service time of a type II customer , $\hat{B_L}(s)$ the LST of the length of the busy period and l the number of type I customers that arrive during service time of type II customer.

Then $B_L = B_L^1 + \cdots B_L^l$,where B_L^j denote the busy period generated by jth type I customers that arrive during X, where $1 \le j \le l$. Proceeding as in the above proof, we get the required result. □

Let T_s'' denote the duration of time the server stays in vacation mode until either he gets back to normal mode through the random clock realization or the WV is interrupted with the number of type I customers in the system hitting N.

Conditional distribution of T_s'' given a type II customer arrives before the random clock expires

We can study this by a phase type distribution with representation (β_4, S_4) where the underlying markov chain has state space $\{0\} \cup \{(i,j) : 1 \leq i \leq N-1; 1 \leq j \leq m\} \cup \{*\}$ where i denotes the number of type I customers in the system, j the service phase and * the absorbing state indicating the vacation expiration. The infinitesimal generator $\mathcal{S}_4$ of $T_s''(t)$ is given by

$$\mathcal{S}_4 = \begin{bmatrix} S_4 & \boldsymbol{S}_4^0 \\ \boldsymbol{0} & 0 \end{bmatrix}, \text{ where,}$$

$$S_4 = \begin{bmatrix} -(\lambda+\eta) & \lambda\alpha & & & & \\ \theta\boldsymbol{T}^0 & \theta T-(\lambda+\eta)I & \lambda I & & & \\ & \theta\boldsymbol{T}^0\boldsymbol{\alpha} & \theta T-(\lambda+\eta)I & \lambda I & & \\ & \ddots & \ddots & \ddots & & \\ & & \theta\boldsymbol{T}^0\boldsymbol{\alpha} & \theta T-(\lambda+\eta)I & \lambda I \\ & & & \theta\boldsymbol{T}^0\boldsymbol{\alpha} & \theta T-(\lambda+\eta)I \end{bmatrix} \tag{c5}$$

and

$$\boldsymbol{S}_4^0 = \begin{bmatrix} \eta \\ \eta\boldsymbol{e}(m) \\ \vdots \\ \eta\boldsymbol{e}(m) \\ (\lambda+\eta)\boldsymbol{e}(m) \end{bmatrix}. \tag{c6}$$

The initial probability vector is given by

$$\boldsymbol{\beta}_4 = (1, \mathbf{0}), \text{ where } \mathbf{0} \text{ is a zero matrix of order } 1 \times (N-1)m. \qquad \text{(c7)}$$

Thus we have the following Lemma.

Lemma 2.3.1. The expected duration of time the server stays in vacation mode until either the server gets back to normal mode through the random clock expiring or the WV is interrupted as the number of type I customers in the system hits N given a type II customer arrives before the random clock expires, is given by $\boldsymbol{\beta}_4(-S_4)^{-2}\boldsymbol{S}_4^0$.

To find the waiting time of a type II customer who joins for service at time x, we have to consider different possibilities depending on the status of server at that time. The server may be in vacation mode, WV mode, normal mode 1 or in normal mode 2. Let Z_2 be the random variable representing the waiting time of a type II customer in the queue. Define $W_2(x) = \text{Prob}(Z_2 \leq x)$ and $W_2^*(s)$ be the corresponding LST.

Case I

Let F_1 denote the event that the system is in the state $(1, 0, 0, v)$, $1 \leq v \leq n$ immediately after arrival of the tagged customer. In this case the waiting time is the sum of time duration from his arrival epoch till the server shifts to normal mode and the time duration of busy period generated by type I customers present at that time, if any. Let $W_2^*(s/F_1)$ denote the corresponding conditional LST of the waiting time.

Then

$$W_2^*(s/F_1) = \boldsymbol{\beta}_4(sI - S_4)^{-1}\boldsymbol{S}_4^0\big[t_0 + \sum_{p=1}^{N}\boldsymbol{\gamma}_p(sI - T_1)^{-1}\boldsymbol{T}_1^0 t_p\big]$$

where t_p, $0 \leq p \leq N$ denote the probabaility that there are p type I customers when the vacation expires, which is given by

$$t_p = \begin{cases} \int_{t=0}^{\infty} \boldsymbol{\beta}_4 (e^{S_4 t})_p \eta dt & \text{if } p = 0 \\ \int_{t=0}^{\infty} \boldsymbol{\beta}_4 (e^{S_4 t})_p \eta \boldsymbol{e}(m) dt & \text{if } 1 \le p \le N-1 \\ \int_{t=0}^{\infty} \boldsymbol{\beta}_4 (e^{S_4 t})_{N-1} \lambda \boldsymbol{e}(m) dt & \text{if } p = N \end{cases}$$

where $(e^{S_4 t})_p$ denote the columns in $e^{S_4 t}$ corresponding to p type I customer states, T_1, $\boldsymbol{T}_1^0$, $\boldsymbol{\gamma}_p$, S_4, $\boldsymbol{S}_4^0$ and $\boldsymbol{\beta}_4$ are given by (c2), (c3),(c4),(c5), (c6) and (c7) respectively.

Case II

Let F_2 denote the event the system is in the state $(b+1, 0, 0, v)$, $b \ge 1$; $1 \le v \le n$ immediately after arrival of the tagged customer. In this case the waiting time is the sum of time duration from his arrival epoch till the server shifts to normal mode, the time duration of busy period generated by type I customers present at that time, if any and the time duration of busy cycles generated by type I customers arriving during the service time of each of the b type II customers. Let $W_2^*(s/F_2)$ denote the corresponding conditional LST of the waiting time.

Then

$$W_2^*(s/F_2) = \boldsymbol{\beta}_4 (sI - S_4)^{-1} \boldsymbol{S}_4^0 \big[t_0 + \sum_{p=1}^{N} \boldsymbol{\gamma}_p (sI - T_1)^{-1} \boldsymbol{T}_1^0 t_p \big] (\hat{B_{c_L}}(s))^b$$

where $\hat{B_{c_L}}(s)$ is given by Theorem 2.3.3.

Case III

Let F_3 denote the event the system is in the state $(b+1, a, 0, u, v)$, $b \ge 0$; $1 \le a \le N-1$; $1 \le u \le m$; $1 \le v \le n$ immediately after arrival of the tagged customer. In this case also the waiting time is the sum of time duration from his arrival epoch till the server shifts to normal mode, the time duration of busy period generated by type I customers present at that time, if any and the time duration of busy cycles generated by type I customers arriving during

the service time of each of the b type II customers. Let $W_2^*(s/F_3)$ denote the corresponding conditional LST.
Then

$$W_2^*(s/F_3) = \boldsymbol{\beta}_a^u (sI - S_4)^{-1} \boldsymbol{S}_4^0 \big[t_0 + \sum_{p=1}^{N} \boldsymbol{\gamma}_p (sI - T_1)^{-1} \boldsymbol{T}_1^0 t_p\big] (\hat{B_{c_L}}(s))^b$$

where $\boldsymbol{\beta}_a^u = (0, \mathbf{0}, \cdots, \mathbf{e}'_u, \cdots \mathbf{0})$, $\boldsymbol{e}'_u$ is in the $(a+1)^{th}$ position and $\mathbf{0}$ denotes zero matrix of order $1 \times m$.

Case IV

Let F_4 denote the event the system is in the state $(b+1, a, 1, u, v)$, $b \geq 0$; $1 \leq a \leq L$; $1 \leq u \leq m$; $1 \leq v \leq n$ immediately after arrival of the tagged customer. In this case the waiting time is the sum of time duration of an a-cycle in which the current service phase is u and time duration of busy cycles generated by type I customers arriving during the service time of each of the b type II customers. Let $W_2^*(s/F_4)$ denote the corresponding conditional LST.
Then

$$W_2^*(s/F_4) = (\boldsymbol{\gamma}_a^u (sI - T_1)^{-1} \boldsymbol{T}_1^0)(\hat{B_{c_L}}(s))^b$$

where $\boldsymbol{\gamma}_a^u = (\mathbf{0}, \cdots, \mathbf{e}'_u, \cdots \mathbf{0})$, where $\boldsymbol{e}'_u$ is in the a^{th} position and $\mathbf{0}$ denotes zero matrix of order $1 \times m$.

Case V

Let F_5 denote the event the system is in the state $(b+1, a, 2, u, v)$, $b \geq 1$; $0 \leq a \leq L$; $1 \leq u \leq m'$; $1 \leq v \leq n$ immediately after arrival of the tagged customer. In this case the waiting time is the sum of the residual service time of the type II customer in service, time duration of the busy period of type I customers generated during the service time of type II customer in service when the tagged customer arrives and time duration of busy cycles generated by type I customers arriving during the service time of each of the $b-1$ type II customers. Let $W_2^*(s/F_5)$ denote the corresponding conditional LST.

Then

$$W_2^*(s/F_5) = (\boldsymbol{e}'_u(sI - T')^{-1}\boldsymbol{T'}^0)(\bar{B}_L(s))(\hat{B_{c_L}}(s))^{b-1}$$

where $\bar{B}_L(s)$ is given by Theorem 2.3.4.

Let $w_{i_1,i_2,j,k,l}$ denote the probabilty that the system is in the state (i_1, i_2, j, k, l) immediately after arrival of the tagged customer. Then,

$$\begin{aligned}
w_{1,0,0,v} &= \frac{d^{(1)}_{v'v}}{\lambda+\eta-d^{(0)}_{v'v'}} x_{0,0,v'}, \text{ for, } 1 \le v, v' \le n \\
w_{b+1,0,0,v} &= \frac{d^{(1)}_{v'v}}{\lambda+\eta-d^{(0)}_{v'v'}} x_{b,0,0,v'}, \text{ for, } b \ge 1,\ 1 \le v, v' \le n \\
w_{b+1,a,0,u,v} &= \frac{d^{(1)}_{v'v}}{\lambda+\eta-\theta T_{uu}-d^{(0)}_{v'v'}} x_{b,a,0,u,v'}, \text{ for, } b \ge 0,\ 1 \le a \le N-1, \\
&\quad 1 \le v, v' \le n \\
w_{b+1,a,1,u,v} &= \frac{d^{(1)}_{v'v}}{\lambda-T_{uu}-d^{(0)}_{v'v'}} x_{b,a,1,u,v'}, \text{ for, } b \ge 0,\ 1 \le a \le L,\ 1 \le u \le m, \\
&\quad 1 \le v, v' \le n \\
w_{b+1,a,2,u,v} &= \frac{d^{(1)}_{v'v}}{\lambda-T_{uu}-d^{(0)}_{v'v'}} x_{b,a,2,u,v'}, \text{ for, } b \ge 1,\ 0 \le a \le L,\ 1 \le u \le m', \\
&\quad 1 \le v, v' \le n
\end{aligned}$$

Thus we have the following Theorem.

Theorem 2.3.5. *The LST of the waiting time of a type II customer is given by*

$$W_2^*(s) = \sum_{v=1}^{n} W_2^*(s/F_1) w_{1,0,0,v} + \sum_{b=1}^{\infty}\sum_{v=1}^{n} W_2^*(s/F_2) w_{b+1,0,0,v} +$$
$$\sum_{b=0}^{\infty}\sum_{a=1}^{N-1}\sum_{u=1}^{m}\sum_{v=1}^{n} W_2^*(s/F_3)(w_{b+1,a,0,u,v}) + \sum_{b=0}^{\infty}\sum_{a=1}^{L}\sum_{u=1}^{m}\sum_{v=1}^{n} W_2^*(s/F_4)(w_{b+1,a,1,u,v}) +$$
$$\sum_{b=1}^{\infty}\sum_{a=0}^{L}\sum_{u=1}^{m'}\sum_{v=1}^{n} W_2^*(s/F_5)(w_{b+1,a,2,u,v}). \quad (2.17)$$

Next we proceed to the analysis of model II.

2.4 Model Description and Mathematical Formulation of model II

Now we consider the case where the server continues to serve at a lower rate until either the vacation clock realizes or the number of type I customers present in the system plus the number of type I customers already served during the current vacation equal to N. All other assumptions are same as in model I.

In this case, $N_1(t), N_2(t), S(t)$, $J(t)$ and $M(t)$ are as defined for model I and we define $K(t)$ to be number of type I customers present in the system + number of type I customers already served during the current vacation at time t.

It is easy to verify that $\{(N_1(t), N_2(t), S(t), K(t), J(t), M(t)) : t \geq 0\}$ is an LIQBD with state space

$$\Omega = \cup_{i=0}^{\infty} l(i)$$

where $l(0) = \{(0, 0, k : 1 \leq k \leq n)\} \cup \{(0, i_2, 0, j_2, j_3, k) : 1 \leq i_2 \leq N-1; i_2 \leq j_2 \leq N-1; 1 \leq j_3 \leq m; 1 \leq k \leq n\} \cup \{(0, i_2, 1, j_3, k) : 1 \leq i_2 \leq L; 1 \leq j_3 \leq m; 1 \leq k \leq n\}$ and for $i_1 \geq 1$,
$l(i_1) = \{(i_1, 0, 0, k) : 1 \leq k \leq n\} \cup \{(i_1, 0, 2, j_3, k) : 1 \leq j_3 \leq m'; 1 \leq k \leq n\} \cup \{(i_1, i_2, 0, j_2, j_3, k) : 1 \leq i_2 \leq N-1; i_2 \leq j_2 \leq N-1; 1 \leq j_3 \leq m; 1 \leq k \leq n\} \cup \{(i_1, i_2, 1, j_3, k) : 1 \leq i_2 \leq L; 1 \leq j_3 \leq m; 1 \leq k \leq n\} \cup \{(i_1, i_2, 2, j_3, k) : 1 \leq i_2 \leq L; 1 \leq j_3 \leq m'; 1 \leq k \leq n\}$
Here also we note that when $N_1(t) = N_2(t) = 0$, server will be on vacation and so $S(t), K(t)$ and $J(t)$ need not be considered. When $N_2(t) = 0$ and $S(t) = 0$, then $K(t)$ and $J(t)$ need not be considered. The only other component in the state vector in both cases would be $M(t)$. Also when $S(t)$=1 or 2, then $K(t)$ need not be considered.

The infinitesimal generator of the above process is

$$\mathcal{Q}_2 = \begin{bmatrix} G_0 & H_0 & & & \\ H_1 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

where G_0 contains transitions within the level 0; H_0 represents transitions from level 0 to level 1; H_1 represents transitions from level 1 to level 0; A_0 represents transitions from level h to level $h+1$ for $h \geq 1$, A_1 represents transitions within the level h for $h \geq 1$ and A_2 represents transitions from level h to level $h-1$ for $h \geq 2$.

The boundary blocks G_0, H_0, H_1 are of orders $\left(n + \frac{mn}{2}(N^2 - N + 2L)\right) \times \left(n + \frac{mn}{2}(N^2 - N + 2L)\right)$, $\left(n + \frac{mn}{2}(N^2 - N + 2L)\right) \times \left((1+m')n + \frac{mn}{2}(N^2 - N + 2L) + Lm'n\right)$, $\left((1+m')n + \frac{mn}{2}(N^2 - N + 2L) + Lm'n\right) \times \left(n + \frac{mn}{2}(N^2 - N + 2L)\right)$ respectively. A_0, A_1, A_2 are square matrices of order $(1+m')n + \frac{mn}{2}(N^2 - N + 2L) + Lm'n$.

Define the entries of $G_{0_{(i_1,j_1,k_1,l_1,m_1)}}^{(i_2,j_2,k_2,l_2,m_2)}$, $H_{0_{(i_1,j_1,k_1,l_1,m_1)}}^{(i_2,j_2,k_2,l_2,m_1)}$, $H_{1_{(i_1,j_1,k_1,l_1,m_1)}}^{(i_2,j_2,k_2,l_2,m_2)}$ as transition submatrices which contains transitions of the form $(0, i_1, j_1, k_1, l_1, m_1) \rightarrow (0, i_2, j_2, k_2, l_2, m_2)$, $(0, i_1, j_1, k_1, l_1, m_1) \rightarrow (1, i_2, j_2, k_2, l_2, m_2)$, $(1, i_1, j_1, k_1, l_1, m_1) \rightarrow (0, i_2, j_2, k_2, l_2, m_2)$ respectively. Define the entries of $A_{0_{(i_1,j_1,k_1,l_1,m_1)}}^{(i_2,j_2,k_2,l_2,m_2)}$, $A_{1_{(i_1,j_1,k_1,l_1,m_1)}}^{(i_2,j_2,k_2,l_2,m_2)}$ and $A_{2_{(i_1,j_1,k_1,l_1,m_1)}}^{(i_2,j_2,k_2,l_2,m_2)}$ as transition submatrices which contains transitions of the form $(h, i_1, j_1, k_1, l_1, m_1) \rightarrow (h+1, i_2, j_2, k_2, l_2, m_2)$, where $h \geq 1$;$(h, i_1, j_1, k_1, l_1, m_1) \rightarrow (h, i_2, j_2, k_2, l_2, m_2)$, where $h \geq 1$ and $(h, i_1, j_1, k_1, l_1, m_1) \rightarrow (h-1, i_2, j_2, k_2, l_2, m_2)$, where $h \geq 1$ respectively. Since none or one event alone could take place in a short interval of time with positive probability, in general, a transition such as $(i_1, j_1, k_1, l_1, m_1, n_1) \rightarrow (i_2, j_2, k_2, l_2, m_2, n_2)$ has positive rate only for exactly one of $i_2, j_2, k_2, l_2, m_2, n_2$ different from

$i_1, j_1, k_1, l_1, m_1, n_1.$

$$G_{0_{(i_1,j_1,k_1,l_1,m_1)}}^{(i_2,j_2,k_2,l_2,m_2)} = \begin{cases} \lambda(\boldsymbol{\alpha} \otimes I_n) & i_1 = 0, i_2 = 1; j_1 = j_2 = 0; k_2 = 1; \\ & 1 \leq l_2 \leq m, 1 \leq m_1, m_2 \leq n \\ \lambda I_{mn} & 1 \leq i_1 \leq N-2, i_2 = i_1 + 1; j_1 = j_2 = 0; \\ & i_1 \leq k_1 \leq N-2, \\ & k_2 = k_1 + 1; 1 \leq l_1, l_2 \leq m; 1 \leq m_1, m_2 \leq n \\ \lambda \boldsymbol{e}(m) \otimes (\boldsymbol{\alpha} \otimes I_n) & 1 \leq i_1 \leq N-1, i_2 = i_1 + 1; j_1 = 0, j_2 = 1; \\ & k_1 = N-1; 1 \leq l_1, l_2 \leq m; 1 \leq m_1, m_2 \leq n \\ \lambda I_{mn} & 1 \leq i_1 \leq L-1, i_2 = i_1 + 1; j_1 = j_2 = 1; \\ & 1 \leq l_1, l_2 \leq m; 1 \leq m_1, m_2 \leq n \\ \eta \boldsymbol{e}(m) \otimes (\boldsymbol{\alpha} \otimes I_n) & 1 \leq i_1 \leq N-1; j_1 = 0, j_2 = 1; \\ & i_1 \leq k_1 \leq N-1; 1 \leq l_1, l_2 \leq m; 1 \leq m_1, m_2 \leq n \\ \theta \boldsymbol{T^0} \otimes I_n & i_1 = 1, i_2 = 0; j_1 = 0, j_2 = 0; 1 \leq k_1 \leq N-1, \\ & 1 \leq l_1 \leq m; 1 \leq m_1, m_2 \leq n \\ \boldsymbol{T}^0 \otimes I_n & i_1 = 1, i_2 = 0; j_1 = 1, j_2 = 0; 1 \leq l_1 \leq m; \\ & 1 \leq m_1, m_2 \leq n \\ \theta \boldsymbol{T^0} \boldsymbol{\alpha} \otimes I_n & 2 \leq i_1 \leq N-1, i_2 = i_1 - 1; j_1 = 0, j_2 = 0; \\ & i_1 \leq k_1 \leq N-1, k_2 = k_1; 1 \leq l_1, l_2 \leq m; \\ & 1 \leq m_1, m_2 \leq n \\ \boldsymbol{T^0} \boldsymbol{\alpha} \otimes I_n & 2 \leq i_1 \leq L, i_2 = i_1 - 1; j_1 = j_2 = 1; \\ & 1 \leq l_1, l_2 \leq m; 1 \leq m_1, m_2 \leq n \\ D_0 - \lambda I_n & i_1 = i_2 = 0; j_1 = j_2 = 0; 1 \leq m_1, m_2 \leq n \\ \theta T \oplus D_0 - (\lambda + \eta) I_{mn} & 1 \leq i_1 \leq N-1, i_2 = i_1; j_1 = j_2 = 0; \\ & i_1 \leq k_1 \leq N-1, k_2 = k_1; 1 \leq l_1, l_2 \leq m; \\ & 1 \leq m_1, m_2 \leq n \\ T \oplus D_0 - \lambda I_{mn} & 1 \leq i_1 \leq L-1, i_2 = i_1; j_1 = j_2 = 1; \\ & 1 \leq l_1, l_2 \leq m; 1 \leq m_1, m_2 \leq n \\ T \oplus D_0 & i_1 = i_2 = L; j_1 = j_2 = 1; 1 \leq l_1, l_2 \leq m; \\ & 1 \leq m_1, m_2 \leq n \end{cases}$$

$$H_{0_{(i_1,j_1,k_1,l_1,m_1)}}^{(i_2,j_2,k_2,l_2,m_2)} = \begin{cases} D_1 & i_1 = 0 = i_2 = 0; j_1 = j_2 = 0; 1 \le m_1, m_2 \le n \\ I_m \otimes D_1 & 1 \le i_1 \le N-1, i_2 = i_1; j_1 = j_2 = 0; i_1 \le k_1 \le N-1, \\ & k_2 = k_1; 1 \le l_1, l_2, \le m; 1 \le m_1, m_2 \le n \\ I_m \otimes D_1 & 1 \le i_1 \le L, i_2 = i_1; j_1 = j_2 = 1; 1 \le l_1, l_2 \le m; \\ & 1 \le m_1, m_2 \le n \end{cases}$$

$$H_{1_{(i_1,j_1,k_1,l_1,m_1)}}^{(i_2,j_2,k_2,l_2,m_2)} = \begin{cases} \boldsymbol{T'}^0 \otimes I_n & i_1 = i_2 = 0; j_1 = 2, j_2 = 0; 1 \le l_1 \le m'; \\ & 1 \le m_1, m_2 \le n \\ \boldsymbol{T'}^0 \boldsymbol{\alpha} \otimes I_n & 1 \le i_1 \le L, i_2 = i_1; j_1 = 2, j_2 = 1; 1 \le l_1 \le m', \\ & 1 \le l_2 \le m; 1 \le m_1, m_2 \le n \end{cases}$$

$$A_{0_{(i_1,j_1,k_1,l_1,m_1)}}^{(i_2,j_2,k_2,l_2,m_2)} = \begin{cases} D_1 & i_1 = i_2 = 0; j_1 = j_2 = 0; 1 \le m_1, m_2 \le n \\ I_{m'} \otimes D_1 & i_1 = i_2 = 0; j_1 = j_2 = 2; 1 \le l_1, l_2 \le m'; \\ & 1 \le m_1, m_2 \le n \\ I_m \otimes D_1 & 1 \le i_1 \le N-1, i_2 = i_1; j_1 = j_2 = 0; i_1 \le k_1 \le N-1, \\ & k_2 = k_1; 1 \le l_1, l_2 \le m; 1 \le m_1, m_2 \le n \\ I_m \otimes D_1 & 1 \le i_1 \le L, i_2 = i_1; j_1 = j_2 = 1; 1 \le l_1, l_2 \le m; \\ & 1 \le m_1, m_2 \le n \\ I_{m'} \otimes D_1 & 1 \le i_1 \le L, i_2 = i_1; j_1 = j_2 = 2; 1 \le l_1, l_2 \le m'; \\ & 1 \le m_1, m_2 \le n \end{cases}$$

$$A_{2_{(i_1,j_1,k_1,l_1,m_1)}}^{(i_2,j_2,k_2,l_2,m_2)} = \begin{cases} \boldsymbol{T'}^0 \boldsymbol{\alpha'} \otimes I_n & i_1 = i_2 = 0; j_1 = j_2 = 2; 1 \le l_1, l_2 \le m'; \\ & 1 \le m_1, m_2 \le n \\ \boldsymbol{T'}^0 \boldsymbol{\alpha} \otimes I_n & 1 \le i_1 \le L, i_2 = i_1; j_1 = 2, j_2 = 1; 1 \le l_1 \le m', \\ & 1 \le l_2 \le m; 1 \le m_1, m_2 \le n \end{cases}$$

$$A_{1(i_1,j_1,k_1,l_1,m_1)}^{(i_2,j_2,k_2,l_2,m_2)} = \begin{cases} \lambda(\boldsymbol{\alpha} \otimes I_n) & i_1 = 0, i_2 = 1; j_1 = j_2 = 0; k_2 = 1; \\ & 1 \leq l_2 \leq m;\ 1 \leq m_1, m_2 \leq n \\ \lambda I_{mn}, & 1 \leq i_1 \leq N-2; i_2 = i_1 + 1; j_1 = j_2 = 0; \\ & i_1 \leq k_1 \leq N-2, k_2 = k_1 + 1; 1 \leq l_1, l_2 \leq m; \\ & 1 \leq m_1, m_2 \leq n \\ \lambda \boldsymbol{e}(m) \otimes (\boldsymbol{\alpha} \otimes I_n) & 1 \leq i_1 \leq N-1; j_1 = 0, j_2 = 1; k_1 = N-1; \\ & 1 \leq l_1, l_2 \leq m; 1 \leq m_1, m_2 \leq n \\ \lambda I_{mn} & 1 \leq i_1 \leq L-1, i_2 = i_1 + 1; j_1 = j_2 = 1; \\ & 1 \leq l_1, l_2 \leq m; 1 \leq m_1, m_2 \leq n \\ \lambda I_{mn} & 0 \leq i_1 \leq L-1, i_2 = i_1 + 1; j_1 = j_2 = 2; \\ & 1 \leq l_1, l_2 \leq m'; 1 \leq m_1, m_2 \leq n \\ \eta(\boldsymbol{\alpha'} \otimes I_n) & i_1 = i_2 = 0; j_1 = 0, j_2 = 2; 1 \leq l_2 \leq m'; \\ & 1 \leq m_1, m_2 \leq n \\ \eta \boldsymbol{e}(m) \otimes (\boldsymbol{\alpha} \otimes I_n) & 1 \leq i_1 \leq N-1, i_2 = i_1; j_1 = 0, j_2 = 1; \\ & i_1 \leq k_1 \leq N-1; 1 \leq l_1, l_2 \leq m; 1 \leq m_1, m_2 \leq n \\ \theta \boldsymbol{T}^0 \otimes I_n & i_1 = 1, i_2 = 0; j_1 = j_2 = 0; 1 \leq k_1 \leq N-1; \\ & 1 \leq l_1 \leq m; 1 \leq m_1, m_2 \leq n \\ \boldsymbol{T}^0 \boldsymbol{\alpha'} \otimes I_n & i_1 = 1, i_2 = 0; j_1 = 1, j_2 = 2; 1 \leq l_1 \leq m, \\ & 1 \leq l_2 \leq m'; 1 \leq m_1, m_2 \leq n \\ \theta \boldsymbol{T}^0 \boldsymbol{\alpha} \otimes I_n & 2 \leq i_1 \leq N-1, i_2 = i_1 - 1; j_1 = j_2 = 0; \\ & i_1 \leq k_1 \leq N-1, \\ & k_2 = k_1; 1 \leq l_1, l_2 \leq m; 1 \leq m_1, m_2 \leq n \\ \boldsymbol{T}^0 \boldsymbol{\alpha} \otimes I_n & 2 \leq i_1 \leq L, i_2 = i_1 - 1; j_1 = j_2 = 1; \\ & 1 \leq l_1, l_2 \leq m; 1 \leq m_1, m_2 \leq n \\ D_0 - (\lambda + \eta) I_n & i_1 = i_2 = 0; j_1 = j_2 = 0; 1 \leq m_1, m_2 \leq n \\ T' \oplus D_0 - \lambda I_{mn} & i_1 = i_2 = 0; j_1 = j_2 = 2; 1 \leq l_1, l_2 \leq m'; \\ & 1 \leq m_1, m_2 \leq n \\ \theta T \oplus D_0 - (\lambda + \eta) I_{mn} & 1 \leq i_1 \leq N-1, i_2 = i_1; j_1 = j_2 = 0; \\ & i_1 \leq k_1 \leq N-1, k_2 = k_1; 1 \leq l_1, l_2 \leq m; \\ & 1 \leq m_1, m_2 \leq n \\ T \oplus D_0 - \lambda I_{mn} & 1 \leq i_1 \leq L-1, i_2 = i_1; j_1 = j_2 = 1; \\ & 1 \leq l_1, l_2 \leq m; 1 \leq m_1, m_2 \leq n \\ T' \oplus D_0 - \lambda I_{mn} & 1 \leq i_1 \leq L-1, i_2 = i_1; j_1 = j_2 = 2; \\ & 1 \leq l_1, l_2 \leq m'; 1 \leq m_1, m_2 \leq n \\ T \oplus D_0 & i_1 = i_2 = L; j_1 = j_2 = 1; 1 \leq l_1, l_2 \leq m; \\ & 1 \leq m_1, m_2 \leq n \\ T' \oplus D_0 & i_1 = i_2 = L; j_1 = j_2 = 2; 1 \leq l_1, l_2 \leq m'; \\ & 1 \leq m_1, m_2 \leq n \end{cases}$$

2.5 Steady State Analysis

First we find the condition for stability of the system under study.

2.5.1 Stability condition

Let $\boldsymbol{\pi} = (\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \ldots, \boldsymbol{\pi}_L)$ denote the steady state probability vector of the generator

$$A = A_0 + A_1 + A_2 = \begin{bmatrix} B_1 & B_2 & & & & & & & & \\ C_1 & E_1 & F_1 & & & & & & & \\ & C_2 & E_2 & F_2 & & & & & & \\ & & \ddots & \ddots & \ddots & & & & & \\ & & & C_{N-1} & E_{N-1} & F_{N-1} & & & & \\ & & & & G' & H & J & & & \\ & & & & & G & H & J & & \\ & & & & & & \ddots & \ddots & \ddots & \\ & & & & & & & G & H & J \\ & & & & & & & & G & K \end{bmatrix}$$

where

$$B_1(k,l) = \begin{cases} D_0 + D_1 - (\lambda+\eta)I_n & k=1, l=1 \\ \eta(\boldsymbol{\alpha'} \otimes I_n) & k=1, l=2 \\ 0 & k=2, l=1 \\ \boldsymbol{T'}^0\boldsymbol{\alpha'} \otimes I_n + T' \oplus D_0 - \lambda I_{m'n} + I_{m'} \otimes D_1 & k=2, l=2 \end{cases}$$

$$B_2(k,l) = \begin{cases} \lambda(\boldsymbol{\alpha} \otimes I_n) & k=1, l=1 \\ \lambda I_{m'n} & k=2, l=2 \\ 0 & \text{otherwise} \end{cases}, C_1(k,l) = \begin{cases} \boldsymbol{e}(N-1) \otimes \left(\theta \boldsymbol{T}^0 \otimes I_n\right) & k=1, l=1 \\ \boldsymbol{T}^0\boldsymbol{\alpha'} \otimes I_n & k=2, l=2 \\ 0 & \text{otherwise} \end{cases}$$

For $2 \leq i \leq N-1$,

$$C_i(k,l) = \begin{cases} I_{N-i} \otimes \left(\theta \boldsymbol{T}^0 \boldsymbol{\alpha} \otimes I_n\right) & k=1, l=2 \\ \boldsymbol{T}^0 \boldsymbol{\alpha} \otimes I_n & k=2, l=3 \\ 0 & \text{otherwise} \end{cases}$$

For $1 \leq i \leq N-1$,

$$E_i(k,l) = \begin{cases} I_{N-i} \otimes \left(\theta T \oplus D_0 - (\lambda+\eta) I_{mn} + I_m \otimes D_1\right) & k=1, l=1 \\ \boldsymbol{e}\big((N-i)m\big) \otimes \left(\eta(\boldsymbol{\alpha} \otimes I_n)\right) & k=1, l=2 \\ T \oplus D_0 - \lambda I_{mn} + I_m \otimes D_1 & k=2, l=2 \\ \boldsymbol{T'}^0 \boldsymbol{\alpha} \otimes I_n & k=3, l=2 \\ T' \oplus D_0 - \lambda I_{m'n} + I_{m'} \otimes D_1 & k=3, l=3 \\ 0 & \text{otherwise} \end{cases}$$

For $1 \leq i \leq N-2$,

$$F_i(k,l) = \begin{cases} \lambda I_{(N-1-i)mn} & k=1, l=1 \\ \lambda \boldsymbol{e}(m) \otimes (\boldsymbol{\alpha} \otimes I_n) & k=2, l=2 \\ \lambda I_{mn} & k=3, l=2 \\ \lambda I_{m'n} & k=4, l=3 \\ 0 & \text{otherwise} \end{cases}, \quad F_{N-1}(k,l) = \begin{cases} \lambda \boldsymbol{e}(m) \otimes (\boldsymbol{\alpha} \otimes I_n) & k=1, l=1 \\ \lambda I_{mn} & k=2, l=1 \\ \lambda I_{m'n} & k=3, l=2 \\ 0 & \text{otherwise} \end{cases}$$

$$G'(k,l) = \begin{cases} \boldsymbol{T}^0 \boldsymbol{\alpha} \otimes I_n & k=1, l=2 \\ 0 & \text{otherwise} \end{cases}, \quad G(k,l) = \begin{cases} \boldsymbol{T}^0 \boldsymbol{\alpha} \otimes I_n & k=1, l=1 \\ 0 & \text{otherwise} \end{cases}$$

$$H(k,l) = \begin{cases} T \oplus D_0 - \lambda I_{mn} + I_m \otimes D_1 & k=l=1 \\ T' \oplus D_0 - \lambda I_{m'n} + I_{m'} \otimes D_1 & k=l=2 \\ \boldsymbol{T'}^0 \boldsymbol{\alpha} \otimes I_n & k=2, l=1 \\ 0 & \text{otherwise} \end{cases}, \quad J(k,l) = \begin{cases} \lambda I_{mn} & k=1, l=1 \\ \lambda I_{m'n} & k=2, l=2 \end{cases}$$

$$K(k,l) = \begin{cases} T \oplus D_0 + I_m \otimes D_1 & k = l = 1 \\ T' \oplus D_0 + I_{m'} \otimes D_1 & k = l = 2 \\ \boldsymbol{T'}^0 \boldsymbol{\alpha} \otimes I_n & k = 2, l = 1 \end{cases}$$

with dimension of B_1, B_2, C_1 be $(n + m'n) \times (n + m'n), (n + m'n) \times \big(Nmn + m'n\big), \big(Nmn+m'n\big) \times (n+m'n)$ respectively. For $2 \leq i \leq N-1$, C_i be of order $\big(((N-i+1)mn+m'n)\big)\times\big(((N-i+2)mn+m'n)\big)$, E_i be a square matrix of order $(N-i+1)mn+m'n$, F_i is of order $\big(((N-i+1)mn+m'n)\big)\times\big(((N-i)mn+m'n)\big)$, G' is of order $(m + m')n \times (2m + m')n$, G, H, J and K are square matrices of order $(m + m')n$.

ie,

$$\boldsymbol{\pi} A = 0, \boldsymbol{\pi e} = 1. \tag{2.18}$$

The $LIQBD$ description of the model indicates that the queueing system is stable (see Neuts [40]) if and only if the left drift exceeds that of right drift. That is,

$$\boldsymbol{\pi} A_0 \mathbf{e} < \boldsymbol{\pi} A_2 \mathbf{e}. \tag{2.19}$$

The vector $\boldsymbol{\pi}$ cannot be obtained directly in terms of the parametres of the model. From (2.18)we get

$$\boldsymbol{\pi}_i = \boldsymbol{\pi}_{i-1} \mathcal{U}_{i-1}, 1 \leq i \leq L \tag{2.20}$$

where

$$\mathcal{U}_0 = -B_2(E_1 + \mathcal{U}_1 C_2)^{-1}$$

$$\mathcal{U}_i = \begin{cases} -F_i(D_{i+1} + U_{i+1}C_{i+2})^{-1} & for\, 1 \leq i \leq N-3 \\ -F_{N-2}(E_{N-1} + \mathcal{U}_{N-1}G)^{-1} & for\, i = N-2 \\ -E_{N-1}(H + \mathcal{U}_N G)^{-1}, & for\, i = N-1 \\ -\lambda(H + \mathcal{U}_{i+1}G)^{-1} & for\, N \leq i \leq L-2 \\ -\lambda J^{-1} & for\, i = L-1. \end{cases}$$

From the normalizing condition $\boldsymbol{\pi e} = 1$ we have

$$\boldsymbol{\pi}_0 \left(\sum_{j=0}^{L-1} \prod_{i=0}^{j} \mathcal{U}_i + I \right) \boldsymbol{e} = 1. \tag{2.21}$$

The inequality (2.19) gives the stability condition as

$$\boldsymbol{\pi}_0 \left[(I_{(1+m')} \otimes D_1)\boldsymbol{e} + \sum_{i=0}^{N-2} \prod_{j=0}^{i} \mathcal{U}_j (I_{((N-i)m+m')} \otimes D_1)\boldsymbol{e} + \sum_{i=N-1}^{L-1} \prod_{j=0}^{i} \mathcal{U}_j (I_{(m+m')} \otimes D_1)\boldsymbol{e} \right]$$
$$< \boldsymbol{\pi}_0 \left[A_{20} + \sum_{i=0}^{N-2} \prod_{j=0}^{i} \mathcal{U}_j A_{2i} + \sum_{i=N+1}^{L-1} \prod_{j=0}^{i} \mathcal{U}_j A_{2(N-1)}) \right] \tag{2.22}$$

where, $\mathcal{A}_{20} = \begin{bmatrix} 0 \\ (\boldsymbol{T'}^0 \boldsymbol{\alpha'} \otimes I)\boldsymbol{e} \end{bmatrix}$, $\mathcal{A}_{2i} = \begin{bmatrix} 0 \\ (\boldsymbol{T'}^0 \boldsymbol{\alpha} \otimes I)\boldsymbol{e} \end{bmatrix}$, $1 \leq i \leq N-2$ and $A_{2(N-1)} = \begin{bmatrix} 0 \\ (\boldsymbol{T'}^0 \boldsymbol{\alpha'} \otimes I)\boldsymbol{e} \end{bmatrix}$, with 0 a zero column vector of order n, $(N-i)mn$ and mn for A_{20}, A_{2i}, $1 \leq i \leq N-2$ and $A_{2(N-1)}$ respectively.

2.5.2 Steady-state probability vector

Assuming that the condition (2.22) is satisfied we proceed to find the steady-state probability of the system state.

Let $\boldsymbol{x}$ be the steady state probability vector of Q. We partition this vector as

$$\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2 \ldots),$$

where $\boldsymbol{x}_0$ is of dimension $n + \frac{mn}{2}(N^2 - N + 2L)$, $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots$ are of dimension $(1+m')n + \frac{mn}{2}(N^2 - N + 2L) + Lm'n$. Under the stability condition, we have

$$\boldsymbol{x}_i = \boldsymbol{x}_1 R^{i-1}, i \geq 2$$

where the matrix R is the minimal nonnegative solution to the matrix quadratic equation

$$R^2A_2 + RA_1 + A_0 = 0$$

and the vectors $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$are obtained by solving the equations

$$\boldsymbol{x}_0G_0 + \boldsymbol{x}_1H_1 = 0 \tag{2.23}$$

$$\boldsymbol{x}_0H_0 + \boldsymbol{x}_1(A_1 + RA_2) = 0 \tag{2.24}$$

subject to the normalizing condition

$$\boldsymbol{x}_0\boldsymbol{e} + \boldsymbol{x}_1(I - R)^{-1}\boldsymbol{e} = 1. \tag{2.25}$$

2.5.3 Distribution of duration of slow service mode

The duration U_{slow}, in slow service mode is defined as the time the server starts in slow service mode (through initiating a WV) until either switching to normal mode through vacation clock realization or with the number of type I customers in the sytem plus number of type I customers already served during the current vacation hitting the threshold value N, $1 \leq N \leq L$ or the number of type I customers hitting 0 before expiration of vacation. We consider the Markov process $U_{slow}(t) = \{(N(t), J(t), K(t)) : t \geq 0\}$ where $N(t)$ is the number of type I customers in the system at time t, $J(t)$ the number of type I customers in the system plus number of type I customers already served during the current vacation and $K(t)$, the service phase at the time t. Thus the state space of the process is $\{(i, j, k) : 1 \leq i \leq N-1; i \leq j \leq N-1; 1 \leq k \leq m\} \cup \{0\} \cup \{*_1\} \cup \{*_2\}$ where 0 denotes the absorbing state indicating that there is no type I customer in the system and $*_1$ denotes the absorbing state indicating the vacation expiration by vacation clock realization and $*_2$ denotes the absorbing state indicating the vacation expiration by the number of type customers in the system plus number of type I customers already served during the current vacation hitting N. The initial probability vector is given by

$$\boldsymbol{\gamma}_1 = \frac{1}{d_1}(w_1, w_2, \cdots, w_m, \mathbf{0})$$

where, for, $1 \leq j \leq m$, w_j and d_1 are defined as in section 2.2.3 and $\mathbf{0}$ is a zero matrix of order $1 \times \frac{(N-2)(N+1)}{2}m$.
The infinitesimal generator $\mathcal{U}_1$ of $U_{slow}(t)$ has the form

$$\mathcal{U}_1 = \begin{bmatrix} U_1 & \boldsymbol{U}_1^{(0)} & \boldsymbol{U}_1^{(1)} & \boldsymbol{U}_1^{(2)} \\ \mathbf{0} & 0 & 0 & 0 \end{bmatrix} \text{ where,}$$

$$U_1 = \begin{bmatrix} L_1 & M_1 & & & \\ K_1 & L_2 & M_2 & & \\ & \ddots & \ddots & \ddots & \\ & & K_{N-3} & L_{N-2} & M_{N-2} \\ & & & K_{N-2} & L_{N-1} \end{bmatrix} \text{ with,}$$

$$K_i = \begin{bmatrix} \mathbf{0} & I_{N-i-1} \otimes (\theta \boldsymbol{T}^0 \boldsymbol{\alpha}) \end{bmatrix}, \text{ for } 1 \leq i \leq N-2$$

where $\mathbf{0}$ is a zero matrix of order $(N-i-1)m \times m$.

$$L_i = I_{N-i} \otimes (\theta T - (\lambda + \eta) I_m), \text{for } 1 \leq i \leq N-1$$

$$M_i = \begin{bmatrix} \lambda I_{(N-i-1)m} \\ \mathbf{0} \end{bmatrix}, \text{for } 1 \leq i \leq N-2$$

where $\mathbf{0}$ is a zero matrix of order $m \times (N-i-1)m$.

$$\boldsymbol{U}_1^{(0)} = \begin{bmatrix} \boldsymbol{e}(N-1) \otimes \theta \boldsymbol{T}^0 \\ \mathbf{0} \end{bmatrix}$$

where $\mathbf{0}$ is a zero matrix of order $\frac{(N-2)(N-1)}{2}m \times 1$.

$$\boldsymbol{U}_1^{(1)} = \begin{bmatrix} \eta\boldsymbol{e}((N-2)m) \\ \eta\boldsymbol{e}(m) \\ \eta\boldsymbol{e}((N-3)m) \\ \eta\boldsymbol{e}(m) \\ \vdots \\ \eta\boldsymbol{e}(m) \\ \eta\boldsymbol{e}(m) \\ \eta\boldsymbol{e}(m) \end{bmatrix}, \boldsymbol{U}_1^{(2)} = \begin{bmatrix} \mathbf{0} \\ \lambda\boldsymbol{e}(m) \\ \mathbf{0} \\ \lambda\boldsymbol{e}(m) \\ \vdots \\ \mathbf{0} \\ \lambda\boldsymbol{e}(m) \\ \lambda\boldsymbol{e}(m) \end{bmatrix}$$

where $\mathbf{0}$'s are zero matrices of order $(N-2)m \times 1, (N-3)m \times 1, \ldots, m \times 1$ respectively.
Thus we have the following Lemma.

Lemma 2.5.1. The expected duration of time the server remains in WV until the number of type I customers reach 0 is given by $\boldsymbol{\gamma}_1(-U_1)^{-2}\boldsymbol{U}_1^{(0)}$.

Define the random variable M_2 as number of returns to 0 type I customer state starting from 0 type I customer state during vacation mode of service before the arrival of a type II customer.

2.5.4 Expected value of M_2

Let U_s denote the duration of slow service until the arrival of a type II customer.

Distribution of U_s

We consider the Markov process $U_s(t) = \{(N(t), J(t), K(t), M(t)) : t \geq 0\}$ where $N(t)$ is the number of type I customers in the system at time t, $J(t)$ the number of type I customers in the system plus number of type I customers already served during the current vacation, $K(t)$ the service phase and $M(t)$ the arrival phase of type II customer at that instant. Thus the state space of the process is $\{(i, j, k, l) : 1 \leq i \leq N-1; i \leq j \leq N-1; 1 \leq k \leq m; 1 \leq l \leq n\} \cup \{0\} \cup \{*_1\} \cup \{*_2\}$ where 0 denotes the absorbing state indicating that there is no type I customer in the system, $*_1$,$*_2$ denote the absorbing states indicating the vacation expiration and arrival of a type II customer respectively. The initial probability vector is given by

$$\boldsymbol{\gamma_2} = (1/d_2)(w_{1,1}, \cdots, w_{1,n}, \cdots, w_{m,1}, \cdots, w_{m,n}, \mathbf{0})$$

where, for, $1 \leq j \leq m$, $1 \leq k \leq n$, $w_{j,k}$ and d_2 are defined as in section 2.2.4 and $\mathbf{0}$ is a zero matrix of order $1 \times \frac{(N-2)(N+1)}{2} mn$. The infinitesimal generator $\mathcal{U}_2$ of $U_s(t)$ has the form

$$\mathcal{U}_2 = \begin{bmatrix} U_2 & \boldsymbol{U}_2^{(0)} & \boldsymbol{U}_2^{(1)} & \boldsymbol{U}_2^{(2)} \\ \mathbf{0} & 0 & 0 & 0 \end{bmatrix} \text{ where,}$$

$$U_2 = \begin{bmatrix} L_1 & M_1 & & & & \\ K_1 & L_2 & M_2 & & & \\ & \ddots & \ddots & \ddots & & \\ & & K_{N-3} & L_{N-2} & M_{N-2} \\ & & & K_{N-2} & L_{N-1} \end{bmatrix} \text{ with,}$$

$$K_i = \begin{bmatrix} \mathbf{0} & I_{N-1-i} \otimes (\theta \boldsymbol{T}^0 \boldsymbol{\alpha} \otimes I) \end{bmatrix}, \text{ for } 1 \leq i \leq N-2,$$

where $\mathbf{0}$ is a zero matrix of order $(N-i-1)mn \times mn$,

$$M_i = \begin{bmatrix} \lambda I_{(N-i-1)mn} \\ \mathbf{0} \end{bmatrix}, \text{ for } 1 \leq i \leq N-2,$$

where $\mathbf{0}$ is a zero matrix of order $mn \times (N-1-i)mn$.

$$L_i = I_{N-i} \otimes (\theta T \oplus D_0 - (\lambda+\eta)I_{mn}), \text{ for } 1 \le i \le N-1$$

$$\boldsymbol{U}_2^{(0)} = \begin{bmatrix} \boldsymbol{e}(N-1) \otimes (\theta \boldsymbol{T}^0 \otimes \boldsymbol{e}(n)) \\ \mathbf{0} \end{bmatrix},$$

$$\boldsymbol{U}_2^{(1)} = \begin{bmatrix} \eta \boldsymbol{e}((N-2)mn) \\ (\lambda+\eta)\boldsymbol{e}(mn) \\ \eta \boldsymbol{e}((N-3)mn) \\ (\lambda+\eta)\boldsymbol{e}(mn) \\ \vdots \\ \eta \boldsymbol{e}(mn) \\ (\lambda+\eta)\boldsymbol{e}(mn) \\ (\lambda+\eta)\boldsymbol{e}(mn) \end{bmatrix}, \boldsymbol{U}_2^{(2)} = \begin{bmatrix} \delta \boldsymbol{e}((N-1)m)) \\ \delta \boldsymbol{e}((N-2)m) \\ \vdots \\ \delta \boldsymbol{e}(m) \end{bmatrix}$$

where $\mathbf{0}$ is a zero matrix of order $\frac{(N-2)(N-1)}{2}mn \times 1$ and δ is given by (c1). Thus we have the following Lemma.

Lemma 2.5.2. The expected duration of time the server remains continuously in WV until the number of type I customers reach 0 and before the arrival of a type II customer is given by $\boldsymbol{\gamma}_2(-U_2)^{-2}\boldsymbol{U}_2^{(0)}$.

Let U_s' denote the duration of time the server starts in slow service mode until either he gets back to normal mode through the vacation expiration or the arrival of a type II customer.

Distribution of U_s'

The distribution of U_s' can be studied as the time until absorption in a continuous time Markov chain with state space $\{(0,l) : 1 \le l \le n\} \cup \{(i,j,k,l) : 1 \le i \le N-1; i \le j \le N-1; 1 \le k \le m; 1 \le l \le n\} \cup \{*_1\} \cup \{*_2\}$, where, i denotes the number of type I customers in the system, j the number of type

I customers in the system plus number of type I customers already served during the current vacation, k, the service phase, l, the arrival phase of type II customer, $*_1$ the absorbing state indicating the vacation expiration and $*_2$ the absorbing state indicating the arrival of a type II customer.

The initial probability vector is given by

$$\boldsymbol{\gamma}_3 = (\lambda/d_2)(\mathbf{0}, w_{1,1}, \cdots, w_{1,n}, \cdots, w_{m,1}, \cdots, w_{m,n}, \mathbf{0})$$

For, $1 \leq j \leq m$, $1 \leq k \leq n$, $w_{j,k}$ and d_2 are defined as in section 2.2.4 and, first $\mathbf{0}$ is a zero matrix of order n and second $\mathbf{0}$ is a zero matrix of order $1 \times \frac{(N-2)(N+1)mn}{2}$.

The infinitesimal generator $\mathcal{U}_3$ of $U_s'(t)$ has the form

$$\mathcal{U}_3 = \begin{bmatrix} U_3 & \boldsymbol{U}_3^{(0)} & \boldsymbol{U}_3^{(1)} \\ \mathbf{0} & 0 & 0 \end{bmatrix} \text{ where,}$$

$$U_3 = \begin{bmatrix} D_0 - \lambda I & M_1 & & & \\ K_1 & L_1 & M_2 & & \\ & \ddots & \ddots & \ddots & \\ & & K_{N-2} & L_{N-2} & M_{N-1} \\ & & & K_{N-1} & L_{N-1} \end{bmatrix} \text{ with,}$$

$M_1 = \begin{bmatrix} \lambda(\alpha \otimes I) & \mathbf{0} \end{bmatrix}$, where $\mathbf{0}$ is a zero matrix of order $n \times (N-2)mn$.

$$K_1 = \begin{bmatrix} \boldsymbol{e}(N-1) \otimes (\theta \boldsymbol{T}^0 \otimes I)) \end{bmatrix}$$

For $2 \leq i \leq N-1$,

$K_i = \begin{bmatrix} \mathbf{0} & I_{N-i} \otimes (\theta \boldsymbol{T}^0 \boldsymbol{\alpha} \otimes I) \end{bmatrix}$, where $\mathbf{0}$ is a zero matrix of order $(N-i)mn \times mn$

and

$$M_i = \begin{bmatrix} \lambda I_{(N-i)mn} \\ \mathbf{0} \end{bmatrix}$$

where $\mathbf{0}$ is a zero matrix of order $mn \times (N-i)mn$.

$$L_i = I_{N-i} \otimes (\theta T \oplus D_0 - (\lambda+\eta) I_{mn}), \text{ for } 1 \leq i \leq N-1,$$

$$\boldsymbol{U}_3^{(0)} = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{e}(N-2) \otimes \eta \boldsymbol{e}(mn) \\ (\lambda+\eta)\boldsymbol{e}(mn) \\ \boldsymbol{e}(N-3) \otimes \eta \boldsymbol{e}(mn) \\ (\lambda+\eta)\boldsymbol{e}(mn) \\ \vdots \\ \eta \boldsymbol{e}(mn) \\ (\lambda+\eta)\boldsymbol{e}(mn) \\ (\lambda+\eta)\boldsymbol{e}(mn) \end{bmatrix}, \boldsymbol{U}_3^{(1)} = \begin{bmatrix} \delta \\ \boldsymbol{e}((N-1)m)) \otimes \delta \\ \boldsymbol{e}((N-2)m) \otimes \delta \\ \vdots \\ \boldsymbol{e}(m) \otimes \delta \end{bmatrix}$$

where, $\mathbf{0}$ is a zero matrix of order $n \times 1$ and δ is given by (c1).
Thus we have the following Lemma.

Lemma 2.5.3. The expected duration of time the server remains in WV with or without hitting zero state of type I customer until the arrival of a type II customer before hitting normal mode is given by $\boldsymbol{\gamma}_3(-U_3)^{-2}\boldsymbol{U}_3^{(1)}$.

Thus we arrive at

Theorem 2.5.1. *The expected number of returns to 0 type I customer state during the vacation mode of service before the arrival of a type II customer, is given by* $\left(\frac{1}{\lambda} + \boldsymbol{\gamma}_3(-U_3)^{-2}\boldsymbol{U}_3^{(1)}\right) / \left(\frac{1}{\lambda} + \boldsymbol{\gamma}_2(-U_2)^{-2}\boldsymbol{U}_2^{(0)}\right)$.

2.6 Waiting Time Analysis

2.6.1 Type I customer

To find the waiting time of a type I customer who joins for service at time x, we have to consider different possibilities depending on the status of server at that time.The server may be on vacation, WV, normal mode 1 or in normal mode 2. Let Z_1 be the random variable representing the waiting time of a type I customer in the queue. Define $W_1(x) = \text{Prob}(Z_1 \leq x)$ and $W_1^*(s)$ be the corresponding LST.

Case I

The tagged customer arrives to the system when the server is on vacation. Suppose E_1 denote the event that the system is in the state $(0, 1, 0, 1, u, v)$, $1 \leq u \leq m$; $1 \leq v \leq n$ or in the state $(n_1, 1, 0, 1, u, v)$, $n_1 \geq 1$; $1 \leq u \leq m$; $1 \leq v \leq n$ immediately after arrival of the tagged customer. Let $W_1^*(s|E_1)$ denote the corresponding LST. Then

$$W_1^*(s|E_1) = 1.$$

Case II

The tagged type I customer arrives to the system when the server is on WV. Suppose that $a+1$ is the position of the tagged customer when he arrives the system. For $1 \leq a \leq N-2$, let E_2 denote the event the system be in the state $(n_1, a+1, 0, t+1, u, v)$, $n_1 \geq 0$; $a \leq t \leq N-2$; $1 \leq u \leq m$; $1 \leq v \leq n$ immediately after arrival of the tagged customer arrives.. Let $W_1^*(s|E_2)$ denote the corresponding LST.

Case (i)

Let E denote the event that the server switches to normal mode due to random clock expiration during the slow service. Then $E = \cup_{i=1}^{i=a+1}(E \cap K_i)$ where K_1 denotes the event the random clock expire during the residual service time of the customer in service and for $2 \leq i \leq a$, K_i denotes the event the random clock expire during the ith service. In these cases, the waiting time of an arbitrary type I customer is the sum of time duration, starting from his arrival epoch till random clock expiration, service time of the customer in service at the time of random clock expiration from the beginning in the normal mode of service and service time of the remaining customers. Let K_{a+1} denotes the event the random clock expires after the ath service. In this case, the waiting time of an arbitrary customer is the sum of the residual service time of the customer in service when the tagged customer arrives and service time of remaining $a-1$ type I customers in slow mode.

Now,

$$P(E/E_2) = \left(\int_{t=0}^{\infty} \boldsymbol{e}_{\frac{a}{2}(2N-a-1)m+(t-a-1)m+u}(\frac{(N-1)Nm}{2})exp(U_1 t)\boldsymbol{U}_1^{(1)} dt \right)$$

where U_1, $\boldsymbol{U}_1^{(1)}$ are as defined in section 2.5.3.
Let $p_{a,u} = \left(e_{\frac{a}{2}(2N-a-1)m+(t-a-1)m+u}(\frac{(N-1)Nm}{2})(-U_1)^{-2}\boldsymbol{U}_1^{(1)}\right)^{-1}$ be the rate of absorption to $\{*_1\}$ from U_1 and $\mu^{(i)}$ denote the expected rate of sum of i service time distributions, each following PH$(\boldsymbol{\alpha}, T)$ from the arrival epoch of the tagged customer. Here, $\mu^{(1)} = \theta\mu_u$ which is the residual service rate when the server is providing slow service in phase u.

$$P(K_1|E, E_2) = \frac{p_{a,u}}{p_{a,u} + \mu^{(1)}},$$

$$P(K_i|E, E_2) = \frac{p_{a,u}}{p_{a,u} + \mu^{(i)}} - \frac{p_{a,u}}{p_{a,u} + \mu^{(i-1)}}, \text{ for } 2 \leq i \leq a,$$

$$P(K_{a+1}/E, E_2) = \frac{\mu^{(a)}}{p_{a,u} + \mu^{(a)}}.$$

Then the conditional LSTs are given by

$$W_1^*(s|E_2, E, K_1) = \left(\frac{\eta}{s+\eta}\right)(\boldsymbol{\alpha}(sI-T)^{-1}\boldsymbol{T}^0)^a,$$

$$W_1^*(s|E_2, E, K_i) = \left(\frac{\eta}{s+\eta}\right)(\boldsymbol{\alpha}(sI - T)^{-1}\boldsymbol{T}^0)^{a-i+1}, \text{ for } 2 \leq i \leq a$$

and

$$W_1^*(s|E_2, E, K_{a+1}) = (\boldsymbol{e}'_u(sI - \theta T)^{-1}\theta\boldsymbol{T}^0)(\boldsymbol{\alpha}(sI - \theta T)^{-1}\theta\boldsymbol{T}^0)^{a-1}.$$

Thus the conditional LST

$$W_1^*(s|E_2, E) = \sum_{i=1}^{a+1} W_1^*(s|E_2, E, K_i)P(K_i|E_2, E).$$

Case(ii)

Let F denote the event "the server switches to normal mode when the number of type I customers in the system plus number of type I customers already served during the current vacation hits N" during the slow service. Then $F = \cup_{i=1}^{i=a+1}(F \cap M_i)$ where M_1 denote the event: the number of type I customers plus number of type I customers already served during the current vacation reaches N during the residual service time. For $2 \leq i \leq a$, M_i denote the event: the number of type I customers in the system plus number of type I customers already served during the current vacation reaches N during the ith customer's service time. In these cases, the waiting time of an arbitrary type I customer is the sum of time duration starting from his arrival epoch till the number of type I customers in the system plus number of type I customers already served during the current vacation hits N, service time of the customer in service at the time of switching to normal mode from the beginning in the normal mode of service and service time of remaining customers. Let M_{a+1} denote the event "the number of type I customers in the system plus number of type I customers already served during the current vacation reaches N after the ath customer's service". In this case, the waiting time of an arbitrary customer is the sum of the residual service time of the customer in service when the tagged customer arrives and service time of remaining $a-1$ type I customers in slow mode.

Now,

$$P(F/E_2) = \left(\int_{t=0}^{\infty} \boldsymbol{e}_{\frac{a}{2}(2N-a-1)m+(t-a-1)m+u}(\frac{(N-1)Nm}{2}) exp(U_1 t) \boldsymbol{U}_1^{(2)} dt \right)$$

where U_1, $\boldsymbol{U}_1^{(2)}$ are as defined in section 2.5.3.
Let $q_{a,u} = \left(\boldsymbol{e}_{\frac{a}{2}(2N-a-1)m+(t-a-1)m+u}(\frac{(N-1)Nm}{2})(-U_1)^{-2} \boldsymbol{U}_1^{(2)} \right)^{-1}$ be the rate of absorption to $\{*_2\}$ from U_1.

$$P(M_1|F, E_2) = \frac{q_{a,u}}{q_{a,u} + \mu^{(1)}},$$

$$P(M_i|F, E_2) = \frac{q_{a,u}}{q_{a,u} + \mu^{(i)}} - \frac{q_{a,u}}{q_{a,u} + \mu^{(i-1)}}, \text{ for } 2 \leq i \leq a$$

and

$$P(M_{a+1}|F, E_2) = \frac{\mu^{(a)}}{q_{a,u} + \mu^{(a)}}$$

The conditional LSTs,

$$W_1^*(s|E_2, F, M_1) = \left(\frac{\lambda}{s+\lambda} \right)^{N-t-1} (\boldsymbol{\alpha}(sI - T)^{-1} \boldsymbol{T}^0)^a,$$

$$W_1^*(s|E_2, F, M_i) = \left(\frac{\lambda}{s+\lambda} \right)^{N-t-1} (\boldsymbol{\alpha}(sI - T)^{-1} \boldsymbol{T}^0)^{a-i+1}, \text{ for } 2 \leq i \leq a$$

and

$$W_1^*(s|E_2, F, M_{a+1}) = (\boldsymbol{e'}_u(sI - \theta T)^{-1} \theta \boldsymbol{T}^0)(\boldsymbol{\alpha}(sI - \theta T)^{-1} \theta \boldsymbol{T}^0)^{a-1}$$

Thus the conditional LST

$$W_1^*(s|E_2, F) = \sum_{i=1}^{a+1} W_1^*(s|E_2, F, M_i)P(M_i|E_2, F)$$

Case (iii)

Let G denote the event that the system becomes empty before vacation expiration.

$$P(G/E_2) = \left(\int_{t=0}^{\infty} \boldsymbol{e}_{\frac{a}{2}(2N-a-1)m+(t-a-1)m+u}(\frac{(N-1)Nm}{2})exp(U_1t)\boldsymbol{U}_1^{(0)}dt\right)$$

where U_1, $\boldsymbol{U}_1^{(0)}$ are as defined in section 2.5.3.

In this case the conditional LST,

$$W_1^*(s/E_2, G) = (\boldsymbol{e}'_u(sI - \theta T)^{-1}\theta T^0)(\boldsymbol{\alpha}(sI - \theta T)^{-1}\theta\boldsymbol{T}^0)^{a-1}.$$

Thus the conditional LST,

$$W_1^*(s/E_2) = W_1^*(s/E_2, E)P(E/E_2) + W_1^*(s/E_2, F)P(F/E_2) + W_1^*(s/E_2, G)P(G/E_2).$$

Case III

Let E_3 denote the event that the customer arrives to the system when the server is in normal mode 1. This case is same as for model I. Let $W_1^*(s/E_3)$ denote the corresponding conditional LST.

Then the conditional LST of the waiting time is given by

$$W_1^*(s|E_3) = (\boldsymbol{e}'_u(sI - T)^{-1}\boldsymbol{T}^0)(\boldsymbol{\alpha}(sI - T)^{-1}\boldsymbol{T}^0)^{a-1}.$$

Case IV

Let E_4 denote the event that the customer arrives to the system when the

server is in normal mode 2. This case is also same as for model I. Let $W_1^*(s/E_4)$ denote the corresponding LST.
Then the conditional LST of the waiting time,

$$W_1^*(s|E_4) = (\boldsymbol{e}'_u(sI - T')^{-1}\boldsymbol{T'}^0)(\boldsymbol{\alpha}(sI - T)^{-1}\boldsymbol{T}^0)^a.$$

Let $w_{i_1,i_2,j_1,j_2,k,l}$ denote the probabilty that the system is in the state $(i_1, i_2, j_1, j_2, k, l)$ immedietly after arrival of the tagged customer. Then,

$$\begin{aligned}
w_{0,1,0,1,u,v} &= \tfrac{\lambda\alpha_u}{\lambda+\eta-d_{vv}^{(0)}} x_{0,0,v}, \text{ for, } 1 \le u \le m,\ 1 \le v \le n \\
w_{n_1,1,0,1,u,v} &= \tfrac{\lambda\alpha_u}{\lambda+\eta-d_{vv}^{(0)}} x_{n_1,0,0,v}, \text{ for, } n_1 \ge 1,\ 1 \le u \le m,\ 1 \le v \le n \\
w_{n_1,a+1,0,t+1,u,v} &= \tfrac{\lambda}{\lambda+\eta-\theta T_{uu}-d_{vv}^{(0)}} x_{n_1,a,0,t,u,v}, \text{ for, } n_1 \ge 0,\ 1 \le a \le N-2, \\
&\quad a \le t \le N-2, 1 \le u \le m,\ 1 \le v \le n \\
w_{n_1,a+1,1,u,v} &= \textstyle\sum_{u'=1}^{m} \tfrac{\lambda\alpha_u}{\lambda+\eta-\theta T_{u'u'}-d_{vv}^{(0)}} x_{n_1,a,0,N-1,u',v} + \tfrac{\lambda}{\lambda-T_{uu}-d_{vv}^{(0)}} \\
&\quad x_{n_1,a,1,u,v}, \text{ for, } n_1 \ge 0,\ 1 \le a \le N-1,\ 1 \le u \le m, \\
&\quad 1 \le v \le n \\
w_{n_1,a+1,1,u,v} &= \tfrac{\lambda}{\lambda-T_{uu}-d_{vv}^{(0)}} x_{n_1,a,1,u,v}, \text{ for, } n_1 \ge 0,\ N \le a \le L-1, \\
&\quad 1 \le u \le m,\ 1 \le v \le n \\
w_{n_1,a+1,2,u,v} &= \tfrac{\lambda}{\lambda-T_{uu}-d_{vv}^{(0)}} x_{n_1,a,2,u,v}, \text{ for, } n_1 \ge 1,\ 0 \le a \le L-1, \\
&\quad 1 \le u \le m',\ 1 \le v \le n
\end{aligned}$$

Thus we have the following Theorem.

Theorem 2.6.1. *The LST of the waiting time of a type I customer is given by*

$$\begin{aligned}
W_1^*(s) = \frac{1}{d}\Big[&\sum_{n_1=0}^{\infty}\sum_{v=1}^{n} w_{n_1,1,0,1,u,v} + \sum_{n_1=0}^{\infty}\sum_{a=1}^{N-2}\sum_{t=a}^{N-2}\sum_{u=1}^{m}\sum_{v=1}^{n} W_1^*(s|E_2)(w_{n_1,a+1,0,t+1,u,v}) \\
&+ \sum_{n_1=0}^{\infty}\sum_{a=1}^{L-1}\sum_{u=1}^{m}\sum_{v=1}^{n} W_1^*(s|E_3)(w_{n_1,a+1,1,u,v}) + \sum_{n_1=1}^{\infty}\sum_{a=0}^{L-1}\sum_{u=1}^{m}\sum_{v=1}^{n} W_1^*(s|E_4)(w_{n_1,a+1,2,u,v})\Big]
\end{aligned} \tag{2.26}$$

where

$$d = \sum_{n_1=0}^{\infty}\sum_{v=1}^{n} w_{n_1,1,0,1,u,v} + \sum_{n_1=0}^{\infty}\sum_{a=1}^{N-2}\sum_{t=a}^{N-2}\sum_{u=1}^{m}\sum_{v=1}^{n} w_{n_1,a+1,0,t+1,u,v} + \\ + \sum_{n_1=0}^{\infty}\sum_{a=1}^{L-1}\sum_{u=1}^{m}\sum_{v=1}^{n} w_{n_1,a+1,1,u,v} + \sum_{n_1=1}^{\infty}\sum_{a=0}^{L-1}\sum_{u=1}^{m}\sum_{v=1}^{n} w_{n_1,a+1,2,u,v}. \quad (2.27)$$

2.6.2 Type II Customer

To find the LST of the waiting time distribution of a type II customer, we have to compute the following distribution.

Let U_s'' be the duration of time the server, starting in vacation, until either he gets back to normal mode through the random clock expiring or the WV is interrupted as the number of type I customers in the system plus number of type I customers already served during the current vacation hits N.

Conditional distribution of U_s'' given a type II customer arrives before the random clock expires

We can study this by a phase type distribution with representation (γ_4, U_4) where the underlying markov chain has state space $\{0\} \cup \{(i, j, k) : 1 \leq i \leq N-1;\ i \leq j \leq N-1;\ 1 \leq k \leq m\} \cup \{*\}$ where i denotes the number of type I customers in the system, j the number of type I customers in the system plus number of type I customers already served during the current vacation, k, the service phase and * denotes the absorbing state indicating the vacation expiration. The infinitesimal generator $\mathcal{U}_4$ of $U_s''(t)$ is given by

$$\mathcal{U}_4 = \begin{bmatrix} U_4 & \boldsymbol{U}_4^0 \\ 0 & 0 \end{bmatrix}$$

where,

$$U_4 = \begin{bmatrix} -(\lambda+\eta) & M_1 & & & \\ K_1 & L_1 & M_2 & & \\ & \ddots & \ddots & \ddots & \\ & & K_{N-2} & L_{N-2} & M_{N-1} \\ & & & K_{N-1} & L_{N-1} \end{bmatrix}. \tag{c8}$$

with

$$M_1 = \begin{bmatrix} \lambda\alpha & 0 \end{bmatrix}, \text{ where } \mathbf{0} \text{ is a zero matrix of order } 1 \times (N-2)m.$$

$$K_1 = \boldsymbol{e}(N-1) \otimes \theta \boldsymbol{T}^0$$

For $2 \leq i \leq N-1$,

$$K_i = \begin{bmatrix} \mathbf{0} & I_{N-i} \otimes (\theta \boldsymbol{T}^0 \boldsymbol{\alpha}) \end{bmatrix}, \text{ where } \mathbf{0} \text{ is a zero matrix of order } (N-i)m \times m$$

and

$$M_i = \begin{bmatrix} \lambda I_{(N-i)m} \\ \mathbf{0} \end{bmatrix}$$

where $\mathbf{0}$ is a zero matrix of order $m \times (N-i)m$.

$$L_i = I_{N-i} \otimes (\theta T \oplus D_0 - (\lambda+\eta) I_m), \text{ for } 1 \leq i \leq N-1$$

$$\boldsymbol{U}_4^0 = \begin{bmatrix} \eta \\ \eta\boldsymbol{e}((N-2)m) \\ (\lambda+\eta)\boldsymbol{e}(m) \\ \vdots \\ \eta\boldsymbol{e}(m) \\ (\lambda+\eta)\boldsymbol{e}(m) \\ (\lambda+\eta)\boldsymbol{e}(m) \end{bmatrix}. \tag{c9}$$

The initial probability vector is given by

$$\gamma_4 = (1, \mathbf{0}), \text{ where } \mathbf{0} \text{ is a zero matrix of order } 1 \times \frac{(N-1)N}{2}m. \tag{c10}$$

Thus we have the following Lemma.

Lemma 2.6.1. The expected duration of time the server stays in vacation mode until either the server gets back to normal mode through the random clock expiring or the WV is interrupted as the number of type I customers in the system plus the number of type I customers already served during the current vacation hits N given a type II customer arrives before the random clock expires, is given by $\boldsymbol{\gamma}_4(-U_4)^{-1}\boldsymbol{e}$.

To find the waiting time of a type II customer who arrives at time x, we have to consider different possibilities depending on the status of server at that time. The server may be in vacation mode, WV mode, normal mode 1 or in normal mode 2. Let Z_2 be the random variable representing the waiting time of a type II customer in the queue. Define $W_2(x) = \text{Prob}(Z_2 \leq x)$ and $W_2^*(s)$ be the corresponding LST.

Case I

Let F_1 denote the event that the system is in the state $(1, 0, 0, v)$ immediately after arrival of the tagged customer. In this case the waiting time is the sum of time duration from his arrival epoch till the server shifts to normal mode and the time duration of busy period generated by type I customers present at that time, if any. Let $W_2^*(s|F_1)$ denote the corresponding conditional LST of the waiting time.

Then

$$W_2^*(s|F_1) = \boldsymbol{\gamma}_4(sI - U_4)^{-1}\boldsymbol{U}_4^0\big[t_0 + \sum_{p=1}^{N}\boldsymbol{\gamma}_p(sI - T_1)^{-1}\boldsymbol{T}_1^0 t_p\big]$$

where t_p, $0 \leq p \leq N$ denote the probabaility that there are p type I customers when the vacation expires which is given by

$$t_p = \begin{cases} \int_{t=0}^{\infty} \boldsymbol{\gamma}_4 (e^{U_4 t})_p \eta dt & \text{if } p = 0 \\ \int_{t=0}^{\infty} \boldsymbol{\gamma}_4 (e^{U_4 t})_p (\boldsymbol{U}_4^0)_p dt & \text{if } 1 \leq p \leq N-2 \\ \int_{t=0}^{\infty} \boldsymbol{\gamma}_4 (e^{U_4 t})_{N-1} \eta \boldsymbol{e}(m) dt & \text{if } p = N-1 \\ \int_{t=0}^{\infty} \boldsymbol{\gamma}_4 (e^{U_4 t})_{N-1} \lambda \boldsymbol{e}(m) dt & \text{if } p = N \end{cases}$$

where $(e^{U_4 t})_p$ denote the columns in $e^{U_4 t}$ corresponding to p type I customer states and $(\boldsymbol{U}_4^0)_p$ the absorbing rates corresponding to p type I customers, T_1, $\boldsymbol{T}_1^0$, $\boldsymbol{\gamma}_p$, U_4, $\boldsymbol{U}_4^0$ and $\boldsymbol{\gamma}_4$ are given by (c2), (c3),(c4),(c8), (c9) and (c10) respectively.

Case II

Let F_2 denote the event that the system is in the state $(b+1, 0, 0, v)$, $b \geq 1$ immediately after arrival of the tagged customer. In this case the waiting time is the sum of time duration from his arrival epoch till the server shifts to normal mode, the time duration of busy period generated by type I customers present at that time, if any and the time duration of busy cycles generated by type I customers arriving during the service time of each of the b type II customers. Let $W_2^*(s|F_2)$ denote the corresponding conditional LST of the waiting time.

Then

$$W_2^*(s|F_2) = \boldsymbol{\gamma}_4 (sI - U_4)^{-1} \boldsymbol{U}_4^0 \big[t_0 + \sum_{p=1}^{N} \boldsymbol{\gamma_p} (sI - T_1)^{-1} \boldsymbol{T}_1^0 t_p\big] (\hat{B_{c_L}}(s))^b$$

where $\hat{B_{c_L}}(s))$ is given by Theorem 2.3.3.

Case III

Let F_3 denote the event that the system is in the state $(b+1, a, 0, t, u, v)$, $b \geq 0$; $1 \leq a \leq N-1$; $a \leq t \leq N-1$ immediately after arrival of the tagged customer. In this case also the waiting time is the sum of time duration from his arrival epoch till the server shifts to normal mode, the time duration of

busy period generated by type I customers present at that time, if any and the time duration of busy cycles generated by type I customers arriving during the service time of each of the b type II customers. Let $W_2^*(s|F_3)$ denote the corresponding conditional LST.
Then

$$W_2^*(s|F_3) = \boldsymbol{\gamma}_{a,t}^u (sI - U_4)^{-1} \boldsymbol{U}_4^0 \big[t_0 + \sum_{p=1}^{N} \boldsymbol{\gamma}_p (sI - T_1)^{-1} \boldsymbol{T}_1^0 t_p\big] (\hat{B_{c_L}}(s))^b$$

where $\boldsymbol{\gamma}_{a,t}^u = (0, \mathbf{0}, \cdots, \boldsymbol{e}'_{t-a+1}(N-a) \otimes \boldsymbol{e}'_u, \mathbf{0})$ with $\boldsymbol{e}'_{t-a+1}(N-a) \otimes \boldsymbol{e}'_u$ is in the $(a+1)^{th}$ position, where $\mathbf{0}'$s are zero matrices of order $(N-1)m, \cdots, (N-a+1)m$, $(N-a-1)m, \cdots, m$ respectively, $\boldsymbol{\gamma}_p = (0, \mathbf{0}, \cdots, \boldsymbol{\alpha}, \mathbf{0}, \cdots, \mathbf{0})$ where α is in the pth position and $\mathbf{0}$ denotes zero matrix of order m.

Case IV

Let F_4 denote the event that the system is in the state $(b+1, a, 1, u, v)$, $b \geq 0$; $1 \leq a \leq L$; $1 \leq u \leq m$; $1 \leq v \leq n$ immediately after arrival of the tagged customer. This case is same as for model I. Let $W_2^*(s|F_4)$ denote the corresponding conditional LST.
Then

$$W_2^*(s|F_4) = (\boldsymbol{\gamma}_a^u (sI - T_1)^{-1} \boldsymbol{T}_1^0)(\hat{B_{c_L}}(s))^b$$

where $\boldsymbol{\gamma}_a^u = (\mathbf{0}, \cdots, \mathbf{e}'_u, \cdots, \mathbf{0})$ where $\boldsymbol{e}'_u$ is in the a^{th} position and $\mathbf{0}$ denotes zero matrix of order m.

Case V

Let F_5 denote the event that the system is in the state $(b+1, a, 2, u, v)$, $b \geq 1$; $0 \leq a \leq L$; $1 \leq u \leq m'$; $1 \leq v \leq n$, immediately after arrival of the tagged customer. This case is also same as for model I. Let $W_2^*(s|F_5)$ denote the corresponding conditional LST.
Then

$$W_2^*(s|F_5) = (\boldsymbol{e}'_u (sI - T')^{-1} \boldsymbol{T'}^0)(\bar{B}_L(s)(\hat{B_{c_L}}(s))^{b-1}$$

$\bar{B}_L(s)$ is given by Theorem 2.3.4.

Let $w_{i_1,i_2,j_1,j_2,k,l}$ denote the probabilty that the system is in the state $(i_1, i_2, j_1, j_2, k, l)$ immediately after arrival of the tagged customer. Then,

$$
\begin{aligned}
w_{1,0,0,v} &= \frac{d^{(1)}_{v'v}}{\lambda+\eta-d^{(0)}_{v'v'}} x_{0,0,v'}, \text{ for, } 1 \leq v, v' \leq n \\
w_{b+1,0,0,v} &= \frac{d^{(1)}_{v'v}}{\lambda+\eta-d^{(0)}_{v'v'}} x_{b,0,0,v'}, \text{ for, } b \geq 1,\ 1 \leq v, v' \leq n \\
w_{b+1,a,0,t,u,v} &= \frac{d^{(1)}_{v'v}}{\lambda+\eta-\theta T_{uu}-d^{(0)}_{v'v'}} x_{b,a,0,t,u,v'}, \text{ for, } b \geq 0,\ 1 \leq a \leq N-1, \\
&\quad a \leq t \leq N-1,\ 1 \leq u \leq m,\ 1 \leq v, v' \leq n \\
w_{b+1,a,1,u,v} &= \frac{d^{(1)}_{v'v}}{\lambda-T_{uu}-d^{(0)}_{v'v'}} x_{b,a,1,u,v'}, \text{ for, } b \geq 0,\ 1 \leq a \leq L,\ 1 \leq u \leq m, \\
&\quad 1 \leq v, v' \leq n \\
w_{b+1,a,2,u,v} &= \frac{d^{(1)}_{v'v}}{\lambda-T_{uu}-d^{(0)}_{v'v'}} x_{b,a,2,u,v'}, \text{ for, } b \geq 1,\ 0 \leq a \leq L,\ 1 \leq u \leq m', \\
&\quad 1 \leq v, v' \leq n
\end{aligned}
$$

We sum up the above discussions in the following.

Theorem 2.6.2. *The LST of the waiting time of a type II customer is given by*

$$
\begin{aligned}
W_2^*(s) = \sum_{v=1}^{n} W_2^*(s|F_1) w_{1,0,0,v} + \sum_{b=1}^{\infty}\sum_{v=1}^{n} W_2^*(s|F_2) w_{b+1,0,0,v} + \sum_{b=0}^{\infty}\sum_{a=1}^{N-1}\sum_{t=a}^{N-1}\sum_{u=1}^{m}\sum_{v=1}^{n} \\
W_2^*(s|F_3)(w_{b+1,a,0,t,u,v}) + \sum_{b=0}^{\infty}\sum_{a=1}^{L}\sum_{u=1}^{m}\sum_{v=1}^{n} W_2^*(s|F_4) w_{b+1,a,1,u,v} + \\
\sum_{b=1}^{\infty}\sum_{a=0}^{L}\sum_{u=1}^{m}\sum_{v=1}^{n} W_2^*(s|F_5) w_{b+1,a,2,u,v}. \quad (2.28)
\end{aligned}
$$

2.7 Numerical Results

2.7.1 Comparison of mean/variance of number of type I and type II customers in the system

We fix $\lambda = 1$, $\theta = 0.6$, $D_0 = (-1)$, $D_1 = (1)$, $\boldsymbol{\alpha} = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $T = \begin{bmatrix} -5 & 5 \\ 0 & -5 \end{bmatrix}$ and $\boldsymbol{\alpha'} = \begin{bmatrix} 0.8 & 0.2 \end{bmatrix}$, $T' = \begin{bmatrix} -2.5 & 0 \\ 2.5 & -5 \end{bmatrix}$.

Table 2.1: Mean/Variance of number of type I customers in the system for model I: Effect of η and N

η	N=1, L=3		N=2, L=4		N=3, L=5		N=4, L=6		N=5, L=7	
	Ens	Vns	Ens	Vns	Ens	Vns	Ens	Vns	Ens	Vns
0.01	0.6934	0.7328	0.8170	0.9367	0.8928	1.1001	0.9494	1.2388	0.9947	1.3680
0.02	0.6934	0.7328	0.8169	0.9366	0.8924	1.0996	0.9482	1.2371	0.9924	1.3638
0.03	0.6934	0.7328	0.8168	0.9365	0.8919	1.0991	0.9470	1.2354	0.9901	1.3597
0.04	0.6934	0.7328	0.8166	0.9364	0.8914	1.0986	0.9458	1.2338	0.9878	1.3557
0.05	0.6934	0.7328	0.8165	0.9363	0.8909	1.0981	0.9446	1.2322	0.9857	1.3518

Table 2.2: Mean/Variance of number of type I customers in the system for model II: Effect of η and N

η	N=1, L=3		N=2, L=4		N=3, L=5		N=4, L=6		N=5, L=7	
	Ens	Vns	Ens	Vns	Ens	Vns	Ens	Vns	Ens	Vns
0.01	0.6934	0.7328	0.8170	0.9367	0.8807	1.0854	0.9200	1.1885	0.9479	1.2653
0.02	0.6934	0.7328	0.8169	0.9366	0.8804	1.0851	0.9194	1.1878	0.9470	1.2640
0.03	0.6934	0.7328	0.8168	0.9365	0.8801	1.0848	0.9188	1.1872	0.9461	1.2627
0.04	0.6934	0.7328	0.8166	0.9364	0.8797	1.0845	0.9183	1.1865	0.9453	1.2615
0.05	0.6934	0.7328	0.8165	0.9363	0.8794	1.0842	0.9177	1.1858	0.9444	1.2602

In Tables 2.1 and 2.2, Ens denote the expected number of type I customers in the system and Vns, its variance. In these tables we look at the values of these measures as functions of N and η. The two models coincide with the classical model in the case $N = 1$. These two models coincide in the case of $N = 2$ also. As expected, in both the models, both mean and variance are non-increasing functions of η (for fixed N) and is also non-decreasing functions of N (for fixed η). The rate of decrease of mean and variance as η grows shows an increasing trend with value of N going up. This happens due to the diminished effect of N as N increases. Also the rate of increase of mean and variance with growth of N decreases as η increases. This is due to the increased effect of η with growth of η. But variance is larger than mean in all cases. When $N \geq 3$, both Ens and Vns are comparitively less in model II than in model I.

For the arrival process of type II customers, we consider the following five sets of matrices for D_0 and D_1.

1. Exponential (EXP)

$D_0 = (-1)$, $D_1 = (1)$

2. Erlang (ERA)

$$D_0 = \begin{bmatrix} -3 & 3 & 0 \\ 0 & -3 & 3 \\ 0 & 0 & -3 \end{bmatrix} .D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

3. Hyperexponential (HEXP)

$$D_0 = \begin{bmatrix} -3.4000 & 0 \\ 0 & -0.8500 \end{bmatrix}, D_1 = \begin{bmatrix} 0.6800 & 2.7200 \\ 0.1700 & 0.6800 \end{bmatrix}.$$

4. MAP with negetive correlation (MNA)

$$D_0 = \begin{bmatrix} -0.8101 & 0.8101 & 0 \\ 0 & -1.3497 & 0 \\ 0 & 0 & -40.5065 \end{bmatrix}, D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0.0810 & 0 & 1.2687 \\ 38.0761 & 0 & 2.4304 \end{bmatrix}$$

5. MAP with positive correlation (MPA)

Table 2.3: Effect of η and N on Mean/Variance of number of type II customers in the sytem for model I

(N, L)	η	EXP		ERA		HEA		MNA		MPA	
		Ens	Vns	Ens	Vns	Ens	Vns	Ens	Vns	Ens	Vns
(1,3)	0.01	2.95	8.11	2.44	4.30	3.14	9.82	3.17	9.15	16.35	693.94
	0.03	2.93	8.05	2.42	4.25	3.12	9.76	3.15	9.09	16.33	693.59
	0.05	2.91	8.00	2.40	4.21	3.11	9.70	3.13	9.03	16.32	693.26
(2,4)	0.01	5.18	19.55	4.63	13.78	5.39	22.02	5.40	20.89	19.49	804.02
	0.03	5.02	18.46	4.47	12.80	5.22	20.90	5.24	19.79	19.33	800.52
	0.05	4.87	17.54	4.32	11.97	5.08	19.94	5.10	18.85	19.19	797.43
(3,5)	0.01	9.57	68.10	9.01	59.18	9.78	71.77	9.80	69.91	24.23	944.18
	0.03	8.67	55.02	8.10	46.70	8.88	58.46	8.89	56.74	23.33	917.61
	0.05	7.97	45.98	7.40	38.13	8.18	49.25	8.19	47.64	22.63	898.11
(4,6)	0.01	17.20	242.85	16.63	228.76	17.41	248.45	17.43	245.42	31.98	1242
	0.03	13.84	151.36	13.51	139.51	14.05	156.12	14.06	153.60	28.61	1101
	0.05	11.72	105.45	11.15	95.01	11.94	109.69	11.95	107.48	26.50	1023
(5,7)	0.01	28.54	717.64	27.96	695.96	28.75	726.08	28.76	721.32	43.35	1890
	0.03	19.59	321.67	19.02	305.95	19.81	327.88	19.82	324.48	34.41	1360
	0.05	15.22	186.29	14.65	173.48	15.43	191.41	15.44	188.67	30.03	1159

$$D_0 = \begin{bmatrix} -0.8101 & 0.8101 & 0 \\ 0 & -1.3497 & 0 \\ 0 & 0 & -40.5065 \end{bmatrix}, D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1.2687 & 0 & 0.0810 \\ 2.4304 & 0 & 38.0761 \end{bmatrix}$$

All these five MAP processes are normalized so as to have an arrival rate of 1. However, these are qualitatively different in that they have different variance and correlation structure. The first three arrival processes, namely EXP, ERA and HEA correspond to renewal processes and so the correlation is 0. The arrival process labeled MNA has correlated arrivals with correlation between two successive interarrival times given by -0.4211 and the arrival process corresponding to the one labelled MPA has a positive correlation with value 0.4211.

For the service time distributions, we consider phase type distributions, $\boldsymbol{\alpha} = \begin{bmatrix} 1 & 0 \end{bmatrix}, T = \begin{bmatrix} -5 & 5 \\ 0 & -5 \end{bmatrix}$ and $\boldsymbol{\alpha'} = \begin{bmatrix} 0.8 & 0.2 \end{bmatrix}, T' = \begin{bmatrix} -2.5 & 0 \\ 2.5 & -5 \end{bmatrix}$. We fix $\lambda = 1$ and $\theta = 0.6$

In this case also, the mean and variance are both non-increasing functions of η (for fixed N) and is a non-decreasing function of N(for fixed η, with

Table 2.4: Effect of η and N on Mean/Variance of number of type II customers in the system for model II

(N, L)	η	EXP		ERA		HEXP		MNA		MPA	
		Ens	Vns	Ens	Vns	Ens	Vns	Ens	Vns	Ens	Vns
(1,3)	0.01	2.95	8.11	2.44	4.30	3.14	9.82	3.17	9.15	16.35	693.94
	0.03	2.93	8.05	2.42	4.25	3.12	9.76	3.15	9.09	16.33	693.59
	0.05	2.91	8.00	2.40	4.21	3.11	9.70	3.13	9.03	16.32	693.26
(2,4)	0.01	5.18	19.55	4.63	13.78	5.39	22.02	5.40	20.89	19.49	804.02
	0.03	5.02	18.46	4.47	12.80	5.22	20.90	5.24	19.79	19.33	800.52
	0.05	4.87	17.54	4.32	11.97	5.08	19.94	5.10	18.85	19.19	797.43
(3,5)	0.01	7.60	41.37	7.04	33.76	7.81	44.54	7.83	42.99	22.26	888.02
	0.03	7.11	36.04	6.55	28.76	7.33	39.10	7.34	37.61	21.77	875.40
	0.05	6.71	32.00	6.14	24.99	6.92	34.95	6.93	33.53	21.37	865.32
(4,6)	0.01	10.24	77.57	9.67	68.12	10.45	81.44	10.46	79.46	25.01	972.90
	0.03	9.20	61.54	8.63	52.78	9.41	65.15	9.43	63.33	23.98	941.38
	0.05	8.41	50.75	7.84	42.52	8.62	54.17	8.63	52.47	23.18	918.76
(5,7)	0.01	13.09	132.02	12.52	120.63	13.31	136.61	13.32	134.19	27.91	1073
	0.03	11.26	94.86	10.68	84.70	11.47	99.00	11.48	96.86	26.07	1009
	0.05	9.96	72.76	9.39	63.45	10.18	76.57	10.19	74.62	24.78	967

reference to Tables 2.3 and 2.4), for the input parameters prescribed. This is the case for all combinations of arrival processes of type II customer. But the rate of change in the case of MPA is much smaller compared to other arrivals. Both mean and variance are significantly larger for MPA indicating the role played by (positively) correlated arrivals. We observe that both mean and variance change significantly as functions of η when N becomes large for both the models. When $N \geq 3$, both Ens and Vns are comparitively less for model II compared to model I.

From Figures 2.1 and 2.2, we note that as λ increases both Ens and Vns of type II customers decrease first but increase after a certain stage for all values of N and for all type II arrival processes for both the models. This happens because as λ increases, the rate of hitting N become faster and the queue length decreases. But when λ reaches a specified value the queue length increases due to diminished effect of N. As N increases this λ value becomes larger and larger.This λ value is different for different type II arrival processes and is the smallest for the arrival process MPA.

2.7.2 Optimal N

Next, we find an optimal N for both the models by constructing a suitable cost function.
Let
C_s: Unit time cost of switching to normal mode
C_h: Holding cost for retaining a type II customer when the server is in vacation/WV
R_s: Rate of switching to normal mode
E_v: Expected number of type II customers in the system till vacation expires.
Then the expected cost per unit time,

$$C = C_s \times R_s + C_h \times E_v$$

For model I,

$$R_s = \sum_{n_1=1}^{\infty}\sum_{k=1}^{n} \eta x_{n_1,0,0,k} + \sum_{n_1=0}^{\infty}\sum_{n_2=1}^{N-1}\sum_{j_2=1}^{m}\sum_{k=1}^{n} \eta x_{n_1,n_2,0,j_2,k} + \sum_{n_1=0}^{\infty}\sum_{j_2=1}^{m}\sum_{k=1}^{n} \lambda x_{n_1,N-1,0,j_2,k} \quad (2.29)$$

and

$$E_v = \sum_{n_1=1}^{\infty}\sum_{k=1}^{n} n_1 x_{n_1,0,0,k} + \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{N-1}\sum_{j_2=1}^{m}\sum_{k=1}^{n} n_1 x_{n_1,n_2,0,j_2,k}$$

For model II,

$$R_s = \sum_{n_1=1}^{\infty}\sum_{l=1}^{n} \eta x_{n_1,0,0,l} + \sum_{n_1=0}^{\infty}\sum_{n_2=1}^{N-1}\sum_{j_2=n_2}^{N-1}\sum_{k=1}^{m}\sum_{l=1}^{n} \eta x_{n_1,n_2,0,j_2,k,l} + \sum_{n_1=0}^{\infty}\sum_{n_2=1}^{N-1}\sum_{k=1}^{m}\sum_{l=1}^{n} \lambda x_{n_1,n_2,0,N-1,0,k,l} \quad (2.30)$$

and

$$E_v = \sum_{n_1=1}^{\infty}\sum_{l=1}^{n} n_1 x_{n_1,0,0,l} + \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{N-1}\sum_{j_2=n_2}^{N-1}\sum_{k=1}^{m}\sum_{l=1}^{n} n_1 x_{n_1,n_2,0,j_2,k,l}$$

For both models we fix $L = 15$, $\theta = 0.1, \lambda = 0.05$, $\eta = 0.001$, $\boldsymbol{\alpha} = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $T = \begin{bmatrix} -5 & 5 \\ 0 & -5 \end{bmatrix}$ and $\boldsymbol{\alpha'} = \begin{bmatrix} 0.8 & 0.2 \end{bmatrix}, T' = \begin{bmatrix} -2.5 & 0 \\ 2.5 & -5 \end{bmatrix}$, $C_s = 3000$ and $C_h = 0.05$.

N	Model I			Model II		
	R_s	E_v	Cost	R_s	E_v	Cost
1	0.0296	11.3725	89.2242	0.0296	11.3725	89.2242
2	0.0049	69.0362	18.2221	0.0049	69.0362	18.2221
3	0.0012	278.8977	**17.6771**	0.0018	194.1026	**15.0271**
4	6.8644×10^{-4}	508.8820	27.5034	9.8500×10^{-4}	352.2230	20.5662
5	6.0488×10^{-4}	578.3485	30.7321	7.3595×10^{-4}	473.2067	25.8682
6	5.9332×10^{-4}	589.9899	31.2795	6.4776×10^{-4}	538.8852	28.8875
7	5.9171×10^{-4}	591.7079	31.3605	6.1432×10^{-4}	569.0909	30.2975
8	5.9149×10^{-4}	591.9660	31.3728	6.0103×10^{-4}	582.1676	30.9115

Table 2.5: Optimal N for EXP type II arrival process

N	Model I			Model II		
	R_s	E_v	Cost	R_s	E_v	Cost
1	0.0296	11.4214	89.2608	0.0296	11.4214	89.2608
2	0.0049	69.0862	18.2303	0.0049	69.0862	18.2303
3	0.0012	278.9481	**17.6811**	0.0018	194.1521	**15.0316**
4	6.8678×10^{-4}	508.8366	27.5022	9.8536×10^{-4}	352.1463	20.5634
5	6.0530×10^{-4}	578.5717	30.7445	7.3622×10^{-4}	472.8003	25.8487
6	5.9375×10^{-4}	590.2368	31.2931	6.4809×10^{-4}	538.5319	28.8709
7	5.9215×10^{-4}	591.9584	31.3744	6.1472×10^{-4}	568.9850	30.2934
8	5.9193×10^{-4}	592.2169	31.3866	6.0148×10^{-4}	582.2466	30.9168

Table 2.6: Optimal N for ERA type II arrival process

From Tables 2.5 to 2.9, we get the expected cost corresponding to different values of N for different type II arrival processes, for model I and model II. For both the models, R_s decreases and E_v increases as N increases. As expected,

N	Model I			Model II		
	R_s	E_v	Cost	R_s	E_v	Cost
1	0.0295	11.3550	89.2110	0.0295	11.3550	89.2110
2	0.0049	69.0182	18.2191	0.0049	69.0182	18.2191
3	0.0012	278.8795	**17.6757**	0.0018	194.0845	**15.0254**
4	6.8630×10^{-4}	508.6858	27.4932	9.8486×10^{-4}	352.2043	20.5648
5	6.0476×10^{-4}	578.2960	30.7291	7.3583×10^{-4}	473.1812	25.8666
6	5.9320×10^{-4}	589.9324	31.2762	6.4767×10^{-4}	539.0376	28.8949
7	5.9159×10^{-4}	591.6498	31.3573	6.1420×10^{-4}	569.1643	30.3008
8	5.9137×10^{-4}	591.9077	31.3695	6.0091×10^{-4}	582.1740	30.9114

Table 2.7: Optimal N for HEXP type II arrival process

N	Model I			Model II		
	R_s	E_v	Cost	R_s	E_v	Cost
1	0.0295	11.3028	89.1856	0.0295	11.3028	89.1856
2	0.0049	68.9648	18.2125	0.0049	68.9648	18.2125
3	0.0012	278.8374	**17.6726**	0.0018	194.0309	**15.0213**
4	6.8609×10^{-4}	508.7016	27.4933	9.8461×10^{-4}	352.1768	20.5627
5	6.0451×10^{-4}	578.2791	30.7275	7.3563×10^{-4}	473.2641	25.8701
6	5.9292×10^{-4}	589.7552	31.2665	6.4742×10^{-4}	538.9180	28.8882
7	5.9131×10^{-4}	591.4750	31.3477	6.1393×10^{-4}	569.0171	30.2927
8	5.9109×10^{-4}	591.7349	31.3600	6.0062×10^{-4}	582.0134	30.9025

Table 2.8: Optimal N for MPA type II arrival process

R_s is higher and E_v is smaller for model II than model I for all type II arrival processes. In all cases we see that as N increases, the expected cost first decreases, reaches a minimum value and then increases. This is due to the fact that, R_s decreases and E_v increases, as N increases. The optimal cost is slightly different for different type II arrival processes, but it corresponds to $N = 3$ in all cases(This may vary according to variation in the parameters). Hence model II performs much better than model I for all type II arrival processes. It may be noted that we assigned small values for λ (Poisson arrival rate of type I customers), η (parameter of vacation clock duration) for reducing R_s value and to get clear distinction in the expected cost between models I and II.

N	Model I			Model II		
	R_s	E_v	Cost	R_s	E_v	Cost
1	0.0295	11.4336	89.1955	0.0295	11.4336	89.1955
2	0.0049	69.0995	18.2198	0.0049	69.0995	18.2198
3	0.0012	278.9731	**17.6795**	0.0018	194.1663	**15.0283**
4	6.8611×10^{-4}	508.7373	27.4952	9.8464×10^{-4}	352.2857	20.5682
5	6.0454×10^{-4}	578.2921	30.7282	7.3564×10^{-4}	473.2473	25.8693
6	5.9296×10^{-4}	589.9148	31.2746	6.4746×10^{-4}	539.1406	28.8994
7	5.9135×10^{-4}	591.6300	31.3556	6.1397×10^{-4}	569.2176	30.3028
8	5.9113×10^{-4}	591.8876	31.3678	6.0066×10^{-4}	582.1987	30.9119

Table 2.9: Optimal N for MNA type II arrival process

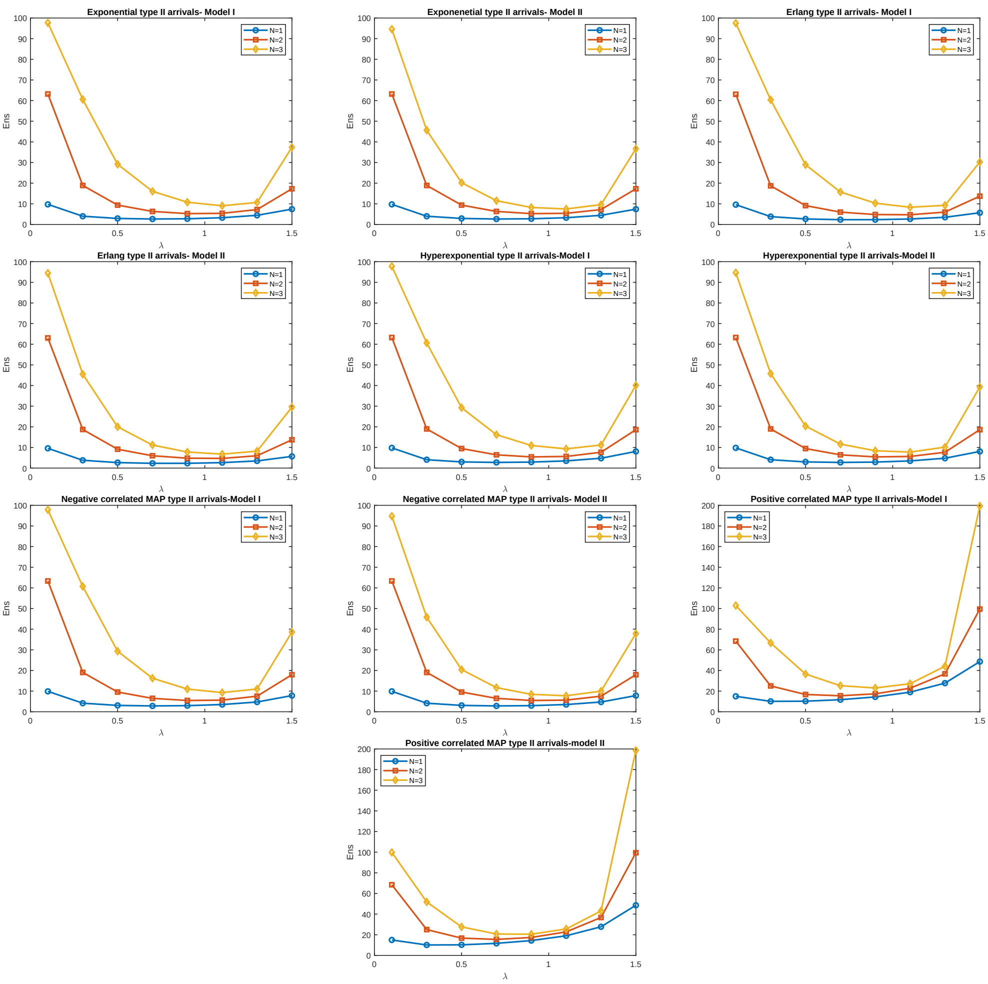

Figure 2.1: Effect of λ on expected number of type II customers in the system

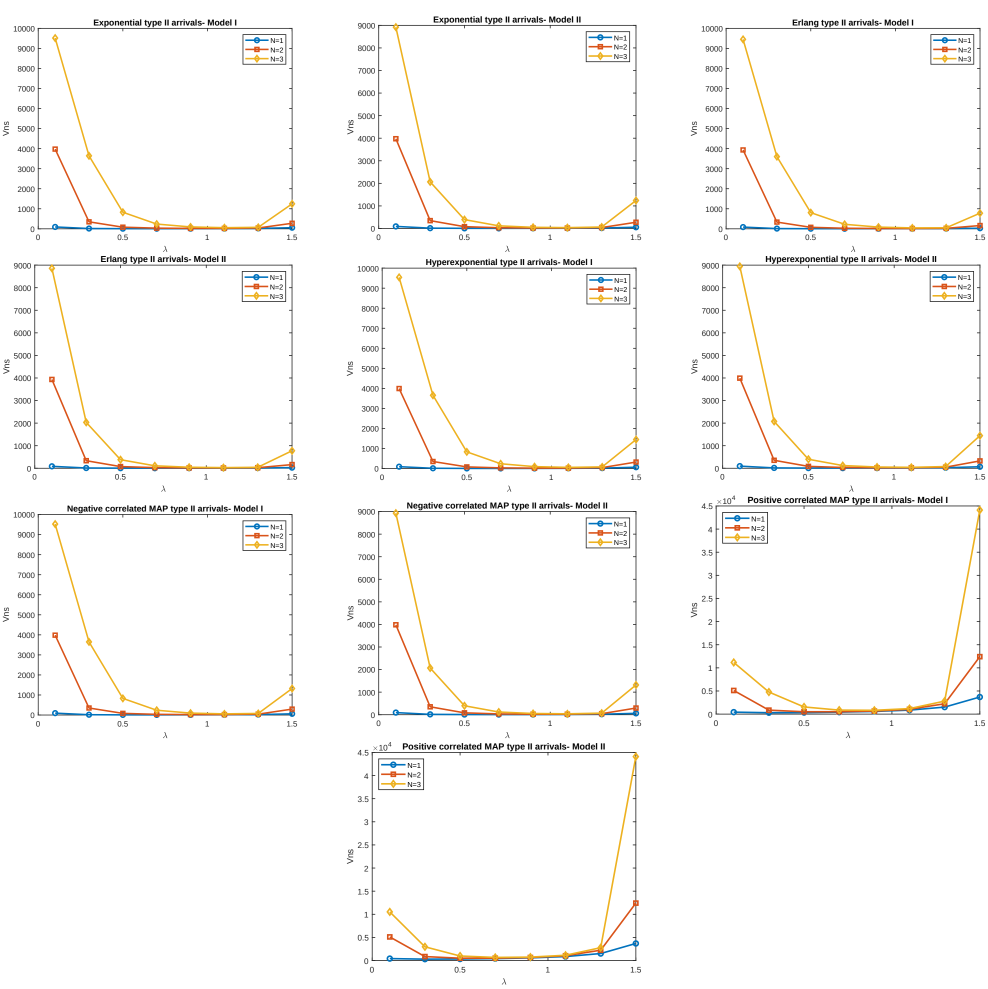

Figure 2.2: Effect of λ on Variance of number of type II customers in the system

Chapter 3

(M, MAP)/(PH, PH)/1 queue with Nonpreemptive priority, Working Interruption and Protection

In the previous chapter we considered working vacation: service is provided to customers at a slower rate during a vacation. In this chapter we consider the case of server providing service when customer is under interruption. We also investigate the effect of providing a protection mechanism to the customer against interruption. We analyze a single server queueing model with two priority classes of customers where the type I customers are assumed to have a non-preemptive priority over type II customers. We consider customer induced interruption during own service. Varghese et al. [50] introduced this new type of interruption in which a customer interrupts own service. Instead of stopping

Some results of this chapter are included in the following paper.
A. Krishnamoorthy, Divya V.: **(M, MAP)/(PH, PH)/1 queue with Nonpreemptive priority, Working Interruption and Protection**,Reliability: Theory and Applications, Vol.13,No.2(49),2018

service completely, the service continues at reduced rate during interruption. However these two models cannot be compared. In Varghese et al. [50], a self interrupted customer goes to the waiting space and stays there until interruption is completed then he moves to another waiting room and wait for his turn for service. However in our model the self interrupted customer is provided service at a reduced rate. The protection for the service of type I customers is provided at the epoch of realization of the clock which starts at the epoch at which the type I customer is taken for service.

There are several real life situations in which this model is suitable. For example, in production process, especially of expensive commodities, it is essential to give protection starting from some stage of manufacture of an item. Thus in a manufacturing process, wherein the item produced has to be protected from variations in power supply; for example: The voltage fluctuation can be considered as an arrival of interruption; this can affect the customer being served or even the server. Thus protection from breakdown of service/damage to customer has to be ensured. Another instance of the model is a patient admitted to hospital for surgery. In this case, he has to be protected from enviornment generated complications.

3.1 Model Description and Mathematical formulation

We consider a single server queue with two priority classes of customers type I and type II with the former arriving according to a Poisson process of rate λ and the latter according to Markovian Arrival Process with representation (D_0, D_1). Service time of both types follow distinct phase type distributions with representations PH($\boldsymbol{\alpha}$,T) of order m_1 and PH($\boldsymbol{\beta}$, S) of order m_2 respectively. The number of type I customers in the system is restricted to a maximum of L. Also type I customers are assumed to have a non-preemptive priority over type II customers. Customer services are subject to interruption by a self induced mechanism. While in interruption arrival of another interruption doesnot affect the customer.The interruptions occur according to Poisson process with rate γ. Instead of stopping the service of that customer completely, it continues at slower rate during interruption. That is, the service time of type I and type II, during an interruption follow phase type distributions with representation PH($\boldsymbol{\alpha}, \theta T$) and PH($\boldsymbol{\beta}, \theta' S$), $0 < \theta, \theta' < 1$ respectively. Thus $\mu = [\boldsymbol{\alpha}(-T)^{-1}\boldsymbol{e}]^{-1}$ is the normal service rate and $\theta\mu$ is the interrupted service rate of type I customers and $\mu' = [\boldsymbol{\beta}(-S)^{-1}\boldsymbol{e}]^{-1}$ and $\theta'\mu'$ are respectively the corresponding rates of normal and interrupted services of type II customers. The server continues to serve at this lower rate until a random clock expires. The duration of interruption is assumed to be exponentially distributed with parameter η. A protection mechanism to diminish the effect of interruptions on type I customers service is arranged. An exponential random clock with mean $\frac{1}{\delta}$ is started simultaneously with each type I service. The protection for the service of type I customers is provided at the epoch of realization of this clock. Type II customers are not provided protection against interruption during their service. Also we assume that the service time of type I customers on activation of protection clock, follows phase type distribution with representation PH($\boldsymbol{\alpha}, \phi T$), $\phi > 1$ and finite.

Let $Q^* = D_0 + D_1$ be the generator matrix of the type II arrival process and $\boldsymbol{\pi^*}$ be its stationary probability vector. Hence $\boldsymbol{\pi^*}$ is the unique (positive) probability vector satisfying $\boldsymbol{\pi^*}Q^* = 0$, $\boldsymbol{\pi}^*\boldsymbol{e} = 1$. The constant $\beta^* = \boldsymbol{\pi^*}D_1\boldsymbol{e}$, referred to as *fundemental rate*, gives the expected number of type II arrivals per unit of time in the stationary version of the MAP. It is assumed that the two arrival processes are mutually independent and are also independent of the service time distributions.

3.1.1 The QBD process

The model described above can be studied as a LIQBD process. First we introduce the followiing notations:

At time t:

$N_1(t)$: number of type II customers in the system

$N_2(t)$: number of type I customers in the system

$$J(t) = \begin{cases} 0, & if\, the\, type\, I\, customer\, in\, service\, is\, unprotected/type\, II\, customer \\ & is\, in\, service \\ 1, & if\, the\, type\, I\, customer\, in\, service\, is\, protected \end{cases}$$

$$K(t) = \begin{cases} 0, & if\;\, the\, server\, provides\, service\, to\, type\, I\, customer\, in\, WI \\ 1, & if\, the\, server\, provides\;\, service\, to\, type\, II\, customer\, in\, WI \\ 2, & if\, the\, server\, provides\, normal\, service\, to\, type\, I\, customer \\ 3, & if\, the\, server\, provides\, normal\, service\, to\, type\, II\, customer \end{cases}$$

$S(t)$: the phase of service when the server is busy

$M(t)$: the phase of arrival of the type II customer.

It is easy to verify that $\{(N_1(t), N_2(t), J(t), K(t), S(t), M(t)) : t \geq 0\}$ is a LIQBD with state space

$l(0) = \{(0, 0, k) : 1 \leq k \leq n\} \cup \{(0, i_2, 0, j_2, k_1, k_2) : 1 \leq i_2 \leq L;\; j_2 = 0\,\text{or}\,2;\; 1 \leq k_1 \leq m_1;\; 1 \leq k_2 \leq n\} \cup \{(0, i_2, 1, 2, k_1, k_2) : 1 \leq i_2 \leq L;\; 1 \leq k_1 \leq m_1;\; 1 \leq k_2 \leq n\}$

For $i_1 \geq 1$,

$\{(i_1, 0, 0, j_2, k_1, k_2) : j_2 = 1\,\text{or}\,3;\; 1 \leq k_1 \leq m_2;\; 1 \leq k_2 \leq n\} \cup \{(i_1, i_2, 0, j_2, k_1, k_2) : 1 \leq i_2 \leq L;\; j_2 = 0\,\text{or}\,2;\; 1 \leq k_1 \leq m_1;\; 1 \leq k_2 \leq n\} \cup \{(i_1, i_2, 0, j_2, k_1, k_2) : 1 \leq i_2 \leq L;\; j_2 = 1\,\text{or}\,3;\; 1 \leq k_1 \leq m_2;\; 1 \leq k_2 \leq n\} \cup \{(i_1, i_2, 1, 2, k_1, k_2) : 1 \leq i - 2 \leq L;\; 1 \leq k_1 \leq m_1;\; 1 \leq k_2 \leq n\}$

The infinitesimal generator of this CTMC is

$$\mathcal{Q}_1 = \begin{bmatrix} B_0 & C_0 & & & \\ B_1 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}.$$

where B_0 contains transitions within the level 0; C_0 represents transitions from level 0 to level 1; B_1 represents transitions from level 1 to level 0; A_0 represents transitions from level g to level $g+1$ for $g \geq 1$, A_1 represents transitions within the level g for $g \geq 1$ and A_2 represents transitions from level g to $g-1$ for $g \geq 2$. The boundary blocks B_0, C_0, B_1 are of orders $n(1+3m_1L) \times n(1+3m_1L)$, $n(1+3m_1L) \times (2m_2n + (3m_1+2m_2)nL)$, $(2m_2n + (3m_1+2m_2)nL) \times n(1+3m_1L)$ respectively. A_0, A_1, A_2 are square matrices of order $2m_2n + (3m_1+2m_2)nL$.
Define the entries of $B_{0_{(h_1,i_1,j_1,k_1,l_1)}}^{(h_2,i_2,j_2,k_2,l_2)}$, $C_{0_{(h_1,i_1,j_1,k_1,l_1)}}^{(h_2,i_2,j_2,k_2,l_2)}$, $B_{1_{(h_1,i_1,j_1,k_1,l_1)}}^{(h_2,i_2,j_2,k_2,l_2)}$ as transition submatrices which contains transitions of the form $(0,h_1,i_1,j_1,k_1,l_1) \to (0,h_2,i_2,j_2,k_2,l_2)$, $(0,h_1,i_1,j_1,k_1,l_1) \to (1,h_2,i_2,j_2,k_2,l_2)$ and $(1,h_1,i_1,j_1,k_1,l_1) \to (0,h_2,i_2,j_2,k_2,l_2)$ respectively. Define the entries of $A_{0_{(h_1,i_1,j_1,k_1,l_1)}}^{(h_2,i_2,j_2,k_2,l_2)}$, $A_{1_{(h_1,i_1,j_1,k_1,l_1)}}^{(h_2,i_2,j_2,k_2,l_2)}$, $A_{2_{(h_1,i_1,j_1,k_1,l_1)}}^{(h_2,i_2,j_2,k_2,l_2)}$ as transition submatrices which contains transitions of the form $(g,h_1,i_1,j_1,k_1,l_1) \to (g+1,h_2,i_2,j_2,k_2,l_2)$, where $g \geq 1$, $(g,h_1,i_1,j_1,k_1,l_1) \to (g,h_2,i_2,j_2,k_2,l_2)$, where $g \geq 1$, $(g,h_1,i_1,j_1,k_1,l_1) \to (g-1,h_2,i_2,j_2,k_2,l_2)$, where $g \geq 2$ respectively. Since none or one event alone could take place in a short interval of time with positive probability, in general, a transition such as $(g_1,h_1,i_1,j_1,k_1,l_1) \to (g_2,h_2,i_2,j_2,k_2,l_2)$ has positive rate only for exactly one of g_1,h_1,i_1,j_1,k_1,l_1 different from g_2,h_2,i_2,j_2,k_2,l_2.

$$
B_{0_{(h_1,i_1,j_1,k_1,l_1)}}^{(h_2,i_2,j_2,k_2,l_2)} = \begin{cases}
\lambda(\boldsymbol{\alpha} \otimes I_n) & h_1 = 0, h_2 = 1; i_2 = 0; j_2 = 2, 1 \le k_2 \le m_1, \\
& 1 \le l_1, l_2 \le n \\
\lambda I_{m_1 n} & 1 \le h_1 \le L-1, h_2 = h_1 + 1; i_1 = i_2 = 0; \\
& j_1 = j_2, j_1 = 0 \text{ or } 2; 1 \le k_1, k_2 \le m_1; \\
& 1 \le l_1, l_2 \le n \\
\lambda I_{m_1 n} & 1 \le h_1 \le L-1, h_2 = h_1 + 1; i_1 = i_2 = 1; \\
& j_1 = j_2 = 2; 1 \le k_1, k_2 \le m_1; 1 \le l_1, l_2 \le n \\
\theta \boldsymbol{T}^0 \otimes I_n & h_1 = 1, h_2 = 0; i_1 = 0; j_1 = 0; 1 \le k_1 \le m_1; \\
& 1 \le l_1, l_2 \le n \\
\boldsymbol{T}^0 \otimes I_n & h_1 = 1, h_2 = 0; i_1 = 0; j_1 = 2; 1 \le k_1 \le m_1; \\
& 1 \le l_1, l_2 \le n \\
\phi \boldsymbol{T}^0 \otimes I_n & h_1 = 1, h_2 = 0; i_1 = 1; j_1 = 2; 1 \le k_1 \le m_1; \\
& 1 \le l_1, l_2 \le n \\
\theta \boldsymbol{T}^0 \boldsymbol{\alpha} \otimes I_n & 2 \le h_1 \le L, h_2 = h_1 - 1; i_1 = i_2 = 0; j_1 = 0, \\
& j_2 = 2; 1 \le k_1, k_2 \le m_1; 1 \le l_1, l_2 \le n \\
\boldsymbol{T}^0 \boldsymbol{\alpha} \otimes I_n & 2 \le h_1 \le L, h_2 = h_1 - 1; i_1 = i_2 = 0; \\
& j_1 = j_2 = 2; 1 \le k_1, k_2 \le m_1; 1 \le l_1, l_2 \le n \\
\phi \boldsymbol{T}^0 \boldsymbol{\alpha} \otimes I_n & 2 \le h_1 \le L, h_2 = h_1 - 1; i_1 = 1, i_2 = 0; \\
& j_1 = j_2 = 2; 1 \le k_1, k_2 \le m_1; 1 \le l_1, l_2 \le n \\
\eta I_{m_1 n} & 1 \le h_1 \le L, h_1 = h_2, i_1 = i_2 = 0; j_1 = 0, \\
& j_2 = 2; 1 \le k_1, k_2 \le m_1; 1 \le l_1, l_2 \le n \\
\gamma I_{m_1 n} & 1 \le h_1 \le L, h_1 = h_2; i_1 = i_2 = 0; j_1 = 2, \\
& j_2 = 0; \le k_1, k_2 \le m_1; 1 \le l_1, l_2 \le n \\
\delta I_{m_1 n} & 1 \le h_1 \le L, h_1 = h_2; i_1 = 0, i_2 = 1; j_1 = 0 \\
& \text{or } 2, j_2 = 2; 1 \le k_1, k_2 \le m_1; 1 \le l_1, l_2 \le n \\
D_0 - \lambda I_n & h_1 = h_2 = 0; 1 \le l_1, l_2 \le n \\
\theta T \oplus D_0 - (\lambda + \eta + \delta) I_{m_1 n} & 1 \le h_1 \le L-1, h_1 = h_2; i_1 = i_2 = 0; \\
& j_1 = j_2 = 0; 1 \le k_1, k_2 \le m_1; 1 \le l_1, l_2 \le n \\
T \oplus D_0 - (\lambda + \gamma + \delta) I_{m_1 n} & 1 \le h_1 \le L-1, h_1 = h_2; i_1 = i_2 = 0; \\
& j_1 = j_2 = 2; 1 \le k_1, k_2 \le m_1; 1 \le l_1, l_2 \le n \\
\phi T \oplus D_0 - \lambda I_{m_1 n} & 1 \le h_1 \le L-1, h_1 = h_2; i_1 = i_2 = 1; \\
& j_1 = j_2 = 2; 1 \le k_1, k_2 \le m_1; 1 \le l_1, l_2 \le n \\
\theta T \oplus D_0 - (\eta + \delta) I_{m_1 n} & h_1 = h_2 = L; i_1 = i_2 = 0; j_1 = j_2 = 0; \\
& 1 \le k_1, k_2 \le m_1; 1 \le l_1, l_2 \le n \\
T \oplus D_0 - (\gamma + \delta) I_{m_1 n} & h_1 = h_2 = L; i_1 = i_2 = 0; j_1 = j_2 = 2; \\
& 1 \le k_1, k_2 \le m_1; 1 \le l_1, l_2 \le n \\
\phi T \oplus D_0 & h_1 = h_2 = L; i_1 = i_2 = 1; j_1 = j_2 = 2; \\
& 1 \le k_1, k_2 \le m_1; 1 \le l_1, l_2 \le n
\end{cases}
$$

$$C_{0_{(h_1,i_1,j_1,k_1,l_1)}}^{(h_2,i_2,j_2,k_2,l_2)} = \begin{cases} \boldsymbol{\beta} \otimes D_1 & h_1 = h_2 = 0; i_2 = 0; j_2 = 3; 1 \le k_2 \le m_2; \\ & 1 \le l_1, l_2 \le n \\ I_{m_1} \otimes D_1 & 1 \le h_1 \le L, h_1 = h_2; i_1 = i_2 = 0; j_1 = j_2 = 0; \\ & 1 \le k_1, k_2, \le m_1; 1 \le l_1, l_2 \le n \\ I_{m_1} \otimes D_1 & 1 \le h_1 \le L, h_1 = h_2; i_1 = i_2 = 0; j_1 = j_2 = 2; \\ & 1 \le k_1, k_2 \le m_1; 1 \le l_1, l_2 \le n \\ I_{m_1} \otimes D_1 & 1 \le h_1 \le L, h_1 = h_2; i_1 = i_2 = 1; j_1 = j_2 = 2; \\ & 1 \le k_1, k_2 \le m_1; 1 \le l_1, l_2 \le n \end{cases}$$

$$B_{1_{(h_1,i_1,j_1,k_1,l_1)}}^{(h_2,i_2,j_2,k_2,l_2)} = \begin{cases} \theta' \boldsymbol{S}^0 \otimes I_n & h_1 = h_2 = 0; i_1 = 0; j_1 = 1; 1 \le k_1 \le m_2, \\ & 1 \le l_1, l_2 \le n \\ \boldsymbol{S}^0 \otimes I_n & h_1 = h_2 = 0; i_1 = 0; j_1 = 3; 1 \le k_1 \le m_2; \\ & 1 \le l_1, l_2 \le n \\ \theta' \boldsymbol{S}^0 \boldsymbol{\alpha} \otimes I_n & h_1 = h_2, 1 \le h_1 \le L; i_1 = i_2 = 0; j_1 = 1, j_2 = 2; \\ & 1 \le k_1 \le m_2, 1 \le k_2 \le m_1; 1 \le l_1, l_2 \le n \\ \boldsymbol{S}^0 \boldsymbol{\alpha} \otimes I_n & h_1 = h_2, 1 \le h_1 \le L; i_1 = i_2 = 0; j_1 = 3, j_2 = 2; \\ & 1 \le k_1 \le m_2, 1 \le k_2 \le m_1; 1 \le l_1, l_2 \le n \end{cases}$$

$$A_{0_{(h_1,i_1,j_1,k_1,l_1)}}^{(h_2,i_2,j_2,k_2,l_2)} = \begin{cases} I_{m_2} \otimes D_1 & i_1 = i_2 = 0; j_1 = j_2 = 1; 1 \le k_1, k_2 \le m_2; \\ & 1 \le l_1, l_2 \le n \\ I_{m_2} \otimes D_1 & i_1 = i_2 = 0; j_1 = j_2 = 3; 1 \le k_1, k_2 \le m_2; \\ & 1 \le l_1, l_2 \le n \\ I_{m_1} \otimes D_1 & 1 \le h_1 \le L, h_1 = h_2; i_1 = i_2 = 0; j_1 = j_2 = 0 \,\text{or}\, 2; \\ & 1 \le k_1, k_2 \le m_1; 1 \le l_1, l_2 \le n \\ I_{m_1} \otimes D_1 & 1 \le h_1 \le L, h_1 = h_2; i_1 = i_2 = 1; j_1 = j_2 = 2; \\ & 1 \le k_1, k_2 \le m_1; 1 \le l_1, l_2 \le n \\ I_{m_2} \otimes D_1 & 1 \le h_1 \le L, h_1 = h_2; i_1 = i_2 = 0; j_1 = j_2 = 1 \,\text{or}\, 3; \\ & 1 \le k_1, k_2 \le m_2; 1 \le l_1, l_2 \le n \end{cases}$$

$$A_{2_{(h_1,i_1,j_1,k_1,l_1)}}^{(h_2,i_2,j_2,k_2,l_2)} = \begin{cases} \theta' \boldsymbol{S}^0 \boldsymbol{\beta} \otimes I_n & h_1 = h_2 = 0; i_1 = i_2 = 0; j_1 = 1, j_2 = 3; 1 \le k_1, k_2 \le m_2; \\ & 1 \le l_1, l_2 \le n \\ \boldsymbol{S}^0 \boldsymbol{\beta} \otimes I_n & h_1 = h_2 = 0; i_1 = i_2 = 0; j_1 = j_2 = 3; 1 \le k_1, k_2 \le m_2; \\ & 1 \le l_1, l_2 \le n \\ \theta' \boldsymbol{S}^0 \boldsymbol{\alpha} \otimes I_n & 1 \le h_1 \le L, h_1 = h_2; i_1 = i_2 = 0; j_1 = 1, j_2 = 2; \\ & 1 \le k_1 \le m_2, 1 \le k_2 \le m_1; 1 \le l_1, l_2 \le n \\ \boldsymbol{S}^0 \boldsymbol{\alpha} \otimes I_n & 1 \le h_1 \le L, h_1 = h_2; i_1 = i_2 = 0; j_1 = 3, j_2 = 2; \\ & 11 \le k_1 \le m_2, \le k_2 \le m_1, 1 \le l_1, l_2 \le n \end{cases}$$

$$A_{1\,(h_1,i_1,j_1,k_1,l_1)}^{(h_2,i_2,j_2,k_2,l_2)} = \begin{cases} \lambda I_{m_1 n} & 1 \le h_1 \le L-1, h_2 = h_1+1; i_1 = i_2 = 0; j_1 = j_2 = 0 \text{ or } 2; \\ & 1 \le k_1, k_2 \le m_1; 1 \le l_1, l_2 \le n \\ \lambda I_{m_2 n} & 0 \le h_1 \le L-1, h_2 = h_1+1; i_1 = i_2 = 0; j_1 = j_2 = 1 \text{ or } 3; \\ & 1 \le k_1, k_2 \le m_2; 1 \le l_1, l_2 \le n \\ \lambda I_{m_1 n} & 1 \le h_1 \le L-1, h_2 = h_1+1; i_1 = i_2 = 1; j_1 = j_2 = 2; \\ & 1 \le k_1, k_2 \le m_1; 1 \le l_1, l_2 \le n \\ \theta \boldsymbol{T}^0 \boldsymbol{\beta} \otimes I_n & h_1 = 1, h_2 = 0; i_1 = i_2 = 0; j_1 = 0, j_2 = 3; 1 \le k_1 \le m_1, \\ & 1 \le k_2 \le m_2; 1 \le l_1, l_2 \le n \\ \boldsymbol{T}^0 \boldsymbol{\beta} \otimes I_n & h_1 = 1, h_2 = 0; i_1 = i_2 = 0; j_1 = 2, j_2 = 3; 1 \le k_1 \le m_1, \\ & 1 \le k_2 \le m_2; 1 \le l_1, l_2 \le n \\ \phi \boldsymbol{T}^0 \boldsymbol{\beta} \otimes I_n & h_1 = 1, h_2 = 0; i_1 = 1, i_2 = 0; j_1 = 2, j_2 = 3; 1 \le k_1 \le m_1, \\ & 1 \le k_2 \le m_2; 1 \le l_1, l_2 \le n \\ \theta \boldsymbol{T}^0 \boldsymbol{\alpha} \otimes I_n & 2 \le h_1 \le L, h_2 = h_1 - 1; i_1 = i_2 = 0; j_1 = 0, j_2 = 2; \\ & 1 \le k_1, k_2 \le m_1; 1 \le l_1, l_2 \le n \\ \boldsymbol{T}^0 \boldsymbol{\alpha} \otimes I_n & 2 \le h_1 \le L, h_2 = h_1 - 1; i_1 = i_2 = 0; j_1 = j_2 = 2; \\ & 1 \le k_1, k_2 \le m_1; 1 \le l_1, l_2 \le n \\ \phi \boldsymbol{T}^0 \boldsymbol{\alpha} \otimes I_n & 2 \le h_1 \le L, h_2 = h_1 - 1; i_1 = i_2 = 1; j_1 = j_2 = 2; \\ & 1 \le k_1, k_2 \le m_1; 1 \le l_1, l_2 \le n \\ \eta I_{m_1 n} & 1 \le h_1 \le L, h_1 = h_2; i_1 = i_2 = 0; j_1 = 0, j_2 = 2; \\ & 1 \le k_1, k_2 \le m_1; 1 \le l_1, l_2 \le n \\ \eta I_{m_2 n} & 0 \le h_1 \le L, h_1 = h_2; i_1 = i_2 = 0; j_1 = 1, j_2 = 3; \\ & 1 \le k_1, k_2 \le m_2; 1 \le l_1, l_2 \le n \\ \gamma I_{m_1 n} & 1 \le h_1 \le L, h_1 = h_2; i_1 = i_2 = 0; j_1 = 2, j_2 = 0; \\ & 1 \le k_1, k_2 \le m_1; 1 \le l_1, l_2 \le n \\ \gamma I_{m_2 n} & 0 \le h_1 \le L, h_1 = h_2; i_1 = i_2 = 0; j_1 = 3, j_2 = 1; \\ & 1 \le k_1, k_2 \le m_2; 1 \le l_1, l_2 \le n \\ \delta I_{m_1 n} & 1 \le h_1 \le L, h_1 = h_2; i_1 = 0, i_2 = 1; j_1 = 0 \text{ or } 2, j_2 = 2; \\ & 1 \le k_1, k_2 \le m_1; 1 \le l_1, l_2 \le n \\ \theta' S \oplus D_0 - (\lambda + \eta) I_{m_2 n} & h_1 = h_2 = 0; i_1 = i_2 = 0; j_1 = j_2 = 1, 1 \le k_1, k_2 \le m_2; \\ & 1 \le l_1, l_2 \le n \\ S \oplus D_0 - (\lambda + \gamma) I_{m_2 n} & h_1 = h_2 = 0; i_1 = i_2 = 0; j_1 = j_2 = 3, 1 \le k_1, k_2 \le m_2; \\ & 1 \le l_1, l_2 \le n \\ \theta T \oplus D_0 - (\lambda + \eta + \delta) I_{m_1 n} & 1 \le h_1 \le L-1, h_1 = h_2; i_1 = i_2 = 0, j_1 = j_2 = 0; \\ & 1 \le k_1, k_2 \le m_1; 1 \le l_1, l_2 \le n \\ \theta' S \oplus D_0 - (\lambda + \eta) I_{m_2 n} & 1 \le h_1 \le L-1, h_1 = h_2; i_1 = i_2 = 0, j_1 = j_2 = 1; \\ & 1 \le k_1, k_2 \le m_2, 1 \le l_1, l_2 \le n \\ T \oplus D_0 - (\lambda + \gamma + \delta) I_{m_1 n} & 1 \le h_1 \le L-1, h_1 = h_2; i_1 = i_2 = 0, j_1 = j_2 = 2, \\ & 1 \le k_1, k_2 \le m_1, 1 \le l_1, l_2 \le n \\ S \oplus D_0 - (\lambda + \gamma) I_{m_2 n} & 1 \le h_1 \le L-1, h_1 = h_2; i_1 = i_2 = 0, j_1 = j_2 = 3, \\ & 1 \le k_1, k_2 \le m_2, 1 \le l_1, l_2 \le n \\ \phi T \oplus D_0 - \lambda I_{m_1 n} & 1 \le h_1 \le L-1, h_1 = h_2; i_1 = i_2 = 1, j_1 = j_2 = 2, \\ & 1 \le k_1, k_2 \le m_1, 1 \le l_1, l_2 \le n \\ \theta T \oplus D_0 - (\eta + \delta) I_{m_1 n} & h_1 = h_2 = L, i_1 = i_2 = 0, j_1 = j_2 = 0, 1 \le k_1, k_2 \le m_1, \\ & 1 \le l_1, l_2 \le n \\ \theta' S \oplus D_0 - \eta I_{m_2 n} & h_1 = h_2 = L, i_1 = i_2 = 0, j_1 = j_2 = 1, 1 \le k_1, k_2 \le m_2, \\ & 1 \le l_1, l_2 \le n \\ T \oplus D_0 - (\gamma + \delta) I_{m_1 n} & h_1 = h_2 = L, i_1 = i_2 = 0, j_1 = j_2 = 2, 1 \le k_1, k_2 \le m_1, \\ & 1 \le l_1, l_2 \le n \\ S \oplus D_0 - \gamma I_{m_2 n} & h_1 = h_2 = L, i_1 = i_2 = 0, j_1 = j_2 = 3, 1 \le k_1, k_2 \le m_2, \\ & 1 \le l_1, l_2 \le n \\ \phi T \oplus D_0 & h_1 = h_2 = L, i_1 = i_2 = 1, j_1 = j_2 = 2, 1 \le k_1, k_2 \le m_1, \\ & 1 \le l_1, l_2 \le n \end{cases}$$

3.2 Steady State Analysis

First we find the condition for stability of the system under study.

3.2.1 Stability condition

Let $\boldsymbol{\pi} = (\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \ldots, \boldsymbol{\pi}_L)$ denote the steady state probability vector of the generator

$$A = A_0 + A_1 + A_2 = \begin{bmatrix} F_0 & F_1 & & & & \\ F_2 & F_3 & \lambda I & & & \\ & F_4 & F_3 & \lambda I & & \\ & & \ddots & \ddots & \ddots & \\ & & F_4 & F_3 & \lambda I & \\ & & & F_4 & F_5 & \end{bmatrix}.$$

ie,

$$\boldsymbol{\pi} A = 0, \boldsymbol{\pi}\boldsymbol{e} = 1. \tag{3.1}$$

In the above,

$$F_0(k,l) = \begin{cases} \theta' S \oplus D_0 - (\lambda+\eta) I_{m_2 n} + I_{m_2} \otimes D_1 & k=1, l=1 \\ \eta I_{m_2 n} + \theta' \boldsymbol{S}^0 \beta \otimes I_n & k=1, l=2 \\ \gamma I m_2 n & k=2, l=1 \\ S \oplus D_0 - (\lambda+\gamma) I_{m_2 n} + \boldsymbol{S}^0 \boldsymbol{\beta} \otimes I_n + I_{m_2} \otimes D_1 & k=2, l=2 \end{cases}$$

$$F_1(k,l) = \begin{cases} \lambda I_{m_2 n} & k=1, l=2 \\ \lambda I_{m_2 n} & k=2, l=4 \\ 0 & \text{otherwise} \end{cases}, \quad F_2(k,l) = \begin{cases} \theta \boldsymbol{T}^0 \boldsymbol{\beta} \otimes I_n & k=1, l=2 \\ \boldsymbol{T}^0 \boldsymbol{\beta} \otimes I_n & k=3, l=2 \\ \phi \boldsymbol{T}^0 \boldsymbol{\beta} \otimes I_n & k=5, l=2 \\ 0 & \text{otherwise} \end{cases}$$

$$F_3(k,l) = \begin{cases} \theta T \oplus D_0 - (\lambda+\eta+\delta)I_{m_1n} + I_{m_1} \otimes D_1 & k=1, l=1 \\ \eta I_{m_1n} & k=1, l=3 \\ \delta I_{m_1n} & k=1, l=5 \\ \theta' S \oplus D_0 - (\lambda+\eta)I_{m_2n} + I_{m_2} \otimes D_1 & k=2, l=2 \\ \theta' \boldsymbol{S}^0 \boldsymbol{\alpha} \otimes I_n & k=2, l=3 \\ \eta I_{m_2n} & k=2, l=4 \\ \gamma I_{m_1n} & k=3, l=1 \\ T \oplus D_0 - (\lambda+\gamma+\delta)I_{m_1n} + I_{m_1} \otimes D_1 & k=3, l=3 \\ \delta I_{m_1n} & k=3, l=5 \\ \gamma I_{m_2n} & k=4, l=2 \\ \boldsymbol{S}^0 \boldsymbol{\alpha} \otimes I_n & k=4, l=3 \\ S \oplus D_0 - (\lambda+\gamma)I_{m_2n} + I_{m_2} \otimes D_1 & k=4, l=4 \\ \phi T \oplus D_0 - \lambda I_{m_1n} + I_{m_1} \otimes D_1 & k=5, l=5 \\ 0 & \text{otherwise} \end{cases}$$

$$F_4(k,l) = \begin{cases} \theta \boldsymbol{T}^0 \boldsymbol{\alpha} \otimes I_n & k=1, l=3 \\ \boldsymbol{T}^0 \boldsymbol{\alpha} \otimes I_n & k=3, l=3 \\ \phi \boldsymbol{T}^0 \boldsymbol{\alpha} \otimes I_n & k=5, l=3 \\ 0 & \text{otherwise} \end{cases},$$

$$F_5(k,l) = \begin{cases} \theta T \oplus D_0 - (\eta+\delta)I_{m_1n} + I_{m_1} \otimes D_1 & k=1, l=1 \\ \eta I_{m_1n} & k=1, l=3 \\ \delta I_{m_1n} & k=1, l=5 \\ \theta' S \oplus D_0 - \eta I_{m_2n} + I_{m_2} \otimes D_1 & k=2, l=2 \\ \theta' \boldsymbol{S}^0 \boldsymbol{\alpha} \otimes I_n & k=2, l=3 \\ \eta I_{m_2n} & k=2, l=4 \\ \gamma I_{m_1n} & k=3, l=1 \\ T \oplus D_0 - (\gamma+\delta)I_{m_1n} + I_{m_1} \otimes D_1 & k=3, l=3 \\ \delta I_{m_1n} & k=3, l=5 \\ \gamma I_{m_2n} & k=4, l=2 \\ \boldsymbol{S}^0 \boldsymbol{\alpha} \otimes I_n & k=4, l=3 \\ S \oplus D_0 - \gamma I_{m_2n} + I_{m_2} \otimes D_1 & k=4, l=4 \\ \phi T \oplus D_0 + I_{m_1} \otimes D_1 & k=5, l=5 \\ 0 & \text{otherwise} \end{cases}$$

with dimensions of F_0, F_1, F_2 be $2m_2n \times 2m_2n$, $2m_2n \times (3m_1 + 2m_2)n$,$(3m_1 + 2m_2)n \times 2m_2n$ respectively. F_3, F_4 and F_5 are square matrices of order $(3m_1 + 2m_2)n$. The $LIQBD$ description of the model indicates that the queueing system is stable (see Neuts [40]) if and only if the left drift exceeds that of right drift. That is,

$$\boldsymbol{\pi} A_0 \mathbf{e} < \boldsymbol{\pi} A_2 \mathbf{e}. \tag{3.2}$$

The vector $\boldsymbol{\pi}$ cannot be obtained directly in terms of the parametres of the model. From (3.1)we get

$$\boldsymbol{\pi}_i = \boldsymbol{\pi}_{i-1} \mathcal{U}_{i-1}, 1 \leq i \leq L \tag{3.3}$$

where

$$\mathcal{U}_0 = -F_1 (F_3 + \mathcal{U}_1 F_4)^{-1}$$

$$\mathcal{U}_i = \begin{cases} -\lambda (F_3 + U_{i+1} F_4)^{-1} & for\ 1 \leq i \leq L-2 \\ -\lambda F_5^{-1} & for\ i = L-1. \end{cases}$$

From the normalizing condition $\boldsymbol{\pi e} = 1$ we have

$$\boldsymbol{\pi}_0 \left(\sum_{j=0}^{L-1} \prod_{i=0}^{j} \mathcal{U}_i + I \right) e = 1. \tag{3.4}$$

The inequality (3.2) gives the stability condition as

$$\boldsymbol{\pi}_0 \left[(I_{(2m_2)} \otimes D_1) \boldsymbol{e} + \sum_{i=0}^{L-1} \prod_{j=0}^{i} \mathcal{U}_j (I_{3m_1+2m_2} \otimes D_1) \boldsymbol{e} \right] <$$

$$\boldsymbol{\pi}_0 \left[[\boldsymbol{e}_1(2)(\theta' \boldsymbol{S}^0 \boldsymbol{\beta} \otimes I) + e_2(2) \boldsymbol{S}^0 \boldsymbol{\beta} \otimes I)] \boldsymbol{e}(m_2 n) + \sum_{i=0}^{L-1} \prod_{j=0}^{i} \mathcal{U}_j [\boldsymbol{e}_2(5) \theta' \boldsymbol{S}^0 \boldsymbol{\alpha} \otimes I) + \boldsymbol{e}_4(5)(\boldsymbol{S}^0 \boldsymbol{\alpha} \otimes I)] \boldsymbol{e}(m_2 n) \right]. \tag{3.5}$$

3.2.2 Steady-state probability vector

Assuming that the condition (3.5) is satisfied we proceed to find the steady-state probability of the system state. Let $\boldsymbol{x}$ be the steady state probability vector of Q. We partition this vector as $\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2 \ldots)$, where $\boldsymbol{x}_0$ is of dimension $n(1 + 3m_1L)$ and $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots$ are each of dimension $n(2m_2 + (3m_1 + 2m_2)L)$. Under the stability condition, we have $\boldsymbol{x}_i = \boldsymbol{x}_1 R^{i-1}, i \geq 2$, where the matrix R is the minimal nonnegative solution to the matrix quadratic equation

$$R^2A_2 + RA_1 + A_0 = 0$$

and the vectors $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$are obtained by solving the equations

$$\boldsymbol{x}_0 B_0 + \boldsymbol{x}_1 B_1 = 0 \tag{3.6}$$

$$\boldsymbol{x}_0 C_0 + \boldsymbol{x}_1 (A_1 + RA_2) = 0 \tag{3.7}$$

subject to the normalizing condition

$$\boldsymbol{x}_0 \boldsymbol{e} + \boldsymbol{x}_1 (I - R)^{-1} \boldsymbol{e} = 1 \tag{3.8}$$

3.2.3 Analysis of service time of a type I customer

The duration of service of a type I customer is a phase type distribution with representation $(\boldsymbol{\alpha'}, S_1)$ where the underlying MC has state space $\{(i, j, k) : i = 0;\ j = 0\,or\,2;\ 1 \leq k \leq m_1\} \cup \{(i, 2, k) : i = 1; 1 \leq k \leq m_1)\} \cup \{*\}$ where i denotes the status of the protection clock, j, the status of the server, k, the service phase and *, the absorbing state indicating service completion. The infinitesimal generator is

$$\mathcal{S}_1 = \begin{bmatrix} S_1 & \boldsymbol{S}_1^0 \\ \mathbf{0} & 0 \end{bmatrix}, \text{ where, } S_1 = \begin{bmatrix} \theta T - (\eta + \delta) I_{m_1} & \eta I_{m_1} & \delta I_{m_1} \\ \gamma I_{m_1} & T - (\gamma + \delta) I_{m_1} & \delta I_{m_1} \\ \mathbf{0} & \mathbf{0} & \phi T \end{bmatrix} \text{ and } \boldsymbol{S}_1^0 = \begin{bmatrix} \theta \boldsymbol{T}^0 \\ \boldsymbol{T}^0 \\ \phi \boldsymbol{T}^0 \end{bmatrix}$$

The initial probability vector is $\boldsymbol{\alpha'} = \begin{bmatrix} \mathbf{0} & \boldsymbol{\alpha} & \mathbf{0} \end{bmatrix}$, , where $\mathbf{0}$ is a zero matrix of order $1 \times m_1$.

Thus the service time distribution of a type I customer is PH$(\boldsymbol{\alpha'}, S_1)$ of order $3m_1n$.

3.2.4 Analysis of service time of a type II customer

The duration of service of a type II customer turn out to be a phase type distribution $(\boldsymbol{\beta'}, S_2)$ where the underlying MC has state space $\{(i, j) : i = 1\, or\, 3; 1 \le j \le m_2\} \cup \{*\}$ where i denotes the status of the server, j, the service phase and *, the absorbing state indicating service completion. The infinitesimal generator is

$$\mathcal{S}_2 = \begin{bmatrix} S_2 & \boldsymbol{S}_2^0 \\ \mathbf{0} & 0 \end{bmatrix}, \text{ where, } S_2 = \begin{bmatrix} \theta' S - \eta I_{m_2} & \eta I_{m_2} \\ \gamma I_{m_2} & S - \gamma I_{m_2} \end{bmatrix} \text{ and } \boldsymbol{S}_2^0 = \begin{bmatrix} \theta' \boldsymbol{S}^0 \\ S^0 \end{bmatrix}$$

The initial probability vector is $\boldsymbol{\beta'} = \begin{bmatrix} \mathbf{0} & \boldsymbol{\alpha} \end{bmatrix}$, where $\mathbf{0}$ is a zero matrix of order $1 \times m_2$. Thus we have the service time distribution of a type II customer is PH$(\boldsymbol{\beta'}, S_2)$ of order $2m_2n$.

3.3 Waiting time analysis

3.3.1 Type I Customer

To find the waiting time of a type I customer who joins for service at time t, we have to consider different possibilities depending on the status of server at that time. Let $W_1(t)$ be the waiting time of a type I customer who arrives at time t and $W_1^*(s)$ be the corresponding LST.

Case I

Suppose that E_1 denote the event the system is in the state $(0, 1, 0, 2, u, v)$, $1 \le u \le m_1$; $1 \le v \le n$ immediately after arrival of the tagged customer. Let $W_1^*(s/E_1)$ denote the corresponding LST.Then

$$W_1^*(s/E_1) = 1$$

Case II

E_2 be the event that the system is in the state $(n_1, a + 1, 0, 0, u, v)$, $n_1 \ge 0$; $1 \le a \le L - 1$; $1 \le u \le m_1$; $1 \le v \le n$, immediately after arrival of the tagged customer. In this case the waiting time is the sum of the residual service time of the type I customer in service when the tagged customer arrives and service time of $a-1$ remaining type I customers. Let $W_1^*(s/E_2)$ represent the corresponding conditonal LST. Then

$$W_1^*(s/E_2) = \big(\boldsymbol{e'}_u(3m_1)(sI - S_1)^{-1}\boldsymbol{S}_1^0\big)\big(\boldsymbol{\alpha'}(sI - S_1)^{-1}\boldsymbol{S}_1^0\big)^{a-1}.$$

Case III

E_3 denote the event: the system is in the state $(n_1, a+1, 0, 2, u, v)$, $n_1 \geq 0$; $1 \leq a \leq L-1$; $1 \leq u \leq m_1$; $1 \leq v \leq n$, immediately after arrival of the tagged customer. In this case the waiting time is the sum of the residual service time of the type I customer in service when the tagged customer arrives and service times of $a-1$ remaining type I customers. With $W_1^*(s/E_3)$ as the corresponding conditonal LST, we have

$$W_1^*(s/E_3) = \big(\boldsymbol{e'}_{m_1+u}(3m_1)(sI - S_1)^{-1}\boldsymbol{S}_1^0\big)\big(\boldsymbol{\alpha'}(sI - S_1)^{-1}\boldsymbol{S}_1^0\big)^{a-1}.$$

Case IV

E_4 denote the event: the system is in the state $(n_1, a+1, 1, 2, u, v)$, $n_1 \geq 0$; $1 \leq a \leq L-1$; $1 \leq u \leq m_1$; $1 \leq v \leq n$, immediately after arrival of the tagged customer. In this case the waiting time is the sum of the residual service time of the type I customer in service when the tagged customer arrives and service times of $a-1$ remaining type I customers. Let $W_1^*(s/E_4)$ represent the corresponding conditonal LST. Then

$$W_1^*(s/E_4) = \big(\boldsymbol{e'}_{2m_1+u}(3m_1)(sI - S_1)^{-1}\boldsymbol{S}_1^0\big)\big(\boldsymbol{\alpha'}(sI - S_1)^{-1}\boldsymbol{S}_1^0\big)^{a-1}.$$

Case V

E_5 denote the event: the system is in the state $(n_1, a+1, 0, 1, u, v)$, $n_1 \geq 1$; $0 \leq a \leq L-1$; $1 \leq u \leq m_2$; $1 \leq v \leq n$, immediately after arrival of the tagged customer. In this case the waiting time is the sum of the residual service time of the type II customer in service when the tagged customer arrives and service times of a remaining type I customers. Let $W_1^*(s/E_5)$ represent the corresponding conditonal LST. Then

$$W_1^*(s/E_5) = \big(\boldsymbol{e'}_u(2m_2)(sI - S_2)^{-1}\boldsymbol{S}_2^0\big)\big(\boldsymbol{\alpha'}(sI - S_1)^{-1}\boldsymbol{S}_1^0\big)^a.$$

Case VI

E_6 denote the event: the system is in the state $(n_1, a+1, 0, 3, u, v)$, $n_1 \geq 1$; $0 \leq a \leq L-1$; $1 \leq u \leq m_2$; $1 \leq v \leq n$, immediately after arrival of the tagged customer. In this case the waiting time is the sum of the residual service time of the type II customer in service when the tagged customer arrives and service times of a remaining type I customers. Let $W^*(s/E_6)$ represent the corresponding conditonal LST. Then

$$W_1^*(s/E_6) = \big(\boldsymbol{e'}_{m_2+u}(2m_2)(sI - S_2)^{-1}\boldsymbol{S}_2^0\big)\big(\boldsymbol{\alpha'}(sI - S_1)^{-1}\boldsymbol{S}_1^0\big)^a.$$

Let $w_{i_1,i_2,j_1,j_2,k,l}$ denote the probabilty that the system is in the state $(i_1, i_2, j_1, j_2, k, l)$ immediately after arrival of the tagged customer. Then,

$$\begin{aligned}
w_{0,1,0,2,u,v} &= \tfrac{\lambda\alpha_u}{\lambda-d_{vv}^0}x_{0,0,v}, \text{ for, } 1 \leq u \leq m,\ 1 \leq v \leq n \\
w_{n_1,a+1,0,0,u,v} &= \tfrac{\lambda}{\lambda+\eta+\delta-\theta T_{uu}-d_{vv}^0}x_{n_1,a,0,0,u,v}, \text{ for, } n_1 \geq 1,\ 1 \leq u \leq m_1, \\
&\quad 1 \leq v \leq n \\
w_{n_1,a+1,0,2,u,v} &= \tfrac{\lambda}{\lambda+\gamma+\delta-T_{uu}-d_{vv}^0}x_{n_1,a,0,2,u,v}, \text{ for, } n_1 \geq 0,\ 1 \leq a \leq L-1, \\
&\quad 1 \leq u \leq m_1,\ 1 \leq v \leq n \\
w_{n_1,a+1,1,2,u,v} &= \tfrac{\lambda}{\lambda-T_{uu}-d_{vv}^0}x_{n_1,a,1,2,u,v}, \text{ for, } n_1 \geq 0,\ 1 \leq a \leq L-1,\ 1 \leq u \leq m_1, \\
&\quad 1 \leq v \leq n \\
w_{n_1,a+1,0,1,u,v} &= \tfrac{\lambda}{\lambda+\eta-\theta' S_{uu}-d_{vv}^0}x_{n_1,a,0,1,u,v}, \text{ for, } n_1 \geq 1,\ 0 \leq a \leq L-1, \\
&\quad 1 \leq u \leq m_2, 1 \leq v \leq n \\
w_{n_1,a+1,0,3,u,v} &= \tfrac{\lambda}{\lambda+\gamma-S_{uu}-d_{vv}^0}x_{n_1,a,0,3,u,v}, \text{ for, } n_1 \geq 1,\ 0 \leq a \leq L-1, \\
&\quad 1 \leq u \leq m_2,\ 1 \leq v \leq n
\end{aligned}$$

Thus we have the following Theorem.

Theorem 3.3.1. *The LST of the waiting time of a type I customer is* *given by*

$$W_1^*(s) = \frac{1}{d}\Big[\sum_{v=1}^{n} w_{0,1,0,2,u,v} + \sum_{n_1=0}^{\infty}\sum_{a=1}^{L-1}\sum_{u=1}^{m_1}\sum_{v=1}^{n} W^*(s/E_2)w_{n_1,a+1,0,0,u,v} + \sum_{n_1=0}^{\infty}\sum_{a=1}^{L-1}\sum_{u=1}^{m_1}\sum_{v=1}^{n} W^*(s/E_3)w_{n_1,a+1,0,2,u,v} + \sum_{n_1=0}^{\infty}\sum_{a=1}^{L-1}\sum_{u=1}^{m_1}\sum_{v=1}^{n} W^*(s/E_4)w_{n_1,a+1,1,2,u,v} + \sum_{n_1=1}^{\infty}\sum_{a=0}^{L-1}\sum_{u=1}^{m_2}\sum_{v=1}^{n} W^*(s/E_5)w_{n_1,a+1,0,1,u,v} + \sum_{n_1=1}^{\infty}\sum_{a=0}^{L-1}\sum_{u=1}^{m_2}\sum_{v=1}^{n} W^*(s/E_6)w_{n_1,a+1,0,3,u,v}\Big] \tag{3.9}$$

where,

$$d = \sum_{v=1}^{n} w_{0,1,0,2,u,v} + \sum_{n_1=0}^{\infty}\sum_{a=1}^{L-1}\sum_{u=1}^{m_1}\sum_{v=1}^{n} w_{n_1,a+1,0,0,u,v} + \sum_{n_1=0}^{\infty}\sum_{a=1}^{L-1}\sum_{u=1}^{m_1}\sum_{v=1}^{n} w_{n_1,a+1,0,2,u,v} + \sum_{n_1=0}^{\infty}\sum_{a=1}^{L-1}\sum_{u=1}^{m_1}\sum_{v=1}^{n} w_{n_1,a+1,1,2,u,v} + \sum_{n_1=1}^{\infty}\sum_{a=0}^{L-1}\sum_{u=1}^{m_2}\sum_{v=1}^{n} w_{n_1,a+1,0,1,u,v} + \sum_{n_1=1}^{\infty}\sum_{a=0}^{L-1}\sum_{u=1}^{m_2}\sum_{v=1}^{n} w_{n_1,a+1,0,3,u,v}$$

3.3.2 Type II customer

To find the LST of the waiting time distribution of a type II customer, we have to compute certain distributions. We proceed to such computations.

Definition 3.3.1. Consider the duration of time with p type I customers in the system at a service commencement epoch of type I customers until the number of type I customers become zero for the first time, we call this a *p-cycle*, denoted by B_p.

Distribution of a p-cycle

This is a phase type distribution with representation $(\boldsymbol{\gamma}_p, T_1)$ where the underlying Markov chain has state space $\{(i, j, k, l) : 1 \leq i \leq L;\ j = 0;\ k =$

$0\,or\,2;\ 1 \leq l \leq m_1\} \cup \{(i,j,k,l) : 1 \leq i \leq L;\ j = 1;\ k = 2;\ 1 \leq l \leq m_1\} \cup \{*\}$ and i, j, k, l and * respectively denote the number of type I customers in the system, the status of the protection clock, the status of the server, the service phase and the absorbing state indicating that the number of type I customers become zero. The infinitesimal generator $\mathcal{T}_1$ of $B_p(t)$ has the form

$$\mathcal{T}_1 = \begin{bmatrix} T_1 & \boldsymbol{T}_1^0 \\ 0 & 0 \end{bmatrix}, \text{where}\, T_1 = \begin{bmatrix} E_1 & \lambda I_{m_1} & & & \\ E_2 & E_1 & \lambda I_{m_1} & & \\ & \ddots & \ddots & \ddots & \\ & & E_2 & E_1 & \lambda I_{m_1} \\ & & & E_2 & E_3 \end{bmatrix}, \boldsymbol{T}_1^0 = \begin{bmatrix} E^0 \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}$$

where

$$E_1 = \begin{bmatrix} \theta T - (\lambda+\eta+\delta)I_{m_1} & \eta I_{m_1} & \delta I_{m_1} \\ \gamma I & T - (\lambda+\gamma+\delta)I_{m_1} & \delta I_{m_1} \\ 0 & 0 & \phi T - \lambda I_{m_1} \end{bmatrix}, E_2 = \begin{bmatrix} 0 & \theta \boldsymbol{T}^0 \boldsymbol{\alpha} & 0 \\ 0 & \boldsymbol{T}^0 \boldsymbol{\alpha} & 0 \\ 0 & \phi \boldsymbol{T}^0 \boldsymbol{\alpha} & 0 \end{bmatrix}.$$

$$E_3 = \begin{bmatrix} \theta T - (\eta+\delta)I_{m_1} & \eta I_{m_1} & \delta I_{m_1} \\ \gamma I_{m_1} & T - (\gamma+\delta)I_{m_1} & \delta I_{m_1} \\ 0 & 0 & \phi T \end{bmatrix} \text{and}\ E^0 = \begin{bmatrix} \theta \boldsymbol{T}^0 \\ \boldsymbol{T}^0 \\ \phi \boldsymbol{T}^0 \end{bmatrix}.$$

The initial probabilty vector is

$$\boldsymbol{\gamma}_p = \begin{bmatrix} \mathbf{0} & \cdots & \mathbf{0} & \boldsymbol{\gamma}' & \mathbf{0} & \cdots & \mathbf{0} \end{bmatrix}, 1 \leq p \leq L$$

where $\mathbf{0}$ s a zero matrix of order $1 \times 3m_1$, with $\boldsymbol{\gamma}' = \begin{bmatrix} \mathbf{0} & \boldsymbol{\alpha} & \mathbf{0} \end{bmatrix}, 1 \leq p \leq L$ is in the pth position and $\mathbf{0}$ is a zero matrix of order $1 \times m_1$.
Thus we have the following Theorem.

Theorem 3.3.2. *The LST of the length of a p-cycle is given by*

$$\boldsymbol{\gamma}_p (sI - T_1)^{-1} \boldsymbol{T}_1^0$$

LST of the busy cycle generated by type I customers arriving during the service time of a type II customer

Theorem 3.3.3. *The LST of the busy cycle generated by type I customers arriving during the service time of a type II customer is given by*

$$\hat{B}_{c_L}(s) = \boldsymbol{\beta'}[(s+\lambda)I - S_2]^{-1}\boldsymbol{S_2}^0 + \sum_{p=1}^{L-1} \boldsymbol{\gamma}_p(sI - T_1)^{-1}\boldsymbol{T}_1^0\lambda^p\boldsymbol{\beta'}[(s+\lambda)I - S_2]^{-(p+1)}\boldsymbol{S}_2^0$$
$$+\boldsymbol{\gamma}_L(sI-T_1)^{-1}\boldsymbol{T}_1^0\boldsymbol{\beta'}[\lambda^{-1}((s+\lambda)I-S_2)]^{-L}[I-\lambda[(s+\lambda)I-S_2]^{-1}]^{-1}[(s+\lambda)I-S_2]^{-1}\boldsymbol{S_2}^0 \tag{3.10}$$

Proof. Replace $\boldsymbol{\alpha'}$ by $\boldsymbol{\beta'}$, T' by S_2 and $\boldsymbol{T'}^0$ by $\boldsymbol{S}_2^0$ in the proof of Theorem 2.3.3. □

LST of the busy period of type I customers generated during the service time of a type II customer

Theorem 3.3.4. *The LST of the busy period generated by type I customers arriving during the service time of a type II customer is given by*

$$\hat{B}_L(s) = \boldsymbol{\beta'}[\lambda I-S_2]^{-1}\boldsymbol{S_2}^0+\sum_{p=1}^{L-1}\boldsymbol{\gamma}_p(sI-T_1)^{-1}\boldsymbol{T}_1^0\lambda^p\boldsymbol{\beta'}[\lambda I-S_2]^{-(p+1)}\boldsymbol{S}_2^0+\boldsymbol{\gamma}_L(sI-T_1)^{-1}$$
$$\boldsymbol{T}_1^0\boldsymbol{\beta'}[\lambda^{-1}(\lambda I - S_2)]^{-L}[I - \lambda[\lambda I - S_2]^{-1}]^{-1}[\lambda I - S_2]^{-1}\boldsymbol{S}_2^0 \tag{3.11}$$

Proof. Replace $\boldsymbol{\alpha'}$ by $\boldsymbol{\beta'}$, T' by S_2 and $\boldsymbol{T'}^0$ by $\boldsymbol{S}_2^0$ in the proof of Theorem 2.3.4. □

Now, to find the waiting time of a type II customer who joins for service at time t, we have to consider different possibilities depending on the status of server at that time. Let $W_2(t)$ be the waiting time of a type II customer who arrives at time t and $W_2^*(s)$ be the corresponding LST.

Case I

Suppose that F_1 denotes the event the system is in the state $(1, 0, 0, 3, u, v)$,

$1 \leq u \leq m_2$; $1 \leq v \leq n$ immediately after arrival of the tagged customer. Let $W_2^*(s/F_1)$ denote the corresponding LST.Then

$$W_2^*(s/F_1) = 1$$

Case II

F_2 be the event that the system is in one of the states $(b{+}1, a, 0, 0, u, v)$, $b \geq 0$; $1 \leq a \leq L$; $1 \leq u \leq m_1$; $1 \leq v \leq n$ immediately after arrival of the tagged customer. In this case, the waiting time is the length of the busy cycle generated by a type I customers starting from his arrival epoch plus lengths of busy cycles of type I customers generated during service times of each of the b type II customers. Let $W_2^*(s/F_2)$ denote the corresponding LST. Then

$$W_2^*(s/F_2) = \boldsymbol{e'}_{(a-1)3m_1+u}(3Lm_1)(sI - T_1)^{-1}\boldsymbol{T}_1^0(\hat{B}_{c_L}(s))^b$$

Case III

F_3 denote the event the system is in one of the states $(b{+}1, a, 0, 2, u, v)$, $b \geq 0$; $1 \leq a \leq L$; $1 \leq u \leq m_1$; $1 \leq v \leq n$ immediately after arrival of the tagged customer. In this case, the waiting time is the length of the busy cycle generated by a type I customers starting from his arrival epoch plus lengths of busy cycles of type I customers generated during service times of each of the b type II customers. Let $W_2^*(s/F_3)$ denote the corresponding LST.Then

$$W_2^*(s/F_3) = \boldsymbol{e'}_{(a-1)3m_1+m_1+u}(3Lm_1)(sI - T_1)^{-1}\boldsymbol{T}_1^0(\hat{B}_{c_L}(s))^b$$

Case IV

F_4 denote the event the system is in one of the states $(b{+}1, a, 1, 2, u, v)$, $b \geq 0$; $1 \leq a \leq L$; $1 \leq u \leq m_1$; $1 \leq v \leq n$ immediately after arrival of the tagged customer. In this case, the waiting time is the length of the busy cycle generated by a type I customers starting from his arrival epoch plus lengths of busy cycles of type I customers generated during service times of each of the b type II customers. Let $W_2^*(s/F_4)$ denote the corresponding LST. Then

$$W_2^*(s/F_4) = \boldsymbol{e'}_{(a-1)3m_1+2m_1+u}(3Lm_1)(sI - T_1)^{-1}\boldsymbol{T}_1^0(\hat{B}_{c_L}(s))^b$$

Case V

F_5 denote the event the system is in one of the states $(b+1, a, 0, 1, u, v)$, $b \geq 1$; $0 \leq a \leq L$; $1 \leq u \leq m_2$; $1 \leq v \leq n$ immediately after arrival of the tagged customer. In this case, the waiting time is the length of residual service time of the type II customer in service plus length of the busy period generated by type I customers arriving during the service time of the type II customer in service plus lengths of busy cycles of type I customers generated during service time of each of the $b-1$ type II customers. Let $W_2^*(s/F_5)$ denote the corresponding LST. Then

$$W_2^*(s/F_5) = \boldsymbol{e'}_u(2m_2)(sI - S_2)^{-1}\boldsymbol{S}_2^0 \hat{B}_L(s)(\hat{B}_{c_L}(s))^{b-1}$$

Case VI

F_6 denote the event the system is in one of the states $(b+1, a, 0, 3, u, v)$, $b \geq 1$; $0 \leq a \leq L$; $1 \leq u \leq m_2$; $1 \leq v \leq n$ immediately after arrival of the tagged customer. In this case the waiting time is the length of residual service time of the type II customer in service plus the length of the busy period generated by type I customers arriving during the service time of the type II customer in service plus lengths of busy cycles of type I customers generated during service time of each of the $b-1$ type II customers. Let $W_2^*(s/F_6)$ denote the corresponding LST.Then

$$W_2^*(s/F_6) = \boldsymbol{e'}_{m_2+u}(2m_2)(sI - S_2)^{-1}\boldsymbol{S}_2^0 \hat{B}_L(s)(\hat{B}_{c_L}(s))^{b-1}$$

Let $w_{i_1,i_2,j_1,j_2,k,l}$ denote the probabilty that the system is in the state

$(i_1, i_2, j_1, j_2, k, l)$ immedietly after arrival of the tagged customer. Then,

$$
\begin{aligned}
w_{1,0,0,3,u,v} &= \tfrac{d_{v'v}{}^{1}\beta_u}{\lambda - d^0_{v'v'}} w_{0,0,v'}, \text{ for, } 1 \le u \le m_2,\ 1 \le v, v' \le n \\
w_{b+1,a,0,0,u,v} &= \tfrac{d_{v'v}{}^{1}}{\lambda+\eta+\delta-\theta T_{uu}-d^0_{v'v'}} w_{b,a,0,0,u,v'}, \text{ for, } b \ge 1,\ 1 \le u \le m_1, \\
&\quad 1 \le v, v' \le n \\
w_{b+1,a,0,2,u,v} &= \tfrac{d_{v'v}{}^{1}}{\lambda+\gamma+\delta-T_{uu}-d^0_{v'v'}} w_{b,a,0,2,u,v'}, \text{ for, } b \ge 0,\ 1 \le a \le L,\ 1 \le u \le m_1, \\
&\quad 1 \le v, v' \le n \\
w_{b+1,a,1,2,u,v} &= \tfrac{d_{v'v}{}^{1}}{\lambda-T_{uu}-d^0_{v'v'}} w_{b,a,1,2,u,v'}, \text{ for, } b \ge 0,\ 1 \le a \le L,\ 1 \le u \le m_1, \\
&\quad 1 \le v, v' \le n \\
w_{b+1,a,0,1,u,v} &= \tfrac{d_{v'v}{}^{1}}{\lambda+\eta-\theta' S_{uu}-d^0_{v'v'}} w_{b,a,0,1,u,v'}, \text{ for, } b \ge 1,\ 0 \le a \le L,\ 1 \le u \le m_2, \\
&\quad 1 \le v, v' \le n \\
w_{b+1,a,0,3,u,v} &= \tfrac{d_{v'v}{}^{1}}{\lambda+\gamma-S_{uu}-d^0_{v'v'}} w_{b,a,0,3,u,v'}, \text{ for, } b \ge 1,\ 0 \le a \le L,\ 1 \le u \le m_2, \\
&\quad 1 \le v, v' \le n
\end{aligned}
$$

Thus we have the following Theorem.

Theorem 3.3.5. *The LST of the waiting time of a type II customer is given by*

$$
\begin{aligned}
W_2^*(s) = &\sum_{v=1}^{n} w_{1,0,0,3,u,v} + \sum_{b=0}^{\infty}\sum_{a=1}^{L}\sum_{u=1}^{m_1}\sum_{v=1}^{n} W^*(s/F_2) w_{b+1,a,0,0,u,v} + \\
&\sum_{b=0}^{\infty}\sum_{a=1}^{L}\sum_{u=1}^{m_1}\sum_{v=1}^{n} W^*(s/F_3) w_{b+1,a,0,2,u,v} + \sum_{b=0}^{\infty}\sum_{a=1}^{L}\sum_{u=1}^{m_1}\sum_{v=1}^{n} W^*(s/F_4) w_{b+1,a,1,2,u,v} + \\
&\sum_{b=1}^{\infty}\sum_{a=0}^{L}\sum_{u=1}^{m_2}\sum_{v=1}^{n} W^*(s/F_5) w_{b+1,a,0,1,u,v} + \sum_{b=1}^{\infty}\sum_{a=0}^{L}\sum_{u=1}^{m_2}\sum_{v=1}^{n} W^*(s/F_6) w_{b+1,a,0,3,u,v}
\end{aligned}
\tag{3.12}
$$

3.4 Expected number of interruptions during a single type I service

3.4.1 Distribution of duration of time till interruptions occur during a single type I service

Consider the Markov process, $\chi_1 = (N(t), J(t), K(t))$, where $N(t)$ denote the number of interruptions upto time t, $J(t)$, status of the server (providing normal or interrupted service) and $K(t)$, the service phase at time t. The state space of the process is given by $\{(0, 2, k) : 1 \leq k \leq m_1\} \cup \{(i, j, k) : i \geq 1;\, j = 0 \,\text{or}\, 2;\, 1 \leq k \leq m_1\} \cup \{*_1\} \cup \{*_2\}$ where $*_1$ denotes the absorbing state indicating the service completion and $*_2$ denotes the absorbing state indicating the realization of protection. The infinitesimal generator of the process is given by

$$\mathcal{U} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \delta\boldsymbol{e}(m_1) & \boldsymbol{T}^0 & T-(\gamma+\delta)I_{m_1} & \gamma I_{m_1} & 0 & 0 & 0 & \cdots \\ \delta\boldsymbol{e}(m_1) & \theta\boldsymbol{T}^0 & 0 & \theta T-(\eta+\delta)I_{m_1} & \eta I_{m_1} & 0 & 0 & \cdots \\ \delta\boldsymbol{e}(m_1) & \boldsymbol{T}^0 & 0 & 0 & T-(\gamma+\delta)I_{m_1} & \gamma I_{m_1} & 0 & \cdots \\ \delta\boldsymbol{e}(m_1) & \theta\boldsymbol{T}^0 & 0 & 0 & 0 & \theta T-(\eta+\delta)I_{m_1} & \eta I_{m_1} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$

3.4.2 Distribution of number of interruptions during a single type I service

Let y_k be the probabaility that the number of interruptions during a single type I service is k. Then y_k is the probabilty that the absorption occurs from the level k for the process χ_1. Hence y_k are given by

$$y_0 = -\boldsymbol{\alpha}(T - (\gamma + \delta)I))^{-1}(\boldsymbol{T}^0 + \delta\boldsymbol{e})$$

and for $k = 1, 2, 3, \ldots$

$$y_k = \boldsymbol{\alpha}(T-(\gamma+\delta)I)^{-1}\gamma I\big((\theta T-(\eta+\delta)I)^{-1}\eta I\,(T-(\gamma+\delta)I)^{-1}\gamma I\big)^{k-1}(\theta T-(\eta+\delta)I)^{-1}$$
$$\big((\theta \boldsymbol{T}^0+\delta \boldsymbol{e})-\eta I\,(T-(\gamma+\delta)I)^{-1}(\boldsymbol{T}^0+\delta\boldsymbol{e})\big) \quad (3.13)$$

Thus we have the following Theorem.

Theorem 3.4.1. *The expected number of interruptions during any particular type I customer service is given by*

$$E(i)=\sum_{k=0}^{\infty}ky_k=\boldsymbol{\alpha}(T-(\gamma+\delta)I)^{-1}\gamma I\left(\left(I-(\theta T-(\eta+\delta)I)^{-1}\eta I\,(T-(\gamma+\delta)I)^{-1}\gamma I\right)\right)^{-2}$$
$$(\theta T-(\eta+\delta)I)^{-1}\big((\theta\boldsymbol{T}^0+\delta\boldsymbol{e})-\eta I\,(T-(\gamma+\delta)I)^{-1}(\boldsymbol{T}^0+\delta\boldsymbol{e})\big). \quad (3.14)$$

3.5 Expected number of interruptions during a single type II service

3.5.1 Distribution of duration of time till interruptions occur during a single type II service

Consider the Markov process, $\chi_2 = (N(t), J(t), K(t))$, where $N(t)$ denote the number of interruptions, $J(t)$, status of the server (providing normal or interrupted service) and $K(t)$, the service phase at time t. The state space of the process of the process is given by $\{(0,3,k) : 1 \le k \le m_2\} \cup \{(i,j,k) : i \ge 1;\ j = 1 \,\text{or}\, 3; 1 \le k \le m_2\} \cup \{*\}$ where $*$ denotes the absorbing state indicating the service completion. The infinitesimal generator of the process is given by

$$\mathcal{U}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \boldsymbol{S}^0 & S-\gamma I_{m_2} & \gamma I_{m_2} & 0 & 0 & 0 & \cdots \\ \theta'\boldsymbol{S}^0 & 0 & \theta' S-\eta I_{m_2} & \eta I_{m_2} & 0 & 0 & \cdots \\ \boldsymbol{S}^0 & 0 & 0 & S-\gamma I_{m_2} & \gamma I_{m_2} & 0 & \cdots \\ \theta'\boldsymbol{S}^0 & 0 & 0 & 0 & \theta' S-\eta I_{m_2} & \eta I_{m_2} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}.$$

3.5.2 Distribution of number of interruptions during a single type II service

Let z_k be the probabaility that the number of interruptions during a single type II service is k. Then z_k is the probabilty that the absorption occurs from the level k for the process χ_2. Hence z_k are given by

$$z_0 = -\boldsymbol{\beta}(S-\gamma I))^{-1}\boldsymbol{S}^0$$

and for $k = 1, 2, 3, \ldots$

$$z_k = \boldsymbol{\beta}(S-\gamma I)^{-1}\gamma I\big((\theta' S-\eta I)^{-1}\eta I\,(S-\gamma I)^{-1}\gamma I\big)^{k-1}(\theta' S-\eta I)^{-1}\big(\theta'\boldsymbol{S}^0- \eta I\,(S-\gamma I)^{-1}\boldsymbol{S}^0\big) \quad (3.15)$$

Thus we have the following Theorem.

Theorem 3.5.1. *The expected number of interruptions during any particular type II customer service is given by*

$$E(i) = \sum_{k=0}^{\infty} k z_k = \boldsymbol{\beta}(S-\gamma I)^{-1}\gamma I\big(I-(\theta' S-\eta I)^{-1}\eta I\,(S-\gamma I)^{-1}\gamma I\big)^{-2} (\theta' S-\eta I)^{-1}\big(\theta'\boldsymbol{S}^0-\eta I\,(S-\gamma I)^{-1}\boldsymbol{S}^0\big). \quad (3.16)$$

3.6 Other Performance Measures

- The probability that the server is idle:

$$p_{idle} = \sum_{v=1}^{n} x_{0,v}.$$

- Mean number of type I customers in the system:

$$E_{nsh} = \sum_{n_1=0}^{\infty}\sum_{n_2=1}^{L}\sum_{u=1}^{m_1}\sum_{v=1}^{n} n_2 x_{n_1,n_2,0,0,u,v} + \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{L}\sum_{u=1}^{m_2}\sum_{v=1}^{n} n_2 x_{n_1,n_2,0,1,u,v} + \sum_{n_1=0}^{\infty}\sum_{n_2=1}^{L}\sum_{u=1}^{m_1}\sum_{v=1}^{n} n_2 x_{n_1,n_2,0,2,u,v} + \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{L}\sum_{u=1}^{m_2}\sum_{v=1}^{n} n_2 x_{n_1,n_2,0,3,u,v} + \sum_{n_1=0}^{\infty}\sum_{n_2=1}^{L}\sum_{u=1}^{m_1}\sum_{v=1}^{n} n_2 x_{n_1,n_2,1,2,u,v}$$

- Mean number of type II customers in the system:

$$E_{nsl} = \sum_{n_1=0}^{\infty} n_1 x_{n_1} \boldsymbol{e}$$

- The fraction of time during which the system is protected:

$$T_p = \sum_{n_1=0}^{\infty}\sum_{n_2=1}^{L}\sum_{u=1}^{m_1}\sum_{v=1}^{n} x_{n_1,n_2,1,2,u,v}$$

- The fraction of time the server is providing service to type I customers during WI:

$$T_{ih} = \sum_{n_1=0}^{\infty}\sum_{n_2=1}^{L}\sum_{u=1}^{m_1}\sum_{v=1}^{n} x_{n_1,n_2,0,0,u,v}$$

- The fraction of time the server is providing service to type II customers during WI:

$$T_{il} = \sum_{n_1=1}^{\infty}\sum_{n_2=0}^{L}\sum_{u=1}^{m_2}\sum_{v=1}^{n} x_{n_1,n_2,0,1,u,v}$$

- The fraction of time the server is providing service to type I customers in normal mode:

$$T_{nh} = \sum_{n_1=0}^{\infty}\sum_{n_2=1}^{L}\sum_{u=1}^{m_1}\sum_{v=1}^{n} x_{n_1,n_2,0,2,u,v}$$

- The fraction of time the server provides service to type II customers in normal mode:

$$T_{nl} = \sum_{n_1=1}^{\infty}\sum_{n_2=0}^{L}\sum_{u=1}^{m_2}\sum_{v=1}^{n} x_{n_1,n_2,0,3,u,v}$$

3.7 Analysis of a cost function

We construct a cost function based on the above performance measures.
Let
C_h: Holding cost for retaining a type I customer
C_l: Holding cost for retaining a type II customer
C_p: Unit time cost of providing service with protection
C_{ih}: Unit time cost of providing service when the server is providing service to type I customer in WI
C_{il}: Unit time cost of providing service when the server is providing service to type II customer in WI
C_{nh}: Unit time cost of providing service when the server is providing service to type I customer in normal mode
C_{nl}: Unit time cost of providing service when the server is providing service to type II customer in normal mode
Then the expected cost per unit time,

$$C = E_{nsh} \times C_h + E_{nsl} \times C_l + T_p \times \phi C_p + T_{ih} \times \theta C_{ih} + T_{il} \times \theta' C_{il} + T_{nh} \times C_{nh} + T_{nl} \times C_{nl}$$

3.8 Numerical Results

For the arrival process of type II customers, we consider the following two sets of matrices for D_0 and D_1 :

1. MAP with negetive correlation (MNA)

$$D_0 = \begin{bmatrix} -0.8101 & 0.8101 & 0 \\ 0 & -1.3497 & 0 \\ 0 & 0 & -40.5065 \end{bmatrix}, D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0.0810 & 0 & 1.2687 \\ 38.0761 & 0 & 2.4304 \end{bmatrix}$$

2. MAP with positive correlation (MPA)

$$D_0 = \begin{bmatrix} -0.8101 & 0.8101 & 0 \\ 0 & -1.3497 & 0 \\ 0 & 0 & -40.5065 \end{bmatrix}, D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 1.2687 & 0 & 0.0810 \\ 2.4304 & 0 & 38.0761 \end{bmatrix}$$

These two MAP processes are normalized so as to have an arrival rate of 1. The arrival process labeled MNA has correlated arrivals with correlation between two successive interarrival times given by -0.4211 and the arrival process corresponding to the one labelled MPA has a positive correlation with value 0.4211.

θ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
E_{nsh}	1.3493	1.2748	1.2194	1.1774	1.1450	1.1193	1.0985	1.0815	1.0672	1.0552
E_{nsl}	49.9733	19.8907	13.2051	10.3241	8.7368	7.7382	7.0548	6.5593	6.1843	5.8910
T_p	0.0334	0.0324	0.0318	0.0313	0.0308	0.0305	0.0302	0.0300	0.0298	0.0296
T_{ih}	0.1298	0.1104	0.0955	0.0838	0.0746	0.0672	0.0611	0.0559	0.0516	0.0479
T_{il}	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863
T_{nh}	0.3924	0.3988	0.4032	0.4063	0.4086	0.4103	0.4116	0.4126	0.4134	0.4141
T_{nl}	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482
C	33.7635	31.2805	30.9839	**30.9648**	31.0063	31.0595	31.1111	31.1575	31.1982	31.2335

Table 3.1: Effect of θ: Fix $L = 3, \theta' = 0, 6, \lambda = 2, \eta = 0.5, \delta = 1, \gamma = 0.6$ and $\phi = 4$

Tables 3.1 to 3.6 contain the effect of different parameters on various performance measures and on the cost function when the arrival process of type II customer is MNA and tables 7 to 12 contain the effect of different parameters on various performance measures and on the cost function when the arrival process of type II customer is MPA.

Table 3.1 indicates the effect of the parameter θ on various performance measures and the cost function.As θ increases, type I customers get faster service during WI and hence E_{nsh} decreases. Then more number of type II customers also get service and hence E_{nsl} also decreases. T_p and T_{ih} also decreases since the expected number of type I customers during WI decreases. As θ increases, T_{il} and T_{nl} remains fixed due to the diminished effect of θ on type II customers and T_{nh} increases due to the fact that the system stays in WI serving type I customers for lesser time and hence it stays more in normal mode serving type I customers. As θ increases, the system cost first decreases, reach an optimal value(30.9648) corresponding to $\theta = 0.4$ and then increases.

ϕ	1	1.5	2	2.5	3	3.5	4	4.5	5
E_{nsh}	1.3572	1.1902	1.1112	1.0658	1.0366	1.0162	1.0013	0.9898	0.9808
E_{nsl}	1.1787×10^4	12.1182	7.6530	6.1872	5.4634	5.0334	4.7491	4.5473	4.3968
T_p	0.1581	0.1087	0.0826	0.0665	0.0557	0.0479	0.0420	0.0374	0.0337
T_{ih}	0.0482	0.0497	0.0504	0.0507	0.0509	0.0511	0.0512	0.0513	0.0513
T_{il}	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863
T_{nh}	0.3591	0.3702	0.3751	0.3778	0.3795	0.3806	0.3814	0.3820	0.3824
T_{nl}	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482
C	1.2112×10^3	34.4147	**34.2737**	34.2923	34.3216	34.3469	34.3673	34.3837	34.3969

Table 3.2: Effect of ϕ: Fix $L = 3, \theta = 0.7, \theta' = 0,6, \lambda = 2, \eta = 0.5, \delta = 1.5$ and $\gamma = 0.6$

Table 3.2 indicates the effect of the parameter ϕ on various performance measures and the cost function. As ϕ increases, the type I customers in protected mode get faster service and hence E_{nsh} decreases. As a result, E_{nsl} also decreases. As expected T_p also decreases. As ϕ increases, T_{ih} and T_{nh} increase since T_p decreases. T_{il} and T_{nl} remains unchanged since ϕ has only a small effect on low priority customers. As ϕ increases, the system cost first decreases, reach an optimal value(34.2737) corresponding to $\phi = 2$ and then increases.

δ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
E_{nsh}	1.3590	1.3225	1.2883	1.2562	1.2260	1.1975	1.1706	1.1452	1.1212	1.0985
E_{nsl}	1071.6	57.1361	29.5220	19.9883	15.1618	12.2491	10.3021	8.9100	7.8661	7.0548
T_p	0.0035	0.0069	0.0102	0.0133	0.0164	0.0193	0.0222	0.0250	0.0276	0.0302
T_{ih}	0.0865	0.0831	0.0798	0.0767	0.0737	0.0709	0.0683	0.0657	0.0633	0.0611
T_{il}	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863
T_{nh}	0.4750	0.4674	0.4599	0.4526	0.4454	0.4384	0.4315	0.4247	0.4181	0.4116
T_{nl}	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482
C	129.7496	29.2871	27.4764	**27.4443**	27.8543	28.4282	29.0719	29.7453	30.4286	31.1111

Table 3.3: Effect of δ: Fix $L = 3, \theta = 0.7, \theta' = 0,6, \lambda = 2, \eta = 0.5, \gamma = 0.6$ and $\phi = 4$

Table 3.3 indicates the effect of the parameter δ on various performance measures and the cost function. As δ increases, protection clock realizes quickly and hence T_p increases, so T_{ih} and T_{nh} decreases. But T_{il} and T_{nl} remains unchanged since δ has only a small effect on low priority customers. In this case also, as δ increases, the system cost first decreases, reach an optimal value(27.4443) corresponding to $\delta = 0.4$ and then increases.

η	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
E_{nsh}	1.1161	1.1112	1.1067	1.1025	1.0985	1.0948	1.0913	1.0880	1.0848	1.0819
E_{nsl}	7.7025	7.5160	7.3475	7.1944	7.0548	6.9270	6.8096	6.7013	6.6012	6.5083
T_p	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302	0.0303	0.0303	0.0303
T_{ih}	0.0663	0.0649	0.0636	0.0623	0.0611	0.0599	0.0587	0.0576	0.0566	0.0555
T_{il}	0.0994	0.0958	0.0924	0.0893	0.0863	0.0836	0.0810	0.0785	0.0762	0.0740
T_{nh}	0.4058	0.4074	0.4089	0.4103	0.4116	0.4129	0.4141	0.4153	0.4165	0.4175
T_{nl}	0.3403	0.3425	0.3445	0.3464	0.3482	0.3499	0.3514	0.3529	0.3543	0.3556
C	31.6461	31.5012	31.3642	31.2344	31.1111	30.9939	30.8823	30.7759	30.6743	30.5772

Table 3.4: Effect of η: Fix $L = 3, \theta = 0.7, \theta' = 0,6, \lambda = 2, \eta = 0.5, \gamma = 0.6$ and $\phi = 4$

Table 3.4 indicates the effect of the parameter η on various performance measures and the cost function. As η increases, the server turns to normal mode quickly. Hence T_{nh} and T_{nl} increase and E_{nsh}, E_{nsl}, T_{ih} and T_{il} decrease. η has only a very small effect on T_p. The cost function decreases as η increases.

θ'	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
E_{nsh}	1.3367	1.1562	1.0535	0.9894	0.9467	0.9166	0.8945	0.8779	0.8649
E_{nsl}	56.6142	10.2087	6.2053	4.7515	4.0095	3.5628	3.2658	3.0547	2.8973
T_p	0.0464	0.0490	0.0504	0.0512	0.0517	0.0521	0.0523	0.0525	0.0527
T_{ih}	0.0369	0.0389	0.0400	0.0406	0.0410	0.0413	0.0415	0.0417	0.0418
T_{il}	0.2089	0.1576	0.1260	0.1049	0.0898	0.0785	0.0697	0.0627	0.0569
T_{nh}	0.3182	0.3357	0.3452	0.3508	0.3544	0.3568	0.3586	0.3599	0.3608
T_{nl}	0.3791	0.3685	0.3622	0.3580	0.3551	0.3529	0.3512	0.3499	0.3488
C	38.4471	**35.5315**	36.0777	36.5074	36.8103	37.0278	37.1887	37.3113	37.4072

Table 3.5: Effect of θ': Fix $L = 3, \theta = 0.7, \lambda = 2, \eta = 0.8, \delta = 2, \gamma = 0.6$ and $\phi = 4$

Table 3.5 indicates the effect of the parameter θ' on various performance measures and the cost function. As expected, T_{il} decreases and hence E_{nsl} and E_{nsh} decrease, T_{ih} , T_{nh} and T_p increase since type I customers have high priority. As a result, T_{nl} decreases. As θ' increases, the system cost first decreases, reach an optimal value(35.5315) corresponding to $\theta' = 0.2$ and then increases.

Table 3.6 indicates the effect of the parameter γ on various performance measures and the cost function. As γ increases, more interruptions occur during service and hence both E_{nsh} and E_{nsl} increases. T_p also increases in a slow rate. As γ increases T_{ih} and T_{il} increase and T_{nh} and T_{nl} decrease since the system stays more time in interruption mode. As γ increases, the cost

γ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
E_{nsh}	0.9997	1.0204	1.0407	1.0604	1.0797	1.0985	1.1169	1.1349	1.1525	1.1697
E_{nsl}	4.4562	4.8646	5.3190	5.8279	6.4019	7.0548	7.8046	8.6747	9.6973	10.9167
T_p	0.0301	0.0301	0.0302	0.0302	0.0302	0.0302	0.0302	0.0303	0.0303	0.0303
T_{ih}	0.0113	0.0220	0.0324	0.0423	0.0519	0.0611	0.0699	0.0784	0.0866	0.0945
T_{il}	0.0160	0.0313	0.0459	0.0599	0.0734	0.0863	0.0988	0.1107	0.1222	0.1333
T_{nh}	0.4586	0.4485	0.4378	0.4293	0.4203	0.4116	0.4033	0.3953	0.3876	0.3801
T_{nl}	0.3904	0.3812	0.3725	0.3640	0.3560	0.3482	0.3407	0.3336	0.3267	0.3200
C	27.0694	27.9294	28.7606	29.5661	30.3486	31.1111	31.8570	32.5901	33.3148	34.0369

Table 3.6: Effect of γ: Fix $L = 3, \theta = 0.7, \lambda = 2, \eta = 0.8, \delta = 2, \gamma = 0.6$ and $\phi = 4$

function increases. Note the sharpness in decrease of the value of E_{nsl} is quite pronounced. However the trend is not seen in table 4 which gives the effect of η.

θ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
E_{nsh}	1.3471	1.2716	1.2167	1.1761	1.1451	1.1208	1.1013	1.0853	1.0721	1.0609
E_{nsl}	343.0679	141.3074	96.1158	76.4713	65.5556	58.6331	53.8616	50.3784	47.7263	45.6412
T_p	0.0334	0.0324	0.0318	0.0313	0.0308	0.0305	0.0302	0.0300	0.0298	0.0296
T_{ih}	0.1298	0.1104	0.0955	0.0838	0.0746	0.0672	0.0611	0.0559	0.0516	0.0479
T_{il}	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863
T_{nh}	0.3924	0.3988	0.4032	0.4063	0.4086	0.4103	0.4116	0.4126	0.4134	0.4141
T_{nl}	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482
C	63.0719	43.4206	39.2737	37.5789	36.6882	36.1497	35.7932	35.5413	35.3548	35.2114

Table 3.7: Effect of θ: Fix $L = 3, \theta' = 0, 6, \lambda = 2, \eta = 0.5, \delta = 1, \gamma = 0.6$ and $\phi = 4$

Table 3.7 indicates the effect of the parameter θ on various performance measures and the cost function. In this case also E_{nsh} and E_{nsl} decreases as θ increases. But the values of E_{nsl} is much high when the arrival process of type II customer is MPA. All other values are same as in the case of MNA. But the cost function decreases as θ increases.

ϕ	1	1.5	2	2.5	3	3.5	4	4.5	5
E_{nsh}	1.3571	1.1900	1.1141	1.0711	1.0436	1.0244	1.0104	0.9996	0.9911
E_{nsl}	4.4374×10^4	90.9874	58.6062	47.8674	42.5211	39.3245	37.1995	35.6852	34.5516
T_p	0.1581	0.1087	0.0826	0.0665	0.0557	0.0479	0.0420	0.0374	0.0337
T_{ih}	0.0482	0.0497	0.0504	0.0507	0.0509	0.0511	0.0512	0.0513	0.0513
T_{il}	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863
T_{nh}	0.3591	0.3702	0.3751	0.3778	0.3795	0.3806	0.3814	0.3820	0.3824
T_{nl}	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482
C	4.4699×10^3	42.3015	39.3705	38.4629	38.0309	37.7801	37.6169	37.5023	37.4175

Table 3.8: Effect of ϕ: Fix $L = 3, \theta = 0.7, \theta' = 0, 6, \lambda = 2, \eta = 0.5, \delta = 1.5$ and $\gamma = 0.6$

Table 3.8 indicates the effect of the parameter ϕ on various performance measures and the cost function. Both E_{nsh} and E_{nsl} decrease as ϕ increases. The cost function and E_{nsl} decreases sharply as ϕ increases from 1 to 1.5. However, with further increase in ϕ value does not produce that decrease in values of cost function and E_{nsl}.

δ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
E_{nsh}	1.3589	1.3214	1.2867	1.2545	1.2244	1.1965	1.1703	1.1458	1.1229	1.1013
E_{nsl}	7519.70	411.63	214.53	146.44	111.95	91.12	77.17	67.19	59.70	53.86
T_p	0.0035	0.0069	0.0102	0.0133	0.0164	0.0193	0.0222	0.0250	0.0276	0.0302
T_{ih}	0.0865	0.0831	0.0798	0.0767	0.0737	0.0709	0.0683	0.0657	0.0633	0.0611
T_{il}	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863	0.0863
T_{nh}	0.4750	0.4674	0.4599	0.4526	0.4454	0.4384	0.4315	0.4247	0.4181	0.4116
T_{nl}	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482	0.3482
C	774.5584	64.7362	45.9761	40.0889	37.5324	36.3143	35.7588	**35.5737**	35.6123	35.7932

Table 3.9: Effect of δ: Fix $L = 3, \theta = 0.7, \theta' = 0,6, \lambda = 2, \eta = 0.5, \gamma = 0.6$ and $\phi = 4$

Table 3.9 indicates the effect of the parameter δ on various performance measures and the cost function. Both E_{nsh} and E_{nsl} decrease as δ increases.In this case, as δ increases, the system cost first decreases, reaches an optimal value(35.5737) corresponding to $\delta = 0.8$ and then increases. Both E_{nsl} and the cost show sharp decrease in their values when δ moves from 0.1 to 0.2. Thereafter the decrease is not that pronounced.

η	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
E_{nsh}	1.1184	1.1136	1.1093	1.1051	1.1013	1.0976	1.0942	1.0910	1.0880	1.0851
E_{nsl}	58.6679	57.2868	56.0367	54.8999	53.8616	52.9096	52.0337	51.2250	50.4761	49.7807
T_p	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302	0.0302	0.0303	0.0303	0.0303
T_{ih}	0.0663	0.0649	0.0636	0.0623	0.0611	0.0599	0.0587	0.0576	0.0566	0.0555
T_{il}	0.0994	0.0958	0.0924	0.0893	0.0863	0.0836	0.0810	0.0785	0.0762	0.0740
T_{nh}	0.4058	0.4074	0.4089	0.4103	0.4116	0.4129	0.4141	0.4153	0.4165	0.4175
T_{nl}	0.3403	0.3425	0.3445	0.3464	0.3482	0.3499	0.3514	0.3529	0.3543	0.3556
C	36.7438	36.4795	36.2344	36.0062	35.7932	35.5936	35.4062	35.2298	35.0634	34.9061

Table 3.10: Effect of η Fix $L = 3, \theta = 0.7, \theta' = 0,6, \lambda = 2, \eta = 0.5, \gamma = 0.6$ and $\phi = 4$

Table 3.10 indicates the effect of the parameter η on various performance measures and the cost function. Both E_{nsh} and E_{nsl} decrease as η increases.The cost fuction decreases as η increases.

Table 3.11 indicates the effect of the parameter θ' on various perfor-

θ'	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
E_{nsh}	1.3380	1.1630	1.0642	1.0027	0.9616	0.9325	0.9111	0.8947	0.8819	0.8716
E_{nsl}	417.6867	79.1550	49.1289	37.9315	32.0828	28.4903	26.0604	24.3078	22.9843	21.9498
T_p	0.0464	0.0490	0.0504	0.0512	0.0517	0.0521	0.0523	0.0525	0.0527	0.0528
T_{ih}	0.0369	0.0389	0.0400	0.0406	0.0410	0.0413	0.0415	0.0417	0.0418	0.0419
T_{il}	0.2089	0.1576	0.1260	0.1049	0.0898	0.0785	0.0697	0.0627	0.0569	0.0521
T_{nh}	0.3182	0.3357	0.3452	0.3508	0.3544	0.3568	0.3586	0.3599	0.3608	0.3616
T_{nl}	0.3791	0.3685	0.3622	0.3580	0.3551	0.3529	0.3512	0.3499	0.3488	0.3479
C	74.5558	42.4295	40.3754	39.8320	39.6251	39.5285	39.4764	39.4450	39.4244	39.4098

Table 3.11: Effect of θ': Fix $L = 3, \theta = 0.7, \lambda = 2, \eta = 0.8, \delta = 2, \gamma = 0.6$ and $\phi = 4$

mance measures and the cost function. Both E_{nsh} and E_{nsl} decrease as θ' increases.The cost fuction decreases as θ' increases, as it is to be expected. However, there is a sharp decrease in value of E_{nsl} when θ' moves from 0.1 to 0.2. For higher values of θ', the initial sharpness in decrease is not seen.

γ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
E_{nsh}	1.0050	1.0251	1.0448	1.0640	1.0829	1.1013	1.1193	1.1369	1.1542	1.1711
E_{nsl}	34.0325	37.1338	40.5924	44.4740	48.8618	53.8616	59.6115	66.2942	74.1569	83.5428
T_p	0.0301	0.0301	0.0302	0.0302	0.0302	0.0302	0.0302	0.0303	0.0303	0.0303
T_{ih}	0.0113	0.0220	0.0324	0.0423	0.0519	0.0611	0.0699	0.0784	0.0866	0.0945
T_{il}	0.0160	0.0313	0.0459	0.0599	0.0734	0.0863	0.0988	0.1107	0.1222	0.1333
T_{nh}	0.4586	0.4485	0.4378	0.4293	0.4203	0.4116	0.4033	0.3953	0.3876	0.3801
T_{nl}	0.3904	0.3812	0.3725	0.3640	0.3560	0.3482	0.3407	0.3336	0.3267	0.3200
C	30.0298	31.1586	32.2900	33.4325	34.5961	35.7932	37.0389	38.3530	39.7616	41.3003

Table 3.12: Effect of γ: Fix $L = 3, \theta = 0.7, \lambda = 2, \eta = 0.8, \delta = 2, \gamma = 0.6$ and $\phi = 4$

Table 3.12 indicates the effect of the parameter γ on various performance measures and the cost function. Both E_{nsh} and E_{nsl} increase as γ increases. As expected, the cost increases as γ increases.

Chapter 4

On a Queueing System with processing of Service items under Vacation and N-policy

The motivation for this chapter is two papers by Kazimirsky[23] and Gabi Hanukov et al. [17]. In those the authors analyzed a single server queue in which the service consists of two independent stages. The first stage can be performed even in the absence of customers, whereas the second stage requires the customer to be present. When there is no customer in the system, the server produces an inventory of first stage called 'preliminary ' services, which is used to reduce customer's overall sojourn times. Hence in those models customer will not have to wait for the entire service to be carried out from the beginning, provided processed item is available at the time the customer is taken for service. Such customers have a shorter service time in comparison to those who encounter the system with no processed item when taken for

Some results of this chapter are included in the following paper.
V. Divya, A. Krishnamoorthy , V. M. Vishnevsky: **On a Queueing System with processing of Service items under Vacation and N-policy** DCCN 2018, CCIS 919, pp. 43-57, Springer Nature Switzerland AG 2018.

service.

Yadin and Naor [51] introduced the concept of N-policy in which the server turns on with the accumulation of N or more customers and turns off when the system is empty. This has the advantage that the length of a busy period becomes larger when server is activated on accumulation of N or more customers, thereby bringing down the expected cost incurred per unit time.

We consider a single server queueing system in which customers arrive according to Markovian Arrival process. When the system is empty, the server goes for vacation and produces inventory for future use during this period. Maximum inventory level is fixed as L. Processing time for each item of inventory follows phase type distribution. The server returns from vacation when there are N customers in the system. The service time follows two distinct phase type distributions according to whether there is no processed item or there are processed items at the beginning of service. Each customer requires an item from inventory for service which is used exclusively for the service of that particular customer only.

4.1 Model Description and Mathematical formulation

We assume that customers arrive at a single server queueing system according to MAP with representation (D_0, D_1) of order n. When the system is empty, the server goes for vacation and produces inventory for future use during this period. The maximum inventory level permitted is L. The inventory processing time follows phase type distribution PH$(\boldsymbol{\alpha}, T)$ of order m_1. These are required for the service of customers - one for each customer. The server returns from vacation when N customers accumulate in the system. The service time follows the PH$(\boldsymbol{\beta}, S)$ distribution of order m_2 when there is no processed item and it follows PH$(\boldsymbol{\gamma}, U)$ of order m_3 when there are processed items with $PH(\boldsymbol{\beta}, S) \underset{st}{\prec} PH(\boldsymbol{\gamma}, U)$.

Let $Q^* = D_0 + D_1$ be the generator matrix of the arrival process and $\boldsymbol{\pi^*}$ be its stationary probability vector. Hence $\boldsymbol{\pi^*}$ is the unique (positive) probability vector satisfying

$$\boldsymbol{\pi^*} Q^* = 0, \ \ \boldsymbol{\pi^*} \boldsymbol{e} = 1.$$

The constant $\beta^* = \boldsymbol{\pi^*} D_1 \boldsymbol{e}$, referred to as *fundemental rate*, gives the expected number of arrivals per unit of time in the stationary version of the MAP. It is assumed that the arrival process is independent of the inventory processing and service process.

4.1.1 The QBD process

The model described above can be studied as a LIQBD process. First we introduce the following notations:
At time t:
$N(t)$: the number of customers in the system
$I(t)$: the number of processed inventory

$$J(t) = \begin{cases} 0, when\, the\, server\, is\, on\, vacation \\ 1, when\, the\, server\, is\, busy\, serving\, customer \end{cases}$$

$K(t)$: the phase of the inventory processing/service process
$M(t)$: the phase of arrival of the customer.
It is easy to verify that $\{(N(t), I(t), J(t), K(t), M(t)) : t \geq 0\}$ is a LIQBD with state space: (i) corresponding to no customer in the system
$l(0) = \{(0, i, 0, k_1, l) : 0 \leq i \leq L-1;\ 1 \leq k_1 \leq m_1;\ 1 \leq l \leq n\} \cup \{(0, L, 0, l) : 1 \leq l \leq n\}$.
(ii) when there are h customers in the system, for $1 \leq h \leq N-1$:
$l(h) = \{(h, i, 0, k_1, l) : 0 \leq i \leq L-1;\ 1 \leq k_1 \leq m_1;\ 1 \leq l \leq n\} \cup \{(h, L, 0, l) : 1 \leq l \leq n\} \cup \{(h, 0, 1, k_2, l) : 1 \leq k_2 \leq m_2;\ 1 \leq l \leq n\} \cup \{(h, i, 1, k_3, l) : 1 \leq i \leq L-N+h;\ 1 \leq k_3 \leq m_3;\ 1 \leq l \leq n\}$(last part only when $L - N + h > 0$)
and (iii)for $h \geq N$:

$l(h) = \{(h,0,1,k_2,l) : 1 \le k_2 \le m_2;\ 1 \le l \le n\} \cup \{(h,i,1,k_3,l) : 1 \le i \le L;\ 1 \le k_3 \le m_3;\ 1 \le l \le n\}$.
The infinitesimal generator of this CTMC is

$$\mathcal{Q} = \begin{bmatrix} E_0 & F_0 & & & & & & & \\ B_1 & E_1 & F_1 & & & & & & \\ & B_2 & E_2 & F_2 & & & & & \\ & & \ddots & \ddots & \ddots & & & & \\ & & B_{N-2} & E_{N-2} & F_{N-2} & & & & \\ & & & B_{N-1} & E_{N-1} & F'_{N-1} & & & \\ & & & & B'_N & A_1 & A_0 & & \\ & & & & & A_2 & A_1 & A_0 & \\ & & & & & & \ddots & \ddots & \ddots \end{bmatrix}.$$

The boundary blocks $E_0, F_0, B_1, F'_{N-1}, B'_N$ are of orders $(Lm_1+1)n \times (Lm_1+1)n$, $(Lm_1+1)n \times \big((Lm_1+1)n + m_2n + (L-N+1)m_3n\big)$, $\big((Lm_1+1)n+m_2n+(L-N+1)m_3n\big) \times (Lm_1+1)n$, $\big((m_1+m_2)n+(L-1)(m_1+m_3)n+n\big) \times (m_2+Lm_3)n$, $(m_2+Lm_3)n \times \big((m_1+m_2)n+(L-1)(m_1+m_3)n+n\big)$, respectively. For $2 \le h \le N-1$, B_h is of order $\big((m_1+m_2)n+(L-N+h)(m_1+m_3)n+(N-h-1)m_1n+n\big) \times \big(m_1+m_2)n+(L-N+h-1)(m_1+m_3)n+(N-h)m_1n+n\big)$. For $1 \le h \le N-1$, E_h is of order $\big((m_1+m_2)n+(L-N+h)(m_1+m_3)n+(N-h-1)m_1n+n\big) \times \big((m_1+m_2)n+(L-N+h)(m_1+m_3)n+(N-h-1)m_1n+n\big)$. For $1 \le h \le N-2$, F_h is of order $\big((m_1+m_2)n+(L-N+h)(m_1+m_3)n+(N-h-1)m_1n+n\big) \times \big((m_1+m_2)n+(L-N+h+1)(m_1+m_3)n+(N-h-2)m_1n+n\big)$. A_0, A_1, A_2 are square matrices of order $(m_2+Lm_3)n$. Define the entries $E_{0_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)}$, $F_{0_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)}$, $B_{1_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)}$, $E_{h_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)}$, $B_{h_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)}$, $F_{h_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)}$, ${F'}_{N-1}{}_{(i_1,j_1,k_1,l_1)}^{(i_2,j_2,k_2,l_2)}$ and ${B'}_N{}_{(i_1,j_1,k_1,l_1)}^{(i_2,j_2,k_2,l_2)}$ as transition submatrices which contains transitions of the form $(0,i_1,j_1,k_1,l_1) \to (0,i_2,j_2,k_2,l_2)$, $(0,i_1,j_1,k_1,l_1) \to (1,i_2,j_2,k_2,l_2)$, $(1,i_1,j_1,k_1,l_1) \to (0,i_2,j_2,k_2,l_2)$, $(h,i_1,j_1,k_1,l_1) \to (h,i_2,j_2,$

$k_2, l_2)$, where $1 \le h \le N-1$; $(h, i_1, j_1, k_1, l_1) \rightarrow (h-1, i_2, j_2, k_2, l_2)$, where $2 \le h \le N-1$; $(h, i_1, j_1, k_1, l_1) \rightarrow (h+1, i_2, j_2, k_2, l_2)$, where $1 \le h \le N-2$; $(N-1, i_1, j_1, k_1, l_1) \rightarrow (N, i_2, j_2, k_2, l_2)$ and $(N, i_1, j_1, k_1, l_1) \rightarrow (N-1, i_2, j_2, k_2, l_2)$ respectively. Define the entries $A_{2_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)}, A_{1_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)}$ and $A_{0_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)}$ as transition submatrices which contains transitions of the form $(h, i_1, j_1, k_1, l_1) \rightarrow (h-1, i_2, j_2, k_2, l_2)$, where $h \ge N+1$; $(h, i_1, j_1, k_1, l_1) \rightarrow (h, i_2, j_2, k_2, l_2)$, for $h \ge N$ and $(h, i_1, j_1, k_1, l_1) \rightarrow (h+1, i_2, j_2, k_2, l_2)$, with $h \ge N$ respectively. Since none or one event alone could take place in a short interval of time with positive probability, in general a transition such as $(h_1, i_1, j_1, k_1, l_1) \rightarrow (h_2, i_2, j_2, k_2, l_2)$ has positive rate only for exactly one of h_1, i_1, j_1, k_1, l_1 different from h_2, i_2, j_2, k_2, l_2.

$$E_{0_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)} = \begin{cases} \boldsymbol{T}^0\boldsymbol{\alpha} \otimes I_n & i_2 = i_1+1, 0 \le i_1 \le L-2; j_1 = j_2 = 0; \\ & 1 \le k_1, k_2 \le m_1; 1 \le l_1, l_2 \le n \\ \boldsymbol{T}^0 \otimes I_n & i_1 = L-1, i_2 = L; j_1 = j_2 = 0; 1 \le k_1, k_2 \le m_1; \\ & 1 \le l_1, l_2 \le n \\ T \oplus D_0 & i_1 = i_2, 0 \le i_1 \le L-1; j_1 = j_2 = 0; 1 \le k_1, k_2 \le m_1; \\ & 1 \le l_1, l_2 \le n \\ D_0 & i_1 = i_2 = L; j_1 = j_2 = 0; 1 \le l_1, l_2 \le n \end{cases}$$

$$F_{0_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)} = \begin{cases} I_{m_1} \otimes D_1 & 0 \le i_1 \le L-1, i_1 = i_2; j_1 = j_2 = 0; 1 \le k_1, k_2, \le m_1; \\ & 1 \le l_1, l_2 \le n \\ D_1 & i_1 = i_2 = L; j_1 = j_2 = 0; 1 \le l_1, l_2 \le n \end{cases}$$

$$B_{1_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)} = \begin{cases} \boldsymbol{S}^0\boldsymbol{\alpha} \otimes I_n & i_1 = i_2 = 0; j_1 = 1, j_2 = 0; 1 \le k_1 \le m_2, \\ & 1 \le k_2 \le m_1; 1 \le l_1, l_2 \le n \\ \boldsymbol{U}^0\boldsymbol{\alpha} \otimes I_n & 1 \le i_1 \le L-N+1, i_2 = i_1 - 1; j_1 = 1, j_2 = 0; \\ & 1 \le k_1 \le m_3, 1 \le k_2 \le m_1; 1 \le l_1, l_2 \le n \end{cases}$$

For $1 \leq h \leq N-1$,

$$E_{h_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)} = \begin{cases} \boldsymbol{T}^0\boldsymbol{\alpha} \otimes I_n & 0 \leq i_1 \leq L-2, i_2 = i_1+1; j_1 = j_2 = 0; \\ & 1 \leq k_1, k_2 \leq m_1; 1 \leq l_1, l_2 \leq n \\ \boldsymbol{T}^0 \otimes I_n & i_1 = L-1, i_2 = L; j_1 = j_2 = 0; \\ & 1 \leq k_1 \leq m_1; 1 \leq l_1, l_2 \leq n \\ T \oplus D_0 & i_1 = i_2, 0 \leq i_1 \leq L-1; j_1 = j_2 = 0; 1 \leq k_1, k_2 \leq m_1; \\ & 1 \leq l_1, l_2 \leq n \\ S \oplus D_0 & i_1 = i_2 = 0; j_1 = j_2 = 1; 1 \leq k_1, k_2 \leq m_2; \\ & 1 \leq l_1, l_2 \leq n \\ U \oplus D_0 & i_1 = i_2, 1 \leq i_1 \leq L-N+h; j_1 = j_2 = 1; \\ & 1 \leq k_1, k_2 \leq m_3; 1 \leq l_1, l_2 \leq n \\ D_0 & i_1 = i_2 = L; j_1 = j_2 = 0; 1 \leq l_1, l_2 \leq n \end{cases}$$

For $2 \leq h \leq N-1$,

$$B_{h_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)} = \begin{cases} \boldsymbol{S}^0\boldsymbol{\beta} \otimes I_n & i_1 = i_2 = 0; j_1 = j_2 = 1; 1 \leq k_1, k_2 \leq m_2; \\ & 1 \leq l_1, l_2 \leq n \\ \boldsymbol{U}^0\boldsymbol{\beta} \otimes I_n & i_1 = 1, i_2 = 0; j_1 = j_2 = 1; 1 \leq k_1 \leq m_3, \\ & 1 \leq k_2 \leq m_2; 1 \leq l_1, l_2 \leq n \\ \boldsymbol{U}^0\boldsymbol{\gamma} \otimes I_n & 2 \leq i_1 \leq L-N+h, i_2 = i_1-1; j_1 = j_2 = 1; \\ & 1 \leq k_1, k_2 \leq m_3; 1 \leq l_1, l_2 \leq n \end{cases}$$

For $1 \leq h \leq N-2$,

$$F_{h_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)} = \begin{cases} I_{m_1} \otimes D_1 & 0 \leq i_1 \leq L-1, i_1 = i_2; j_1 = j_2 = 0; 1 \leq k_1, k_2 \leq m_1; \\ & 1 \leq l_1, l_2 \leq n \\ I_{m_2} \otimes D_1 & i_2 = i_1 = 0; j_1 = j_2 = 1; 1 \leq k_1, k_2 \leq m_2, 1 \leq l_1, l_2 \leq n \\ I_{m_3} \otimes D_1 & i_2 = i_1, 1 \leq i_1 \leq L-N+h; j_1 = j_2 = 1; \\ & 1 \leq k_1, k_2 \leq m_3, 1 \leq l_1, l_2 \leq n \\ D_1 & i_1 = i_2 = L; j_1 = j_2 = 0; 1 \leq k_1, k_2 \leq m_1; 1 \leq l_1, l_2 \leq n \end{cases}$$

$$F'_{N-1}{}^{(i_2,j_2,k_2,l_2)}_{(i_1,j_1,k_1,l_1)} = \begin{cases} \boldsymbol{e}(m_1)\otimes(\boldsymbol{\beta}\otimes D_1) & i_1=i_2=0; j_1=0, j_2=1; 1\le k_1\le m_1, \\ & 1\le k_2\le m_2; 1\le l_1,l_2\le n \\ I_{m_2}\otimes D_1 & i_2=i_1=0; j_1=j_2=1; 1\le k_1,k_2\le m_2; \\ & 1\le l_1,l_2\le n \\ I_{m_3}\otimes D_1 & i_2=i_1, 0\le i_1\le L-1; j_1=j_2=1;, \\ & 1\le k_1,k_2\le m_3, 1\le l_1,l_2\le n \\ \boldsymbol{e}(m_1)\otimes(\boldsymbol{\gamma}\otimes D_1) & 1\le i_1\le L-1; j_1=0, j_2=1; 1\le k_1\le m_1, \\ & 1\le k_2\le m_3; 1\le l_1,l_2\le n \\ \boldsymbol{\gamma}\otimes D_1 & i_1=i_2=L; j_1=0, j_2=1; 1\le k_1\le m_1, \\ & 1\le k_2\le m_3; 1\le l_1,l_2\le n \end{cases}$$

$$B'_N{}^{(i_2,j_2,k_2,l_2)}_{(i_1,j_1,k_1,l_1)} = \begin{cases} \boldsymbol{S}^0\boldsymbol{\beta}\otimes I_n & i_1=i_2=0; j_1=j_2=1; 1\le k_1,k_2\le m_2,; \\ & 1\le l_1,l_2\le n \\ \boldsymbol{U}^0\boldsymbol{\beta}\otimes I_n & i_1=1, i_2=0; j_1=j_2=1; 1\le k_1\le m_3, \\ & 1\le k_2\le m_2; 1\le l_1,l_2\le n \\ \boldsymbol{U}^0\boldsymbol{\gamma}\otimes I_n & 2\le i_1\le L, i_2=i_1-1; j_1=j_2=1; 1\le k_1,k_2\le m_3; \\ & 1\le l_1,l_2\le n \end{cases}$$

$$A_{2_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)} = \begin{cases} \boldsymbol{S}^0\boldsymbol{\beta}\otimes I_n & i_1=i_2=0; j_1=j_2=1; 1\le k_1,k_2\le m_2; \\ & 1\le l_1,l_2\le n \\ \boldsymbol{U}^0\boldsymbol{\beta}\otimes I_n & i_1=1, i_2=0; j_1=j_2=1; 1\le k_1\le m_3, 1\le k_2\le m_2; \\ & 1\le l_1,l_2\le n \\ \boldsymbol{U}^0\boldsymbol{\gamma}\otimes I_n & i_2=i_1-1, 2\le i_2\le L; j_1=j_2=1; 1\le k_1,k_2\le m_3; \\ & 1\le l_1,l_2\le n \end{cases}$$

$$A_{1_{(h,i_1,j_1,k_1,l_1)}}^{(h,i_2,j_2,k_2,l_2)} = \begin{cases} S\oplus D_0 & i_1=i_2=0; j_1=j_2=1; 1\le k_1,k_2\le m_2; 1\le l_1,l_2\le n \\ U\oplus D_0 & i_1=i_2, 1\le i_1\le L; j_1=j_2=1; 1\le k_1,k_2\le m_3; \\ & 1\le l_1,l_2\le n \end{cases}$$

$$A_{0_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)} = \begin{cases} I_{m_2} \otimes D_1 & i_1 = i_2 = 0; j_1 = j_2 = 1; 1 \leq k_1, k_2 \leq m_2; \\ & 1 \leq l_1, l_2 \leq n \\ I_{m_3} \otimes D_1 & i_1 = i_2, 1 \leq i_1 \leq L; j_1 = j_2 = 1; 1 \leq k_1, k_2 \leq m_3; \\ & 1 \leq l_1, l_2 \leq n \end{cases}$$

Next we proceed for the steady state analysis of the system described.

4.2 Steady State Analysis

To this end we first obtain the

4.2.1 Stability condition

The stabilty condition for the system is given by

Lemma 4.2.1. The system is stable iff

$$\boldsymbol{\pi}^* D_1 \boldsymbol{e} < (\boldsymbol{\beta}(-S)^{-1}\boldsymbol{e})^{-1} \tag{4.1}$$

4.2.2 Steady-state probability vector

Assuming that the condition (4.1) is satisfied we proceed to find the steady-state probability of the system state.

Let $\boldsymbol{x}$ be the steady state probability vector of Q. We partition this vector as

$$\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2 \ldots),$$

where $\boldsymbol{x}_0$ is of dimension $(Lm_1 + 1)n$, $\boldsymbol{x}_h$, $1 \leq h \leq N-1$ are of dimension $(m_1+m_2)n + (L-N+h)(m_1+m_3)n + (N-h-1)m_1n + n$ and $\boldsymbol{x}_N, \boldsymbol{x}_{N+1} \ldots$ are of dimension $(m_2 + Lm_3)n$. Under the stability condition, we have

$$\boldsymbol{x}_{N+i} = \boldsymbol{x}_N R^i, i \geq 1$$

where the matrix R is the minimal nonnegative solution to the matrix quadratic equation

$$R^2A_2 + RA_1 + A_0 = 0$$

and the vectors $\boldsymbol{x}_0, \boldsymbol{x}_1, \cdots, \boldsymbol{x}_N \ldots$are obtained by solving the equations

$$\boldsymbol{x}_0E_0 + \boldsymbol{x}_1B_1 = 0 \tag{4.2}$$

$$\boldsymbol{x}_0F_0 + \boldsymbol{x}_1E_1 + \boldsymbol{x}_2B_2 = 0 \tag{4.3}$$

$$\boldsymbol{x}_{i-1}F_{i-1} + \boldsymbol{x}_iE_i + \boldsymbol{x}_{i+1}B_{i+1} = 0, \text{ for } 2 \leq i \leq N-2 \tag{4.4}$$

$$\boldsymbol{x}_{N-2}F_{N-2} + \boldsymbol{x}_{N-1}E_{N-1} + \boldsymbol{x}_NB_{N'} = 0 \tag{4.5}$$

$$\boldsymbol{x}_{N-1}F'_{N-1} + \boldsymbol{x}_N(A_1 + RA_2) = 0 \tag{4.6}$$

(4.7)

subject to the normalizing condition

$$\sum_{i=0}^{N-1} \boldsymbol{x}_i\boldsymbol{e} + \boldsymbol{x}_N(I-R)^{-1}\boldsymbol{e} = 1 \tag{4.8}$$

Remark:

Our model reduces to Hanukov et al. [17] if we assume $N = 1$, restrict MAP arrival process to Poisson process of rate λ, phase type processing time to exponential distribution of mean duration $\frac{1}{\mu_1}$, phase type service time when there is no processed item to two exponential stages with mean durations $\frac{1}{\mu_2}$ and $\frac{1}{\mu_3}$ and phase type service time when there is processed item to a single exponential stage of mean duration $\frac{1}{\mu_3}$. Clearly the stability condition

$$\boldsymbol{\pi^*}D_1\boldsymbol{e} < (\boldsymbol{\beta}(-S)^{-1}\boldsymbol{e})^{-1}$$

reduces to

$$\frac{1}{\lambda} > \frac{1}{\mu_2} + \frac{1}{\mu_3}$$

which is the stability condition for Hanukov et al. [17] model. Also steady state vectors in our model with the above restrictions coincides with that in

Hanukov et al. [17].

4.2.3 Distribution of time till the number of customers hit N or the inventory level reaches L

We consider the Markov process $\{N(t), I(t), J(t), K(t)\}$ with state space $\{(h,i,j,k) : 0 \leq h \leq N-1; 0 \leq i \leq L-1; 1 \leq j \leq m_1; 1 \leq k \leq n\} \cup \{*_1\} \cup \{*_2\}$ where $*_1$ denotes the absorbing state indicating the inventory level hitting L and $*_2$ denotes the absorbing state indicating the number of customers reaching N. The infinitesimal generator of the process is

$$\mathcal{V}_1 = \begin{bmatrix} V_1 & \boldsymbol{V}_1^{(0)} & \boldsymbol{V}_1^{(1)} \\ \mathbf{0} & 0 & 0 \end{bmatrix} \text{where}, V_1 = \begin{bmatrix} E & I_{Lm_1} \otimes D_1 & & \\ \ddots & \ddots & & \\ & E & I_{Lm_1} \otimes D_1 \\ & & E \end{bmatrix},$$

$$\boldsymbol{V}_1^{(0)} = \begin{bmatrix} \boldsymbol{e}_L(L) \otimes (\boldsymbol{T}^0 \otimes \boldsymbol{e}(n)) \\ \vdots \\ \boldsymbol{e}_L(L) \otimes (\boldsymbol{T}^0 \otimes \boldsymbol{e}(n)) \end{bmatrix}, \boldsymbol{V}_1^{(1)} = \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \boldsymbol{e}(Lm_1) \otimes \delta \end{bmatrix}$$

with

$$E = \begin{bmatrix} T \oplus D_0 & \boldsymbol{T}^0\boldsymbol{\alpha} \otimes I_n & & \\ \ddots & \ddots & & \\ & T \oplus D_0 & \boldsymbol{T}^0\boldsymbol{\alpha} \otimes I_n \\ & & T \oplus D_0 \end{bmatrix} \text{and}\, \delta = \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_n \end{bmatrix},$$

with δ_i representing the sum of ith row of the D_1 matrix.

The initial probability vector is

$$\boldsymbol{\psi}_1 = (1/d_1)(x_{0,0,0,1,1}, \cdots, x_{0,0,0,1,n}, \cdots, x_{0,0,0,m_1,1}, \cdots, x_{0,0,0,m_1,n}, \mathbf{0})$$

where

$$d_1 = \sum_{l=1}^{n} \sum_{k=1}^{m_1} x_{0,0,0,k,l}$$

and $\mathbf{0}$ is a zero matrix of order $1 \times \big((N-1)Lm_1n + (L-1)m_1n\big)$.
Thus we have the following Lemma.

Lemma 4.2.2. The expected duration of time till the inventory level reaches L before the number of customers hit N is given by $\boldsymbol{\psi}_1(-V_1)^{-2}\boldsymbol{V}_1^{(0)}$ and the expected duration of time till the number of customers hit N before the inventory level reaches L is given by $\boldsymbol{\psi}_1(-V_1)^{-2}\boldsymbol{V}_1^{(1)}$.

4.2.4 Distribution of idle time

Case (i)

Suppose that the number of customers become N only after the inventory level hits L. The probabaility for this event is the probability for absorption of PH$(\boldsymbol{\psi}_1, V_1)$ to $*_1$. In this case, we can study this conditional distribution by a phase type distribution PH$(\boldsymbol{\psi}_2, V_2)$ where the underlying Markov process has state space $\{(h, L, 0, l) : 0 \leq h \leq N-1;\ 1 \leq l \leq n\} \cup \{*\}$ where $*$ denotes the absorbing state indicating that the number of customers hitting N. The infinitesimal generator is

$$\boldsymbol{V_2} = \begin{bmatrix} V_2 & \boldsymbol{V_2}^0 \\ \mathbf{0} & 0 \end{bmatrix}, \text{ where, } V_2 = \begin{bmatrix} D_0 & D_1 & & \\ \ddots & \ddots & & \\ & D_0 & D_1 & \\ & & D_0 & \end{bmatrix}, \boldsymbol{V}_2^0 = \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \delta \end{bmatrix}$$

where $\delta = \begin{bmatrix} \delta_1 \\ \vdots \\ \delta_n \end{bmatrix}$ with δ_i representing the sum of ith row of the D_1 matrix.
The initial probability vector is

$$\boldsymbol{\psi}_2 = (1/d_2)(v_{0,L,0,1}, \cdots, v_{0,L,0,n}, \cdots, v_{N-1,L,0,1}, \cdots, v_{N-1,L,0,n})$$

where, for $0 \leq h \leq N-2, 1 \leq l \leq n$,

$$v_{h,L,0,l} = \sum_{k=1}^{m_1} \frac{\eta_k}{\sum_{l \neq l'} d^0_{ll'} + \delta_l + \sum_{k \neq k'} T_{kk'} + \eta_k} x_{h,L-1,0,k,l}$$

and, for $h = N-1$, $1 \leq l \leq n$,

$$v_{N-1,L,0,l} = \sum_{k=1}^{m_1} \frac{\eta_k}{\sum_{l \neq l'} d^0_{ll'} + \sum_{k \neq k'} T_{kk'} + \eta_k} x_{N-1,L-1,0,k,l},$$

with, $d_2 = \sum_{h=0}^{N-1} \sum_{l=1}^{n} v_{h,L,0,l}$.
Here, η_k represents the absorption rate from phase k in PH$(\boldsymbol{\alpha}, T)$, $T_{kk''}$ represent the kk'th entry of T, $d^0_{ll'}$ represent the transition rates from the phase l to the phase l' without arrival and δ_l represent the lth row sum of D_1 matrix.

Case(ii) Suppose that the number of customers become N before the inventory level hits L. The probabaility for this event is the probability for absorption of PH$(\boldsymbol{\psi_1}, V_1)$ to $*_2$. In this case, the idle time=0.
Thus we have the following Theorem.

Theorem 4.2.1. *The LST of the distribution of the idle time is given by*

$$\left(\boldsymbol{\psi}_2(sI - V_2)^{-1} V_2^0\right) \left(\int_{t=0}^{\infty} \boldsymbol{\psi}_1 e^{V_1 t} \boldsymbol{V}_1^{(0)} dt\right)$$

4.2.5 Distribution of time until the number of customers hit N

We can study this by a phase type distribution PH(ψ_3, V_3) where the underlying Markov process has state space $\{(h,i,j,k) : 0 \leq h \leq N-1; 0 \leq i \leq L-1; 1 \leq j \leq m_1; 1 \leq k \leq n\} \cup \{(h,L,k) : 0 \leq h \leq N-1; 1 \leq k \leq n\} \cup \{*\}$ where $*$ denotes the absorbing state indicating the number of customers reaching N. The infinitesimal generator is

$$\mathcal{V}_3 = \begin{bmatrix} V_3 & \boldsymbol{V}_3^0 \\ \mathbf{0} & 0 \end{bmatrix},$$

where

$$V_3 = \begin{bmatrix} F & I_{Lm_1+1} \otimes D_1 & & \\ \ddots & \ddots & & \\ & F & I_{Lm_1+1} \otimes D_1 \\ & & F \end{bmatrix}, \boldsymbol{V}_3^0 = \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \boldsymbol{e}(Lm_1+1) \otimes \delta \end{bmatrix}$$

with

$$F = \begin{bmatrix} T \oplus D_0 & \boldsymbol{T}^0\boldsymbol{\alpha} \otimes I_n & & & \\ \ddots & \ddots & & & \\ & T \oplus D_0 & \boldsymbol{T}^0\boldsymbol{\alpha} \otimes I_n & \\ & & T \oplus D_0 & \boldsymbol{T}^0 \otimes I_n \\ & & & D_0 \end{bmatrix}$$

The initial probability vector is

$$\boldsymbol{\psi}_3 = (1/d_1(x_{0,0,0,1,1}, \cdots, x_{0,0,0,1,n}, \cdots, x_{0,0,0,m_1,1}, \cdots, x_{0,0,0,m_1,n}, \mathbf{0})$$

where

$$d_1 = \sum_{l=1}^{n} \sum_{k=1}^{m_1} x_{0,0,0,k,l}$$

and $\mathbf{0}$ is a zero matrix of order $1 \times \big((N-1)(Lm_1+1)n + ((L-1)m_1+1)n\big)$. Thus we have the following Lemma.

Lemma 4.2.3. The distribution of time from the epoch the processing starts until the number of customers hit N is a phase type with representation $\mathrm{PH}(\psi_3, V_3)$.

4.2.6 Distribution of number of inventory processed before the arrival of first customer

To compute the above distribution, first we find the following:

Distribution of processing time till the arrival of first customer

Consider the Markov process with state space $\{(i,j,k) : 0 \le i \le L-1; 1 \le j \le m_1; 1 \le k \le n\} \cup \{(L,k) : 1 \le l \le n\} \cup \{*\}$, where i denotes the number of items in the inventory, j, the phase of inventory processing, k, the arrival phase of customer, *, the absorbing state indicating the arrival of a customer. The infinitesimal generator of the process is given by

$$\mathcal{V}_4 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ \boldsymbol{e}(m_1)\otimes\delta & T\oplus D_0 & \boldsymbol{T}^0\boldsymbol{\alpha}\otimes I_n & 0 & 0 & 0 \\ \boldsymbol{e}(m_1)\otimes\delta & 0 & T\oplus D_0 & \boldsymbol{T}^0\boldsymbol{\alpha}\otimes I_n & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \boldsymbol{e}(m_1)\otimes\delta & 0 & 0 & T\oplus D_0 & \boldsymbol{T}^0\boldsymbol{\alpha}\otimes I_n & 0 \\ \boldsymbol{e}(m_1)\otimes\delta & 0 & 0 & 0 & T\oplus D_0 & \boldsymbol{T}^0\otimes I_n \\ \delta & 0 & 0 & 0 & 0 & D_0 \end{bmatrix}.$$

The initial probability is given by

$$\boldsymbol{\psi}_4 = \frac{1}{d_1}(x_{0,0,0,1,1},\dots,x_{0,0,0,1,n},\dots,x_{0,0,0,m_1,1},\dots x_{0,0,0,m_1,n},\mathbf{0})$$

where $\mathbf{0}$ is a zero matrix of order $1\times((L-1)m_1+1)n$.
Let Y denote the number of items processed before the first arrival and y_k be the probabaility that k items are processed before an arrival. Then y_k is the probabilty that the absorption occurs from the level k for the process. Hence y_k are given by

$$y_0 = -\boldsymbol{\alpha}(T\oplus D_0)^{-1}(\boldsymbol{e}(m_1)\otimes\delta)$$

For $k = 1,2,3,\dots L-1$

$$y_k = (-1)^{k+1}\boldsymbol{\alpha}\left((T \oplus D_0)^{-1}(\boldsymbol{T}^0\boldsymbol{\alpha} \otimes I_n)\right)^k (T \oplus D_0)^{-1}(\boldsymbol{e}(m_1) \otimes \delta)$$

and

$$y_L = (-1)^{L+1}\boldsymbol{\alpha}\left((T \oplus D_0)^{-1}(\boldsymbol{T}^0\boldsymbol{\alpha} \otimes I_n)\right)^{L-1} (T \oplus D_0)^{-1}(\boldsymbol{T}^0 \otimes I_n)D_0^{-1}\delta$$

Thus we have the following Lemma.

Lemma 4.2.4. The distribution of number of inventory processed before the arrival of first customer is given by $P(Y = k) = y_k$.

Definition 4.2.1. Starting up with the epoch of departure of a customer leaving behind no customer in the system until the next epoch at which no customer is left at a service completion epoch is called a *busy cycle*.

4.2.7 Distribution of Busy Cycle

First we assume that $L > N$.
The distribution of duration of busy cycle can be studied by a continuous time Markov chain with state space $\{(h, i, 0, k, l) : 0 \leq h \leq N-1;\ 0 \leq i \leq L-1;\ 1 \leq k \leq m_1;\ 1 \leq l \leq n\} \cup \{(h, L, 0, l) : 0 \leq h \leq N-1;\ 1 \leq l \leq n\} \cup \{(h, i, 1, k, l) : 1 \leq h \leq M;\ i = 0;\ 1 \leq k \leq m_2;\ 1 \leq l \leq n\} \cup \{(h, i, 1, k, l) : 1 \leq h \leq N-1;\ 1 \leq i \leq L-N+h;\ 1 \leq k \leq m_3;\ 1 \leq l \leq n\} \cup \{(h, i, 1, k, l) : N \leq h \leq M;\ 1 \leq i \leq L;\ 1 \leq k \leq m_3;\ 1 \leq l \leq n\} \cup \{*\}$, where $(h, i, 0, k, l)$ denote the states that correspond to the server being in vacation with h customers in the system,i, items in the inventory, k, processing phase and l,the arrival phase, $(h, L, 0, l)$ denote the states that correspond to the server being in vacation with h customers in the system, L, items in the inventory and l,the arrival phase, $(h, i, 1, k, l)$ denote the states that correspond to the server being in normal mode with h customers in the system,i, items in the inventory, k,

service phase and l,the arrival phase, $*$ denote the absorbing state indicating that the number of customers become zero by a service completion and M is chosen in such a way that $P\left(\sum_{h=0}^{M} \boldsymbol{x}_h \boldsymbol{e} > 1-\epsilon\right) \to 0$ for every $\epsilon > 0$. Then the distribution of a busy cycle can be studied by a phase type distribution PH(ϕ, B), whose infinitesimal generator is given by

$$\mathcal{B} = \begin{bmatrix} B & \boldsymbol{B}^0 \\ \boldsymbol{0} & 0 \end{bmatrix} \text{ where, } B = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \end{bmatrix}$$

Now,

$$B_{11} = \begin{bmatrix} F & I_{Lm_1+1} \otimes D_1 & & \\ & \ddots & \ddots & \\ & & F & I_{Lm_1+1} \otimes D_1 \\ & & & F \end{bmatrix},$$

with

$$F = \begin{bmatrix} T \oplus D_0 & \boldsymbol{T}^0\boldsymbol{\alpha} \otimes I_n & & & \\ \ddots & \ddots & & & \\ & T \oplus D_0 & \boldsymbol{T}^0\boldsymbol{\alpha} \otimes I_n & & \\ & & T \oplus D_0 & \boldsymbol{T}^0 \otimes I_n \\ & & & D_0 \end{bmatrix}$$

$$B_{12} = \boldsymbol{e}_N(N)\boldsymbol{e}'_N(M) \otimes B'_{12},$$

where,

$$B'_{12} = \begin{bmatrix} \boldsymbol{e}(m_1) \otimes (\boldsymbol{\beta} \otimes D_1) & & & \\ & I_{L-1} \otimes (\boldsymbol{e}(m_1) \otimes (\boldsymbol{\gamma} \otimes D_1) & & \\ & & \ddots & \\ & & & \boldsymbol{\gamma} \otimes D_1 \end{bmatrix}$$

$$B_{22} = \begin{bmatrix} E_1 & F_1 & & & & & & \\ G_1 & E_2 & F_2 & & & & & \\ & \ddots & \ddots & \ddots & & & & \\ & & G_{N-2} & E_{N-1} & F_{N-1} & & & \\ & & & G_{N-1} & E_N & I \otimes D_1 & & \\ & & & \ddots & \ddots & \ddots & & \\ & & & & G_{M-2} & E_{M-1} & I \otimes D_1 \\ & & & & & G_{M-1} & E'_M \end{bmatrix}$$

For $1 \leq h \leq N-1$,

$$G_h = \begin{bmatrix} \boldsymbol{S}^0\boldsymbol{\beta} \otimes I_n & 0 \\ \boldsymbol{U}^0\boldsymbol{\beta} \otimes I_n & 0 \\ 0 & I_{L-N+h} \otimes (\boldsymbol{U}^0\boldsymbol{\gamma} \otimes I_n) \end{bmatrix}, E_h = \begin{bmatrix} S \oplus D_0 & \\ & I_{L-N+h} \otimes (U \oplus D_0) \end{bmatrix}$$

and

$$F_h = \begin{bmatrix} I_{m_2+(L-N+h)m_3} \otimes D_1 & 0 \end{bmatrix}.$$

$$\text{For } N \leq h \leq M-1,\ E_h = \begin{bmatrix} S \oplus D_0 & \\ & I_L \otimes (U \oplus D_0) \end{bmatrix}$$

$$\text{For } h \geq N,\ G_h = \begin{bmatrix} \boldsymbol{S}^0\boldsymbol{\beta} \otimes I_n & 0 & 0 \\ \boldsymbol{U}^0\boldsymbol{\beta} \otimes I_n & 0 & 0 \\ 0 & I_{L-1} \otimes (\boldsymbol{U}^0\boldsymbol{\gamma} \otimes I_n) & 0 \end{bmatrix}$$

$$E'_M = \begin{bmatrix} E_{M_0} & \\ & I_L \otimes (E_{M_1}) \end{bmatrix}, \text{ where}$$

$E_{M_0} = S \oplus D_0 - I_{m_2} \otimes \Delta$ and $E_{M_1} = U \oplus D_0 - I_{m_3} \otimes \Delta$, with

$$\Delta = \begin{bmatrix} \delta_1 & & \\ & \ddots & \\ & & \delta_n \end{bmatrix}$$

and

$$\boldsymbol{B}^0 = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{B}^{00} \end{bmatrix}, \text{ with}, \boldsymbol{B}^{00} = \begin{bmatrix} B_{000} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}, \text{ where}, \boldsymbol{B}^{000} = \begin{bmatrix} \boldsymbol{S}^0 \otimes \boldsymbol{e}(n) \\ \boldsymbol{e}(L-N+1) \otimes (\boldsymbol{U}^0 \otimes \boldsymbol{e}(n)) \end{bmatrix}$$

The initial probability vector is

$$\boldsymbol{\phi} = (\boldsymbol{\phi'}, \mathbf{0})$$

where, $\boldsymbol{\phi'} = \frac{1}{d_3}(w_{0,0,0,1,1}, \cdots, w_{0,0,0,m_1,n}, \cdots, w_{0,L-1,0,1,1}, \cdots, w_{0,L-1,0,m_1,n}, \mathbf{0})$, with

$$d_3 = \sum_{i=0}^{L-1} \sum_{k'=1}^{m_1} \sum_{l=1}^{n} w_{0,i,0,k',l}$$

.

For $1 \leq k' \leq m_1$; $1 \leq l \leq n$,

$$w_{0,0,0,k',l} = \sum_{k=1}^{m_2} \frac{\sigma_k \alpha_{k'}}{\delta_l + \sum_{l \neq l'} d^0_{ll'} + \sigma_k + \sum_{k \neq k''} S_{kk''}} x_{1,0,1,k,l} + \sum_{k=1}^{m_3} \frac{\tau_k \alpha_{k'}}{\delta_l + \sum_{l \neq l'} d^0_{ll'} + \tau_k + \sum_{k \neq k''} U_{kk''}} x_{1,1,1,k,l}, \quad (4.9)$$

For $1 \leq i \leq L-1$; $1 \leq k' \leq m_1$, $1 \leq l \leq n$,

$$w_{0,i,0,k',l} = \sum_{k=1}^{m_3} \frac{\tau_k \alpha_{k'}}{\delta_l + \sum_{l \neq l'} d^0_{ll'} + \tau_k + \sum_{k \neq k''} U_{kk''}} x_{1,i+1,1,k,l},$$

where σ_k, τ_k represent the absorption rates from service phase k in PH$(\boldsymbol{\beta}, S)$ and PH$(\boldsymbol{\gamma}, U)$ respectively, $S_{kk''}$, $U_{kk''}$ represent the kk''th entry of S and U respectively, $\alpha_{k'}$ represents the probability that the processing of item starts in phase k', $d^0_{ll'}$ represent the transition rates from the phase l to the phase l' without arrival and δ_l represent the lth row sum of D_1 matrix.
From the above discussions we have the following.

Theorem 4.2.2. *The LST of the distribution of a busy cycle in which no item is left in the inventory is given by*

$$\hat{B_{C_1}}(s) = \boldsymbol{\phi}(sI - B)^{-1} I'(\boldsymbol{B}^0)'$$

where, I' denote the columns of identity matrix corresponding to the 1 customer level with number of items in the inventory 0 and 1 and

$$(\boldsymbol{B}^0)' = \begin{bmatrix} \boldsymbol{S}^0 \otimes \boldsymbol{e}(n) \\ \boldsymbol{U}^0 \otimes \boldsymbol{e}(n) \end{bmatrix}$$

Theorem 4.2.3. *The LST of the distribution of a busy cycle in which atleast one item is left in the inventory is given by*

$$\hat{B_{C_2}}(s) = \boldsymbol{\phi}(sI - B)^{-1} I''(\boldsymbol{B}^0)''$$

where, I'' denote the columns of identity matrix corresponding to 1 customer level with number of items in the inventory > 1 and

$$(\boldsymbol{B}^0)'' = \boldsymbol{e}(L - N) \otimes (\boldsymbol{U}^0 \otimes \boldsymbol{e}(n))$$

Theorem 4.2.4. *For stationary MAP, the expected number of busy cycles in which at least one inventory left in an interval of length t is given by*

$$\left(t/(\boldsymbol{\phi}(-B)^{-1}\boldsymbol{e})\right) \left(\hat{B_{C_2}}'(0) / \left(\hat{B_{C_1}}'(0) + \hat{B_{C_2}}'(0)\right)\right)$$

4.3 Numerical Results

We fix $\boldsymbol{\alpha} = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $\boldsymbol{\beta} = \begin{bmatrix} 1 & 0 \end{bmatrix}$ and $\boldsymbol{\gamma} = \begin{bmatrix} 0.8 & 0.2 \end{bmatrix}$, $T = \begin{bmatrix} -3 & 3 \\ 0 & -3 \end{bmatrix}$,
$S = \begin{bmatrix} -4 & 4 \\ 0 & -4 \end{bmatrix}$, $U = \begin{bmatrix} -2 & 2 \\ 0 & -2 \end{bmatrix}$ and $D_0 = -1$, $D_1 = 1$.

For these input parameters we get the system characteristics as given in Table 4.1. The behaviour of the performance characteristics is on expected lines. Let E denote Expected Idle time, SD, standard deviation of Idle time, CV, Coefficient of Variation of Idle time.

Table 4.1: Mean/Standard Deviation/Coefficient of Variation of idle time of the server

$L \downarrow N \rightarrow$	2			3			4		
	E	SD	CV	E	SD	CV	E	SD	CV
2	0.90	1.20	1.33	1.47	1.52	1.03	2.00	1.79	0.90
3	0.63	1.07	1.71	1.15	1.43	1.25	1.78	1.77	1.00
4	0.42	0.92	2.19	0.86	1.31	1.52	1.44	1.68	1.17
5	0.27	0.76	2.80	0.63	1.16	1.86	1.12	1.56	1.39

Chapter 5

On a Queueing System with Processing of Service Items under Vacation and N-policy with Impatient Customers

In this chapter we extend the queueing model considered in the previous chapter to the case where the customers are impatient. In addition we formulate a strategic game corresponding to the problem and investigate the individual, social and system optimal strategies by introducing appropriate costs associated with certain system parameters.

Next we turn to further details of this chapter. We consider a single server queueing system in which customers arrive according to Markovian Arrival process. When the system is empty, the server goes for vacation and produces inventory for future use during this period. Maximum inventory that can be

Some results of this chapter are included in the following paper.
Divya V.,Vishnevsky, V.M., Kozyrev, D., A. Krishnamoorthy: **On a Queueing System with Processing of Service Items under Vacation and N-policy with Impatient Customers** (communicated).

held is L. Inventory processing time follows phase type distribution. Server returns from vacation when N customers accumulate in the system. Service time of customers follow two distinct phase type distributions according as there is no processed item or there are processed items at the beginning of service.The customers join the queue with probability p or balk with probability $1-p$. Further customers while waiting for service, may become impatient and renege after a random time period which is exponentially distributed. Somewhat related work is by Wang and Zhang [53]. Whereas they follow replenishment policy through external sources in the context of queueing-inventory, we investigate the system in which the item is processed by the server himself. Further, in Wang and Zhang model, the server has to stay idle when inventory level drops to zero; in the present model the server processes the item and serves the customer if at a service commencement epoch the item is not available.

5.1 Model Description and Mathematical formulation

We assume that customers arrive at a single server queueing system according to MAP with representation (D_0, D_1) of order n. At the end of a service if the system is left with no customer, the server goes for vacation and produces inventory for future use during this period. Maximum number of such items that can be held is restricted to L. Processing time for each item in the inventory follows phase type distribution PH$(\boldsymbol{\alpha}, T)$ of order m_1. Server returns from vacation when there are N customers in the system. The service time follows PH$(\boldsymbol{\beta}, S)$ of order m_2 when there is no processed item and it follows PH$(\boldsymbol{\gamma}, U)$ of order m_3 when there are processed items. Customers join the queue with probability p or balk with probability $1-p$. Also the customers waiting for service may become impatient and renege after a random time period which is exponentially distributed with parameter $(n-1)\phi$, $n \geq 1$, where n is the number of customers in the system.

Let $Q^* = D_0 + D_1$ be the generator matrix of the arrival process and $\boldsymbol{\pi^*}$ be its stationary probability vector. Hence $\boldsymbol{\pi^*}$ is the unique (positive) probability vector satisfying

$$\boldsymbol{\pi^*} Q^* = 0, \ \boldsymbol{\pi^*} \boldsymbol{e} = 1.$$

The quantity $\beta^* = \boldsymbol{\pi^*} D_1 \boldsymbol{e}$, referred to as *fundemental rate*, gives the expected number of arrivals per unit time in the stationary version of the MAP. It is assumed that the arrival process is independent of the inventory processing and service process.

5.1.1 The QBD process

The model described above can be studied as a level dependent quasi-birth-and-death (LDQBD) process. First we introduce the followiing notations:
At time t:
$N(t)$: the number of customers in the system
$I(t)$: the number of processed inventory

$$J(t) = \begin{cases} 0, if \ the \ server \ is \ on \ vacation \\ 1, if \ the \ server \ is \ busy \ serving \ customer \end{cases}$$

$K(t)$: the phase of the inventory processing/service process
$M(t)$: the phase of arrival of customer.
It is easy to verify that $\{(N(t), I(t), J(t), K(t), M(t)) : t \geq 0\}$ is LDQBD with state space
$l(0) = \{(0, i, 0, k_1, l) : 0 \leq i \leq L-1; 1 \leq k_1 \leq m_1, \ 1 \leq l \leq n\} \cup \{(0, L, 0, l) : 1 \leq l \leq n\}$
For $1 \leq h \leq N-1$,
$l(h) = \{(h, i, 0, k_1, l) : 0 \leq i \leq L-1; 1 \leq k_1 \leq m_1; 1 \leq l \leq n\} \cup \{(h, L, 0, l) : 1 \leq l \leq n\} \cup \{(h, 0, 1, k_2, l) : 1 \leq k_2 \leq m_2; 1 \leq l \leq n\} \cup \{(h, i, 1, k_3, l) : 1 \leq i \leq L; 1 \leq k_3 \leq m_3; 1 \leq l \leq n\}$
and for $h \geq N$,

$l(h) = \{(h, 0, 1, k_2, l) : 1 \leq k_2 \leq m_2;\ 1 \leq l \leq n\} \cup \{(h, i, 1, k_3, l) : 1 \leq i \leq L;\ 1 \leq k_3 \leq m_3;\ 1 \leq l \leq n\}$.

Note that if there is no customer in the system and level of processed item is L, then also $J(t) = 0$, indicating that server is idle. Further at the start of a new cycle, the server stays idle until N customers accumumulate.

The infinitesimal generator of this CTMC is

$$\bar{Q} = \begin{bmatrix} B_0 & C_0 & & & & & & & \\ B_1 & E_1 & I\otimes pD_1 & & & & & & \\ & B_2 & E_2 & I\otimes pD_1 & & & & & \\ & & \ddots & \ddots & \ddots & & & & \\ & & B_{N-2} & E_{N-2} & I\otimes pD_1 & & & & \\ & & & B_{N-1} & E_{N-1} & F & & & \\ & & & & B'_N & A_1^{(N)} & A_0{}^{(N)} & & \\ & & & & & A_2{}^{(N+1)} & A_1{}^{(N+1)} & A_0{}^{(N+1)} & \\ & & & & & & A_2{}^{(N+2)} & A_1{}^{(N+2)} & A_0{}^{(N+2)} \\ & & & & & & \ddots & \ddots & \ddots \end{bmatrix}$$

The boundary blocks B_0, C_0, B_1 are of orders $(Lm_1+1)n\times(Lm_1+1)n$, $(Lm_1+1)n\times\big((m_1+m_2)n+(L-1)(m_1+m_3)n+(1+m_3)n\big)$, $\big((m_1+m_2)n+(L-1)(m_1+m_3)n+(1+m_3)n\big)\times(Lm_1+1)n$ respectively. For $2 \leq h \leq N-1$, B_h and for $1 \leq h \leq N-1$, E_h are square matrices of order $(m_1+m_2)n+(L-1)(m_1+m_3)n+(1+m_3)n$. F and B'_N are of orders $\big((m_1+m_2)n+(L-1)(m_1+m_3)n+(1+m_3)n\big)\times(m_2+Lm_3)n$ and $(m_2+Lm_3)n\times\big((m_1+m_2)n+(L-1)(m_1+m_3)n+(1+m_3)n\big)$ respectively. For $h \geq N$, $A_0{}^{(h)}, A_1{}^{(h)}$ and for $h \geq N+1$, $A_2{}^{(h)}$ are square matrices of order $(m_2 + Lm_3)n$. Define the entries $B_{0_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)}$, $C_{0_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)}$ and $B_{1_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)}$ as transition submatrices which contain transitions of the form $(0, i_1, j_1, k_1, l_1) \to (0, i_2, j_2, k_2, l_2), (0, i_1, j_1, k_1, l_1) \to (1, i_2, j_2, k_2, l_2)$ and $(1, i_1, j_1, k_1, l_1) \to (0, i_2, j_2, k_2, l_2)$ respectively. Define $E_{h_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)}$, $B_{h_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)}$, F and B'_N as transition submatrices which contain transitions of the form $(h, i_1, j_1, k_1, l_1) \to (h, i_2, j_2, k_2, l_2)$, where $1 \leq h \leq N-1$, $(h, i_1, j_1, k_1, l_1) \to$

$(h-1, i_2, j_2, k_2, l_2)$, where $2 \leq h \leq N-1$, $(N-1, i_1, j_1, k_1, l_1) \to (N, i_2, j_2, k_2, l_2)$ and $(N, i_1, j_1, k_1, l_1) \to (N-1, i_2, j_2, k_2, l_2)$ respectively. Define the entries ${A_2}^{(h)}{}^{(i_2,j_2,k_2,l_2)}_{(i_1,j_1,k_1,l_1)}$, ${A_1}^{(h)}{}^{(i_2,j_2,k_2,l_2)}_{(i_1,j_1,k_1,l_1)}$ and ${A_0}^{(h)}{}^{(i_2,j_2,k_2,l_2)}_{(i_1,j_1,k_1,l_1)}$ as transition submatrices which contain transitions of the form $(h, i_1, j_1, k_1, l_1) \to (h-1, i_2, j_2, k_2, l_2)$, where $h \geq N+1$, $(h, i_1, j_1, k_1, l_1) \to (h, i_2, j_2, k_2, l_2)$ and $(h, i_1, j_1, k_1, l_1) \to (h+1, i_2, j_2, k_2, l_2)$, where $h \geq N$ respectively. Since none or one event alone could take place in a short interval of time with positive probability, in general, a transition such as $(h_1, i_1, j_1, k_1, l_1) \to (h_2, i_2, j_2, k_2, l_2)$ has positive rate only for exactly one of h_1, i_1, j_1, k_1, l_1 different from h_2, i_2, j_2, k_2, l_2.

$$B_{0_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)} = \begin{cases} \boldsymbol{T}^0\boldsymbol{\alpha} \otimes I_n & i_2 = i_1+1, 0 \leq i_1 \leq L-2; j_1 = j_2 = 0; 1 \leq k_1, k_2 \leq m_1; \\ & 1 \leq l_1, l_2 \leq n \\ \boldsymbol{T}^0 \otimes I_n & i_1 = L-1, i_2 = L; j_1 = j_2 = 0; 1 \leq k_1, k_2 \leq m_1; \\ & 1 \leq l_1, l_2 \leq n \\ T \oplus \Delta & i_1 = i_2, 0 \leq i_1 \leq L-1; j_1 = j_2 = 0; 1 \leq k_1, k_2 \leq m_1; \\ & 1 \leq l_1, l_2 \leq n \\ \Delta & i_1 = i_2 = L; j_1 = j_2 = 0; 1 \leq l_1, l_2 \leq n \end{cases}$$

where

$$\Delta = D_0 + (1-p) \begin{bmatrix} \delta_1 & & & \\ & \delta_2 & & \\ & & \ddots & \\ & & & \delta_n \end{bmatrix}.$$

$$C_{0_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)} = \begin{cases} I_{m_1} \otimes pD_1 & 0 \leq i_1 \leq L-1; i_1 = i_2; j_1 = j_2 = 0; 1 \leq k_1, k_2, \leq m_1; \\ & 1 \leq l_1, l_2 \leq n \\ pD_1 & i_2 = i_1 = L; j_1 = j_2 = 0; 1 \leq l_1, l_2 \leq n \end{cases}$$

$$B_{1_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)} = \begin{cases} \boldsymbol{S}^0\boldsymbol{\alpha} \otimes I_n & i_1 = i_2 = 0; j_1 = 1, j_2 = 0; 1 \leq k_1 \leq m_2, \\ & 1 \leq k_2 \leq m_1; 1 \leq l_1, l_2 \leq n \\ \boldsymbol{U}^0\boldsymbol{\alpha} \otimes I_n & 1 \leq i_1 \leq L; i_2 = i_1 - 1; j_1 = 1, j_2 = 0;; 1 \leq k_1 \leq m_3, \\ & 1 \leq k_2 \leq m_1; 1 \leq l_1, l_2 \leq n \end{cases}$$

For $1 \leq h \leq N-1$,

$$E_{h_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)} = \begin{cases} \boldsymbol{T}^0\boldsymbol{\alpha} \otimes I_n & 0 \le i_1 \le L-2, i_2 = i_1+1; j_1 = j_2 = 0; \\ & 1 \le k_1, k_2 \le m_1; 1 \le l_1, l_2 \le n \\ \boldsymbol{T}^0 \otimes I_n & i_1 = L-1, i_2 = L; j_1 = j_2 = 0; \\ & 1 \le k_1 \le m_1; 1 \le l_1, l_2 \le n \\ T \oplus \Delta - (h-1)\phi I_{m_1 n} & i_1 = i_2, 0 \le i_1 \le L-1; j_1 = j_2 = 0; \\ & 1 \le k_1, k_2 \le m_1; 1 \le l_1, l_2 \le n \\ S \oplus \Delta - (h-1)\phi I_{m_1 n} & i_1 = i_2 = 0, j_1 = j_2 = 1, 1 \le k_1, k_2 \le m_2, \\ & 1 \le l_1, l_2 \le n \\ U \oplus \Delta - (h-1)\phi I_{m_1 n} & i_1 = i_2, 1 \le i_1 \le L; j_1 = j_2 = 1, 1 \le k_1, k_2 \le m_3, \\ & 1 \le l_1, l_2 \le n \\ \Delta - (h-1)\phi I_n & i_1 = i_2 = L; j_1 = j_2 = 0; 1 \le l_1, l_2 \le n \end{cases}$$

For $2 \le h \le N-1$,

$$B_{h_{(i_1,j_1,k_1,l_1)}}^{(i_2,j_2,k_2,l_2)} = \begin{cases} (h-1)\phi I_{m_1 n} & 0 \le i_1 \le L-1, i_1 = i_2; ; j_1 = j_2 = 0; \\ & 1 \le k_1, k_2 \le m_1; 1 \le l_1, l_2 \le n \\ (h-1)\phi I_n & i_1 = i_2 = L; j_1 = j_2 = 0; 1 \le k_1, k_2 \le m_1; \\ & 1 \le l_1, l_2 \le n \\ \boldsymbol{S}^0\boldsymbol{\beta} \otimes I_n + (h-1)\phi I_{m_2 n} & i_1 = i_2 = 0; j_1 = j_2 = 1; 1 \le k_1, k_2 \le m_2; \\ & 1 \le l_1, l_2 \le n \\ (h-1)\phi I_{m_3 n} & 1 \le i_1 \le L, i_1 = i_2; j_1 = j_2 = 1; 1 \le k_1, k_2 \le m_3; \\ & 1 \le l_1, l_2 \le n \\ \boldsymbol{U}^0\boldsymbol{\beta} \otimes I_n & i_1 = 1, i_2 = 0; j_1 = j_2 = 1; 1 \le k_1 \le m_3, \\ & 1 \le k_2 \le m_2; 1 \le l_1, l_2 \le n \\ \boldsymbol{U}^0\boldsymbol{\gamma} \otimes I_n & 2 \le i_1 \le L, i_2 = i_1 - 1; j_1 = j_2 = 1; \\ & 1 \le k_1, k_2 \le m_3; 1 \le l_1, l_2 \le n \end{cases}$$

$$F_{(i_1,j_1,k_1,l_1)}^{(i_2,j_2,k_2,l_2)} = \begin{cases} \boldsymbol{e}(m_1) \otimes (\boldsymbol{\beta} \otimes pD_1) & i_1 = i_2 = 0; j_1 = 0, j_2 = 1; 1 \le k_1 \le m_1, 1 \le k_2 \le m_2; \\ & 1 \le l_1, l_2 \le n \\ I_{m_2} \otimes pD_1 & i_2 = i_1 = 0; j_1 = j_2 = 1; 1 \le k_1, k_2 \le m_2, 1 \le l_1, l_2 \le n \\ I_{m_3} \otimes pD_1 & i_2 = i_1, 1 \le i_1 \le L; j_1 = j_2 = 1; 1 \le k_1, k_2 \le m_2, \\ & 1 \le l_1, l_2 \le n \\ \boldsymbol{e}(m_1) \otimes (\boldsymbol{\gamma} \otimes pD_1) & 1 \le i_1 \le L-1; j_1 = 0, j_2 = 1; 1 \le k_1 \le m_1, 1 \le k_2 \le m_3; \\ & 1 \le l_1, l_2 \le n \\ \boldsymbol{\gamma} \otimes pD_1 & i_1 = i_2 = L; j_1 = 0, j_2 = 1; 1 \le k_1 \le m_1, 1 \le k_2 \le m_3; \\ & 1 \le l_1, l_2 \le n \end{cases}$$

$$B'^{(i_2,j_2,k_2,l_2)}_{N(i_1,j_1,k_1,l_1)} = \begin{cases} \boldsymbol{S}^0\boldsymbol{\beta} \otimes I_n + (N-1)\phi I_{m_2n} & i_1 = i_2 = 0; j_1 = j_2 = 1; 1 \le k_1, k_2 \le m_2; \\ & 1 \le l_1, l_2 \le n \\ \boldsymbol{U}^0\boldsymbol{\beta} \otimes I_n & i_1 = 1, i_2 = 0; j_1 = j_2 = 1; 1 \le k_1 \le m_3, \\ & 1 \le k_2 \le m_2; 1 \le l_1, l_2 \le n \\ \boldsymbol{U}^0\boldsymbol{\gamma} \otimes I_n & 2 \le i_1 \le L, i_2 = i_1 - 1; j_1 = j_2 = 1; \\ & 1 \le k_1, k_2 \le m_3; 1 \le l_1, l_2 \le n \\ (N-1)\phi I_{m_3n} & 1 \le i_1 \le L, i_2 = i_1; j_1 = j_2 = 1; \\ & 1 \le k_1, k_2 \le m_3; 1 \le l_1, l_2 \le n \end{cases}$$

For $h \ge N+1$,

$$A_2{}^{(h)}{}^{(i_2,j_2,k_2,l_2)}_{(i_1,j_1,k_1,l_1)} = \begin{cases} \boldsymbol{S}^0\boldsymbol{\beta} \otimes I_n + (h-1)\phi I_{m_2n} & i_1 = i_2 = 0; j_1 = j_2 = 1; 1 \le k_1, k_2 \le m_2; \\ & 1 \le l_1, l_2 \le n \\ \boldsymbol{U}^0\boldsymbol{\beta} \otimes I_n & i_1 = 1, i_2 = 0; j_1 = j_2 = 1; 1 \le k_1 \le m_3, \\ & 1 \le k_2 \le m_2; 1 \le l_1, l_2 \le n \\ \boldsymbol{U}^0\boldsymbol{\gamma} \otimes I_n & i_2 = i_1 - 1, 2 \le i_2 \le L; j_1 = j_2 = 1; \\ & 1 \le k_1, k_2 \le m_3; 1 \le l_1, l_2 \le n \\ (h-1)\phi I_{m_3n} & i_2 = i_1, 1 \le i_2 \le L; j_1 = j_2 = 1; \\ & 1 \le k_1, k_2 \le m_3; 1 \le l_1, l_2 \le n \end{cases}$$

For $h \ge N$,

$$A_1{}^{(h)}{}^{(i_2,j_2,k_2,l_2)}_{(i_1,j_1,k_1,l_1)} = \begin{cases} S \oplus \Delta - (h-1)\phi I_{m_2n} & i_1 = i_2 = 0, j_1 = j_2 = 1, 1 \le k_1, k_2 \le m_2; \\ & 1 \le l_1, l_2 \le n \\ U \oplus \Delta - (h-1)\phi I_{m_3n} & i_1 = i_2, 1 \le i_1 \le L; j_1 = j_2 = 1, \\ & 1 \le k_1, k_2 \le m_3, 1 \le l_1, l_2 \le n \end{cases}$$

$$A_0{}^{(h)}{}^{(i_2,j_2,k_2,l_2)}_{(i_1,j_1,k_1,l_1)} = \begin{cases} I_{m_2} \otimes pD_1 & i_1 = i_2 = 0; j_1 = j_2 = 1; 1 \le k_1, k_2 \le m_2; 1 \le l_1, l_2 \le n \\ I_{m_3} \otimes pD_1 & i_1 = i_2, 1 \le i_1 \le L; j_1 = j_2 = 1; 1 \le k_1, k_2 \le m_3; \\ & 1 \le l_1, l_2 \le n \end{cases}$$

Remarks: When $L = 0$ (that is, no item processed during vacation) the problem discussed reduces to classical N-policy.

5.2 Steady State Analysis

First we find the condition for stability of the system under study.

5.2.1 Stability condition

Lemma 5.2.1. The system under consideration is stable.

Proof. We use the following result to prove this.

Proposition(Tweedie) Let $\{X(t)\}$ be a Markov process with discrete state space $\mathcal{S}$ and rates of transition q_{sr}, $s, r \in \mathcal{S}$, $\sum_r q_{sr} = 0$. Assume that there exist
1. a function $\psi(s)$, $s \in \mathcal{S}$, which is bounded from below (this function is said to be a Lyapunov or test function);
2. a positive number ϵ such that:

- variables $y_s = \sum_{r \neq s} q_{sr}(\psi(r) - \psi(s)) < \infty$ for all $s \in S$;
- $y_s \leq -\epsilon$ for all $s \in S$ except perhaps a finite number of states.

Then the process $\{X(t)\}$ is regular and ergodic.

For the model under discussion, we consider the following test function:

$$\psi(s) = \psi(h, i, j, k, l) = h$$

. The mean drifts

$$\begin{aligned} y_s &= \sum_{r \neq s} q_{sr} \left(\psi(r) - \psi(s)\right) \\ &= q_{s,s+1} - q_{s,s-1} \end{aligned} \tag{5.1}$$

We have $q_{s,s+1} = r_1$, say (a constant) and $q_{s,s-1} = r_2 + (s-1)\phi$, where r_2 is a constant.
Hence from (5.1), $y_s = r_1 - r_2 - (s-1)\phi$, which depends only on the level s.

Now,

$$\lim_{s\to\infty} y_s = -\infty.$$

Thus the assumptions of Tweedie's result hold and hence the Markov process under cosideration is regular and ergodic (see Falin and Templeton [11]). Hence the system is stable. □

Next we proceed to find the steady-state probability of the system state.

5.2.2 Steady-state probability vector

By finite truncation method we get steady state vectors of the LDQBD approximately. In this method, we truncate the infinitesimal genarator at a finite level K. The level K is chosen in such a way that probability of customer loss due to truncation is small. To get an appropriate level,say , K_f, we start with an initial value for K and increasing it in unit steps until a properly chosen cut-off criterion is satisfied. Here, we use the algorithm by Artalejo et al.[1], the steps of which are explained below.
With K as cut-off level, the modified generator is

$$\bar{Q}_{\mathcal{K}} = \begin{bmatrix} B_0 & C_0 & & & & & & & & \\ B_1 & E_1 & I\otimes pD_1 & & & & & & & \\ & B_2 & E_2 & I\otimes pD_1 & & & & & & \\ & & \ddots & \ddots & \ddots & & & & & \\ & & B_{N-2} & E_{N-2} & I\otimes pD_1 & & & & & \\ & & & B_{N-1} & E_{N-1} & F & & & & \\ & & & & B'_N & A_1^{(N)} & {A_0}^{(N)} & & & \\ & & & & & {A_2}^{(N+1)} & {A_1}^{(N+1)} & {A_0}^{(N+1)} & & \\ & & & & & \ddots & \ddots & \ddots & & \\ & & & & & & {A_2}^{(K-1)} & {A_1}^{(K-1)} & {A_0}^{(K-1)} \\ & & & & & & & {A_2}^{(K)} & \theta^{(K)} \end{bmatrix}.$$

where $\theta_K = {A_1}^{(K)} + {A_0}^{(K)}$. Let $\bar{\pi}$ be the stationary distribution of $\bar{Q}(K)$ which satisfies

$$\begin{aligned} \bar{\boldsymbol{\pi}}\bar{Q}(K) &= 0 \\ \bar{\boldsymbol{\pi}}\boldsymbol{e} &= 1 \end{aligned} \tag{5.2}$$

where $\bar{\boldsymbol{\pi}} = [\bar{\boldsymbol{\pi}}(0), \bar{\boldsymbol{\pi}}(1), \ldots, \bar{\boldsymbol{\pi}}(K)]$. Define $\boldsymbol{y} = [\boldsymbol{y}_0(K), \boldsymbol{y}_1(K)]$ with

$$\begin{aligned}
\boldsymbol{y}_0(K) &= [\bar{\boldsymbol{\pi}}(0), \bar{\boldsymbol{\pi}}(1), \ldots, \bar{\boldsymbol{\pi}}(K-1)], \\
\boldsymbol{y}_1(K) &= \bar{\boldsymbol{\pi}}(K).
\end{aligned}$$

Now $\boldsymbol{y}(K,i) = \bar{\boldsymbol{\pi}}(i)$, $0 \leq i \leq K$. Here $\boldsymbol{y}_0(K)$ is a row vector of dimension $m = (Lm_1+1)n + (N-1)[m_2 n + L(m_1+m_3)n + n] + (K-N)(m_2+Lm_3)n$ and $\boldsymbol{y}_1(K)$ is a row vector of dimension $(m_2+Lm_3)n$. Now from ((5.2)), we have

$$[\boldsymbol{y}_0(K), \boldsymbol{y}_1(K)] \begin{bmatrix} H_{00}(K) & H_{01}(K) \\ H_{10}(K) & H_{11}(K) \end{bmatrix} = [\mathbf{0}_m, \mathbf{0}_{(m_2+Lm_3)n}] \tag{5.3}$$

where $H_{00}(K)$ is obtained from $\bar{Q}(K)$ by deleting the last column matrices and last row matrices. $H_{01}(K) = [0, 0, \ldots, 0, A_0^{(K-1)}]^T$, $H_{10}(K) = [0, 0, \ldots, 0, A_2^{(K)}]$ and $H_{11}(K) = \theta^{(K)}$. These are block structured matrices with $K \times K, K \times 1, 1 \times K$ and 1×1 blocks respectively. 0_m and $0_(m_2 + Lm_3)n$ are row vectors of dimensions m and $(m_2+Lm_3)n$ respectively, with all entries equal to zero. From (5.3), we get

$$\boldsymbol{y}_1(K) H_{10}(K) H_{00}^{-1}(K) = -\boldsymbol{y}_0(K) \tag{5.4}$$

$$\boldsymbol{y}_1(K)[H_{11}(K) - H_{10}(K) H_{00}^{-1}(K) H_{01}(K)] = \mathbf{0}_{(m_2+Lm_3)n} \tag{5.5}$$

Also we have

$$H_{00}(K) = \begin{bmatrix} H_{00}(K-1) & H_{01}(K-1) \\ J_0(K-1) & J_1(K-1) \end{bmatrix}$$

where

$$\begin{aligned}
J_0(K-1) &= [0, \ldots, 0, A_2^{(K-1)}] \\
J_1(K-1) &= A_1^{(K-1)}
\end{aligned}$$

The inverse of matrix $H_{00}(K)$ can be determined using Theorem 4.2.4 in Hunter [19] as

$$H_{00}^{-1}(K) = \begin{bmatrix} M_{00}(K) & M_{01}(K) \\ M_{10}(K) & M_{11}(K) \end{bmatrix}$$

where

$$\begin{aligned}
M_{00}(K) &= [H_{00}(K-1) - H_{01}(K-1)J_1^{-1}(K-1)J_0(K-1)]^{-1}, \\
M_{01}(K) &= -J_1{}^{-1}(K-1)J_0(K-1)M_{00}(K), \\
M_{11}(K) &= [J_1(K-1) - J_0(K-1)H_{00}^{-1}(K-1)H_{01}(K-1)]^{-1}, \\
M_{01}(K) &= -H_{00}^{-1}(K-1)H_{01}(K-1)M_{11}(K)
\end{aligned}$$

Now we can see that the structure of the block matrices $H_{01}(K-1)$ and $J_0(K-1)$ simplify the above set of equations. We have $H_{00}^{-1}(K-1)H_{01}(K-1) = \begin{bmatrix} M_{01}(K-1) \\ M_{11}(K-1) \end{bmatrix} A_0{}^{(K-2)}$. Also $J_0(K-1)H_{00}^{-1}(K-1)H_{01}(K-1) = A_2^{(K-1)}M_{11}(K-1)A_0^{(K-2)}$.
By Example 4.2.2 Hunter [19], we have $(X + AYB)^{-1} = X^{-1} - X^{-1}A(Y^{-1} + BX^{-1}A)^{-1}BX^{-1}$. Then we have

$$(X + AYB)^{-1} = [I - X^{-1}A(Y^{-1} + BX^{-1}A)^{-1}B]X^{-1}.$$

Here, we have $X = H_{00}(K-1)$, $A = -H_{01}(K-1), Y = J_1^{-1}(K-1)$ and $B = J)_0(K-1)$. Finally, we get

$$\begin{aligned}
M_{00}(K) &= [I - M_{01}(K)J_0(K-1)]H_{00}^{-1}(K-1) \\
M_{11}(K) &= [J_1(K-1) - A_2{}^{(K-1)}M_{11}(K-1)A_0{}^{(K-2)}]^{-1}, \\
M_{01}(K) &= -\begin{bmatrix} M_{01}(K-1) \\ M_{11}(K-1) \end{bmatrix} A_0{}^{(K-2)}M_{11}(K) \\
M_{10}(K) &= -J_1^{-1}(K-1)J_0(K-1)M_{00}(K)
\end{aligned}$$

Thus the computation of the vector $\boldsymbol{y}_1(K)$ reduces to solving the system of equations (5.5) subject to the normalizing condition

$$\bar{\boldsymbol{\pi}}(K)[\boldsymbol{e} - H_{10}(K)H_{00}^{-1}(K)\boldsymbol{e}] = 1$$

The vector $\boldsymbol{y}_0(K)$ can be solved by substituting $\boldsymbol{y}_1(K)$ in (5.4). To get the cut-off value, successive increments of K are made, starting from $N+2$ and we stop at the point $K = K_c$ when

$$max_{0 \leq i \leq K_c} ||\boldsymbol{y}(K_c, i) - \boldsymbol{y}(K_c - 1, i)||_\infty < \epsilon$$

where $\epsilon > 0$ is infinitesimal quantity and $||.||_\infty$ is the infinity norm (see Goswami and Selavaraju[15]).

5.2.3 Distribution of time until the number of customers hit N

We show that this is a phase type distribution where the underlying Markov process has state space $\{(h, i, j, k) : 0 \leq h \leq N-1; 0 \leq i \leq L-1; 1 \leq j \leq m_1; 1 \leq k \leq n\} \cup \{(h, L, k) : 0 \leq h \leq N-1; 1 \leq k \leq n\} \cup \{*\}$ where $*$ denotes the absorbing state indicating the number of customers reaching N. The infinitesimal generator is

$$\mathcal{V}_1 = \begin{bmatrix} V_1 & \boldsymbol{V}_1^{(0)} \\ \boldsymbol{0} & 0 \end{bmatrix},$$

where

$$V_1 = \begin{bmatrix} F_0 & I_{Lm_1+1} \otimes pD_1 & & & \\ & F_0 & I_{Lm_1+1} \otimes pD_1 & & \\ & G_2 & F_2 & I_{Lm_1+1} \otimes pD_1 & \\ & \ddots & \ddots & \ddots & \\ & & G_{N-2} & FN-2 & I_{Lm_1+1} \otimes pD_1 \\ & & & G_{N-1} & F_{N-1} \end{bmatrix}$$

,

$$\boldsymbol{V}_1^0 = \begin{bmatrix} \mathbf{0} \\ \vdots \\ \mathbf{0} \\ \boldsymbol{e}(Lm_1+1)\otimes p\delta \end{bmatrix} \text{ with } F_0 = \begin{bmatrix} T\oplus\Delta & \boldsymbol{T}^0\boldsymbol{\alpha}\otimes I_n & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & T\oplus\Delta & \boldsymbol{T}^0\boldsymbol{\alpha}\otimes I_n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & T\oplus\Delta & \boldsymbol{T}^0\otimes I_n \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \Delta \end{bmatrix}$$

For $2 \le h \le N-1$,

$$G_h = (h-1)\phi I_{(Lm_1+1)n}$$

and

$$F_h = \begin{bmatrix} T\oplus\Delta-(h-1)\phi I_{m_1n} & \boldsymbol{T}^0\boldsymbol{\alpha}\otimes I_n & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & T\oplus\Delta-(h-1)\phi I_{m_1n} & \boldsymbol{T}^0\boldsymbol{\alpha}\otimes I_n & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & T\oplus\Delta-(h-1)\phi I_{m_1n} & \boldsymbol{T}^0\otimes I_n \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \Delta-(h-1)\phi I_{m_1n} \end{bmatrix}$$

The initial probability vector is

$$\psi_1 = (\frac{1}{d_1})(w_{0,0,1,1},\cdots,w_{0,0,1,n},\cdots,w_{0,0,m_1,1},\cdots,w_{0,0,m_1,n}\cdots,w_{0,L-1,m_1,1},\cdots,$$
$$w_{0,L-1,m_1,n},\mathbf{0}) \quad (5.6)$$

where,

$$w_{0,0,k,l} = \sum_{k'=1}^{m_2}\frac{\sigma'_k\alpha_k}{-d_{ll}^{(0)}-S_{k'k'}}x_{1,0,1,k',l} + \sum_{k'=1}^{m_3}\frac{\tau'_k\alpha_k}{-d_{ll}^{(0)}-U_{k'k'}}x_{1,1,1,k',l},$$

$$w_{0,i,k,l} = \sum_{k'=1}^{m_3}\frac{\tau'_k\alpha_k}{-d_{ll}^{(0)}-U_{k'k'}}x_{1,i+1,1,k',l}, \text{ with } 1 \le i \le L-1.$$

and

$$d_1 = \sum_{l=1}^{n}\sum_{i=0}^{L-1}\sum_{k=1}^{m_1} w_{0,i,k,l}$$

where $\mathbf{0}$ is a zero matrix of order $1 \times ((N-1)Lm_1n + n)$.
Here, σ'_k represents the absorption rate to phase k' from PH($\boldsymbol{\beta}, S$), τ'_k represents the absorption rate to phase k' from PH($\boldsymbol{\gamma}, U$), $S_{k'k'}$ represent the $k'k'$th entry of S, $U_{k'k'}$ represent the $k'k'$th entry of U and $d^{(0)}_{ll}$ represent the diagonal entry in lth row of D_0.

5.2.4 Some other Performance Measures

- Probability that the server is idle,

$$P_{idle} = \sum_{h=0}^{N-1}\sum_{l=1}^{n} x_{h,L,0,l}$$

- Expected number of customers in the system,

$$E_s = \sum_{h=1}^{N-1}\sum_{i=0}^{L-1}\sum_{k=1}^{m_1}\sum_{l=1}^{n} hx_{h,i,0,k,l} + \sum_{h=1}^{N-1}\sum_{l=1}^{n} hx_{h,L,0,l} + \sum_{h=1}^{\infty}\sum_{k=1}^{m_2}\sum_{l=1}^{n} hx_{h,0,1,k,l} + \sum_{h=1}^{\infty}\sum_{i=1}^{L}\sum_{k=1}^{m_3}\sum_{l=1}^{n} hx_{h,i,1,k,l} \quad (5.7)$$

- Expected number of items in the inventory,

$$E_{it} = \sum_{h=0}^{N-1}\sum_{i=1}^{L-1}\sum_{k=1}^{m_1}\sum_{l=1}^{n} ix_{h,i,0,k,l} + \sum_{h=0}^{N-1}\sum_{l=1}^{n} Lx_{h,L,0,l} + \sum_{h=1}^{\infty}\sum_{i=1}^{L}\sum_{k=1}^{m_3}\sum_{l=1}^{n} ix_{h,i,1,k,l} \quad (5.8)$$

- Expected rate at which the inventory processing is switched on,

$$E_{ipo} = \sum_{k=1}^{m_2}\sum_{l=1}^{n} \sigma_k x_{1,0,1,k,l} + \sum_{i=1}^{L}\sum_{k=1}^{m_3}\sum_{l=1}^{n} \tau_k x_{1,i,1,k,l}$$

5.3 special cases

1. $p = 1$, $\phi = 0$
   In this case, the present model reduces to Divya et al.[8]. We see that the model can be studied as a LIQBD process.

2. $\phi = 0$
   In this case also, the model can be studied as a LIQBD process with obvious modifications in Divya et al.[8].

From now on we concentrate in the case $\phi = 0$.

First, we find the LST of the waiting time distribution.

5.3.1 Waiting Time Analysis

To find the waiting time of a customer who joins for service at time t, we have to consider different possibilities depending on the status of server at that time.The server may be on vacation or in normal mode. Let $W(t)$ be the waiting time of a customer in the system who arrives at time t and $W^*(s)$ be the corresponding LST.

Case I(Vacation mode)

Let E_1 denote the event that the tagged customer immedietly after his arrival finds the system in the state $(h'+1, i', 0, k', l')$ or in the state $(h'+1, L, 0, l')$, where $0 \leq h' \leq N-2$; $0 \leq i' \leq L-1$; $1 \leq k' \leq m_1$; $1 \leq l_1 \leq n$.

In this case, the waiting time is the time until absorption in a Markov process whose state space is given by $\{(h, i, k_1, l) : 1 \leq h \leq N-1; 0 \leq i \leq L-1; 1 \leq k_1 \leq m_1; 1 \leq l \leq n\} \cup \{(h, L, l) : 1 \leq h \leq N-1; 1 \leq l \leq n\} \cup \{(h^*, 0, k_2) : 1 \leq h^* \leq N-1; 1 \leq k_2 \leq m_2\} \cup \{(h^*, i, k_3) : 1 \leq h^* \leq N-1; 1 \leq i \leq L; 1 \leq k_3 \leq m_3\} \cup \{*\}$ where (h, i, k_1, l) denote the states that correspond to the server being in vacation with h customers in the system ,i,items in inventory, k_1,the processing phase and l,the arrival phase, (h, L, l) denote the state that correspond to the server being in vacation mode with h customers in the system, L items in inventory and l, the arrival phase. $(h^*, 0, k_2)$ denote the states that correspond to the tagged customer being in the position h^* when the server is in normal mode, k_2, the service phase when there is no processed item, (h^*, i, k_3) denote the states that correspond to the tagged customer being in position h^* when the server is in normal mode with i processed items in the inventory and k_3 denote the service phase and $*$ denote the absorbing state indicating the service completion of the tagged customer. Thus the conditional waiting time can be studied by a phase type distribution with representation PH$(\boldsymbol{\psi}_1, W_1)$ where

$$W_1 = \begin{bmatrix} M_{11} & M_{12} \\ 0 & M_{22} \end{bmatrix}, \boldsymbol{W}_1^0 = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{M}^0 \end{bmatrix},$$

where

$$\boldsymbol{M}^0 = \begin{bmatrix} \boldsymbol{M}^{00} \\ \mathbf{0} \end{bmatrix}, \text{ with } \boldsymbol{M}^{00} = \begin{bmatrix} \boldsymbol{S}^0 \\ \boldsymbol{e}(L) \otimes \boldsymbol{U}^0 \end{bmatrix}$$

$$M_{11} = \begin{bmatrix} E & I_{Lm_1+1} \otimes D_1 & & \\ & \ddots & \ddots & \\ & & E & I_{Lm_1+1} \otimes D_1 \\ & & & E \end{bmatrix},$$

where

$$E = \begin{bmatrix} T \oplus \Delta & \boldsymbol{T}^0 \boldsymbol{\alpha} \otimes I_n & & & \\ \ddots & \ddots & & & \\ & T \oplus \Delta & \boldsymbol{T}^0 \boldsymbol{\alpha} \otimes I_n & & \\ & & T \oplus \Delta & \boldsymbol{T}^0 \otimes I_n \\ & & & \Delta \end{bmatrix}$$

$$M_{12} = e_{N-1}(N-1)\boldsymbol{e}'_{h_1}(N-1) \otimes F,$$

where

$$F = \begin{bmatrix} \boldsymbol{e}(m_1) \otimes (p\delta \otimes \boldsymbol{\beta}) & & & \\ & \boldsymbol{e}(m_1) \otimes (p\delta \otimes \boldsymbol{\gamma}) & & \\ & & \ddots & \\ & & & p\delta \otimes \boldsymbol{\gamma} \end{bmatrix}$$

$$M_{22} = \begin{bmatrix} G & & & \\ H & G & & \\ & \ddots & \ddots & \\ & & H & G \end{bmatrix},$$

where

$$G = \begin{bmatrix} S & \\ & I_L \otimes U \end{bmatrix}, \; H = \begin{bmatrix} \boldsymbol{S}^0\boldsymbol{\beta} & 0 & 0 \\ \boldsymbol{U}^0\boldsymbol{\beta} & 0 & 0 \\ 0 & I_{L-1} \otimes (\boldsymbol{U}^0\boldsymbol{\gamma}) & 0 \end{bmatrix}$$

Thus the conditional LST,

$$W^*(s|E_1) = \boldsymbol{\psi}_1(sI - W_1)^{-1}\boldsymbol{W}_1^0,$$

where $\boldsymbol{\psi}_1$ is the initial probabilty vector which ensures that the Markov chain always starts from the level h.

Case II(Normal mode)

Let E_2 denote the event that the tagged customer immedietly after his joining finds the system in the state $(h'+1, 0, 1, k'', l')$, where $h' \geq 1$; $1 \leq k'' \leq m_2$; $1 \leq l' \leq n$ or in the state $(h'+1, i', 1, k''', l')$, where $1 \leq h' \leq N-1$; $1 \leq i' \leq L-N+h'$; $1 \leq k''' \leq m_3$; $1 \leq l' \leq n$ or in the state $(h'+1, i', 1, k''', l')$, where $h' \geq N$; $1 \leq i' \leq L$; $1 \leq k''' \leq m_3$; $1 \leq l' \leq n$.

In this case, the waiting time is the time until absorption in a Markov process whose state space is given by $\{(h, 0, k) : 2 \leq h \leq K;\ 1 \leq k \leq m_2\} \cup \{(h, i, k) : 2 \leq h \leq N-1;\ 1 \leq i \leq L-N+h;\ 1 \leq k \leq m_3\} \cup \{(h, i, k) : N \leq h \leq K;\ 1 \leq i \leq L;\ 1 \leq k \leq m_3\} \cup \{*\}$ where $(h, 0, k)$ denote the states that correspond to the server being in normal mode with h customers in the system, service phase k when there is no processed item, (h, i, k) denote the states that correspond to the server being in normal mode with h customers in the system, service phase k when there are i processed items and $*$ denote the absorbing state indicating the service completion of the tagged customer and K is chosen in such a way that $P\left(\sum_{h=0}^{K} \boldsymbol{x}_h\boldsymbol{e} > 1-\epsilon\right) \to 0$ for every $\epsilon > 0$. Thus the conditional waiting time can be studied by a truncated phase type distribution with representation PH($\boldsymbol{\psi}_2, W_2$) where

$$W_2 = \begin{bmatrix} G_1 & & & & & & & \\ H_1 & G_2 & & & & & & \\ & \ddots & \ddots & & & & & \\ & & H_{N-2} & G_{N-1} & & & & \\ & & & H_{N-1} & G & & & \\ & & & & H & G & & \\ & & & & & \ddots & \ddots & \\ & & & & & & H & G \end{bmatrix}, \boldsymbol{W}_2^0 = \begin{bmatrix} E^0 \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix},$$

where

$$\boldsymbol{E}^0 = \begin{bmatrix} \boldsymbol{S}^0 \\ \boldsymbol{e}(L-N+1) \otimes \boldsymbol{U}^0 \end{bmatrix},$$

For $1 \leq h \leq N-1$,

$$G_h = \begin{bmatrix} S & \\ & I_{L-N+h} \otimes U \end{bmatrix}, H_h = \begin{bmatrix} \boldsymbol{S}^0\boldsymbol{\beta} & 0 \\ \boldsymbol{U}^0\boldsymbol{\beta} & 0 \\ 0 & I_{L-N+h} \otimes \boldsymbol{U}^0\boldsymbol{\gamma} \end{bmatrix}$$

$$E = \begin{bmatrix} S & \\ & I_L \otimes U \end{bmatrix}, F = \begin{bmatrix} \boldsymbol{S}^0\boldsymbol{\beta} & 0 & 0 \\ \boldsymbol{U}^0\boldsymbol{\beta} & 0 & 0 \\ 0 & I_{L-1} \otimes \boldsymbol{U}^0\boldsymbol{\gamma} & 0 \end{bmatrix}$$

Thus the conditional LST,

$$W^*(s|E_2) = \psi_2(sI - W_2)^{-1}{W_2}^0$$

where $\boldsymbol{\psi}_2$ is the initial probabilty vector which ensures that the Markov chain always starts from the level h.

Let $w_{h,i,j,k,l}$ and $w_{h,L,0,l}$ denote the probabaility that the tagged customer finds the system in the state (h, i, j, k, l) and $(h, L, 0, l)$ respectively immedietly after his arrival. Then

$$
\begin{array}{lcll}
w_{h,i,0,k_1,l} & = & \sum_{l'=1}^{n} \frac{pd^{(1)}_{l'l}}{-d^{(0)}_{l'l'}-(1-p)\delta_{l'}-T_{k_1k_1}} x_{h-1,i,0,k_1,l'}, & 1 \leq h \leq N-1,\ 0 \leq i \leq L-1, \\
 & & & 1 \leq k_1 \leq m_1,\ 1 \leq l \leq n \\
w_{h,L,0,l} & = & \sum_{l'=1}^{n} \frac{pd^{(1)}_{l'l}}{-d^{(0)}_{l'l'}-(1-p)\delta_{l'}} x_{h-1,L,0,l'}, & 1 \leq h \leq N-1, 1 \leq l \leq n \\
w_{h,0,1,k_2,l} & = & \sum_{l'=1}^{n} \frac{pd^{(1)}_{l'l}}{-d^{(0)}_{l'l'}-(1-p)\delta_{l'}-S_{k_2k_2}} x_{h-1,0,1,k_2,l'}, & 2 \leq h \leq N-1 \text{ or } h \geq N+1, \\
 & & & 1 \leq k_2 \leq m_2, 1 \leq l \leq n \\
w_{N,0,1,k_2,l} & = & \sum_{l'=1}^{n} \sum_{k_1=1}^{m_1} \frac{pd^{(1)}_{l'l}\beta_{k_2}}{-d^{(0)}_{l'l'}-(1-p)\delta_{l'}-T_{k_1k_1}} x_{N-1,0,0,k_1,l'} & \\
 & & + \sum_{l'=1}^{n} \frac{pd^{(1)}_{l'l}}{-d^{(0)}_{l'l'}-(1-p)\delta_{l'}-S_{k_2k_2}} x_{N-1,0,1,k_2,l'}, & 1 \leq k_2 \leq m_2,\ 1 \leq l \leq n \\
w_{h,i,1,k_3,l} & = & \sum_{l'=1}^{n} \frac{pd^{(1)}_{l'l}}{-d^{(0)}_{l'l'}-(1-p)\delta_{l'}-U_{k_3k_3}} x_{h-1,i,1,k_3,l'}, & 2 \leq h \leq N-1,\ 1 \leq L-N+h-1, \\
 & & & 1 \leq k_3 \leq m_3,\ 1 \leq l \leq n \\
w_{h,i,1,k_3,l} & = & \sum_{l'=1}^{n} \frac{pd^{(1)}_{l'l}}{-d^{(0)}_{l'l'}-(1-p)\delta_{l'}-U_{k_3k_3}} x_{h-1,i,1,k_3,l'}, & h \geq N+1,\ 1 \leq i \leq L, \\
 & & & 1 \leq k_3 \leq m_3,\ 1 \leq l \leq n \\
w_{N,i,1,k_3,l} & = & \sum_{l'=1}^{n} \sum_{k_1=1}^{m_1} \frac{pd^{(1)}_{l'l}\gamma_{k_3}}{-d^{(0)}_{l'l'}-(1-p)\delta_{l'}-T_{k_1k_1}} x_{N-1,i,0,k_1,l'} & \\
 & & + \sum_{l'=1}^{n} \frac{pd^{(1)}_{l'l}}{-d^{(0)}_{l'l'}-(1-p)\delta_{l'}-S_{k_2k_2}} x_{N-1,i,1,k_2,l'}, & 1 \leq i \leq L,\ 1 \leq k_2 \leq m_2,\ 1 \leq l \leq n
\end{array}
$$

Thus we have the following Theorem.

Theorem 5.3.1. *The LST of the waiting time is given by*

$$
\begin{aligned}
W^*(s) = \frac{1}{d_2}\Big[&\sum_{h'=1}^{N-1}\sum_{i'=0}^{L-1}\sum_{k'=1}^{m_1}\sum_{l'=1}^{n} \psi_1(sI-W_1)^{-1}W_1^0 w_{h',i',0,k',l'} + \sum_{h'=1}^{N-1}\sum_{l'=1}^{n} \psi_1(sI-W_1)^{-1}W_1^0 w_{h',L,0,l'} \\
&+ \sum_{h'=1}^{\infty}\sum_{k''=1}^{m_2}\sum_{l'=1}^{n} \psi_2(sI-W_2)^{-1}W_2^0 w_{h',0,1,k'',l'} + \sum_{h'=1}^{N}\sum_{i'=1}^{L-N+h'-1}\sum_{k'''=1}^{m_3}\sum_{l'=1}^{n} \psi_2(sI-W_2)^{-1}W_2^0 w_{h',i',1,k''',l'} \\
&+ \sum_{h'=N+1}^{\infty}\sum_{i'=1}^{L}\sum_{k'''=1}^{m_3}\sum_{l'=1}^{n} \psi_2(sI-W_2)^{-1}W_2^0 w_{h',i',1,k''',l'}\Big]
\end{aligned} \tag{5.9}
$$

where

$$d_2 = \sum_{h'=1}^{N-1}\sum_{i'=0}^{L-1}\sum_{k'=1}^{m_1}\sum_{l'=1}^{n} w_{h',i',0,k',l'} + \sum_{h'=1}^{N-1}\sum_{l'=1}^{n} w_{h',L,0,l'} + \sum_{h'=1}^{\infty}\sum_{k''=1}^{m_2}\sum_{l'=1}^{n} w_{h',0,1,k'',l'} + \sum_{h'=1}^{N}\sum_{i'=1}^{L-N+h'-1}\sum_{k'''=1}^{m_3}\sum_{l'=1}^{n} w_{h',i',1,k''',l'} + \sum_{h'=N+1}^{\infty}\sum_{i'=1}^{L}\sum_{k'''=1}^{m_3}\sum_{l'=1}^{n} w_{h',i',1,k''',l'} \quad (5.10)$$

Now,we assume that each customer receives a reward of R units after service completion and he has to pay a price $q\,(0 \leq q < R)$ for an item. Let h_w denote the waiting cost per unit time of a customer in the system.

5.3.2 Individual equillibrium strategy

Define

$$F_1(p) = R - q - h_w E(W).$$

We shall find an equillibrium strategy according to which the customers join the system.

5.3.3 Revenue maximization

This is concerned with pricing of the item served to the customer. We have to find an optimal price q to maximize the revenue of the server given by

$$F_2(q) = p_e q \boldsymbol{\pi}^* D_1 \boldsymbol{e} - h_1 E_s - h_2 E_{it} - cE_{ipo}$$

where
h_1 : holding cost/customer/unit time
h_2 : holding cost/item/unit time
c : switching on cost of inventory processing each time it is turned on.
p_e : Individual equillibrium strategy corresponding to q.

5.3.4 Social optimal strategy

Next we consider social optimal strategy. For a given price q and a joining probability p, the surplus of all customers is $S_1 = \boldsymbol{\pi}^* pD_1 e(R - q - h_w E(w))$ and the server revenue is $S_2 = pq\boldsymbol{\pi}^* D_1 e - h_1 E_s - h_2 E_{it} - cE_{ipo}$.
Therefore, the expected social welfare per unit time is,

$$\begin{aligned} F_3(p) &= S_1 + S_2 \\ &= \pi^* pD_1 e(R - h_w E(W)) - h_1 E_s - h_2 E_{it} - cE_{ipo} \end{aligned}$$

5.3.5 Numerical results

Fix $N = 3$, $L = 2, \alpha = \beta = \begin{bmatrix} 1 & 0 \end{bmatrix}, \gamma = \begin{bmatrix} 0.8 & 0.2 \end{bmatrix}$, $T = \begin{bmatrix} -50 & 50 \\ 0 & -50 \end{bmatrix}$, $S = \begin{bmatrix} -80 & 80 \\ 0 & -80 \end{bmatrix}$, $U = \begin{bmatrix} -150 & 150 \\ 0 & -150 \end{bmatrix}, R = 75$, $q = 60$, $h_w = 50, h_1 = 2, h_2 = 1, c = 30$.

We find the individual optimum and social optimum corresponding to the above parameters.

p	E_{wt}	E_s	E_{it}	E_{ipo}	F_1	F_3
0.1	0.5045	1.0248	1.8794	0.6445	-10.2243	76.2876
0.2	0.2571	1.0514	1.7601	1.2429	2.1430	238.8498
0.3	0.1762	1.0806	1.6421	1.7921	6.1878	339.5609
0.4	0.1368	1.1129	1.5254	2.2882	8.1599	472.8811
0.5	0.1138	1.1486	1.4102	2.7271	9.3077	607.5573
0.6	0.0991	1.1881	1.2965	3.1041	10.0449	743.7425
0.7	0.0891	1.2316	1.1848	3.4149	10.5461	881.5515
0.8	0.0821	1.2794	1.0752	3.6551	10.8954	1021.0406
0.9	0.0773	1.3325	0.9680	3.8207	11.1356	1162.1853
1	0.0743	1.3925	0.8635	3.9081	11.2870	1304.8471

Table 5.1: Effect of p on various performance measures, when $D_0 = (-20)$, $D_1 = (20)$

p	E_{wt}	E_s	E_{it}	E_{ipo}	F_1	F_3
0.1	0.4052	1.0312	1.8494	0.7985	-5.2584	108.9862
0.2	0.2084	1.0657	1.7009	1.5239	4.5807	273.3550
0.3	0.1446	1.1045	1.5545	2.1694	7.7701	439.4306
0.4	0.1138	1.1486	1.4102	2.7271	9.3077	607.5573
0.5	0.0962	1.1986	1.2684	3.1882	10.1881	778.0391
0.6	0.0853	1.2549	1.1297	3.5441	10.7364	951.0852
0.7	0.0783	1.3187	0.9946	3.7865	11.0843	1126.7485
0.8	0.0743	1.3925	0.8635	3.9081	11.2869	1304.8471
0.9	0.0728	1.4827	0.7371	3.9027	11.3611	1484.8406
1	0.0741	1.6035	0.6156	3.7651	11.2950	1665.6008

Table 5.2: Effect of p on various performance measures, when $D_0 = (-25)$, $D_1 = (25)$

p	E_{wt}	E_s	E_{it}	E_{ipo}	F_1	F_3
0.1	0.3392	1.0378	1.8196	0.9496	-1.9586	141.7398
0.2	0.1762	1.0806	1.6421	1.7921	6.1878	339.5609
0.3	0.1240	1.1303	1.4676	2.5151	8.8023	540.0395
0.4	0.0991	1.1881	1.2965	3.1041	10.0449	743.7425
0.5	0.0853	1.2549	1.1297	3.5441	10.7365	951.0857
0.6	0.0773	1.3324	0.9680	3.8207	11.1356	1162.1853
0.7	0.0734	1.4260	0.8124	3.9215	11.3322	1376.6660
0.8	0.0732	1.5501	0.6636	3.8363	11.3401	1593.3114
0.9	0.0777	1.7424	0.5222	3.5570	11.1132	1809.3387
1	0.0898	2.1045	0.3886	3.0780	10.5080	2018.3028

Table 5.3: Effect of p on various performance measures, when $D_0 = (-30)$, $D_1 = (30)$

In Tables 5.1, 5.2 and 5.3, E_{wt} denotes the expected waiting time of an arbitrary customer. We can see that the E_{wt} decreases as p increases upto some p' (shown in bold letters) and after that it increases. This is due to the effect of N-policy. As p increases (upto p'), the number of customers in the system hit N more fast so that the server stops processing of service items and start serving customers and hence E_{wt} decreases. When p becomes p', E_{wt} starts increasing due to the diminished effect of N. Hence F_1 increases as p increases upto p' and after that it decreases. As we expect, E_s increases as p increases. As p increases, E_{it} decreases, since larger number of customers are served in a cycle. E_{ipo} increases upto p', as p increases. This is due to the effect of N-policy. As p increases, the number of customers in the system hit N more rapidly and hence customers leave the system quickly sothat the server can switch on to processing at a faster rate. When p increases beyond p', E_{ipo} decreases as p increases due to the diminished effect of N.

From Tables 5.1, 5.2 and 5.3, we can see that F_1 is strictly increasing on [0,p'] and strictly decreasing on [p',1]. Thus,

1. If $F_1(p') \leq 0$, then $F_1(p) \leq 0$ for all $p \in [0, 1]$. In this case, the maximum benefit is negative which implies that customers do not join the system even if there is no customer in the system.

2. If $F_1(0) > 0$ and $F_1(1) > 0$, then $F_1(p) > 0$ for all $p \in [0, 1]$. In this case, the customers prefer to join the system, because the minimal benefit is positive.

3. If, $F_1(p') \geq 0$ and $F_1(0) < 0$, $\exists\ p_e \in [0, p']$ such that $F_1(p_e) = 0$.

4. If $F_1(p') \geq 0$ and $F_1(1) < 0$, $\exists\ p_e \in [p', 1]$ such that $F_1(p_e) = 0$.

5. If $F_1(p') \geq 0$, $F_1(0) < 0$ and $F_1(1) < 0$ then $\exists\ p_e \in [0, p']$ such that $F_1(p_e) = 0$ and $p'_e \in [p', 1]$ such that $F_1(p'_e) = 0$.

Hence, if, either of the cases 3,4 and 5 happen, then the customers are indifferent between joining and balking the system. Suppose that, case 3 holds. Then the above discussions imply that when the joining probability p adopted by other customers is greater than p_e, the expected net benefit of an arriving customer is positive provided he joins, thus the unique best response is 1. Conversely, the unique best response is 0 if $p < p_e$ because then the expected net benefit is negative. If $p = p_e$, every strategy is the best response since the expected net benefit is always 0. This behaviour illustrates a situation that an individuals best response is an increasing function of the strategy selected by other customers. Therefore, we expect a crowd situation in this case due to the effect of N-policy.

Next, suppose that, case 4 holds. Then the above discussions imply that when the joining probability p adopted by other customers is smaller than p_e, the expected net benefit of an arriving customer is positive provided he joins, thus the unique best response is 1. Conversely, the unique best response is 0 if $p > p_e$ because then the expected net benefit is negative. If $p = p_e$, every strategy is the best response since the expected net benefit is always 0. This behaviour illustrates a situation that an individuals best response is a decreasing function of the strategy selected by other customers. Therefore, we can avoid a crowd situation. This is due to the diminished effect of N-policy.

Next, suppose that case 5 holds, then the above discussions imply that when the joining probability p adopted by other customers is greater than p_e and less than p'_e, the expected net benefit of an arriving customer is positive provided he joins, thus the unique best response is 1. Conversely, the unique best response is 0 if $p < p_e$ or $p > p'_e$ because then the expected net benefit is negative. If $p = p_e$ or $p = p'_e$, every strategy is the best response since the expected net benefit is always 0.

$p \downarrow q \rightarrow$	10	20	30	40	50	60	70	75
0.02	-50.11	-60.11	-70.11	-80.11	-90.11	-100.11	-110.11	-120.11
0.1	39.78	29.78	19.78	9.78	-0.22	-10.22	-20.22	-30.22
0.2	52.14	42.14	32.14	22.14	12.14	2.14	-7.86	-17.86
0.3	56.19	46.19	36.19	26.19	16.19	6.19	-3.81	-13.81
0.4	58.16	48.16	38.16	28.16	18.16	8.16	-1.84	-11.84
0.5	59.31	49.31	39.31	29.31	19.31	9.31	-0.69	-10.69
0.6	60.04	50.04	40.04	30.04	20.04	10.04	0.04	-9.96
0.7	60.55	50.55	40.55	30.55	20.55	10.55	0.55	-9.45
0.8	60.90	50.90	40.90	30.90	20.90	10.90	0.90	-9.1
0.9	61.14	51.14	41.14	31.14	21.14	11.14	1.14	-8.86
1	61.29	51.29	41.29	31.29	21.29	11.29	1.29	-8.71

Table 5.4: Individual optimum when $D_0 = (-20)$, $D_1 = (20)$

$p \downarrow q \rightarrow$	10	20	30	40	50	60	70	75
0.02	-25.12	-35.12	-45.12	-55.12	-65.12	-75.12	-85.12	-95.12
0.1	44.74	34.74	24.74	14.74	4.74	-5.26	-15.26	-25.26
0.2	54.58	44.58	34.58	24.58	14.58	4.58	-5.42	-15.42
0.3	57.77	47.77	37.77	27.77	17.77	7.77	-2.23	-12.23
0.4	59.31	49.31	39.31	29.31	19.31	9.31	-0.69	-10.69
0.5	60.19	50.19	40.19	30.19	20.19	10.19	0.19	-9.81
0.6	60.74	50.74	40.74	30.74	20.74	10.74	0.74	-9.26
0.7	61.08	51.08	41.08	31.08	21.08	11.08	1.08	-8.92
0.8	61.29	51.29	41.29	31.29	21.29	11.29	1.29	-8.71
0.9	61.36	51.36	41.36	31.36	21.36	11.36	1.36	-8.64
1	61.30	51.30	41.30	31.30	21.30	11.30	1.30	-8.70

Table 5.5: Individual optimum when $D_0 = (-25)$, $D_1 = (25)$

$p \downarrow q \rightarrow$	10	20	30	40	50	60	70	75
0.02	-8.46	-18.46	-28.46	-38.46	-48.46	-58.46	-68.46	-78.46
0.1	48.04	38.04	28.04	18.04	8.04	-1.96	-11.96	-21.96
0.2	56.19	46.19	36.19	26.19	16.19	6.19	-3.81	-13.81
0.3	58.80	48.80	38.80	28.80	18.80	8.80	-1.20	-11.20
0.4	60.05	50.05	40.05	30.05	20.05	10.05	0.05	-9.95
0.5	60.74	50.74	40.74	30.74	20.74	10.74	0.74	-8.26
0.6	61.14	51.14	41.14	31.14	21.14	11.14	1.14	-8.86
0.7	61.33	51.33	41.33	31.33	21.33	11.33	1.33	-8.67
0.8	61.34	51.34	41.34	31.34	21.34	11.34	1.34	-8.86
0.9	61.12	51.12	41.12	31.12	21.12	11.12	1.12	-8.88
1	60.51	50.51	40.51	30.51	20.51	10.51	0.51	-9.49

Table 5.6: Individual optimum when $D_0 = (-30)$, $D_1 = (30)$

From Tables 5.4, 5.5 and 5.6, we get the values of F_1 corresponding to different values of p and q when the arrival rates are 20, 25 and 30 respectively.

In our experiment, $\exists$ a q' such that $F_1(p') \geq 0$, $F_1(0) < 0, F_1(1) \geq 0$ and $\exists$ exactly one equillibrium p_e in $(0, p']$ for all $q \in [0, q')$ where $0 < q' < R$ (in Table 5.6, $q' = 70.51$). Also p_e is strictly increasing for all q in $[0, q')$(in Fig 5.1, $(p_e, 0)$ corresponding to different q's are plotted using squares). This is due to the effect of N-policy. Also, $\exists$ q'', where $q' \leq q'' < R$ such that when

$q' \leq q \leq q''$, $\exists\ p_e \in [0, p']$ and $p'_e \in [p', 1]$ such that p_e is strictly increasing and p'_e is strictly decreasing in $[q', q'']$ (in Table 5.6, $q'' = 71.34$). This case is shown in Fig 5.2. When $q \in (q'', R]$, $F_1(p) < 0$ for $p \in [0, 1]$ and there is no equillibrium probability. Hence, if q increases (upto q'), more customers are supposed to join the queue, since the server can start service only if the number of customers in the system hit N. When q increases from q'' to R, customers do not join the system since the maximum benefit is negative.

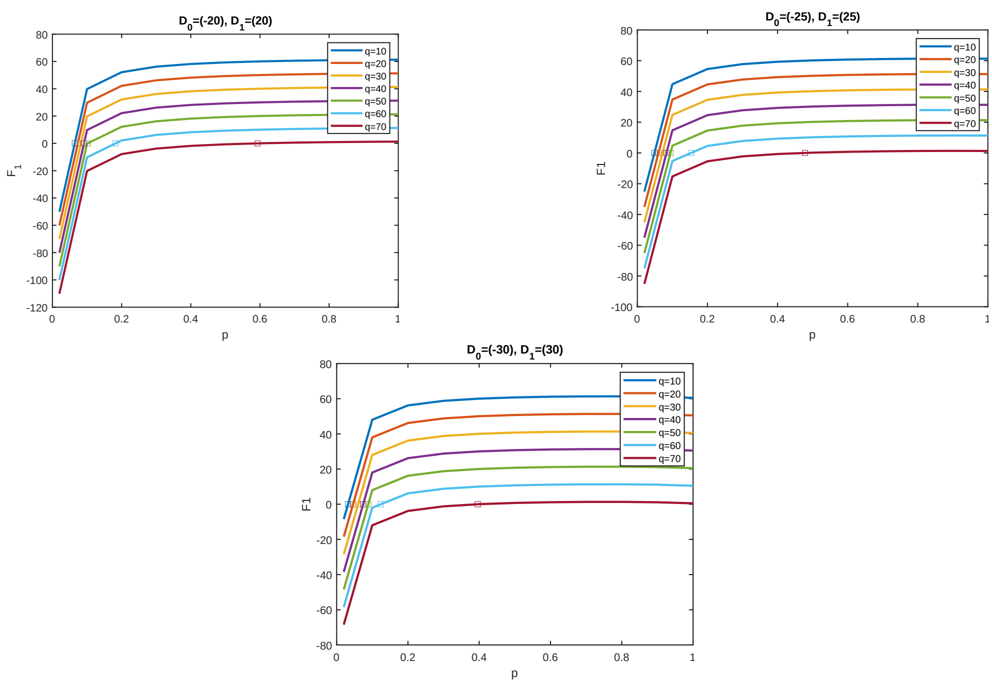

Figure 5.1: Effect of $q\,(< q')$ on individual equillibrium strategy

Fig 5.1 shows individual equillibrium probabilities p_e as q varies $(0 < q < q')$, corresponding to different arrival rates. We can see that p_e increases as q increases for the three different arrival rates. But p_e decreases as arrival rate increases. Fig 5.2 shows individual equillibrium probabilities p_e, p'_e as q varies $(q' \leq q \leq q'')$ corresponding to different arrival rates. We see that p_e increases

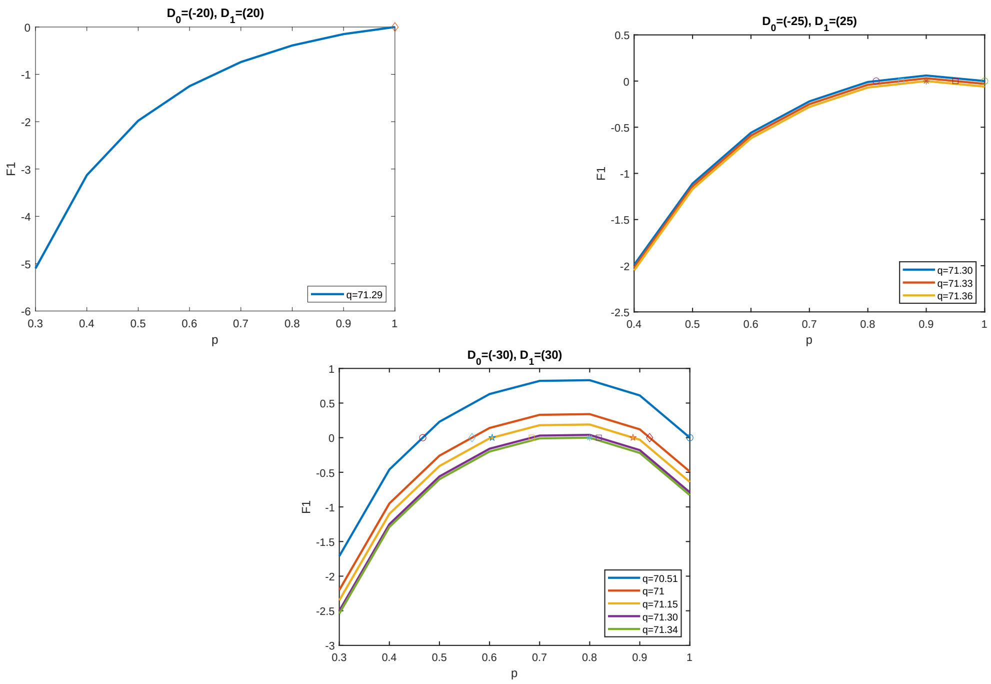

Figure 5.2: Effect of $q \, (q' \leq q \leq q'')$ on individual equillibrium strategy

and p_e' decreases as q increases and coincides when $q = q''$ for three different arrival rates.

Tables 5.7 and 5.8 show the effect of q on revenue of the server. Here, we see that F_2 decreases as q increases upto q'. This happens because when q increases upto q', p_e increases and hence the rate of hitting N becomes faster so that E_{ipo} increases. But we see that when q increases in $[q', q'']$, after a certain q-value, revenue function increases if p_e is the joining probability. This is due to the diminished effect of N-policy. Here, in all the cases, maximum revenue occur corresponding to $q = 10$ and the revenue decreases if a higher q is levied upto q'. But when q increases beyond q', after a certain q-value, revenue increases if a higher q is levied.

Again, from Tables 5.1, 5.2 and 5.3, we can see that F_3 increases as p

q	$D_0 = (-20)$, $D_1 = (20)$		$D_0 = (-25)$, $D_1 = (25)$		$D_0 = (-30)$, $D_1 = (30)$	
	p_e	F_2	p_e	F_2	p_e	F_2
10	0.0646	-15.31	0.0327	-11.22	0.0320	-12.45
20	0.0735	-16.82	0.0477	-14.45	0.0461	-16.08
30	0.0824	-18.32	0.0628	-17.67	0.0603	-19.67
40	0.0913	-19.82	0.0778	-20.81	0.0745	-23.20
50	0.1018	-21.56	0.0929	-23.92	0.0886	-26.65
60	0.1827	-34.50	0.1535	-35.91	0.1240	-35.01
70	0.5932	-84.23	0.4784	-84.63	0.3960	-84.29

Table 5.7: Revenue Maximization ($0 < q < q'$)

D_0, D_1	q	p_e	F_2	p'_e	F_2
(-20),(20)	71.29($q' = q''$)	1	-100.89	1	-100.89
(-25),(25)	71.30(q')	0.8143	-100.76	1	-91.78
	71.33	0.8571	-99.87	0.9500	-95.52
	71.36(q'')	0.9000	-98.28	0.9000	-98.28
(-30),(30)	70.51(q')	0.4667	-92.10	1	-66.92
	71	0.5650	-98.98	0.9197	-80.85
	71.15	0.6053	-100.39	0.8864	-85.567
	71.30	0.6842	-100.66	0.8182	-93.25
	71.34(q'')	0.8000	-94.85	0.8000	-94.85

Table 5.8: Revenue Maximization ($q' \leq q \leq q'$)

inreases. But the rate of increase decreases as p increases. Here, the social optimum corresponds to $p = 1$ (p_s) in all cases.

5.4 Special case: The system in normal mode

5.4.1 Waiting time Analysis

To find the waiting time of a customer who joins for service at an epoch in the long run, we have to consider different possibilities depending on the status of server at that time. Let E denote the event the system is working in normal mode. Let $W(t|E)$ be the conditional waiting time of a customer who arrives at time t and $W^*(s|E)$ be the corresponding conditional LST.

Let $w_{h,i,j,k,l}$ denote the probabaility that the tagged customer finds the system in the state (h,i,j,k,l) immediately after his arrival when the system is in normal mode.
Then

$$\begin{array}{lcll} w_{h,0,1,k,l} & = & \sum_{l'=1}^{n} \frac{pd^{(1)}_{l'l}}{-d^{(0)}_{l'l'}-(1-p)\delta_{l'}-S_{kk}} x_{h-1,0,1,k,l'}, & 2 \leq h \leq N-1 \,\text{or}\, h \geq N+1, \\ & & & 1 \leq k \leq m_2,\, 1 \leq l \leq n \\ w_{N,0,1,k,l} & = & \sum_{l'=1}^{n} \frac{pd^{(1)}_{l'l}}{-d^{(0)}_{l'l'}-(1-p)\delta_{l'}-S_{kk}} x_{N-1,0,1,k,l'}, & 1 \leq k \leq m_2,\, 1 \leq l \leq n \\ w_{h,i,1,k,l} & = & \sum_{l'=1}^{n} \frac{pd^{(1)}_{l'l}}{-d^{(0)}_{l'l'}-(1-p)\delta_{l'}-U_{kk}} x_{h-1,i,1,k,l'}, & 2 \leq h \leq N-1,\, 1 \leq L-N+h-1, \\ & & & 1 \leq k \leq m_3,\, 1 \leq l \leq n \\ w_{h,i,1,k,l} & = & \sum_{l'=1}^{n} \frac{pd^{(1)}_{l'l}}{-d^{(0)}_{l'l'}-(1-p)\delta_{l'}-U_{kk}} x_{h-1,i,1,k,l'}, & h \geq N+1,\, 1 \leq i \leq L,\, 1 \leq k \leq m_3, \\ & & & 1 \leq l \leq n \\ w_{N,i,1,k,l} & = & \sum_{l'=1}^{n} \frac{pd^{(1)}_{l'l}}{-d^{(0)}_{l'l'}-(1-p)\delta_{l'}-S_{kk}} x_{N-1,i,1,k,l'}, & 1 \leq i \leq L,\, 1 \leq k_2 \leq m_2,\, 1 \leq l \leq n \end{array}$$

Case I: $L \leq N$

Case (1)

Let E_1 denote the event that the tagged customer immediately after his arrival finds the system in the state $(r+1,0,0,1,k,l)$, where $r \geq 1$; $1 \leq k \leq m_2$; $1 \leq l \leq n$. In this case, processed item is not available to any customer. Thus waiting time is the sum of residual service time and r service time each following $PH(\boldsymbol{\beta}, S)$.

$$W^*(s|E, E_1) = \boldsymbol{e'}_u (sI-S)^{-1}\boldsymbol{S}^0(\boldsymbol{\beta}(sI-S)^{-1}\boldsymbol{S}^0)^r$$

Case (2)

Let E_2 denote the event that the tagged customer immediately after his arrival finds the system in the state $(r+1,i,1,k,l)$, where $1 \leq r \leq N-1$; $1 \leq i \leq L-N+r$; $1 \leq k \leq m_3$; $1 \leq l \leq n$. In this case, processed item is available to i customers. Thus waiting time is the sum of residual service time and $i-1$

service time each following PH($\boldsymbol{\gamma}, U$) and $r+1-i$ service time each following PH($\boldsymbol{\beta}, S$).

$$W^*(s|E,E_2) = \boldsymbol{e'}_u(sI-U)^{-1}\boldsymbol{U}^0(\boldsymbol{\gamma}(sI-U)^{-1}\boldsymbol{U}^0)^{i-1}(\boldsymbol{\beta}(sI-S)^{-1}\boldsymbol{S}^0)^{r+1-i}$$

Case (3)

Let E_3 denote the event that the tagged customer immediately after his arrival finds the system in the state $(r+1,i,1,k,l)$, where $r \geq N$; $1 \leq i \leq L$; $1 \leq k \leq m_3$; $1 \leq l \leq n$. In this case, processed item is available to i customers. Thus waiting time is the sum of residual service time and $i-1$ service time each following PH($\boldsymbol{\gamma}, U$) and $r+1-i$ service time each following PH($\boldsymbol{\beta}, S$).

$$W^*(s|E,E_3) = \boldsymbol{e'}_u(sI-U)^{-1}\boldsymbol{U}^0(\boldsymbol{\gamma}(sI-U)^{-1}\boldsymbol{U}^0)^{i-1}(\boldsymbol{\beta}(sI-S)^{-1}\boldsymbol{S}^0)^{r+1-i}$$

Thus the conditional LST of the waiting time,

$$W^*(s|E) = \frac{1}{d_3}\Big[\sum_{r=1}^{\infty}\sum_{k=1}^{m_2}\sum_{l=1}^{n} W^*(s|E,E_1)w_{r+1,0,1,k,l} + \sum_{r=1}^{N-1}\sum_{i=1}^{L-N+r}\sum_{k=1}^{m_3}\sum_{l=1}^{n} W^*(s|E,E_2) w_{r+1,i,1,k,l} + \sum_{r=N}^{\infty}\sum_{i=1}^{L}\sum_{k=1}^{m_3}\sum_{l=1}^{n} W^*(s|E,E_3)w_{r+1,i,1,k,l}\Big] \quad (5.11)$$

where

$$d_3 = \sum_{r=1}^{\infty}\sum_{k=1}^{m_2}\sum_{l=1}^{n} w_{r+1,0,1,k,l} + \sum_{r=1}^{N-1}\sum_{i=1}^{L-N+r}\sum_{k=1}^{m_3}\sum_{l=1}^{n} w_{r+1,i,1,k,l} + \sum_{r=N}^{\infty}\sum_{i=1}^{L}\sum_{k=1}^{m_3}\sum_{l=1}^{n} w_{r+1,i,1,k,l} \quad (5.12)$$

Case II: $L > N$

Case(1)

Let F_1 denote the event that the tagged customer immediately after his arrival finds the system in the state $(r+1, 0, 1, k, l)$, where $r \geq 1$; $1 \leq k \leq m_2$; $1 \leq l \leq n$. In this case, processed item is not available to any customer. Thus waiting time is the sum of residual service time and r service time each following PH($\boldsymbol{\beta}, S$).

$$W^*(s|E, F_1) = \boldsymbol{e'}_u(sI - S)^{-1}\boldsymbol{S}^0(\boldsymbol{\beta}(sI - S)^{-1}\boldsymbol{S}^0)^r$$

Case(2)

Let F_2 denote the event that the tagged customer immediately after his arrival finds the system in the state $(r+1, i, 1, k, l)$, where $1 \leq r \leq N-1$; $1 \leq i \leq L-N+r$; $1 \leq k \leq m_3$; $1 \leq l \leq n$.

Case(i), $1 \leq i < r+1$

In this case, processed item is available to i customers. Thus the conditional LST,

$$W^*(s|E, F_2) = \boldsymbol{e'}_u(sI - U)^{-1}\boldsymbol{U}^0(\boldsymbol{\gamma}(sI - U)^{-1}\boldsymbol{U}^0)^{i-1}(\boldsymbol{\beta}(sI - S)^{-1}\boldsymbol{S}^0)^{r+1-i}$$

Case(ii), $r+1 \leq i \leq L-N+r$

In this case, processed item is available to all the $r+1$ customers. Thus the conditional LST,

$$W^*(s|E, F_2) = \boldsymbol{e'}_u(sI - U)^{-1}\boldsymbol{U}^0(\boldsymbol{\gamma}(sI - U)^{-1}\boldsymbol{U}^0)^r$$

Case(3)

Let F_3 denote the event that the tagged customer immediately after his arrival finds the system in the state $(r+1, i, 1, k, l)$, where $r \geq N$; $1 \leq i \leq L$; $1 \leq k \leq m_3$; $1 \leq l \leq n$.

Case (i), $N \leq r \leq L$

Case(a), $1 \leq i < r+1$

In this case, processed item is available to i customers. Thus the conditional LST,

$$W^*(s|E, F_3) = \boldsymbol{e'}_u(sI - U)^{-1}\boldsymbol{U}^0(\boldsymbol{\gamma}(sI - U)^{-1}\boldsymbol{U}^0)^{i-1}(\boldsymbol{\beta}(sI - S)^{-1}\boldsymbol{S}^0)^{r+1-i}$$

Case(b), $r+1 \leq i \leq L$

In this case, processed item is available to all the $r+1$ customers. Thus the conditional LST,

$$W^*(s|E, F_3) = \boldsymbol{e'}_u(sI - U)^{-1}\boldsymbol{U}^0(\boldsymbol{\gamma}(sI - U)^{-1}\boldsymbol{U}^0)^{r}$$

Case (ii), $r \geq L+1$

In this case, processed item is available to i customers. Thus the conditional LST,

$$W^*(s|E, F_3) = \boldsymbol{e'}_u(sI - U)^{-1}\boldsymbol{U}^0(\boldsymbol{\gamma}(sI - U)^{-1}\boldsymbol{U}^0)^{i-1}(\boldsymbol{\beta}(sI - S)^{-1}\boldsymbol{S}^0)^{r+1-i}$$

Thus the conditional LST of the waiting time,

$$W^*(s|E) = \frac{1}{d_4}\Big[\sum_{r=1}^{\infty}\sum_{k=1}^{m_2}\sum_{l=1}^{n} W^*(s|E, F_1)w_{r+1,0,1,k,l} + \sum_{r=1}^{N-1}\sum_{i=1}^{L-N+r}\sum_{k=1}^{m_3}\sum_{l=1}^{n} W^*(s|E, F_2) w_{r+1,i,1,k,l} + \sum_{r=N}^{\infty}\sum_{i=1}^{L}\sum_{k=1}^{m_3}\sum_{l=1}^{n} W^*(s|E, F_3))w_{r+1,i,1,k,l}\Big] \quad (5.13)$$

where

$$d_4 = \sum_{r=1}^{\infty}\sum_{k=1}^{m_2}\sum_{l=1}^{n} w_{r+1,0,1,k,l} + \sum_{r=1}^{N-1}\sum_{i=1}^{L-N+r}\sum_{k=1}^{m_3}\sum_{l=1}^{n} w_{r+1,i,1,k,l} + \sum_{r=N}^{\infty}\sum_{i=1}^{L}\sum_{k=1}^{m_3}\sum_{l=1}^{n} w_{r+1,i,1,k,l} \quad (5.14)$$

Fix $N = 3$, $L = 2, \boldsymbol{\alpha} = \boldsymbol{\beta} = \begin{bmatrix} 1 & 0 \end{bmatrix}, \boldsymbol{\gamma} = \begin{bmatrix} 0.8 & 0.2 \end{bmatrix}$, $T = \begin{bmatrix} -50 & 50 \\ 0 & -50 \end{bmatrix}$, $S = \begin{bmatrix} -80 & 80 \\ 0 & -80 \end{bmatrix}$, $U = \begin{bmatrix} -150 & 150 \\ 0 & -150 \end{bmatrix}, R = 75$, $q = 60$, $h_w = 50, h_1 = 2, h_2 = 1, c = 30$.

p	E_{wt}	E_s	E_{it}	E_{ipo}	F_1	F_3
0.1	0.0485	1.5214	0.6972	19.3813	12.5747	-440.0284
0.2	0.0499	1.5604	0.6564	18.3266	12.5034	-263.5605
0.3	0.0512	1.6081	0.6117	17.2386	12.4386	-86.3546
0.4	0.0524	1.6657	0.5640	16.1240	12.3798	91.4217
0.5	0.0535	1.7341	0.5145	14.9939	12.3259	269.4597
0.6	0.0545	1.8144	0.4645	13.8595	12.2749	447.4200
0.7	0.0555	1.9081	0.4153	12.7308	12.2237	624.9757
0.8	0.0566	2.0174	0.3677	11.6154	12.1681	801.8239
0.9	0.0579	2.1455	0.3224	10.5189	12.1027	977.6669
1	0.0596	2.2971	0.2796	9.4448	12.0200	1152.1810

Table 5.9: Effect of p on various performance measures, when $D_0 = (-20)$, $D_1 = (20)$

From Table 5.9, we see that E_{wt} increases as p increases. This happens since when the system is working in normal mode, the number of customers accumulating in the system increases with increasing value of p. As p increases F_1 decreases consequent to increase in E_{wt}. As we expect, E_s increases as p increases. As p increases, E_{it} decreases, since larger number of customers get served in a cycle. E_{ipo} decreases as p increases. This happens due to the fact that when p increases more customers accumulate in the system and hence customers leave the system slowly so that the server switch on to processing at a slower rate. Also from Table 5.9, we see that F_3 increases as p increases. Thus the social optimum corresponds to $p = 1$.

Here when $q < 72.02$, the expected net benefit is always positive. When q increases beyond 72.02, (see Table 5.9), we can find a $p_e \in [0, 1]$ such that $F_1(p_e) = 0$ and p_e is decreasing (see Fig 5.3). Here when the joining probability p adopted by other customers is smaller than p_e, the expected net benefit of an

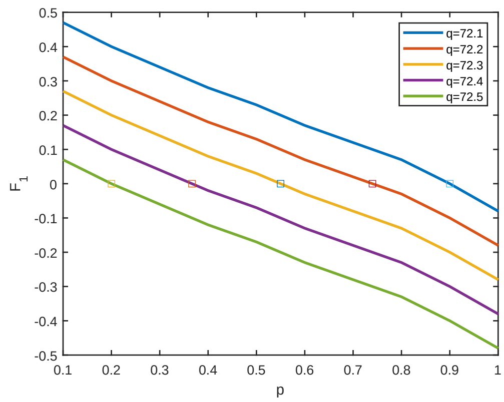

Figure 5.3: Effect of q on individual equillibrium strategy when $D_0 = (-20), D_1 = (20)$

arriving customer is positive provided he joins. Thus the unique best response is 1. Conversely, the unique best response is 0 if $p > p_e$ since, the expected net benefit is negative. If $p = p_e$, every strategy is the best response since the expected net benefit is always 0. This behaviour illustrates a situation that an individuals best response is a decreasing function of the strategy selected by other customers. Therefore, we can avoid a crowd situation.

q	p_e	F_2
72.1	0.9000	-302.1809
72.2	0.7400	-357.9769
72.3	0.5500	-425.8322
72.4	0.3667	-491.4621
72.5	0.2000	-549.3741

Table 5.10: Effect of q on Revenue function

Also, in this case revenue function F_2 decreases as q increases. This happens due to the fact that as q increases, the equillibrium probability p_e decreases and hence E_{ipo} increases (see Table 5.10).

Chapter 6

A Two-Server Queueing System with Processing of Service Items by a Server

This chapter is concerned with a two server queueing system in which Server 1 (S_1) provides service alone, whereas Server 2 (S_2) provides service and also processes the item required (we call this additional item or inventory) to serve the customers. Each customer requires exactly one additional item for his service. In the absence of this additional item service cannot be provided. Therefore S_2 keeps processing the item until it hits a threshold value L. At this epoch he switches to serve customers, if any waiting. However, when the additional item level reduces to s, S_2 returns to process items. His service rate is higher than that of S_1; both servers provide service according to phase type

1. Presented in the International Conference on Advances in Applied Probability and Stochastic Processes organised by the Centre for Research, Department of Mathematics, CMS College, Kottayam held from 07-10 January 2019.
2. Some results of this chapter are included in the following paper.
A. Krishnamoorthy, Divya V.: **A Two-Server Queueing System with Processing of Service Items by a Server** (communicated).

distributed random variable. Processing of each additional item requires a Phase type distributed amount of time, independent of the arrival and service processes.

6.1 Model Description and Mathematical formulation

We consider a two-server queueing system in which the customers arrive according to Markovian Arrival Process with representation (D_0, D_1) of order n. Each customer is to be provided with a processed item at the end of his service. S_1 is always available to the customers provided processed item is available, whereas S_2 produces items for service(inventory) for future use whenever the inventory level drops to a threshold s. Until the inventory level reaches L, (the maximum permitted level) he does not provide service to customers. The inventory processing time follows phase type distribution PH$(\boldsymbol{\alpha}, T)$ of order m_1. After processing L items, S_2 starts serving customers if any waiting; else stays idle. S_1 is dedicated to service only. Servers provide service only if there are processed items. Also, when a customer arrives to an empty system, S_1 provides him service and S_2 remains idle even he is not engaged in processing the inventory. The service time at S_2 follows phase type distribution PH$(\boldsymbol{\beta}, S)$ of order m_2 and that at S_1 follows phase type distribution PH$(\boldsymbol{\beta}, \theta S)$ of order m_2, $0 < \theta < 1$. If the inventory level drops to level s after a service completion by S_2, then he starts processing items. If the inventory level drops to level s due to a service completion by S_1, then the customer served by S_2 is shifted to S_1 for the remaining part of his service and S_2 goes for processing items. The arrival process is independent of the inventory processing and service process.

6.1.1 The QBD process

The model described above can be studied as a LIQBD process. First we introduce the following notations:
At time t:
$N(t)$: the number of customers in the system
$I(t)$: the number of processed items in the inventory

$$J(t) : \text{status of } S_2 = \begin{cases} 0, \text{ when } S_2 \text{ is processing items} \\ 1, \text{ when } S_2 \text{ is serving a customer} \end{cases}$$

$$K_1(t) = \begin{cases} \text{processing/service phase of } S_2 \\ 0, \text{ when } S_2 \text{ is idle} \end{cases}$$

$$K_2(t) = \begin{cases} \text{service phase of } S_1 \\ 0, \text{ when } S_1 \text{ is idle} \end{cases}$$

$M(t)$: the phase of arrival of the customer.

It is easy to verify that $\{(N(t), I(t), J(t), K_1(t), K_2(t), M(t)) : t \geq 0\}$ is a LIQBD with state space:

(i) no customer in the system
$l(0) = \{(0, i, 0, k_1, 0, p) : 0 \leq i \leq L-1; 1 \leq k_1 \leq m_1; 1 \leq p \leq n\} \cup \{(0, i, 0, 0, p) : s+1 \leq i \leq L; 1 \leq p \leq n\}$
(ii) when there is 1 customer in the system
$l(1) = \{(1, 0, 0, k_1, 0, p) : 1 \leq k_1 \leq m_1; 1 \leq p \leq n\} \cup \{(1, i, 0, k_1, k_2, p) : 1 \leq i \leq L-1; 1 \leq k_1 \leq m_1; 1 \leq k_2 \leq m_2; 1 \leq p \leq n\} \cup \{(1, i, 0, k_2, p) : s+1 \leq i \leq L; 1 \leq k_2 \leq m_2; 1 \leq p \leq n\} \cup \{(1, i, 1, k_1, 0, p) : s+1 \leq i \leq L-1; 1 \leq k_1 \leq m_2; 1 \leq p \leq n\}$
(iii) when there are h customers in the system, $h \geq 2$:
$l(h) = (h, 0, 0, k_1, 0, p) : 1 \leq k_1 \leq m_1; 1 \leq p \leq n\} \cup \{(h, i, 0, k_1, k_2, p) : 1 \leq i \leq L-1; 1 \leq k_1 \leq m_1; 1 \leq k_2 \leq m_2; 1 \leq p \leq n\} \cup \{(h, i, 1, k_1, k_2, p) : s+1 \leq i \leq$

$L; 1 \leq k_1, k_2 \leq m_2; 1 \leq p \leq n\}$
Note that when $K_1(t) = 0$, $J(t)$ need not be considered.
The infinitesimal generator of this CTMC is

$$\bar{Q} = \begin{bmatrix} A_{00} & A_{01} & & & & \\ A_{10} & A_{11} & A_{12} & & & \\ & A_{21} & A_1 & A_0 & & \\ & & A_2 & A_1 & A_0 & \\ & & & \ddots & \ddots & \ddots \end{bmatrix}.$$

where $A_{00}, A_{01}, A_{10}, A_{11}, A_{12}, A_{21}$ represent transitions within level 0, from level 0 to level 1, from level 1 to level 0, within level 1, from level 1 to level 2, from level 2 to level 1 respectively; A_0 represents transitions from level h to level $h+1$ for $h \geq 2$, A_1 represents transitions within the level h for $h \geq 2$ and A_2 represents transitions from level h to $h-1$ for $h \geq 3$. The boundary blocks $A_{00}, A_{01}, A_{10}, A_{11}, A_{12}, A_{21}$ are of orders $(s+1)m_1n+(L-s-1)(1+m_1)n+n$, $((s+1)m_1n+(L-s-1)(1+m_1)n+n) \times (m_1n+sm_1m_2n+(L-s-1)(2m_2+m_1m_2)n+m_2n)$,$(m_1n+sm_1m_2n+(L-s-1)(2m_2+m_1m_2)n+m_2n) \times ((s+1)m_1n+(L-s-1)(1+m_1)n+n)$, $m_1n+sm_1m_2n+(L-s-1)(2m_2+m_1m_2)n+m_2n$, $(m_1n+sm_1m_2n+(L-s-1)(2m_2+m_1m_2)n+m_2n) \times (m_1n+sm_1m_2n+(L-s-1)(m_1m_2n+{m_2}^2n)+{m_2}^2n)$, $(m_1n+sm_1m_2n+(L-s-1)(m_1m_2n+{m_2}^2n)+{m_2}^2n) \times (m_1n+sm_1m_2n+(L-s-1)(2m_2+m_1m_2)n+m_2n)$ respectively. A_0, A_1, A_2 are square matrices of order $m_1n+sm_1m_2n+(L-s-1)(m_1m_2n+{m_2}^2n)+{m_2}^2n$.

Define the entries of $A_{pq_{(h_1,i_1,j_1,k_1,l_1)}}^{(h_2,i_2,j_2,k_2,l_2)}$ as transition submatrices which contains transitions of the form $(p,h_1,i_1,j_1,k_1,l_1) \rightarrow (q,h_2,i_2,j_2,k_2,l_2)$, where $q=0$ or 1, when $p=0$; $q=0,1$ or 2, when $p=1$ and $q=1$, when $p=2$. Define the entries of $A_{0_{(h_1,i_1,j_1,k_1,l_1)}}^{(h_2,i_2,j_2,k_2,l_2)}$, $A_{1_{(h_1,i_1,j_1,k_1,l_1)}}^{(h_2,i_2,j_2,k_2,l_2)}$, $A_{2_{(h_1,i_1,j_1,k_1,l_1)}}^{(h_2,i_2,j_2,k_2,l_2)}$ as transition submatrices which contains transitions of the form $(g,h_1,i_1,j_1,k_1,l_1) \rightarrow (g+1,h_2,i_2,j_2,k_2,l_2)$, where $g \geq 2$; $(g,h_1,i_1,j_1,k_1,l_1) \rightarrow (g,h_2,i_2,j_2,k_2,l_2)$, where $g \geq 2$; $(g, h_1, i_1, j_1, k_1, l_1) \to (g-1, h_2, i_2, j_2, k_2, l_2)$, where $g \geq 3$ respectively. Since none or one event alone could take place in a short interval of time with positive probability, in general, a transition such as $(g_1, h_1, i_1, j_1, k_1, l_1) \to (g_2, h_2, i_2, j_2, k_2, l_2)$ has positive rate only for exactly one of $g_2, h_2, i_2, j_2, k_2, l_2$ different from $g_1, h_1, i_1, j_1, k_1, l_1$.

$$A_{00_{(i_1,j_1,k_1,k_2,l_1)}}^{(i_2,j_2,k_1',k_2',l_2)} = \begin{cases} \boldsymbol{T}^0\boldsymbol{\alpha} \otimes I_n & i_2 = i_1+1, 0 \leq i_1 \leq L-2; j_1 = j_2 = 0; 1 \leq k_1, k_1' \leq m_1; \\ & k_2 = k_2' = 0; 1 \leq l_1, l_2 \leq n \\ \boldsymbol{T}^0 \otimes I_n & i_1 = L-1, i_2 = L; j_1 = j_2 = 0; 1 \leq k_1 \leq m_1, \\ & k_1' = 0; k_2 = k_2' = 0; 1 \leq l_1, l_2 \leq n \\ T \oplus D_0 & i_1 = i_2, 0 \leq i_1 \leq L-1; j_1 = j_2 = 0; 1 \leq k_1, k_1' \leq m_1; \\ & k_2 = k_2' = 0; 1 \leq l_1, l_2 \leq n \\ D_0 & i_1 = i_2, s+1 \leq i_1 \leq L; k_1 = k_1' = 0; k_2 = k_2' = 0; \\ & 1 \leq l_1, l_2 \leq n \end{cases}$$

$$A_{01_{(i_1,j_1,k_1,k_2,l_1)}}^{(i_2,j_2,k_1',k_2',l_2)} = \begin{cases} I_{m_1} \otimes D_1 & i_1 = i_2 = 0; j_1 = j_2 = 0; 1 \leq k_1, k_1' \leq m_1; \\ & k_2 = k_2' = 0; 1 \leq l_1, l_2 \leq n \\ I_{m_1} \otimes (\boldsymbol{\beta} \otimes D_1) & i_1 = i_2, 1 \leq i_1 \leq L-1; j_1 = j_2 = 0; \\ & 1 \leq k_1, k_1' \leq m_1; k_2 = 0; \\ & 1 \leq k_2' \leq m_2; 1 \leq l_1, l_2 \leq n \\ \boldsymbol{\beta} \otimes D_1 & i_1 = i_2, s+1 \leq i_1 \leq L; k_1 = k_1' = 0; \\ & k_2 = 0; 1 \leq k_2' \leq m_2; 1 \leq l_1, l_2 \leq n \end{cases}$$

$$A_{10_{(i_1,j_1,k_1,k_2,l_1)}}^{(i_2,j_2,k'_1,k_2,l_2)} = \begin{cases} I_{m_1} \otimes (\theta \boldsymbol{S}^0 \otimes I_n) & i_2 = i_1 - 1, 1 \le i_1 \le L-1; j_1 = j_2 = 0; \\ & 1 \le k_1, k'_1 \le m_1, 1 \le k_2 \le m_2; k'_2 = 0; \\ & 1 \le l_1, l_2 \le n \\ \theta \boldsymbol{S}^0 \boldsymbol{\alpha} \otimes I_n & i_1 = s+1, i_2 = s; j_2 = 0; k_1 = 0, 1 \le k'_1 \le m_1; \\ & 1 \le k_2 \le m_2; k'_2 = 0; 1 \le l_1, l_2 \le n \\ \theta \boldsymbol{S}^0 \otimes I_n & i_2 = i_1 - 1, s+2 \le i_1 \le L; k_1 = k'_1 = 0; \\ & 1 \le k_2 \le m_2; k'_2 = 0; 1 \le l_1, l_2 \le n \\ \boldsymbol{S}^0 \boldsymbol{\alpha} \otimes I_n & i_1 = s+1, i_2 = s; j_1 = 1, j_2 = 0; 1 \le k_1 \le m_2, \\ & 1 \le k'_1 \le m_1; k_2 = k'_2 = 0; 1 \le l_1, l_2 \le n \\ \boldsymbol{S}^0 \otimes I_n & i_2 = i_1 - 1, s+2 \le i_1 \le L-1; j_1 = 1; \\ & 1 \le k_1 \le m_2, k'_1 = 0; k_2 = k'_2 = 0; \\ & 1 \le l_1, l_2 \le n \end{cases}$$

$$A_{11_{(i_1,j_1,k_1,k_2,l_1)}}^{(i_2,j_2,k'_1,k'_2,l_2)} = \begin{cases} \boldsymbol{T}^0 (\boldsymbol{\alpha} \otimes \boldsymbol{\beta}) \otimes I_n & i_1 = 0, i_2 = 1; j_1 = j_2 = 0; 1 \le k_1, k'_1 \le m_1; \\ & k_2 = 0, 1 \le k'_2 \le m_2; 1 \le l_1, l_2 \le n \\ \boldsymbol{T}^0 \boldsymbol{\alpha} \otimes I_{m_2 n} & 1 \le i_1 \le L-2, i_2 = i_1 + 1; j_1 = j_2 = 0; \\ & 1 \le k_1, k'_1 \le m_1; 1 \le k_2, k'_2 \le m_2; 1 \le l_1, l_2 \le n \\ \boldsymbol{T}^0 \otimes I_{m_2 n} & i_1 = L-1, i_2 = L; j_1 = 0; 1 \le k_1 \le m_1, k'_1 = 0; \\ & 1 \le k_2, k'_2 \le m_2; 1 \le l_1, l_2 \le n \\ T \oplus D_0 & i_1 = i_2 = 0; j_1 = j_2 = 0; 1 \le k_1, k'_1 \le m_1; \\ & k_2 = k'_2 = 0; 1 \le l_1, l_2 \le n \\ \theta S \oplus D_0 & i_1 = i_2, s+1 \le i_1 \le L; k_1 = k'_1 = 0; \\ & 1 \le k_2, k'_2 \le m_2; 1 \le l_1, l_2 \le n \\ S \oplus D_0 & i_1 = i_2, s+1 \le i_1 \le L-1; j_1 = j_2 = 1; \\ & 1 \le k_1, k'_1 \le m_2; k_2 = k'_2 = 0; \\ & 1 \le l_1, l_2 \le n \\ T \oplus \theta S \oplus D_0 & i_1 = i_2, 1 \le i_1 \le L-1; j_1 = j_2 = 0; \\ & 1 \le k_1, k'_1 \le m_1; 1 \le k_2, k'_2 \le m_2; 1 \le l_1, l_2 \le n \end{cases}$$

$$A_{12_{(i_1,j_1,k_1,k_2,l_1)}}^{(i_2,j_2,k_1',k_2',l_2)} = \begin{cases} I_{m_1} \otimes D_1 & i_1 = i_2 = 0; j_1 = j_2 = 0; 1 \leq k_1, k_1' \leq m_1; \\ & k_2 = k_2' = 0; 1 \leq l_1, l_2 \leq n \\ I_{m_1 m_2} \otimes D_1 & i_1 = i_2, 1 \leq i_1 \leq L-1; j_1 = j_2 = 0; 1 \leq k_1, k_1' \leq m_1; \\ & 1 \leq k_2, k_2' \leq m_2; 1 \leq l_1, l_2 \leq n \\ \boldsymbol{\beta} \otimes (I_{m_2} \otimes D_1) & i_1 = i_2, s+1 \leq i_1 \leq L; j_2 = 1; k_1 = 0, 1 \leq k_1' \leq m_2; \\ & 1 \leq k_2, k_2' \leq m_2; 1 \leq l_1, l_2 \leq n \\ I_{m_2} \otimes (\boldsymbol{\beta} \otimes D_1) & i_1 = i_2, s+1 \leq i_1 \leq L-1; j_1 = j_2 = 1; \\ & 1 \leq k_1, k_1' \leq m_2; k_2 = 0, 1 \leq k_2' \leq m_2; \\ & 1 \leq l_1, l_2 \leq n \end{cases}$$

$$A_{21_{(i_1,j_1,k_1,k_2,l_1)}}^{(i_2,j_2,k_1',k_2,l_2)} = \begin{cases} I_{m_1} \otimes (\theta \boldsymbol{S}^0 \otimes I_n) & i_1 = 1, i_2 = 0; j_1 = j_2 = 0; 1 \leq k_1, k_1' \leq m_1, \\ & 1 \leq k_2 \leq m_2, k_2' = 0; 1 \leq l_1, l_2 \leq n \\ I_{m_1} \otimes (\theta \boldsymbol{S}^0 \boldsymbol{\beta} \otimes I_n) & i_2 = i_1 - 1, 2 \leq i_1 \leq L-1; j_1 = j_2 = 0; \\ & 1 \leq k_1, k_1' \leq m_1; 1 \leq k_2, k_2' \leq m_2; \\ & 1 \leq l_1, l_2 \leq n \\ \theta \boldsymbol{S}^0 \otimes I_{m_2 n} & i_2 = i_1 - 1, s+2 \leq i_1 \leq L; j_1 = j_2 = 1; \\ & 1 \leq k_1, k_1' \leq m_2; 1 \leq k_2 \leq m_2; k_2' = 0; \\ & 1 \leq l_1, l_2 \leq n \\ \boldsymbol{S}^0 \otimes I_{m_2 n} & i_2 = i_1 - 1, s+2 \leq i_1 \leq L; j_1 = 1; 1 \leq k_1 \leq m_2, \\ & k_1' = 0; 1 \leq k_2, k_2' \leq m_2; 1 \leq l_1, l_2 \leq n \\ \boldsymbol{S}^0 \boldsymbol{\alpha} \otimes I_{m_2 n} + \mathcal{B} & i_1 = s+1, i_2 = s; j_1 = 1, j_2 = 0; 1 \leq k_1 \leq m_2; \\ & 1 \leq k_1' \leq m_1; 1 \leq k_2, k_2' \leq m_2; 1 \leq l_1, l_2 \leq n \end{cases}$$

where,

$$\mathcal{B} = \begin{bmatrix} \boldsymbol{\alpha} \otimes B_1 \\ \boldsymbol{\alpha} \otimes B_2 \\ \vdots \\ \boldsymbol{\alpha} \otimes B_{m_2} \end{bmatrix}$$

where

$$B_{m_i} = \left[\begin{array}{cccc} 0 & \cdots & \theta \boldsymbol{S}^0 \otimes I_n & \cdots 0 \end{array}\right], \text{where } \theta S^0 \otimes I_n \text{ is in the } i^{th} \text{position}$$

$$A_{0_{(i_1,j_1,k_1,k_2,l_1)}}^{(i_2,j_2,k_1',k_2',l_2)} = \begin{cases} I_{m_1} \otimes D_1 & i_1 = i_2 = 0; j_1 = j_2 = 0; \\ & 1 \leq k_1, k_1' \leq m_1; k_2 = k_2' = 0; 1 \leq l_1, l_2 \leq n \\ I_{m_1 m_2} \otimes D_1 & i_1 = i_2, 1 \leq i_1 \leq L-1; j_1 = j_2 = 0; 1 \leq k_1, k_1' \leq m_1; \\ & 1 \leq k_2, k_2' \leq m_2; 1 \leq l_1, l_2 \leq n \\ I_{m_2^2} \otimes D_1 & i_1 = i_2, s+1 \leq i_1 \leq L; j_1 = j_2 = 1; 1 \leq k_1, k_1' \leq m_2; \\ & 1 \leq k_2, k_2' \leq m_2; 1 \leq l_1, l_2 \leq n \end{cases}$$

$$A_{1_{(i_1,j_1,k_1,k_2,l_1)}}^{(i_2,j_2,k_1',k_2',l_2)} = \begin{cases} \boldsymbol{T}^0(\boldsymbol{\alpha} \otimes \boldsymbol{\beta}) \otimes I_n & i_1 = 0, i_2 = 1; j_1 = j_2 = 0; 1 \leq k_1, k_1' \leq m_1; \\ & k_2 = 0, 1 \leq k_2' \leq m_2; 1 \leq l_1, l_2 \leq n \\ \boldsymbol{T}^0 \boldsymbol{\alpha} \otimes I_{m_2 n} & 1 \leq i_1 \leq L-2, i_2 = i_1 + 1; j_1 = j_2 = 0; \\ & 1 \leq k_1, k_1' \leq m_1; 1 \leq k_2, k_2' \leq m_2; \\ & 1 \leq l_1, l_2 \leq n \\ \boldsymbol{T}^0 \boldsymbol{\beta} \otimes I_{m_2 n} & i_1 = L-1, i_2 = L; j_1 = 0, j_2 = 1; 1 \leq k_1 \leq m_1, \\ & 1 \leq k_1' \leq m_2; 1 \leq k_2, k_2' \leq m_2; \\ & 1 \leq l_1, l_2 \leq n \\ T \oplus D_0 & i_1 = i_2 = 0; j_1 = j_2 = 0; 1 \leq k_1, k_1' \leq m_1; \\ & k_2 = k_2' = 0; 1 \leq l_1, l_2 \leq n \\ T \oplus \theta S \oplus D_0 & i_1 = i_2, 1 \leq i_1 \leq L-1; j_1 = j_2 = 0; \\ & 1 \leq k_1, k_1' \leq m_1; 1 \leq k_2, k_2' \leq m_2; \\ & 1 \leq l_1, l_2 \leq n \\ S \oplus \theta S \oplus D_0 & i_1 = i_2, s+1 \leq i_1 \leq L; j_1 = j_2 = 1; 1 \leq k_1, k_1' \leq m_2; \\ & 1 \leq k_2, k_2' \leq m_2; 1 \leq l_1, l_2 \leq n \end{cases}$$

$$A_{2_{(i_1,j_1,k_1,k_2,l_1)}}^{(i_2,j_2,k_1',k_2,l_2)} = \begin{cases} I_{m_1} \otimes (\theta \boldsymbol{S}^0 \otimes I_n) & i_1 = 1, i_2 = 0; j_1 = j_2 = 0; \\ & 1 \le k_1, k_1' \le m_1, 1 \le k_2 \le m_2, \\ & k_2' = 0; 1 \le l_1, l_2 \le n \\ I_{m_1} \otimes (\theta \boldsymbol{S}^0 \boldsymbol{\beta} \otimes I_n) & i_2 = i_1 - 1, 2 \le i_1 \le L-1; \\ & j_1 = j_2 = 0; 1 \le k_1, k_1' \le m_1; \\ & 1 \le k_2, k_2' \le m_2; 1 \le l_1, l_2 \le n \\ \boldsymbol{S}^0 \boldsymbol{\alpha} \otimes I_{m_2 n} + \mathcal{B} & i_1 = s+1, i_2 = s; j_1 = 1, j_2 = 0; \\ & 1 \le k_1 \le m_2; 1 \le k_1' \le m_1; \\ & 1 \le k_2, k_2' \le m_2; 1 \le l_1, l_2 \le n \\ I_{m_1} \otimes (\theta \boldsymbol{S}^0 \boldsymbol{\beta} \otimes I_n) + \boldsymbol{S}^0 \boldsymbol{\beta} \otimes I_{m_2 n} & i_2 = i_1 - 1, s+2 \le i_1 \le L; \\ & j_1 = j_2 = 1; 1 \le k_1, k_1' \le m_2; \\ & 1 \le k_2, k_2' \le m_2; 1 \le l_1, l_2 \le n \end{cases}$$

Next we proceed for the steady state analysis of the system described.

6.2 Steady State Analysis

To this end we first obtain the

6.2.1 Stability condition

Let $\boldsymbol{\pi} = (\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, \ldots, \boldsymbol{\pi}_L)$ denote the steady state probability vector of the generator

$$A = A_0 + A_1 + A_2 = \begin{bmatrix} F_0 & F_1 & & & & & & & & & & \\ F_2 & F_3 & F_4 & & & & & & & & & \\ & F_5 & F_3 & F_4 & & & & & & & & \\ & & \ddots & \ddots & \ddots & & & & & & & \\ & & F_5 & F_3 & F_4 & & & & & & & \\ & & & F_5 & F_3 & F_6 & & & & & & \\ & & & & F_7 & F_8 & F_9 & & & & & \\ & & & & & F_{10} & F_8 & F_9 & & & & \\ & & & & & & \ddots & \ddots & \ddots & & & \\ & & & & & & & F_{10} & F_8 & F_9 & & \\ & & & & & & & & F_{10} & F_8 & F_{11} & \\ & & & & & & & & & F_{12} & F_{13} & \end{bmatrix}.$$

Then $\boldsymbol{\pi}$ satisfies

$$\boldsymbol{\pi} A = 0, \boldsymbol{\pi} e = 1. \qquad (6.1)$$

The LIQBD description of the model indicates that the queueing system is stable (see Neuts[40]) if and only if the left drift exceeds that of right drift. That is,

$$\boldsymbol{\pi} A_0 \mathbf{e} < \boldsymbol{\pi} A_2 \mathbf{e} \qquad (6.2)$$

The vector $\boldsymbol{\pi}$ cannot be obtained directly in terms of the parameters of the model. From (6.1)we get

$$\pi_i = \pi_{i-1} \mathcal{U}_{i-1}, 1 \leq i \leq L \qquad (6.3)$$

where

$$\mathcal{U}_i = \begin{cases} -F_1(F_3 + \mathcal{U}_1 F_5)^{-1} & for\, i = 0 \\ -F_4(F_3 + U_{i+1} F_5)^{-1} & for\, 1 \leq i \leq s-2 \\ -F_4(F_3 + \mathcal{U}_s F_7)^{-1} & for\, i = s-1 \\ -F_6(F_8 + \mathcal{U}_{s+1} F_{10})^{-1} & for\, i = s \\ -F_9(F_8 + \mathcal{U}_{i+1} F_{10})^{-1} & for\, s+1 \leq i \leq L-3 \\ -F_9(F_8 + U_{L-1} F_{12}) & for\, i = L-2 \\ -F_{11}(F_{13})^{-1} & for\, i = L-1 \end{cases}$$

From the normalizing condition $\boldsymbol{\pi e} = 1$ we have

$$\boldsymbol{\pi}_0 \left(\sum_{j=0}^{L-1} \prod_{i=0}^{j} \mathcal{U}_i + I \right) \boldsymbol{e} = 1 \tag{6.4}$$

We get $\boldsymbol{\pi}_0$ by solving (6.1) and (6.4). Substituting (6.3) and (6.4) in (6.2) gives the stability condition as

$$\boldsymbol{\pi}_0 \left[(I_{m_1} \otimes D_1)\boldsymbol{e} + \sum_{j=1}^{s} \prod_{i=0}^{j} \mathcal{U}_i (I_{m_1 m_2} \otimes D_1)\boldsymbol{e} + \sum_{j=s+1}^{L-1} \prod_{i=0}^{j} \mathcal{U}_i (I_{m_1 m_2} \otimes D_1)\boldsymbol{e} + \prod_{i=0}^{L-1} \mathcal{U}_i (I_{m_2{}^2} \otimes D_1)\boldsymbol{e} \right] <$$
$$\boldsymbol{\pi}_0 \left[\sum_{j=1}^{s} \prod_{i=0}^{j} \mathcal{U}_i (\boldsymbol{e}(m_1) \otimes (\theta \boldsymbol{S}^0 \otimes I_n)\boldsymbol{e}) + \sum_{j=s+1}^{L-1} \prod_{i=0}^{j} \mathcal{U}_i A'_2 \boldsymbol{e} + + \prod_{i=0}^{L-1} \mathcal{U}_i (\boldsymbol{e}(m_1) \otimes (\theta \boldsymbol{S}^0 \otimes I_n) + (\boldsymbol{S}^0 \otimes I_{m_2 n}))\boldsymbol{e}) \right] \tag{6.5}$$

where

$$A'_2 = \begin{bmatrix} \boldsymbol{e}(m_1) \otimes (\theta \boldsymbol{S}^0 \otimes I_n) \\ \boldsymbol{e}(m_1) \otimes (\theta \boldsymbol{S}^0 \otimes I_n) + (\boldsymbol{S}^0 \otimes I_{m_2 n}) \end{bmatrix}$$

6.2.2 Steady-state probability vector

Assuming that the condition (6.5) is satisfied we proceed to find the steady-state probability of the system state.

Let $\boldsymbol{x}$ be the steady state probability vector of $\bar{Q}$. We partition this vector as

$$\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \boldsymbol{x}_2 \ldots),$$

where $\boldsymbol{x}_0$ is of dimension $(s+1)m_1n + (L-s-1)(1+m_1)n + n$, $\boldsymbol{x}_1$ is of dimension $m_1n + sm_1m_2n + (L-s-1)(2m_2+m_1m_2)n + m_2n$, $\boldsymbol{x}_2, \boldsymbol{x}_3, \ldots$ are of dimension $m_1n + sm_1m_2n + (L-s-1)(m_1m_2n + {m_2}^2n) + {m_2}^2n$. Under the stability condition, we have

$$\boldsymbol{x}_i = \boldsymbol{x}_2 R^{i-2}, i \geq 3$$

where the matrix R is the minimal nonnegative solution to the matrix quadratic equation

$$R^2A_2 + RA_1 + A_0 = 0$$

and the vectors $\boldsymbol{x}_0$, $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ are obtained by solving the equations

$$\boldsymbol{x}_0A_{00} + \boldsymbol{x}_1A_{10} = 0 \quad (6.6)$$

$$\boldsymbol{x}_0A_{01} + \boldsymbol{x}_1A_{11} + \boldsymbol{x}_2A_{21} = 0 \quad (6.7)$$

$$\boldsymbol{x}_1A_{12} + \boldsymbol{x}_2(A_1 + RA_2) = 0 \quad (6.8)$$

subject to the normalizing condition

$$\boldsymbol{x}_0\boldsymbol{e} + \boldsymbol{x}_1\boldsymbol{e} + \boldsymbol{x}_2(I-R)^{-1}\boldsymbol{e} = 1 \quad (6.9)$$

6.3 Level crossing problems

6.3.1 Distribution of number of downcrossings from inventory level s to $s-1$ before hitting $s+1$

To find this distribution, first we find the the distribution of duration of time till down crossing from s to $s-1$ occur before hitting $s+1$. This can be studied as the time until absorption in the continuous time Markov chain,$\chi_1 = \{(N_1(t), N_2(t), I(t), K_1(t), K_2(t), K_3(t))\}$ where $N_1(t)$ denotes the number of down crossings from s to $s-1$, $N_2(t)$, the number of customers in the system,

$I(t)$, the number of processed items, $K_1(t)$, processing phase of S_2, $K_2(t)$, the service phase of S_1, $K_3(t)$, the phase of the customer arrival process at time t.

The state space of the process is $\{(i,0,k,l_1,0,p) : i \geq 0; 0 \leq k \leq s; 1 \leq l_1 \leq m_1; 1 \leq p \leq n\} \cup \{(i,j,0,l_1,0,p) : i \geq 0; 1 \leq j \leq M; 1 \leq l_1 \leq m_1; 1 \leq p \leq n\} \cup \{(i,j,k,l_1,l_2,p) : i \geq 0; 1 \leq j \leq M; 1 \leq k \leq s; 1 \leq l_1 \leq m_1; 1 \leq l_2 \leq m_2; 1 \leq p \leq n\} \cup \{*\}$ where $*$ denote the absorbing state indicating the hitting of level $s+1$. Here $M(\epsilon)$ is chosen in such a way that $P\left(\sum_{h=0}^{M(\epsilon)} \boldsymbol{x}_h \boldsymbol{e} > 1-\epsilon\right) \to 0$ for every $\epsilon > 0$.
The infinitesimal generator of the process is given by

$$\mathcal{U} = \begin{bmatrix} 0 & 0 & 0 & \cdots & & & \\ \boldsymbol{E}^0 & B & C & & & & \\ \boldsymbol{E}^0 & & B & C & & & \\ \vdots & & & \ddots & \ddots & & \\ \boldsymbol{E}^0 & & & & B & C & \\ \vdots & & & & & \ddots & \ddots \end{bmatrix}.$$

where

$$B = \begin{bmatrix} F_1 & G_1 & & & & \\ H_1 & F_2 & G_2 & & & \\ & H_2 & F_2 & G_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & H_2 & F_2 & G_2 \\ & & & & & F_3 \end{bmatrix}$$

with

$$F_1 = \begin{bmatrix} T \oplus D_0 & \boldsymbol{T}^0 \boldsymbol{\alpha} \otimes I_n & & \\ \ddots & \ddots & & \\ & T \oplus D_0 & \boldsymbol{T}^0 \boldsymbol{\alpha} \otimes I_n & \\ & & T \oplus D_0 & \end{bmatrix}, G_1 = \begin{bmatrix} I_{m_1} \otimes D_1 & \\ & I_s \otimes (I_{m_1} \otimes (\boldsymbol{\beta} \otimes D_1)) \end{bmatrix},$$

$$H_1 = \begin{bmatrix} 0 & 0 \\ I_s \otimes (I_{m_1} \otimes (\theta \boldsymbol{S}^0 \otimes I_n)) & 0 \end{bmatrix}, \; F_2 = \begin{bmatrix} T \oplus D_0 & \boldsymbol{T}^0(\boldsymbol{\alpha} \otimes \boldsymbol{\beta}) \otimes I_n & & & \\ & T \oplus \theta S \oplus D_0 & \boldsymbol{T}^0\boldsymbol{\alpha} \otimes I_{m_2 n} & & \\ & & \ddots & \ddots & \\ & & & T \oplus \theta S \oplus D_0 & \boldsymbol{T}^0\boldsymbol{\alpha} \otimes I_{m_2 n} \\ & & & & T \oplus \theta S \oplus D_0 \end{bmatrix},$$

$$G_2 = I_{m_1 + s m_1 m_2} \otimes D_1, \; H_2 = \begin{bmatrix} 0 & 0 \\ I_{m_1} \otimes (\theta \boldsymbol{S}^0 \otimes I_n) & 0 \\ 0 & I_{s-1} \otimes (I_{m_1} \otimes (\theta \boldsymbol{S}^0 \boldsymbol{\beta} \otimes I_n)) \end{bmatrix},$$

$$F_3 = \begin{bmatrix} T \oplus D_0 - I_{m_1} \otimes \Delta & \boldsymbol{T}^0(\boldsymbol{\alpha} \otimes \boldsymbol{\beta}) \otimes I_n & & & \\ & T \oplus \theta S \oplus D_0 - I_{m_1 m_2} \otimes \Delta & \boldsymbol{T}^0\boldsymbol{\alpha} \otimes I_{m_2 n} & & \\ & & \ddots & \ddots & \\ & & & T \oplus \theta S \oplus D_0 - I_{m_1 m_2} \otimes \Delta & \boldsymbol{T}^0\boldsymbol{\alpha} \otimes I_{m_2 n} \\ & & & & T \oplus \theta S \oplus D_0 - I_{m_1 m_2} \otimes \Delta \end{bmatrix}$$

with

$$\Delta = \begin{bmatrix} \delta_1 & & \\ & \ddots & \\ & & \delta_n \end{bmatrix}.$$

$$C = \begin{bmatrix} 0 & \cdots 0 \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots 0 \cdots & 0 \\ 0 & \cdots C' \cdots & 0 \end{bmatrix}, \text{where}, C' = \begin{bmatrix} 0 & 0 \\ I_{m_1} \otimes (\theta \boldsymbol{S}^0 \otimes I_n) & 0 \\ 0 & I_{s-1} \otimes (I_{m_1} \otimes (\theta \boldsymbol{S}^0 \boldsymbol{\beta} \otimes I_n)) \end{bmatrix}$$

$$\boldsymbol{E}^0 = \begin{bmatrix} \boldsymbol{E}_1^0 \\ \boldsymbol{e}(M) \otimes \boldsymbol{E}_2^0 \end{bmatrix}$$

with

$$\boldsymbol{E}_1^0 = \begin{bmatrix} \mathbf{0} \\ \vdots \\ \boldsymbol{T}^0 \otimes \boldsymbol{e}(n) \end{bmatrix}, \; \boldsymbol{E}_2^0 = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{T}^0 \otimes \boldsymbol{e}(m_2 n) \end{bmatrix}$$

Let y_k, $k = 0, 1, \cdots$ be the probability that the number of downcrossings from inventory level s to $s - 1$ is k. Then y_k is the probabilty that the absorption occurs from the level k for the process χ_1. Hence y_k are given by

$$y_0 = \boldsymbol{\gamma}_1(-B)^{-1}\boldsymbol{E}^0$$

and for $k = 1, 2, 3, \ldots$

$$y_k = \boldsymbol{\gamma}_1((-B)^{-1}C)^k(-B)^{-1}\boldsymbol{E}^0$$

where,

$$\boldsymbol{\gamma}_1 = (1/d)(\boldsymbol{x}_{0,0,0,1,0,1}, \cdots, \boldsymbol{x}_{0,s,0,m_1,0,n}, \cdots, \boldsymbol{x}_{M,0,0,1,0,1}, \cdots, \boldsymbol{x}_{M,s,0,m_1,m_2,n})$$

with

$$d = \sum_{i=0}^{s}\sum_{k_1=1}^{m_1}\sum_{p=1}^{n}\boldsymbol{x}_{0,i,0,k_1,0,p} + \sum_{h=1}^{M}\sum_{i=0}^{s}\sum_{k_1=1}^{m_1}\sum_{k_2=1}^{m_2}\sum_{p=1}^{n}\boldsymbol{x}_{h,i,0,k_1,k_2,p}$$

Thus we arrive at the Lemma.

Lemma 6.3.1. The expected number of downcrossings from inventory level s to $s-1$ before hitting $s+1$ is

$$E(i) = \sum_{k=0}^{\infty} k y_k$$

6.3.2 Distribution of number of upcrossings of inventory level from s to $s+1$ before hitting $s-1$

To find this distribution, first we find the the distribution of duration of time till upcrossing from s to $s+1$ occur before hitting $s-1$. This again can be studied as the time until absorption in a continuous time the Markov chain $\chi_2 = \{(N_1(t), N_2(t), I(t), J(t), K_1(t), K_2(t), K_3(t))\}$ where $N_1(t)$ denotes the number of upcrossings from s to $s+1$, $N_2(t)$, the number of customers in the system, $I(t)$, number of processed items, $J(t)$, status of S_2, $K_1(t)$, processing/service phase of S_2, $K_2(t)$, the service phase of S_1, $K_3(t)$, the arrival phase at time t.

The state space of the process is $\{(h,0,j,0,k_1,0,l) : h \geq 0;\ s \leq j \leq L-1;\ 1 \leq k_1 \leq m_1;\ 1 \leq l \leq n\} \cup \{(h,0,j,0,0,l) : h \geq 0;\ s+1 \leq j \leq L;\ 1 \leq l \leq n\} \cup \{(h,i,j,0,k_1,k_2,l) : h \geq 0;\ 1 \leq i \leq M;\ s \leq j \leq L-1;\ 1 \leq k_1 \leq m_1;\ 1 \leq k_2 \leq m_2;\ 1 \leq l \leq n\} \cup \{(h,1,j,0,k_2,l) : s+1 \leq j \leq L;\ 1 \leq k_2 \leq m_2;\ 1 \leq l \leq n\} \cup \{(h,1,j,1,k_1,0,l) : h \geq 0;\ s+1 \leq j \leq L-1;\ 1 \leq k_1 \leq m_2;\ 1 \leq l \leq n\} \cup \{(h,i,j,1,k_1,k_2,l) : h \geq 0;\ 2 \leq i \leq M;\ s+1 \leq j \leq L;\ 1 \leq k_1,k_2 \leq m_2;\ 1 \leq l \leq n\} \cup \{*\}$ where $*$ denote the absorbing state indicating the hitting of level $s+1$. Here $M(\epsilon)$ is chosen in such a way that $P\left(\sum_{h=0}^{M(\epsilon)} \boldsymbol{x}_h \boldsymbol{e} > 1-\epsilon\right) \to 0$ for every $\epsilon > 0$.

Let z_k, $k = 0, 1, \cdots$ be the probability that the number of upcrossings from inventory level s to $s+1$ is k. Then z_k is the probabilty that the absorption occurs from the level k for the process χ_2.

Proceeding on similar lines as in the proof of Lemma 6.3.1, we arrive at Lemma.

Lemma 6.3.2. The expected number of upcrossings from inventory level s to $s+1$ before hitting $s-1$ is

$$E(i) = \sum_{k=0}^{\infty} k z_k$$

6.4 Performance Measures

1. Expected number of customers in the system, $E_s = \sum_{h=1}^{\infty} h \boldsymbol{x}_h \boldsymbol{e}$

2. Expected number of processed items in the inventory,

$$E_{it} = \sum_{i=1}^{L-1}\sum_{k_1=1}^{m_1}\sum_{p=1}^{n} i x_{0,i,0,k_1,0,p} + \sum_{i=s+1}^{L}\sum_{p=1}^{n} i x_{0,i,0,0,p} +$$
$$\sum_{h=1}^{\infty}\sum_{i=1}^{L-1}\sum_{k_1=1}^{m_1}\sum_{k_2=1}^{m_2}\sum_{p=1}^{n} i x_{h,i,0,k_1,k_2,p} + \sum_{i=s+1}^{L}\sum_{k_2=1}^{m_2}\sum_{p=1}^{n} x_{1,i,0,k_2,p} +$$
$$\sum_{i=s+1}^{L-1}\sum_{k_1=1}^{m_2}\sum_{p=1}^{n} i x_{1,i,1,k_1,0,p} + \sum_{h=2}^{\infty}\sum_{i=s+1}^{L}\sum_{k_1=1}^{m_2}\sum_{k_2=1}^{m_2}\sum_{p=1}^{n} i x_{h,i,1,k_1,k_2,p}$$

3. Expected rate at which the inventory processing is switched on,

$$R_{ipo} = \sum_{k_1=1}^{m_2}\sum_{p=1}^{n} \sigma_{k_1} x_{1,s+1,1,k_1,0,p} + \sum_{k_2=1}^{m_2}\sum_{p=1}^{n} \theta\sigma_{k_2} x_{1,s+1,0,k_2,p} +$$
$$\sum_{h=2}^{\infty}\sum_{k_1=1}^{m_2}\sum_{k_2=1}^{m_2}\sum_{p=1}^{n} (\theta\sigma_{k_2} + \sigma_{k_1}) x_{h,s+1,1,k_1,k_2,p} \quad (6.10)$$

4. Expected rate of switching of S_2 to service mode,

$$R_{sn} = \sum_{k_2=1}^{m_2}\sum_{i=s+1}^{L}\sum_{p=1}^{n}\sum_{p'=1}^{n} d_{pp'}^{(1)} x_{1,i,1,0,k_2,p} +$$
$$\sum_{h=2}^{\infty}\sum_{k_1=1}^{m_1}\sum_{k_2=1}^{m_2}\sum_{p=1}^{n} \eta_{k_1} x_{h,L-1,0,k_1,k_2,p} \quad (6.11)$$

6.5 Analysis of a cost function

We construct a cost function based on the above performance measures.
Let

c_1: Unit time cost for switching on inventory processing

c_2: Unit time cost for switching of S_2 to service mode

h_1: Unit time cost for holding a customer
h_2: Unit time cost for holding an item in inventory
Then the expected cost per unit time,

$$C = c_1 R_{ipo} + c_2 R_{sn} + h_1 E_s + h_2 E_{it}$$

6.6 Numerical Experiments

We find optimal s and optimal L by using the above cost function.

We fix $\boldsymbol{\alpha} = \begin{bmatrix} 0.9 & 0.1 \end{bmatrix}, T = \begin{bmatrix} -4 & 4 \\ 0 & -4 \end{bmatrix}, \boldsymbol{\beta} = \begin{bmatrix} 0.8 & 0.2 \end{bmatrix}, S = \begin{bmatrix} -3 & 3 \\ 0 & -3 \end{bmatrix}$,
$\theta = 0.6, c_1 = 100, c_2 = 5, h_1 = 30$ and $h_2 = 1$.

For the arrival process of type II customers, we consider the following five set of matrices for $D0$ and $D1$.

1. Exponential (EXP)

$$D_0 = (-1),\ D_1 = (1)$$

2. Erlang (ERA)

$$D_0 = \begin{bmatrix} -3 & 3 & 0 \\ 0 & -3 & 3 \\ 0 & 0 & -3 \end{bmatrix}, D_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 3 & 0 & 0 \end{bmatrix}$$

3. Hyperexponential (HEXP)

$$D_0 = \begin{bmatrix} -3.4000 & 0 \\ 0 & -0.8500 \end{bmatrix}, D_1 = \begin{bmatrix} 0.6800 & 2.7200 \\ 0.1700 & 0.6800 \end{bmatrix}$$

4. MAP with negative correlation (MNA)

$$D0 = \begin{bmatrix} -0.8101 & 0.8101 & 0 \\ 0 & -1.3497 & 0 \\ 0 & 0 & -40.5065 \end{bmatrix}, D1 = \begin{bmatrix} 0 & 0 & 0 \\ 0.0810 & 0 & 1.2687 \\ 38.0761 & 0 & 2.4304 \end{bmatrix}$$

5. MAP with positive correlation (MPA)

$$D0 = \begin{bmatrix} -0.8101 & 0.8101 & 0 \\ 0 & -1.3497 & 0 \\ 0 & 0 & -40.5065 \end{bmatrix}, D1 = \begin{bmatrix} 0 & 0 & 0 \\ 1.2687 & 0 & 0.0810 \\ 2.4304 & 0 & 38.0761 \end{bmatrix}$$

All these five MAP processes are normalized so as to have an arrival rate of 1. However, these are qualitatively different in that they have different variance and correlation structure. The first three arrival processes, namely EXP, ERA and HEA correspond to renewal processes and so the correlation is 0. The arrival process labeled MNA has correlated arrivals with correlation between two successive interarrival times given by -0.4211 and the arrival process corresponding to the one labelled MPA has a positive correlation with value 0.4211.

Tables 6.1 to 6.5 indicate the effect of the parameter s on various performance measures and the cost function corresponding to different arrival processses when L is fixed. In the following we summarize the observations based on these tables.

We see that R_{ipo} increases when s increases. This happens because when s increases, the inventory level reaches s more rapidly from above. R_{sn} also increases as s increases. This is due to the fact that when s increases, S_2 is switched on to procesing at a faster rate and hence the inventory level reaches to maximum value L at a faster rate and as a result S_2 switched on to service mode if customers are waiting. E_s decreases as s increases. This happens since when s increases both R_{ipo} and R_{sn} increase and as a result customers get service at a faster rate. E_{it} increases as s increases. This is because when s increases, S_2 is switched on to processing mode at a faster rate. The cost funtion first decreases reaches a minimum value and then increases for all arrival processes. The optimal cost varies for different arrival processes (see Fig 6.1). It is the highest for MPA. This shows the effect of positive correlation.

s	2	3	4	5	6	7	8	9	10
R_{ipo}	0.035	0.037	0.039	0.042	0.045	0.049	0.053	0.058	0.064
R_{sn}	0.178	0.180	0.181	0.182	0.184	0.186	0.188	0.191	0.194
E_s	1.984	1.952	1.923	1.894	1.866	1.837	1.808	1.779	1.750
E_{it}	10.467	10.976	11.485	11.994	12.500	13.005	13.507	14.005	14.497
C	74.336	74.105	73.990	73.918	**73.883**	73.894	73.963	74.110	74.340

Table 6.1: Effect of s: Fix $L = 20$ and arrival process as EXP

s	2	3	4	5	6	7	8	9	10
R_{ipo}	0.034	0.036	0.038	0.041	0.045	0.048	0.053	0.058	0.064
R_{sn}	0.199	0.201	0.202	0.204	0.206	0.209	0.212	0.215	0.219
E_s	1.553	1.527	1.501	1.475	1.448	1.421	1.393	1.365	1.336
E_{it}	10.487	11.001	11.516	12.031	12.546	13.061	13.576	14.089	14.602
C	61.475	61.420	**61.414**	61.432	61.475	61.553	61.680	61.872	62.155

Table 6.2: Effect of s: Fix $L = 20$ and arrival process as ERA

s	2	3	4	5	6	7	8	9	10
R_{ipo}	0.035	0.037	0.039	0.042	0.045	0.049	0.053	0.058	0.064
R_{sn}	0.171	0.172	0.173	0.175	0.176	0.178	0.180	0.183	0.186
E_s	2.152	2.119	2.090	2.060	2.032	2.003	1.975	1.947	1.920
E_{it}	10.457	10.963	11.469	11.975	12.478	12.978	13.474	13.966	14.451
C	79.319	79.051	78.923	78.848	**78.815**	78.831	78.909	79.068	79.333

Table 6.3: Effect of s: Fix $L = 20$ and arrival process as HEXP

s	2	3	4	5	6	7	8	9	10
R_{ipo}	0.033	0.035	0.036	0.040	0.043	0.046	0.050	0.055	0.060
R_{sn}	0.076	0.078	0.079	0.081	0.083	0.085	0.088	0.092	0.095
E_s	16.697	16.645	16.631	16.629	16.630	16.633	16.635	16.637	16.639
E_{it}	10.644	11.122	11.605	12.090	12.573	13.054	13.533	14.009	14.480
C	515.265	**514.383**	514.689	515.370	516.188	517.078	518.028	519.044	520.144

Table 6.4: Effect of s: Fix $L = 20$ and arrival process as MPA

s	2	3	4	5	6	7	8	9	10
R_{ipo}	0.035	0.037	0.040	0.042	0.046	0.049	0.054	0059	0.065
R_{sn}	0.208	0.209	0.211	0.212	0.214	0.216	0.219	0.222	0.225
E_s	2.100	2.068	2.037	2.008	1.9778	1.949	1.918	1.890	1.858
E_{it}	10.418	10.918	11.427	11.924	12.430	12.918	13.419	13.892	14.381
C	77.951	77.702	77.546	77.460	**77.380**	77.383	77.399	77.542	77.729

Table 6.5: Effect of s: Fix $L = 20$ and arrival process as MNA

Tables 6.6 to 6.10 indicate the effect of the parameter L on various performance measures and the cost function when s is fixed. We summarize the observations based on these tables below.

R_{ipo} decreases as L increases. This is due to the fact that the level s is attained at a slower rate. R_{sn} also decreases as L increases. This happens since L is attained at a slower rate. E_s increases as L increases. This happens since when L increases both R_{ipo} and R_{sn} decrease and as a result customers get service at a slower rate. E_{it} increases as L increases since more items are processed at a stretch. The cost funtion first decreases reaches a minimum value and then increases for all arrival processes. The optimal cost varies for different arrival processes.(see Fig 6.2) It is the highest for MPA. This shows the effect of positive correlation.

L	8	9	10	11	12	13	14	15	16	17	18
R_{ipo}	0.131	0.109	0.093	0.081	0.072	0.064	0.058	0.053	0.049	0.045	0.042
R_{sn}	0.227	0.216	0.208	0.202	0.197	0.194	0.191	0.188	0.186	0.184	0.182
E_s	1.617	1.641	1.667	1.695	1.723	1.751	1.780	1.809	1.838	1.867	1.895
E_{it}	5.090	5.597	6.096	6.591	7.083	7.573	8.062	8.549	9.035	9.521	10.006
C	67.86	66.80	**66.43**	66.52	66.90	67.48	68.21	69.04	69.96	70.93	71.96

Table 6.6: Effect of L: Fix $s = 3$ and arrival process as EXP

L	8	9	10	11	12	13	14	15	16	17	18
R_{ipo}	0.134	0.111	0.094	0.081	0.072	0.064	0.058	0.053	0.048	0.045	0.041
R_{sn}	0.255	0.244	0.235	0.229	0.223	0.219	0.215	0.212	0.209	0.206	0.204
E_s	1.187	1.216	1.247	1.277	1.307	1.336	1.365	1.393	1.421	1.448	1.475
E_{it}	5.208	5.694	6.178	6.660	7.142	7.624	8.106	8.588	9.070	9.552	10.035
C	55.51	54.46	**54.12**	54.22	54.61	55.19	55.90	56.70	57.57	58.49	59.44

Table 6.7: Effect of L: Fix $s = 3$ and arrival process as ERA

L	8	9	10	11	12	13	14	15	16	17	18
R_{ipo}	0.130	0.108	0.092	0.081	0.071	0.064	0.058	0.053	0.049	0.045	0.042
R_{sn}	0.219	0.208	0.200	0.194	0.189	0.186	0.183	0.180	0.178	0.176	0.175
E_s	1.795	1.817	1.842	1.867	1.894	1.921	1.949	1.977	2.005	2.033	2.062
E_{it}	5.048	5.559	6.063	6.561	7.056	7.549	8.040	8.529	9.017	9.504	9.991
C	73.02	71.93	**71.54**	71.59	71.94	72.49	73.20	74.01	74.92	75.88	76.90

Table 6.8: Effect of L: Fix $s = 3$ and arrival process as HEXP

L	8	9	10	11	12	13	14	15	16	17	18
R_{ipo}	0.122	0.101	0.087	0.076	0.067	0.060	0.055	0.050	0.046	0.043	0.040
R_{sn}	0.139	0.124	0.114	0.106	0.100	0.096	0.092	0.088	0.085	0.083	0.081
E_s	16.71	16.70	16.69	16.69	16.68	16.68	16.67	16.67	16.66	16.66	16.65
E_{it}	5.033	5.551	6.063	6.573	7.080	7.587	8.093	8.599	9.104	9.609	10.113
C	519.3	517.3	516.1	515.3	514.7	514.4	514.2	514.1	**514.0**	514.0	514.1

Table 6.9: Effect of L: Fix $s = 3$ and arrival process as MPA

L	8	9	10	11	12	13	14	15	16	17	18
R_{ipo}	0.132	0.111	0.094	0.082	0.072	0.065	0.059	0.054	0.049	0.046	0.042
R_{sn}	0.264	0.250	0.242	0.235	0.230	0.225	0.222	0.219	0.216	0.214	0.212
E_s	1.723	1.745	1.776	1.800	1.833	1.860	1.891	1.919	1.950	1.979	2.009
E_{it}	4.940	5.488	5.979	6.505	6.987	7.500	7.980	8.483	8.964	9.461	9.942
C	71.12	70.14	**69.81**	69.90	70.32	70.89	71.67	72.50	73.46	74.44	75.51

Table 6.10: Effect of L: Fix $s = 3$ and arrival process as MNA

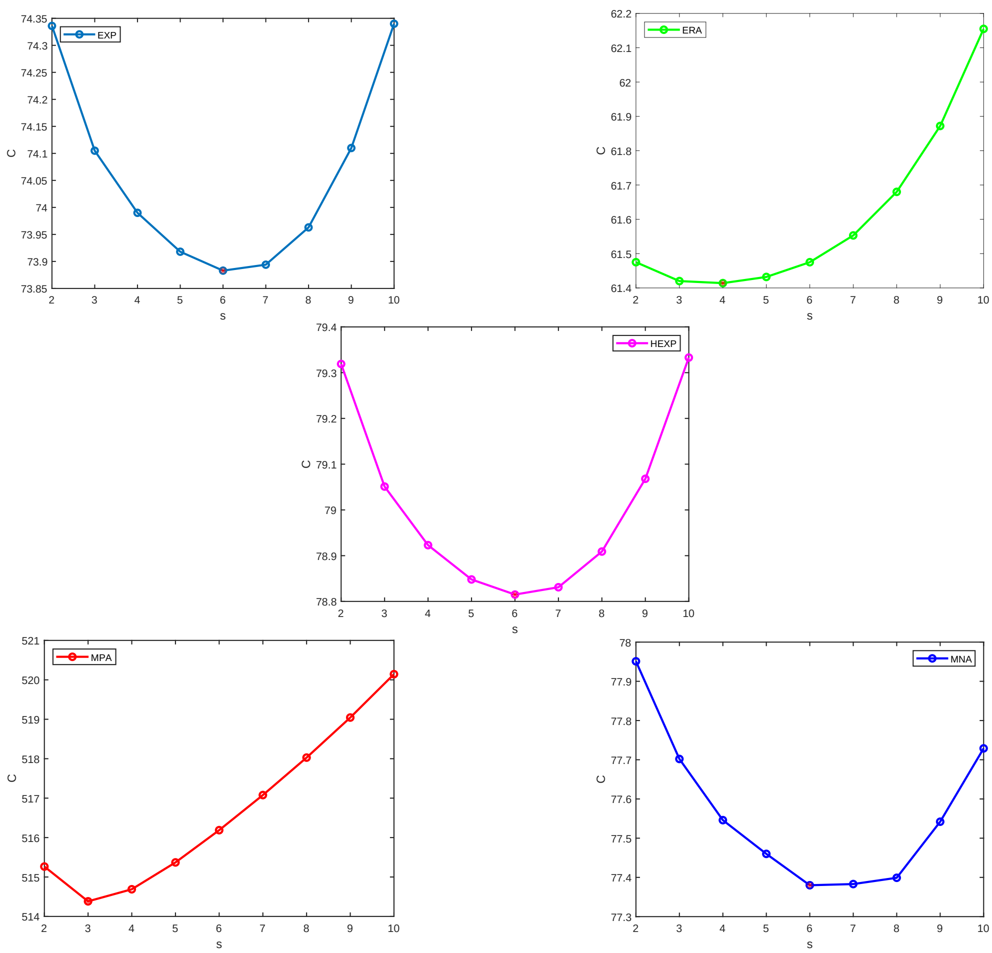

Figure 6.1: Effect of s when $L = 20$

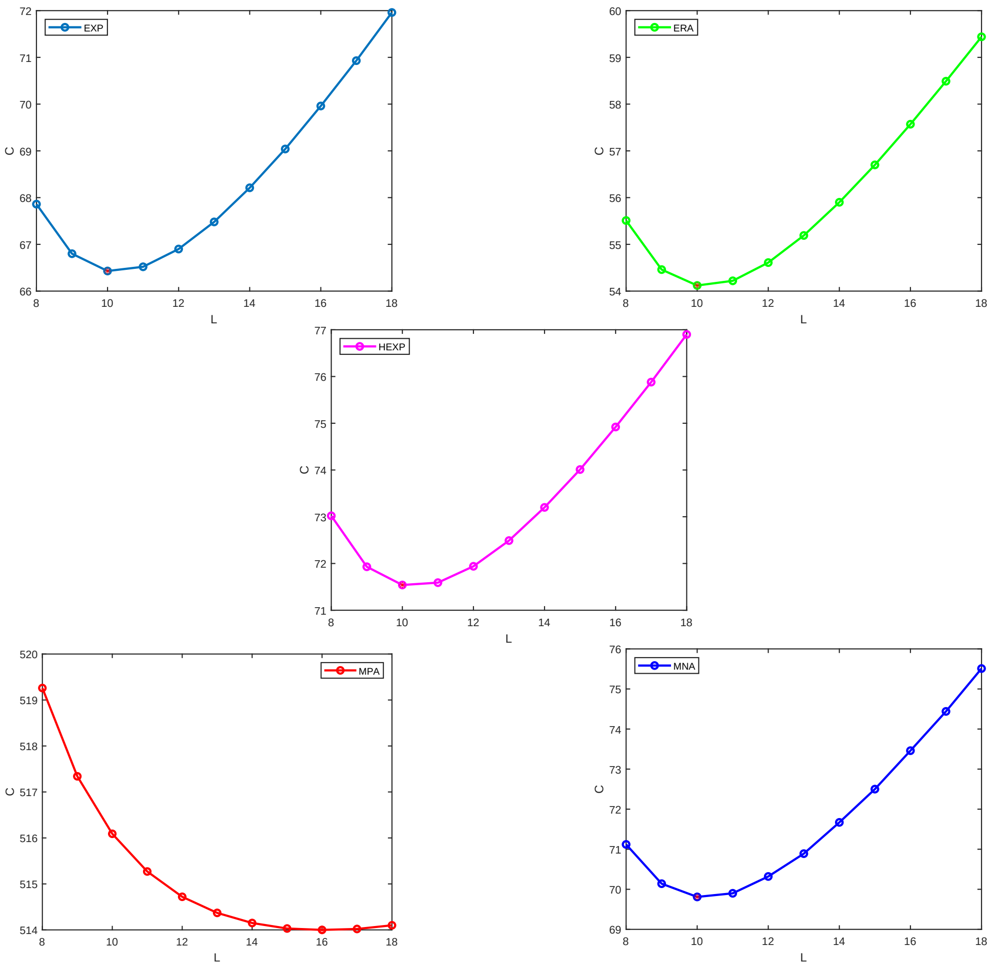

Figure 6.2: Effect of L when $s = 3$

Concluding remarks and suggestions for future study

In this thesis we discussed some queueing models with working vacation, working interruption and processing of service items by identifying the underlying continuous time Markov chains. In the following we give a sketch of our findings in this thesis:

In chapter 2, we considered two (M, MAP)/(PH, PH)/1 queues with non-preemptive priority and exponentially distributed working vacation under N-policy. Based on two distinct definitions of N-policies, we studied the distribution of the duration of slow service mode without any break, expected number of returns to 0 type I customer state, starting from 0 type I customer state during vacation mode of service before the arrival of a type II customer and the distribution of a p-cycle in normal mode. Also we provided LSTs of busy cycle, busy period of type I customers generated during the service time of a type II customer. For the waiting time distributions of both type I and type II customers, we provided an analysis using LST. We also performed some numerical experiments to find the mean and variance of the number of both type I and type II customers in the system and optimal N for both models by constructing a cost function. We compared the two queueing models with non-preemptive service and exponentially distributed working vacations and N-policy. These models were analyzed under the assumption of stability. Numerical experiments were carried out to find the superior one. It is possible to extend the arrival process of type I customers to MAP. In a future work we propose to extend the models discussed here to the case in which the type II customers are impatient. This will lead to the problem of finding individual optimal strategy of type II customers, maximum revenue of the server and social optimal strategy. Also, the extension of the models discussed to multi-server case is proposed to be taken up.

In chapter 3, we considered a (M,MAP)/(PH,PH)/1 queue with non preemptive priority, exponentially distributed working interruptions and protection. We analysed the distribution of service time of type I and type II customers and the distribution of a p-cycle. Also we provided LSTs of busy cycle, busy period of type I customers generated during the service time of a type II customer. For the waiting time distributions of type I and type II customers, we provided an analysis using LST and the matrix analytic method. We also performed some numerical experiments to evaluate some performance measures and also found optimal values using a cost function. Extension of the model discussed to multi-server is proposed to be taken up in a future study.

In chapter 4, we considered a MAP/(PH,PH)/1 queue with processing of service items under Vacation and N-policy. We obtained the distribution of time till the number of customers hit N or the inventory level reaches L, distribution of idle time, the distribution of time until the number of customers hit N and also the distribution of number of inventory processed before the arrival of first customer. Also we provided the distribution of a busy cycle, LSTs of busy cycles in which no item is left in the inventory and also that for atleast one item left in the inventory. We performed some numerical experiments to evaluate the expected idle time, standard deviation and coefficient of varaiation of idle time of the server .

In chapter 5, we considered a MAP/(PH,PH)/1 queue with processing of service items under Vacation and N-policy with impatient customers. We found the distribution of time until the number of customers hit N. Several system performance characteristics were computed. LST of the waiting time distribution for the case of no reneging was derived. Also we performed some numerical experiments for computing individual optimal strategy, maximum revenue to the server and social optimal strategy for the special case of no reneging.

In chapter 6, we considered a MAP/(PH,PH)/2 queue with processing of service items by a server. We analyzed the model in steady state by Matrix Analytic Method and also derived some important distributions. Also we provided some numerical experiments to find the optimal values of L and s. We propose to extend this model to multi-server in a future study.

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List of Publications

1. *A. Krishnamoorthy, Divya V.*: **(M,MAP)/(PH,PH)/1 queue with Nonpreemptive Priority, Working Interruption and Protection**, Reliability:Theory and Applications, Vol.13,No.2(49), June 2018.

2. *Divya, V.,Krishnamoorthy,A., Vishnevsky, V. M.*: **On a Queueing System with processing of Service items under Vacation and N-policy**, DCCN 2018, CCIS 919, pp. 43-57, Springer Nature Switzerland AG 2018.

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4. *Divya, V., Vishnevsky, V. M., Kozyrev, D., Krishnamoorthy,A.*: **On a Queueing system with Processing of Service Items under Vacation and N-policy with impatient customers** (communicated).

5. *A. Krishnamoorthy, Divya V.*: **A Two-Server Queueing System with Processing of Service Items by a Server** (communicated).

Papers presented

1. Krishnamoorthy,A., Divya,V.,*Comparison of two Queueing Models with Working vacation and N-policy*, International Conference in Stochastic

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