BIVARIATE AND CONDITIONAL PARTIAL MOMENTS: SOME PROPERTIES AND APPLICATIONS

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By

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To My Loving Parents and Teachers

CERTIFICATE

Certified that the thesis entitled **"Bivariate and Conditional Partial Moments: Some Properties and Applications"** is a bonafide record of work done by Mr. Vipin N under my guidance in the Department of Statistics, Cochin University of Science and Technology, Cochin-22, Kerala, India and that no part of it has been included anywhere previously for the award of any degree or title.

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Certified that all the relevant corrections and modifications suggested by the audience during the pre-synopsis seminar and recommended by the Doctoral Committee of the candidate has been incorporated in the thesis.

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DECLARATION

This thesis contains no material which has been accepted for the award of any other Degree or Diploma in any University and to the best of my knowledge and belief, it contains no material previously published by any other person, except where due references are made in the text of the thesis.

Cochin 682 022 **Vipin. N**

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Vipin N

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List of Acronyms

CDF	Cumulative Distribution Function
pdf	probability density function
sf	survival function
rv	Random variable
i.i.d.	independent and identically distributed
MSE	Mean Squared Error
MRLF	Mean Residual Life Function
UPM	Upper Partial Moments
LPM	Lower Partial Moments
CHR	Cumulative Hazard Rate
CUPM	Conditional Upper Partial Moment
CLPM	Conditional Lower Partial Moment
MLPRI	Mean Left Proportional Residual Income
ES	Expected Shortfall

CES	Conditional Expected Shortfall
VaR	Value at Risk
CVaR	Conditional Value at Risk
TCE	Tail Conditional Expectation
BRFR	Bivariate Reversed Failure Rate
BMRL	Bivariate Mean Residual Life
BRMRLF	Bivariate Reversed Mean Residual Life
PSLD	Positive Stop Loss Dependence
NSLD	Negative Stop Loss Dependence
F-G-M	Farlie-Gumbel-Morgenstern
SNBP	Sankaran and Nair Bivariate Pareto
IBHR-1	Increasing Bivariate Hazard Rate-1
IBHR-2	Increasing Bivariate Hazard Rate-2
DBMRL	Decreasing Bivariate Mean Residual Life
BIFR	Bivariate Increasing Failure Rate
BDMRL	Bivariate Decreasing Mean Residual Life
CFSD	Conditional First order stochastic Dominance
CSSD	Conditional Second order stochastic Dominance
CTSD	Conditional Third order stochastic Dominance

List of Symbols

$\mathbf{I}$	$[0, 1]$
$\mathbb{R}$	Real line
$\mathbb{R}_n$	n-dimensional Euclidean space
$\mathbb{R}_2^+$	Positive octant in the 2-dimensional Euclidean space
x^+	$max(x, 0)$
$f(\cdot)$	pdf
$F(\cdot)$	CDF
$R(\cdot)$	sf
$h(\cdot)$	Failure rate
$H(\cdot)$	CHR
$\bar{h}(\cdot)$	Reversed failure rate
$h^*(\cdot\|\cdot)$	Failure rate of conditional survival models
$m(\cdot)$	MRL
$\bar{m}(\cdot)$	Reversed MRL

$p_r(t)$	r^{th} order UPM about a point t
$l_r(t)$	r^{th} order LPM about a point t
$\phi_r(t_1\|t_2)$	r^{th} order UPM of the conditional specified model
$\psi_r(t_1\|t_2)$	r^{th} order UPM of the conditional survival model
$\delta_r(t_1\|X_2 \leq t_2)$	r^{th} order CLPM
$B(a,b)$	Incomplete beta function
$U(0,1)$	Continuous standard uniform distribution
$I(\cdot)$	Indicator function
${}_2F_1(a,b;c;z)$	Gauss hypergeometric function
$(a)_k$	Pochhammer symbol
${}_1F_1(a;b;z)$	Kummer confluent hypergeometric function
$\Gamma(a,z)$	Incomplete gamma function
$E_n(z)$	Exponential integral function

Chapter 1

Introduction

The origin of the scientific word *moment* is basically from the discipline physics. In mathematics, moment is a quantitative measure of the shape of a set of points. If the points represent a probability density, then the zeroth moment is the total probability (i.e. one), the first moment is the mean, the second central moment is the variance, the third central moment is the skewness, and the fourth central moment (with normalization and shift) is the kurtosis. It is closely related to the concept of 'moment of inertia' in physics. Mathematically, for an univariate continuous random variable (rv) X defined on the real line $\mathbb{R}$ with an absolute continuous Cumulative Distribution Function (CDF) $F(\cdot)$ of any probability distribution and let t be a centering point or a reference level, then the r^{th} moment of X about t is given by the Riemann–Stieltjes integral

$$E\left[(X-t)^r\right] = \int\limits_{-\infty}^{\infty} (x-t)^r dF\left(x\right). \tag{1.1}$$

For any positive integer r, the r^{th} moment about zero of a probability density function (pdf) $f(.)$ is the expected value of X^r and is called a *raw moment* or *crude moment* of X. For the second and higher moments, the *central moment* (i.e. the moments about the mean, with t being the mean) provides clearer information about the distribution's shape and the second central moment is the variance of the rv X, which quantifies the amount of dispersion in the data around the mean.

Since moments are closely connected to different parameters of an underline population, there are various types of moments of the rv X defined in the literature. For example, the r^{th} inverse moment about zero by $E(X^{-r})$ and the r^{th} logarithmic moment about zero by $E(log(X))^r$ etc. Among these the one which is very popular in different applied problems such as reliability analysis, actuarial studies and income (poverty) studies are the *partial moments*. Partial moments are more appropriate than the usual conventional moments when the investigator knows information about the occurrence of an event is only from (after) a particular reference point, say t. In such situations, (1.1) reduces to either

$$p_r(t) = \int_t^\infty (x-t)^r dF(x) \tag{1.2}$$

or

$$l_r(t) = \int_{-\infty}^t (t-x)^r dF(x). \tag{1.3}$$

(1.2) and (1.3) are generally known as Upper Partial Moment and Lower Partial Moment denoted by UPM and LPM respectively. Partial moments are also referred to as the *'one-sided moments'* since they use exactly one side information (i.e. either upward or downward) from the data, about a reference level t. UPM is popular in reliability studies whereas LPM is widely used in calculating the moments in the reversed time $(-\infty, t)$. Lower partial moments (LPM) are also used in portfolio theory and in areas involving financial risk, which addresses the problem of managing risky investment policies with the object of maximizing returns. Further, LPM also plays an important role in the analysis of risks in income (poverty) studies.

Variance is a classical measure of dispersion. Partial moments are the measures that generalize the variance. For example, $r = 2$ in (1.2) and (1.3) gives a new measure called the *Target Semi-Variance (TSV)*, TSV is a measure of dispersion of the outcomes below the target return. A special case of the TSV arises when the target return is the expected value. The resulting measure is called the *semivariance*. Semivariance is a measure of dispersion of all observations that fall below (or above) the mean or target value of a data set. Semivariance is the average squared deviations of values that are lesser (greater) than the mean. Semivariance is similar to variance; however, it only considers observations below (above) the mean. In portfolio or asset analysis, semivariance provides a measure for downside risk. Unlike standard deviation and variance that provide measures of volatility, semivariance only looks at the negative (positive) fluctuations of an asset. By neutralizing all values below (above) the mean, or an investor's target return, semivariance estimates the average loss that a portfolio could incur.

Upside risk is the chance that an asset or investment will increase in value beyond your expectations. The concept of upside risk serves a number of useful purposes. It can be a red flag that a particular fund or investment manager is taking excessive risks. In other words, upside risk allows you to assess both potential losses and gains with risk approximations. An upside risk is an outcome better than the benchmark, while a downside risk is an outcome that is worse than the benchmark. For example, if we lend money to someone the natural choice of benchmark outcome is that they will pay back, but there is a risk that they won't. In this case the risk is bad thing, and there is no outcome better than the benchmark outcome. The only risk is a downside one, though a different choice of benchmark could change this. Other situations have a more obvious upside. Suppose a company launches a new product and makes plans for production based on an estimate of sales in the first three months. The natural benchmark level is the estimate made and used for planning, but there is a risk of lower sales and a risk of higher sales. The risk of lower sales is the downside, while the risk of higher sales is the upside.

Stochastic modelling is a very popular and powerful technique in most scientific studies to understand the basic characteristics of a random phenomenon. One of the basic problems in such situations is to identify the underlying stochastic model that is supposed to generate the observations. When data is the only input material for the model selection, in-order to choose an appropriate model, usually the analyst start with a general class of probability distributions and finally select an appropriate member from the class, which is considered to be as the best fitted model for the corresponding data set. The term 'best fitted model' is a very important and relative one. Since the overall fit of the model is usually quantified with aids of the following tools such as (i) graphical techniques which mainly includes Probability- Probability plots (P-P plots) and Quantile- Quantile plots (Q-Q plots) or (ii) statistical testing of hypothesis procedures such as Kolmogrov-Smirnov test, Cramer-Von Moses test, Anderson-Darling test, the χ^2 or likelihood ratio tests or using the (iii) informations criterion's such as Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), Devian Information Criterion (DIC) or Focused Information Criterion (FIC) etc. However, the major dilemma the analyst must face while using these techniques is that the different models have different tail behaviours and the sample size of the data sets may not be sufficiently large enough to detect such differences and more over most of the aforementioned goodness of fit tests are at best only approximate evaluations.

Generally it is not easy to isolate all the physical causes that contribute individually or collectively to the generation of data and to mathematically account for each. The task of determining the correct stochastic model representing the given data becomes very difficult. A standard practice in such contexts is to ascertain the physical properties of the process generating the observations, express them by means of mathematical equations or inequalities and then solve them to obtain the model. Characterizations of probability distributions play a vital role in modelling and analysis of statistical data. The tool that enables the exact determination of a probability model is the 'characterization theorem'. If for a random variable X, there exists a family of distributions say, F such that a distribution belongs to F implies that X has the property P. The characterization theorem makes a conclusion that if X exhibits P then the distribution belongs to F, which in general implies that F is the only model having this designated property P. Thus, if the analyst is able to translate the physical characteristics of the system in terms of different statistical measures such as Failure Rate (FR), Mean Residual Life (MRL) function etc. and if there exist a probability model which is characterized by such property, the model selection procedure can be satisfactorily accomplished.

Multivariate data commonly arise in most of the scientific investigations and accordingly multivariate distributions are employed for modelling and analysis of data. Much of the early work in the literature on analysis of bivariate (multivariate) data was focused on bivariate and multivariate normal distributions as there had been a tendency to regard all distributions as normal. However, normal distribution is inappropriate in cases where the data exhibit multi-modality or skewness and hence significant developments have been made with regard to non-normal distributions. Bivariate (Multivariate) distributions with non-normal marginals arise in many fields. For example in lifetime data analysis, the variables of interest are non-negative that often have skewed marginal distributions like exponential, Pareto and Weibull distributions. In reliability, multivariate lifetime data arise when each study subject may experience several events. For instance, the sequence of tumour recurrences, the occurrence of blindness in both eyes and the onset of a genetic disease among family members etc. For various non-normal bivariate (multivariate) distributions, one may refer to Balakrishnan and Lai (2009).

With the wide applicability of partial moments in different fields such as risk analysis, actuarial science, forensic science, reliability modeling, survival analysis etc, the study of the same based on residual and past lifetime are of greater interest among researchers. It has different kinds of interpretations in different contexts. When $r = 0$, then $p_0(t)$ and $l_0(t)$, coincide the survival and distribution function of X respectively. Similarly when $r = 1$ one can relate these partial moments with the popular risk measures such as Conditional Expected Shortfall (CES) and Value-at-Risk (VAR) or with the income (poverty) inequality measures such as income gap ratio, Mean Proportional Residual Income (MPRI) or mean deprivation. For $r = 2$ and $t = E(X)$ the partial moments are called upper or lower semivariance of X. The square root of the lower semivariance can be used to replace the standard deviation in the definition of the Sharpe ratio or in the Markowitz criterion.

Much research on the use and applications of partial moments in higher dimensions has not been reported in the literature. Motivated by these facts, the present work is focused on deriving various characteristic properties of partial moments of bivariate and conditional random variables and its applications in different fields such as life-length , actuarial and income and poverty studies. An overview of the thesis is as follows. The thesis is organized into eight chapters. After this introductory chapter, in Chapter 2, we have pointed out the relevance and scope of the study along with a review of literature. We have also given a brief introduction to the concept of partial moments and their usage and interpretations in different fields. The rest of the chapters are organized as follows:

In Chapter 3, we further explore the concept of BUPM in the context of life-length studies. We prove characterizations to some important bivariate distributions such as bivariate Gumbel's exponential and Pareto model etc. We derive an identity for bivariate distributions when BUPM takes the form of a general class of distributions which contains many important moment relationships, and a generalization of the result due to Lin (2003).

There are many practical situations where the access to conditional distributions are more likely than to their joint distribution. Accordingly in Chapter 4 we study upper partial moments in the conditional setup. It is shown that the conditional upper partial moments determine the corresponding distribution uniquely. The relationships with reliability measures such as conditional hazard rate and mean residual life are obtained. Characterizations results based on conditional upper partial moments for some well known bivariate lifetime distributions are derived. A concept that has applications in economics is income gap ratio which is used for developing indices of affluence and poverty (Sen (1988)). Characterizations of conditional upper partial moments using income gap ratio are also obtained. Finally, non-parametric estimators for conditional partial moments are introduced which are validated using simulated and real data sets.

In Chapter 5, we extend the notion of LPM to the conditional set up and study its usefulness in the context of stochastic modeling. The relationship between various measures in reliability studies, income (poverty) studies, and risk analysis are also proved. Apart from the concept of income gap ratio another measure of interest that has applications in actuarial studies is the Expected Shortfall. Comparisons of the two bivariate returns based on Conditional Expected Shortfall using the quantile version of CLPMs are also discussed. Finally, a non-parametric estimator for conditional LPM is proposed and has been validated through simulated and real data sets.

In most of the practical situations often the investigator cannot record the sampling units with equal probability. The notion of weighted distribution introduced by Rao (1965) gives a unified approach for modeling such biased sampling situations. Motivated by this fact in Chapter 6, we investigate various properties of partial moments in the context of bivariate weighted models. We also study different dependence notions for weighted models using partial moments.

In many statistical models, the assumption of independence between two or more variables is often due to convenience rather than to the problem at hand. The study of dependence between variables can be done through copula functions. Motivated by this, in Chapter 7 we aims at extending the concept of UPM to the bivariate case based on copula function and study its various properties. The relationship between survival copula and first-order bivariate partial moments are established. We also investigate some applications of quantile-based conditional upper partial moments in the context of reliability, actuarial and income (poverty) studies.

Finally, in Chapter 8, we summarizes major conclusions of the present study along with discussions on future research problems on this topic.

Chapter 2

Basic concepts and review of literature

2.1 Introduction

In this section we present the definition and properties of partial moments, and review its applications in different applied fields.

Definition 2.1.1. Let $X : (\Omega, \mathscr{A}, P) \to \left(\overline{\mathbb{R}}, \overline{\mathscr{B}}\right)$ be a numerical random variable that is integrable with respect to P, where $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ denotes the extended set of real numbers and $\overline{\mathscr{B}} := \sigma\left(\mathscr{B} \cup \{\{-\infty\}, \{+\infty\}\}\right)$ denotes the Borel σ-algebra on $\overline{\mathbb{R}}$. Let $n \in \mathbb{N}$, then we define the n^{th} moment of X about a point t as

$$E\left[(X-t)^n\right] = \int_{-\infty}^{\infty} (x-t)^n dP. \tag{2.1}$$

If X is non-negative and $E(X^n)$ is finite, then $E(X^n)$ is called the n^{th} raw moment and $E([X - E(X)]^n)$ the n^{th} central moment of X.

However, there are practical situations in which the information is available only from a point t onwards or till a point t, then (2.1) modifies to the notion of partial moments or one-sided moments. When we are interested only in the left tail or in the right tail (losses or gains), the lower partial moments or the upper partial moments of the distribution of a rv are of importance.

2.1.1 Upper partial moments

Let X be a non-negative continuous random variable with probability density function $f(.)$ and cumulative distribution function $F(.)$. Then the r^{th} degree Upper Partial Moment (UPM) about a point t is defined as

$$\begin{aligned} p_r(t) &= E[(X-t)_+]^r, r = 0, 1, 2, ... \\ &= \int_t^\infty (x-t)^r dF(x), \end{aligned} \tag{2.2}$$

where

$$(X-t)_+ = max\,(X-t, 0) = \begin{cases} X-t & if \quad X \geq t \\ 0 & if \quad X < t \end{cases} \tag{2.3}$$

represents the amount by which X exceeds a threshold *t*.

The random variable $(X-t)_+$ used in defining partial moments are meaningful in the study of personal incomes. When X represents the income of an individual and *t* is the tax exemption level, then $(X-t)_+$ represents the taxable income. The income which fall short of tax exemption level *t* is of no effect in the computation of taxes and therefore treated as zero. Thus the study of partial moments is useful in analyzing measurements that exceed a threshold level without truncating the distribution at *t*. The quantity $(X-t)_+$ is interpreted as the residual life in life length studies (see Lin (2003)). In actuarial sciences, $(X-t)_+$

represents the financial loss incurred by an insurance company, called *risks*. $p_r(t)$ is usually referred to as the r^{th} degree *stop-loss transform* of the risk X and is a standard measure of dangerousness of X (see Cheng and Pai (2003)).

In the context of risk analysis $p_r(t)$ has different meaningful interpretation (i.e. unexpected gain). When $r = 2$, the measure corresponds to the Target semi-variance (TSV). Since TSV consider only deviations of returns from a target t, it is a more appropriate measure of risk than the traditional measure standard deviation, which considers both the positive and negative deviations from the expected return. Hence, returns above the threshold, t, are seen by investors as unexpected gain. Further it more related with the measures expected shortfall and Value-at-Risk (VaR).

Chong (1977) has characterized the exponential and geometric distributions by the properties of partial means. Gupta and Gupta (1983) studied partial moments in the discrete set-up. Gupta (2007) and Sunoj (2004) obtained partial moments and their properties in respect of length-biased and equilibrium distributions. Sankaran and Nair (2004) introduced partial moments for bivariate case and studied characterizations of bivariate discrete models using the discrete bivariate upper partial moment, also known as the bivariate factorial moments. Sunoj and Maya (2008) studied the properties of lower partial moments in stochastic modeling. Kundu and Nanda (2010) studied some properties of partial moments of the inactivity time. A study on certain reliability aspects and applications of quantile-based partial moments is available in Nair et al. (2013b). Nair et al. (2013b) studied the quantile-based univariate stop-loss transform, deriving certain characterizations to some well-known probability models and its applications in the context of income study. Kundu and Sarkar (2017) characterized some continuous distributions based on partial moments of the inactivity time. Recently, Sunoj and Vipin (2017) studied the conditional partial moments and obtained some characteristic properties and its usefulness in certain applied problems.

2.1.1.1 Bivariate upper partial moments

There have been several attempts to generalize the partial moments to higher dimensions. Hürlimann (2002) and Sankaran and Nair (2004) defined the bivariate version of univariate UPMs defined in (2.2) and studied their properties including some characterizations.

Definition 2.1.2. Let $\boldsymbol{X} = (X_1, X_2)$ be a non-negative random vector admitting an absolutely continuous distribution function $F(t_1, t_2)$ with respect to a Lebesgue measure in the positive octant $\mathbb{R}_2^+ = \{(t_1, t_2)|t_1, t_2 > 0\}$ of the two dimensional Euclidean space $\mathbb{R}_2$. Assume that $E(X_1^r X_2^s)$ is finite for any two positive integers r and s. Then the $(r, s)^{th}$ Bivariate Upper Partial Moment denoted by BUPM is defined as

$$\begin{aligned} p_{r,s}\left(t_1, t_2\right) &= E\left[(X_1 - t_1)_+^r (X_2 - t_2)_+^s\right], r, s = 0, 1, 2, \ldots \\ &= \int_{t_1}^{\infty}\int_{t_2}^{\infty} (x_1 - t_1)^r (x_2 - t_2)^s f(x_1, x_2) dx_1 dx_2. \end{aligned} \tag{2.4}$$

Hürlimann (2002) studied (2.4) in the context of actuarial studies whereas Sankaran and Nair (2004) gave more emphasis with related to reliability analysis by mainly focusing on characterizing bivariate discrete probability models using the discrete version of (2.4) known as *bivariate factorial moments*.

Let $(X_i, Y_i),\ i = 1, 2, ...n$ be n independent and identically distributed pairs of lifetimes. A natural estimator of BUPM (2.4) is the empirical estimator proposed by Sankaran and Nair (2004), given as

$$\widehat{p}_{r,s}(t_1,t_2) = \frac{1}{n}\sum_{i=1}^{n}(X_i - t_1)^r(Y_i - t_2)^s I\left(X_i > t_1, Y_i > t_2\right). \tag{2.5}$$

The estimator (2.5) is the $(r,s)^{th}$ moment of the observations surviving beyond some threshold values (t_1,t_2). The strong consistency and asymptotic normality properties of the estimator (2.5) are given in Sankaran and Nair (2004).

2.1.2 Lower partial moments

The concept of risk plays an important role in many studies of economics, business and insurance. Consider a portfolio with a random return X and assume that individual has a target return t. An outcome larger than t is non-risky and desirable, then the individual faces only a one-sided risk called the downside risk that occurs when X falls short of t (Adams and Montesi (1995)). An important risk measure widely used in this context is the Lower Partial Moment (LPM) (see Unser (2000), Sunoj and Maya (2008) and the references therein). It provides a measure of a specified minimum return (target return) that may not be earned by a financial investment. Lower Partial Moments (LPMs) have several advantages over the traditional measure of risk, the variance (see Bawa (1975), Fishburn (1977) etc). A mathematical definition of LPM is as follows. Let X be a non-negative continuous random variable with probability density function f and cumulative distribution function F. Then the r^{th} univariate lower partial moment (LPM) about a point t is defined as

$$l_r(t) \;=\; E[(X-t)_-]^r, r = 0,1,2,... \tag{2.6}$$

where

$$(X-t)_{-} = max\,(t-X,0) = \begin{cases} t-X & if \quad X \leq t \\ \\ 0 & if \quad X > t. \end{cases} \tag{2.7}$$

In terms of the distribution function,

$$l_r(t) \;\; = \;\; \int_{-\infty}^{t} (t-x)^r dF(x). \tag{2.8}$$

Note that LPM is a function of the underlying distribution function and it is an increasing function of the target return *t*. Differentiating (2.8) successively *r* times with respect to t gives

$$F(t) = \left(\frac{1}{r!}\right) \frac{d^r}{dt^r} \left(l_r\left(t\right)\right). \tag{2.9}$$

(2.9) implies that the LPM, $l_r(t)$, determines the distribution uniquely.

Since LPMs consider only negative deviations of returns from a target *t*, it is a more appropriate measure of risk than the standard deviation, which considers both the positive and negative deviations from the expected return. Hence, returns below the threshold, *t*, are seen by investors as loss. This (loss) threshold level, might be the rate of inflation, the real interest rate, the return on a benchmark index, the risk-free rate etc. In reliability theory, the LPM is known as the r^{th} order partial mean inactivity time (past lifetime). Due to the importance of LPM in many fields such as risk analysis, actuarial science, forensic science, reliability modeling, survival analysis etc, the study of partial moments and its higher orders based on past lifetime are of greater interest among researchers (see Kundu and Sarkar (2017) and the references therein). Also, as both LPMs and poverty indices focus on the lower part of the distribution (such as incomes of the poor), LPM is an efficient tool to measure poverty indices in the income analysis (see

Sunoj and Maya (2008)).

Since the both upper and lower partial moments have its own interpretation and applications in different fields such as reliability modelling, actuarial studies and income (poverty) studies, in the following sections we give a brief review on basic reliability, risk and income (poverty) concepts useful in the present study.

2.2 Some basic concepts in reliability theory - Univariate case

The term 'reliability' corresponds to the probability that an equipment or unit will perform the required function under the conditions specified for its operations for a given period of time. The primary concern in reliability theory is to understand the patterns in which failures occur, for different mechanisms and under varying operating environments, as a function of its age. This is accomplished by identifying the probability distribution of the lifetime represented by a non-negative random variable X . Accordingly, several concepts have been developed that help in evaluating the effect of age, based on the distribution function of the lifetime random variable X and the residual life. In this section, we recall the definitions and properties of some popular reliability measures, that are used in the subsequent chapters.

2.2.1 Reliability function

Let $a = \inf\{t|F(t) > 0\}$ and $b = \sup\{t|F(t) < 1\}$ be such that (a, b), $-\infty \leq a < b \leq \infty$ is the interval support of the rv X. The reliability function or survival function (sf) of a rv X, denoted by $R(\cdot)$ is defined as

$$R(t) = P(X > t) = 1 - F(t),$$

where $F(\cdot)$ is the distribution function (df) of X. It gives the probability of failure free operation for a time period greater than t.

The survival function $R(.)$ of X can be written in terms of $p_r(t)$ as $R\,(t) = \frac{(-1)^r}{r!}\frac{d^r p_r(t)}{dt^r}$ (see Navarro et al. (1998), Sunoj (2004)).

2.2.2 Failure rate

The failure rate also known as the hazard rate of a rv X, denoted by $h(\cdot)$, is defined as

$$h(t) = \lim_{\Delta t \to 0} \frac{P[t \leq X < t + \Delta t | X > t]}{\Delta t}. \tag{2.10}$$

The failure rate $h(t)$, measures the instantaneous rate of failure at time t, given that the component survives at least up to time t. $h(t)\Delta t$ represents the approximate probability of failure in the interval $[t, t+\Delta t)$, given the component survived up to time t, provided Δt is very small. Kotz and Shanbhag (1980) defined failure rate as the Radon Nikodym derivative with respect to Lebesgue measure on $\{t : F(t) < 1\}$, of the hazard measure $H(B) = \int_B \frac{dF(t)}{[1-F(t)]}$ for every Borel set B of the form $(-\infty, L)$, where $L = \inf\{t : F(t) = 1\}$. If $f(\cdot)$ is the pdf of X, (2.10) can be equivalently written as

$$h(t) = \frac{f(t)}{R(t)} = -\frac{d}{dt} \log R(t).$$

$h(\cdot)$ uniquely determines the sf $R(\cdot)$ through the relationship

$$R(t) = \exp\left(-\int_0^x h(u)du\right) = \exp(-H(t)),$$

where $H(t) = \int_0^x h(u)du$ is known as cumulative hazard rate. The hazard rate and the UPM are connected by the relation

$$-\frac{d}{dt}\log\left(p_0(t)\right) = h\left(t\right).$$

The concept of hazard rate is widely used for characterizing lifetime distributions. For example, constancy of hazard rate is a characteristic property of exponential distribution (Galambos and Kotz (1978)). A large volume of literature is available on characterizations and other properties of hazard rate function (see, for example, Barlow et al. (1963), Nanda and Shaked (2001), Nair and Asha (2004), Nanda (2010), Noughabi et al. (2013) and references therein).

2.2.3 Reversed hazard rate

Barlow et al. (1963) proposed reversed hazard rate function for a rv X, denoted by $\bar{h}(\cdot)$ and is defined as

$$\bar{h}(t) = \lim_{\Delta t \to 0} \frac{P[t - \Delta t < X \leq t | X \leq t]}{\Delta t}.$$

$\bar{h}(t)$ measures the instantaneous rate of failure of a unit at time t, given that it failed before time t. Thus, $\bar{h}(t)\Delta t$ gives the probability that the unit failed in an infinitesimal interval $(t - \Delta t, t]$, given that it failed before t. If the pdf $f(\cdot)$ exists, the above equation can be expressed as

$$\bar{h}(t) = \frac{f(t)}{F(t)} = \frac{d}{dx}\log F(t).$$

Keilson and Sumita (1982) shown that $\bar{h}(\cdot)$ determines the CDF through the relationship

$$F(t) = \exp\left(-\int_t^b \bar{h}(u)du\right) = \exp(-\bar{H}(t)),$$

where $\bar{H}(t) = \int_t^b \bar{h}(u)du$ denotes the cumulative reversed hazard rate.

Finkelstein (2002) established the relationship between $\bar{h}(\cdot)$ and $h(\cdot)$ as

$$\bar{h}(t) = \frac{h(t)}{\exp\left(\int_0^t h(u)du\right) - 1}.$$

For more details on reversed hazard rate one can refer to Gupta and Nanda (2001), Nanda and Shaked (2001), Nair and Asha (2004), Bartoszewicz and Skolimowska (2004), Chandra and Roy (2005), Nair et al. (2005), Sunoj and Maya (2006), Sankaran et al. (2007) and Kundu and Ghosh (2017) .

2.2.4 Mean residual life function

For a rv X with $E(X) < \infty$, the Mean Residual Life Function (MRLF) denoted by $m(\cdot)$, defined by Swartz (1973) as

$$m(t) = E(X - t|X > t). \tag{2.11}$$

$m(t)$ measures the average residual life of a component which has survived a time t. If the df $F(\cdot)$ is continuous with respect to Lebesgue measure, (2.11) becomes

$$m(t) = \frac{1}{R(t)} \int_t^\infty R(u)du.$$

$m(\cdot)$ uniquely determines the underlying distribution through the relationship

$$R(t) = \frac{m(0)}{m(t)} \exp\left[-\int_0^t \frac{1}{m(u)} du\right].$$

Model identification can be done easily by knowing the functional form of $m(\cdot)$. For example, characterization of distribution using the linear form of $m(\cdot)$ is available in Hall and Wellner (1981). MRLF is related to the failure rate by the equation

$$h(t) = \frac{1 + m'(t)}{m(t)}.$$

Bryson and Siddiqui (1969) proved that increasing hazard rate of a component implies decreasing MRLF of that component. The Mean Residual Life (MRL) function and the UPM are connected by the relation

$$\frac{p_1(t)}{p_0(t)} = E\left(X - t | X > t\right).$$

For more properties on $m(\cdot)$, one could refer to Hall and Wellner (1981), Mukherjee and Roy (1986), Nanda (2010), Gupta (2016a) and references therein.

2.2.5 Reversed mean residual life function

The reversed mean residual life function is an analogous concept of MRLF but defined for the past lifetime $(t - X | X \leq t)$, given by

$$\bar{m}(t) = E(t - X | X \leq t).$$

It measures the average past lifetime of a rv which failed at time t. It is also known as mean inactivity time or mean past lifetime in reliability. If the df $F(\cdot)$ is continuous with respect to Lebesgue measure, $\bar{m}(\cdot)$ can be written as

$$\bar{m}(t) = \frac{1}{F(t)} \int_0^t F(u)du.$$

The reversed mean residual life time is related to reversed hazard rate through the relationship,

$$\bar{h}(t) = \frac{1 - \bar{m}'(t)}{\bar{m}(t)}.$$

Like $m(\cdot)$, $\bar{m}(\cdot)$ also uniquely determines the underlying df by the relationship (Chandra and Roy (2001)),

$$F(t) = \exp\left(-\int_t^\infty \frac{1 - \bar{r}'(u)}{\bar{m}(u)} du\right).$$

For more details on reversed mean residual life function, we refer to Kayid and Ahmad (2004), Ahmad and Kayid (2005), Gandotra et al. (2011), Kayid and Izadkhah (2014), Kundu and Ghosh (2017) and references therein.

2.2.6 Vitality function

Kupka and Loo (1989) introduced the concept of vitality function as a Borel-measurable function on the real line as

$$v(t) = E\left(X \,|X > t\,\right) = \frac{1}{R(t)} \int_t^\infty uf(u)du. \tag{2.12}$$

Clearly, (2.12) measures the expected life of a component, when it has survived t units of time. The vitality function is closely related to MRLF by the relationship

$$v(t) = x + m(t)$$

and

$$v'(t) = m(t)h(t),$$

where $v'(t)$ is the derivative of $v(t)$. Due to the one-to-one relationship between $m(\cdot)$ and $v(\cdot)$, the vitality function uniquely determines the underlying distribution. Moreover, the vitality function and the UPM are connected by the relation, $\frac{p_1(0)}{p_0(t)} = E\left(X|X > t\right)$.

2.3 Bivariate case

Let $\boldsymbol{X} = (X_1, X_2)$ be a random vector defined on $\mathbb{R}_2 = (-\infty, \infty) \times (-\infty, \infty)$. Then joint (bivariate) df of (X_1, X_2) is defined as $F(t_1, t_2) = P(X_1 \leq t_1, X_2 \leq t_2)$. It satisfies the following properties:

(i) $\lim\limits_{t_1 \to -\infty} \lim\limits_{t_2 \to -\infty} F(t_1, t_2) = \lim\limits_{t_1 \to -\infty} F(t_1, t_2) = \lim\limits_{t_2 \to -\infty} F(t_1, t_2) = 0$,

(ii) $\lim\limits_{t_1 \to \infty} \lim\limits_{t_2 \to \infty} F(t_1, t_2) = 1$,

(iii) If $a < b$ and $c < d$, then $F(a, c) < F(b, d)$,

(iv) If $a > t_1$ and $b > t_2$, then $F(a, b) - F(a, t_2) - F(t_1, b) + F(t_1, t_2) \geq 0$.

The bivariate sf of (X_1, X_2) is defined as $R(t_1, t_2) = P(X_1 > t_1, X_2 > t_2)$. $R(t_1, t_2)$ is related to $F(t_1, t_2)$ by the equation

$$R(t_1, t_2) = 1 - \lim_{t_2 \to \infty} F(t_1, t_2) - \lim_{t_1 \to \infty} F(t_1, t_2) + F(t_1, t_2).$$

If $F(t_1, t_2)$ is absolutely continuous and if the second order derivative exists then the joint density function $f(t_1, t_2)$ can be defined as

$$f(t_1, t_2) = \frac{\partial^2 R(t_1, t_2)}{\partial t_1 \partial t_2} = \frac{\partial^2 F(t_1, t_2)}{\partial t_1 \partial t_2}.$$

2.3.1 Bivariate hazard rate

A straightforward extension of univariate hazard rate (or failure rate) to the bivariate case is due to Basu (1971), defined as a scalar failure rate,

$$k(t_1, t_2) = \frac{f(t_1, t_2)}{R(t_1, t_2)}.$$

Puri and Rubin (1974) characterized a mixture of exponential distributions by the constancy $k(t_1, t_2) = c$ for $t_1 > 0$ and $t_2 > 0$. However, in general $k(t_1, t_2)$ does not determine a bivariate distribution uniquely. For more properties, see Yang and Nachlas (2001), Finkelstein (2003) and Finkelstein and Esaulova (2005).

An alternative and a more popular definition on bivariate hazard rate is due to Johnson and Kotz (1975) who proposed a vector-valued bivariate failure rate,

$$h(t_1, t_2) = (h_1(t_1, t_2), h_2(t_1, t_2)),$$

where

$$h_i(t_1, t_2) = -\frac{\partial}{\partial t_i} \log R(t_1, t_2), \quad i = 1, 2,$$

is the instantaneous failure rate of X_i at time t_i given that X_i was alive at time t_i and that X_{3-i} survived beyond time t_{3-i}, $i = 1, 2$. Unlike $k(t_1, t_2)$, $h(t_1, t_2)$ uniquely determines the df (see Marshall et al. (2011) and Shanbhag and Kotz (1987)) through the expression

$$R(t_1, t_2) = \exp\left[-\int_0^{t_1} h_1(u, 0)du - \int_0^{t_2} h_2(t_1, v)dv\right]$$

or

$$R(t_1, t_2) = \exp\left[-\int_0^{t_1} h_1(u, t_2)du - \int_0^{t_2} h_2(0, v)dv\right].$$

Some characterizations of probability models based on $h(t_1, t_2)$ can be found in Navarro and Ruiz (2004), Kotz et al. (2007) and Navarro et al. (2007).

Some other versions of failure rate in bivariate set up are also available in literature, for example Cox (1972), Marshall (1975), Shaked and Shanthikumar (1987), Basu and Sun (1997), Finkelstein (2003) and references therein.

2.3.2 Bivariate reversed hazard rate

Motivated with the wide applicability of bivariate failure rate due to Johnson and Kotz (1975), Roy (2002a) proposed a vector-valued reversed hazard rate. Let (X_1, X_2) be a random vector with joint df $F(t_1, t_2)$ and $F_i(\cdot)$ denotes the marginal df of $X_i, i = 1, 2$. The support of (X_1, X_2) be $D = [0, b_1] \times [0, b_2]$ where (b_1, b_2) is such that $F(b_1, b_2) < 1$ then the bivariate reversed failure rate is defined as

$$\bar{h}(t_1, t_2) = (\bar{h}_i(t_1, t_2), \bar{h}_i(t_1, t_2)),$$

where

$$\begin{aligned}\bar{h}_i(t_1, t_2) &= \lim_{\Delta t_i \to 0} \frac{P\left(t_i - \Delta t_i \le X_i \le t_i | X_1 \le t_1, X_2 \le t_2\right)}{\Delta t_i} \\ &= \frac{\partial}{\partial t_i} \log F(t_1, t_2), i = 1, 2.\end{aligned}$$

Here $\bar{h}_1(t_1, t_2)\Delta t_1$, represents the probability of failure of the first component in the interval $(t_1 - \Delta t_1, t_1]$ given that it has failed before t_1 and the second has failed before t_2. The interpretation for $\bar{h}_2(t_1, t_2)$ is similar.

$\bar{h}(t_1, t_2)$ uniquely determine $F(t_1, t_2)$ by the relationships

$$F(t_1, t_2) = \exp\left[-\int_{t_1}^{b_1} \bar{h}_1(u, b_2)du - \int_{t_2}^{b_2} \bar{h}_2(t_1, v)dv\right] \tag{2.13}$$

or

$$F(t_1, t_2) = \exp\left[-\int_{t_1}^{b_1} \bar{h}_1(u, t_2)du - \int_{t_2}^{b_2} \bar{h}_2(b_1, v)dv\right]. \tag{2.14}$$

For more details on bivariate reversed hazard rate we refer to Sankaran and Gleeja (2006, 2008), Asha and Rejeesh (2007, 2009), Domma (2011) and Kundu and Kundu (2017).

2.3.3 Bivariate mean residual life function

Buchanan and Singpurwalla (1977) introduced a bivariate MRLF as

$$e(t_1, t_2) = \frac{1}{R(t_1, t_2)}\int_0^\infty \int_0^\infty P[X_1 > t_1 + x_1, X_2 > t_2 + x_2]dx_1dx_2,\ t_i > 0, i = 1, 2.$$

Even if $e(t_1, t_2)$ is a direct extension of univariate MRLF, it does not uniquely determine the underlying distribution.

An alternative definition to bivariate MRLF is provided by Shanbhag and Kotz (1987) and Arnold and Zahedi (1988) as follows. Let (X_1, X_2) be a random vector on $\mathbb{R}_2^+ = \{(t_1, t_2)|t_i > 0, i = 1, 2\}$ with joint df $F(t_1, t_2)$ and let (L_1, L_2) be the vector of extended real numbers such that $L_i = \inf\{t|F_i(t_i) = 1\}$ where $F_i(\cdot)$ is the df of X_i. Further let $E(X_i) < \infty$, for $i = 1, 2$. The vector-valued Borel-measurable function $m(t_1, t_2)$ on $\mathbb{R}_2^+$ is given by

$$\begin{aligned} m(t_1, t_2) &= (m_1(t_1, t_2), m_2(t_1, t_2)) \\ &= (E(X_1 - t_1 | X_1 > t_1, X_2 > t_2), E(X_2 - t_2 | X_1 > t_1, X_2 > t_2)), \end{aligned}$$

for all $(X_1, X_2) \in \mathbb{R}_2^+$, $t_i < L_i$, $i = 1, 2$, is called the bivariate mean residual life function. When (X_1, X_2) is continuous and non-negative, the components of bivariate MRLF are given by

$$m_1(t_1, t_2) = E(X_1 - t_1 | X_1 > t_1, X_2 > t_2) = \frac{1}{R(t_1, t_2)} \int_{t_1}^{\infty} R(u, t_2) du \tag{2.15}$$

and

$$m_2(t_1, t_2) = E(X_2 - t_2 | X_1 > t_1, X_2 > t_2) = \frac{1}{R(t_1, t_2)} \int_{t_2}^{\infty} R(t_1, v) dv. \tag{2.16}$$

Unlike $e(t_1, t_2)$, the bivariate MRLF $m(t_1, t_2)$ uniquely determines the distribution through the identities (Nair and Nair (1988))

$$R(t_1, t_2) = \frac{m_1(0,0) m_2(t_1, 0)}{m_1(t_1, 0) m_2(t_1, t_2)} \exp\left[- \int_0^{t_1} \frac{du}{m_1(u, 0)} - \int_0^{t_2} \frac{dv}{m_2(t_1, v)} \right]$$

or

$$R(t_1, t_2) = \frac{m_1(0, t_2) m_2(0,0)}{m_1(t_1, t_2) m_2(0, t_2)} \exp\left[- \int_0^{t_2} \frac{dv}{m_2(0, v)} - \int_0^{t_1} \frac{du}{m_1(u, t_2)} \right].$$

Similar to the relationship between failure rate and MRLF in the univariate case, the bivariate MRLF is related to bivariate failure rate by

$$h_i(t_1, t_2) = \frac{1 + \frac{\partial}{\partial t_i} m_i(t_1, t_2)}{m_i(t_1, t_2)}, \quad i = 1, 2.$$

For more applications of bivariate mean residual life function we refer to Sankaran and Nair (1993c), Roy (2002b), Nair et al. (2004) and Sunoj and Vipin (2017).

2.3.4 Covariance residual life function

An important aspect to be considered while modeling bivariate data on life times is the dependency structure between them, which can be measured in terms of the covariance. Since covariance between lifetimes can also be studied in terms of their residual lives, a discussion of covariance of residual lives becomes relevant. Based on this idea Nair et al. (2004) introduced the concept of covariance residual life function of (X_1, X_2) as follows.

Let (X_1, X_2) be a random vector that takes values in the positive octant $\mathbb{R}_2^+ = \{(t_1, t_2)\,|t_1 > 0, t_2 > 0\}$ of the two dimensional space with absolutely continuous survival function $R(t_1, t_2) = P[X_1 > t_1, X_2 > t_2]$ and density function $f(t_1, t_2)$ with $E(X_1 X_2) < \infty$. Then following Nair et al. (2004), the product moment residual life function is

$$M\,(t_1, t_2) = E\,[(X_1 - t_1)\,(X_2 - t_2)\,|X_1 > t_1, X_2 > t_2]$$

and the covariance residual life of (X_1, X_2) will be

$$C\,(t_1, t_2) = M\,(t_1, t_2) - m_1\,(t_1, t_2)\,m_2\,(t_1, t_2)$$

where $m_i(t_1, t_2), i = 1, 2$ is the i^{th} component of the BMRL vector as given in (2.15) and (2.16).

2.3.5 Bivariate reversed mean residual life function

A vector-valued bivariate reversed mean residual life function is proposed by Nair and Asha (2008). Let (X_1, X_2) be a random vector defined on $\mathbb{R}_2$ with joint df $F(t_1, t_2)$ and marginal df $F_i(\cdot)$, $i = 1, 2$, $E(X_1, X_2) < \infty$ and let (a_1, a_2) and (b_1, b_2) be vectors of real numbers such that $a_i = \inf\{t|F_i(t) > 0\}$ and $b_i = \sup\{t|F_i(t) < 1\}$ then bivariate reversed mean residual life function is defined as a Borel-measurable function

$$\bar{r}(t_1, t_2) = (\bar{r}_1(t_1, t_2), \bar{r}_2(t_1, t_2)),$$

where

$$\bar{r}_1(t_1, t_2) = E(t_1 - X_1|(X_1 \leq t_1, X_2 \leq t_2) = \frac{1}{F(t_1, t_2)} \int_{a_1}^{t_1} F(u, t_2)du$$

and

$$\bar{r}_2(t_1, t_2) = E(t_2 - X_2|(X_1 \leq t_1, X_2 \leq t_2) = \frac{1}{F(t_1, t_2)} \int_{a_2}^{t_2} F(t_1, v)dv.$$

The bivariate reversed mean residual life function uniquely determines the underlying distribution through the relationships

$$F(t_1, t_2) = \frac{\bar{r}_1(b_1, b_2)\bar{r}_2(t_1, b_2)}{\bar{r}_1(t_1, b_2)\bar{r}_2(t_1, t_2)} \exp\left(-\int_{t_1}^{b_1} \frac{du}{\bar{r}_1(u, b_2)} - \int_{t_2}^{b_2} \frac{dv}{\bar{r}_2(t_1, v)}\right)$$

and

$$F(t_1, t_2) = \frac{\bar{r}_1(b_1, t_2)\bar{r}_2(b_1, b_2)}{\bar{r}_1(t_1, t_2)\bar{r}_2(b_1, t_2)} \exp\left(-\int_{t_1}^{b_1} \frac{du}{\bar{r}_1(u, t_2)} - \int_{t_2}^{b_2} \frac{dv}{\bar{r}_2(b_1, v)}\right).$$

Further, bivariate reversed mean residual life function is related to bivariate reversed hazard rate by

$$\bar{h}_i(t_1,t_2) = \frac{1 - \frac{\partial}{\partial t_i}\bar{r}_i(t_1,t_2)}{\bar{r}_i(t_1,t_2)}, \quad i = 1,2.$$

For more properties and results based on bivariate reversed mean residual life function, we refer to Kayid (2006), Asha and Rejeesh (2009) and Ghosh and Kundu (2017).

2.3.6 Bivariate vitality function

Kupka and Loo (1989) have employed a new method of measuring the phenomenon of ageing with the aid of vitality function which is the expectation of a random variable X conditioned on $X > t$. The properties of vitality function and its relationship to the other ageing concepts were discussed in Section 2.2.6. Sankaran and Nair (1991) extend the notion of vitality function to the bivariate case and point out some of its applications in the analysis of lifetime data.

Let $\boldsymbol{X} = (X_1, X_2)$ be a bivariate random vector in the support of $\{(t_1,t_2)|a_i \leq t_i \leq b_i\}, i = 1,2$ for $a_i \geq -\infty$ and $b_i \leq +\infty$, with survival function $R(t_1,t_2)$. For values of $t_i < b_i$ such that $P[X \geq x] > 0$ and $t_i^+ = max(0,t_i)$ satisfying $E(X_i^+) < \infty$, the vector-valued function,

$$v\left(t_1,t_2\right) = \left(v_1\left(t_1,t_2\right), v_2\left(t_1,t_2\right)\right),$$

where, $v_i\left(t_1,t_2\right) = E\left[X_i|X_i \geq t_i, X_j \geq t_j\right], i,j = 1,2, i \neq j$ is called the bivariate vitality function of $\boldsymbol{X}$.

2.3.7 Bivariate variance residual life

For a rv X with $E(X) < \infty$, the Variance Residual Life Function (VRLF) denoted by $V(\cdot)$, defined as

$$V(t) = Var(X - t|X > t)$$

See Launer (1984), Gupta et al. (1987), Gupta (1987), Sankaran and Nair (1993b) and Gupta and Kirmani (2000). The Variance residual life function (VRLF) and the UPM are connected by, $V(t) = \frac{p_2(t) - p_1^2(t)}{p_0(t)}$. Unlike the MRLF, VRLF does not determine the underlying life distribution uniquely.

Let (X_1, X_2) be a bivariate random vector admitting absolutely continuous distribution function with respect to Lebesgue measure in the positive octant that takes values in the positive octant $\mathbb{R}_2^+ = \{(t_1, t_2)\,|t_1 > 0, t_2 > 0\}$ of the two dimensional Euclidean space $\mathbb{R}_2$ and having survival function $R(t_1, t_2) = P[X_1 > t_1, X_2 > t_2]$. Assume that $E(X_i^2) < \infty, i = 1, 2$. Then

$$V(t_1, t_2) = (V_1(t_1, t_2), V_2(t_1, t_2)),$$

where, $V_i(t_1, t_2) = E\left[(X_i - t_i)^2 | X_1 \geq t_1, X_2 \geq t_2\right] - m_i^2(t_1, t_2), i = 1, 2.$ is defined as the bivariate variance residual life and $V_i(t_1, t_2)$ as its components by Sankaran and Nair (1993b).

Using BVRLF and BMRLF, Gupta and Kirmani (2000) defined the residual coefficient of variation in the bivariate case as the vector $(CV_1(t_1, t_2), CV_2(t_1, t_2))$, where,

$$CV_i(t_1, t_2) = \frac{\sqrt{V_i(t_1, t_2)}}{m_i(t_1, t_2)}, i = 1, 2.$$

2.4 Bivariate ageing classes

In this section we consider some bivariate ageing classes. A detailed study of the various ageing classes and their properties is available in Lai and Xie (2006). Let (X_1, X_2) denote a bivariate random vector with joint density function $f(t_1, t_2)$ and joint survival function $R(t_1, t_2)$. Then, the random vector (X_1, X_2) is said to have

(i) Increasing Bivariate Hazard Rate-1 (IBHR-1) if $\frac{R(t_1+s,t_2+s)}{R(t_1,t_2)}$ is decreasing in t_1, t_2 for all $s > 0$.

(ii) Increasing Bivariate Hazard Rate-2 (IBHR-2) if $\frac{R(t_1+s_1,t_2+s_2)}{R(t_1,t_2)}$ is decreasing in t_1, t_2 for all s_1, $s_2 > 0$.

(iii) Decreasing Bivariate Mean Residual Life (DBMRL) if $m_1(t_1, t_2)$ is decreasing in t_1 for all t_2 and $m_2(t_1, t_2)$ is decreasing in t_2 for all t_1.

(iv) bivariate Decreasing Mean Residual Life-I (DMRL-I) if for all $t \geq 0$ for which $R(t,t) > 0$, $\frac{\int_t^\infty \int_t^\infty R(x,y)dxdy}{R(t,t)}$ is non-increasing in t.

(v) bivariate (DMRL-II) if for all $t \geq 0$ for which $R(t,t) > 0$, $\frac{\int_t^\infty \int_t^\infty R(x,x)dx}{R(t,t)}$ is non increasing in t.

Apart from the above definitions, for an exchangeable random vector (X_1, X_2), Bassan et al. (2002) defined the bivariate ageing notions such as Bivariate Increasing Failure Rate (BIFR) and Bivariate Decreasing Mean Residual Life (BDMRL). An exchangeable random vector (X_1, X_2) is said to have

(vi) Bivariate IHR (BIFR) distribution in the strong sense, if and only if the ratio $\frac{R(t_1+s,t_2)}{R(t_2+s,t_1)}$ is increasing in s for $t_1 < t_2$.

(vii) Bivariate Decreasing Mean Residual Life (BDMRL) distribution if for $t_1 < t_2$ if $E((X_1 - t_1)|X_1 > t_1, X_2 > t_2) \geq E((X_2 - t_2)|X_1 > t_1, X_2 > t_2)$.

2.5 Conditionally specified models

It is inherently difficult to visualise bivariate distributions. Conditional densities can be easily visualised unlike marginal or joint densities. For example, in some human population it is reasonable to visualise the unimodal distribution of heights for a given weight with the mode of the conditional distribution varying monotonically with the weight. In a similar way a unimodal distribution of weights for a given height can be easily visualised with the mode varying monotonically with the height. But it is not so easy to visualise the appropriate joint distributions without certain assertion. A variety of transformation are being used to characterize the joint df. Joint characteristic function, joint moment generating function, and joint hazard function are some among them. They are well defined and will determine the joint df uniquely.

To determine the joint df, the knowledge of the marginals is inadequate. But if we incorporate conditional specification instead of marginal specification or together with marginal specification then the picture brightens. Sometimes one could characterize joint distribution in this way, *i.e.* the knowledge of one marginal density say $f_{X_1}(\cdot)$ and the conditional density of X_2 given X_1 will completely specify the joint density function $f_{X_1X_2}(\cdot)$ of a bivariate rv. Alternatively one may specify the distribution solely in terms of the features of two families of conditional densities. This approach is called conditional specification of the joint distribution. For works on conditionally specified models one can refer to Arnold et al. (1999) and references therein.

2.6 Conditional survival models

In conditionally specified bivariate distribution, joint density $f_{X_1X_2}(\cdot)$ has been referred with all conditionals of X_1 given $X_2 = t_2$ belonging to a particular parametric family and all conditionals of X_2 given $X_1 = t_1$, belonging to another parametric family. In the case of bivariate survival models, component survival *i.e.* on events such as $\{X_1 > t_1\}$ and $\{X_2 > t_2\}$ have been conditioned. For works on conditional survival models we refer to Arnold (1995).

2.7 Proportional hazards rate model

Proportional hazards rate model, more popularly known as Cox proportional hazards model was proposed by Cox (1972). Let X and Y be two rvs with pdfs f and g, sfs R and $\bar{G}$ and hazard rates h_X and h_Y respectively, then X and Y are said to satisfy proportional hazards rate (PHR) model if they satisfy the relationship

$$h_Y(t) = \theta h_X(t) \quad \text{or equivalently} \quad \bar{G}(t) = (R(t))^{\theta},$$

where $\theta > 0$, is a constant, with the pdf $g(t) = \theta(R(t))^{\theta-1}f(t)$. Proportional hazards model has been used to model failure time data in reliability and survival analysis. Studies related to PHR model could be found in Clayton and Cuzick (1985), Ebrahimi and Kirmani (1996), Kundu and Gupta (2004), Nair and Gupta (2007), Sankaran and Sreeja (2007), Dewan and Sudheesh (2009), Nair et al. (2018a) and the references therein.

2.7.1 Conditional proportional hazards rate model

A popular model for modeling the effects of covariates on survival is the celebrated *Cox-proportional hazards model (PHR)*. Let X and Y be two random variables with the same support $(0, \infty)$ and with hazard rate functions $h_X = \frac{f}{R}$ and $h_Y = \frac{g}{G}$, respectively. Then X and Y satisfy the PHR model when $h_Y(t) = \theta h_X(t)$, for all t (see Cox (1959)). This relationship is also equivalent to $G(t) = (F(t))^{\theta}$, for all t. The PHR model is extended to conditional models as follows. The random vectors (X_1, X_2) and (Y_1, Y_2) satisfy the Conditional Proportional Hazards Rate (CPHR) model (see Sankaran and Sreeja (2007)) when the corresponding conditional hazard rate functions of $(X_i|X_j = t_j)$ and $(Y_i|Y_j = t_j)$ satisfy

$$h_{(Y_i|Y_j=t_j)}(t_i|t_j) = \theta_i(t_j)h_{(X_i|X_j=t_j)}(t_i|t_j),$$

for $i, j = 1, 2; i \neq j$ and $t_i, t_j \geq 0$, or equivalently $\bar{G}_i(t_i|t_j) = (R_i(t_i|t_j))^{\theta_i(t_j)}$, where $\bar{G}_i(t_i|t_j) = P(Y_i > t_i|Y_j = t_j)$ and $R_i(t_i|t_j) = S_i(t_i|t_j) = P(X_i > t_i|X_j = t_j)$. For conditional survival models, CPHR model becomes

$$h_{(Y_i|Y_j>t_j)}(t_i|t_j) = \delta_i(t_j)h_{(X_i|X_j>t_j)}(t_i|t_j),$$

for $i, j = 1, 2; i \neq j$ and $t_i, t_j \geq 0$. This is equivalent to $\bar{G}^*_i(t_i|t_j) = (R^*_i(t_i|t_j))^{\delta_i(t_j)}$, where $\bar{G}^*_i(t_i|t_j) = P(Y_i > t_i|Y_j > t_j)$ and $R^*_i(t_i|t_j) = R_i(t_i|t_j) = P(X_i > t_i|X_j > t_j)$.

2.8 Weighted distributions

Let $\boldsymbol{X} = (X_1, ..., X_p)$ is a p-dimensional random vector with probability density function $f(\boldsymbol{x}; \boldsymbol{\theta})$ where $\theta \in \Theta$ and $\Theta \in R^q$ is parameter space. Suppose that a realization $\boldsymbol{x}$ of $\boldsymbol{X}$ under $f(\boldsymbol{x}; \boldsymbol{\theta})$ enters the investigator's record with probability proportional to $w(\boldsymbol{x}, \boldsymbol{\beta}) > 0$, so that

$$\frac{P\left(\text{Rec}ording|\boldsymbol{X} = \boldsymbol{x}\right)}{P\left(\text{Rec}ording|\boldsymbol{X} = \boldsymbol{y}\right)} = \frac{w\left(\boldsymbol{x}, \boldsymbol{\beta}\right)}{w\left(\boldsymbol{y}, \boldsymbol{\beta}\right)}.$$

Here, the recording (weight) function, $w(\boldsymbol{x}, \boldsymbol{\beta})$ is non-negative function with parameter representing the recording mechanism and $\boldsymbol{\beta}$ may be known or unknown parameter. Clearly, the recorded $\boldsymbol{x}$ is not an observation on $\boldsymbol{X}$, but on a weighted random vector $\boldsymbol{X}^w$, with probability density function

$$f^w(\boldsymbol{x};\boldsymbol{\theta},\boldsymbol{\beta}) = \frac{w(\boldsymbol{x},\boldsymbol{\beta})\, f(\boldsymbol{x},\boldsymbol{\theta})}{E\left[w(\boldsymbol{X},\boldsymbol{\beta})\right]}, \tag{2.17}$$

called the weighted distribution, and the corresponding mechanism of recording observations is called weighted sampling. Weighted sampling occurs when the usual random sample of a population of interest is not available, due to the data having unequal probabilities of entering the sample.

The concept of weighted distributions can be traced from the studies by Fisher (1934) on how methods of ascertainment can influence the form of distribution of recorded observations. However, Rao (1965) identified the need for a unifying the concept of weighted distributions and studied various sampling situations that can be modeled by weighted distributions. These situations arise when the recorded observations cannot be considered as a random sample from the original distributions, such as non-observability of some events or damage occurred to the original observation resulting in reduced value, or the adoption of a sampling mechanism which gives unequal chances to the units in the original.

A mathematical definition of a weighted distribution is obtained by considering a probability space $(\Omega, \Im, P)$ and a rv $X : \Omega \to H$, where $H = (a, b)$ is an interval on the real line with $a > 0$ and $b(> a)$ can be finite or infinite. When the df $F(\cdot)$ of X is absolutely continuous with pdf $f(\cdot)$ and $w(\cdot)$, a non-negative function satisfying $\mu^w = E(w(X)) < \infty$, then the rv X^w with pdf

$$f^w(t) = \frac{w(t)}{\mu^w} f(t), \quad a < t < b,$$

is said to have weighted distribution, corresponding to the distribution of X. The definition in the discrete case is similar.

Depending on the selection of weight function $w(\cdot)$, we have different weighted distributions. For example, when $w(t) = t$, then X^w is called the length-biased rv X^L with pdf,

$$f^L(t) = \frac{t}{\mu} f(t), \quad a < t < b,$$

where $\mu = E(X) < \infty$. Length-biased sampling is usually adopted when a suitable sampling frame is absent. In length-biased sampling items are selected at a rate proportional to its length, so that larger values of the quantity being measured are sampled with higher probabilities. In such situations, the possible bias due to the nature of data collection process can be utilized to connect the population parameters to that of the sampling distribution. That is, if we know the choice mechanism behind the biased sample, then the process of inference on population parameters is easier. Length-biased sampling has wide variety of applications on various topics such as reliability theory, survival analysis, population studies and clinical trials. For a more details on various aspects of length-biased sampling one can refer to Fisher (1934), Rao (1965), Neel and Schull (1966), Eberhardt (1968), Zelen (1971), Cook and Martin (1974), Patil and Rao (1977, 1978), Eberhardt (1978), Sankaran and Nair (1993d), Sen and Khattree (1996), Oluyede (1999, 2000), Van et al. (2000), Sunoj (2004), Bar-Lev and Schouten (2004), Kersey and Oluyede (2013) and Das and Kundu (2016).

When the weight is inversely proportional to length of unit of interest, we use $w(t) = \frac{1}{x}$, called inversed length-biased distribution (see Barmi and Simonoff (2000)). Barmi and Simonoff (2000) proposed a transformation-based technique for the density estimation of weighted distributions and used length-biased and inverse length-biased sampling for the study.

Some of the known and important distributions in statistics and applied probability can be expressed as weighted distributions. Equilibrium distributions, residual-life distributions, distribution of order statistics, proportional hazards models (see Gupta and Kirmani (1990), Bartoszewicz and Skolimowska (2004)) are some of the examples and are given in Table 2.1. Thus the theory of weighted distributions is appropriate whenever these distributions are applied. For more details on applications and recent works of weighted distributions, we refer to Gupta and Kirmani (1990), Jones (1991), Navarro et al. (2001), Sunoj and Maya (2006), Di Crescenzo and Longobardi (2006), Maya and Sunoj (2008), Navarro et al. (2014), Jarrahiferiz et al. (2016) and Sunoj and Vipin (2017).

2.8.1 Bivariate weighted distributions

The wide applicability of weighted distributions in the univariate case has motivated many researchers to extend the concept of weighted distribution to higher dimensions. Let $\boldsymbol{X} = (X_1, X_2)$ be a bivariate random vector in the support of $(a_1, b_1) \times (a_2, b_2)$, $b_i > a_i$, $i = 1, 2$ where (a_i, b_i) is an interval on the real line with absolutely continuous df $F(t_1, t_2)$, and pdf $f(t_1, t_2)$. By defining $w(t_1, t_2)$ as a non-negative weight function satisfying $E(w(X_1, X_2)) < \infty$, Mahfoud and Patil (1982) defined bivariate weighted distribution as the distribution of the random vector

Table 2.1: Special cases of weighted distributions

$w(t)$	**Distribution**	$f^w(t)$
$\frac{1}{h(t)}$	Equilibrium distribution	$\frac{R(t)}{E(X)}$
$[R(t)]^{\theta-1}, \theta > 0$	Proportional hazards model	$\theta\,[R(t)]^{\theta-1} f(t)$
$[F(t)]^{\theta-1}, \theta > 0$	Proportional reversed hazards model	$\theta\,[F(t)]^{\theta-1} f(t)$
$\frac{f(t+x)}{f(t)}$	Residual life distribution	$\frac{f(t+x)}{R(x)}$
$\frac{f(x-t)}{f(t)}, x > t$	Reversed residual life distribution	$\frac{f(x-t)}{F(x)}$
$[F(t)]^{j-1}[R(t)]^{n-j}$, $j = 1, 2, ..., n$	Distribution of j^{th} order statistics	$\frac{n!f(t)}{(j-1)!(n-j)!}[F(t)]^{j-1}[R(t)]^{n-j}$
$[-\log R(t)]^{n-1}$	Distribution of upper record value	$\frac{[-\log R(t)]^{n-1}}{(n-1)!} f(t)$
$[-\log F(t)]^{n-1}$	Distribution of lower record value	$\frac{[-\log F(t)]^{n-1}}{(n-1)!} f(t)$

$(X_1^w, X_2^w)'$ with pdf

$$f^w(t_1, t_2) = \frac{w(t_1, t_2)}{E(w(X_1, X_2))} f(t_1, t_2), \quad a_i < t_i < b_i, \ i = 1, 2. \tag{2.18}$$

For more properties of bivariate weighted distributions one can refer to Nair and Sunoj (2003), Sunoj and Sankaran (2005), Navarro et al. (2006), Arnold et al. (2016), Alavi (2017), Kayal and Sunoj (2017) and references therein.

Jain and Nanda (1995) extended the definition to the p - variate case. Let $\boldsymbol{X} = (X_1, X_2, ..., X_P)'$ be a p - dimensional non-negative random vector with pdf $f(\boldsymbol{x})$ and $\boldsymbol{X}^w = (X_1^w, X_2^w, ..., X_p^w)'$ be the multivariate weighted version of $\boldsymbol{X}$ such that the weight function $w(\boldsymbol{x})[w : \boldsymbol{X} \to A \subseteq \mathbb{R}^+$, where $\mathbb{R}^+$ denotes the positive real line] is non-negative with finite and nonzero expectation. Then the multivariate weighted density corresponding to $f(\boldsymbol{x})$ is given by

$$f^w(\boldsymbol{t}) \;=\; \frac{w(\boldsymbol{t})f(\boldsymbol{t})}{E(w(\boldsymbol{X}))}. \tag{2.19}$$

For more recent works, see Navarro et al. (2006), Kim (2008) and Kim (2010a,b).

2.9 Quantile function

Many of the probability models used in the literature may not have a tractable distribution function. In such cases, an alternative approach for modelling and analysis of statistical data is through the quantile function. Quantile function has many interesting properties that are not shared by the distribution functions. The quantile function $Q(u)$ of the rv X is defined as,

$$Q(u) = F^{-1}(u) = \inf\{t : F(t) \geq u\}, \tag{2.20}$$

for $-\infty < t < \infty$ and $0 \leq u \leq 1$. We assume that X is a non-negative random variable with absolutely continuous distribution function $F(.)$ and probability density function $f(.)$. When $F(.)$ is continuous, from (2.20) $FQ(u) = u$, where $FQ(u)$ represents the composite function $F(Q(u))$. We take $Q(0) = 0$ generally, and an adjustment has to be made in the results when $Q(0) > 0$. The mean of the distribution assumed to be finite, is

$$\mu = \int_0^1 Q\left(p\right) dp$$

which is same as

$$\int_0^1 \left(1-p\right) q\left(p\right) dp$$

where $q(u) = \frac{d}{du}(Q(u))$ is the quantile density function. When $F(t)$ is strictly increasing, $f(t) > 0$ so that the quantile density function exists by $f(Q(u))q(u) = 1$.

Using (2.20) Nair and Sankaran (2011) and Nair et al. (2013a,b) introduced quantile-based univariate lower and upper partial moments and studied its usefulness in the context of risk analysis, lifelength and income (poverty) studies.

2.10 Copula

Sklar (1959) introduced the notion of copula to study the relationship between a multidimensional probability function and its lower dimensional marginals. Sklar, who first used the word "copula"in the mathematical or statistical sense through his theorem. Copulas were initially used in the development of the theory of probabilistic metric spaces. Later, the concept was used to define non-parametric measures of dependence between random variables, and since then, it began to play an important role in probability and mathematical statistics.

A copula is a function which "couples"a multivariate distribution function to its one-dimensional marginal distribution functions. They provide a general method for binding several univariate marginal distributions together to form a multivariate distribution. Over the past forty years, copulas have played an important role in several areas of statistics. Copulas are considered to be highly appealing in the non-Gaussian setup as they can capture dependence more broadly than the standard multivariate normal framework. Following Clayton (1978), several families of single parameter copula models have been proposed for analyzing survival data. For more discussions, the reader is referred to the books by Joe (1997) and Nelsen (2007). Firstly, we introduce the formal definition of the two-dimensional copula function. The multi-dimensional definition is similar.

Consider a random vector $\boldsymbol{X} = (X_1, X_2)$. Suppose its marginals are continuous, i.e. $F_i(t_i) = Pr[X_i \leq t_i]$ for $i = 1, 2$, are continuous functions. By applying the probability integral transform to each component, $U_1 = F_1(X_1)$ and $U_2 = F_2(X_2)$ have uniform distributed marginals. The copula of (X_1, X_2) is defined as the joint cumulative distribution function of (U_1, U_2) namely:

$$C^*(u_1, u_2) = P\left[U_1 \leq u_1, U_2 \leq u_2\right]. \tag{2.21}$$

The copula C^* contains all the information on the dependence structure between the components of (X_1, X_2), whereas the marginal cumulative distribution functions $F_1(t_1)$ and $F_2(t_2)$ contain all the information on the marginal distributions.

The formula (2.21) can be rewritten using of the inverse functions $F_i^{-1}(t_i)$ for $i = 1, 2$ as

$$C^*(u_1, u_2) = P\left[X_1 \leq F_1^{-1}(u_1), X_2 \leq F_2^{-1}(u_2)\right].$$

A two-dimensional copula is a function $C^*(u_1, u_2) : [0, 1] \times [0, 1] \to [0, 1]$ with the following properties:

(i) $C^*(u_1, u_2)$ is grounded, i.e., for every (u_1, u_2) in $[0, 1] \times [0, 1] \to [0, 1]$, $C(u_1, u_2) = 0$ if at least one coordinate is 0.

(ii) $C^*(u_1, u_2)$ is two-increasing, i.e., for every a and b in $[0, 1]$ such that $a < b$, the C^*-volume $V_{C^*}([a, b])$ of the box $[a, b]$ is positive.

(iii) $C^*(u_1, 1) = u_1$ and $C^*(1, u_2) = u_2$ for every $(u_1, u_2) \in [0, 1] \times [0, 1]$.

Clearly, a copula is a function which assigns any point in the unit square $[0, 1] \times$

$[0, 1]$ to a number in the interval $[0, 1]$. From a probabilistic point of view, a copula function is a joint distribution whose marginal distributions are uniform. Next, we state the famous Sklar (1959) theorem which links the univariate marginals and the multivariate dependence structure.

Theorem 2.10.1 (Sklar). *Let $F(t_1, t_2)$ be a bivariate distribution function with marginals $F_1(t_1)$ and $F_2(t_2)$. Then, there exists a bivariate copula C^* such that for all (t_1, t_2) in $\mathbb{R}_2$*

$$F(t_1, t_2) = C^*(F_1(t_1), F_2(t_2)). \tag{2.22}$$

If $F_1(.)$ and $F_2(.)$ are both continuous, then copula C^* is uniquely defined. Conversely, if C^* is a bivariate copula and $F_1(.)$ and $F_2(.)$ are probability distribution functions, then the function $F(., .)$ defined by (2.22) is a bivariate distribution function with marginals $F_1(\cdot)$ and $F_2(\cdot)$.

2.10.1 Survival copula

Let $\boldsymbol{X} = (X_1, X_2)$ be a non-negative random vector with continuous survival function $R(t_1, t_2)$ and marginal survival functions $R_i(t_i) = P(X_i > t_i),\ i = 1, 2$ which are continuous and strictly decreasing. Then the survival copula $C(u, v)$ of $\boldsymbol{X}$ is a mapping $C(u, v) : [0, 1] \times [0, 1] \rightarrow [0, 1]$, defined by (Nelsen (2007))

$$C(u, v) = R\left(R_1^{-1}(u), R_2^{-1}(v)\right),$$

where R_1^{-1}, R_2^{-1} are the usual inverse of the marginal sf's R_1 and R_2 respectively. Alternatively, the joint survival function

$$R(t_1, t_2) = C(R_1(t_1), R_2(t_2)).$$

The survival copula satisfies the following properties

(i) $C(u,1) = u$, $C(1,v) = v$ and $C(u,0) = 0 = C(0,v)$.

(ii) For every u_1, u_2, v_1, v_2 in $[0,1]$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$,

$$C(u_1, v_1) + C(u_2, v_2) - C(u_1, v_2) - C(u_2, v_1) \geq 0.$$

The survival copula C couples the joint survival function to its univariate marginals analogous to a copula connects the joint distribution function to its marginals.

There exists a link between the survival copula C and the copula C^*, given by

$$C(u_1, u_2) = u_1 + u_2 - 1 + C^*(1 - u_1, 1 - u_2). \tag{2.23}$$

2.11 Risk measures

In the context of actuarial theory, a non-negative random variable X represents the random amount that an insurance company pay to a policyholder, in case of claim. The comparison of risks is generally carried out through measures such as the Value-at-Risk (VaR) and Expected Shortfall (ES). One can easily relate partial moments with these measures in the situation where the analyst considers both the right and left tail behavior of the distribution of the asset returns.

2.11.1 Value-at-Risk

Among the popular risk measures existing in the literature a well-known risk measure is the Value-at-Risk (VaR), defined as a quantile of the distribution of the random loss at level $\alpha \in (0,1)$ and it has become a benchmark in fields such as economics, insurance and finance. VaR at level α indicates the maximum possible

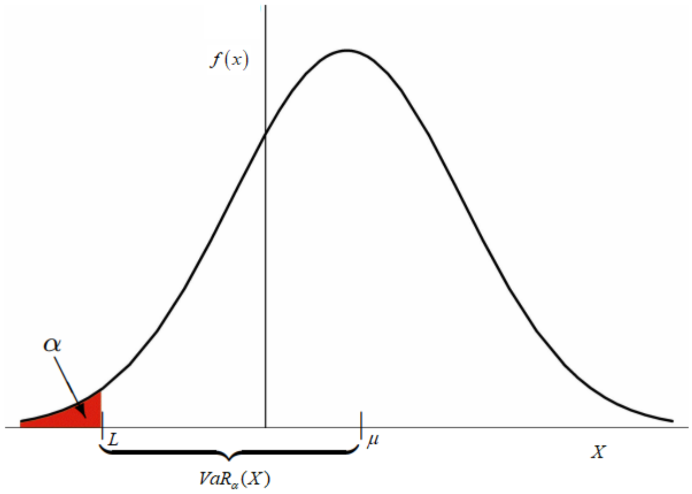

Figure 2.1: The Value-at-Risk as a risk measure

loss, when α percent of the left tail distribution is ignored.

Let X be a rv which denotes the return on the investment based on some arbitrary assets with a df F. The VaR of X at the confidence level $\alpha \in (0,1)$ denoted by $\mathrm{VaR}_\alpha(X)$ is the smallest number y such that the probability that $Y = -X$ does not exceed y is at least $1-\alpha$. Mathematically, $VaR_\alpha(X)$ is the $(1-\alpha)$-quantile of Y, i.e.,

$$\mathrm{VaR}_\alpha(X) = \inf\{x \in \mathbb{R} : F_X(x) > \alpha\} = F_Y^{-1}(1-\alpha).$$

In general,

$$\mathrm{VaR}_\alpha(X) = L - \mu,$$

where μ is the mean of the distribution and L is the value such that $P(X \leq L) = \alpha$. Thus, the risk is measured as the maximum deviation from the mean when the left tail of the distribution is ignored. This fact is graphically represented in Figure 2.1.

When the right tail of the returns are of interest, the $VaR_X(\alpha)$ is the larger risk for the $100\alpha\%$ of the situations and gives the maximum risk, for a fixed time horizon, for the $100\alpha\%$ of the cases. The main disadvantage of this function is that the VaR does not provide information about the thickness of the upper tail of the distribution, thus some other measures have been considered for this purpose (see Belzunce et al. (2012b), Belzunce and Martínez-Riquelme (2017)).

2.11.2 Expected shortfall

A measure of risk is coherent if it simultaneously satisfies the following properties: sub-additivity, monotonicity, positive homogeneity, and translation invariance (see Artzner et al. (1999) and Delbaen (2002)). As described in Acerbi and Tasche (2002), sub-additivity may be violated by the VaR.

Although popular in financial applications, because it gives a lower bound on the loss made in the worst percent of the cases during a prespecified period, the VaR is therefore not a coherent measure of risk. In addition, the practical usefulness of the VaR is limited by the fact that it tells us nothing about the potential size of the loss in the worst-case scenario. Even before the introduction of the VaR, the expected value of the left tail of the returns on a risky asset has been proposed as an alternative measure of risk. This quantity, variously known as the expected shortfall or the tail conditional expectation or the tail conditional mean, measures the loss that one may expect to make in the worst percent of the cases.

For a non-negative continuous rv with an absolute continuous distribution function F, and a corresponding quantile function $Q(\alpha)$ as defined in (2.20), the

α-level expected shortfall of X with $0 < \alpha < 1$ is defined by

$$\tau(\alpha) = E(X|X \leq Q(\alpha)) = \frac{1}{\alpha} \int_{-\infty}^{Q(\alpha)} u \, dF(u). \tag{2.24}$$

The measure (2.24) is of left tail interest. The concept can be similarly defined for the right tail events interest as the right-spread function defined by, $\tau_X^*(\alpha) = p_1(F^{-1}(\alpha)) = E\left[(X - F^{-1}(\alpha))_+\right] = \int_{F^{-1}(\alpha)}^{\infty} R(u) \, du.$

Given a random variable X, with distribution function $F(.)$, the stop-loss function is defined by the first order UPM as $p_1(F^{-1}(\alpha)) = E\left[(X - F^{-1}(\alpha))_+\right] = \int_{F^{-1}(\alpha)}^{\infty} R(u) \, du$. It has greater importance in actuarial studies. If the random variable X denotes the random risk for an insurance company, it is very common that the company passes a part of it to a reinsurance company. In particular, the first company bears the whole risk as long as it is less than a fixed value t (called retention), and if $X > t$ the reinsurance company will take over the amount $X - t$. The expected cost for the reinsurance company, $E[(X - t)_+]$, is called the net premium.

As the VaR at a fixed level only gives local information about the underlying distribution, a method to escape from this shortcoming is to consider the expected shortfall over some quantile. Expected shortfall at probability level α is the stop-loss premium with retention $VaR_X(\alpha)$. Specifically,

$$\tau_X^*(\alpha) = E[(X - VaR_X(\alpha))_+] = p_1(VaR_X(\alpha)).$$

When the right tail of the events are of interest, a dual measure corresponding to (2.24) is known by the name Conditional Tail Expectation (CTE). Denuit et al. (2006) defines the CTE as the conditional expected loss given that the loss exceeds its VaR:

$$CTE_X(\alpha) = E\left[X|X > VaR_X(\alpha)\right] \tag{2.25}$$

(2.25) is a quantitative measure for the 'average loss in the worst $100(1-\alpha)\%$ cases'.

2.12 Income and poverty measures

The partial moments LPM and UPM have a meaningful interpretation in comparing income inequality and deprivation of a population. Although poverty is studied mostly using income distributions there is equal interest in knowing the level of affluence in a population. Since the affluence indices and upper partial moment are mainly focusing on the upper part of the distribution, UPM's can be considered as a useful tool for finding the affluence level of a population. For instance, in univariate case Sen (1988), Belzunce et al. (1998) have developed the methodology to analyze the the inequality incomes among the rich individuals and proposed indices for their measurement.

2.12.1 Income-gap ratio

A concept that has applications in economics is income-gap ratio which is used for developing indices of affluence and poverty (Sen (1988)). For a non-negative random variable X, Sen (1988) has defined a measure of income-gap ratio namely

$$\beta(t) = 1 - \frac{t}{E(X|X > t)}, \tag{2.26}$$

where the point t is the level above which the population is considered to be affluent. Clearly, (2.26) is a measure defined in the upper part of the distribution and therefore useful in identifying the affluence of a population.

LPMs are considered as a useful tool in poverty and income studies as both LPMs and poverty indices are focusing on the lower part of the distribution (see Sunoj and Maya (2008)). In income studies, if the objective is of interest is to study about the poverty or deprivation of a population a dual measure of (2.26) can be considered. The analogue of (2.26) in left-tail a useful index to measure the level of poverty is the income-gap ratio, given by $\beta^*(t) = t - X$.

Abdul-Sathar et al. (2007) extended (2.26) into bivariate setup. The income-gap ratio for the truncated random variable $X_i|X_j > t_j; i, j = 1, 2; i \neq j$ is defined as (Abdul-Sathar et al. (2007))

$$\beta_i\left(t_i, t_j\right) = 1 - \frac{t_i}{v_i\left(t_i, t_j\right)}, \tag{2.27}$$

where $v_i\left(t_1, t_2\right) = E(X_i|X_1 > t_1, X_2 > t_2)$ is the i^{th} component of the bivariate vitality function defined by Sankaran and Nair (1991).

2.12.2 Mean left proportional residual income

Associated with income-gap ratio another measure useful in income studies is the mean left proportional residual income (MLPRI) due to Belzunce et al. (1998). For the income distribution left truncated at t, the left proportional residual income (LPRI) is the ratio $\frac{(X|X>t)}{t}$. the corresponding MLPRI is defined by $\gamma(t) = \frac{E(X|X>t)}{t}$. Recently, Sankaran et al. (2015) extended MLPRI to the bivariate case and studied its properties, as a vector

$$\begin{aligned}
(\gamma_1(t_1,t_2),\gamma_2(t_1,t_2)) &= \left(E\left(\frac{X_1}{t_1}|X_1>t_1,X_2>t_2\right),E\left(\frac{X_2}{t_2}|X_1>t_1,X_2>t_2\right)\right)\\
&= \left(\frac{v_1(t_1,t_2)}{t_1},\frac{v_2(t_1,t_2)}{t_2}\right). \qquad (2.28)
\end{aligned}$$

Chapter 3

Some properties of bivariate upper partial moments*

3.1 Introduction

The bivariate extension of UPM is due to Hürlimann (2002) and Sankaran and Nair (2004). For $r, s = 0, 1, 2, \ldots, (r, s)^{th}$ order BUPM of a random vector (X_1, X_2) is defined as (Hürlimann (2002))

$$p_{r,s}(t_1, t_2) = E\left[(X_1 - t_1)_+^r (X_2 - t_2)_+^s\right], t_1, t_2 \in \mathbb{R}.$$

The contents of the present chapter evolves as an extension of the work done by Sankaran and Nair (2004) with the aim of studying and formulating new characteristic properties of higher-order BUPMs (bivaraite stop-loss transforms).

The organization of the present chapter as follows. In section 3.2 we study the relationships between the BUPMs and some important reliability measures.

*Contents of this Chapter have been accepted as entitled "On characterizations of some bivariate continuous distributions by properties of higher-degree bivariate stop-loss transforms", *Communications in Statistics–Theory and Methods* (see Nair et al. (2018b)).

Section 3.3 studies about new characterization results using BUPMs. A new bivariate distribution is proposed by extending the characterizing identity of univariate UPM due to Lin (2003) to the bivariate case. Characterization theorems that extend the result of Abraham et al. (2007) are also established to identify the bivariate Pareto law. Finally, to illustrate the theoretical results established in the present chapter a real data analysis has been carried out in Section 3.4.

3.2 Some properties of BUPMs in the context of life-length studies

The important properties of BUPMs listed by Sankaran and Nair (2004) are :

(i) $p_{0,0}\left(t_1, t_2\right) = R\left(t_1, t_2\right)$

(ii) $p_{r,s}\left(t_1, t_2\right) = r \int\limits_{t_1}^{\infty} p_{r-1,s}\left(x_1, t_2\right) dx_1 = s \int\limits_{t_2}^{\infty} p_{r,s-1}\left(t_1, x_2\right) dx_2$

(iii) $R\left(t_1, t_2\right) = \frac{(-1)^{r+s}}{r!s!} \frac{\partial^{+s} p_{r,s}(t_1,t_2)}{\partial t_1^r \partial t_2^s}, r+s>1$

(iv) $p_{r,s}(t_1, t_2)$ is decreasing in $t_i, i = 1, 2.$ and for fixed t_1 and t_2, $p_{r,s}(t_1, t_2)$ is increasing in $r(s)$.

Property (iii) shows that the partial moment $p_{r,s}$ determines the distribution of (X_1, X_2) for any pair $(r, s), r, s = 0, 1, 2, \ldots$ (A more general definition is obtained if r, s are positive reals, but then property (iii) does not hold) and property (ii) are recurrence relations.

It is to be noted that the survival function of the bivariate random variable $\boldsymbol{X}$ can be obtained from a $(r, s)^{th}$ order BUPM by the uniqueness theorem given in property (iii). Moreover, the survival function can also obtained from a $(r, s)^{th}$

BUPM by $p_{0,0}(t_1,t_2) = -\frac{\partial p_{1,0}(t_1,t_2)}{\partial t_1} = -\frac{\partial p_{0,1}(t_1,t_2)}{\partial t_2}$. The concepts and the results in reliability analysis can be restated in terms of partial moments. This points out to the usefulness of partial moments in reliability modelling. In the sequel we present some important reliability measures using bivariate partial moments in Table 3.2.

Apart from obtaining different reliability measures in terms of partial moments it would also worthwhile to study the relationship between partial moments and various ageing concepts useful in life length studies. Hence we establish some important ageing concepts in the bivariate case in terms of BUPMs, given in Table 3.1.

Table 3.1: Definitions of bivariate aging classes in terms of BUPMs

Sl. No	Bivariate Ageing Class	Definition
1	IBHR-1	$\frac{p_{0,0}(t_1+s,t_2+s)}{p_{0,0}(t_1,t_2)}$ is decreasing in t_1, t_2 for all $s>0$
2	IBHR-2	$\frac{p_{0,0}(t_1+s_1,t_2+s_2)}{p_{0,0}(t_1,t_2)}$ is decreasing in t_1, t_2 for all s_1, $s_2>0$
3	DBMRL	$\frac{p_{1,0}(t_1,t_2)}{p_{0,0}(t_1,t_2)}$ is decreasing in t_1 for all t_2 and $\frac{p_{0,1}(t_1,t_2)}{p_{0,0}(t_1,t_2)}$ is decreasing in t_2 for all t_1
4	BDMRL-I	$\frac{p_{1,1}(t,t)}{p_{0,0}(t,t)}$ is decreasing in t for all $t \geq 0$
5	BDMRL-II	$\frac{\int_t^\infty p_{1,1}(x,x)dx}{p_{0,0}(t,t)}$ is decreasing in t for all $t \geq 0$
6	BIFR	$\frac{p_{0,0}(t_1+s,t_2)}{p_{0,0}(t_2+s,t_1)}$ is decreasing in s, for all $t_1 < t_2$
7	BDMRL	$\frac{p_{1,0}(t_1,t_2)}{p_{0,0}(t_1,t_2)} < \frac{p_{0,1}(t_1,t_2)}{p_{0,0}(t_1,t_2)}$, for all $t_1 < t_2$

Table 3.2: Relationship connecting bivariate reliability measures and bivariate upper partial moments

Sl. No	Measure	Partial moment representation
1	Failure rate (Johnson and Kotz (1975))	$\left(\frac{\partial}{\partial t_1}\left(\frac{\partial \log p_{1,0}(t_1,t_2)}{\partial t_1}\right), \frac{\partial}{\partial t_2}\left(\frac{\partial \log p_{0,1}(t_1,t_2)}{\partial t_2}\right)\right)$
2	Mean residual life (Arnold and Zahedi (1988))	$\left(\frac{p_{1,0}(t_1,t_2)}{p_{0,0}(t_1,t_2)}, \frac{p_{0,1}(t_1,t_2)}{p_{0,0}(t_1,t_2)}\right)$
3	Variance residual life (Sankaran and Nair (1993b))	$\left(\left(\frac{p_{2,0}(t_1,t_2)}{p_{0,0}(t_1,t_2)} - \left(\frac{p_{1,0}(t_1,t_2)}{p_{0,0}(t_1,t_2)}\right)^2\right), \left(\frac{p_{0,2}(t_1,t_2)}{p_{0,0}(t_1,t_2)} - \left(\frac{p_{0,1}(t_1,t_2)}{p_{0,0}(t_1,t_2)}\right)^2\right)\right)$
4	Vitality function (Sankaran and Nair (1991))	$\left(t_1 + \frac{p_{1,0}(t_1,t_2)}{p_{0,0}(t_1,t_2)}, t_2 + \frac{p_{0,1}(t_1,t_2)}{p_{0,0}(t_1,t_2)}\right)$
5	Product moment residual life (Nair et al. (2004))	$\frac{p_{1,1}(t_1,t_2)}{p_{0,0}(t_1,t_2)}$
6	Covariance residual life (Nair et al. (2004))	$\frac{(p_{1,1}(t_1,t_2) - p_{1,0}(t_1,t_2)p_{0,1}(t_1,t_2))}{p_{0,0}(t_1,t_2)}$
7	Residual coefficient of variation (Gupta and Kirmani (2000))	$\left(\frac{\left(\frac{p_{2,0}(t_1,t_2)}{p_{0,0}(t_1,t_2)} - \left(\frac{p_{1,0}(t_1,t_2)}{p_{0,0}(t_1,t_2)}\right)^2\right)^{\frac{1}{2}}}{\frac{p_{1,0}(t_1,t_2)}{p_{0,0}(t_1,t_2)}}, \frac{\left(\frac{p_{0,2}(t_1,t_2)}{p_{0,0}(t_1,t_2)} - \left(\frac{p_{0,1}(t_1,t_2)}{p_{0,0}(t_1,t_2)}\right)^2\right)^{\frac{1}{2}}}{\frac{p_{0,1}(t_1,t_2)}{p_{0,0}(t_1,t_2)}}\right)$

3.3 Characterization results

Chong (1977) characterized the exponential and geometric distributions by the property

$$E(X-t-s)_+ E\,(X) = E(X-t)_+ E(X-s)_+ \tag{3.1}$$

for all $t, s > 0$. Associated with a lifetime rv X and about a point t, Lin (2003) pointed out that three are three kinds of residual life those are

(i) $(X-t)_+$ as given in Stoyan and Daley (1983)

(ii) $X-t|X>t$ (Hall and Wellner (1981))

(iii) excess life in renewal theory (see Nair and Sankaran (2010b)).

Denoting by $h(x) = E(X-x)_+$, Lin (2003) modified Chong's Theorem as

$$h(x)h(y) = \alpha h(x+y)$$

for all $x, y > 0$ and $\alpha > 0$ a constant if and only if

$$F(x) = 1 - b\ exp\left(-\frac{b\,x}{\alpha}\right), x \geq 0, b = R\,(0)\,. \tag{3.2}$$

We extend this and some other theorems in his paper in the Theorems 3.3.1 through 3.3.4.

Theorem 3.3.1. *Let* (X_1, X_2) *be a non-negative random vector with survival function* $R\,(x_1, x_2)$ *such that* $E\,(X_i) < \infty, i = 1, 2$. *Then the property*

$$\begin{aligned} &E\left[(X_i - t_i)_+ | X_{3-i} > t_{3-i}\right] E\left[(X_i - s_i)_+ | X_{3-i} > t_{3-i}\right] \\ &\qquad = E\left[X_i | X_{3-i} > t_{3-i}\right] E\left[(X_i - t_i - s_i)_+ | X_{3-i} > t_{3-i}\right] \end{aligned} \tag{3.3}$$

holds for all $t_i, s_i \geq 0,\ i = 1, 2$ *if and only if*

$$R(x_1, x_2) = b_1 b_2\ exp\left[-\frac{b_1\ x_1}{\alpha_1} - \frac{b_2\ x_2}{\alpha_2} - \theta x_1 x_2\right],\ x_1, x_2 \geq 0,$$
$$0 < b_i < 1, \alpha_i > 0, 0 \leq \theta \leq \frac{b_1 b_2}{\alpha_1 \alpha_2}. \tag{3.4}$$

Proof. 'if part'. When the distribution is (3.4), the conditional survival function of X_1 given $X_2 > t_2$ is

$$R_1\left(x_1 | X_2 > t_2\right) = b_1\ exp\left[-\left(\frac{b_1}{\alpha_1} + \theta t_2\right) x_1\right]$$

and hence

$$E\left[\left(X_1 - t_1\right)_+ | X_2 > t_2\right] = \frac{b_1}{b_1 \alpha_1^{-1} + \theta t_2} \exp\left[-\left(\frac{b_1}{\alpha_1} + \theta t_2\right) t_1\right]. \tag{3.5}$$

Thus all the expectations in (3.3), have the similar values as in (3.5). Substitution of these values in (3.3) proves (3.3) for $i = 1$ and a similar proof holds when $i = 2$.

'only if part'. For this we assume (3.3) for $i = 1$ and write it as

$$\int_{t_1}^{\infty} R_1\left(x_1 | X_2 > t_2\right) dx_1 \int_{s_1}^{\infty} R_1\left(x_1 | X_2 > t_2\right) dx_1$$
$$= \int_{0}^{\infty} R_1\left(x_1 | X_2 > t_2\right) dx_1 \int_{t_1 + s_1}^{\infty} R_1\left(x_1 | X_2 > t_2\right) dx_1.$$

Denoting $\alpha_1\left(t_2\right) = \int_{0}^{\infty} R_1\left(x_1 | X_2 > t_2\right) dx_1$, the last equation becomes

$$\bar{G}\left(t_1, t_2\right) \bar{G}\left(s_1, t_2\right) = \bar{G}\left(t_1 + s_1, t_2\right), \tag{3.6}$$

where

$$\bar{G}(t_1,t_2)=\frac{\int\limits_{t_1}^{\infty}R_1(x_1|X_2>t_2)\,dx_1}{\alpha_1(t_2)}. \tag{3.7}$$

We observe that (3.6) is a Cauchy functional equation for a given t_2, whose unique continuous solution is $\bar{G}(t_1,t_2)=\exp[-a_1(t_2)\,t_1]\,,a_1(t_2)>0$. This gives

$$R_1(t_1|X_2>t_2)=b_1(t_2)\exp\left[-\frac{b_1(t_2)}{\alpha_1(t_2)}t_1\right], \tag{3.8}$$

$b_1(t_2)=\alpha_1(t_2)a_1(t_2)$. In the same manner taking $i=2$ in (3.3) we can write the survival function of X_2 given $X_1>t_1$ as

$$R_2(t_2|X_1>t_1)=b_2(t_1)\exp\left[-\frac{b_2(t_1)}{\alpha_2(t_1)}t_2\right] \tag{3.9}$$

with $\alpha_2(t_1)=E(X_2|X_1>t_1)$. Setting $t_1=0$

$$R_2(t_2)=b_2\exp\left[-\frac{b_2}{\alpha_2}t_2\right],b_2=b_2(0)\,,\alpha_2=\alpha_2(0)\,. \tag{3.10}$$

Combining (3.8) and (3.10),

$$R(t_1,t_2)=b_2\,b_1(t_2)\exp\left[-\frac{b_1(t_2)}{\alpha_1(t_2)}t_1-\frac{b_2}{\alpha_2}t_2\right]. \tag{3.11}$$

Similarly

$$R(t_1,t_2)=b_1\,b_2(t_1)\exp\left[-\frac{b_2(t_1)}{\alpha_2(t_1)}t_2-\frac{b_1}{\alpha_1}t_1\right],b_1=b_1(0)\,,\alpha_1=\alpha_1(0)\,.$$

Equating the expressions for $R(t_1,t_2)$ leads to the functional equation

$$b_2b_1(t_2)\exp\left[-\frac{b_1(t_2)}{\alpha_1(t_2)}t_1-\frac{b_2}{\alpha_2}t_2\right]=b_1\,b_2(t_1)\exp\left[-\frac{b_2(t_1)}{\alpha_2(t_1)}t_2-\frac{b_1}{\alpha_1}t_1\right].$$

or equivalently

$$t_1\left(\frac{b_1}{\alpha_1}-\frac{b_1(t_2)}{\alpha_1(t_2)}\right)+\log b_2+\log b_1(t_2)=t_2\left(\frac{b_2}{\alpha_2}-\frac{b_2(t_1)}{\alpha_2(t_1)}\right)+\log b_1+\log b_2(t_1). \quad (3.12)$$

Since the left (right) side is linear in $t_1(t_2)$ the right (left) also must be linear in $t_1(t_2)$. This gives for some constants A_i, B_i, C_i and $D_i, i=1,2$,

$$\log b_1(t_2)=A_1+A_2t_2, \quad \frac{b_1}{\alpha_1}-\frac{b_1(t_2)}{\alpha_1(t_2)}=B_1+B_2t_2$$

$$\log b_2(t_1)=C_1+C_2t_1 \text{ and } \frac{b_2}{\alpha_2}-\frac{b_2(t_1)}{\alpha_2(t_1)}=D_1+D_2t_1.$$

Substituting in the identity (3.12) and equating like terms

$$A_2=D_1,\ B_1=C_2 \text{ and } B_2=D_2.$$

As $t_2\to 0$ in $\frac{b_1}{\alpha_1}-\frac{b_1(t_2)}{\alpha_1(t_2)}$, $B_1=0$ and hence $C_2=0$. Thus $\log b_2(t_1)$ is a constant for all t_1 which gives $b_2(t_1)=b_2$. Similarly $b_1(t_2)=b_1$. With these solutions,

$$\exp\left[-\frac{b_1}{\alpha_1(t_2)}t_1-\frac{b_2}{\alpha_2}t_2\right]=\exp\left[-\frac{b_2}{\alpha_2(t_1)}t_2-\frac{b_1}{\alpha_1}t_1\right]$$

and

$$\left(\frac{b_2}{\alpha_2(t_1)}-\frac{b_2}{\alpha_2}\right)\frac{1}{t_1}=\left(\frac{b_1}{\alpha_1(t_2)}-\frac{b_1}{\alpha_1}\right)\frac{1}{t_2}.$$

For the last equation to hold for all t_1, t_2 either side must be a constant, say θ. Thus

$$\alpha_2(t_1)=\frac{\alpha_2 b_2}{b_2+\alpha_2\theta t_1}$$

and

$$\alpha_1(t_2)=\frac{\alpha_1 b_1}{b_1+\alpha_1\theta t_2}.$$

Substituting in (3.11) we have (3.4) to complete the proof. (The distribution (3.4) has probability mass of $1 - b_i$ at $X_i = 0$ and therefore it is distinct from the Gumbel's form. Also $R(0,0) = b_1 b_2$, the marginal distributions are not exponential.) □

Remark 3.3.1. Stoyan and Daley (1983) points out that $(X - x)_+$ can be treated as the residual life of a device of life length X. With this interpretation, the importance of the theorem is that the given residual lives at two smaller ages $t_i, s_i > 0$, we can compute the future residual lives at $mt_i + ns_i$ for all positive integers without observing the latter.

Remark 3.3.2. When X_1 and X_2 are positive random variables we have a characterization of the Gumbel's bivariate exponential distribution

$$R(t_1, t_2) = exp\left[-\frac{t_1}{\alpha_1} - \frac{t_2}{\alpha_2} - \theta\frac{t_1}{\alpha_1}\frac{t_2}{\alpha_2}\right], t_1, t_2 > 0. \tag{3.13}$$

Remark 3.3.3. Setting $t_2 = 0$, Theorem 1 in Lin (2003) follows for the random variable X_1. As a further special case assume $X_1 > 0$ to obtain the characterization of the univariate exponential.

Remark 3.3.4. When X_1 and X_2 are two risks associated with an individual or a company, the dependence between them plays an important role in risk analysis. We say that X_1 and X_2 are positively stop-loss dependent if

$$E\left[(X_i - t_i)_+ | X_{3-i} > t_{3-i}\right] \geq E(X_i - t_i)_+, i = 1, 2, t_1, t_2 \geq 0$$

and negatively dependent if the inequality is reversed. Thus the expectations considered in (3.3) are relevant in this context. Note that the Gumbel distribution in (3.13) is negatively stop-loss dependent.

Theorem 3.3.1 allows an extension to the multivariate case in which $\boldsymbol{X} = (X_1, X_2, ..., X_n)$, $\boldsymbol{x} = (x_1, x_2, ..., x_n)$, $\boldsymbol{t} = t_1, t_2, ..., t_n$, $\boldsymbol{X}^{(i)} = (X_1, ..., X_{i-1}, X_{i+1}, ..., X_n)$ and similarly $\boldsymbol{x}^{(i)}$ and $\boldsymbol{t}^{(i)}$. The distribution has survival function

$$R(\boldsymbol{x}) = b_1 b_2 ... b_n exp\left[-\sum_{i=1}^{n} \frac{b_i}{\alpha_i} x_i - \sum_{i<j=1}^{n} \frac{b_{ij}}{\alpha_{ij}} x_i x_j - ... \frac{b_{12...n}}{\alpha_{12...n}} x_1 x_2 ... x_n\right], \ \boldsymbol{x} \geq 0,$$

all b's lying in $(0,1)$, α's positive. The characteristic property is

$$\begin{aligned} &E\left[(X_i - t_i)_+ | \boldsymbol{X}^{(i)} > \boldsymbol{t}^{(i)}\right] E\left[(X_i - s_i)_+ | \boldsymbol{X}^{(i)} > \boldsymbol{t}^{(i)}\right] \\ &\qquad = E\left(X_i | \boldsymbol{X}^{(i)} > t^{(i)}\right) E\left[(X_i - t_i - s_i)_+ | \boldsymbol{X}^{(i)} > \boldsymbol{t}^{(i)}\right] \end{aligned}$$

for $i = 1, 2, ..., n$, and the vector inequalities are understood component-wise. We prove the result by induction starting with $n = 2$ established in Theorem 3.3.1. For $n \geq 3$, we have $(n-1)$ equations instead of (3.12) and all of them are solved by same type of arguments. Similar extensions can be given to results in Theorem 3.3.2 through 3.3.5.

Theorem 3.3.2. *Under the conditions on (X_1, X_2) in Theorem 3.3.1, (X_1, X_2) follows distribution* (3.4) *if and only if*

$$E^r\left[(X_i - t_i)_+ | X_{3-i} > t_{3-i}\right] = \alpha_i^{r-1}(t_{3-i}) E\left[(X_i - rt_i)_+ | X_{3-i} > t_{3-i}\right] \qquad (3.14)$$

holds for all $t_i > 0, \ i = 1, 2$ and any $r > 1$.

Proof. 'if part'. Denote by

$$A_1(t_1, t_2) = \frac{E\left[(X_1 - t_1)_+ | X_2 > t_2\right]}{\alpha_1(t_2)} = \frac{\int_{t_1}^{\infty} R_1(x_1 | X_2 > t_2) dx_1}{\alpha_1(t_2)},$$

so that $A_1(0,t_2) = 1$ and $\dfrac{dA_1'}{dt_1} = -\dfrac{b_1}{\alpha_1(t_2)}$, treating t_2 as a constant and $b_1 = F_1(0)$. By MacLaurin expansion

$$A_1(t_1,t_2) = 1 - \frac{b\,t_1}{\alpha_1(t_2)} + o\,(t_1)$$

and also from (3.14)

$$A_1(t_1,t_2)^r = A_1(rt_1,t_2).$$

Thus

$$\begin{aligned} A_1(t_1,t_2) &= A_1\left(\frac{t_1}{r},t_2\right)^r = A_1\left(\frac{t_1}{r^k},t_2\right)^{r^k}, r > 1 \\ &= 1 - \frac{b_1 t_1}{\alpha_1(t_2) r^k} + o\left(\frac{t_1}{r^k}\right) \end{aligned}$$

which tends to $\exp\left[\frac{b_1 t_1}{\alpha_1(t_2)}\right]$ as $k \to \infty$. Thus

$$\int_{t_1}^{\infty} R_1\,(x_1|X_2 > t_2) dx_1 = \alpha_1(t_2) \exp\left[-\frac{b_1 t_1}{\alpha_1(t_2)}\right]$$

giving

$$R_1\,(t_1|X_2 > t_2) = b_1 \exp\left[-\frac{b_1}{\alpha_1(t_2)}\right].$$

Similarly

$$R_2\,(t_2|X_1 > t_1) = b_2 \exp\left[-\frac{b_2}{\alpha_2(t_1)}\right], b_2 = R_2(0).$$

The rest of the proof is the same as in Theorem 3.3.1 and accordingly (X_1, X_2) has distribution (3.4).

The 'only if part', follows from the expressions

$$E^r\left[(X_i - t_i)_+ | X_{3-i} > t_{3-i}\right] = \left(\frac{b_i}{b_i\alpha_i^{-1} + \theta t_{3-i}}\right)^r \exp\left[-rt_i\left(\frac{b_i}{\alpha_i} + \theta t_{3-i}\right)\right]$$

$$E\left[(X_i - t_i r)_+ | X_{3-i} > t_{3-i}\right] = \left(\frac{b_i}{b_i\alpha_i^{-1} + \theta t_{3-i}}\right) \exp\left[-rt_i\left(\frac{b_i}{\alpha_i} + \theta t_{3-i}\right)\right]$$

and

$$\alpha_i\left(t_{3-i}\right) = \left(\frac{b_i}{b_i\alpha_i^{-1} + \theta t_{3-i}}\right).$$

□

Remark 3.3.5. Remark 3.3.2 hold for this case also. The residual life at a higher age can be written in terms of those of a lower one using this theorem also, but in a different manner compared to Theorem 3.3.1.

Remark 3.3.6. The advantage of Theorem 3.3.2 over Theorem 3.3.1 is that one needs only t_1 and t_2 in the former where as the latter needs t_1, t_2, s_1 and s_2 to verify the characteristic property.

Remark 3.3.7. The Theorem extends to the multivariate case to characterize $R(\boldsymbol{x})$ given above by the property

$$E^r\left[(X_i - t_i)_+ | \boldsymbol{X}^{(i)} > \boldsymbol{t}^{(i)}\right] = \alpha_i^{r-1}\left(\boldsymbol{t}^{(i)}\right) E\left[(X_i - rt_i)_+ | \boldsymbol{X}^{(i)} > \boldsymbol{t}^{(i)}\right].$$

Theorem 3.3.3. *Let* (X_1, X_2) *be a non-negative random variable with* $E(X_i) < \infty, i = 1, 2$. *Then*

$$E\left[(X_i - t_i)_+ | X_{3-i} > t_{3-i}\right] = \alpha_i\left(t_{3-i}\right) R_i\left(t_i | X_{3-i} > t_{3-i}\right) \tag{3.15}$$

holds for all $t_i > 0,\ i = 1, 2$ *and* $\alpha_i(t_{3-i}) > 0$ *if and only if* (X_1, X_2) *has distribution* (3.4).

Proof. When the distribution is (3.4), we see from (3.5) that (3.15) holds for $i = 1$. The proof is similar for $i = 2$. This proves 'if' part.

Conversely, let (3.15) be true for $i = 1$. Since $X_1 \geq 0$, we have $0 \leq E(X_1) < \infty$ and

$$E\left(X_1\right) = \alpha_1\left(t_2\right) R_1\left(0|X_2 > t_2\right).$$

This means that if $E(X_1) = 0$, the probability of X_1 is concentrated at $X_1 = 0$ and hence (3.15) is trivially satisfied for $i = 1$. The case of $i = 2$ is similar. When $E(X_1) > 0$,

$$\int_{t_1}^{\infty} R_1\left(x_1|X_2 > t_2\right) dx_1 = \alpha_1(t_2) R_1\left(t_1|X_2 > t_2\right).$$

This is equivalent to

$$\frac{d}{dt_1} \log \int_{t_1}^{\infty} R_1\left(x_1|X_2 > t_2\right) dx_1 = -[\alpha_1(t_2)]^{-1}$$

which solves into

$$R_1\left(t_1|X_2 > t_2\right) = \alpha_1(t_2) \exp\left[-\frac{b_1(t_2)}{\alpha_1(t_2)} t_1\right],$$

the expression in (3.8). For $i = 2$, we have also (3.9). The rest of the proof follows from the steps in Theorem 3.3.1. □

We generalize Theorem 3.3.2 to higher order moments.

Theorem 3.3.4. *Let* (X_1, X_2) *be a non-negative random vector with* $E(X_i^n) < \infty, i =$

$1, 2$. *Then the property*

$$E^r\left[(X_i - t_i)_+^n \,|X_{3-i} > t_{3-i}\right] = E[X_i^n|X_{3-i} > t_{3-i}]^{r-1} E\left[(X_i - t_i r)_+^n \,|X_{3-i} > t_{3-i}\right] \tag{3.16}$$

for any $r > 1$ *and integer* $n \geq 2$ *and* $t_i \geq 0,\ i = 1, 2$ *if and only if* (X_1, X_2) *has survival function*

$$R(x_1, x_2) = \frac{b_1^n b_2^n}{(n!)^2 (\alpha_1\alpha_2)^{n-1}}\ exp\left[-\frac{b_1\ x_1}{\alpha_1} - \frac{b_2\ x_2}{\alpha_2} - \theta x_1 x_2\right],\ x_1, x_2 \geq 0,$$
$$b_i, \alpha_i > 0, 0 \leq \theta \leq \frac{b_1 b_2}{\alpha_1\alpha_2}, \tag{3.17}$$

provided $0 < \frac{b_1^n b_2^n}{(n!)^2(\alpha_1\alpha_2)^{n-1}} < 1$.

Proof. When the distribution is (3.17),

$$E^r\left[(X_i - t_i)_+^n \,|X_{3-i} > t_{3-i}\right] = \frac{e^{-\left(\frac{b_i}{\alpha_i} + \theta t_{3-i}\right)tr}}{\left(\frac{b_i}{\alpha_i} + \theta t_{3-i}\right)^{rn}} (n!)^r$$

$$E^{r-1}\left[X_i|X_{3-i} > t_{3-i}\right] = \frac{(n!)^{r-1}}{\left(\frac{b_i}{\alpha_i} + \theta t_{3-i}\right)^{n(r-1)}}$$

and

$$E\left[(X_i - t_i r)_+|X_{3-i} > t_{3-i}\right] = \frac{n!}{\left(\frac{b_i}{\alpha_i} + \theta t_{3-i}\right)^n} e^{-\left(\frac{b_i}{\alpha_i} + \theta t_{3-i}\right)tr}.$$

This proves (3.16). To prove the sufficiency part, for $i = 1, 2$, we first note that

$$\begin{aligned} E\left[(X_i - t_i)_+^n \,|X_{3-i} > t_{3-i}\right] &= \int_{t_i}^{\infty} (x_i - t_i)^n f_i\left(x_i|X_{3-i} > t_{3-i}\right) dx_i \\ &= n\int_{t_i}^{\infty} (x_i - t_i)^{n-1} R_i\left(x_i|X_{3-i} > t_{3-i}\right) dx_i. \end{aligned} \tag{3.18}$$

Consider $A\left(t_1, t_2\right)=\frac{E\left[\left(X_1-t_1\right)_{+}^n \mid X_2>t_2\right]}{E\left[X_1^n \mid X_2>t_2\right]}$, where t_2 is fixed. Then,

$$\begin{aligned}
A^r\left(t_1, t_2\right) &= \frac{E\left[\left(X_1-t_1\right)_{+}^n \left|X_2>t_2\right]^r\right.}{E^r\left[X_1^n \mid X_2>t_2\right]}=\frac{E\left[\left(X_1-t_1 r\right)_{+}^n \left|X_2>t_2\right]\right.}{E\left[X_1^n \mid X_2>t_2\right]} \\
&= A(t_1 r, t_2),
\end{aligned} \tag{3.19}$$

because of (3.16). Now, $A(0, t_2)=1$ and

$$\begin{aligned}
\frac{dA\left(t_1, t_2\right)}{dt_1} &= \frac{-n\left(n-1\right)}{E\left[X_1^n \mid X_2>t_2\right]} \int_{t_i}^{\infty}\left(x_1-t_1\right)^{n-2} R_1\left(x_1 \mid X_2>t_2\right) dx_1 \\
&= \frac{-n E\left[\left(X_1-t_1\right)_{+}^{n-1} \left|X_2>t_2\right]\right.}{E\left[X_1^n \mid X_2>t_2\right]},
\end{aligned}$$

so that

$$\left.\frac{dA\left(t_1, t_2\right)}{dt_1}\right|_{t_1=0}=A^{\prime}\left(0, t_2\right)=\frac{-b_1\left(t_2\right)}{\alpha_1\left(t_2\right)}, b_1\left(t_2\right) \ =n E\left[X_1^{n-1} \mid X_2>t_2\right]$$

and

$$\alpha_1\left(t_2\right)=E\left[X_1^n \mid X_2>t_2\right].$$

Thus $A(t_1, t_2)$ admits the MacLaurin expansion

$$A(t_1, t_2)=1-\frac{b_1\left(t_2\right)}{\alpha_1\left(t_2\right)} t_1+o(t_1).$$

By virtue of (3.19), proceeding as in Theorem 2.2

$$A(t_1, t_2)=\exp \left[-\frac{b_1\left(t_2\right)}{\alpha_1\left(t_2\right)} t_1\right].$$

From (3.18),

$$n \int_{t_i}^{\infty} (x_1 - t_1)^{n-1} R_1 (x_1 | X_2 > t_2) dx_1 = b_1 (t_2) A(t_1, t_2) = b_1 (t_2) \exp \left[-\frac{b_1 (t_2)}{\alpha_1 (t_2)} t_1 \right].$$

Differentiating the last expression *n* times with respect to t_1,

$$R_1 (t_1 | X_2 > t_2) = \frac{b_1^n (t_2)}{n! [\alpha_1 (t_2)]^{n-1}} \exp \left[-\frac{b_1 (t_2)}{\alpha_1 (t_2)} t_1 \right]. \tag{3.20}$$

Similarly

$$R_2 (t_2 | X_1 > t_1) = \frac{b_2^n (t_1)}{n! [\alpha_2 (t_1)]^{n-1}} \exp \left[-\frac{b_2 (t_1)}{\alpha_2 (t_1)} t_2 \right]. \tag{3.21}$$

Setting $t_2 = 0$ ($t_1 = 0$) in (3.20) ((3.21)) we find $P[X_1 > t_1]$ ($P[X_2 > t_2]$) and then

$$\begin{aligned} R (t_1, t_2) &= \frac{(b_1 (t_2) b_2)^n}{(n!)^2 [\alpha_1 (t_2) \alpha_2]^{n-1}} \exp \left[-\frac{b_1 (t_2)}{\alpha_1 (t_2)} t_1 - \frac{b_2}{\alpha_2} t_2 \right] \\ &= \frac{(b_2 (t_1) b_1)^n}{(n!)^2 [\alpha_2 (t_1) \alpha_1]^{n-1}} \exp \left[-\frac{b_2 (t_2)}{\alpha_2 (t_1)} t_2 - \frac{b_1}{\alpha_1} t_1 \right] \end{aligned} \tag{3.22}$$

where $b_i = b_i(0)$ and $\alpha_i = \alpha_i(0), i = 1, 2$. Equating the two, leads to

$$\frac{(b_1 (t_2) b_2)^n}{(n!)^2 [\alpha_1 (t_2) \alpha_2]^{n-1}} \exp \left[-\frac{b_1 (t_2)}{\alpha_1 (t_2)} t_1 - \frac{b_2}{\alpha_2} t_2 \right] =$$

$$\frac{(b_2 (t_1) b_1)^n}{(n!)^2 [\alpha_2 (t_1) \alpha_1]^{n-1}} \exp \left[-\frac{b_2 (t_1)}{\alpha_2 (t_1)} t_2 - \frac{b_1}{\alpha_1} t_1 \right]$$

As in Theorem 3.3.1, using the linearity of t_1 and t_2 on either side, we find

$$\frac{b_2^n (t_1)}{[\alpha_2 (t_1)]^{n-1}} = \frac{b_2^n}{\alpha_2^{n-1}}$$

and likewise, $\frac{b_1^n (t_2)}{[\alpha_1 (t_2)]^{n-1}} = \frac{b_1^n}{\alpha_1^{n-1}}$. Substituting these in (3.22) we have the reduced form

$$\frac{b_1 (t_2)}{\alpha_1 (t_2)} t_1 - \frac{b_2}{\alpha_2} t_2 = \frac{b_2 (t_1)}{\alpha_2 (t_1)} t_2 - \frac{b_1}{\alpha_1} t_1$$

with solutions

$$\frac{b_i\left(t_{3-i}\right)}{\alpha_i\left(t_{3-i}\right)}=\theta t_{3-i}+\frac{b_i}{\alpha_i}, i=1,2$$

using the method in Theorem 3.3.2. This leads to (3.17) and the proof of the theorem is complete. □

Remark 3.3.8. An important observation is that Theorem 3.3.4 is valid only for $n \geq 2$ and hence Theorem 3.3.2 cannot be deduced from it.

Remark 3.3.9. The property characterizes the Gumbel's bivariate exponential distribution if and only if

$$\frac{\left(b_1 b_2\right)^n}{(n!)^2\left(\alpha_1 \alpha_2\right)^{n-1}}=1$$

in which case X_1 and X_2 are positive random variables with exponential distributions and $\frac{b_i^n}{n!\alpha_i^{n-1}}=1, i=1,2$.

Abraham et al. (2007) have found characterization of Pareto distribution with the sf, $R(t)=\left(\frac{k}{t}\right)^a$, where $t \geq k>0$, where k, a are constants and $a>r$ for some positive integer r by the following property

$$p_r(t)\, p_r(s)=p_r(1)\, p_r(ts) \text{ for all } t, s>1.$$

In the following theorem we extend this property in to the bivariate case.

Theorem 3.3.5. *Let (X_1, X_2) be a non-negative random vector in the support of $[1, \infty) \times [1, \infty)$ with $E(X_i)<\infty, i=1,2$. Then (X_1, X_2) follows the bivariate distribution with survival function*

$$R\left(t_1, t_2\right)=b_1 b_2 t_1^{-\frac{b_1}{\alpha_1}-1} t_2^{-\frac{b_2}{\alpha_2}-\frac{\log t_1}{\theta}-1}, \tag{3.23}$$

where $t_1, t_2 \geq 1, b_1, b_2, \alpha_1, \alpha_2 > 0, 0 \leq \theta \leq \frac{b_1 b_2}{\alpha_1 \alpha_2}, b_1, b_2 < 1$, *if and only if*

$$E\left[(X_i - t_i)_+ | X_{3-i} > t_{3-i}\right] E\left[(X_i - s_i)_+ | X_{3-i} > t_{3-i}\right]$$
$$= E\left[X_i | X_{3-i} > t_{3-i}\right] E\left[(X_i - t_i s_i)_+ | X_{3-i} > t_{3-i}\right] \tag{3.24}$$

for $i = 1, 2$ *and all* $t_i \geq 1$.

Proof. Assuming (3.23)

$$E\left[(X_i - t_i)_+ | X_{3-i} > t_{3-i}\right] = \frac{b_i {t_i}^{-\frac{b_i}{\alpha_i} - \frac{\log t_{3-i}}{\theta}}}{\frac{b_i}{\alpha_i} + \frac{\log t_{3-i}}{\theta}}. \tag{3.25}$$

Setting $t_i = s_i, 0, t_i s_i$ in (3.25) the property (3.24) is verified. On the other hand, assuming (3.24), we have as before for $i = 1$,

$$\bar{G}(t_1, t_2)\bar{G}(s_1, t_2) = \bar{G}(t_1 s_1, t_2) \tag{3.26}$$

with t_2 known, where

$$\bar{G}\left(t_1, t_2\right) = \frac{\int\limits_{t_1}^{\infty} R_1\left(x_1 | X_2 > t_2\right) dx_1}{\alpha_1\left(t_2\right)}, \ \alpha_1\left(t_2\right) = E\left(X_1 | X_2 > t_2\right).$$

The only continuous solution to (3.26) that provides a survival function is

$$\bar{G}\left(t_1, t_2\right) = {t_1}^{-a_1(t_2)}.$$

This gives

$$\int\limits_{t_1}^{\infty} R_1\left(x_1 | X_2 > t_2\right) dx_1 = \alpha_1\left(t_2\right) {t_1}^{-a_1(t_2)}$$

and

$$R\left(x_1|X_2>t_2\right)=b_1\left(t_2\right)t_1^{-\frac{b_1(t_2)}{\alpha_1(t_2)}-1},b_1\left(t_2\right)=a_1\left(t_2\right)\alpha_1\left(t_2\right). \tag{3.27}$$

Similarly

$$R\left(x_2|X_1>t_1\right)=b_2\left(t_1\right)t_2^{-\frac{b_2(t_1)}{\alpha_1(t_2)}-1}. \tag{3.28}$$

Equations (3.27) and (3.28) lead to

$$\begin{aligned}R\left(t_1,t_2\right)&=b_1t_1^{-\frac{b_1}{\alpha_1}-1}b_2\left(t_1\right)t_2^{-\frac{b_2(t_1)}{\alpha_2(t_1)}-1}\\&=b_2t_2^{-\frac{b_2}{\alpha_2}-1}b_1\left(t_2\right)t_1^{-\frac{b_1(t_2)}{\alpha_1(t_2)}-1},b_i=b_i(1),\alpha_i=\alpha_i(1).\end{aligned} \tag{3.29}$$

Arguing as before, $b_2(t_1)=b_2$ and $b_1(t_2)=b_1$. Thus (3.29) simplifies to

$$t_1^{-\frac{b_1}{\alpha_1}+\frac{b_1}{\alpha_1(t_2)}}=t_2^{-\frac{b_2}{\alpha_2}+\frac{b_2}{\alpha_2(t_1)}}$$

or

$$\frac{\log t_1}{-\frac{b_2}{\alpha_2}+\frac{b_2}{\alpha_1(t_1)}}=\frac{\log t_2}{-\frac{b_1}{\alpha_1}+\frac{b_2}{\alpha_1(t_2)}}.$$

The solution of the last equation is

$$\frac{b_{3-i}}{\alpha_{3-i}\left(t_i\right)}=\frac{\log t_i}{\theta}+\frac{b_{3-i}}{\alpha_{3-i}}.$$

Substituting into (3.29) we recover (3.23). □

Remark 3.3.10. The marginal distribution of X_i is $R_i\left(t_i\right)=b_it_i^{\frac{b_i}{\alpha_i}t_i-1},t_i\geq 1$ with $R_i(1)=b_i,i=1,2$. When $X_1>1$ and $X_2>1$, $b_1=1,b_2=1$ so that $R\left(t_1,t_2\right)=t_1^{\frac{b_1}{\alpha_1}-1}t_2^{\frac{b_2}{\alpha_2}-\frac{\log t_2}{\theta}-1}$, which is a bivariate Pareto law with Pareto marginals.

Remark 3.3.11. The result in Theorems 3.3.1 through 3.3.6 continues to hold good even if the support is bounded below in both the arguments.

Theorem 3.3.6. *Let* (X_1,X_2) *be a random vector in the support of* $(0,b_1)\times(0,b_2),b_1,b_2\leq$

∞ *such that* $E(X_1^r X_2^s) < \infty$ *for some non-negative integer values r and s. Then the partial moment of* (X_1, X_2) *satisfy*

$$a_2\,(s+1)\,\frac{\partial^{r+s+1}p_{r+1,s}\,(t_1,t_2)}{\partial t_1^{r+1}\partial t_2^{s}} = a_1\,(r+1)\,\frac{\partial^{r+s+1}p_{r,s+1}\,(t_1,t_2)}{\partial t_1^{r}\partial t_2^{s+1}} \tag{3.30}$$

for all $t_1, t_2 > 0$ *and positive constants* a_1 *and* a_2 *if and only if* (X_1, X_2) *has a survival function of the form*

$$R\,(t_1,t_2) = g\,(a_1t_1 + a_2t_2) \tag{3.31}$$

where g *satisfies the conditions for a bivariate survival function.*

Proof. First assume that (3.31) holds. Then from the uniqueness theorem of $p_{r,s}(t_1, t_2)$ in Sankaran and Nair (2004),

$$R\,(t_1,t_2) = \frac{(-1)^{r+s}}{r!s!}\frac{\partial^{r+s}p_{r,s}\,(t_1,t_2)}{\partial t_1^{r}\partial t_2^{s}}, \tag{3.32}$$

we can write

$$\frac{(-1)^{r+s}}{(r+1)!s!}\frac{\partial^{r+s+1}p_{r+1,s}\,(t_1,t_2)}{\partial t_1^{r+1}\partial t_2^{s}} = a_1 g^{'}\,(a_1t_1 + a_2t_2)$$

and

$$\frac{(-1)^{r+s}}{r!(s+1)!}\frac{\partial^{r+s+1}p_{r,s+1}\,(t_1,t_2)}{\partial t_1^{r}\partial t_2^{s+1}} = a_2 g^{'}\,(a_1t_1 + a_2t_2)\,.$$

From the last two equations (3.30) follows. To prove the 'only if' part we note that (3.30) is equivalent to

$$a_2\frac{\partial R}{\partial t_1} = a_1\frac{\partial R}{\partial t_2}. \tag{3.33}$$

To solve the partial differential equation (3.40), we set $u = a_1t_1 + a_2t_2$ and $v = t_1$. Then

$$\frac{\partial R\,(t_1,t_2)}{\partial t_1} = \frac{\partial}{\partial u}R\left(\frac{u - a_2t_2}{a_1}, t_2\right) = 0$$

whose general solution is of the form $g(u)$ or $R(t_1,t_2) = g(a_1t_1 + a_2t_2)$ in which g satisfies the conditions for a bivariate survival function. This completes the proof. □

Remark 3.3.12. When $r = s = 1$ we have the simple case

$$a_2 \frac{\partial^3 p_{2,1}(t_1,t_2)}{\partial t_1^2 \partial t_2} = a_1 \frac{\partial^3 p_{1,2}(t_1,t_2)}{\partial t_1 \partial t_2^2}$$

which is equivalent to

$$a_2 \frac{\partial p_{1,0}}{\partial t_1} = a_1 \frac{\partial p_{0,1}}{\partial t_2}$$

also characterizes the distribution.

Remark 3.3.13. Some of the distributions belonging to the class (3.31) are

(a) the bivariate exponential

$$R(t_1,t_2) = e^{-(a_1t_1+a_2t_2)}, a_1 > 0, a_2 > 0, t_1, t_2 > 0$$

with independent marginals $R_i(t_i) = e^{-a_it_i}$ with $p_{r,s}(t_1,t_2) = \frac{r!s!}{{a_1}^r {a_2}^r}$

(b) the bivariate Lomax

$$R(t_1,t_2) = (1 + a_1t_1 + a_2t_2)^{-c}, a_1, a_2, c > 0;$$

with marginals $R_i(t_i) = (1 + a_it_i)^{-c}$ and

$$p_{r,s}(t_1,t_2) = \frac{r!s!}{a_1^r a_2^s (c-1)\dots(c-s-r)} (1 + a_1t_1 + a_2t_2)^{r+s-c}, c > r + s$$

and

(c) the bivariate re-scaled beta

$$R\left(t_1,t_2\right)=\left(1-a_1t_1-a_2t_2\right)^b, a_1,a_2,b>0; 0<t_1<\frac{1}{a_1}, 0<t_2<\frac{1-a_1t_1}{a_2}$$

having $R_i(t_i)=(1-a_it_i)^b$ and

$$p_{r,s}(t_1,t_2)=\frac{r!s!}{a_1^ra_2^s\left(b+1\right)\ldots\left(b+r+s\right)}\left(1-a_1t_1-a_2t_2\right)^{b-s-r}, b>r+s.$$

Remark 3.3.14. The derivatives of partial moments, also known as the *stop-loss moment rates* have a significant role in risk analysis and reliability modelling.

If the interest is relationships among $p_{r,s}$ and not their rates, we have the following theorem.

Theorem 3.3.7. *If* (X_1,X_2) *is as in Theorem 3.3.6, the property*

$$a_1(s+1)p_{r+1,s}(t_1,t_2)=a_2(r+1)p_{r,s+1}(t_1,t_2) \tag{3.34}$$

holds for all t_1,t_2 *and non-negative integers r,s if and only if the partial moments are of the form*

$$p_{r+1,s}(t_1,t_2)=A(a_1t_1+a_2t_2), a_1,a_2>0. \tag{3.35}$$

The survival function of (X_1,X_2) *in the case has the form* (3.31).

Proof. We have

$$a_1(s+1)p_{r+1,s}(t_1,t_2)=a_1(s+1)\left(r+1\right)\int_{t_1}^{\infty}p_{r,s}(x_1,t_2)dx_1$$

and

$$a_2(r+1)p_{r,s+1}(t_1,t_2)=a_2(s+1)\left(r+1\right)\int_{t_2}^{\infty}p_{r,s}(t_1,x_2)dx_2,$$

or

$$a_1 \int_{t_1}^{\infty} p_{r,s}(x_1, t_2)dx_1 = a_2 \int_{t_2}^{\infty} p_{r,s}(t_1, x_2)dx_2 \tag{3.36}$$

on assuming (3.34) to be true. we can write (3.36) as

$$a_1 \frac{\partial B(t_1, t_2)}{\partial t_2} = a_2 \frac{\partial B(t_1, t_2)}{\partial t_1}, \tag{3.37}$$

where

$$B(t_1, t_2) = \int_{t_1}^{\infty} \int_{t_2}^{\infty} p_{r,s}(x_1, x_2)dx_1 dx_2. \tag{3.38}$$

The partial differential equation (3.37) can be solved in the same manner as in Theorem 3.3.6 to leave

$$B(t_1, t_2) = c(a_1 t_1 + a_2 t_2), a_1, a_2 > 0.$$

Also (3.38) gives

$$p_{r,s}(t_1, t_2) = a_1 a_2 c^{''}(a_1 t_1 + a_2 t_2)$$

which is of the form (3.35). The converse part follow from the representation (3.32). □

In the following theorem we characterize bivariate exponential mixture with independent marginals using bivariate partial moments.

Theorem 3.3.8. *The ratio*

$$\frac{p_{r,s}(t_1, t_2)}{p_{r-1,s-1}(t_1, t_2)} = K; K > 0 \tag{3.39}$$

if and only if (X_1, X_2) *has density function*

$$f(t_1, t_2) = \left(\frac{K}{rs}\right) \int_0^\infty \int_0^\infty exp\left[-\lambda_1 t_1 - \lambda_2 t_2\right] d\mu, \tag{3.40}$$

for all $t_1, t_2 \geq 0$ *and positive integers* r, s*, where* μ *is a probability measure on the set* $\left[\lambda_1\lambda_2 = \left(\frac{K}{rs}\right), \lambda_1, \lambda_2 > 0\right]$.

Proof. When (3.39) holds

$$p_{r,s}\left(t_1, t_2\right) = K p_{r-1,s-1}\left(t_1, t_2\right).$$

Differentiating with respect to $t_1(t_2)$, $r(s)$ times, we have

$$\frac{\partial^{r+s}}{\partial t_1^r \partial t_2^s} p_{r,s}(t_1, t_2) = K \frac{\partial^{r+s}}{\partial t_1^r \partial t_2^s} p_{r-1,s-1}(t_1, t_2)$$

which can be written as

$$R\left(t_1, t_2\right) = \frac{rs}{K} f(t_1, t_2)$$

Now from Theorem 1 in (Puri and Rubin (1974)), $f(t_1, t_2)$ has solution (3.40). By direct calculation $\frac{f(t_1,t_2)}{R(t_1,t_2)} = \frac{rs}{K}$ so that (3.40) implies (3.39). □

3.4 Application

In this section we illustrate the use of the characterization theorem in carrying out a preliminary diagnostic check to see whether a given set of observations follow the Gumbel's bivariae exponential distribution. If $X_1$and X_2 are positive, for $r = 2$ in Theorem 3.3.2, the ratios

$$\frac{\alpha_i^{r-1}\left(t_{3-i}\right) E\left[\left(X_i - 2t_1\right)_+ | X_{3-i} > t_{3-i}\right]}{E^2\left[\left(X_i - t_1\right)_+ | X_{3-i} > t_{3-i}\right]}, i = 1, 2 \tag{3.41}$$

Table 3.3: First failure times of transmission pumps (X_1, X_2) on DQG-66A Caterpillar Tractors

Tractor Number	X_1	X_2
1	1641	850
2	5556	1607
3	5421	2225
4	3168	3223
5	1534	3379
6	6367	3832
7	9460	3871
8	6679	4142
9	6142	4300
10	5995	4789
11	3953	6310
12	6922	6310
13	4210	6378
14	5161	6449
15	4732	6949

are unity ensures that (X_1, X_2) follows Gumbel's bivariate exponential distribution. Accordingly for a given random sample, if the estimates of (3.41) for all t_1, t_2 has only small fluctuations around unity the data provides reasonable support to the assumed model. We consider the data in Table 3.3 on first failure times of transmission (X_1) and the transmission pump (X_2) on DQG-66A Caterpillar tractors reported in Barlow and Proschan (1976). Using the same data, Ebrahimi and Zahedi (1989) have shown that the failure times (X_1, X_2) do not follow a bivariate Gumbel's exponential distribution based on a goodness of fit test for the distribution against the alternative that (X_1, X_2) is bivariate new better than used in expectation.

When the points are $(x_{1i}, x_{2i}), i = 1, 2 \ldots, n$ the partial moments are estimated by

$$p(a_i, t_{3-i}) = \frac{1}{n}\sum_{i=1}^{n}\big((x_{1i} - a_i)_+|X_{2i} > t_2\big)I(x_{1i} > a_i),$$

where $a_i = 0, t_i, 2t_i$ and $I(x_{1i} > a_i) = 1$ or 0 according as $x_{1i} > a_i$ or not. The above estimators are unbiased and consistent. The estimates of various functions are presented in Table 3.4 through Table 3.6 for admissible values of (t_1, t_2).

Table 3.4: Estimated values of $\alpha_i(t_{3-i})$

$t_2 \backslash t_1$	1500	1600	1800	2000	2200	2500	3000
800	5129.4	5027.13	4917.73	4917.73	4917.73	4917.73	4917.73
1500	5020	4917.73	4917.73	4917.73	4917.73	4917.73	4917.73
2000	4649.6	4547.33	4547.33	4547.33	4547.33	4547.33	4547.33
3000	4288.2	4185.93	4185.93	4185.93	4185.93	4185.93	4185.93
4000	2919.6	2919.6	2919.6	2919.6	2919.6	2919.6	2919.6
5000	1665.2	1665.2	1665.2	1665.2	1665.2	1665.2	1665.2
6000	1665.2	1665.2	1665.2	1665.2	1665.2	1665.2	1665.2

Table 3.5: Estimated values of $E\left[(X_i - 2t_i)_+|X_{3-i} > t_{3-i}\right]$

$t_2 \backslash t_1$	1500	1600	1800	2000	2200	2500	3000
800	2317.73	2146.53	1826.53	1509.67	1229	846.867	371.333
1500	2317.73	2146.53	1826.53	1509.67	1229	846.867	371.333
2000	2147.33	1989.47	1696.13	1405.93	1151.93	809.8	371.333
3000	1985.93	1841.4	1574.73	1311.2	1083.87	781.733	371.333
4000	1319.6	1212.93	999.6	789.4	615.4	393.267	116.2
5000	665.2	598.533	465.2	335	241	138.867	61.4667
6000	665.2	598.533	465.2	335	241	138.867	61.4667

Table 3.6: Estimated values of $E^2\left[(X_i - t_i)_+ | X_{3-i} > t_{3-i}\right]$

$t_2 \backslash t_1$	1500	1600	1800	2000	2200	2500	3000
800	1.31725×10^7	1.24877×10^7	1.12744×10^7	1.01404×10^7	9.06652×10^6	7.56837×10^6	5.37189×10^6
1500	1.31044×10^7	1.24684×10^7	1.12744×10^7	1.01404×10^7	9.06652×10^6	7.56837×10^6	5.37189×10^6
2000	1.12198×10^7	1.06755×10^7	9.65552×10^6	8.68677×10^6	7.76923×10^6	6.48891×10^6	4.61104×10^6
3000	9.53698×10^6	9.07576×10^6	8.21357×10^6	7.39441×10^6	6.61827×10^6	5.53473×10^6	3.94393×10^6
4000	4.4927×10^6	4.26946×10^6	3.84003×10^6	3.43336×10^6	3.04945×10^6	2.51624×10^6	1.74134×10^6
5000	1.35769×10^6	1.28112×10^6	1.13465×10^6	997069	868375	692002	442491
6000	1.35769×10^6	1.28112×10^6	1.13465×10^6	997069	868375	692002	442491

Finally, the ratios (3.41) computed from the above estimates given in Table 3.7 shows that all the values depart substantially from unity. Thus we conclude that the observations do not support the distribution as Gumbel's bivarite exponential, reaffirm the conclusion made by Ebrahimi and Zahedi (1989) based on a different method.

Table 3.7: Estimated values of $\frac{\alpha_i^{r-1}(t_{3-i})E\left[(X_i-2t_1)_+|X_{3-i}>t_{3-i}\right]}{E^2\left[(X_i-t_1)_+|X_{3-i}>t_{3-i}\right]}$

$t_2\backslash t_1$	1500	1600	1800	2000	2200	2500	3000
800	0.902527	0.86412	0.79671	0.732134	0.666617	0.550272	0.33994
1500	0.887871	0.846624	0.79671	0.732134	0.666617	0.550272	0.33994
2000	0.889875	0.847435	0.798806	0.735975	0.674227	0.567496	0.366203
3000	0.892954	0.849293	0.802541	0.742263	0.685526	0.591228	0.394119
4000	0.857547	0.829445	0.760002	0.671276	0.589196	0.456308	0.194825
5000	0.815864	0.777972	0.682722	0.559482	0.462142	0.334162	0.231314
6000	0.815864	0.777972	0.682722	0.559482	0.462142	0.334162	0.231314

Chapter 4

Some properties of conditional upper partial moments in the context of stochastic modeling*

4.1 Introduction

Let $\boldsymbol{X} = (X_1, X_2)$ be a non-negative random vector admitting an absolutely continuous distribution function $F(x_1, x_2)$ with respect to a Lebesgue measure. Also let $(X|\mathscr{A})$ denote the conditioned random variable, where the conditioning event $\mathscr{A}$ is treated in two ways. First, is of the type $(X_1|X_2 = t_2)$ (commonly known as conditionally specified models) and the second event is of the type $(X_1|X_2 > t_2)$ (commonly known as conditional survival models). In order to justify the practical importance and usefulness of these two types of conditioning, consider the following examples given in Gupta (2008).

(i) Suppose that a system consisting of two components with survival times

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(X_1, X_2) which are dependent. The first type of conditioning pertains to the situation in which first component is still working given that the second fails at the time t_2; while second type can be related to a situation where first component is working given that the second is also working after the point t_2.

(ii) (Danish twins data): Let (X_1, X_2) denote the survival times of twins. Then the first type of conditioning pertains to the life length of twin 1 when twin 2 dies at age t_2 while the second pertains to the situation where we are interested in the life length of twin 1 when twin 2 is still alive at age t_2 (see Hougaard (2000)).

(iii) Let (X_1, X_2) denote the survival times of a couple. In the insurance business, in order to determine the premium on a particular policy, it is important to know whether the other spouse has died or is surviving at a particular time.

The organization of the present chapter is as follows. In Section 4.2 we introduce the concept of upper partial moments under two types of conditioning and study its properties. The Section 4.3 deals with the characterizations of bivariate distributions based on two types of upper partial moments. In Section 4.4, characterization results based on conditional upper partial moments and measures in income studies are proved. We propose a nonparametric estimator for the partial moment in Section 4.5. Sections 4.6 and 4.7 presents the results of a simulation study along with applications to real datasets.

4.2 Conditional upper partial moments

In this section, we propose and study the two types conditional upper partial moments (i.e upper partial moment for conditional specified and conditional survival models) and its properties in the context of stochastic modelling.

4.2.1 Upper partial moments of conditionally specified models

In this section, we consider upper partial moments based on conditioning of first type for the random variables $(X_1|X_2 = t_2)$ and $(X_2|X_1 = t_1)$ and study its properties.

Definition 4.2.1. Let $\boldsymbol{X} = (X_1, X_2)$ be a non-negative random vector admitting an absolutely continuous distribution function F with respect to a Lebesgue measure in the positive octant $\mathbb{R}_2^+ = \{(x_1, x_2)|x_1, x_2 > 0\}$ of the two dimensional Euclidean space $\mathbb{R}_2$. Assume that $E(X_1^r X_2^s), r, s = 0, 1, 2, ...$ is finite. Then for $i, j = 1, 2$ and $i \neq j$, the conditional upper partial moment of the first kind is defined as

$$\begin{aligned} \phi_r(t_i|t_j) = \phi^r_{X_i|X_j}(t_i|t_j) &= E\left[(X_i - t_i)^r_+|X_j = t_j\right], r = 0, 1, 2, ... \\ &= \int_{t_i}^{\infty} (x_i - t_i)^r f_i(x_i|t_j) dx_i, \end{aligned} \tag{4.1}$$

where $(X_i - t_i)_+$ is defined as in (2.3) and $f_i(x_i|t_j) = f_{X_i|X_j}(x_i|t_j)$.

For convenience, we always assume that X_1 and X_2 have finite moments of any degree. Then for $i, j = 1, 2$ and $i \neq j$, the conditional upper partial moment of the first type, $\phi_r(t_i|t_j)$ has the following properties, (i) $\phi^1_{X_i|X_j}(0|t_j) = \phi_1(0|t_j) = E\left(X_i|X_j = t_j\right)$, (ii) Both $\phi_r(t_i|t_j)$ are of decreasing nature and both are continuous in $t_i \geq 0$ and (iii) as $t_i \to \infty$, $\phi_r(t_i|t_j) \to 0$.

Let the conditional survival functions for the first type of conditioning are defined by $S_i(t_i|t_j) = P(X_i > t_i|X_j = t_j)$. Then (4.1) will be,

$$\phi_r(t_i|t_j) = r \int_{t_i}^{\infty} (x_i - t_i)^{r-1} S_i(x_i|t_j) dx_i. \tag{4.2}$$

Also, from (4.1), we have $\phi_0(t_i|t_j) = S_i(t_i|t_j)$. Further, we can easily relate $\phi_r(t_1|t_2)$ and $\phi_r(t_2|t_1)$ with conditional failure rates of $X_1|X_2 = t_2$ and $X_2|X_1 = t_1$ defined by $h_i(t_i|t_j) = -\frac{\partial}{\partial t_i} \ln S_i(t_i|t_j),\ i, j = 1, 2, i \neq j$ as

$$\phi_r(t_i|t_j) = \int_{t_i}^{\infty} (x_i - t_i)^r h_i(x_i|t_j) \exp\{-H_i(x_i|t_j)\}\, dx_i,$$

where $H_i(.|t_j)$ denotes the conditional cumulative failure (hazard) function corresponding to $h_i(t_i|t_j)$. On the other hand, from (4.2), we have

$$h_i(t_i|t_j) = -\frac{\partial}{\partial t_i} \ln \phi_1'(t_i|t_j),$$

where ${\phi'}_1(t_i|t_j) = \frac{\partial}{\partial t_i}\phi_1(t_i|t_j)$.

Let $r(t_1, t_2) = (r_1(t_1|t_2), r_2(t_1|t_2))$ denote the vector valued conditional mean residual life function for the random variables $(X_1|X_2 = t_2)$ and $(X_2|X_1 = t_1)$, where $r_i(t_i|t_j) = E\left(X_i - t_i|X_i > t_i, X_j = t_j\right) = \frac{1}{S_i(t_i|t_j)} \int_{t_i}^{\infty} S_i(x_i|t_j)dx_i$. Then

$$\phi_1(t_i|t_j) = r_i(t_i|t_j)S_i(t_i|t_j).$$

Also, we have $r_i(t_i|t_j) = \left(-\frac{\partial}{\partial t_i} \ln \phi_1(t_i|t_j)\right)^{-1}$.

In the following theorem we show that the conditional upper partial moments determine the corresponding conditional distributions uniquely.

Theorem 4.2.1. *The conditional partial moment $\phi_r(t_i|t_j), i, j = 1, 2; i \neq j$ determines the corresponding conditional distribution uniquely using the relationship*

$$r!(-1)^r S_i(t_i|t_j) = \frac{\partial^r}{\partial {t_i}^r}\left(\phi_r(t_i|t_j)\right).$$

Proof. Let the conditional survival function defined in (4.2), $S_i(t_i|t_j)$ be absolutely continuous. Differentiating (4.2) with respect to t_i, r times, we obtain the required form (see Gupta and Gupta (1983)). □

Recurrence relationships are generally useful for finding higher order moments using its preceding moments, as it does not require direct computation which often needs large number of steps. Also, it provides the functional relationship between two consecutive moments. Based on this idea, in the following theorem we obtain a recurrence relationship satisfied by the conditionally specified upper partial moments, that gives the nature of relationship between two consecutive conditional upper partial moments.

Theorem 4.2.2. *The conditional partial moment $\phi_r(t_i|t_j), i = 1, 2; i \neq j$ satisfies the recurrence relationship given by, $\frac{\partial}{\partial t_i}\phi_r(t_i|t_j) + r\phi_{r-1}(t_i|t_j) = 0$.*

Proof. From (4.2), we have

$$\phi_{r-1}(t_i|t_j) = (r-1)\int_{t_i}^{\infty}(x_i - t_i)^{r-2}S_i(x_i|t_j)dx_i.$$

Taking logarithm on both sides of (4.2), we get

$$\ln[\phi_r(t_i|t_j)] = \ln\left[r\int_{t_i}^{\infty}(x_i - t_i)^{r-1}S_i(x_i|t_j)dx_i\right].$$

Differentiating with respect to t_i, we obtain the required relationship. □

4.2.1.1 Upper partial moments for the minimum and maximum

The distributions of maximum and minimum of the random variables X_1 and X_2 are of much importance in reliability studies and survival analysis. For example, maximum of all component's lifetimes gives the total lifetime of the parallel system and for the series system, the minimum gives the total lifetime. We define the upper partial moments for minimum and maximum of two random variables. Consider a two component system with lifetimes (X_1, X_2) with probability density function f. Let $T_1 = min(X_1, X_2)$ and $T_2 = max(X_1, X_2)$ denote the random variables of minimum and maximum, respectively. Gupta and Gupta (2001) defined the density function of T_1 and T_2 as

$$f_{T_1}(u) = f_{X_2}(u)P(X_1 > u|X_2 = u) + f_{X_1}(u)P(X_2 > u|X_1 = u) \tag{4.3}$$

and

$$f_{T_2}(u) = f_{X_1}(u)P(X_2 < u|X_1 = u) + f_{X_2}(u)P(X_1 < u|X_2 = u) \tag{4.4}$$

Then the upper partial moments of minimum and maximum can be defined as follows

$$l_r(t) = E\left[(T_1 - t)_+\right]^r = \int_t^{\infty} (u-t)^r f_{T_1}(u)du$$

and

$$h_r(t) = E\left[(T_2 - t)_+\right]^r = \int_t^{\infty} (u-t)^r f_{T_2}(u)du,$$

where $(T_i - t)_+; i = 1, 2$ is defined as in (2.3). Now the following theorem is immediate.

Theorem 4.2.3. *The upper partial moments of minimum and maximum $l_r(t)$ and $h_r(t)$ are connected by the relation*

$$h_r(t) + l_r(t) = p_r\left(t\right) + q_r\left(t\right), \tag{4.5}$$

where $p_r(t)$ and $q_r(t)$ are the univariate upper partial moments for the random variables X_1 and X_2 as defined in (1.1).

Proof. From (4.3) and (4.4) we have $f_{T_1}(u) + f_{T_2}(u) = f_{X_1}(u) + f_{X_2}(u)$, from which the expression given in (4.5) is immediate. □

From Gupta (2016b) we have,

$$P\left(T_2 > x_2 | T_1 = x_1\right) = \frac{\int_y^\infty f_{X_1,X_2}(x_1,u)du + \int_y^\infty f_{X_1,X_2}(u,x_1)du}{f_{T_1}(x_1)}$$

In this case, the conditional partial moment will be,

$$\phi_r(t) = \int_t^\infty (x_2 - t)^r f_{T_2>x_2|T_1=x_1}(x_2|x_1)dx_2 \tag{4.6}$$

Example 4.2.1. Let (X_1, X_2) follows the bivariate density, $f(x_1, x_2) = e^{-(x_1+x_2)}$, $x_1 > 0,\ x_2 > 0$, with $f_{T_2>x_2|T_1=x_1}(x_2|x_1) = e^{x_1-x_2}$. Then from (4.6), it follows that $\phi_r(t) = \Gamma(r+1)e^{x_1-t}$. Here the two upper partial moments satisfies the relation $l_r(t) + h_r(t) = 2e^{-t}\Gamma(r+1)$, and the ratio $\frac{h_r(t)}{l_r(t)}$ is of convex nature.

4.2.2 Upper partial moments of conditionally survival models

From the reliability point of view, it is more important to study the conditional distributions of the random variables $X_1|X_2 > t_2$ and $X_2|X_1 > t_1$, usually known as conditionally survival models. Now we define the conditional partial moment of second kind as follows.

Definition 4.2.2. Let $\boldsymbol{X} = (X_1, X_2)$ be a non-negative random vector admitting an absolutely continuous distribution function F with respect to a Lebesgue measure in the positive octant $\mathbb{R}_2^+ = \{(x_1, x_2)|x_1, x_2 > 0\}$ of the two dimensional Euclidean space $\mathbb{R}_2$. Assume that $E(X_1^r X_2^s)$ is finite. Then the conditional partial moment of the second kind is defined as

$$\begin{aligned}\psi_r(t_i|t_j) = \psi^r_{X_i|X_j>t_j}(t_i|t_j) &= E\left[(X_i - t_i)_+^r | X_j > t_j\right], r = 0, 1, 2, ...\\ &= \int_{t_i}^\infty (x_i - t_i)^r f_i^*(x_i|t_j)dx_i; i, j = 1, 2; i \neq j. \end{aligned} \tag{4.7}$$

We always assume that X_1 and X_2 have finite moments of any degree. Also, $\psi_r(t_i|t_j)$ has the following properties, (i) $\psi^1_{X_i|X_j}(0|t_j) = \psi_1(0|t_j) = E\left(X_i|X_j > t_j\right)$, (ii) Both $\psi_r(t_i|t_j)$ are of decreasing nature and both are continuous in $t_i \geq 0$ and (iii) as $t_i \to \infty$, $\psi_r(t_i|t_j) \to 0$.

Denoting $R_i(t_i|t_j) = P(X_i > t_i|X_j > t_j)$ as the conditional survival function such that $R(t_1, t_2) = P(X_1 > t_1, X_2 > t_2)$, the joint survival function, then for $i, j = 1, 2$ and $i \neq j$, (4.7) implies that,

$$\psi_r(t_i|t_j) = r \int_{t_i}^{\infty} (x_i - t_i)^{r-1} R_i(x_i|t_j) dx_i. \tag{4.8}$$

Also, we have $\psi_0(t_i|t_j) = R_i(t_i|t_j)$. Further, when the two events $(X_1 > t_1)$ and $(X_2 > t_2)$ are independent, then $\psi_r(t_i|t_j) = p_r(t_i)$, the univariate partial moment defined in (2.2). It is easy relate $\psi_r(t_1)$ and $\psi_r(t_2)$ with the vector valued conditional hazard rate function defined by $h^*(t_1, t_2) = (h^*_1(t_1, t_2), h^*_2(t_1, t_2))$, where $h^*_i(t_i, t_j) = -\frac{\partial}{\partial t_i} \ln R_i(t_i|t_j) \quad i, j = 1, 2$ and $i \neq j$ given by $h^*_i(t_i, t_j) = -\frac{\partial}{\partial t_i} \ln \psi'_1(t_i|t_j)$, where $\psi'{}_1(t_i|t_j) = \frac{\partial}{\partial t_i}\psi_1(t_i|t_j)$. From (4.8) we also obtain $\psi_r(t_i|t_j) = \int_{t_i}^{\infty} (x_i - t_i)^r h^*_i(x_i, t_j) \exp\left\{-H^*_i(x_i|t_j)\right\} dx_i$, where $H^*_i(.|t_j)$ denotes the conditional cumulative failure (hazard) function corresponding to $h^*_i(t_1, t_2) i, j = 1, 2, i \neq j$.

If we denote $r^*(t_1, t_2) = (r^*_1(t_1, t_2), r^*_2(t_1, t_2))$ the vector valued mean residual life function, where

$$r^*_i(t_1, t_2) = E\left(X_i - t_i|X_1 > t_1, X_2 > t_2\right) = \frac{1}{R(t_1, t_2)} \int_{t_i}^{\infty} R(x_i, t_j) dx_i, i, j = 1, 2, i \neq j,$$

then we have $\psi_1(t_i|t_j) = r^*_i(t_1, t_2) R(t_1, t_2)$. Also, $r^*_i(t_1, t_2) = \left(-\frac{\partial}{\partial t_i} \ln \psi_1(t_i|t_j)\right)^{-1}$.

Like in the case of conditionally specified models, the conditional partial moment $\psi_r(t_i|t_j)$ uniquely determines the corresponding conditional distribution.

Theorem 4.2.4. *The conditional partial moment $\psi_r(t_i|t_j), i,j = 1,2; i \neq j$ determines the corresponding conditional distribution uniquely through relationship*

$$r!\,(-1)^r R_i(t_i|t_j) = \frac{\partial^r}{\partial t_i^r}\left(\psi_r(t_i|t_j)\right).$$

Following result is a recurrence relationship satisfied by the conditional partial moment $\psi_r(t_i|t_j)$.

Theorem 4.2.5. *The conditional upper partial moments $\psi_r(t_i|t_j), i = 1,2; i \neq j$ satisfies the recurrence relationship, $\frac{\partial}{\partial t_i}\psi_r(t_i|t_j) + r\psi_{r-1}(t_i|t_j) = 0$.*

4.2.3 Examples of conditional upper partial moments

Here we provide some examples of conditional upper partial moments for certain bivariate lifetime distributions.

4.2.3.1 Bivariate exponential distribution - Gumbel Type I

Suppose (X_1, X_2) has a bivariate Gumbel Type I exponential distribution. Then for some $0 \leq \theta \leq 1$,

$$f(x_1,x_2) = e^{-(x_1+x_2+\theta x_1 x_2)}\left[(1+\theta x_1)(1+\theta x_2) - \theta\right],\ x_1 \geq 0, x_2 \geq 0,\ 0 \leq \theta \leq 1, \tag{4.9}$$

with conditional densities $f_i(x_i|x_j) = e^{-x_i(1+\theta x_j)}\left[(1+\theta x_1)(1+\theta x_2) - \theta\right]$ (see Balakrishnan and Lai (2009)) and $f_i^*(x_i|x_j) = e^{-x_i(1+\theta x_j)}\ (1+\theta x_j)$. The conditional upper partial moments (4.1) and (4.7) are respectively given by

$$\phi_r(t_i|t_j) = \frac{\Gamma\,(r+1)}{(1+\theta t_j)^{r+1}} e^{-t_i(1+\theta t_j)}\left[(1+\theta t_1)\ (1+\theta t_2) + r\theta\right], i = 1,2; i \neq j$$

and

$$\psi_r(t_i|t_j) = \frac{\Gamma\,(r+1)}{(1+\theta t_j)^r} e^{-t_i(1+\theta t_j)}, i = 1, 2; i \neq j.$$

4.2.3.2 Bivariate exponential distribution- Gumbel Type II

This model is a special case of Farlie-Gumbel-Morgenstern's bivariate distributions with marginal distributions which are both standard exponential. The joint density function is

$$f(x_1, x_2) = e^{-x_1 - x_2} \left[1 + \alpha \left(2e^{-x_1} - 1\right) \left(2e^{-x_2} - 1\right)\right], \quad x_1, x_2 > 0, \quad |\alpha| < 1. \tag{4.10}$$

The conditional densities are given by

$$f_i(x_i|x_j) = e^{-x_i} \left[1 + \alpha \left(2e^{-x_1} - 1\right) \left(2e^{-x_2} - 1\right)\right]$$

and

$$f_i^*(x_i|x_j) = e^{-x_i} \left\{\alpha \left(2e^{-x_1} - 1\right) \left(e^{-x_2} - 1\right) + 1\right\}.$$

Then we have,

$$\phi_r(t_i|t_j) = \frac{\Gamma\,(r+1)}{2^r\ e^{2t_i}} \left[\alpha \left(2e^{-t_j} - 1\right) + 2^r e^{t_i} \left(1 - \alpha \left(2e^{-t_j} - 1\right)\right)\right], i = 1, 2; i \neq j$$

and

$$\psi_r(t_i|t_j) = \frac{\Gamma\,(r+1)}{2^r\ e^{2t_i}} \left[\alpha \left(e^{-t_j} - 1\right) + 2^r e^{t_i} \left(1 - \alpha \left(e^{-t_j} - 1\right)\right)\right], i = 1, 2; i \neq j.$$

4.2.3.3 Bivariate Pareto Type I

A bivariate distribution with joint density function (Mardia (1962))

$$f(x_1,x_2) = (\alpha+1)\alpha(\theta_1\theta_2)^{\alpha+1}(\theta_2 x_1 + \theta_1 x_2 - \theta_1\theta_2)^{-(\alpha+2)}, x_i \geq \theta_i > 0,\ i = 1,2,\ \alpha > 0. \tag{4.11}$$

is called a bivariate Pareto distribution of the first kind. Then we have

$$f_i(x_i|x_j) = (\alpha+1)\ {\theta_i}^{\alpha+1}\theta_j\ \frac{(\theta_j x_i + \theta_i x_j - \theta_1\theta_2)^{-(\alpha+2)}}{(x_j - \theta_j)^{-(\alpha+1)}}, i = 1,2; i \neq j$$

and

$$f_i^*(x_i|x_j) = \alpha\ \theta_j\ \frac{(\theta_j x_i + \theta_i x_j - \theta_1\theta_2)^{-(\alpha+1)}}{(\theta_i x_j - \theta_1\theta_2)^{-\alpha}}, i = 1,2; i \neq j$$

respectively. The corresponding conditional upper partial moments will be

$$\phi_r\left(t_i|t_j\right) = \frac{\Gamma(\alpha - r + 1)\Gamma(r+1)(\alpha+1)\theta_i^{\alpha+1}\theta_j^{-r}}{\Gamma(\alpha+2)(t_j - \theta_j)^{-(\alpha+1)}}\left(\theta_j t_i + \theta_i t_j - \theta_1\theta_2\right)^{r-\alpha-1},$$

where $i = 1,2; i \neq j, \alpha - r > -1, r > -1$ and $\alpha > -2$ and

$$\psi_r\left(t_i|t_j\right) = \frac{\Gamma(\alpha - r)\Gamma(r+1)\alpha\theta_j^{-r}}{\Gamma(\alpha+1)(\theta_i t_j - \theta_1\theta_2)^{-\alpha}}\left(\theta_j t_i + \theta_i t_j - \theta_1\theta_2\right)^{r-\alpha}, i = 1,2; i \neq j,$$

where $\alpha > r, r > -1$ and $\alpha > -1$.

4.2.3.4 Bivariate Lomax distribution

Consider a bivariate Lomax distribution with survival function (Sankaran and Nair (2004))

$$R(x_1,x_2) = (1 + a_1 x_1 + a_2 x_2)^{-b};\ x_1, x_2 > 0, \quad a_1, a_2 > 0,\ b > 1. \tag{4.12}$$

Then $f_i(x_i|x_j) = \frac{a_i(b+1)(1+a_1x_1+a_2x_2)^{-(b+2)}}{(1+a_jx_j)^{-(b+1)}}$ and $f_i^*(x_i|x_j) = a_i b\frac{(1+a_1x_1+a_2x_2)^{-(b+1)}}{(1+a_jx_j)^{-b}}$. Then for $i = 1,2; i \neq j$, the corresponding conditional upper partial moments are given by

$$\phi_r(t_i|t_j) = \frac{(b+1)\,\Gamma\,(b+1-r)\,\Gamma\,(r+1)}{\Gamma\,(b+2)} a_i^{-r}(1+a_jt_j)^{b+1}(1+a_1t_1+a_2t_2)^{r-(b+1)},$$

where $b+1 > r$ and $b > 2$ and

$$\psi_r(t_i|t_j) = \frac{b\Gamma\,(b-r)\,\Gamma\,(r+1)}{\Gamma\,(b+1)} a_i^{-r}(1+a_jt_j)^{b}(1+a_1t_1+a_2t_2)^{r-b},$$

where $b > r$ and $b > -1$.

4.2.3.5 Conditional proportional hazard models

The random vectors (X_1, X_2) and (Y_1, Y_2) satisfy the conditional proportional hazard rate (CPHR) model (see Sankaran and Sreeja (2007)) when the corresponding conditional hazard rate functions of $(X_i|X_j = t_j)$ and $(Y_i|Y_j = t_j)$ satisfy

$$h_{(Y_i|Y_j=t_j)}(t_i|t_j) = \theta_i(t_j)h_{(X_i|X_j=t_j)}(t_i|t_j), \tag{4.13}$$

for $i, j = 1,2; i \neq j$ and $t_i, t_j \geq 0$, or equivalently $\bar{G}_i(t_i|t_j) = (\bar{F}_i(t_i|t_j))^{\theta_i(t_j)}$, where $\bar{G}_i(t_i|t_j) = P(Y_i > t_i|Y_j = t_j)$ and $\bar{F}_i(t_i|t_j) = S_i(t_i|t_j) = P(X_i > t_i|X_j = t_j)$. For conditional survival models, CPHR model becomes

$$h_{(Y_i|Y_j>t_j)}(t_i|t_j) = \delta_i(t_j)h_{(X_i|X_j>t_j)}(t_i|t_j), \tag{4.14}$$

for $i, j = 1,2; i \neq j$ and $t_i, t_j \geq 0$. This is equivalent to $\bar{G}_i^*(t_i|t_j) = (\bar{F}_i^*(t_i|t_j))^{\delta_i(t_j)}$, where $\bar{G}_i^*(t_i|t_j) = P(Y_i > t_i|Y_j > t_j)$ and $\bar{F}_i^*(t_i|t_j) = R_i(t_i|t_j) = P(X_i > t_i|X_j > t_j)$.

The corresponding upper partial moments then becomes

$$\phi_r(t_i|t_j) = \theta(t_j)E\left[(X_i - t_i)^r (S_i(X_i|t_j))^{\theta_i(t_j)-1}\right], i, j = 1, 2, i \neq j$$

and

$$\psi_r(t_i|t_j) = \delta_i(t_j)E\left[(X_i - t_i)^r (R_i(X_i|t_j))^{\delta_i(t_j)-1}\right], i, j = 1, 2, i \neq j.$$

Suppose (X_1, X_2) and (Y_1, Y_2) be two random vectors with their corresponding CUPMs $\phi_r(t_i|t_j)$ and $\phi^*_r(t_i|t_j)$ satisfying the CPHR model (4.13) for $i, j = 1, 2, i \neq j$. Then, it follows that

$$\begin{aligned} h_{Y_i|Y_j}(t_i|t_j) = \theta_i(t_j)\, h_{X_i|X_j}(t_i|t_j) \quad &\Leftrightarrow \quad \frac{\partial}{\partial t_i}\big(\ln \bar{G}_i(t_i|t_j)\big) = \theta_i(t_j)\frac{\partial}{\partial t_i}(\ln S_i(t_i|t_j)) \\ &\Leftrightarrow \quad \frac{\partial}{\partial t_i}\left(\ln\left(\frac{\partial^r}{\partial t_j^r}\phi^*_r(t_i|t_j)\right)\right) \\ &\qquad\qquad = \theta_i(t_j)\frac{\partial}{\partial t_i}\left(\ln\left(\frac{\partial^r}{\partial t_j^r}\phi_r(t_i|t_j)\right)\right) \\ &\Leftrightarrow \quad \frac{\nabla_1^{r+1}(\phi^*_r(t_i|t_j))}{\nabla_1^r(\phi^*_r(t_i|t_j))} = \theta_i(t_j)\frac{\nabla_1^{r+1}(\phi_r(t_i|t_j))}{\nabla_1^r(\phi_r(t_i|t_j))}, \end{aligned} \tag{4.15}$$

where $\nabla_i^{r+1}, i = 1, 2$ denotes the $(r+1)^{th}$ partial derivative of the function with respect to t_i defined by $\nabla_i^{r+1} = \frac{\partial^{r+1}}{\partial t_i^{r+1}}, i = 1, 2.$

Similarly for $i, j = 1, 2$ and $i \neq j$, the CPHR model (4.14) can be restated in terms of the second type of conditional upper partial moments as follows

$$h_{Y_i|Y_j>t_j}(t_i|t_j) = \theta_i(t_j)\, h_{X_i|X_j>t_j}(t_i|t_j) \Leftrightarrow \frac{\nabla_1^{r+1}(\psi^*_r(t_i|t_j))}{\nabla_1^r(\psi^*_r(t_i|t_j))} = \theta_i(t_j)\frac{\nabla_1^{r+1}(\psi_r(t_i|t_j))}{\nabla_1^r(\psi_r(t_i|t_j))}.$$

4.3 Characterizations of bivariate distributions based on conditional upper partial moments

In this section, we prove some characterization theorems based on conditional upper partial moments

Theorem 4.3.1. *The condition* $\phi_r(t_i|t_j) = \psi_r(t_i|t_j), i, j = 1, 2; i \neq j$, *holds if and only if the joint distribution of* (X_1, X_2) *is the product of two independent exponential random variables with probability density function*

$$f(x_1, x_2) = e^{-\lambda_1 x_1 - \lambda_2 x_2}, \lambda_1, \lambda_2, x_1, x_2 > 0. \tag{4.16}$$

Proof. Assume that the condition $\phi_r(t_i|t_j) = \psi_r(t_i|t_j), i = 1, 2$, holds true. Differentiating it with respect to t_i, r times and using Theorem (4.2.1) and Theorem (4.2.4), we get $S_i(t_i|t_j) = R_i(t_i|t_j)$. Now talking logarithm on both sides and differentiating with respect to t_i, we have $h_i(t_i|t_j) = h_i^*(t_i|t_j)$, implies that $R_i(t_i|t_j) = e^{-\lambda_i t_i}, i = 1, 2$, and therefore (4.16). □

Theorem 4.3.2. *Let* $\boldsymbol{X} = (X_1, X_2)$ *be a non-negative random vector admitting an absolutely continuous distribution function* $F(x_1, x_2)$. *Then the relationship*

$$\frac{\phi_r(t_i|t_j)}{\phi_{r-1}(t_i|t_j)} = \frac{\psi_r(t_i|t_j)}{\psi_{r-1}(t_i|t_j)} = r, \ for \ i, j = 1, 2 \ and \ i \neq j \tag{4.17}$$

holds true if and only if (X_1, X_2) *follows* (4.16) *with* $\lambda_i = 1, i = 1, 2$.

Proof. First consider the conditional specified case. Note that (4.17) implies

$$\phi_r(t_i|t_j) = r \ \phi_{r-1}(t_i|t_j).$$

By virtue of definition, we have

$$r\int_{t_i}^{\infty}(x_i-t_i)^{r-1}S_i(x_i|t_j)dx_i = r(r-1)\int_{t_i}^{\infty}(x_i-t_i)^{r-2}S_i(x_i|t_j)dx_i \tag{4.18}$$

Differentiating (4.18) with respect to t_i, $r-1$ times and after some algebraic simplification, we have, $h_i\left(t_i|t_j\right)=1$. Now using the formula

$$f_i(x_i|x_j) = h_i(x_i|x_j)\exp\left\{-\int_0^{x_i} h_i(u|x_j)du\right\}, \tag{4.19}$$

completes the first part of the proof.

To prove the conditional survival case, from (4.17) we have

$$\psi_r(t_i|t_j) = r\ \psi_{r-1}(t_i|t_j)$$

and equivalently

$$r\int_{t_i}^{\infty}(x_i-t_i)^{r-1}R_i(x_i|t_j)dx_i = r(r-1)\int_{t_i}^{\infty}(x_i-t_i)^{r-2}R_i(x_i|t_j)dx_i \tag{4.20}$$

Differentiating (4.20) with respect to t_i, $r-1$ times and after some algebraic simplification, we have, ${h_i}^*\left(t_i|t_j\right)=1$. Now using the formula for vector valued failure rate by Johnson and Kotz (1975)

$$R\left(x_1,x_2\right) = \exp\left[-\int_0^{x_1} h_1\left(u,x_2\right)du - \int_0^{x_2} h_2\left(0,u\right)du\right] \tag{4.21}$$

and

$$R\left(x_1,x_2\right) = \exp\left[-\int_0^{x_1} h_1\left(u,0\right)du - \int_0^{x_2} h_2\left(x_1,u\right)du\right] \tag{4.22}$$

we obtain the required form. The necessary part is straightforward. □

In the following theorem we characterize bivariate distribution with Pareto conditionals given in Arnold (1987) using the ratio of two consecutive upper partial moments of the first type.

Theorem 4.3.3. *For $i, j = 1, 2$ and $i \neq j$, the ratio of two consecutive conditional upper partial moments satisfies the relationship,*

$$\frac{\phi_{r+1}(t_i|t_j)}{\phi_r(t_i|t_j)} = A\,t_i + B_i(t_j); A > 0, \tag{4.23}$$

where $B_i(t_j)$ is function of t_j only, if and only if (X_1, X_2) follows bivariate distribution with Pareto conditionals given in Arnold (1987) with the joint pdf

$$f\,(x_1, x_2) = K_1(1 + a_1x_1 + a_2x_2 + bx_1x_2)^{-m},\ K_1, a_1, a_2, b > 0, m > 2,\ x_1, x_2 > 0 \tag{4.24}$$

Proof. Suppose that (4.23) holds true. Then, $\phi_{r+1}(t_i|t_j) = \phi_r(t_i|t_j)(At_i + B_i(t_j))$ differentiating both sides of with respect to t_i, r times and on simplification, we obtain, $h_i(t_i|t_j) = \frac{At_i+B_i(t_j)}{C}$, where, $C = (1 + A)\ (r + 1)$. The remaining part of the proof is similar to the proof of Theorem 4.1 in Sunoj and Linu (2012). Using the relation $h_i(t_i|t_j) = \frac{f_i(t_i|t_j)}{S_i(t_i|t_j)} = \frac{-f(t_1,t_2)}{\frac{\partial}{\partial t_j}R(t_1,t_2)}$, we have

$$f\,(t_1, t_2) = m_i(t_j)[At_i + B_i(t_j)]^{\frac{-(C+A)}{A}},\ \ C \neq 0 \tag{4.25}$$

Proceeding in the same way as in the proof of Theorem 4.1 given in Sunoj and Linu (2012) we finally get,

$$f\,(t_1, t_2) = m_2(0)[B_2(0)]^{\frac{-(C+A)}{A}} \left[1 + \frac{At_1}{B_1(0)} + \frac{At_2}{B_2(0)} + \delta At_1t_2\right]^{\frac{-(C+A)}{A}}$$

where, δ = a constant, which is in the form of bivariate Pareto given in (4.24), with $K_1 = m_2(0)[B_2(0)]^{\frac{-(C+A)}{A}}$, $a_1 = \frac{A}{B_1(0)}$, $a_2 = \frac{A}{B_2(0)}$, $b = \delta A$ and $m = \frac{A+C}{A}$.

To prove the converse part, using (4.1) the conditional partial moment $\phi_r(t_i|t_j)$ for the density (4.24), will be

$$\phi_r(t_i|t_j) = \frac{(m-1)\left(1+a_it_i+a_jt_j+bt_it_j\right)^{r-m+1}}{\left(1+a_jt_j\right)^{1-m}\left(a_i+bt_j\right)^r} B\left(r+1, m-r-1\right)$$

So that $\frac{\phi_{r+1}(t_i|t_j)}{\phi_r(t_i|t_j)} = \frac{r+1}{m-r-2}\left[t_i + \frac{1+a_jt_j}{a_i+bt_j}\right]$, which is of the form (4.23). Hence the proof. □

Theorem 4.3.4. *For $i, j = 1, 2$ and $i \neq j$, the ratio of two consecutive conditional upper partial moments satisfies the relationship,*

$$\frac{\phi_{r+1}(t_i|t_j)}{\phi_r(t_i|t_j)} = \frac{r+1}{A_i + Bt_j}; A_i, B > 0 \tag{4.26}$$

if and only if (X_1, X_2) follows bivariate distribution with exponential conditionals given in Arnold and Strauss (1988) with the joint pdf

$$f\left(x_1, x_2\right) = K_2 \exp\left(-\lambda_1 x_1 - \lambda_2 x_2 - \theta x_1 x_2\right), \, K_2, \lambda_1, \lambda_2, \theta, \, x_1, x_2 > 0. \tag{4.27}$$

Proof. Suppose the relationship (4.26) holds true. Then we have,

$$\phi_{r+1}(t_i|t_j) = \frac{r+1}{A_i + Bt_j}\phi_r(t_i|t_j) \tag{4.28}$$

Differentiating both sides of (4.28) with respect to t_i, $r+1$ times and rearranging we have

$$h_i\left(t_i|t_j\right) = A_i + Bt_j \tag{4.29}$$

We have the relation,

$$h_i(t_i|t_j) = \frac{f_i(t_i|t_j)}{S_i(t_i|t_j)} = \frac{-f\left(t_1, t_2\right)}{\frac{\partial}{\partial t_j} R\left(t_1, t_2\right)} \tag{4.30}$$

From (4.29) and (4.30), we get

$$\frac{-\partial R(t_1,t_2)}{\partial t_j}(A_i+Bt_j)=f(t_1,t_2)$$

Differentiating both sides with respect to t_i, we have

$$(A_i+Bt_j)=\frac{-\partial \log f(t_1,t_2)}{\partial t_i}$$

Integrating both sides with respect to t_i yield

$$\log f(t_1,t_2)=-(A_i+Bt_j)\,t_i+m_i(t_j)\,.$$

Equivalently

$$f(t_1,t_2)=e^{-(A_i+Bt_j)t_i+m_i(t_j)}. \tag{4.31}$$

Equating for $i=1, j=2$ and $i=2, j=1$ in (4.31), we obtain

$$e^{-(A_1+Bt_2)t_1+m_1(t_2)}=e^{-(A_2+Bt_1)t_2+m_2(t_1)}. \tag{4.32}$$

As $t_1\to 0$ in (4.32), we have

$$e^{m_1(t_2)}=e^{-A_2t_2+m_2(0)}. \tag{4.33}$$

Similarly as $t_2\to 0$ in (4.32), we get

$$e^{m_2(t_1)}=e^{-A_1t_1+m_1(0)} \tag{4.34}$$

Using the expression of $e^{m_1(t_2)}$, the joint pdf $f(t_1,t_2)$ in equation (4.31) for $i=1$ becomes

$$f(t_1,t_2)=e^{-(A_1+Bt_2)t_1}e^{-A_2t_2+m_2(0)} \tag{4.35}$$

Hence, the joint pdf $f(t_1, t_2)$ in (4.35) becomes

$$f\left(t_1, t_2\right) = e^{m_2(0)} e^{-(A_1 t_1 + A_2 t_2 + B t_1 t_2)}, \tag{4.36}$$

which is in the form of equation (4.27) with $K_2 = e^{m_2(0)}, \lambda_1 = A_1, \lambda_2 = A_2$ and $\theta = B$.

To prove the converse part, for the density (4.27) the conditional partial moment will be of the form

$$\phi_r(t_i|t_j) = \frac{\Gamma\left(r+1\right)}{\left(1+\theta t_j\right)^r} e^{-t_i(1+\theta t_j)}$$

and

$$\frac{\phi_{r+1}(t_i|t_j)}{\phi_r(t_i|t_j)} = \frac{r+1}{\lambda_i + \theta t_j}, i, j = 1, 2, i \neq j.$$

Hence the proof. □

Theorem 4.3.5. *For $i, j = 1, 2$ and $i \neq j$, the relationship*

$$\psi_1(t_i|t_j)\ \psi_1(s_i|t_j) = \psi_1(t_i + s_i|t_j) \tag{4.37}$$

holds true if and only if (X_1, X_2) follows (4.16) *with $\lambda_i = 1, i = 1, 2$.*

Proof. Suppose (4.37) holds. Then, for $i, j = 1, 2$ and $i \neq j$ we have

$$\int_{t_i}^{\infty} R_i(x_i|t_j) dx_i \int_{s_i}^{\infty} R_i(x_i|t_j) dx_i = \int_{t_i+s_i}^{\infty} R_i(x_i|t_j) dx_i.$$

Differentiating both sides with respect to s_i , we get

$$\int_{t_i}^{\infty} R_i(x_i|t_j) dx_i\ R_i(s_i|t_j) = R_i(s_i + t_i|t_j).$$

Differentiating both sides with respect to t_i , we obtain

$$R_i\,(t_i|t_j)\;R_i\,(s_i|t_j) = -\frac{\partial}{\partial t_i}\,(R_i(t_i+s_i|t_j))\,.$$

$$\Rightarrow R_i\,(t_i|t_j)\;\;R_i\,(s_i|t_j) = f_i^*(t_i+s_i|t_j).$$

Substituting $s_i = 0$ gives, $h_i^*\,(t_i,t_j) = 1$. Now using (4.21) and (4.22), reduces to

$$R\,(x_1,x_2) = \exp\left[-(x_1+x_2)\right],$$

which proves the theorem. Since for the density (4.16) with $\lambda_i = 1, i = 1,2$, the first conditional partial moment is of the form $\psi_1(t_i|t_j) = e^{-t_i}$, for $i,j = 1,2$ and $i \neq j$, the proof of converse part is straightforward. □

In the following theorem we characterize bivariate exponential Gumbel Type I given in (4.9) using the ratio of two consecutive conditional upper partial moments.

Theorem 4.3.6. *The ratio of two consecutive conditional upper partial moments of the second type satisfies the condition*

$$\frac{\psi_{r+1}(t_i|t_j)}{\psi_r(t_i|t_j)} = \frac{r+1}{1+Bt_j}, i,j = 1,2; i \neq j,\; -1 \leq B \leq 1, \tag{4.38}$$

if and only if (X_1, X_2) *follows bivariate exponential Gumbel Type I in* (4.9).

Proof. From (4.38), we have, $\psi_{r+1}(t_i|t_j) = \frac{r+1}{1+Bt_j}\psi_r(t_i|t_j)$. Differentiating both sides of equation with respect to t_i, $r+1$ times and on simplification, we have, $h_i{}^*\,(t_i|t_j) = 1+Bt_j$. Now using the formula for vector valued failure rate by Johnson and Kotz (1975) defined in (4.21) and (4.22) we have,

$$R(x_1,x_2) = exp\left\{-\left(x_1+x_2+Bx_1x_2\right)\right\},$$

which is in the form of the joint reliability function of the pdf (4.9) with $\theta = B$. Now using the expression of $\psi_r(t_i|t_j)$ given in the section 4.2.3.2, the proof of the converse part is straightforward. □

Theorem 4.3.7. *For $i, j = 1, 2$ and $i \neq j$, the ratio of two consecutive conditional upper partial moments satisfies the relationship*

$$\frac{\psi_{r+1}(t_i|t_j)}{\psi_r(t_i|t_j)} = K\left(t_i + B_i(t_j)\right);\ K > 0 \tag{4.39}$$

if and only if (X_1, X_2) follows bivariate distribution with Pareto conditionals with the joint survival function given by

$$R\left(x_1, x_2\right) = \left(1 + a_1 x_1 + a_2 x_2 + b x_1 x_2\right)^{-m}; a_1, a_2, m, x_1, x_2 > 0; 0 < b \leq \left(m+1\right) a_1 a_2. \tag{4.40}$$

Proof. Suppose (4.39) holds. Then, $\psi_{r+1}(t_i|t_j) = K\left(t_i + B_i(t_j)\right)\psi_r(t_i|t_j)$. Differentiating both sides $r+1$ times with respect to t_i and on some simplification, we obtain

$$h_i^*(t_1, t_2) = \frac{(r+1)\,(K+1)}{K\left(t_i + B_i(t_j)\right)}. \tag{4.41}$$

Using (4.21) and (4.22) we get

$$R(x_1, x_2) = \left\{\frac{x_1 + B_1\left(0\right)}{B_1\left(0\right)}\ \frac{x_2 + B_2\left(x_1\right)}{B_2\left(x_1\right)}\right\}^{-c} \tag{4.42}$$

and

$$R(x_1, x_2) = \left\{\frac{x_1 + B_1\left(x_2\right)}{B_1\left(x_2\right)}\ \frac{x_2 + B_2\left(0\right)}{B_2\left(0\right)}\right\}^{-c}, \tag{4.43}$$

where $c = \frac{(1+K)(r+1)}{K}$. Now the characterization follows from Roy (1989).

Conversely, for the density given in (4.40), using (4.8) we have,

$$\psi_r(t_i|t_j) = \frac{\Gamma\left(r+1\right)\ \Gamma\left(m-r\right)}{\Gamma\left(m+1\right)}\frac{m(1+a_j t_j)^m}{\left(a_i + b t_j\right)^r}\left(1 + a_i t_i + a_j t_j + b t_i t_j\right)^{r-m}$$

and $\frac{\psi_{r+1}(t_i|t_j)}{\psi_r(t_i|t_j)} = \frac{r+1}{m-r-1}\left[t_i + \frac{1+a_j t_j}{a_i+bt_j}\right]$, which is of the form (4.39). Hence the proof.

□

4.4 Applications of conditional upper partial moments in income studies

The measurement and comparison of income among individuals in a society is a problem that has been attracting the interest of a lot of researchers in economics and statistics. The conditional partial moment is a useful tool to find some indices in income and poverty studies. For instance, consider a random vector $\boldsymbol{X} = (X_1, X_2)$, where X_1 denotes the household income, X_2 the wealth of family and t_1 denotes the level of income above which the family is considered affluent. Then the conditional partial moment represent the average residual income beyond the affluence level. One can also consider similar other situations like, let $\boldsymbol{X} = (X_1, X_2)$ represent two attributes of income, say income from the land and income from the employment etc. Also, important inequality measures used in income studies can be expressed in terms of the conditional upper partial moments. For example, income gap ratio $\beta(t)$ is a popular affluence index used in income studies to measure the intricate relationship of poverty and affluence in a population. The monotonic behavior of the income gap ratio can be used to find a suitable choice of model of income (see Abdul-Sathar et al. (2007) and Belzunce et al. (1998) etc). Then the income gap ratio for the truncated random variable $X_i|X_j > t_j; i,j = 1,2; i \neq j$ defined by Abdul-Sathar et al. (2007) as

$$\beta_i\left(t_i, t_j\right) = 1 - \frac{t_i}{v_i\left(t_i, t_j\right)}, \tag{4.44}$$

where $v_i(t_1, t_2) = E(X_i|X_1 > t_1, X_2 > t_2)$ is the i^{th} component of the bivariate vitality function defined in Sankaran and Nair (1991). In terms of the conditional upper partial moments, we have the relationship

$$v_i(t_i, t_j) = t_i + \frac{\psi_1(t_i|t_j)}{\psi_0(t_i|t_j)}. \tag{4.45}$$

Combining (4.44) and (4.45), we obtain

$$\beta_i(t_i, t_j) = \frac{\psi_1(t_i|t_j)}{t_i\psi_0(t_i|t_j) + \psi_1(t_i|t_j)}. \tag{4.46}$$

Further, when the two events $(X_1 > t_1)$ and $(X_2 > t_2)$ are independent, then $\psi_r(t_i|t_j) = p_r(t_i)$, so that

$$\beta_i(t_i, t_j) = \frac{p_1(t_i)}{t_i\, p_0(t_i) + p_1(t_i)}$$

and

$$\beta_i(t_i, t_j) = \frac{r(t_i)\, R(t_i)}{t_i R(t_i) + r(t_i)\, R(t_i)} = \frac{r(t_i)}{1 + r(t_i)},$$

where $r(t_i)$ and $R(t_i)$ are the univariate mean residual life function and univariate survival function respectively (see Sunoj (2004)).

Theorem 4.4.1. *For $i, j = 1, 2$ and $i \neq j$ the bivariate income gap ratio satisfy the relationship*

$$\frac{1 - \beta_i(t_i, t_j)}{K\beta_i(t_i, t_j) - 1} = t_i\, B_i(t_j)\,, K > 0 \tag{4.47}$$

if and only if (X_1, X_2) belongs to bivariate distribution with Pareto conditionals defined in (4.40).

Proof. Let the the relationship (4.47) holds true. Then using (4.44) and on simplifications, we get

$$\frac{\psi_1(t_i|t_j)}{\psi_0(t_i|t_j)} = \frac{t_i + B_i(t_j)}{K - 1}$$

$$\Rightarrow (K-1)\int_{t_i}^{\infty} R_2\left(u|t_j\right)du = R_2\left(t_i|t_j\right)\left(t_i\, + B_i\left(t_j\right)\right).$$

Differentiating both sides of the above equation with respect to t_i and simplifying we have $h_i^*(t_1,t_2) = \frac{K}{t_i+B_i(t_j)}$. The rest of the proof is similar to the proof of Theorem 4.3.7. Converse part of the proof is direct. □

In the next theorem we characterize the bivariate exponential distribution (Arnold (1995)), with conditional density of the form

$$R_i(x_i|x_j) = \exp\left[-x_i\left(\alpha x_j + \beta\right)\right];\; i,j = 1,2; i \neq j \tag{4.48}$$

using the bivariate income gap ratio $\beta\left(t_1,t_2\right)$ defined in (4.44).

Theorem 4.4.2. *The bivariate income gap ratio satisfy the relationship*

$$\beta_i\left(t_i,t_j\right) = \frac{1}{1+(\alpha\; t_j+\beta)t_i}, i,j = 1,2; i \neq j \tag{4.49}$$

if and only if (X_1,X_2) *belongs to bivariate exponential distribution defined in* (4.48).

Proof. Suppose (4.49) holds. Using (4.44) we get $h_i^*(t_1,t_2) = \alpha t_j + \beta$. Now using $R_i(t_i|t_j) = \exp\left\{-\int_0^{t_i} h_i^*(x_1,x_2)dx_1\right\}$, we have the theorem. Proof of the converse part is direct. □

Corollary 4.4.1. *The Gumbel Type I distribution defined in* (4.9) *is a particular case of the distribution* (4.48) *for the values $\alpha = \theta$ and $\beta = 1$. In this case, the bivariate income gap ratio of the form*

$$\beta_i\left(t_i,t_j\right) = \frac{1}{1+(\theta t_j+1)t_i}, i = 1,2; i \neq j$$

characterizes the Gumbel Type I distribution.

Corollary 4.4.2. *The bivariate income gap ratio is of the form* $\beta_i(t_1,t_2) = \frac{1}{1+t_i}, i = 1,2$ *if and only if* (X_1, X_2) *follows independent exponential given in* (4.16) *with* $\lambda_i = 1, i = 1, 2$.

Remark 4.4.1. The useful measure in income studies is the mean left proportional residual income (MLPRI) due to Belzunce et al. (1998) (see section 2.12.2) can be obtained using the CUPMs, through the relationship

$$(\gamma_1(t_1,t_2), \gamma_2(t_1,t_2)) = \left(1 + \frac{\psi_1(t_1|t_2)}{t_1\psi_0(t_1|t_2)}, 1 + \frac{\psi_1(t_2|t_1)}{t_2\psi_0(t_2|t_1)}\right).$$

Since there exists one to one relationship between MLPRI, bivariate income gap ratio and vitality function, the theorems in Section 4.4 can be useful in deriving new characterizations to MLPRI. Hence separate statements of these results omitted.

4.5 Estimator of the conditional upper partial moments

In this section we propose non-parametric estimator for the conditional partial moment $\psi_r(t_1|t_2)$. Let $(X_i, Y_i),\ i = 1, 2, ...n$ be n independent and identically distributed pairs of lifetimes with survival function $R(t_1, t_2)$. The estimators for $\phi_r(t_1|t_2)$ and $\psi_r(t_1|t_2)$ are respectively defined by

$$\widehat{\phi}_r(t_1|t_2) = \frac{1}{n}\sum_{i=1}^{n}(X_i - t_1)^r I(X_i > t_1, Y_i = t_2) \tag{4.50}$$

and

$$\widehat{\psi}_r(t_1|t_2) = \frac{1}{n}\sum_{i=1}^{n}(X_i - t_1)^r I(X_i > t_1, Y_i > t_2) \tag{4.51}$$

where $I(.)$ is the usual indicator function. The estimator in (4.50) and (4.51) are the r^{th} sample moment of the observations from $X_i > t_1|Y_i = t_2$ and $X_i > t_1|Y_i > t_2$ respectively. Kulkarni and Rattihalli (2002) and Sankaran and Nair (2004) discussed the unbiasedness and consistency properties of (4.51).

4.6 Simulation study and estimation of CUPMs from the real data sets

In this section, we present the results of a simulation study and illustrate the properties of empirical estimator $\widehat{\psi}_r(t_1|t_2)$ by analyzing two real data sets, (i) recurrence time of bladder tumor given in Kulkarni and Rattihalli (2002) and (ii) personal wealth data obtained from the website of Internal Revenue Service (IRS), United States respectively.

4.6.1 Simulation study

To study the performance of the estimators $\widehat{\phi}_r(t_1|t_2)$ and $\widehat{\psi}_r(t_1|t_2)$, we carried out a series of 1000 simulations each of size n ($n = 10, 100$ and 1000) from (i) a bivariate distribution with Pareto conditionals (4.24) given in Arnold (1987) with the joint pdf

$$f(t_1, t_2) = K_1(1 + a_1t_1 + a_2t_2 + bt_1t_2)^{-m}, \; K_1, a_1, a_2, b > 0, \; m > 2, \; t_1, t_2 > 0$$

for $a_1 = a_2 = b = 1$ and $m = 8$ and (ii) from a bivariate Gumbel's exponential distribution (4.9) with joint CDF

$$F(t_1, t_2) = 1 - e^{-t_1} - e^{-t_2} + e^{-t_1-t_2-\theta t_1 t_2}, t_1, t_2 > 0, 0 \leq \theta \leq 1$$

for $\theta = 0.5$. The corresponding CUPMs for the above distributions will be

$$\phi_r(t_1|t_2) = \frac{(m-1)\,(1+a_1t_1+a_2t_2+bt_1t_2)^{r-m+1}}{(1+a_2t_2)^{1-m}(a_1+bt_2)^r} B\,(r+1, m-r-1)$$

and

$$\psi_r(t_1|t_2) = \frac{\Gamma\,(r+1)}{(1+\theta t_2)^r} e^{-t_1(1+\theta t_2)}.$$

The performance of the empirical estimators $\widehat{\phi}_r(t_1|t_2)$ and $\widehat{\psi}_r(t_1|t_2)$ obtained from simulation study are given in Table 4.1 and Table 4.2. The results of simulation studies shows that bias and MSE of the proposed empirical estimators $\widehat{\phi}_r(t_1|t_2)$ and $\widehat{\psi}_r(t_1|t_2)$, decreases with increasing sample sizes.

4.6.2 Estimation of CUPMs from the real dataset

In this section, we further illustrate the performance of the non-parametric estimator of $\psi_r(t_1|t_2)$ by analyzing a real dataset.

4.6.2.1 Cancer recurrent data

The cancer recurrence data is taken from Kulkarni and Rattihalli (2002). The data consist of observations on patients having bladder tumer when they entered the trial. These tumors were removed and patients were given a treatment. At subsequent follow-up visits any tumors found were removed and the treatment was continued. The variables observed are X = time to first recurrence of tumor (in months) and Y = time to second recurrence of a tumor (in months).

Table 4.3 to Table 4.6 provides the estimates of $\psi_r(t_1|t_2)$ for different values of r. It is easy to see that $\widehat{\psi}_r(t_1|t_2)$ is decreasing in t_1 and t_2. Further, for fixed t_1 and t_2, $\widehat{\psi}_r(t_1|t_2)$ is increasing in r.

Table 4.1: Performance of $\widehat{\phi}_r(t_1|t_2)$ for bivariate Pareto model (4.24) with $a_1 = a_2 = b = 1$ and $m = 8$.

r	t_1	t_2	Bias×10^3			MSE×10^3		
			$n = 10$	$n = 100$	$n = 1000$	$n = 10$	$n = 100$	$n = 1000$
0	0.1	0.1	0.6419	0.6919	0.1239	23.6100	2.8046	0.5083
	0.3	0.3	-7.4663	-0.8763	-0.2313	13.4921	1.2912	0.1566
	0.3	1.5	-2.3663	-1.5963	0.4897	13.1766	1.3571	0.1527
	1.4	1.4	-0.0803	-0.0603	0.0427	0.2056	0.0215	0.0023
1	0.1	0.1	1.2783	0.0622	0.0276	4.1906	0.4299	0.0691
	0.3	0.3	0.5963	0.1932	-0.1348	1.6710	0.1850	-0.1348
	0.3	1.5	-0.7759	0.1633	-0.2439	1.5532	0.1626	0.0174
	1.4	1.4	0.6491	-0.0623	0.0224	0.1547	0.0067	0.0009
2	0.1	0.1	0.9346	-0.1139	-0.1630	5.3176	1.0517	0.0704
	0.3	0.3	-1.3333	-0.2267	0.1245	1.5284	0.2351	0.0321
	0.3	1.5	-1.5812	-0.3714	-0.0978	1.2358	0.2251	0.0262
	1.4	1.4	-0.1725	0.0230	0.0473	0.1121	0.0233	0.0065
3	0.1	0.1	1.9172	-0.3117	0.1160	26.7658	6.9466	0.3874
	0.3	0.3	-0.6577	-1.7043	0.0124	7.1449	1.2766	0.3186
	0.3	1.5	-3.3308	-0.1636	0.1315	1.1613	0.3240	0.0388
	1.4	1.4	-0.4473	0.0189	-0.3207	0.2913	0.2436	0.0450

Table 4.2: Performance of $\widehat{\psi}_r(t_1|t_2)$ for bivariate Gumbel's exponential with $\theta = 0.5$

r	t_1	t_2	Bias			MSE		
			$n = 10$	$n = 100$	$n = 1000$	$n = 10$	$n = 100$	$n = 1000$
0	0.05	0.05	-0.06059	-0.06059	-0.06059	0.003671	0.003671	0.003671
	0.25	0.25	-0.43022	-0.43022	-0.43022	0.185087	0.185087	0.185087
	0.5	0.5	-0.82623	-0.82623	-0.82623	0.68265	0.68265	0.68265
	0.75	0.75	-0.97163	-0.97163	-0.97163	0.944069	0.944069	0.944069
	1	1	-0.99752	-0.99752	-0.99752	0.995049	0.995049	0.995049
1	0.05	0.05	0.166698	0.17755	0.179632	0.051788	0.037243	0.033437
	0.25	0.25	0.159434	0.164419	0.165095	0.028134	0.027722	0.027393
	0.5	0.5	0.040282	0.040409	0.040999	0.001837	0.001688	0.00169
	0.75	0.75	0.005298	0.005261	0.005397	3.88E-05	3.12E-05	2.96E-05
	1	1	0.000407	0.000378	0.000386	1.8E-07	2.98E-07	1.71E-07
2	0.05	0.05	0.272924	0.283445	0.28851	0.307852	0.140006	0.095359
	0.25	0.25	0.141681	0.146492	0.146909	0.026111	0.023259	0.021915
	0.5	0.5	0.022921	0.02288	0.023381	0.000745	0.000579	0.000556
	0.75	0.75	0.002236	0.002151	0.002283	9.76E-06	6.88E-06	5.43E-06
	1	1	0.000137	0.000122	0.000127	1.89E-08	5.78E-08	2.73E-08
3	0.05	0.05	0.710201	0.691505	0.695735	4.488793	1.705568	0.749468
	0.25	0.25	0.193954	0.194968	0.196203	0.062791	0.050543	0.040409
	0.5	0.5	0.019523	0.019578	0.019976	0.000966	0.000489	0.000421
	0.75	0.75	0.001424	0.001329	0.001449	5.65E-06	4.60E-06	2.32E-06
	1	1	6.88E-05	6.09E-05	6.19E-05	4.74E-09	1.87E-08	1.35E-08

Table 4.3: Estimates of the conditional partial moment $\psi_r(t_1|t_2)$ for the cancer recurrence data when $r = 0$.

$t_1 \backslash t_2$	0	2	10	12	14	15	17
0	1.000000	1.000000	0.736842	0.684211	0.526316	0.368421	0.157895
2	0.789474	0.789474	0.631579	0.473684	0.473684	0.263158	0.105263
3	0.473684	0.473684	0.473684	0.421053	0.315789	0.210526	0.105263
5	0.421053	0.421053	0.421053	0.368421	0.315789	0.210526	0.105263
7	0.368421	0.368421	0.368421	0.315789	0.315789	0.210526	0.105263
9	0.263158	0.263158	0.263158	0.263158	0.263158	0.157895	0.105263
10	0.210526	0.210526	0.210526	0.210526	0.210526	0.157895	0.105263

Table 4.4: Estimates of the conditional partial moment $\psi_r(t_1|t_2)$ for the cancer recurrence data when $r = 1$.

$t_1 \backslash t_2$	0	2	10	12	14	15	17
0	7.05263	7.05263	6.10526	5.63158	5.36842	3.78947	2.42105
2	5.05263	5.05263	4.63158	4.26316	4.10526	3.05263	2.10526
3	4.26316	4.26316	4.05263	3.73684	3.63158	2.78947	2
5	3.31579	3.31579	3.21053	3	3	2.36842	1.78947
7	2.47368	2.47368	2.47368	2.36842	2.36842	1.94737	1.57895
9	1.73684	1.73684	1.73684	1.73684	1.73684	1.52632	1.36842
10	1.47368	1.47368	1.47368	1.47368	1.47368	1.36842	1.26316

4.7 Application of CUPM in data modelling

Pareto distributions have been extensively employed for modelling and analysis of statistical data under different contexts. Originally, the distribution was first proposed as a model to explain the allocation of income among individuals. Later, various forms of the Pareto distribution have been formulated for modelling and analysis of data from engineering, environment, geology, hydrology etc. These diverse applications of the Pareto distributions lead researchers to develop different kinds of bivariate (multivariate) Pareto distributions. Accordingly, Mardia (1962)

Table 4.5: Estimates of the conditional partial moment $\psi_r(t_1|t_2)$ for the cancer recurrence data when $r = 2$.

$t_1 \backslash t_2$	0	2	10	12	14	15	17
0	91.2632	91.2632	87.0526	82.7895	81.4737	67.4737	54.9474
2	67.0526	67.0526	65.5789	63	62.5263	53.7895	45.8947
3	57.7368	57.7368	56.8947	55	54.7895	47.9474	41.7895
5	42.5789	42.5789	42.3684	41.5263	41.5263	37.6316	34.2105
7	31	31	31	30.7895	30.7895	29	27.4737
9	22.5789	22.5789	22.5789	22.5789	22.5789	22.0526	21.5789
10	19.3684	19.3684	19.3684	19.3684	19.3684	19.1579	18.9474

Table 4.6: Estimates of the conditional partial moment $\psi_r(t_1|t_2)$ for the cancer recurrence data when $r = 3$.

$t_1 \backslash t_2$	0	2	10	12	14	15	17
0	1717.05	1717.05	1694.32	1655.95	1649.37	1502.53	1371.37
2	1246.11	1246.11	1239.37	1221.32	1219.89	1140.21	1069.47
3	1059.32	1059.32	1055.95	1044.58	1044.16	987.737	938
5	760.263	760.263	759.842	756.474	756.474	731.842	710.421
7	541.211	541.211	541.211	540.789	540.789	532.789	525.789
9	381.947	381.947	381.947	381.947	381.947	380.474	379.053
10	319.158	319.158	319.158	319.158	319.158	318.737	318.316

introduced two types of bivariate (multivariate) Pareto models which are referred as bivariate Pareto distributions of first kind and second kind respectively. Since then there has been a lot of works in the form, alternative derivation of bivariate Pareto models, their extensions, inference, characterizations and applications to a variety of fields. Various types of bivariate(multivariate) Pareto distributions discussed and studied in literature include those of Lindley and Singpurwalla (1986), Arnold (1990), Sankaran and Nair (1993a), Balakrishnan and Lai (2009), Sankaran and Kundu (2014) and Arnold (2015).

In this section we illustrate the use of characterization theorems in ascertaining whether a given set of observations follow a specified distribution. In Theorem 4.3.7, we have proved that the ratios of the two consecutive CUPMs of the second type characterizes the bivariate Pareto model (SNBP model) introduced by Sankaran and Nair (1993a) of the form

$$R\left(t_1, t_2\right) = \left(1 + a_1 t_1 + a_2 t_2 + b t_1 t_2\right)^{-m},$$

where $a_1, a_2, m, t_1, t_2 > 0, 0 < b \leq (m+1)\, a_1 a_2$ through the relationships

$$\frac{\psi_{r+1}(t_i|t_j)}{\psi_r(t_i|t_j)} = \frac{r+1}{m-r-1}\left[t_i + \frac{1 + a_j t_j}{a_i + b t_j}\right],\ i = 1, 2, i \neq j. \tag{4.52}$$

where $\psi_r\left(t_i|t_j\right) = \psi_{X_i|X_j>t_j}\left(t_i|t_j\right), i, j = 1, 2, i \neq j$ is the r^{th} partial moment defined for the conditional survival models $X_i|X_j > t_j\ i = 1, 2, i \neq j$. In particular, for X_1 and X_2 are positive with $r \geq 1$, the quantity

$$\frac{\psi_{r+1}(t_i|t_j)}{\psi_r(t_i|t_j)}\left[\frac{r+1}{m-r-1}\left[t_i + \frac{1 + a_j t_j}{a_i + b t_j}\right]\right]^{-1}\ i = 1, 2, i \neq j \tag{4.53}$$

equal to unity ensures (X_1, X_2) follows bivariate Pareto model. Thus for a given random sample, if the estimates of (4.53) for all t_1, t_2 has only small fluctuations around unity it is reasonable to expect the SNBP model.

In order to illustrate the theorem, we proceed as follows. We estimates the values of the ratio by using

$$\frac{\hat{\psi}_{r+1}(t_1|t_2)}{\hat{\psi}_r(t_1|t_2)}\left[\frac{r+1}{m-r-1}\left[t_1 + \frac{1 + \hat{a}_2 t_2}{\hat{a}_1 + \hat{b} t_2}\right]\right]^{-1} \tag{4.54}$$

for admissible values of (t_1, t_2). If the ratios computed from the above estimate are very much closed to unity, then one can conclude that the observations can be modeled by a SNBP model. Now we illustrate the use of Theorem 4.3.7 by using the stiffness data set (Table 4.7) given in Wichern and Johnson (1992) that represents two different measurements of stiffness viz., 'shock' and 'vibration' of 30 boards. The first measurement (shock) involves sending a shock wave down the board and the second measurement (vibration) is determined while vibrating the board. Recently, Sankaran and Kundu (2014) used this data set to explain the SNBP model and observed that unlike SNBP model, the independent Pareto marginals provides better fit. Hence, in order to estimate the quantity in the RHS of (4.52) we also use the maximum likelihood estimates of the parameters obtained by Sankaran and Kundu (2014) for the SNBP model.

The estimates of various functions are presented in Table 4.8 through Table 4.9 for admissible values of (t_1, t_2). Finally, the ratios 4.54 computed from the above estimates given in Table 4.10 shows that all the values depart substantially from unity. Thus we conclude that the observations do not support their distribution as SNBP model, which reaffirm the conclusion made by Sankaran and Kundu (2014).

Table 4.7: Stiffness data

No.	Shock	Vibration	No.	Shock	Vibration	No.	Shock	Vibration
1	1889	1651	2	2403	2048	3	2119	1700
4	1645	1627	5	1976	1916	6	1712	1713
7	1943	1685	8	2104	1820	9	2983	2794
10	1745	1600	11	1710	1591	12	2046	1907
13	1840	1841	14	1867	1685	15	1859	1649
16	1954	2149	17	1325	1170	18	1419	1371
19	1828	1634	20	1725	1594	21	2276	2189
22	1899	1614	23	1633	1513	24	2061	1867
25	1856	1493	26	1727	1412	27	2168	1896
28	1655	1675	29	2326	2301	30	1490	1382

Table 4.8: Estimates of the $\frac{\psi_{r+1}(t_1|t_2)}{\psi_r(t_1|t_2)}$ for the stiffness data when $r = 1$

$t_2 \backslash t_1$	1300	1400	1600	1700	1800	1900	2000	2100
1300	775.58	703.98	590.14	563.19	544.73	552.95	551.16	570.91
1400	786.19	708.49	590.14	563.19	544.73	552.95	551.16	570.91
1500	802.87	725.19	605.56	574.28	550.05	552.95	551.16	570.91
1600	844.66	763.55	627.92	580.49	550.05	552.95	551.16	570.91
1700	969.45	884.75	731.91	671.07	617.73	581.78	572.89	577.22
1800	992.79	905.20	742.53	672.53	617.73	581.78	572.89	577.22
1900	1092.01	1004.66	841.13	768.79	707.96	667.70	645.66	600.47
2100	1213.52	1126.60	962.75	888.74	823.62	772.90	724.76	670.62

Table 4.9: Values of $\frac{r+1}{\hat{m}-r-1}\left[t_1+\frac{1+\hat{a}_2 t_2}{\hat{a}_1+\hat{b}t_2}\right]$ for $r=1, \hat{a}_1=0.0321, \hat{a}_2=0.0292, \hat{b}=0.0154$ and $\hat{m}=18.3438$.

$t_2 \backslash t_1$	1300	1400	1600	1700	1800	1900	2000	2100
1300	159.32	171.56	196.03	208.27	220.51	232.74	244.98	257.22
1400	159.32	171.56	196.03	208.27	220.50	232.74	244.98	257.22
1500	159.32	171.56	196.03	208.27	220.50	232.74	244.98	257.22
1600	159.32	171.56	196.03	208.27	220.50	232.74	244.98	257.22
1700	159.32	171.56	196.03	208.27	220.50	232.74	244.98	257.22
1800	159.32	171.56	196.03	208.27	220.50	232.74	244.98	257.21
1900	159.32	171.56	196.03	208.27	220.50	232.74	244.98	257.21
2100	159.32	171.55	196.03	208.27	220.50	232.74	244.98	257.21

Table 4.10: Estimated values of $\frac{\hat{\psi}_{r+1}(t_1|t_2)}{\hat{\psi}_r(t_1|t_2)}\left[\frac{r+1}{m-r-1}\left[t_1+\frac{1+\hat{a}_2 t_2}{\hat{a}_1+\hat{b}t_2}\right]\right]^{-1}$ for the stiffness data when $r=1$.

$t_2 \backslash t_1$	1300	1400	1600	1700	1800	1900	2000	2100
1300	4.868	4.103	3.010	2.704	2.470	2.376	2.250	2.220
1400	4.935	4.130	3.010	2.704	2.470	2.376	2.250	2.220
1500	5.039	4.227	3.089	2.757	2.495	2.376	2.250	2.220
1600	5.302	4.451	3.203	2.787	2.495	2.376	2.250	2.220
1700	6.085	5.157	3.734	3.222	2.801	2.500	2.339	2.244
1800	6.232	5.276	3.788	3.229	2.801	2.500	2.339	2.244
1900	6.854	5.856	4.291	3.691	3.211	2.869	2.636	2.335
2100	7.617	6.567	4.911	4.267	3.735	3.321	2.958	2.607

Chapter 5

On conditional lower partial moments and its applications*

5.1 Introduction

As explained in the previous chapter, there are many practical situations where the use of conditional distributions is more available than the joint distribution. It is well known that, the marginal densities cannot determine the joint density uniquely unless the variables are independent. Similar to the conditional survival models discussed in Chapter 4, the determination of the joint distribution of $\boldsymbol{X} = (X_1, X_2)$, when conditional distributions of $(X_1|X_2 \leq t_2)$ and $(X_2|X_1 \leq t_1)$ are known, has also been an important problem studied by many (see Arnold et al. (1999), Ghosh and Kundu (2018) etc).

In the present chapter, we introduce and study Conditional Lower Partial Moments (CLPMs) and study it in the context of reliability modeling, risk analysis

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and income (poverty) studies. The organization of the present chapter is as follows. In Section 5.2, we introduce CLPM. In Section 5.3, characterization results based on CLPM are proved. Section 5.4 study some applications of CLPMs in risk analysis and income studies and obtained certain partial ordering based on it. We propose a nonparametric estimator for the partial moment in Section 5.5. Section 5.6 presents the results of a simulation study along with an application to a real data set.

5.2 Conditional lower partial moments

In this section, we define lower partial moments of conditional distributions and study its important properties.

Definition 5.2.1. Let $\boldsymbol{X} = (X_1, X_2)$ be a bivariate random vector admitting an absolutely continuous distribution function F with respect to a Lebesgue measure in the two dimensional Euclidean space $\mathbb{R}_2$. Let $F_i(t_i|t_j) = P(X_i \leq t_i|X_j \leq t_j)$ denote the conditional distribution function of X_i given $X_j \leq t_j$, $i, j = 1, 2; i \neq j$. Then the Conditional Lower Partial Moment (CLPM) is defined as

$$\begin{aligned} \delta_r(t_i|X_j \leq t_j) &= E\left[(X_i - t_i)_-^r|X_j \leq t_j\right], r = 0, 1, 2, ... \\ &= \int_{-\infty}^{t_i} (t_i - x_i)^r f_i(x_i|t_j) dx_i, \end{aligned} \tag{5.1}$$

where $(X_i - t_i)_-$ is defined as in (2.7) and $f_i(.|t_j)$ is the conditional pdf corresponding to $F_i(.|t_j)$.

Further, $\delta_r(t_i|X_j \leq t_j), i = 1, 2, i \neq j$, possess the following properties,

(i) $\delta_0(t_i|X_j \leq t_j) = F_i\,(t_i|t_j)$.

(ii) for a nonnegative random vector (X_1, X_2) and for fixed t_j, both $\delta_r(t_i|X_j \leq t_j)$

are non-decreasing and continuous in $t_i \geq 0,\ i,j = 1,2, i \neq j$ for all values of r.

(iii) as $t_i \to \infty$, $\delta_r(t_i|X_j \leq t_j) \to \infty$ and $\delta_0(t_i|X_j \leq t_j) \to 1,\ i,j = 1,2, i \neq j$.

Applying integration by parts in (5.1) yields,

$$\delta_r(t_i|X_j \leq t_j) = r \int_{-\infty}^{t_i} (t_i - x_i)^{r-1} F_i(x_i|t_j) dx_i,\ i,j = 1,2, i \neq j. \tag{5.2}$$

Theorem 5.2.1. *For any positive integer r, $\delta_r(t_i|X_j \leq t_j), i,j = 1,2$; $i \neq j$ determines the corresponding conditional distribution uniquely through the relationship*

$$r!\ F_i\left(t_i|t_j\right) = \frac{\partial^r}{\partial t_i^r}\left(\delta_r(t_i|X_j \leq t_j)\right). \tag{5.3}$$

Proof. Differentiating both sides of (5.1) successively r times with respect to t_i gives (5.3). □

Assume $\boldsymbol{X} = (X_1, X_2)$ is a nonnegative random vector with an absolutely joint distribution function $F(x_1, x_2)$ and marginal distribution functions $F_i(x_i), i = 1,2$. Let $a = (a_1, a_2)$ and $b = (b_1, b_2)$ be two vectors of real numbers satisfying $a_i = \inf\{x|F_i(x) > 0\}, i = 1,2$ and $b_i = \sup(x|F_i(x) < 1), i = 1,2$. The variables X_1 and X_2 are thought of as lifetimes of married couples, failure times of two component system, etc. Then two important functions of common use in reversed time (inactivity time) are (i) Bivariate reversed hazard rate (BRHR) due to Roy (2002a) given in section 2.3.2 and (ii) Bivariate reversed mean residual life (BRMRL) defined in section 2.3.5 proposed by Nair and Asha (2008).

From (5.2), it an be easily seen that $\bar{h}_i(t_i, t_j) = \frac{\partial}{\partial t_i} \ln\left(\delta_1^{'}(t_i|X_j \leq t_j)\right)$, where $\delta_1^{'}(t_i|X_j \leq t_j) = \frac{\partial}{\partial t_i}\delta_1(t_i|X_j \leq t_j)$. Also, $\bar{m}_i(t_i, t_j) = \frac{\delta_1(t_i|X_j \leq t_j)}{\delta_0(t_i|X_j \leq t_j)}$, or equivalently $\bar{m}_i(t_i, t_j) = \left(\frac{\partial}{\partial t_i} \ln \delta_1(t_i|X_j \leq t_j)\right)^{-1}$. Thus the important concepts and results defined in past lifetime study using BRHR and BRMRL can be restated in terms of the CLPMs.

5.2.1 Examples: Conditional lower partial moments

Now we provide examples of CLPM for some popular bivariate distributions.

Example 5.2.1. (Normal conditionals): Suppose that the joint density of (X_1, X_2) is bivariate normal with joint density

$$\begin{aligned} f(x_1, x_2) &= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{\frac{-1}{2(1-\rho^2)}\left(\left(\frac{x_1-\mu_1}{\sigma_1}\right)^2 + \left(\frac{x_2-\mu_2}{\sigma_2}\right)^2\right.\right. \\ &\left.\left. -2\rho\left(\frac{x_1-\mu_1}{\sigma_1}\right)\left(\frac{x_2-\mu_2}{\sigma_2}\right)\right)\right\}, -\infty < x_i < \infty, \end{aligned}$$

where $\mu_i \in R, \sigma_i > 0$, and $-1 \leq \rho \leq 1$. Then we have,

$$f_1(x_1|x_2) = \frac{1}{\sqrt{2\pi}\sigma_1\sqrt{1-\rho^2}}\left(-\frac{\left(x_1 - \left(\mu_1 + \sigma_1\rho\left(\frac{x_2-\mu_2}{\sigma_2}\right)\right)\right)^2}{2\sigma_1^2(1-\rho^2)}\right)$$

and

$$\begin{aligned} \delta_r(t_1|X_2 \leq t_2) &= \frac{1}{2}K_1K_3^{r/2}\left(\sqrt{K_3}\Gamma\left(\frac{r+1}{2}\right){}_1F_1\left(-\frac{r}{2};\frac{1}{2};-\frac{K_2^2}{K_3}\right)\right. \\ &\left. +K_2r\Gamma\left(\frac{r}{2}\right){}_1F_1\left(\frac{1-r}{2};\frac{3}{2};-\frac{K_2^2}{K_3}\right)\right) \end{aligned}$$

where, $K_1 = \frac{1}{\sqrt{2\pi}\sigma_1\sqrt{1-\rho^2}}$, $K_2 = t_1 - \left(\mu_1 + \frac{\sigma_1}{\sigma_2}\rho(t_2-\mu_2)\right)$, $K_3 = 2\sigma_1^2(1-\rho^2)$ and ${}_1F_1(a;b;z)$ denotes the Kummer confluent hypergeometric function defined by ${}_1F_1(a;b;z) = \sum\limits_{k=0}^{\infty}\frac{(a)_k}{(b)_k}\frac{z^k}{k!}$, where $(a)_k$ represent the Pochhammer symbol defined by $(a)_k = a(a-1)\dots(a-k+1)$. The case of $\delta_r(t_2|X_1 \leq t_1)$ is similar.

Example 5.2.2. (Pareto conditionals): Suppose that the joint density of (X_1, X_2)

follows bivariate Pareto with joint density

$$f\left(x_1,x_2\right)=\frac{\alpha\left(\alpha+1\right)}{\sigma_1\sigma_2}\left(1+\frac{x_1}{\sigma_1}+\frac{x_2}{\sigma_2}\right)^{-(\alpha+2)},x_i,\sigma_i,\alpha\geq 0,i=1,2.$$

and

$$f_1\left(x_1|x_2\right)=\frac{\alpha}{\sigma_1\left(1+\frac{x_2}{\sigma_2}\right)}\left(1+\frac{x_1}{\sigma_1\left(1+\frac{x_2}{\sigma_2}\right)}\right)^{-(\alpha+2)}.$$

Then,

$$\delta_r(t_1|X_2\leq t_2)=\frac{\alpha t_1^{r+1}{}_2F_1\left(1,\alpha+2;r+2;-\frac{t_1}{\sigma_1\left(1+\frac{t_2}{\sigma_2}\right)}\right)}{\left(r+1\right)\sigma_1\left(1+\frac{t_2}{\sigma_2}\right)},$$

where ${}_2F_1(a,b;c;z)$ denotes the Gauss hypergeometric function defined by,

$${}_2F_1(a,b;c;z)=\sum_{k=0}^{\infty}\frac{\left(a\right)_k\left(b\right)z^k}{\left(b\right)_k k!},$$

with $\left(a\right)_k$ denote the Pochhammer symbol.

Example 5.2.3. (Centered Normal conditionals distribution): Suppose that the joint density of (X_1,X_2) is $f\left(x_1,x_2\right)=Ke^{-\left(x_1^2+x_2^2+cx_1^2x_2^2\right)};c>0$, where K is a normalizing constant. Then $f_1\left(x_1|x_2\right)=\frac{\left(1+cx_2^2\right)^{1/2}}{\sqrt{2\pi}}e^{-\left(1+cx_2^2\right)x_1^2}$, and consequently,

$$\delta_r(t_1|X_2\leq t_2)=\frac{2^{-r-\frac{3}{2}}\left(-\mathrm{ct}_2^2-1\right)^{-r}e^{-2\left(\mathrm{ct}_2^2+1\right)t_1}\left(\Gamma\left(r+1,-2t_1\left(\mathrm{ct}_2^2+1\right)\right)-\Gamma(r+1)\right)}{\sqrt{\pi}\sqrt{\mathrm{ct}_2^2+1}},$$

where, $\Gamma(a,z)$ is the incomplete gamma function defined by $\Gamma(a,z)=\int_z^{\infty}t^{a-1}e^{-t}dt$.

Example 5.2.4. (Exponential conditionals): Consider the bivariate exponential conditionals distribution (Arnold and Strauss (1988)), with the joint density function,

$$f\left(x_1,x_2\right)=Ce^{-(x_1+x_2+ax_1x_2)};\ x_1,x_2,a,C>0,$$

such that

$$f_1\left(x_1|x_2\right) = \left(1 + ax_2\right) e^{-(1+ax_2)x_1}.$$

Then,

$$\delta_r(t_1|X_2 \leq t_2) = e^{-t_1(at_2+1)}(-at_2-1)^{-r}\left(\Gamma\left(r+1, -t_1\left(at_2+1\right)\right) - \Gamma(r+1)\right)$$

Example 5.2.5. (Ali-Mikhail-Haq family of bivariate distributions, (Ali et al. (1978)):

$$F\left(x_1, x_2\right) = \frac{x_1}{(1-\alpha(1-x_1)(1-x_2))}, 0 \leq x_1, x_2 \leq 1, -1 \leq \alpha \leq 1.$$

Consequently, $\delta_r(t_1|X_2 \leq t_2) = \frac{t^{r+1}\left(r\Gamma(r+1)_2F_1\left(1,1;r+2;\frac{t_1(t_2-1)\alpha}{(t_2-1)\alpha+1}\right)-1\right)}{\alpha(t_1-1)(t_2-1)-1}$, where ${}_2F_1(a,b;c;z)$ denotes the Gauss hypergeometric function.

In the following theorems we examine some properties of CLPM.

Theorem 5.2.2. *The variables X_1 and X_2 are independent if and only if*

$$\delta_r(t_i|X_j \leq t_j) = l_r\left(t_i\right), i, j = 1, 2, i \neq j.$$

Theorem 5.2.3. *The CLPM, $\delta_r(t_i|X_j \leq t_j)$ $i, j = 1, 2, i \neq j$ satisfies the following recurrence relationship,*

$$\frac{\partial}{\partial t_i}\delta_r(t_i|X_j \leq t_j) - r\delta_{r-1}(t_i|X_j \leq t_j) = 0.$$

Proof. From (5.2), we have $\delta_r(t_i|X_j \leq t_j) = (r-1)\int_{t_i}^{\infty}(t_i - x_i)^{r-2}F_i(x_i|t_j)dx_i$. Taking logarithm on both sides of (5.2), we get

$$\ln[\delta_r(t_i|X_j \leq t_j)] = \ln\left[r\int_{-\infty}^{t_i}(t_i - x_i)^{r-1}F_i(x_i|t_j)dx_i\right]. \tag{5.4}$$

Differentiating (5.4) with respect to t_i, we obtain the required relationship. □

In the following theorem we obtain a relationship connecting CLPM and LPM.

Theorem 5.2.4. *Let $\boldsymbol{X} = (X_1, X_2)$ be a non-negative random vector admitting an absolutely continuous distribution function $F(x_1, x_2)$ with respect to a Lebesgue measure in the two dimensional Euclidean space $\mathbb{R}_2$. Assume that $E(X_1X_2)$ is finite. Then for $i, j = 1, 2$ and $i \neq j$, the CLPM, $\delta_r(t_i|X_j \leq t_j)$ and the LPM, $l_r(t_i)$ satisfies the relationship*

$$\delta_r(t_i|X_j \leq t_j)l_r(t_j) - \delta_r(t_j|X_i \leq t_i)l_r(t_i) = 0. \tag{5.5}$$

Proof. The proof follows immediately from the definition of CLPM and from the following relation,

$$F_i(x_i|x_j)\, F_j(x_j) = F_j(x_j|x_i)\, F_i(x_i)\,,\ i, j = 1, 2, i \neq j.$$

Multiplying both sides by $r(t_i - x_i)^{r-1}$ and integrating with respect to x_i, we have,

$$F_j(x_j) \int_{t_i}^{\infty} r(t_i - x_i)^{r-1} F_i(x_i|x_j)\, dx_i = F_j(x_j|x_i) \int_{t_i}^{\infty} r(t_i - x_i)^{r-1} F_i(x_i)\, dx_i,$$

$$\Leftrightarrow F_j(x_j)\, \delta_r(t_i|X_j \leq t_j) = F_j(x_j|x_i)\, l_r(t_i).$$

Now multiplying both sides by $r(t_j - x_j)^{r-1}$ and integrating with respect to x_j, yields (5.5). □

Theorem 5.2.5. *Let $\delta_r(t_i|X_j \leq t_j), i, j = 1, 2$ and $i \neq j$, denotes the CLPM as defined in* (5.1)*. Then, for every $r \geq 1$, $\delta_r(t_i|X_j \leq t_j)$ is a convex function of t_i.*

Proof. The proof is similar to the proof given in Bawa (1978) and Hogan and Warren (1972), since for all values of r, $(t_i - x)^r$ will always be a convex function in t_i. Then the theorem follows from the linearity property of conditional expectation. □

The following example illustrates Theorem 5.2.5.

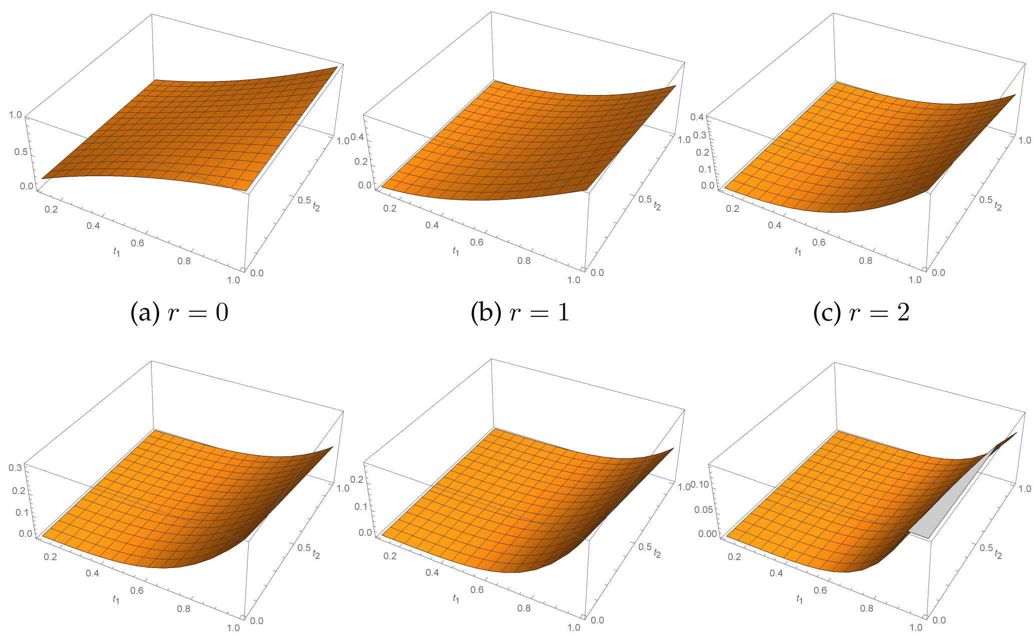
Example 5.2.6. (Farlie–Gumbel–Morgenstern Distribution): Consider the one-parameter Farlie–Gumbel–Morgenstern (F-G-M) family of distributions with uniform marginal, given by the pdf,

$$f(x_1, x_2) = 1 + \alpha(1 - 2x_1)(1 - 2x_2)\ , 0 \leq x_i \leq 1,\ -1 \leq \alpha \leq 1, i = 1, 2. \tag{5.6}$$

Then $f_1(x_1|x_2) = 1 + \alpha(1 - 2x_1)(1 - 2x_2)$ and the CLPM

$$\delta_r(t_i|X_j \leq t_j) = \frac{t_i^{r+1}}{r+1}\left(1 + \alpha(1 - 2t_j) + \frac{2\alpha t_i(2t_j - 1)}{r+2}\right),\ i, j = 1, 2,\ i \neq j.$$

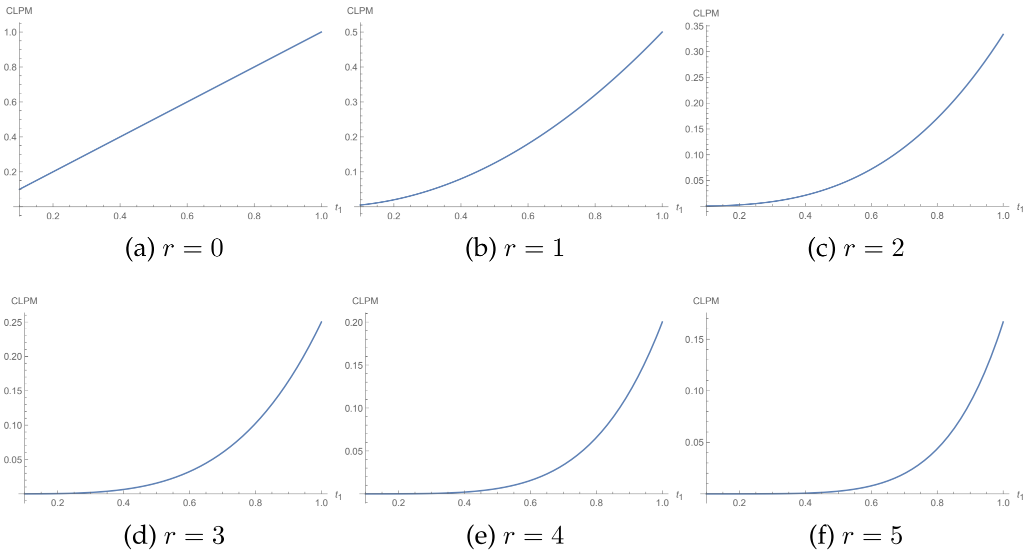
Clearly, $\delta_r(t_i|X_j \leq t_j)$ is non-decreasing in t_i for fixed t_j and non-decreasing in r for fixed t_i and t_j, $i, j = 1, 2$ and $i \neq j$, which are illustrated in Figure 5.1 and Figure 5.2 respectively.



(a) $r = 0$ (b) $r = 1$ (c) $r = 2$

(d) $r = 3$ (e) $r = 4$ (f) $r = 5$

Figure 5.1: Surface plots of CLPM for F-G-M distribution (5.6) with $\alpha = 0.5$ for different values of r.



(a) $r = 0$ (b) $r = 1$ (c) $r = 2$

(d) $r = 3$ (e) $r = 4$ (f) $r = 5$

Figure 5.2: Plot of $\delta_r(t_1|X_2 \leq t_2)$ for F-G-M distribution (5.6) with $\alpha = 0.5$ and $t_2 = 0.5$ for different values of r.

5.3 Characterization of bivariate distributions using CLPM

A standard practice adopted in statistical modellng is to ascertain the physical properties of the process, express them by means of equations or inequalities and then solve them to obtain the model. Accordingly, in the present section, we pay attention on identifying certain characterization relationships based on CLPMs to model some important probability distributions.

Theorem 5.3.1. *The CLPM satisfies the following linear equation*

$$\delta_r(t_i|X_j \leq t_j) = K t_i \delta_{r-1}(t_i|X_j \leq t_j),\ 0 < K < 1 \tag{5.7}$$

if and only if (X_1, X_2) *follows a bivariate uniform distribution with CDF*

$$F(t_1, t_2) = \frac{t_1 t_2}{b\ d},\ 0 \leq t_1 \leq b, 0 \leq t_2 \leq d. \tag{5.8}$$

Proof. Assume that (5.7) holds. Now using (5.1), we have $\delta_r(t_i|X_j \leq t_j) = \frac{t_i^{r+1}}{b(r+1)}$, $i = 1, 2, i \neq j$, so that $\frac{\delta_r(t_i|X_j \leq t_j)}{\delta_{r-1}(t_i|X_j \leq t_j)} = \frac{rt_i}{r+1}$, $i = 1, 2, i \neq j$, thus (5.7). Conversely, suppose that (5.7) holds true. Differentiating both sides of (5.7) with respect to t_i, r times and rearranging the terms, we have $\bar{h}_i\,(t_1, t_2) = \frac{1}{t_i}$. Now making use use of equations (2.13) and (2.14) due to Roy (2002a), the model (5.8) follows. □

The bivariate power distribution is of great importance in income studies and reliability modeling for past lifetime (see Nair and Asha (2008)). In the following theorem we prove characterizations to bivariate power and bivariate uniform models based on relationship between two consecutive CLPMs.

Theorem 5.3.2. *The ratio of the two consecutive CLPMs satisfies the relationship*

$$\frac{\delta_r(t_i|X_j \leq t_j)}{\delta_{r-1}(t_i|X_j \leq t_j)} = \frac{rt_i}{K + r + B(t_j)},\ i, j = 1, 2,\ i \neq j, \tag{5.9}$$

where $B(t_j)$ is a function depending only on t_j, if and only if $X = (X_1, X_2)$ follows bivariate uniform distribution

$$F(t_1, t_2) = t_2\, t_1^{1+\theta \log t_2},\ 0 < t_1, t_2 < 1 \tag{5.10}$$

for $K = 1$ and bivariate power distribution

$$F(t_1, t_2) = t_2^{2k_2-1} t_1^{2k_1-1+\theta \log t_2},\ 0 < t_1, t_2 < 1 \tag{5.11}$$

for $K > 0$.

Proof. Let (5.9) holds true. Then we have,

$$\delta_r(t_i|X_j \leq t_j)\,(K + r + B(t_j)) = rt_i\delta_{r-1}(t_i|X_j \leq t_j), i, j = 1, 2\ \ i \neq j,$$

Differentiating both sides with respect to t_i, r times and rearranging the terms, we have $\bar{h}_i\,(t_1, t_2) = \frac{K+B(t_j)}{t_i}$. Now making use use of (2.13) and (2.14) due to Roy (2002a), we have the result. For the distribution (5.10) using (5.1) gives

$$\delta_r(t_i|X_j \leq t_j) = \frac{r!\,\Gamma\,(2+\theta\log t_j)\,t_i^{1+\theta\log t_j+r}}{\Gamma\,(2+\theta\log t_j+r)}\ i,j=1,2,\ i\neq j$$

and for the distribution (5.11), we have

$$\delta_r(t_i|X_j \leq t_j) = \frac{r!\,\Gamma\,(2k_1+\theta\log t_j)\,t_i^{2k_1-1+\theta\log t_j+r}}{\Gamma\,(2k_1+\theta\log t_j+r)}\ i,j=1,2,\ i\neq j,$$

so that (5.9) follows immediately. Hence the proof. □

Now we prove a characterization to bivariate Type 3 extreme-value distribution using CLPMs.

Theorem 5.3.3. *Let (X_1, X_2) be a bivariate random vector in the support of $(-\infty, b_1) \times (-\infty, b_2)$ with $b_i < \infty$ admitting an absolutely continuous distribution function $F(x_1, x_2)$, then the ratio of the two consecutive CLPMs is of the form*

$$\frac{\delta_r(t_i|X_j \leq t_j)}{\delta_{r-1}(t_i|X_j \leq t_j)} = \frac{r}{\alpha(t_j)},\ \ i,j=1,2\ \ i\neq j, \tag{5.12}$$

where, $\alpha(t_j)$ is some function of t_j only, if and only if $X = (X_1, X_2)$ follows bivariate Type 3 extreme-value distribution (Nair and Asha (2008)) given by the joint CDF,

$$F(t_1,t_2) = e^{[c_1(t_1-b_1)+c_2(t_2-b_2)+\theta(t_1-b_1)(t_2-b_2)]},\ -\infty < t_i < b_i < \infty, c_i > 0. \tag{5.13}$$

Proof. Using (5.1), for (5.13) yields $\delta_r(t_i|X_j \leq t_j) = \frac{r!\,e^{\left[c_i+\theta\left(t_j-b_j\right)\right](t_i-b_i)}}{(c_i+\theta(t_j-b_j))^r}$, $i=1,2, i\neq j$, so that $\frac{\delta_r(t_i|X_j\leq t_j)}{\delta_{r-1}(t_i|X_j\leq t_j)} = \frac{r}{c_i+\theta(t_j-b_j)}$, $i=1,2, i\neq j$ and (5.12) holds. Conversely assuming (5.12), differentiating both sides with respect to t_i, r times and rearranging the terms, we have $\bar{h}_i\,(t_1,t_2) = \alpha(t_j)$. the rest part of the proof follows from (2.13) and (2.14) due to Roy (2002a). □

5.4 Applications of conditional lower partial moments

5.4.1 Income and poverty studies

Lower partial moments are considered as a useful tool in poverty and income studies as both LPMs and poverty indices are focusing on the lower part of the distribution (see Sunoj and Maya (2008)). In income studies, a useful index to measure the level of poverty is the income-gap ratio, given by $\beta^*(t) = t - X$. We define bivariate income-gap ratio in terms of CLPMs as a vector, $\beta^*(t_1, t_2) = (\beta_1^*(t_1, t_2), \beta_2^*(t_1, t_2))$, where the i^{th} component is given by

$$\beta_i^*(t_1, t_2) = \frac{\delta_1(t_i|X_j \leq t_j)}{t_i \delta_0(t_i|X_j \leq t_j)},\ i, j = 1, 2; i \neq j. \tag{5.14}$$

The applications of Pareto distribution in economics, reliability etc. are well known. The first author to systematically study k-dimensional Pareto distributions was Mardia (1962). In the following theorem, we characterize bivariate Pareto distribution of the first kind (Mardia (1962)) using (5.14).

Theorem 5.4.1. *A non-negative random vector (X_1, X_2) follows bivariate Pareto distribution of the first kind (Mardia (1962)) with the joint density,*

$$f(x_1, x_2) = (\alpha+1)\alpha(\theta_1\theta_2)^{\alpha+1}(\theta_2 x_1 + \theta_1 x_2 - \theta_1\theta_2)^{-(\alpha+2)}, x_i \geq \theta_i > 0,\ i = 1, 2,\ \alpha > 0. \tag{5.15}$$

if and only if the bivariate income-gap ratio satisfy the relationship

$$\frac{\beta_i^*(t_i, t_j)}{\beta_i^*(t_i, t_j) - 1} = K + t_i A_i(t_j), K > 0, i, j = 1, 2; i \neq j \tag{5.16}$$

Proof. Suppose (5.16) holds true. Then, from (5.14), we have, $\beta_i^*(t_i, t_j) = \frac{\bar{m}_i(t_i, t_j)}{t_i}$ and $\bar{m}_i(t_i, t_j) = \frac{t_1(K + t_i A_i(t_j))}{K + t_i A_i(t_j) - 1}$. Now the results follow from Nair and Asha (2008).

To prove the converse part, for the density (5.15), we have

$$f_i(x_i|x_j) = (\alpha+1)\ {\theta_i}^{\alpha+1}\theta_j\ \frac{(\theta_j x_i + \theta_i x_j - \theta_1\theta_2)^{-(\alpha+2)}}{(x_j-\theta_j)^{-(\alpha+1)}}, i,j = 1,2; i \neq j.$$

Now (5.1) yields, $\delta_r(t_i|X_j \leq t_j) = \frac{(\alpha+1)\theta_i^{\alpha+1}\theta_j}{(\theta_i t_j - \theta_i\theta_j)^{-\alpha}} \frac{t_1^{r+1}((\theta_i t_j - \theta_i\theta_j)(r+2)+\theta_j t_i)}{(r+1)(r+2)}$, $i = 1, 2, i \neq j$, proceeds to get the required form (5.16). □

Logistic distribution or log-logistic distributions are sometimes found useful in modelling income data. The next theorem gives a characterization to bivariate logistic distribution based on conditional income-gap ratio defined in (5.14).

Theorem 5.4.2. *If $\boldsymbol{X} = (X_1, X_2)$ is a random vector in the support $\mathbb{R}_2$ with absolutely continuous distribution function $F(\cdot)$ and $E(X_1X_2) < \infty$, then $\boldsymbol{X}$ follows bivariate logistic distribution defined by the joint CDF*

$$F(x_1,x_2) = \left(1 + e^{-x_1} + e^{-x_2}\right)^{-1}, -\infty < x_1, x_2 < \infty, \tag{5.17}$$

if and only if for all (t_1, t_2) in $\mathbb{R}^2$, the bivariate income-gap ratio satisfies,

$$\beta_i^*(t_1,t_2) = \frac{\log\left(e^{t_i}\left(e^{-t_j}+1\right)+1\right)}{t_i}\frac{\left(e^{-t_i}\left(e^{-t_j}+1\right)+e^{-t_i}\right)}{e^{-t_j}+1}, i,j = 1,2,\ i \neq j. \tag{5.18}$$

Proof. Assuming the distribution to be (5.25), using (5.1), we have

$$\delta_r(t_i|X_j \leq t_j) = -r!\ \mathrm{Li}_r\left(-e^{t_i}\left(1+e^{-t_j}\right)\right), i = 1,2, i \neq j \tag{5.19}$$

where $\mathrm{Li}_n(z)$ is the polylogarithm function defined by $\mathrm{Li}_n(z) = \sum_{k=1}^{\infty}\frac{z^k}{k^n}$. Now (5.18) follows from (5.19) and (5.14).

To prove the converse, we note that from (5.14)

$$\bar{m}_i\left(t_1, t_2\right) = t_i \beta_i^*\left(t_1, t_2\right) = \log\left(\frac{1+e^{-t_i}+e^{-t_j}}{e^{-t_j}}\right)\left(\frac{1+e^{-t_i}+e^{-t_j}}{1+e^{-t_j}}\right), i, j = 1, 2; i \neq j$$

The rest part of the proof directly follows from Nair and Asha (2008). □

It has been shown by many authors (see Brzezinski (2014)) about the importance of power function distribution in income studies. The following theorem provides a characterization to bivariate power distribution using conditional income gap ratio (5.14).

Theorem 5.4.3. *If* (X_1, X_2) *is a random vector in the support* $(0, b_1) \times (0, b_2), b_i < \infty, i = 1, 2$ *with absolutely continuous distribution function* $F(\cdot)$ *and* $E(X_1 X_2) < \infty$, *then*

$$\beta_i^*\left(t_1, t_2\right) = a_i\left(t_j\right), i, j = 1, 2,\ i \neq j \tag{5.20}$$

for a non-negative function $a_i(.)$ *if and only if* (X_1, X_2) *follows bivariate power function distribution with joint CDF*

$$F(t_1, t_2) = \left(\frac{t_1}{b_1}\right)^{c_1}\left(\frac{t_2}{b_2}\right)^{c_2+\theta\log\left(\frac{t_1}{b_1}\right)}, t_i \in (0, b_i); c_i > 0, i = 1, 2; \theta \leq 0, \tag{5.21}$$

Proof. For the distribution, (5.21), using (5.1), we have

$$\delta_r(t_i|X_j \leq t_j) = \frac{r!\left(\frac{1}{b_i}\right)^{c_i+\theta\log\left(\frac{t_j}{b_j}\right)} t_i^{c_i+\theta\log\left(\frac{t_j}{b_j}\right)+r}\,\Gamma\left(\theta\log\left(\frac{t_j}{b_j}\right)+c_i+1\right)}{\Gamma\left(c_i+\theta\log\left(\frac{t_j}{b_j}\right)+r+1\right)}, i, j = 1, 2; i \neq j. \tag{5.22}$$

Now (5.20) follows from (5.14) and (5.22).

To prove the converse, assume, (5.20) is true. Then, $\bar{m}_i\left(t_1, t_2\right) = t_i \beta_i^*\left(t_1, t_2\right) = t_i\, a_i\left(t_j\right), i, j = 1, 2; i \neq j$. The remaining part of the proof follows directly from

Nair and Asha (2008). □

For modelling extremal events of insurance and finance, extreme-value distributions are of importance. The following theorem characterizes bivariate Type 3 extreme-value distributions based on the conditional income-gap ratio.

Theorem 5.4.4. *Let (X_1, X_2) be a bivariate random vector in the support of $(-\infty, b_1) \times (-\infty, b_2)$ with $b_i < \infty$ admitting an absolutely continuous distribution function $F(\cdot)$. Then*

$$\beta_i^* (t_1, t_2) = \frac{a_i (t_j)}{t_i}, i, j = 1, 2; i \neq j \tag{5.23}$$

if and only if (X_1, X_2) follows a bivariate Type 3 extreme-value distribution defined in (5.13).

Proof. Using (5.2), for (5.13) we have, $\delta_r(t_i|X_j \leq t_j) = \frac{r!\, e^{\left[c_i+\theta\left(t_j-b_j\right)\right](t_i-b_i)}}{(c_i+\theta(t_j-b_j))^r}$. So that $\beta_i^* (t_1, t_2) = \frac{1}{(c_i+\theta(t_j-b_j))t_i}$ and (5.23) holds. Conversely assuming (5.23) to be true, we have $\bar{m}_i (t_1, t_2) = t_i \beta_i^* (t_1, t_2) = a_i (t_j), i, j = 1, 2; i \neq j$. Now the results follows from Nair and Asha (2008). □

5.4.2 Risk Analysis

The LPM measure risk by negative deviations of the random return, according to a minimal acceptable return or return threshold or target return. In the present section we consider an important situation where auxiliary information about random (assets) return, Y is provided by a predictor, X.

Let $F(y|x) = P(Y \leq y|X \leq x)$ and $Q_{Y|X\leq x}(\alpha|x) = \inf\{y : F(y|x) \geq \alpha\}, \alpha \in (0, 1)$, be respectively denote the conditional CDF and conditional quantile function (CQF) of $Y|X \leq x$ (see Parzen et al. (2004) and Belzunce et al. (2012a)). Assuming $F(\cdot|x)$ is absolutely continuous and strictly increasing in x, we have

$F\left(Q_{Y|X\leq x}\left(\alpha|x\right)\right)=\alpha$ and $Q\left(\alpha|x\right)=Q_{Y|X\leq x}\left(\alpha|x\right)=F^{-1}_{Y|X\leq x}\left(\alpha|x\right)$.

Then, the zeroth order CLPM, $\delta_0(y|X\leq x)=P\left(Y\leq y|X\leq x\right)$ corresponds to the *conditional expected loss* i.e the conditional (shortfall) probability that the expected income (return) will be lower than target. The conditional Value-at-Risk denoted by $CVaR$ for a variable of interest Y will be the α^{th} quantile of the conditional distribution of $Y|X\leq x$, i.e.

$$CVaR_{Y|X\leq x}\left(\alpha\right)=\inf\left\{y:\delta_0(y|X\leq x)\geq\alpha\right\},\alpha\in\left(0,1\right).$$

Another important measure useful in risk analysis is the *Conditional Expected Shortfall (CES)*, which measures the conditional expected loss knowing that the losses are larger than a given quantile. similar to the definition of CES given by Peracchi and Tanase (2008) one can define the CES for $Y|X\leq x$ as,

$$\begin{aligned}\tau\left(\alpha|x\right)&=E\left(Y|Y\leq Q\left(\alpha|x\right),X\leq x\right)\\&=\frac{1}{\alpha}\int\limits_{-\infty}^{\alpha}Q\left(u|x\right)du=Q\left(\alpha|x\right)-\frac{1}{\alpha}\int\limits_{-\infty}^{Q(\alpha|x)}F\left(y|x\right)dy,\end{aligned}$$

setting $t_1=Q\left(\alpha|x\right)$ in (5.1), one can relate the conditional expected shortfall (CES) with the first order CLPM, $\delta_1(y|X\leq x)$ as

$$\tau\left(\alpha|x\right)=Q\left(\alpha|x\right)-\frac{\delta_1\left(Q\left(\alpha|x\right)\right)}{\alpha},\tag{5.24}$$

where, $\delta_1\left(Q\left(\alpha|x\right)\right)=\int\limits_0^{\alpha}\left(Q\left(\alpha|x\right)-Q\left(u|x\right)\right)du=\alpha Q\left(\alpha|x\right)-\int\limits_0^{\alpha}Q\left(u|x\right)du$. Further, it is clear that $\tau\left(\alpha|x\right)$ will be always non-decreasing function in $\alpha\in(0,1)$ and negative, since the two functions in the RHS of (5.24) is non-decreasing in α and $Q\left(\alpha|x\right)<\frac{\delta_1(Q(\alpha|x))}{\alpha}$.

Let $U = Y_1|X \leq x$ and $V = Y_2|X \leq x$ be conditional return on two options with quantile functions $Q(\alpha|x)$ and $W(\alpha|x)$ respectively, and let $Q_n(\alpha|x) = \int_0^\alpha Q_{n-1}(p|x)dp$, $n = 1, 2, \ldots$ with $Q_0(\alpha|x) = Q(\alpha|x)$ (see Muliere and Scarsini (1989)). Similarly we define $W_n(\alpha|x)$. Then, the following theorem provides the importance of first order CLPM in comparing two returns in terms of conditional expected shortfalls.

Theorem 5.4.5. *Let $(\tau_U(\alpha|x), \delta_1(Q(\alpha|x)))$ and $(\tau_V(\alpha|x), \gamma_1(W(\delta|x)))$ denotes the CES and First order CLPM pair for the two random returns U and V respectively. Then, for all $\alpha \in (0,1)$,*

$$\tau_U(\alpha|x) \leq \tau_V(\alpha|x) \Leftrightarrow \delta_1(Q(\alpha|x)) \leq \gamma_1(W(\alpha|x)).$$

if and only if the function $\alpha^{-1}[Q_1(\alpha|x) - W_1(\alpha|x)]$ is a decreasing function of α.

Proof.

$$\begin{aligned}
\tau_U(\alpha|x) \leq \tau_V(\alpha|x) &\Leftrightarrow \frac{1}{\alpha}\int_{-\infty}^{\alpha} Q(u|x)du \leq \frac{1}{\alpha}\int_{-\infty}^{\alpha} W(u|x)du \\
&\Leftrightarrow \frac{Q_1(\alpha|x)}{\alpha} \leq \frac{W_1(\alpha|x)}{\alpha} \\
&\Leftrightarrow \alpha^{-1}[Q_1(\alpha|x) - W_1(\alpha|x)] \leq 0 \\
&\Leftrightarrow \frac{1}{\alpha^2}\left[\alpha(Q(\alpha|x) - W(\alpha|x)) - \left(\int_0^\alpha Q(u|x)du \right.\right. \\
&\qquad\qquad \left.\left. - \int_0^\alpha W(u|x)du\right)\right] \leq 0 \\
&\Leftrightarrow \alpha Q(\alpha|x) - \int_0^\alpha Q(u|x)du \leq \alpha W(\alpha|x) - \int_0^\alpha W(u|x)du \\
&\Leftrightarrow \delta_1(Q(\alpha|x)) \leq \gamma_1(W(\alpha|x)).
\end{aligned}$$

□

Note that, $\delta_1(y|X \leq x) = \int\limits_{-\infty}^{y} F(u|x)du$ gives $\frac{\partial^2 \delta_1(y|X \leq x)}{\partial y^2} = f(y|x) > 0$, shows that $\delta_1(y|X \leq x)$ is convex. The following theorem examines the monotonic properties of CES.

Theorem 5.4.6. *Let $\tau(\alpha|x)$ be the α-level CES, $\alpha \in (0,1)$, and $\delta_1(Q(\alpha|x))$ denote the first order CLPM fixed at α^{th} quantile, $Q(\alpha|x)$ with corresponding conditional quantile density function, $q(\alpha|x)$. Then $\tau(\alpha|x)$ is*

(a) non-decreasing if and only if $\tau(\alpha_1|x) > \tau(\alpha_2|x)$ for $\alpha_1 > \alpha_2$, $\forall \alpha_i \in (0,1), i = 1, 2$.

(b) concave if and only if $\frac{\delta_1(Q(\alpha|x))}{q(\alpha|x)} \geq \frac{\alpha^2}{2}$.

(c) convex if and only if $\frac{\delta_1(Q(\alpha|x))}{q(\alpha|x)} \leq \frac{\alpha^2}{2}$.

Proof. The proof immediately follows from the definition of CES given in (5.24). □

Example 5.4.1. Consider the bivariate logistic distribution specified by the joint CDF,

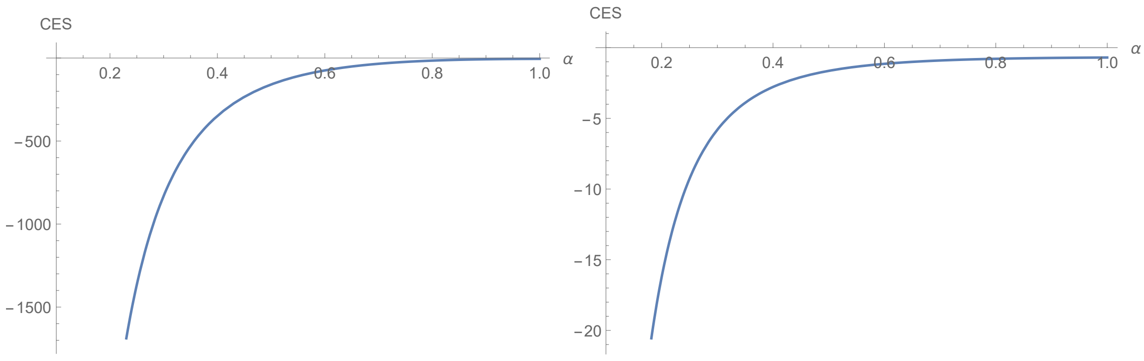
$$F(y,x) = \left(1 + e^{-y} + e^{-x}\right)^{-1}, \; -\infty < y, x < \infty \tag{5.25}$$

Then, $\delta_r(t|X \leq x) = -r! \; \mathrm{Li}_r\left(-e^t\left(1 + e^{-x}\right)\right)$, where $\mathrm{Li}_n(z)$ is the polylogarithm function defined by $\mathrm{Li}_n(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^n}$ and $\delta_1(t|X \leq x) = \log\left(e^t\left(e^{-x}+1\right)+1\right)$. From (5.24) the CES will be,

$$\tau(\alpha|x) = \frac{1}{\alpha} e^{-x} \left(1 - \alpha - e^x \left(\alpha + \log\left(\left(e^{-x}+1\right) e^{-\frac{(\alpha-1)e^{-x}(e^x+1)}{\alpha}} + 1\right) - 1\right)\right).$$

The plot of the CES, $\tau(\alpha|x)$ is further depicted in Figure 3 for different values of x.

When there are multiple options of risky investments a criterion for deciding the best is by introducing an order of preference among them. Stochastic dom-



(a) $x = -5$ (b) $x = 5$

Figure 5.3: Plot of CES for bivariate Logistic distribution (5.25).

inance criteria play an important role in this (see Levy (2015)). One can define Stochastic dominance in terms of the CLPMs as follows.

Definition 5.4.1. Let $U = Y_1|X \leq x$ and $V = Y_2|X \leq x$ be conditional return on two options with corresponding distribution functions $F\left(y_1|x\right)$ and $G\left(y_2|x\right)$ respectively. Let $\delta_r\left(t|X \leq x\right)$ and $\gamma_r\left(t|X \leq x\right)$ and denote conditional partial moments of order *r* for the random variables U and V respectively. Then F dominates G conditionally (FDGC) by first, second and third order respectively denoted by CFSD,CSSD and CTSD, if and only if

(a) Conditional First order stochastic Dominance (CFSD) :

$$\delta_0\left(t|X \leq x\right) \leq \gamma_0\left(t|X \leq x\right)$$

(b) Conditional Second order stochastic Dominance (CSSD) :

$$\delta_1\left(t|X \leq x\right) \leq \gamma_1\left(t|X \leq x\right)$$

(c) Conditional Third order stochastic Dominance (CTSD) :

$$\int_{-\infty}^{t} \delta_1 (v|X \leq x) dv \leq \int_{-\infty}^{t} \gamma_1 (v|X \leq x) dv, \forall t, x.$$

Remark 5.4.1. Note that the definitions (a)-(c) can be defined in terms of conditional CDF as follows.

(i) CFSD : $F_{Y_1|X \leq x}(t|x) \leq G_{Y_2|X \leq x}(t|x), \forall t, x.$

(ii) CSSD : $\int_{-\infty}^{t} F_{Y_1|X \leq x}(u|x) du \leq \int_{-\infty}^{t} G_{Y_2|X \leq x}(u|x) du, \ \forall t, x$

(iii) CTSD : $\int_{-\infty}^{t} \int_{-\infty}^{v} \left[G_{Y_2|X \leq x}(u|x) - F_{Y_1|X \leq x}(u|x) \right] du dv \geq 0, \ \forall t, x$

The computations involving CDFs are often complex and may cause analytical problems in certain cases. An alternative is a quantile approach. Accordingly, we define the following quantile-based conditional stochastic dominance as follows.

Definition 5.4.2. Let $U = Y_1|X$ and $V = Y_2|X$ be conditional return on two options with quantile functions $Q(\alpha|x)$ and $W(\alpha|x)$ respectively. Let, $Q_n(\alpha|x) = \int_0^{\alpha} Q_{n-1}(p|x) dp, \ n = 1, 2, \ldots$ with $Q_0(\alpha|x) = Q(\alpha|x)$. Similarly we define $W_n(\alpha|x)$. Then U dominates V by

(a) first order conditional stochastic dominance $(U \geq_{CFSD} V)$ if $Q(\alpha|x) \geq W(\alpha|x)$ for all α, with strict inequality for at least one α.

(b) second order conditional stochastic dominance, $(U \geq_{CSSD} V)$ if $Q_1(\alpha|x) \geq W_1(\alpha|x)$ for all α, with strict inequality for at least one α.

It is to be noted that for every n,

$$Q_{n-1}(\alpha|x) \geq W_{n-1}(\alpha|x) \Rightarrow Q_n(\alpha|x) \geq W_n(\alpha|x).$$

However, the converse of the above definition need not be true. We now obtain the conditions under which the converse holds.

Theorem 5.4.7. *If* $Q_n(\alpha|x) \geq W_n(\alpha|x)$ *and* $\frac{\int_0^\alpha Q_{n-1}(p|x)dp}{\int_0^\alpha W_{n-1}(p|x)dp}$, $n = 1, 2, 3, \ldots$ *is increasing then* $Q_{n-1}(\alpha|x) \geq W_{n-1}(\alpha|x)$.

Proof. $Q_n(\alpha|x) \geq W_n(\alpha|x) \Rightarrow \frac{Q_n(\alpha|x)}{W_n(\alpha|x)} \geq 1$. Since, $\frac{\int_0^\alpha Q_{n-1}(p|x)dp}{\int_0^\alpha W_{n-1}(p|x)dp}$ is increasing, by differentiation, $Q_{n-1}(\alpha|x) \int_0^\alpha W_{n-1}(p|x)dp - W_{n-1}(\alpha|x) \int_0^\alpha Q_{n-1}(p|x)dp \geq 0$.

$$\Rightarrow \frac{Q_{n-1}(\alpha|x)}{W_{n-1}(\alpha|x)} \geq \frac{\int_0^\alpha Q_{n-1}(p|x)dp}{\int_0^\alpha W_{n-1}(p|x)dp} \Leftrightarrow \frac{Q_{n-1}(\alpha|x)}{W_{n-1}(\alpha|x)} \geq 1.$$

□

In the following theorem we illustrate some sufficient conditions to establish stochastic dominance for conditional random variables using quantile-based CLPMs.

Theorem 5.4.8. *Let* $\delta_r(t)$ *and* $\gamma_r(t)$ *and denote the CLPMs of order r of U and V respectively. Then suppose,*

(a) if $\frac{Q_1(\alpha|x)}{W_1(\alpha|x)}$ *is increasing in* α *and* $\delta_1(t|X \leq x) \leq \gamma_1(t|X \leq x)$, *then* $U \geq_{CFSD} V$.

(b) if $\frac{Q_2(\alpha|x)}{W_2(\alpha|x)}$ *is increasing in* α *and* $\delta_2(t|X \leq x) \leq \gamma_2(t|X \leq x)$, *then* $U \geq_{CSSD} V$.

Proof. Note that,

$$\begin{aligned} \delta_1(t|X \leq x) \leq \gamma_1(t|X \leq x) \quad &\Leftrightarrow \quad \int_0^\alpha Q(p|x)dp \geq \int_0^\alpha W(p|x)dp \\ &\Leftrightarrow \quad Q_1(\alpha|x) \geq W_1(\alpha|x), \end{aligned}$$

then *(a)* follows from Theorem 5.4.5. Similarly from the definition of CTSD (Remark 5.4.1) (Levy (1992)) we have,

$$\begin{aligned}
\delta_2(t|X\le x)\le\gamma_2(t|X\le x) \quad &\Leftrightarrow \quad \int_{-\infty}^{t}\int_{-\infty}^{v} G_{Y_2|X}(u|x)dudv \ge \int_{-\infty}^{t}\int_{-\infty}^{v} F_{Y_2|X}(u|x)dudv \\
&\Leftrightarrow \quad \int_{0}^{\alpha}\int_{0}^{v} Q(p|x)dpdv \ge \int_{0}^{\alpha}\int_{0}^{v} W(p|x)dpdv \\
&\Leftrightarrow \quad \int_{0}^{\alpha} Q_1(p|x)dp \ge \int_{0}^{\alpha} W_1(p|x)dp \\
&\Leftrightarrow \quad Q_2(\alpha|x)\ge W_2(\alpha|x),
\end{aligned}$$

and rest part of the proof of *(b)* follows from Theorem 5.4.5. □

Let $\delta_r(t|X\le x)$ and $\gamma_r(t|X\le x)$ denote the CLPMs of order *r* for the random variables *U* and *V* and let the corresponding quantile formulations be $P_r(u)$ and $A_r(u)$ respectively. Using the quantile-based LPM given in Nair and Sankaran (2011), the first and second order quantile-based CLPMs can be defined as

$$P_1(\alpha|x)=\delta_1(Q(\alpha|x))=\alpha Q(\alpha|x)-\int_{0}^{\alpha} Q(u|x)du \tag{5.26}$$

and

$$P_2(\alpha|x)=\delta_2(Q(\alpha|x))=\int_{0}^{\alpha} Q^2(u|x)du+2Q(\alpha|x)P_1(\alpha|x)-\alpha Q^2(\alpha|x) \tag{5.27}$$

In the following theorem, we present a partial ordering based on first and second order quantile-based CLPMs defined in (5.26) and(5.27) respectively.

Theorem 5.4.9. *Let $U=Y_1|X\le x$ and $V=Y_2|X\le x$ denote two random variables with quantile functions $Q(\alpha|x)$ and $W(\alpha|x)$ respectively. Also let $P_r(\alpha|x)$ and $A_r(\alpha|x)$ denote the quantile based CLPMs of order r for the random variables U and V. Then we say that* *(a) U is smaller than V in the quantile based first CLPM denoted by $U \leq_{FCLP} V$ if $P_1(\alpha|x) \leq A_1(\alpha|x)$ for all $0 < \alpha < 1$.*

(b) U is smaller than V in the quantile based second CLPM denoted by $U \leq_{SCLP} V$ if $P_2(\alpha|x) \leq A_2(\alpha|x)$ for all $0 < \alpha < 1$.

The ordering of U and V in terms of $\delta_r(t|X \leq x)$ and $\gamma_r(t|X \leq x)$ need not mean that similar order preserved for $P_r(u)$ and $A_r(u)$. The following example justifies this property.

Example 5.4.2. Let (Y_1, X) follows bivariate uniform distribution with joint CDF

$$F(y_1, x) = y_1 x, 0 \leq y_1, x \leq 1 \tag{5.28}$$

and (Y_2, X) follows bivariate power distribution with joint CDF

$$G(y_2, x) = \sqrt{y_2 x}, 0 \leq y_2, x \leq 1. \tag{5.29}$$

Then, $F(y_1|x) = y_1$ and $G(y_2|x) = \sqrt{y_2}$. Also $Q(\alpha|x) = \alpha$ and $W(\alpha|x) = \alpha^2$. The two CLPMs corresponding to (5.28) and (5.29) will be,

$$\delta_r(t|X \leq x) = \frac{rt^{r+1}}{r(r+1)} \tag{5.30}$$

and

$$\gamma_r(t|X \leq x) = \frac{\sqrt{\pi} r t^{r+\frac{1}{2}} \Gamma(r)}{2\Gamma\left(r+\frac{3}{2}\right)} \tag{5.31}$$

respectively. Then,

(i) from (5.30) and (5.31), $\delta_1(t|X \leq x) = \frac{t^2}{2}$ and $\gamma_1(t|X \leq x) = \frac{2t^{3/2}}{3}$. Hence,

$$\delta_1(t|X \leq x) \leq \gamma_1(t|X \leq x)$$

for all $0 < t < 1$. Also for (5.28) and (5.29) the first quantile based CLPMs will be, $P_1(\alpha|x) = \frac{\alpha^2}{2}$ and $A_1(\alpha|x) = \frac{2\alpha^3}{3}$ respectively. Note that these two functions cross at $\alpha = \frac{3}{4}$. Hence they are not ordered.

(ii) Similarly from (5.30) and (5.31), $\delta_2(t|X \leq x) = \frac{t^3}{3}$ and $\gamma_1(t|X \leq x) = \frac{8t^{5/2}}{15}$, and thus

$$\delta_2(t|X \leq x) \leq \gamma_2(t|X \leq x)$$

for all $0 < t < 1$. On the other hand, for (5.28) and (5.29), $P_2(\alpha|x) = \frac{\alpha^3}{3}$ and $A_1(\alpha|x) = \frac{8\alpha^5}{15}$. Here, the two functions cross at $\alpha = \sqrt{\frac{5}{8}}$. Hence they are not ordered.

The above observation motivates us to propose the following necessary and sufficient conditions for which the quantile-based first and second order CLPMs admit a partial order.

Theorem 5.4.10. *Let $U = Y_1|X \leq x$ and $V = Y_2|X \leq x$ denote two random variables with quantile functions $Q(\alpha|x)$ and $W(\alpha|x)$ and quantile based CLPMs of order r, $P_r(\alpha|x)$ and $A_r(\alpha|x)$ respectively. Then, for all α, $\alpha \in (0,1)$*

(a) $U \leq_{FCLP} V$ if and only if $\alpha^{-1}[Q_1(\alpha|x) - W_1(\alpha|x)]$ is a decreasing function of α.

(b) $U \leq_{SCLP} V$ if and only if

$$\alpha^2 [V(Y_1|Y_1 \leq Q(\alpha|x), X \leq x) - V(Y_2|Y_2 \leq W(\alpha|x), X \leq x)]$$

is a decreasing function of α.

Proof. The proof *(a)* follows exactly similar to the proof of Theorem 5.4.5. To prove *(b)*, note that

$$\begin{aligned}
V\left(Y_1|Y_1 \leq Q\left(\alpha|x\right), X \leq x\right) &= E\left(Y_1^2|Y_1 \leq Q\left(\alpha|x\right), X \leq x\right) \\
&\quad - E^2\left(Y_1|Y_1 \leq Q\left(\alpha|x\right), X \leq x\right) \\
&= \frac{1}{\alpha}\int_0^\alpha Q^2\left(u|x\right)du - \left(\frac{1}{\alpha}\int_0^\alpha Q\left(u|x\right)du\right)^2 \\
&= \frac{1}{\alpha^2}\left(\alpha\int_0^\alpha Q^2\left(u|x\right)du - \left(\int_0^\alpha Q\left(u|x\right)du\right)^2\right).
\end{aligned}$$

Also from (5.27),

$$\begin{aligned}
P_2\left(\alpha|x\right) &= \int_0^\alpha Q^2\left(u|x\right)du + 2Q\left(\alpha|x\right)P_1\left(\alpha|x\right) - \alpha Q^2\left(\alpha|x\right) \\
&= \int_0^\alpha Q^2\left(u|x\right)du + 2Q\left(\alpha|x\right)\int_0^\alpha \left(Q\left(\alpha|x\right) - Q\left(u|x\right)\right)du - \alpha Q^2\left(\alpha|x\right) \\
&= \int_0^\alpha Q^2\left(u|x\right)du + \alpha Q^2\left(\alpha|x\right) - 2Q\left(\alpha|x\right)\int_0^\alpha Q\left(u|x\right)du \\
&= \frac{\partial}{\partial\alpha}\left[\alpha\int_0^\alpha Q^2\left(u|x\right)du - \left(\int_0^\alpha Q\left(u|x\right)du\right)^2\right] \\
&= \frac{\partial}{\partial\alpha}\left[\alpha^2\left(V\left(Y_1|Y_1 \leq Q\left(\alpha|x\right), X \leq x\right)\right)\right]
\end{aligned}$$

Hence,

$$\begin{aligned}
P_2\left(\alpha|x\right) &\leq A_2\left(\alpha|x\right) \\
\Leftrightarrow \frac{\partial}{\partial\alpha}\left[\alpha^2\left(V\left(Y_1|Y_1 \leq Q\left(\alpha|x\right), X \leq x\right)\right)\right] &\leq \frac{\partial}{\partial\alpha}\left[\alpha^2\left(V\left(Y_2|Y_2 \leq W\left(\alpha|x\right), X \leq x\right)\right)\right]
\end{aligned}$$

$$\Leftrightarrow \frac{\partial}{\partial\alpha}\left[\alpha^2\left(V\left(Y_1|Y_1 \leq Q\left(\alpha|x\right), X \leq x\right) - V\left(Y_2|Y_2 \leq W\left(\alpha|x\right), X \leq x\right)\right)\right] \leq 0,$$

implies that $\alpha^2\left[V\left(Y_1|Y_1 \leq Q\left(\alpha|x\right), X \leq x\right) - V\left(Y_2|Y_2 \leq W\left(\alpha|x\right), X \leq x\right)\right]$ is a decreasing function of α. Hence the proof. $\square$

5.5 Estimator of the conditional lower partial moment

We have discussed various properties of CLPMs in the context of reliability modeling, risk analysis and income (poverty) studies. To use these results in practice one require the estimators of $\delta_r(t_i|X_j \leq t_j), i, j = 1, 2, i \neq j$. In this section we propose a non-parametric estimator for the conditional partial moment $\delta_r(t_i|X_j \leq t_j)$. Let $(X_{1_i}, X_{2_i}),\ i = 1, 2, ...n$ be n independent and identically distributed pairs of observations with distribution function $F(\cdot, \cdot)$. The estimator for $\delta_r(t_1|X_2 \leq t_2)$ is defined by

$$\widehat{\delta}_r(t_1|X_2 \leq t_2) = \frac{1}{n}\sum_{i=1}^{n}(t_1 - X_{1_i})^r I\left(X_{1_i} \leq t_1, X_{2_i} \leq t_2\right) \tag{5.32}$$

where $I(\cdot)$ denotes the indicator function. Similarly one can define $\widehat{\delta}_r(t_2|X_1 \leq t_1)$. The estimator in (5.32) is the r^{th} sample moment of the observations from $X_{1_i} \leq t_1|X_{2_i} \leq t_2$. The consistency and asymptotic properties of the estimator $\widehat{\delta}_r(t_1|X_2 \leq t_2)$ can be proved similar to the estimators proposed by Kulkarni and Rattihalli (2002).

5.6 Simulation study and analysis of a real data set

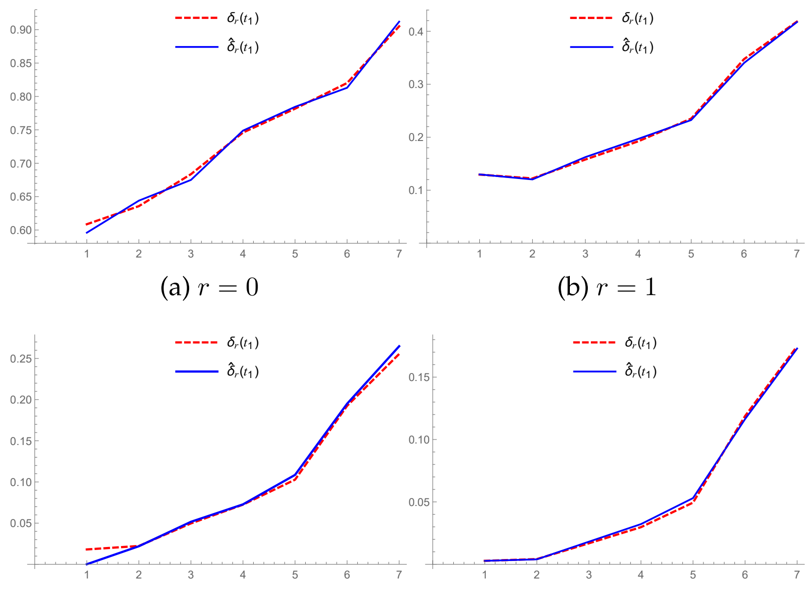
We present here the results of a simulation study and illustrate the properties of empirical estimator $\delta_r(t_i|X_j \leq t_j), i, j = 1, 2, i \neq j$ by analyzing the system reliability data given in ReliaSoft (2003).

5.6.1 Simulation study

To study the performance of the estimator $\widehat{\delta}_r(t_1|X_2 \leq t_2)$, we carried out a series of 1000 simulations each of size n ($n = 10, 100$ and 1000) from a bivariate power distribution with the joint CDF $F(x_1, x_2) = x_2^{2k_2-1} x_1^{2k_1-1+\theta \log x_2}$, $0 < x_1, x_2 < 1$ for $k_1 = k_2 = 1$ and $\theta = 0.5$. The CLPM for the distribution will be

$$\delta_r(t_1|X_2 \leq t_2) = \frac{r!\, \Gamma\left(2k_1 + \theta \log t_2\right)\, t_1^{2k_1-1+\theta \log t_2 + r}}{\Gamma\left(2k_1 + \theta \log t_2 + r\right)}.$$

The performance of the empirical estimator $\widehat{\delta}_r(t_1|X_2 \leq t_2)$ obtained from simulation study is given in Table 5.1 and graphically plotted in Figure 5.4. The results of simulation studies shows that bias and MSE of the proposed empirical estimator $\widehat{\delta}_r(t_1|X_2 \leq t_2)$, decreases with increasing sample sizes. Also the MSE decreases as t_1 and t_2 increases. $\hat{\delta}_r(t_2|X_1 \leq t_1)$ can be estimated in a similar manner.



(a) $r = 0$ (b) $r = 1$

(c) $r = 2$ (d) $r = 3$

Figure 5.4: Performance of $\widehat{\delta}_r(t_1|X_2 \leq t_2)$ for bivariate Power distribution (5.11) with $K_1 = K_2 = \theta = 0.5$.

Table 5.1: Performance of $\widehat{\delta}_r(t_1|X_2 \leq t_2)$ for bivariate Power distribution (5.11) with $K_1 = K_2 = 1$ and $\theta = 0.5$.

r	t_1	t_2	Bias$\times 10^3$			MSE$\times 10^3$		
			$n=10$	$n=100$	$n=1000$	$n=10$	$n=100$	$n=1000$
0	0.2	0.2	7.6948	0.1948	3.3748	18.0152	2.5128	0.9129
	0.4	0.4	-9.6594	2.54056	4.1506	23.3923	4.35301	1.7888
	0.5	0.5	8.2314	16.2314	4.2614	24.9318	3.6815	1.5681
	0.6	0.6	-8.6192	2.2808	5.2708	20.1493	3.4074	1.2078
	0.8	0.8	-7.1672	2.2327	5.4127	13.7824	1.5892	0.4473
	0.9	0.9	6.9907	-1.7092	1.3307	7.1048	0.861	0.1754
1	0.2	0.2	3.0051	2.3399	0.9140	0.8725	0.1424	0.0659
	0.4	0.4	4.5281	5.1027	2.7722	2.4701	0.8563	0.6103
	0.5	0.5	4.6341	5.8129	2.8685	4.5850	1.3007	0.9850
	0.6	0.6	-2.7193	3.0457	3.2079	6.3422	1.7500	1.3833
	0.8	0.8	-7.2615	1.6969	3.1211	8.5525	2.6666	2.1155
	0.9	0.9	-0.9841	3.4273	4.8067	8.9388	2.9524	2.3862
2	0.2	0.2	-0.0003	0.0062	0.2511	0.0000	0.0069	0.0035
	0.4	0.4	1.9302	2.3695	1.1690	0.5084	0.1600	0.1253
	0.5	0.5	0.3379	1.9013	1.9568	1.1005	0.4032	0.3231
	0.6	0.6	5.7896	3.2270	2.5524	1.9985	0.7949	0.6745
	0.8	0.8	2.5602	4.5522	3.8733	5.2803	2.4371	2.0397
	0.9	0.9	9.3798	5.0054	6.3244	9.121	3.6764	3.134
3	0.2	0.2	1.2861	-0.0149	0.0572	0.0814	0.0003	0.0002
	0.4	0.4	0.0168	0.3605	0.3500	0.0181	0.0271	0.0228
	0.5	0.5	2.4531	0.9066	1.0250	0.2658	0.1035	0.0922
	0.6	0.6	3.6207	0.7479	1.2857	0.6693	0.3256	0.2825
	0.8	0.8	-2.4299	3.7394	3.8465	3.6838	1.7326	1.5753
	0.9	0.9	-1.8403	4.2085	6.2142	8.1337	3.5918	3.1161

5.6.2 Analysis of real data

In this section, we obtain the non-parametric estimator of $\delta_r(t_1|X_2 \le t_2)$ for the system reliability data given in ReliaSoft (2003). The data set consists of a parallel system containing two motors without censoring. The system configuration was made in such a way that when both motors are functioning, the load is shared between them. If one of the motors fail, the entire load is then shifted to the surviving motor. The system fails when both motors fail. Sutar and Naik-Nimbalkar (2014) analyzed this data to model the load sharing effect using accelerated failure time (AFT) model. We further analyzed this data to validate the properties of the proposed empirical estimator $\widehat{\delta}_r(t_1|X_2 \le t_2)$. Table 5.2 shows the time to failure data for 18 systems that contain the two motors.

Table 5.3 provides the estimates of $\delta_r(t_1|X_2 \le t_2)$ for different values of r. It is easy to see that $\widehat{\delta}_r(t_1|X_2 \le t_2)$ is non-decreasing in t_1 and t_2 for all values of r. Further, for fixed t_1 and t_2, $\widehat{\delta}_r(t_1|X_2 \le t_2)$ is non-decreasing in r.

Table 5.2: Data set for two motors in a load sharing configuration

System	Time to failure for Motor A (Days)	Time to failure for Motor B (Days)	Event
1	102	65	B Failed First
2	84	148	A Failed First
3	88	202	A Failed First
4	156	121	B Failed First
5	148	123	B Failed First
6	139	150	A Failed First
7	245	156	B Failed First
8	235	172	B Failed First
9	220	192	B Failed First
10	207	214	A Failed First
11	250	212	B Failed First
12	212	220	A Failed First
13	213	265	A Failed First
14	220	275	A Failed First
15	243	300	A Failed First
16	300	248	B Failed First
17	257	330	A Failed First
18	263	350	A Failed First

Table 5.3: Estimates of $\delta_r(t_1|X_2 \leq t_2)$ for load sharing data for different values of *r*

r	t_1	t_2					
		100	150	200	250	300	350
0	100	0	0.055556	0.055556	0.111111	0.111111	0.111111
	150	0.055556	0.222222	0.222222	0.277778	0.277778	0.277778
	200	0.055556	0.277778	0.277778	0.333333	0.333333	0.333333
	250	0.055556	0.277778	0.444444	0.666667	0.833333	0.833333
	300	0.055556	0.277778	0.444444	0.722222	0.888889	1
	350	0.055556	0.277778	0.444444	0.722222	0.888889	1
1	100	0	0.888889	0.888889	1.55556	1.55556	1.55556
	150	2.66667	7.05556	7.05556	10.5	10.5	10.5
	200	5.44444	20.6111	20.6111	26.8333	26.8333	26.8333
	250	8.22222	34.5	37.2778	50.7778	54.8889	54.8889
	300	11	48.3889	59.5	84.1111	96.5556	101
	350	13.7778	62.2778	81.7222	120.222	141	151
2	100	0	14.2222	14.2222	22.2222	22.2222	22.2222
	150	128	376.944	376.944	590.5	590.5	590.5
	200	533.556	1745.61	1745.61	2442.5	2442.5	2442.5
	250	1216.89	4501.17	4565.06	6206	6334.78	6334.78
	300	2178	8645.61	9403.94	12950.4	13907	14085.8
	350	3416.89	14178.9	16465.1	23167.1	25784.8	26685.8
3	100	0	227.556	227.556	323.556	323.556	323.556
	150	6144	22190.4	22190.4	35430.8	35430.8	35430.8
	200	52288.4	164159	164159	242211	242211	242211
	250	180100	615306	617001	860662	864995	864995
	300	431244	1583950	1636900	2255730	2331050	2338280
	350	847388	3278430	3549300	4919410	5252370	5333640

Chapter 6

Upper partial moments of bivariate weighted models*

6.1 Introduction

The concept of weighted distribution was popularized by Rao (1965) in connection with modeling statistical data in situations where the usual practice of employing standard distributions for the purpose was not found appropriate. Based on this idea, the objectives of the present chapter are two-fold. That is, to study more reliability aspects of bivariate and conditional upper partial moments in the context of weighted models as studied respectively in Chapter 3 and Chapter 4.

The present chapter is unfolded as follows. Following the introductory part, in Section 6.2, we introduce bivariate Weighted Upper Partial Moments (BWUPMs) and obtained various properties of BWUPMs. In Section 6.3, we study the bivariate equilibrium models using BWUPM. An important property known as 'stop-loss dependence', which is very useful in the context of actuarial analysis

*Contents of this chapter have been communicated to an International Journal.

is further explored in this section. In section 6.4 we investigate some important dependence concepts useful in life-length and actuarial studies and obtain its relationships with BUPM's. Finally in Section 6.5, we study properties of conditional upper partial moments in the context of weighted models.

6.2 Bivariate weighted upper partial moments

Recalling from chapter 3 that Sankaran and Nair (2004) and Hürlimann (2002) defined the $(r,s)^{th}$ Bivariate Upper Partial Moments (BUPMs) as

$$\begin{aligned} p_{r,s}(t_1,t_2) &= E\left[(X_1-t_1)^r(X_2-t_2)^s I(X_1>t_1,X_2>t_2)\right] \\ &= \int_{t_1}^{\infty}\int_{t_2}^{\infty}(x_1-t_1)^r(x_2-t_2)^s f(x_1,x_2)dx_1dx_2. \end{aligned} \tag{6.1}$$

Using (2.18) and (6.1), the partial moments for bivariate distributions under weighting is defined as follows.

Definition 6.2.1. Let $\boldsymbol{X}=(X_1,X_2)$ be a non-negative random vector admitting an absolutely continuous distribution function $F(x_1,x_2)$ with respect to a Lebesgue measure in the positive octant $\mathbb{R}_2^+=\{(x_1,x_2)|x_1,x_2>0\}$ of the two-dimensional Euclidean space $\mathbb{R}_2$. Assume that for $r,s=0,1,2,\ldots,$ $E(X_1^rX_2^s)$ is finite. Then the $(r,s)^{th}$ Bivariate Weighted Upper Partial Moment (BWUPM) corresponding to the weighted random vector $\boldsymbol{X}^w=(X_1^w,X_2^w)$ is defined as

$$\begin{aligned} p_{r,s}^w(t_1,t_2) &= \int_{t_1}^{\infty}\int_{t_2}^{\infty}(x_1-t_1)^r(x_2-t_2)^s f^w(x_1,x_2)dx_1dx_2 \\ &= \int_{t_1}^{\infty}\int_{t_2}^{\infty}(x_1-t_1)^r(x_2-t_2)^s\frac{w(x_1,x_2)}{\mu_w(x_1,x_2)}f(x_1,x_2)dx_1dx_2. \end{aligned} \tag{6.2}$$

Then $R^w(t_1,t_2)=\frac{A(t_1,t_2)}{\mu_w}R(t_1,t_2)$ provides the bivariate reliability function associated with (X_1^w, X_2^w), where, $A(t_1, t_2) = E(w(X_1, X_2) | X_1 > t_1, X_2 > t_2)$ and $\mu_w = E(w(X_1, X_2))$ (Navarro et al. (2006)). Table 6.1 provides the structural forms of $p_{r,s}^w(t_1, t_2)$ for some popular bivariate weight functions.

Analogous to (6.1), BWUPM's also uniquely determines the corresponding bivariate weighted distribution.

Theorem 6.2.1. *For any two positive integers (r, s), $p_{r,s}^w(t_1, t_2)$ determines the corresponding bivariate weighted distribution uniquely through the relationship,*

$$\frac{\partial^{r+s}}{\partial t_1^r \partial t_2^s} p_{r,s}^w(t_1, t_2) = (-1)^r (-1)^s r!\, s!\, R^w(t_1, t_2). \tag{6.3}$$

Proof. From (6.2), it is easy to see that

$$p_{r,s}^w(t_1, t_2) = r\, s \int_{t_1}^{\infty} \int_{t_2}^{\infty} (x_1 - t_1)^{r-1} (x_2 - t_2)^{s-1} R^w(x_1, x_2) dx_1 dx_2, \tag{6.4}$$

Now differentiating both sides of (6.4) successively r, s times we have the required result. □

Among the different weight functions available in the bivariate case, an important one is $w_1(x_1, x_2) = x_1 x_2$, known as the area sampling or its size-biased version $w(x_1, x_2) = x_1^{\alpha_1} x_2^{\alpha_2}$. In agriculture it is often encounter to estimate the average yield produced in unit area, then the weighted model with $w(x_1, x_2) = x_1 x_2$ corresponds to the situation where the probability of choosing a specified sample is proportional to its area (Nair and Sunoj (2003)). In this context, we have the following theorem on BWUPM.

Theorem 6.2.2. *The $BWUPM$ of the weighted model $f^w(x_1, x_2)$ with the weight func-*

Table 6.1: Bivariate weighted partial moments for different weight functions

Sl. No	$w(x_1, x_2)$	$p^w_{r,s}(t_1, t_2)$
1	x_1	$\frac{p_{r+1,s}(t_1,t_2)+t_1 p_{r,s}(t_1,t_2)}{E(X_1)}$
2	x_2	$\frac{p_{r,s+1}(t_1,t_2)+t_2 p_{r,s}(t_1,t_2)}{E(X_2)}$
3	$x_1 + x_2$	$\frac{(t_1+t_2)p_{r,s}(t_1,t_2)+p_{r+1,s}(t_1,t_2)+p_{r,s+1}(t_1,t_2)}{E(X_1+X_2)}$
4	$x_1 x_2$	$\frac{p_{r+1,s+1}(t_1,t_2)+t_2 p_{r+1,s}(t_1,t_2)+t_1 p_{r,s+1}(t_1,t_2)+t_1 t_2 p_{r,s}(t_1,t_2)}{E(X_1 X_2)}$
5	$x_2 - x_1, x_1 \leq x_2$	$\frac{(t_2-t_1)p_{r,s}(t_1,t_2)+p_{r,s+1}(t_1,t_2)-p_{r+1,s}(t_1,t_2)}{E(X_2-X_1)}$
6	$max(x_1, x_2)$	$\begin{cases} \frac{p_{r+1,s}(t_1,t_2)+t_1 p_{r,s}(t_1,t_2)}{E(X_1)}, & t_1 > t_2 \\ \frac{p_{r,s+1}(t_1,t_2)+t_2 p_{r,s}(t_1,t_2)}{E(X_2)}, & t_1 < t_2 \end{cases}$
7	$min(x_1, x_2)$	$\begin{cases} \frac{p_{r,s+1}(t_1,t_2)+t_2 p_{r,s}(t_1,t_2)}{E(X_2)}, & t_1 > t_2 \\ \frac{p_{r+1,s}(t_1,t_2)+t_1 p_{r,s}(t_1,t_2)}{E(X_1)}, & t_1 < t_2 \end{cases}$

tion $w(x_1, x_2) = x_1^{\alpha_1}\, x_2^{\alpha_2}$, *satisfies the relationship*

$$p^w_{r,s}\,(t_1, t_2) = \sum_{u=0}^{\alpha_1} \sum_{v=0}^{\alpha_2} \begin{pmatrix} \alpha_2 \\ v \end{pmatrix} \begin{pmatrix} \alpha_1 \\ u \end{pmatrix} t_1^u\; t_2^v\; p_{m,n}\,(t_1, t_2),$$

where $\begin{pmatrix} n \\ r \end{pmatrix} = \frac{n!}{r!\;(n-r)!}$, $m = r + \alpha_1 - u$ *and* $n = s + \alpha_2 - v.$

Proof. The proof directly follows from the binomial expansion of the terms in the definition of BWUPM. □

The next theorem examines the monotonicity of bivariate partial moments under weighting.

Theorem 6.2.3. *For the random vector* (X_1^w, X_2^w)*, the bivariate weighted partial moment,* $p_{r,s}^w(t_1, t_2)$ *is*

(a) decreasing in $t_i, i = 1, 2$ *for fixed value of* r *and* s.

(b) increasing in r *or* s *for fixed values values of* t_1 *and* t_2 *if both* $x_1 - t_1 > 1$ *and* $x_2 - t_2 > 1$ *holds true and decreasing in* r *or* s *for fixed values values of* t_1 *and* t_2 *if either* $0 < x_1 - t_1 < 1$ *or* $0 < x_2 - t_2 < 1$ *or both holds true.*

Proof. To prove (a), from (6.2) we have

$$\frac{\partial p_{r,s}^w(t_1, t_2)}{\partial t_1} = -r p_{r-1,s}^w(t_1, t_2) \tag{6.5}$$

and

$$\frac{\partial p_{r,s}^w(t_1, t_2)}{\partial t_2} = -s p_{r,s-1}^w(t_1, t_2)\,. \tag{6.6}$$

From (6.5) and (6.6) it follows that $p_{r,s}^w(t_1, t_2)$ is decreasing in $t_i, i = 1, 2$ irrespective of the weight function $w(x_1, x_2)$ for all values of r and s.

In order to prove (b), note that since $r(s)$ takes positive integer values, $p_{r,s}^w(t_1, t_2)$ is increasing in $r(s)$ if and only if $p_{r,s}^w(t_1, t_2) - p_{r-1,s}^w(t_1, t_2)$ is non-negative for all values of r ($p_{r,s}^w(t_1, t_2) - p_{r,s-1}^w(t_1, t_2)$ is non-negative for all values of s). Now consider,

$$p_{r,s}^w(t_1, t_2) - p_{r-1,s}^w(t_1, t_2) = \int_{t_1}^{\infty}\int_{t_2}^{\infty} (x_1 - t_1)^{r-1}(x_2 - t_2)^s(x_1 - t_1 - 1) f^w(x_1, x_2) dx_1 dx_2 \tag{6.7}$$

and

$$p^w_{r,s}\,(t_1,t_2) - p^w_{r,s-1}\,(t_1,t_2) = \int\limits_{t_1}^{\infty}\int\limits_{t_2}^{\infty} (x_1-t_1)^r(x_2-t_2)^{s-1}(x_2-t_2-1)f^w(x_1,x_2)dx_1dx_2 \tag{6.8}$$

Hence (b) follows from (6.6) and (6.7). □

Arnold and Nagaraja (1991) have shown that the independence of X_1 and X_2 is equivalent to the independence of X_1^w and X_2^w, and equivalent to the weight function $w(x_1,x_2) = w(x_1)w(x_2)$. Then, we have the following result.

Theorem 6.2.4. *For an arbitrary weight function $w(x_1,x_2)$, any two of the following statements together imply the third.*

(a) X_1 and X_2 are independent.

(b) X_1^w and X_2^w are independent.

(c) $w(x_1,x_2)$ is of the form $w(x_1)w(x_2)$ for $(x_1,x_2) \in S_{x_1} \times S_{x_2}$, the Cartesian product space of S_{x_1} and S_{x_2}

(d) $p^w_{r,s}\,(t_1,t_2) = p^w_r\,(t_1)\,p^w_s\,(t_2)$, where $p^w_r\,(t_1)$ and $p^w_s\,(t_2)$ are the univariate partial moments corresponding to the weighted random variables X_1^w and X_2^w respectively.

The assumption of independence is seldom valid in practice. For example, a two components system always exhibit dependence among its component's lifetimes. Thus it is natural to assume a dependence among the components of a system. There are various notions of bivariate dependence that are used in literature (see Shaked and Shanthikumar (2007)). A popular one is the Positive (Negative) Quadrant Dependence (PQD (NQD)) property. A random vector $\boldsymbol{X} = (X_1,X_2)$ satisfies the Positive (Negative) Quadrant Dependence denoted by PQD (NQD) if for all values of $t_1, t_2 > 0$,

$$P(X_1 > t_1, X_2 > t_2) \geq (\leq) P(X_1 > t_1) P(X_2 > t_2). \tag{6.9}$$

Equivalently, the bivariate partial moments, (X_1, X_2) satisfies the Positive Quadrant Dependence (PQD) property if and only if

$$p_{r,s}\left(t_1, t_2\right) \geq (\leq)\, p_r\left(t_1\right) p_s\left(t_2\right) \tag{6.10}$$

for all values of r, s and $t_i > 0, i = 1, 2$.

To this effect we have the following theorem.

Theorem 6.2.5. *Assume that $\boldsymbol{X}$ has PQD (NQD) property. Then $\boldsymbol{X^w}$ has property,*

$$p^w_{r,s}\left(t_1, t_2\right) \geq (\leq)\, p^w_r\left(t_1\right) p^w_s\left(t_2\right). \tag{6.11}$$

if and only if

(i) $\boldsymbol{X^w}$ *has the joint pdf* $f^w\left(x_1, x_2\right) = \frac{w_1(x_1) w_2(x_2) f(x_1, x_2)}{E(w_1(X_1) w_2(X_2))}$ *with* $E\left(w_1\left(X_1\right) w_2\left(X_2\right)\right) = E\left(w_1\left(X_1\right)\right) E\left(w_2\left(X_2\right)\right)$.

(ii) $\boldsymbol{X^w}$ *possess PQD (NQD) property.*

Proof. Let the bivariate random vector $\boldsymbol{X} = (X_1, X_2)$ satisfies the PQD (NQD) property. Hence $R(t_1,t_2) \geq (\leq) R_1(t_1) R_2(t_2) \iff f(t_1,t_2) \geq (\leq) f_1(t_1) f_2(t_2)$.

$$\iff \frac{w_1(x_1)w_2(x_2)f(x_1,x_2)}{E(w_1(X_1)w_2(X_2))} \geq (\leq) \frac{w_1(x_1)f_1(x_1)}{E(w_1(X_1))}\frac{w_2(x_2)f_2(x_2)}{E(w_2(X_2))}$$

$$\iff \int_{t_1}^{\infty}\int_{t_2}^{\infty}(x_1-t_1)^r(x_2-t_2)^s\frac{w_1(x_1)w_2(x_2)f(x_1,x_2)}{E(w_1(X_1)w_2(X_2))}dx_1dx_2$$

$$\geq (\leq) \int_{t_1}^{\infty}(x_1-t_1)^r\frac{w_1(x_1)f_1(x_1)}{E(w_1(X_1))}dx_1\int_{t_2}^{\infty}(x_2-t_2)^r\frac{w_2(x_2)f_2(x_2)}{E(w_2(X_2))}dx_2$$

$$\iff p^w_{r,s}(t_1,t_2) \geq (\leq)\ p^w_r(t_1)\, p^w_s(t_2)$$

$$\iff R^w(t_1,t_2) \geq (\leq)\ R^w_1(t_1)\, R^w_2(t_2)$$

Hence the theorem. □

Corresponding to a non-negative bivariate random vector (X_1^w, X_2^w) Navarro et al. (2006) defined the bivariate reliability function associated with (2.18) as

$$R^w(t_1,t_2) = \frac{m_w(t_1,t_2)}{E(w(X_1,X_2))}R(t_1,t_2),$$

where $m_w(.)$ is the generalized conditional expectation function defined as

$$m_w(t_1,t_2) = E(w(X_1,X_2)\,|X_1 > t_1, X_2 > t_2).$$

Now in terms of the BWUPM we have,

$$R^w(t_1,t_2) = \frac{p^w_{0,0}(t_1,t_2)}{p^w_{0,0}(0,0)}$$

Basu (1971) defined the bivariate scalar failure rate by $k(t_1,t_2) = \frac{f(t_1,t_2)}{R(t_1,t_2)}$. Accordingly, from the uniqueness theorem of BWUPM an extension of $k(t_1,t_2)$ into weighted set-up is defined as follows

$$k^w(t_1,t_2)=\frac{\frac{\partial^{r+s+2}}{\partial t_1^{r+1}\partial t_2^{s+1}}\left(p_{r,s}^w(t_1,t_2)\right)}{\frac{\partial^{r+s}}{\partial t_1^r\partial t_2^s}\left(p_{r,s}^w(t_1,t_2)\right)}$$

The following theorem gives the relation connecting the BUPM and BWUPM for the residual life distribution in bivariate set-up.

Theorem 6.2.6. *The survival function of a bivariate non-negative random vector* (X_1, X_2) *satisfies the relationship*

$$R(t_1,t_2)=\frac{p_{r,s}(2t_1,2t_2)}{p_{r,s}^w(t_1,t_2)} \tag{6.12}$$

if it follows a bivariate residual life distribution defined by the p.d.f

$$f^w(x_1,x_2)=\frac{f(x_1+t_1,x_2+t_2)}{R(t_1,t_2)}.$$

6.3 Equilibrium models

The concept of equilibrium distribution plays an important role in survival analysis, reliability and insurance studies (see Gupta (2007), Navarro and Sarabia (2010)). The equilibrium distribution (or stationary renewal distribution) arises as the limiting distribution of the forward recurrence time in a renewal process. There has been many attempts to extend the concept into higher dimensions (see Gupta and Sankaran (1998), Navarro et al. (2006) and Nair and Preeth (2008)). Navarro et al. (2006) defined the bivariate equilibrium distribution as

$$f^{eq}(t_1,t_2)=\frac{R(t_1,t_2)}{E(X_1X_2)} \tag{6.13}$$

Now the following implication is direct.

Theorem 6.3.1. *The $BWUPM$ of a bivariate random vector $\boldsymbol{X}$ satisfies the relationship*

$$\frac{p_{r,s}^{w}\left(t_1,t_2\right)}{p_{r,s}\left(t_1,t_2\right)}=E\left(X_1X_2\right)$$

if it follows a bivariate equilibrium distribution defined by the p.d.f (6.13)

6.3.1 Bivariate equilibrium distributions of order *n*

Let $\boldsymbol{X}=(X_1,X_2)$ denotes a non-negative random vector defined on the positive octant $\mathbb{R}_2^{+}=\{(x_1,x_2)|x_1,x_2>0\}$ of the two dimensional Euclidean space $\mathbb{R}_2$ with the absolutely continuous survival function $R(t_1,t_2)$ and let $(X_{n,1},X_{n,2})$ be a random vector having survival function $R_n\left(t_1,t_2\right)$ defined by,

$$R_n\left(t_1,t_2\right)=\frac{\int_{t_1}^{\infty}\int_{t_2}^{\infty}R_{n-1}\left(u,v\right)dudv}{\int_{0}^{\infty}\int_{0}^{\infty}R_{n-1}\left(u,v\right)dudv},n=1,2,\ldots$$

with $R_0\left(t_1,t_2\right)=R\left(t_1,t_2\right)$ and the density function

$$f_n\left(t_1,t_2\right)=\frac{R_{n-1}\left(t_1,t_2\right)}{\mu_{n-1:2}},$$

where $\mu_{n,2}=E\left(X_{n,1}X_{n,2}\right)=\int_{0}^{\infty}\int_{0}^{\infty}R_n\left(u,v\right)dudv<\infty$. Then $(X_{n,1},X_{n,2})$ is said to have a bivariate equilibrium distribution of order n based on $R\left(t_1,t_2\right)$ (Nair and Preeth (2008)).

In the following theorem, we obtain the bivariate equilibrium distribution of order n based on BUPM, a bivariate extension of Theorem 3.2 given in Gupta (2007). It provides an alternative method of construction of the bivariate equilibrium distribution of order n, using BUPM.

Theorem 6.3.2. *The survival function of the bivariate equilibrium distribution of order n is given by*

$$R_n(t_1,t_2) = \frac{p_{n,n}(t_1,t_2)}{(n!)^2 \prod\limits_{i=0}^{n-1} \mu_{i:2}} \tag{6.14}$$

Proof. Setting $r = s = n$ in (6.1), we have

$$\begin{aligned}
p_{n,n}(t_1,t_2) &= E\left[(X_1-t_1)_+^n (X_2-t_2)_+^n\right] \\
&= \int_{t_1}^{\infty}\int_{t_2}^{\infty} (x_1-t_1)^n (x_2-t_2)^n f(x_1,x_2)\, dx_1 dx_2 \\
&= n^2 \int_{t_1}^{\infty}\int_{t_2}^{\infty} (x_1-t_1)^{n-1} (x_2-t_2)^{n-1} R(x_1,x_2)\, dx_1 dx_2 \\
&= n^2 \mu_{0:2} \int_{t_1}^{\infty}\int_{t_2}^{\infty} (x_1-t_1)^{n-1} (x_2-t_2)^{n-1} f_1(x_1,x_2)\, dx_1 dx_2 \\
&= n^2 (n-1)^2 \mu_{0:2} \int_{t_1}^{\infty}\int_{t_2}^{\infty} (x_1-t_1)^{n-2} (x_2-t_2)^{n-2} R_1(x_1,x_2)\, dx_1 dx_2 \\
&= n^2 (n-1)^2 \mu_{0:2}\, \mu_{1:2} \int_{t_1}^{\infty}\int_{t_2}^{\infty} (x_1-t_1)^{n-2} (x_2-t_2)^{n-2} f_2(x_1,x_2)\, dx_1 dx_2.
\end{aligned}$$

Proceeding in this manner, yields

$$\begin{aligned}
p_{n,n}(t_1,t_2) &= (n!)^2 \mu_{0:2}\, \mu_{1:2} \ldots \mu_{n-1:2} \int_{t_1}^{\infty}\int_{t_2}^{\infty} f_{n-1}(x_1,x_2)\, dx_1 dx_2 \\
&= (n!)^2 \prod_{i=0}^{n-1} \mu_{i:2}\, R_n(t_1,t_2),
\end{aligned}$$

completes the proof. □

6.3.2 Stop-loss dependence

In actuarial sciences, $(X-t)_+$ represents the financial loss incurred by an insurance company, called *risks*. The r^{th} moment $E[(X-t)_+]^r$ is usually referred to as the r^{th} degree stop-loss transform of the risk X and is a standard measure of dangerousness of X (see Cheng and Pai (2003)). The study of the impact of dependence among risks has become a major topic in actuarial science, in which the assumption of mutual independence of risks is often violated in practice. Actuaries intuitively feel that positive correlations between individual risks reveal a more dangerous situation compared to independence (Denuit et al. (2001)). In this context, Denuit et al. (2006) introduced the concept known as the Positive Stop-Loss Dependence (PSLD). We consider now the first two Stop-Loss Dependence (PSLD) orders for our study.

Definition 6.3.1. Two random variables X_1 and X_2 are said to be positively stop-loss dependent (PSLD,) if the inequalities

$$E\left[(X_1-t_1)_+|X_2>t_2\right]\geq E\left[(X_1-t_1)_+\right]$$

and

$$E\left[(X_2-t_2)_+|X_1>t_1\right]\geq E\left[(X_2-t_2)_+\right]$$

hold for all $t_1,t_2\in\mathbb{R}^+$.

Definition 6.3.2. Let $\psi_r\left(t_i|t_j\right);\, i,j=1,2, i\neq j$ and $p_r(t)$ denote the r^{th} CUPM (see Sunoj and Vipin (2017)) and r^{th} univariate partial moments respectively. Then the two random variables X_1 and X_2 are said to be

(i) First order Positively (Negatively) Stop-Loss Dependent (denoted by FPSLD

(FNSLD)), if the inequalities

$$\psi_1\left(t_i|t_j\right) \geq (\leq)\; p_1\left(t_i\right),\; i,j = 1,2, i \neq j.$$

hold for all $t_i,\; i = 1,2 \in \mathbb{R}^+$.

(ii) Second order Positively (Negatively) Stop-Loss Dependent (denoted by SPSLD (SNSLD)), if the inequalities

$$\psi_2\left(t_i|t_j\right) \geq (\leq)\; p_2\left(t_i\right),\; i,j = 1,2, i \neq j.$$

hold for all $t_i,\; i = 1,2 \in \mathbb{R}^+$.

Clearly, the above two stop loss dependence concepts imply each other. i.e X_1 and X_2 is FPSLD (FNSLD)) $\Leftrightarrow$ X_1 and X_2 is SPSLD (SNSLD)).

Let $R^w(t_1,t_2)$ denotes the survival function corresponding to (X_1^w, X_2^w), with marginal survival functions of X_1^w and X_2^w respectively by, $R_1^w\left(t_1\right) = \frac{A_1(t_1)}{\mu_w} R_1\left(t_1\right)$ and $R_2^w\left(t_2\right) = \frac{A_2(t_2)}{\mu_w} R_2\left(t_2\right)$, where $A_1\left(t_1\right) = A\left(t_1, 0\right)$ and $A_2\left(t_2\right) = A\left(0, t_2\right)$. In the next theorem, obtain the relationship between $\boldsymbol{X}$ and $\boldsymbol{X^w}$ based on the PSLD (NSLD) property for any general weight function $w(x_1,x_2)$.

Theorem 6.3.3. *If $\boldsymbol{X}$ is PSLD (NSLD), then $\boldsymbol{X^w}$ is also PSLD (NSLD) when* $\frac{A(t_1,t_2)\mu_w}{A_1(t_1)A_2(t_2)} \leq (\geq)\, 1$.

Proof. We prove the general case here that is for $r = 1,2$. Suppose

$$\begin{aligned} \boldsymbol{X} = (X_1, X_2) \text{ is PSLD (NSLD)} \quad &\Leftrightarrow \quad \psi_r\left(t_i|t_j\right) \geq (\leq)\; p_r\left(t_i\right),\; i,j = 1,2, i \neq j \\ &\Leftrightarrow \quad \frac{R_1\left(t_1\right) R_2\left(t_2\right)}{R\left(t_1,t_2\right)} \leq (\geq) 1 \end{aligned} \tag{6.15}$$

Now consider,

$$\psi_r^w(t_i|t_j) \geq (\leq)\, p_r^w(t_i),\ i,j=1,2, i\neq j, \tag{6.16}$$

where $\psi_r^w(t_i|t_j), i,j = 1,2, i \neq j$ is the r^{th} order CUPM for the weighted models. Differentiating both sides of (6.16) with respect to t_i, r times and after some simplification yields

$$\begin{aligned} &\Leftrightarrow\ R_i^w(t_i|t_j) \geq (\leq) R_i^w(t_i),\ \ i,j=1,2, i\neq j \\ &\Leftrightarrow\ \frac{A(t_1,t_2)\,\mu_w}{A_1(t_1)\,A_2(t_2)} \leq (\geq) \frac{R_1(t_1)\,R_2(t_2)}{R(t_1,t_2)} \end{aligned} \tag{6.17}$$

Now (6.15) and (6.17) together implies that $\boldsymbol{X^W}$ is PSLD (NSLD) if $\frac{A(t_1,t_2)\mu_w}{A_1(t_1)A_2(t_2)} \leq (\geq) 1$. □

6.4 Dependence measures

In this section, we consider some importance dependence useful in life-length studies and obtain its relationships with BUPM's.

6.4.1 Expectation dependence

Expectation dependence (Wright (1987)) is an important concept in fields such as finance, insurance and asset pricing.

Definition 6.4.1 (Wright (1987))**.** Consider two random variables X_1 and X_2 defined on $[a_1, b_1] \times [a_2, b_2], -\infty < a_i < \infty, -\infty < b_i < \infty, i = 1, 2$, respectively. Then the random variable X_1 is Positive Expectation Dependent on X_2 denoted by $PED(X_1|X_2)$ if

$$E[X_1] \geq E[X_1|X_2 \leq t_2] \ \text{ for all } \ t_2. \tag{6.18}$$

The condition (6.18) states that when the knowledge of X_2 is small (i.e., below the threshold t_2) increases the expected value of X_1. In the following we investigate the usefulness and applications of expectation dependence for bivariate weighted models. Let us assume that the components X_1^w and X_2^w exhibits some dependence. Then, we have the following result.

Theorem 6.4.1. *Let* $\boldsymbol{X^w} = (X_1^w, X_2^w)$ *be a non-negative bivariate dependent weighted random vector defined on* $[0,\infty] \times [0,\infty]$. *Then*

$$E[X_1^w] \geq E[X_1^w|X_2^w \leq t_2] \Leftrightarrow E[X_1^w] \leq E[X_1^w|X_2^w > t_2] \text{ for all } t_2. \tag{6.19}$$

Proof. Note that

$$E\left(X_1^w\right) = E\left(X_1^w|X_2^w \leq t_2\right) F_2^w\left(t_2\right) + E\left(X_1^w|X_2^w > t_2\right) R_2^w\left(t_2\right), \tag{6.20}$$

where $F_2^w(.)$ and $R_2^w(.)$ are the distribution function and the survival function corresponding to the random variable X_2^w respectively. Also we have,

$$E\left(X_1^w\right) = E\left(X_1^w\right) F_2^w\left(t_2\right) + E\left(X_1^w\right) R_2^w\left(t_2\right). \tag{6.21}$$

From (6.20) and (6.21) it follows that

$$R_2^w\left(t_2\right)\left[E\left(X_1^w|X_2^w > t_2\right) - E\left(X_1^w\right)\right] = F_2^w\left(t_2\right)\left[E\left(X_1^w\right) - E\left(X_1^w|X_2^w \leq t_2\right)\right] \tag{6.22}$$

Now applying the definition (6.18), from (6.22) it follows that

$$E[X_1^w] \leq E[X_1^w|X_2^w > t_2] \text{ for all } t_2.$$

Hence the result. □

Theorem 6.4.2. *Let $\boldsymbol{X^w} = (X_1^w, X_2^w)$ be a non-negative bivariate dependent weighted random vector defined on $[0,\infty] \times [0,\infty]$. Then*

$$(X_1^w, X_2^w) \text{ is PSLD} \Leftrightarrow PED(X_1^w|X_2^w) \Leftrightarrow Corr\left[X_1^w, I\left[X_2^w > t_2\right]\right] \geq 0, \ \text{for all} \ t_2 \tag{6.23}$$

where $I[.]$ is the usual indicator function.

Proof.

$$\begin{aligned}
(X_1^w, X_2^w) \text{ is PSLD} \ &\Leftrightarrow \ \psi_r^w(t_1|t_2) \geq p_r^w(t_1) \quad \text{for all } \ t_2 \\
&\Leftrightarrow \ R_1^w(t_1|t_2) \geq R_1^w(t_1), \quad \text{for all } \ t_2 \\
&\Leftrightarrow \ E[X_1^w] \leq E[X_1^w|X_2^w > t_2], \quad \text{for all } \ t_2 \\
&\Leftrightarrow \ PED(X_1^w|X_2^w), \quad \text{for all } \ t_2.
\end{aligned}$$

Now consider

$$\begin{aligned}
PED(X_1^w|X_2^w) \ \text{ for all } \ t_2 \ &\Leftrightarrow \ E\left(X_1^w|X_2^w > t_2\right) \geq E\left(X_1^w\right) \quad \text{for all } \ t_2 \\
&\Leftrightarrow \ R_2^w(t_2)\left[E\left(X_1^w|X_2^w > t_2\right) - E\left(X_1^w\right)\right] \geq 0, \quad \text{for all } \ t_2 \\
&\Leftrightarrow \ E\left(X_1^w I\left[X_2^w > t_2\right]\right) - E\left(X_1^w\right) E\left(I\left[X_2^w > t_2\right]\right) \geq 0, \\
&\qquad\qquad \text{for all } \ t_2 \\
&\Leftrightarrow \ Cov\left[X_1^w, I\left[X_2^w > t_2\right]\right] \geq 0, \quad \text{for all } \ t_2 \\
&\Leftrightarrow \ Corr\left[X_1^w, I\left[X_2^w > t_2\right]\right] \geq 0, \quad \text{for all } t_2
\end{aligned} \tag{6.24}$$

□

The above result shows that the two positive stop loss dependence concepts (i.e FPSLD and SPSLD) and first degree positive expectation dependence are equivalent to the positive correlation between X_1^w and $I[X_2^w > t_2]$.

6.4.2 Stop-loss distance for weighted models

In actuarial studies, models are often compared by finding bounds for differences in probabilities, that is, (integrated) differences between their respective probability density functions or distribution functions. However, this approach is often questionable when tackling insurance problems. Instead, the focus should be on the resulting premiums, in particular stop-loss premiums, $p_1(t) = E\,(X-t)_+$. The reason for this is obvious: small variations in the probabilities of events hardly ever influence the decision made by insurance management. Any difference between calculated premiums, however, will be directly visible (see Denuit et al. (2006)). Motivated with this, Gerber (1979) introduced the concept of stop-loss distance which is defined as follows

Definition 6.4.2. Given two rvs X_1 and X_2, the stop loss distance d_{SL} is defined as

$$d_{SL}\left(X_1, X_2\right) = \sup_{t\in\mathbb{R}^+} \left|p_1\left(t\right) - q_1\left(t\right)\right|, \tag{6.25}$$

where $p_1(.)$ and $q_1(.)$ are the first-order partial moments corresponding to the random variables X_1 and X_2 respectively.

An important stochastic order that lies closely in association with the stop-loss distance is the *stop-loss order* which is defined as follows

Definition 6.4.3. Given two random variables X_1 and X_2, X_1 is said to precede X_2 in the stop-loss order, written as $X_1 \leq_{SL} X_2$, if $p_1(t) \leq q_1(t)$ for all $t \in \mathbb{R}^+$.

In the following theorems, we compare weighted random variables based on the stop-loss order.

Theorem 6.4.3. *Let X_1^w and X_2^w be two weighted random variables. Then*

$$X_1^w \leq_{SL} X_2^w \Leftrightarrow X_1 \leq_{SL} X_2$$

if and only if $\frac{m_{w_1}(t)}{\mu_{w_1}} \leq \frac{m_{w_2}(t)}{\mu_{w_2}}$, *where* $m_{w_i}(t) = E(w(X_i)|X_i > t)$, $i = 1, 2$ *and* $0 < \mu_{w_i} = E(w(X_i)) < \infty,\ i = 1, 2.$

Proof. Let $p_1^w(t)$ and $q_1^w(t)$ denote the partial moments corresponding to the univariate weighted random variables X_1^w and X_2^w respectively. Then, by virtue of Definition 6.4.3, we have

$$\begin{aligned}
X_1^w \leq_{SL} X_2^w &\Leftrightarrow p_1^w(t) \leq q_1^w(t) \\
&\Leftrightarrow R_1^w(t) \leq R_2^w(t) \\
&\Leftrightarrow \frac{E(w(X_1)|X_1 > t)}{E(w(X_1))} R_1(t) \leq \frac{E(w(X_2)|X_2 > t)}{E(w(X_2))} R_2(t) \\
&\Leftrightarrow R_1(t) \leq R_2(t) \\
&\Leftrightarrow X_1 \leq_{SL} X_2.
\end{aligned}$$

□

Theorem 6.4.4. *Let* X_1^w, X_2^w *and* X_3^w *be three univariate weighted random variables. Then*

$$X_1^w \leq_{SL} X_2^w \leq_{SL} X_3^w \Rightarrow d_{SL}(X_1^w, X_2^w) \leq d_{SL}(X_1^w, X_3^w.)$$

Proof. From definition (6.25),

$$\begin{aligned}
d_{SL}(X_1^w, X_2^w) &= \sup_{t \in R^+} \left| E\left[(X_1^w - t)_+\right] - E\left[(X_2^w - t)_+\right]\right| \\
&\leq \sup_{t \in R^+} \left| E\left[(X_1^w - t)_+\right] - E\left[(X_3^w - t)_+\right]\right| \\
&= d_{SL}(X_1^w, X_3^w).
\end{aligned}$$

Hence the result. □

6.4.3 Positive (negative) dependence/association measures

In this section, we propose alternative definitions to positive (negative) dependence of a bivariate density and association measure due to Clayton (1978) using BUPM's. First, we consider the positive (negative) dependence based on the totally positive order 2 condition using BUPM's.

Definition 6.4.4. Let $\mathscr{X}$ and $\mathscr{Y}$ be subsets of real line. A function $f(x, y)$ is said to be totally positive order 2 (TP_2) (reverse regular of order 2 (RR_2)) if

$$f\left(x_1, y_1\right) f\left(x_2, y_2\right) \geq (\leq)\, f\left(x_1, y_2\right) f\left(x_2, y_1\right) \tag{6.26}$$

for all $x_1 \leq x_2$ in $\mathscr{X}$ and $y_1 \leq y_2$ in $\mathscr{Y}$.

Multiplying both sides of (6.26) by $(x_1 - t_1)^r (y_1 - u_1)^s (x_2 - t_2)^r (y_2 - u_2)^s$ and integrating both sides by the respective variables, Definition 6.4.4 can also be restated in terms of BUPMs as follows.

Definition 6.4.5. A joint probability density function $f(x, y)$ is said to be totally positive order 2 (TP_2) (reverse regular of order 2 (RR_2)) if

$$p_{r,s}\left(t_1, u_1\right) p_{r,s}\left(t_2, u_2\right) \geq (\leq)\, p_{r,s}\left(t_1, u_2\right) p_{r,s}\left(t_2, u_1\right) \tag{6.27}$$

for all $t_1 \leq t_2$ in $\mathscr{X}$ and $u_1 \leq u_2$ in $\mathscr{Y}$.

Example 6.4.1 (Bivariate Farlie-Gumbel-Morgenstern (FGM))**.** Let $\boldsymbol{X}$ follows a bivariate Farlie-Gumbel-Morgenstern (FGM) distribution defined by the joint CDF

$$F\left(x_1, x_2\right) = x_1 x_2 + \alpha x_1 x_2 (1 - x_1)(1 - x_2), 0 \leq x_1 \leq 1, , 0 \leq x_2 \leq 1, -1 \leq \alpha \leq 1.$$

Now applying the definition (6.1) it follows that

$$\begin{aligned} p_{r,s}(t_1,t_2) &= \frac{rs}{36}\left(t_2\left(18+t_2(2\alpha t_2-3\alpha+9)\right)\right)+(1-t_1)^r(1-t_2)^s \\ &+\frac{rs}{36}\left(\alpha+2\alpha(t_2-1)^2(2t_2+1)t_1^3-3(t_2-1)^2t_1^2(\alpha+2\alpha t_2-3)\right) \\ &-\frac{rs}{36}\left(18(t_2^2-1)t_1-27\right). \end{aligned} \tag{6.28}$$

Now consider the simplest case, $r=s=1$, then from (6.28) it follows that

$$p_{1,1}\left(t_1,t_2\right)=\frac{1}{36}(t_1-1)^2(t_2-1)^2\left(\alpha+2\alpha\left(t_2+t_1\left(2t_2+1\right)\right)+9\right). \tag{6.29}$$

Then it is easy to verify that the terms defined in the inequality (6.27) and the first order BUPM given in (6.29) together implies that $\boldsymbol{X}$ is $TP_2(RR_2)$ when $\alpha \geq (\leq)\, 0$ for all $t_1 \leq t_2$ in $\mathscr{X}$ and $u_1 \leq u_2$ in $\mathscr{Y}$.

Analogously one can define the $TP_2\,(RR_2)$ property for the weighted p.d.f $f^w(x,y)$ using BWUPM as follows

$$p^w_{r,s}\left(t_1,u_1\right)p^w_{r,s}\left(t_2,u_2\right) \geq (\leq)\, p^w_{r,s}\left(t_1,u_2\right)p^w_{r,s}\left(t_2,u_1\right) \tag{6.30}$$

for all $t_1 \leq t_2$ in $\mathscr{X}$ and $u_1 \leq u_2$ in $\mathscr{Y}$.

Next we consider the well-known measure of association due to Clayton (1978), defined as

$$\theta\left(t_1,t_2\right)=\frac{f\left(t_1,t_2\right)R\left(t_1,t_2\right)}{R_1\left(t_1,t_2\right)R_2\left(t_1,t_2\right)},$$

where $R_1\left(t_1,t_2\right)=\frac{\partial R(t_1,t_2)}{\partial t_1}$ and $R_2\left(t_1,t_2\right)=\frac{\partial R(t_1,t_2)}{\partial t_2}$. When X_1 and X_2 are positively (negatively) associated, $\theta(t_1,t_2)>(<)1$ and $\theta(t_1,t_2)=1$ implies independence of X_1 and X_2. Gupta (2001) further proved that $\theta(t_1,t_2)$ can be expressed as

$$\theta\left(t_1,t_2\right)=\frac{h_{X_1|X_2=t_2}\left(t_1\right)}{h_1\left(t_1,t_2\right)},$$

where $h_1(t_1,t_2)=h_{X_1|X_2>t_2}(t_1)$ is the first component of the vector valued failure rate defined by Johnson and Kotz (1975). In terms of the BUPMs, we have the equivalent representation

$$\theta\left(t_1,t_2\right)=\frac{\nabla_1^{r+1}\left(p_{r,0}^{*}\left(t_1,t_2\right)\right)}{\nabla_1^{r+1}\left(p_{r,0}^{*}\left(t_1,t_2\right)\right)}\frac{\nabla_1^{r}\left(p_{r,0}\left(t_1,t_2\right)\right)}{\nabla_1^{r}\left(p_{r,0}\left(t_1,t_2\right)\right)},$$

where $p_{r,0}^{*}\left(t_1,t_2\right)$ and $p_{r,0}\left(t_1,t_2\right)$ are the $(r,0)^{th}$ BUPM corresponding to the survival functions $S(t_1,t_2)=P(X_1>t_1,X_2=t_2)$ and $R(t_1,t_2)=P(X_1>t_1,X_2>t_2)$ respectively and where $\nabla_i^r(f)$ denotes the r^{th} partial derivative of the function f with respect to t_i defined by $\nabla_i^r(f)=\frac{\partial^r}{\partial t_i^r}, i=1,2$. Extension of this measure into the weighted set-up is also straight forward.

From Gupta (2001) we also have $\theta\left(t_1,t_2\right)\geq(\leq)1$ if and only if $h_1(t_1,t_2)$ is increasing (decreasing) in x_2. In this direction we have the following theorem.

Theorem 6.4.5. *$\theta(t_1^w,t_2^w)>(<)\,1$ or X_1^w and X_2^w are positively (negatively) associated if and only if $\left[-\frac{\nabla_1^{r+1}\left(p_{r,0}^{w}(t_1,t_2)\right)}{\nabla_1^{r}\left(p_{r,0}^{w}(t_1,t_2)\right)}\right]$ is increasing (decreasing) in t_2.*

Proof. $\theta(t_1^w,t_2^w)>(<)\,1\Leftrightarrow h_1^w(t_1,t_2)$ is increasing (decreasing) in t_2.

$$\begin{aligned}\theta^w(t_1,t_2)>(<)\,1 &\iff h_1^w(t_1,t_2)\text{ is increasing (decreasing) in }t_2\\ &\iff -\tfrac{\partial}{\partial t_1}\ln R^w\left(t_1,t_2\right)\text{ is increasing (decreasing) in }t_2\\ &\iff -\tfrac{\partial}{\partial t_1}\left(\ln\left(\tfrac{\partial^r}{\partial t_1^r}p_{r,0}^{w}\left(t_1,t_2\right)\right)\right)\text{ is increasing (decreasing) in }t_2\\ &\iff -\tfrac{\nabla_1^{r+1}\left(p_{r,0}^{w}(t_1,t_2)\right)}{\nabla_1^{r}\left(p_{r,0}^{w}(t_1,t_2)\right)}\text{ is increasing (decreasing) in }t_2.\end{aligned}$$

□

6.5 Conditional upper partial moments for weighted models

In this section we study properties of CUPMs in the context of weighted models. For a bivariate weighted random vector $\boldsymbol{X^W}$ and analogous to (4.1) and (4.7), the CUPMs can be defined as

$$\phi_r^w(t_i|t_j) = \int_{t_i}^{\infty} (x_i - t_i)^r f_i^w(x_i|t_j)dx_i \tag{6.31}$$

and

$$\psi_r^w(t_i|t_j) = \int_{t_i}^{\infty} (x_i - t_i)^r f_i^{*w}(x_i|t_j)dx_i, \tag{6.32}$$

where $f_i^w(x_i|t_j) = \frac{w(x_i)}{E(w(X_i)|X_j=t_j)} f_i(x_i|t_j)$ and $f_i^{*w}(x_i|t_j) = \frac{w(x_i)}{E(w(X_i)|X_j>t_j)} f_i^*(x_i|t_j), i,j = 1,2, i \neq j$ denote the weighted densities for the conditionally specified and survival models. An important weighted model is the length-biased models, when the weight is proportional to the length of the units used i.e., $w(t_i) = t_i, i = 1,2$. A detailed survey of literature on length-biased and equilibrium models we refer to Gupta and Kirmani (1990). For the length-biased case, the conditional partial moment in (6.31) reduces to

$$\phi_r^w(t_i|t_j) = \frac{1}{\phi_1(0|t_j)} \left(\phi_{r+1}(t_i|t_j) + t_i \ \phi_r(t_i|t_j)\right), \ i,j = 1,2, i \neq j. \tag{6.33}$$

In addition, for the equilibrium random variable X^E with probability density function $f^E(x) = \frac{\bar{F}(x)}{E(X)}$, the partial moment is given by

$$p_r^E\left(t\right) = \frac{p_{r+1}\left(t\right)}{\left(r+1\right)E\left(X\right)}, \tag{6.34}$$

(Sunoj (2004)) such that $0 < E(X) < \infty$. Gupta and Sankaran (1998) defined the equilibrium density for conditional random variable $X_i|X_j > t_j$ is given by

$f_i^E(x_i|X_j > t_j) = \frac{R_i(x_i|t_j)}{E(X_i|X_j>t_j)},\ i,j = 1,2, i \neq j$. In this case, the conditional partial moment of the second type using (6.32) becomes

$$\psi_r^w(t_i|t_j) = \frac{\psi_{r+1}(t_i|t_j)}{(r+1)\,\psi_1(0|t_j)},\ i = 1,2. \tag{6.35}$$

Sankaran and Sreeja (2007) proved that the two random vectors (X_1, X_2) and (Y_1, Y_2) satisfy the conditional proportional hazard rate (CPHR) model, when their respective conditional hazard rate functions satisfy

$$h_{X_i^w|X_j^w}\,(t_i|t_j) = \theta_i\,(t_j)\,h_{X_i|X_j}\,(t_i|t_j)\,;\ \ i,j = 1,2, i \neq j, \tag{6.36}$$

where $\theta_i\,(t_j)$ is a nonnegative function of t_j only for $i,j = 1,2, i \neq j$. Now based on the observation (4.15) from chapter 4, we have the following theorem.

Theorem 6.5.1. *Let (X_1^w, X_2^w) be a random vector which has the bivariate weighted distribution associated to (X_1, X_2) and to two nonnegative and differentiable functions w_1 and w_2. Let us assume that the support of (X_1, X_2) is $S = (l,\infty) \times (l,\infty)$ for $l \geq 0$. Then,*

$$\frac{\nabla_1^{r+1}\left(\phi_r^w\,(t_i|t_j)\right)}{\nabla_1^{r}\left(\phi_r^w\,(t_i|t_j)\right)} = \theta_i\,(t_j)\,\frac{\nabla_1^{r+1}\left(\phi_r\,(t_i|t_j)\right)}{\nabla_1^{r}\left(\phi_r\,(t_i|t_j)\right)}, i,j = 1,2, i \neq j \tag{6.37}$$

if and only if (X_1^w, X_2^w) and (X_1, X_2) satisfy the CPHR model (6.36).

6.5.1 Characterization results for weighted models using conditional upper partial moments

We now prove some characterization results for popular bivariate distributions using conditional partial moments in the context of length-biased and equilibrium distributions.

Theorem 6.5.2. *For a random vector $\boldsymbol{X} = (X_1, X_2)$ with $E(X_i|X_j = t_j) < \infty$,*

$$\frac{\nabla_1^{r+1}\left(\phi_r^w\left(t_i|t_j\right)\right)}{\nabla_1^{r}\left(\phi_r^w\left(t_i|t_j\right)\right)} = C_i\,\frac{\nabla_1^{r+1}\left(\phi_r\left(t_i|t_j\right)\right)}{\nabla_1^{r}\left(\phi_r\left(t_i|t_j\right)\right)} \tag{6.38}$$

for all t_i and t_j, where C_i is a constant independent of t_i and t_j, $i, j = 1, 2. i \neq j$ if and only if $\boldsymbol{X}$ are independent Pareto I variables specified by the joint pdf

$$f(x_1, x_2) = C\,x_1^{-\alpha_1} x_2^{-\alpha_2}, x_1, x_2 > 1, \alpha_1, \alpha_2 > 0.$$

Proof. Following the similar steps of the Theorem 6.5.1, (6.38) implies that

$$h^w\left(t_i|t_j\right) = C_i h\left(t_i|t_j\right), i, j = 1, 2; i \neq j.$$

Now the proof of theorem directly follows from the proof of Theorem 2.1 of Sunoj and Sankaran (2005). □

Theorem 6.5.3. *For $i, j = 1, 2$ and $i \neq j$, the conditional partial moments of length-biased and original models satisfy the relationship*

$$\frac{\phi_r^w(t_i|t_j)\;\phi_1(0|t_j)}{\phi_r(t_i|t_j)} = (A+1)\,t_i + B_i(t_j),\;\; A > 0, \tag{6.39}$$

if and only if (X_1, X_2) follows bivariate distribution with Pareto conditionals given in Arnold (1987) with the joint pdf (4.24).

Proof. Assume that (6.39) holds. Now using (6.33) we have

$$\frac{\phi_r^w(t_i|t_j)\;\phi_1(0|t_j)}{\phi_r(t_i|t_j)} = \frac{\phi_{r+1}(t_i|t_j)}{\phi_r(t_i|t_j)} + t_i = (A+1)\,t_i + B_i(t_j).$$

Equivalently

$$\frac{\phi_{r+1}(t_i|t_j)}{\phi_r(t_i|t_j)} = At_i + B_i(t_j)$$

The rest of the proof is similar to that of Theorem 4.3.3. The proof of converse part is direct. □

Theorem 6.5.4. *For $i, j = 1, 2$ and $i \neq j$, the ratio of conditional partial moments of length biased and original models satisfy the relationship*

$$\frac{\phi_r^w(t_i|t_j)}{\phi_r(t_i|t_j)}\phi_1(0|t_j) = r + 1 + t_i, \tag{6.40}$$

if and only if the joint density of (X_1, X_2) is defined in (4.16) *with $\lambda_i = 1, i = 1, 2$.*

Theorem 6.5.5. *For $i, j = 1, 2$ and $i \neq j$ the relationship*

$$\frac{\psi_r^w(t_i|t_j)\ \psi_1(0|t_j)}{\psi_r(t_i|t_j)} = \frac{1}{\lambda}; \lambda > 0 \tag{6.41}$$

holds if and only if the joint density of (X_1, X_2) follows

$$f(x_1, x_2) = e^{-\lambda(x_1+x_2)}\ ,\ x_1, x_2 > 0. \tag{6.42}$$

Proof. Assume that (6.41) holds, then from (6.39) we get

$$\frac{\psi_r^w(t_i|t_j)\ \psi_1(0|t_j)}{\psi_r(t_i|t_j)} = \frac{\psi_{r+1}(t_i|t_j)}{\psi_r(t_i|t_j)(r+1)}. \tag{6.43}$$

Differentiating both sides of $\frac{\psi_{r+1}(t_i|t_j)}{\psi_r(t_i|t_j)(r+1)} = \frac{1}{\lambda}$ with respect to t_i, $r+1$ times and on simplification, we obtain $h_i^*(t_1, t_2) = \lambda$. Substituting in (4.21) and (4.22) yield (6.42). □

Theorem 6.5.6. *For $i, j = 1, 2$ and $i \neq j$ the CUPMs of equilibrium and original models satisfy the relationship*

$$\frac{\psi_r^w(t_i|t_j)\ \psi_1(0|t_j)}{\psi_r(t_i|t_j)} = \frac{1}{1 + Kt_j}, -1 \leq K \leq 1 \tag{6.44}$$

if and only if the joint density of (X_1, X_2) follows bivariate Gumbel Type I exponential *distribution* (4.9).

Proof. From (6.35) and (6.44), we obtain

$$\frac{\psi_{r+1}(t_i|t_j)}{\psi_r(t_i|t_j)(r+1)} = \frac{1}{1+Kt_j}. \tag{6.45}$$

Equivalently

$$(\psi_{r+1}(t_i|t_j))\,(1+Kt_j) = \psi_r(t_i|t_j)(r+1).$$

Differentiating both sides with respect to t_i, $r+1$ times and on simplification, we get $h_i^*(t_1,t_2) = 1+Kt_j$. Now using (4.21) and (4.22), we have $R(t_1,t_2) = exp\{-(t_1+t_2+Kt_1t_2)\}$, the joint survival function of (4.9) with $\theta = K$. The proof of the converse part is straightforward. □

Theorem 6.5.7. *For $i,j = 1,2$ and $i \neq j$ the relationship*

$$\frac{\psi_r^w(t_i|t_j)\;\psi_1(0|t_j)}{\psi_r(t_i|t_j)} = \frac{1}{K}\,(t_i + B_i(t_j))\,, K > 0 \tag{6.46}$$

if and only if (X_1, X_2) follows bivariate distribution with Pareto conditionals given in (4.40).

Proof. From (6.35) and (6.46), we get

$$\frac{\psi_{r+1}(t_i|t_j)}{\psi_r(t_i|t_j)(r+1)} = \frac{1}{K}\,(t_i + B_i(t_j)) \tag{6.47}$$

and

$$\frac{\psi_{r+1}(t_i|t_j)}{\psi_r(t_i|t_j)} = K^*\,(t_i + B_i(t_j))\,, \tag{6.48}$$

where $K^* = \frac{(r+1)}{K}$. The remaining proof directly follows from Theorem 4.3.7. □

Chapter 7

The role of copula-based upper partial moments in stochastic modelling*

7.1 Introduction

The role of copulas in the analysis of lifetime data has been emphasised either implicitly or explicitly during the past thirty years. This can be seen from the works of various researchers like Romeo et al. (2006), Kaishev et al. (2007), Pellerey (2008) and Navarro and Spizzichino (2010). The methodology adopted to analyse bivariate data in these works is to infer the copula directly from the observations or by appealing to reliability functions like the bivariate hazard rate or mean residual life based on the survival function to identify the appropriate copula. It is to be noted that for the various bivariate (multivariate) distributions discussed in literature, measures of association often appear to be of great importance. The theory of copulas provides a flexible tool for identifying the nature

*Contents of this chapter have been communicated to an International Journal.

and extent of dependence in multivariate models.

The modelling and analysis of statistical data using copula has been extensively studied in literature. One could refer to Schweizer and Sklar (2011), Joe (1997), Nelsen (2007), McNeil et al. (2015) and Salvadori et al. (2007). A question that usually arises in the study is the choice of the functional form of the copula. The selection of the copula for a given dataset depends on the range of dependence among the variables. In real life situations we come across a large number of datasets with positive dependence and modelling them using many well known copulas were found in literature. Copulas like Clayton and Frank copulas incorporate strong positive dependence, independence and strong negative dependence. However, Gumbel copula can only incorporate independence and positive association. Motivated by this, we introduce a stochastic order for PQD (NQD) concepts in-terms of copula-based BUPMs.

Partial moments are extensively used in the field of analysis of risks. Most of the real world problems are defined in higher dimensions. The present chapter aims extending the partial moments to the bivariate case through copula function (Nelsen (2007)) and study its various properties. Applications of BUPM and CUPMs in income studies and analysis of risks are also explored. The relationship between survival copula and first-order bivariate partial moments are established. We also investigate some applications of conditional partial moments in the context of reliability, actuarial and income (poverty) studies.

Many of the probability models used in the literature may not have a tractable distribution function. In such cases, an alternative approach for modelling and analysis of statistical data is through the quantile function. Quantile function has many interesting properties that are not shared by the distribution functions. The quantile function $Q(u)$ of X is defined as,

$$Q(u) = F^{-1}(u) = \inf\{x : F(x) \geq u\}, 0 \leq u \leq 1, \tag{7.1}$$

for $-\infty < x < \infty$. Nair et al. (2013b) defined a quantile-based r^{th} order univariate stop-loss transform as

$$\mathrm{P}_r\left(u\right) = p_r\left(Q\left(u\right)\right) = \int_u^1 \left(Q\left(p\right) - Q\left(u\right)\right)^r dp. \tag{7.2}$$

They established various relationships between the first-order quantile-based stop-loss transform $\mathrm{P}_1\left(u\right)$ and distribution properties such as measure of location, dispersion, skewness and kurtosis. Nair et al. (2013b) also obtained the relationships between $\mathrm{P}_1\left(u\right)$ and certain other important income measures such as income gap ratio, Lorenz Curve, Gini Index, Pietra Index, Bonferroni curve etc. Recently, some applications of quantile-based lower partial moments were studied by Sunoj and Vipin (2017). Motivated with this, we further examine some applications of conditional partial moments using quantile functions.

The objectives of the work in the present chapter are manifold. Firstly we define the bivariate upper partial moments using survival copula and prove certain properties in section 7.2. An important ordering property known as tail monotonicity and survival copula with standard exponential marginals are explored in this section using copula-based BUPM. Applications of quantile-based conditional upper partial moments in the context of system reliability studies, actuarial science and income (poverty) studies are studied in Section 7.3. .

7.2 Copula-based bivariate upper partial moments

Let $\boldsymbol{X} = (X_1, X_2)$ be a non-negative random vector with continuous survival function $R(t_1, t_2)$ and marginal survival functions $R_i(t_i) = P(X_i > t_i),\ i = 1, 2$ which are continuous and strictly decreasing. Then the survival copula $C(u, v)$ of $\boldsymbol{X}$ is a mapping $C(u, v) : [0, 1] \times [0, 1] \to [0, 1]$, defined by (Nelsen (2007))

$$C(u, v) = R\left(R_1^{-1}(u), R_2^{-1}(v)\right),$$

where R_1^{-1}, R_2^{-1} are the usual inverse of R_1 and R_2 respectively. Alternatively, the joint survival function

$$R\left(t_1, t_2\right) = C\left(R_1\left(t_1\right), R_2\left(t_2\right)\right).$$

The survival copula satisfies the following properties

(i) $C(u, 1) = u$, $C(1, v) = v$ and $C(u, 0) = 0 = C(0, v)$.

(ii) for every u_1, u_2, v_1, v_2 in $[0, 1]$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$,

$$C\left(u_1, v_1\right) + C\left(u_2, v_2\right) - C\left(u_1, v_2\right) - C\left(u_2, v_1\right) \geq 0.$$

Applying the probability transforms,

$$u = R_1(t_1) \Rightarrow t_1 = R_1^{-1}\left(u\right) \quad \text{and} \quad v = R_2(t_2) \Rightarrow t_2 = R_2^{-1}\left(v\right)$$

in (2.4), the copula-based bivariate partial moments becomes

$$\begin{aligned} P_{r,s}(u,v) &= p_{r,s}\left(R_1^{-1}(u), R_2^{-1}(v)\right) \\ &= rs\int_0^u\int_0^v \left(R_1^{-1}(p)-R_1^{-1}(u)\right)^{r-1}\left(R_2^{-1}(q)-R_1^{-1}(v)\right)^{s-1} \\ &\qquad C(p,q)\frac{dR_1^{-1}(p)}{dp}dp\frac{dR_2^{-1}(q)}{dq}dq. \end{aligned} \tag{7.3}$$

$P_{r,s}(u,v)$ provides the $(r,s)^{th}$ copula-based BUPM. We now consider the simplest case when $r=1$ and $s=1$ in (7.3),

$$P_{1,1}(u,v)=\int_0^u\int_0^v C(p,q)\frac{dR_1^{-1}(p)}{dp}dp\frac{dR_2^{-1}(q)}{dq}dq. \tag{7.4}$$

Differentiating both sides of (7.4) with respect to u and v respectively, we have,

$$\frac{\partial^2 P_{1,1}(u,v)}{\partial u\partial v}=C(u,v)\frac{dR_1^{-1}(u)}{du}\frac{dR_2^{-1}(v)}{dv}. \tag{7.5}$$

Also from (7.3), we get

$$P_{1,0}(u,1)=p_{1,0}\left(R_1^{-1}(u),0\right)=\int_{t_1}^{\infty} R_1(u)du=-\int_0^u p\frac{dR_1^{-1}(p)}{dp}dp \tag{7.6}$$

and

$$P_{0,1}(1,v)=p_{0,1}\left(0,R_2^{-1}(v)\right)=\int_{t_2}^{\infty} R_2(u)du=-\int_0^v q\frac{dR_2^{-1}(q)}{dq}dq, \tag{7.7}$$

where, (7.6) and (7.7) implies that, $-\frac{1}{u}\frac{dP_{1,0}(u,1)}{du}=\frac{dR_1^{-1}(u)}{du}$ and $-\frac{1}{v}\frac{dP_{0,1}(1,v)}{dv}=\frac{dR_2^{-1}(v)}{dv}$. Substituting in (7.5), we have the identity

$$\frac{dP_{1,0}(u,1)}{du}\frac{dP_{0,1}(1,v)}{dv}C(u,v)=uv\frac{\partial^2 P_{1,1}(u,v)}{\partial u\partial v}, \tag{7.8}$$

and hence uniquely determines the bivariate copula. (7.8) provides a alternative method for the construction of survival copula.

Example 7.2.1. Let (X_1, X_2) be a non-negative random vector with an absolute continuous survival function $R(t_1, t_2)$. Then from (7.8) it follows that for the choice of $P_{1,1}(u,v) = u^2 + v^2 + uv$ and $P_{1,0}(u,v) = uv = P_{0,1}(u,v)$ yields the product survival copula, $C(u,v) = uv$.

Remark 7.2.1. Even if the first-order copula-based BUPM, $P_{1,1}(u,v)$ uniquely determines the survival copula $C(u,v)$ using (7.8), however, it requires the knowledge of its corresponding marginals $P_{1,0}(u,v)$ and $P_{0,1}(u,v)$. This motivate us to explore an alternative to identify a functional relationship which could easily identify $C(u,v)$ without the knowledge of $P_{1,0}(u,v)$ and $P_{0,1}(u,v)$.

Defining the copula-based marginal partial moments $p_{1,0}(t_1, t_2)$ and $p_{0,1}(t_1, t_2)$ by,

$$P_{1,0}(u,v) = -\int_0^u C(p,v) \frac{dR_1^{-1}(p)}{dp} dp \tag{7.9}$$

and

$$P_{0,1}(u,v) = -\int_0^v C(u,q) \frac{dR_2^{-1}(q)}{dq} dq, \tag{7.10}$$

in the following theorem, we prove that both $P_{1,0}(u,v)$ and $P_{0,1}(u,v)$ determines the corresponding copula uniquely.

Theorem 7.2.1. *The bivariate copula-based upper partial moments $P_{1,0}(u,v)$ and $P_{0,1}(u,v)$ uniquely determine the survival copula through the relationships*

$$\frac{dP_{1,0}(u,1)}{du} C(u,v) = u \frac{\partial P_{1,0}(u,v)}{\partial u} \tag{7.11}$$

and

$$\frac{dP_{0,1}(1,v)}{dv}C(u,v) = v\frac{\partial P_{0,1}(u,v)}{\partial v} \tag{7.12}$$

respectively.

Remark 7.2.2. Theorem 7.2.1 enable one to determine the survival copula with the knowledge of either $P_{1,0}(u,v)$ or $P_{0,1}(u,v)$.

Example 7.2.2. Let $P_{1,0}(u,v) = vu^v \frac{(\alpha+\beta v)^{\frac{v-1}{v}}}{(\alpha u+\beta v)^{\frac{1}{v}}}$, $\alpha, \beta > 0$. Then from (7.11), we have a new survival copula $C(u,v) = u^v v \left(\frac{\alpha+\beta v}{\alpha u+\beta v}\right)^{\frac{v-1}{v}}$, and displayed in Figure 7.1.

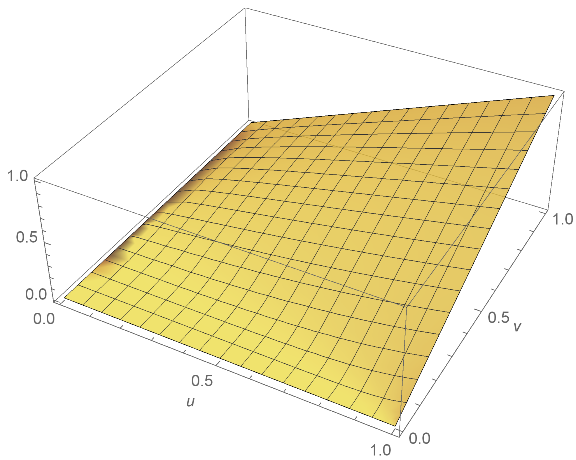


Figure 7.1: Plot of the survival copula $C(u,v) = u^v v \left(\frac{\alpha+\beta v}{\alpha u+\beta v}\right)^{\frac{v-1}{v}}$, $0 < u, v < 1$, for $\alpha = \beta = 0.5$.

The assumption of independence is seldom valid in practice. An important notion that has been widely used in this context is the Positive (Negative) Quadrant Dependence (PQD (NQD)) property. As mentioned in (6.10) in section 6.2, interms of the UPM's, a random vector (X_1, X_2) satisfies the Positive (Negative) Quadrant Dependence denoted by PQD (NQD) if for all values of $t_1, t_2 > 0$,

$$p_{r,s}(t_1,t_2) \geq (\leq)\, p_r(t_1)\, p_s(t_2)$$

for all values of r, s and $t_i > 0, i = 1, 2$.

To this effect we have the following theorem.

Theorem 7.2.2. *A random vector* (X_1, X_2) *is PQD (NQD) if and only if*

$$\frac{P_{1,0}\left(u,v\right)}{P_{1,0}\left(u,1\right)} \geq \left(\leq\right) v \tag{7.13}$$

or

$$\frac{P_{0,1}\left(u,v\right)}{P_{0,1}\left(1,v\right)} \geq \left(\leq\right) u. \tag{7.14}$$

Proof. Applying the probability transforms $u = R_1(t_1) \Rightarrow t_1 = R_1^{-1}\left(u\right)$ and $v = R_2(t_2) \Rightarrow t_2 = R_2^{-1}\left(v\right)$ in (6.9), we have

$$C\left(u,v\right) \geq \left(\leq\right) u\, v. \tag{7.15}$$

Applying (7.11) and (7.12) in (7.15) yields

$$\frac{\partial P_{1,0}\left(u,v\right)}{\partial u} \geq \left(\leq\right) v \frac{dP_{1,0}\left(u,1\right)}{du}$$

and

$$\frac{\partial P_{0,1}\left(u,v\right)}{\partial v} \geq \left(\leq\right) u \frac{dP_{0,1}\left(1,v\right)}{dv}.$$

Integrating both sides of the first inequality by u and second inequality by v yields the required result. □

7.2.1 Tail monotonicity

When X_1 and X_2 are continuous, an important positive dependence concept is the *tail monotonicity*, where the *left (right) tail decreasing (increasing)* are of importance. In the sequel we identify the relationships connecting first-order copula-based partial moment and *right tail increasing* property.

Definition 7.2.1. Let X_1 and X_2 be two random variables. Then,

(i) X_2 is right tail increasing in X_1 denoted by $RTI(X_2|X_1)$ if $P(X_2 > t_2|X_1 > t_1)$ is a nondecreasing function of t_1 for all t_2.

(ii) X_1 is right tail increasing in X_2 denoted by $RTI(X_1|X_2)$ if $P(X_1 > t_1|X_2 > t_2)$ is a nondecreasing function of t_2 for all t_1.

Equivalently, Nelsen (2007) has defined *right tail increasing* property in terms of copula as follows.

Theorem 7.2.3. *Let X_1 and X_2 be continuous random variables with copula C^*. Then*

*(i) $RTI(X_1|X_2)$ if and only if for any v in **I**, $\frac{1-u-v+C^*(u,v)}{1-u}$ is nondecreasing in u.*

*(ii) $RTI(X_2|X_1)$ if and only if for any u in **I**, $\frac{1-u-v+C^*(u,v)}{1-v}$ is nondecreasing in v.*

In the following theorem we obtain relationships connecting right tail increasing and copula-based partial moments.

Theorem 7.2.4. *Let (X_1, X_2) be a random vector with common survival copula C. Then, (X_1, X_2) is said to be*

(i) $RTI(X_1|X_2)$ if and only if the ratio $\frac{\frac{\partial P_{1,0}(u,v)}{\partial u}}{\frac{dP_{1,0}(u,1)}{du}}$ is nondecreasing in $1-u$.

(ii) $RTI(X_2|X_1)$ if and only if the ratio $\frac{\frac{\partial P_{0,1}(u,v)}{\partial u}}{\frac{dP_{0,1}(1,v)}{du}}$ is nondecreasing in $1-v$.

Proof. We only provide the proof of the (i) part of the theorem. The proof of the second part is analogous. Let $u = 1-p$ and $v = 1-q$, where $p, q \in \mathbf{I}$. Now from (7.11), it follows that

$$C(u,v) = \frac{u\frac{\partial P_{1,0}(u,v)}{\partial u}}{\frac{dP_{1,0}(u,1)}{du}} \quad \Rightarrow \quad C(1-u,1-v) = \frac{(1-u)\frac{\partial P_{1,0}(1-u,1-v)}{\partial u}}{\frac{dP_{1,0}(1-u,1)}{du}}$$
$$\Rightarrow \quad \frac{C(1-u,1-v)}{(1-u)} = \frac{\frac{\partial P_{1,0}(1-u,1-v)}{\partial u}}{\frac{dP_{1,0}(1-u,1)}{du}}. \tag{7.16}$$

Now from the relation connecting the copula C^* and the corresponding survival copula C, we have,

$$C\left(u,v\right)=u+v-1+C^*\left(1-u,1-v\right).$$

Equivalently,

$$C\left(1-u,1-v\right)=1-u-v+C^*\left(u,v\right)$$

Now let us assume, for any $v \in \mathbf{I}$, $\frac{\frac{\partial P_{1,0}(u,v)}{\partial u}}{\frac{dP_{1,0}(u,1)}{du}}$ is nondecreasing in $1-u$. Then,

$$\begin{aligned}
\frac{\frac{\partial P_{1,0}(u,v)}{\partial u}}{\frac{dP_{1,0}(u,1)}{du}} \text{ is nondecreasing in } 1-u \quad &\Longleftrightarrow \frac{\frac{\partial P_{1,0}(1-u,1-v)}{\partial u}}{\frac{dP_{1,0}(1-u,1)}{du}} \text{ is nondecreasing in } u \\
&\Longleftrightarrow \frac{C(1-u,1-v)}{(1-u)} \text{ is nondecreasing in } u \\
&\Longleftrightarrow \frac{1-u-v+C^*(u,v)}{1-u} \text{ is nondecreasing in } u \\
&\Longleftrightarrow \frac{1-u-v+C^*(u,v)}{1-u} \text{ is nondecreasing in } u \\
&\Longleftrightarrow RTI(X_2|X_1) \text{ for any } v \in \mathbf{I}.
\end{aligned}$$

Hence the theorem. □

Remark 7.2.3. Even if the survival copula $C(u,v)$ for a given $P_{1,1}\left(u,v\right)$, $P_{1,0}\left(u,v\right)$ and $P_{0,1}\left(u,v\right)$ can be evaluated, characterization of $C(u,v)$ cannot be accomplished without the knowledge of the form of the marginals.

7.2.2 Survival copula with standard exponential marginals

For a survival copula with standard exponential marginals, we define the copula-based first-order partial moment as

$$L_{1,1}(u,v)=\int_0^u\int_0^v \frac{C(p,q)}{pq}dpdq,$$

$$L_{1,0}(u,v)=\int_0^u \frac{C(p,v)}{p}dp \tag{7.17}$$

and

$$L_{0,1}(u,v)=\int_0^v \frac{C(u,q)}{q}dq. \tag{7.18}$$

Now, $L_{1,1}(u,v)$, $L_{1,0}(u,v)$ and $L_{0,1}(u,v)$ determine $C(u,v)$ uniquely through the identities,

$$C(u,v)=uv\frac{\partial^2}{\partial u\partial v}L_{1,1}(u,v),$$

$$C(u,v)=u\frac{\partial}{\partial u}L_{1,0}(u,v) \tag{7.19}$$

and

$$C(u,v)=v\frac{\partial}{\partial v}L_{0,1}(u,v). \tag{7.20}$$

Further, $P_{1,1}(u,v)$, $P_{1,0}(u,v)$ and $P_{0,1}(u,v)$ are connected with $L_{1,1}(u,v)$, $L_{1,0}(u,v)$ and $L_{0,1}(u,v)$ through,

$$\left(\frac{\partial^2}{\partial u\partial v}L_{1,1}(u,v)\right)\left(\frac{dP_{1,0}(u,1)}{du}\frac{dP_{0,1}(1,v)}{dv}\right)=\frac{\partial^2 P_{1,1}(u,v)}{\partial u\partial v},$$

$$\frac{\partial}{\partial u}L_{1,0}(u,v)\frac{dP_{1,0}(u,1)}{du}=\frac{\partial P_{1,0}(u,v)}{\partial u}$$

and

$$\frac{\partial}{\partial u}L_{0,1}(u,v)\frac{dP_{0,1}(1,v)}{dv}=\frac{\partial P_{0,1}(u,v)}{\partial v}.$$

The form of $L_{1,0}(u,v)$ and $L_{0,1}(u,v)$ for some popular copulas are listed in Table 7.1.

Table 7.1: Examples of $L_{1,0}(u,v)$ and $L_{0,1}(u,v)$ for some families of survival copulas

$C(u,v)$	$L_{1,0}(u,v)$	$L_{0,1}(u,v)$
Product:		
uv	uv	uv
Gumbel-Barnett:		
$u\,v\,exp\left[-\beta\,\log u\,\log v\right]$, $0 \leq \beta \leq 1$	$\frac{v\,u^{1-\beta\,\log v}}{1-\beta\,\log v}$	$\frac{u\,v^{1-\beta\,\log u}}{1-\beta\,\log u}$
F-G-M:		
$uv\left[1+\theta\,(1-u)\,(1-v)\right]$, $-1 \leq \theta \leq 1$	$uv\left[1+\theta\,(1-v)\left(1-\frac{u}{2}\right)\right]$	$uv\left[1+\theta\,(1-u)\left(1-\frac{v}{2}\right)\right]$
Ali–Mikhail–Haq:		
$\frac{uv}{1+(1-u)(1-v)}$	$\frac{v}{v-1}\log\left(\frac{u+v-uv-2}{v-2}\right)$	$\frac{u}{u-1}\log\left(\frac{u+v-uv-2}{v-2}\right)$
Clayton:		
$\left(u^{-1}+v^{-1}-1\right)^{-1}$	$\frac{v\log\left(\frac{u}{v}(1-v)+1\right)}{1-v}$	$\frac{u\log\left(\frac{v}{u}(1-u)+1\right)}{1-u}$

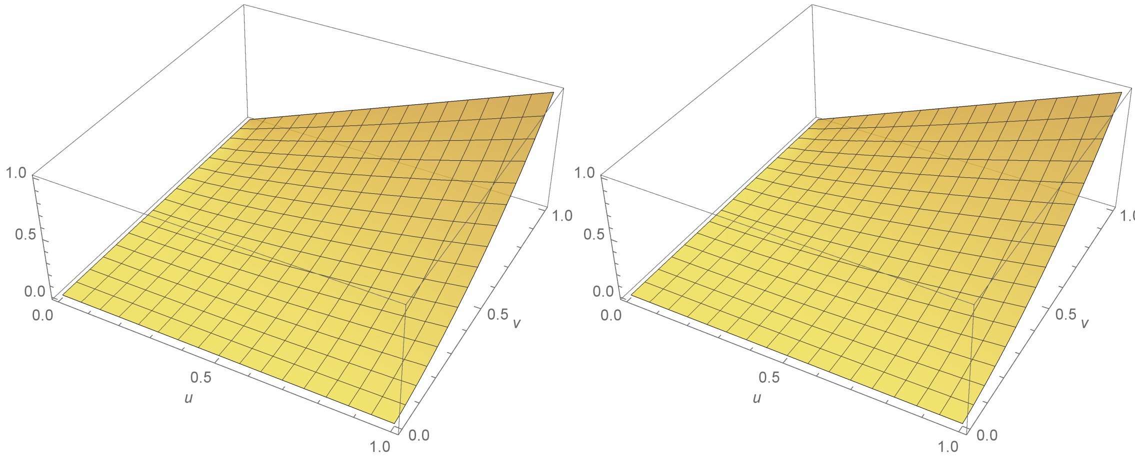
In the following theorem we characterize the Gumbel-Barnett survival copula using (7.17) and (7.18).

Theorem 7.2.5. *The marginal partial moments are of the form*

$$\left(L_{1,0}\left(u,v\right), L_{0,1}\left(u,v\right)\right) = \left(\frac{v\,u^{B_1(v)}}{B_1\left(v\right)}, \frac{u\,v^{B_2(u)}}{B_2\left(u\right)}\right) \tag{7.21}$$

where $B_1(.)$ does not depend on u and $B_2(.)$ does not depend on v if and only if the survival copula is Gumbel-Barnett.

Proof. To prove the sufficient part, we assume that (7.21) holds true. Then, using



(a) $L_{1,0}(u,v)$ (b) $L_{0,1}(u,v)$

Figure 7.2: Plot of $L_{1,0}(u,v)$ and $L_{0,1}(u,v)$ for Gumbel-Barnett survival copula with $\beta = 0.5$.

(7.19) and (7.20) lead to the functional equation

$$v\ u^{B_1(v)} = u\ v^{B_2(u)}, \tag{7.22}$$

which is equivalent to

$$u^{\frac{1}{B_2(u)-1}} = v^{\frac{1}{B_1(v)-1}}.$$

The solution is

$$u^{\frac{1}{B_2(u)-1}} = k = v^{\frac{1}{B_1(v)-1}}$$

giving

$$B_2\left(u\right) = \frac{1-\log u}{\beta}, B_1\left(v\right) = \frac{1-\log\ v}{\beta}, \beta = -\log\ k$$

substituting in (7.22), we obtain the Gumbel-Barnett copula.

The proof of the necessary part directly follows from (7.17), (7.18) and Table 7.1. □

Theorem 7.2.6. *The condition* $C\left(u,v\right) = L_{1,1}\left(u,v\right) = L_{1,0}\left(u,v\right) = L_{0,1}\left(u,v\right)$ *holds* *true if and only if the survival copula is the product survival copula,* $C(u, v) = uv$.

7.3 Applications

In this section we explore some applications of quantile-based conditional partial moments in the context of risk analysis and income and poverty studies.

7.3.1 Applications in system reliability studies

Now illustrate the use of representing partial moments using copulas in the view point of system reliability studies. The distributions of maximum and minimum of the random variables X_1 and X_2 are of much importance in reliability studies and survival analysis. For example, maximum of all component's lifetimes gives the total lifetime of the parallel system and for the series system, the minimum gives the total lifetime. Consider a two component system with lifetimes (X_1, X_2). Let $T_1 = min(X_1, X_2)$ and $T_2 = max(X_1, X_2)$ denote the random variables of minimum and maximum respectively. If the component lifetimes are statistically independent and identically distributed, the properties of such systems are well understood. However, in real life, component lifetimes are often statistically dependent because they share the same environment. In the sequel, we investigate some important properties of the series systems when lifetimes of components are dependent which have the common marginal distribution.

7.3.1.1 Series system

Consider a series system with the system lifetime denoted by the random variable T_1 which consisting of two dependent and identically distributed components lifetimes X_1 and X_2 having the same marginals. Then, from Navarro et al. (2013), the reliability function of the system is given by

$$R_{T_1}(t) = P(T_1 > t) = \delta_C(R(t)), t > 0, \tag{7.23}$$

where $\delta_C(.)$ is defined as the diagonal section of the survival copula C. From (7.23) it follows that

$$f_{T_1}(t) = -\frac{d\delta_C(R(t))}{dt} = f(t)\,\delta_C'(R(t)), t > 0, \tag{7.24}$$

where $\delta'_C(.)$ denotes the partial derivative of $\delta_C(.)$. Hence for any positive integer r, partial moments for the Lifetime of the series system becomes

$$p_r(t) = r\int_t^{\infty} (x-t)^{r-1} R_{T_1}(x)dx. \tag{7.25}$$

Now applying the probability integral transformation $u = R_1(t_1) \Rightarrow t_1 = R_1^{-1}(u)$ and $v = R_2(t_2) \Rightarrow t_2 = R_2^{-1}(v)$ in (7.25), partial moments for the Lifetime of the series system in terms of copula can be expressed as

$$p_r(R^{-1}(p)) = P_r(u) = -r\int_0^u \left(R^{-1}(p) - R^{-1}(u)\right)^{r-1} \delta_C(p)\frac{dR^{-1}(p)}{dp}dp. \tag{7.26}$$

Confining to the case $r = 1$, (7.26) becomes

$$P_1(u) = -\int_0^u \delta_C(p)\frac{dR^{-1}(p)}{dp}dp. \tag{7.27}$$

Assuming the marginal lifetimes follows standard exponential distribution, (7.27) becomes

$$P_1(u) = \int_0^u \frac{\delta_C(p)}{p}dp. \tag{7.28}$$

Hence one can determine the diagonal section of a survival copula from the first order partial moment using the relationship

$$u\frac{dP_1(u)}{du} = \delta_C\left(u\right) \tag{7.29}$$

Example 7.3.1. Consider a series system consisting of two components with system lifetime T_1 and marginals are distributed as standard exponentials. Let (X_1, X_2) follows

(i) Gumbel's exponential distribution specified by the joint survival function

$$R(t_1, t_2) = e^{(t_1+t_2+\theta t_1 t_2)}, t_1, t_2 > 0, 0 \leq \theta \leq 1.$$

Then the corresponding survival copula function is $C(u, v) = u\, v\, e^{-\theta\, log u\, log v}$ and the diagonal section is $\delta_C(u) = u^2\, e^{-\theta\, (log u)^2},\ 0 \leq u \leq 1$ for all $0 \leq \theta \leq 1$. Then a direct application of (7.28) gives

$$P_1\left(u\right) = \frac{\sqrt{\pi}e^{1/\theta}\left(\text{Erf}\left(\frac{\theta\log(u)-1}{\sqrt{\theta}}\right)+1\right)}{2\sqrt{\theta}}, \tag{7.30}$$

where $\text{Erf}\left(u\right) = \frac{2}{\sqrt{\pi}}\int_0^u e^{-p^2}dp.$

(ii) Farlie-Gumbel-Morgenstern (F-G-M) with joint reliability function given by

$$C\left(R\left(t_1\right), R\left(t_2\right)\right) = R\left(t_1\right)R\left(t_2\right)\left[1+\theta R\left(t_1\right)R\left(t_2\right)\right].$$

Then the FGM survival copula function is

$$C(u, v) = u\; v\left[1+\theta\left(1-u\right)\left(1-v\right)\right], -1 \leq \theta \leq 1$$

and the diagonal section of the copula will be $\delta_C(u) = u^2(1+\theta(1-u)^2), 0 \leq$

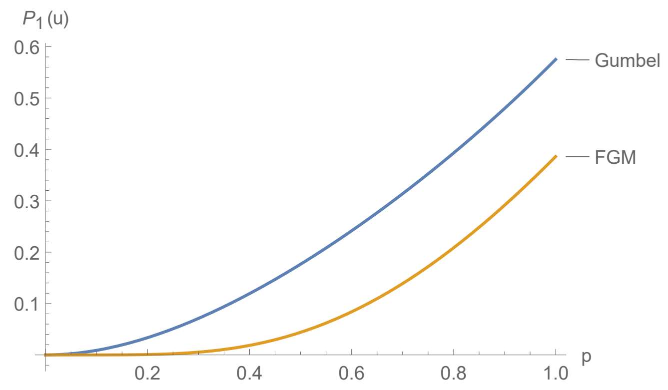


Figure 7.3: Plot of $P_1(u)$ for bivariate Gumbels exponential and F-G-M copula when $\theta = 0.5$.

$u \leq 1$. Hence, from (7.28) it follows that

$$P_1(u) = \frac{\theta u^4}{4} - \frac{2\theta u^3}{3} + \frac{\theta u^2}{2} + \frac{u^2}{2}. \tag{7.31}$$

Figure 3 present the plot of $P_1(u)$ for Bivariate Gumbels exponential ((7.30)) and F-G-M ((7.31)) copula for $\theta = 0.5$. From the figure it is clear that $P_1(u)$ is of convex nature.

Navarro et al. (2013) discussed that the failure rate function of T_1 can be identified using the identity

$$h_T(t) = \frac{\delta'_C(R(t))}{\delta_C(R(t))} f(t) = \alpha(R(t))\, h(t), \tag{7.32}$$

where $\alpha(u) = u\frac{\delta'_C(u)}{\delta_C(u)}$, $0 < u < 1$ and $h(.)$ is the marginal failure rate function.

Now we present the following theorem which states the stochastic orders in terms of $P_1(u)$.

Theorem 7.3.1. *Let $T_1 = min\{X_1, X_2\}$ and $T_2 = min\{Y_1, Y_2\}$ denotes the lifetimes of two series systems with two dependent and identically distributed component lifetimes having the common absolutely continuous reliability functions R and S respectively. Also* *let $(P_1(u), \delta_{C_1}(u))$ and $(Q_1(u), \delta_{C_2}(u))$ denote the first order partial moments together with the corresponding diagonal section of the survival copulas C_1 and C_2 respectively. Then,*

(i) $T_1 \leq_{ST} (\geq_{ST}) T_2$ for all R if and only if $P_1(u) \leq (\geq) Q_1(u)$ in $(0,1)$.

(ii) If $X_1 \leq_{HR} Y_1$ and $\frac{d}{du}\left(\ln\left(\frac{dP_1(u)}{du}\right)\right) \geq \frac{d}{du}\left(\ln\left(\frac{dQ_1(u)}{du}\right)\right)$ in $(0,1)$ then $T_1 \leq_{HR} T_2$.

Proof. Note that two random variables X is said to be smaller than Y in the usual stochastic order (denoted by $X \leq_{ST} Y$) if $R(t) \leq S(t)$ for all t, where R and S are the survival functions corresponding to X and Y respectively. Now from (7.29) it follows that

$$\begin{aligned} P_1(u) \leq (\geq) Q_1(u) \text{ in } (0,1) \quad &\Longleftrightarrow \frac{dP_1(u)}{du} \leq (\geq) \frac{dQ_1(u)}{du} \text{ in } (0,1) \\ &\Longleftrightarrow u\frac{dP_1(u)}{du} \leq (\geq) u\frac{dQ_1(u)}{du} \text{ in } (0,1) \\ &\Longleftrightarrow \delta_{C_1}(u) \leq (\geq) \delta_{C_2}(u) \text{ in } (0,1). \end{aligned}$$

Hence (i) follows from (7.23) and from the definition of the stochastic order.

To prove (ii), since $X_1 \leq_{HR} Y_1$, we have, $\frac{f(t)}{R(t)} \leq \frac{g(t)}{S(t)}$, where $f(t) = -\frac{dR(t)}{dt}$ and $g(t) = -\frac{dS(t)}{dt}$. Moreover, $X_1 \leq_{HR} Y_1 \Leftrightarrow X_1 \leq_{ST} Y_1$, yields $R(t) \leq S(t)$. Hence,

$$\begin{aligned} \frac{d}{du}\left(\ln\left(\frac{dP_1(u)}{du}\right)\right) \geq \frac{d}{du}\left(\ln\left(\frac{dQ_1(u)}{du}\right)\right) \quad &\Longleftrightarrow \frac{u\frac{d^2}{du^2}P_1(u)}{\frac{d}{du}P_1(u)} + 1 \geq \frac{u\frac{d^2}{du^2}Q_1(u)}{\frac{d}{du}Q_1(u)} + 1, \; u \in (0,1) \\ &\Longleftrightarrow u\frac{\delta'_{C_1}(u)}{\delta_{C_1}(u)} \geq u\frac{\delta'_{C_2}(u)}{\delta_{C_2}(u)}, \; u \in (0,1) \\ &\Longleftrightarrow \alpha(R(t)) \geq \alpha(S(t)). \end{aligned}$$

Now using the fact that $\alpha(.)$ is non negative in $(0,1)$ and from (7.32), we have

$$h_{T_1}(t) = \frac{f(t)}{R(t)}\alpha(R(t)) \geq \frac{g(t)}{S(t)}\alpha(S(t)) = h_{T_2}(t).$$

Hence the result. □

7.3.2 Applications in income studies

Abdul-Sathar et al. (2007) defined the income-gap ratio for the bivariate random vector $\beta(t_1, t_2) = (\beta_1(t_1, t_2), \beta_2(t_1, t_2))$, where the i^{th} component of the random vector is defined by

$$\beta_i(t_i, t_j) = 1 - \frac{t_i}{v_i(t_i, t_j)}, i, j = 1, 2, i \neq j, \tag{7.33}$$

where $v_i(t_1, t_2) = E(X_i|X_1 > t_1, X_2 > t_2)$ is the i^{th} component of the bivariate vitality function defined in Sankaran and Nair (1991). Using the definition of BUPM, (7.33) becomes

$$\beta_1(t_1, t_2) = \frac{p_{1,0}(t_1, t_2)}{t_1\ p_{0,0}(t_1, t_2) + p_{1,0}(t_1, t_2)} \tag{7.34}$$

and

$$\beta_2(t_1, t_2) = \frac{p_{0,1}(t_1, t_2)}{t_2\ p_{0,0}(t_1, t_2) + p_{0,1}(t_1, t_2)}. \tag{7.35}$$

Let $m(t_1, t_2) = (m_1(t_1, t_2), m_2(t_1, t_2))$ denotes the vector-valued bivariate mean residual life (BMRL) of Arnold and Zahedi (1988), where $m_i(t_1, t_2) = E(X_i - t_i|X_1 > t_1, X_2 > t_2)$, $i = 1, 2$. Employing the same probability transforms, $u = R_1(t_1) \Rightarrow t_1 = R_1^{-1}(u)$ and $v = R_2(t_2) \Rightarrow t_2 = R_2^{-1}(v)$, that we used for defining the bivariate copula based stop-loss transform, Nair et al. (2017) defined the analogue of bivariate mean residual function of (X_1, X_2) using survival copula as the vector $(M_1(u, v), M_2(u, v))$, where

$$M_1(u, v) = m_1\left(R_1^{-1}(u), R_2^{-1}(v)\right) = \frac{-1}{C(u, v)} \int_0^u C(p, v) \frac{dR_1^{-1}(p)}{dp} dp$$

and

$$M_2(u,v) = m_2\left(R_1^{-1}(u), R_2^{-1}(v)\right) = \frac{-1}{C(u,v)} \int_0^v C(u,q) \frac{dR_2^{-1}(q)}{dq} dq.$$

Similarly, the bivariate quantile version of (7.34) and (7.35) will be

$$\mathrm{B}_1(u,v) = \beta_1\left(R_1^{-1}(u), R_2^{-1}(v)\right) = \frac{P_{1,0}(u,v)}{R_1^{-1}(u)\ C(u,v) + P_{1,0}(u,v)} \tag{7.36}$$

and

$$\mathrm{B}_2(u,v) = \beta_2\left(R_1^{-1}(u), R_2^{-1}(v)\right) = \frac{P_{0,1}(u,v)}{R_2^{-1}(v)\ C(u,v) + P_{0,1}(u,v)} \tag{7.37}$$

Now, we present the following theorem, which can be used to compare two bivariate income-gap ratios of two different populations.

Theorem 7.3.2. *Let* (X_1, X_2) *and* (Y_1, Y_2) *denote two random vectors with the boundary line vectors* (t_1, t_2) *with corresponding two survival copula's* $C(u,v)$ *and* $C^*(u,v)$ *and the corresponding copula-based income-gap ratio vectors be* $(\mathrm{B}_1(u,v), \mathrm{B}_2(u,v))$ *and* $(B_1^*(u,v), \mathrm{B}_2^*(u,v))$*. Then,*

$$\mathrm{B}_1(u,v) \leq (\geq) \mathrm{B}_1^*(u,v) \Leftrightarrow M_1(u,v) \leq (\geq) M_1^*(u,v), \tag{7.38}$$

and

$$\mathrm{B}_2(u,v) \leq (\geq) \mathrm{B}_2^*(u,v) \Leftrightarrow M_2(u,v) \leq (\geq) M_2^*(u,v) \tag{7.39}$$

where, $(\mathrm{M}_1(u,v), \mathrm{M}_2(u,v))$ *and* $(M_1^*(u,v), \mathrm{M}_2^*(u,v))$ *denote the vector-valued bivariate mean residual quantile functions corresponding to* (X_1, X_2) *and* (Y_1, Y_2) *respectively.*

Proof. Suppose $\mathrm{B}_1(u,v) \leq (\geq) \mathrm{B}_1^*(u,v)$ holds true. Then from the definition (7.34) it follows that

$$\mathrm{B}_1(u,v) \leq (\geq)\ \mathrm{B}_1^*(u,v) \quad \Leftrightarrow \quad \frac{P_{1,0}(u,v)}{R_1^{-1}(u)\ C(u,v) + P_{1,0}(u,v)} \leq (\geq) \frac{Q_{1,0}(u,v)}{S_1^{-1}(u)\ C^*(u,v) + Q_{1,0}(u,v)}. \tag{7.40}$$

where, $P_{1,0}(u,v)$ and $Q_{1,0}(u,v)$ denote the quantile based stop-loss transforms corresponding to (X_1, X_2) and (Y_1, Y_2) respectively. Now from (7.40), we have,

$$\mathrm{B}_1(u,v) \leq (\geq)\mathrm{B}_1^*(u,v) \Leftrightarrow \frac{R_1^{-1}(u)}{M_1(u,v)} \geq (\leq)\frac{S_1^{-1}(u)}{M_1^*(u,v)}$$

and since (t_1, t_2) denote the boundary line points, the marginals should be equal. i.e., $R_i^{-1}(u) = S_i^{-1}(u), i = 1,2$ for all values of u. Hence we have (7.38).

In a similar lines using (7.35) and employing the probability transforms, one can also prove that,

$$\mathrm{B}_2(u,v) \leq (\geq)\mathrm{B}_2^*(u,v) \Leftrightarrow \frac{R_2^{-1}(v)}{M_2(u,v)} \geq (\leq)\frac{S_2^{-1}(v)}{M_2^*(u,v)}.$$

Hence the theorem. □

As discussed earlier chapters, the measure MLPRI due to Belzunce et al. (1998) has a significant role in income studies. Hence applying the same probability transforms a copula based version of MLPRI is given by the vector

$$(\mathit{\Gamma}_1(u,v), \mathit{\Gamma}_2(u,v)) = \left(1 - \frac{M_1(u,v)}{R_1^{-1}(u)}, 1 - \frac{M_2(u,v)}{R_2^{-1}(v)}\right).$$

Now the following theorem is immediate.

Theorem 7.3.3. *Let (X_1, X_2) and (Y_1, Y_2) denote the two random vectors with the boundary points (t_1, t_2) and with two distribution functions $F(x_1, x_2)$ and $G(y_1, y_2)$ also let the corresponding MLPRI vectors are denoted by $(\mathit{\Gamma}_1(u, v), \mathit{\Gamma}_2(u, v))$ and $(\mathit{\Gamma}_1^*(u, v), \mathit{\Gamma}_2^*(u, v))$. Then, for $i = 1, 2$,*

$$\mathit{\Gamma}_i(u, v) \leq (\geq)\ \mathit{\Gamma}_i^*(u, v) \Leftrightarrow M_i(u, v) \leq (\geq)\ M_i^*(u, v).$$

The comparison of CES in terms of first order CUPM which is defined in Theorem 5.1 has a different interpretation in the comparison of deprivation in the context of incomes and wealth (see Duclos and Araar (2007), Belzunce et al. (2012a) and the references there in). The indicator defined by Duclos and Araar (2007),

$$\delta(p, q) = \left(F^{-1}(q) - F^{-1}(p)\right)_+ = \begin{cases} F^{-1}(q) - F^{-1}(p) & \text{if} \quad F^{-1}(q) \geq F^{-1}(p) \\ 0 & \text{if} \quad F^{-1}(q) \leq F^{-1}(p) \end{cases},$$

where $0 \leq p \leq 1$ and $0 \leq q \leq 1$ gives the relative deprivation of an individual with income $F^{-1}(p)$, while comparing with an another individual having income $F^{-1}(q)$ and the expected relative deprivation of an individual at rank p is given by

$$\overline{\delta}(p) = \int_0^1 \delta(p, q)\, dq = E\left[\left(X - F^{-1}(p)\right)^+\right]. \tag{7.41}$$

Extending the concept into bivariate set-up, the measure (7.41) can be used to compare the relative deprivation among two populations, say, (X_1, Y_1) and (X_2, Y_2). Then, for the conditioned random variable $X_1 > t_1 | Y_1 > t_2$, , the α-level conditional expected relative deprivation will be of the form (7.43). Hence the Theorem 7.3.4 can be used to compare the conditional expected relative deprivation among two different populations.

7.3.3 Comparisons of risks in actuarial studies

Partial moments are extensively used in actuarial science for the analysis of risks. Let the non-negative random variable X represents the random amount that an insurance company will pay to a policy holder in case of claim. Two measures that are very popular for the comparison of risks are: (i) the *Value-at-Risk (VaR)* and (ii) the *Expected Shortfall (ES)*.

The *VaR* is given by $VaR[X;\alpha] = F^{-1}(\alpha), \alpha \in (0,1)$. The expected shortfall is the right-spread function defined by, $\tau_X(\alpha) = p_1\left(F^{-1}(\alpha)\right) = E\left[\left(X - F^{-1}(\alpha)\right)_+\right] = \int\limits_{F^{-1}(\alpha)}^{\infty} R(u)\,du.$

In some situations, insurance companies do not have only one policy for some policyholders but have two policies. Suppose of a policyholder can have a policy to insure the car and another policy to ensure the house, with random claims X and Y (see Belzunce et al. (2012a)). It is clear that X and Y should exhibit some kind of dependence, say, a positive dependence. Let us consider two risky situations A and B for policyholders, with random risks (X_1, Y_1) and (X_2, Y_2). Then the random variables of interest are $X_1|Y_1 > t_2$ and $Y_1|X_1 > t_2$ or $X_2|Y_2 > t_2$ and $Y_2|X_2 > t_2$, which enables to compare the two risks such that the other risk is being above certain threshold value.

Let $R(t_1|t_2) = P(X_1 > t_1|X_2 > t_2)$ and $Q(\alpha|t_2) = \inf\{t_1 : R(t_1|t_2) \leq 1-\alpha\}$, $\alpha \in (0,1)$ be respectively the conditional survival and quantile functions of $X_1 > t_1|Y_1 > t_2$, where $Q(\alpha|t_2) = Q_{X_1|X_2>t_2}(\alpha|t_2) = F^{-1}_{X_1|X_2>t_2}(\alpha|t_2)$. Assume that $F(.|t_2)$ is absolutely continuous and strictly increasing in t_1. Then from the definition of the r^{th} order conditional upper partial moment (CUPM) given by

$$\psi_r(t_1|t_2) = r\int_{t_1}^{\infty}(x_1-t_1)^{r-1}R(x_1|t_2)dx_1,$$

one can define the quantile based r^{th} order conditional stop-loss transform as,

$$\psi_r\left(Q\left(\alpha|t_2\right)\right) = \int\limits_{\alpha}^{1}\left(Q\left(u|t_2\right) - Q\left(\alpha|t_2\right)\right)^r du.$$

Then, for $r = 1$,

$$\begin{aligned}\psi_1\left(Q\left(\alpha|t_2\right)\right) &= \int\limits_{\alpha}^{1}\left(Q\left(u|t_2\right) - Q\left(\alpha|t_2\right)\right)du \\ &= \int\limits_{\alpha}^{1}Q\left(u|t_2\right)du - \left(1-\alpha\right)Q\left(\alpha|t_2\right).\end{aligned}$$

The conditional value-at-risk will be the α^{th} quantile of the conditional distribution of $X_1 > t_1|X_2 > t_2$, i.e. $CVaR\left(Y\right) = \inf\left\{t_1 : \psi_0(t_1|t_2) \leq 1-\alpha\right\}, \alpha \in (0,1)$. Now, consider the conditional mean of $X_1|X_1 > t_1, X_2 > t_2$ is defined by

$$E\left(X_1|X_1 > t_1, X_2 > t_2\right) = t_1 + \frac{1}{R\left(t_1,t_2\right)}\int\limits_{t_1}^{\infty}R\left(u,t_2\right)du = t_1 + \frac{p_{1,0}\left(t_1,t_2\right)}{p_{0,0}\left(t_1,t_2\right)}.$$

Then, the α-level Conditional Expected Shortfall (CES), $\tau(\alpha|t_2)$ of $(X_1|X_1 > t_1, X_2 > t_2)$ is obtained by setting $t_1 = Q(\alpha|t_2)$. Thus in terms of conditional quantile function, we rewrite

$$E\left(X_1|X_1 > t_1, X_2 > t_2\right) = Q(\alpha|t_2) + \frac{p_{1,0}\left(Q(\alpha|t_2),t_2\right)}{p_{0,0}\left(Q(\alpha|t_2),t_2\right)}.$$

Moreover, we have,

$$E\left(X_1|X_1>t_1,X_2>t_2\right)=\frac{1}{R\left(t_1|t_2\right)}\int_{t_1}^{\infty}udF_{X_1|X_2>t_2}\left(u|t_2\right) \tag{7.42}$$

and by setting $t_1=Q(\alpha|t_2)$ in (7.42) yields

$$\tau(\alpha|t_2)=\frac{1}{\alpha-1}\int_{\alpha}^{1}Q(u|t_2)du. \tag{7.43}$$

Let $U=X_1|Y_1>t_2$ and $V=X_2|Y_2>t_2$ be conditional risk on two options with quantile functions $Q\left(\alpha|t_2\right)$ and $W\left(\alpha|t_2\right)$ respectively. Then, the following theorem provides the importance of quantile-based first order CUPM in comparing two risks in terms of conditional expected shortfalls.

Theorem 7.3.4. *Let $\left(\tau_U\left(\alpha|t_2\right),\psi_1\left(Q\left(\alpha|t_2\right)\right)\right)$ and $\left(\tau_V\left(\alpha|t_2\right),\gamma_1\left(W\left(\delta|t_2\right)\right)\right)$ denotes the CES and First order $CUPM$ pair for U and V respectively. Then, for all $\alpha\in(0,1)$,*

$$\tau_U\left(\alpha|t_2\right)\leq\tau_V\left(\alpha|t_2\right)\Leftrightarrow\psi_1\left(Q\left(\alpha|t_2\right)\right)\leq\gamma_1\left(W\left(\alpha|t_2\right)\right).$$

if and only if the function $\frac{1}{1-\alpha}\left(\int_{\alpha}^{1}Q\left(u|t_2\right)du-\int_{\alpha}^{1}W\left(u|t_2\right)du\right)$ is a decreasing function of α.

Proof.

$$\begin{aligned}
\tau_U(\alpha|x) \leq \tau_V(\alpha|x) \quad &\Leftrightarrow \quad \frac{1}{1-\alpha}\int_{\alpha}^{1} Q(u|t_2)du \leq \frac{1}{1-\alpha}\int_{\alpha}^{1} W(u|t_2)du \\
&\Leftrightarrow \quad \frac{1}{1-\alpha}\left[\int_{\alpha}^{1} Q(u|t_2)du - \int_{\alpha}^{1} W(u|t_2)du\right] \leq 0 \\
&\Leftrightarrow \quad \frac{\left[-Q(\alpha|t_2) + W(\alpha|t_2)\right]}{(1-\alpha)} \\
&\qquad\qquad + \frac{\left[\int_{\alpha}^{1} Q(u|t_2)du - \int_{\alpha}^{1} W(u|t_2)du\right]}{(1-\alpha)^2} \leq 0 \\
&\Leftrightarrow \quad (1-\alpha)W(\alpha|t_2) - \int_{\alpha}^{1} W(u|t_2)du \\
&\qquad\qquad \leq (1-\alpha)Q(\alpha|t_2) - \int_{\alpha}^{1} Q(u|t_2)du \\
&\Leftrightarrow \quad \psi_1(Q(\alpha|t_2)) \leq \gamma_1(W(\alpha|t_2)).
\end{aligned}$$

□

Chapter 8

Conclusion and future work

The notion of 'moment' in statistical sense is a specific quantitative measure of the shape of a set of points. Since the moments are closely connected to the different parameters of an underline population, there are various type of moments of the rv X are defined in the literature. Among these a popular one is the 'partial moments'. Partial moments are the specific quantitative measure for the data when the investigator is more interested in the tail events. For example, in order to analyse the affluence (poverty) of a population, the investigator has to fix a level of reference for the population and consider the portions of the population which falls above or below the reference level. In the context of actuarial studies, this threshold level, might be the rate of inflation, the real interest rate, the return on a benchmark index or the risk-free rate etc. If the analyst focus on the risky returns of an investment from an arbitrary assets, there correspond the upside and downside risks. Upside risk refers to the events in which the target returns falls above the reference level t (i.e. an unexpected gain situation). As a dual the actuary face a downside-risk situation when the returns falls shorts of t (i.e. an unexpected loss situation). These upward and downside risk situations in the context of life-length studies are termed as the expected residual life and inactivity times (reversed lifetime) respectively. Hence if we apply the usual moments to analyse the shape of the points in these kind of situations there is a high chance of obtaining misleading values as the estimates of population characteristics involving moments such as measure of central tendency, measure of variation, peakedness and shape. In such situation its better to deal with the respective upper and lower partial moments accordingly. Due to the wide applications of partial moments in different fields, such as risk analysis, actuarial science, forensic science, reliability modeling, survival analysis, etc., the study of partial moments and its higher orders based on residual and past lifetime are of greater interest among researchers.

In many statistical models, the assumption of independence between two or more variables is often due to convenience rather than to the problem at hand. In developing stochastic models for analyzing multivariate data characterization results for important probability models plays a crucial role. Since Characterization results implicates the facts under the imposed conditions the distribution F is the only probability distribution satisfying the designated property P. Hence one kind of theorem is developed for any such probability model in-terms of certain functional relationships between the measure of our interest, the model identification becomes much simpler. If the structure of the relationship is very simple and easy to interpret, tho model identification will also be very easy. Motivated by these facts the present study focused on developing characterization results for bivariate probability models using bivariate and conditional versions of partial moments.

There has been a various attempts to use partial moments as a useful tool in developing characterization theorem for probability models in the univariate frame work. For review see Chong (1977), Nair and Hitha (1990), Sunoj (2004), Abraham et al. (2007), Sunoj and Maya (2008), Nair et al. (2013a), Nair et al. (2013b), etc. In the present work we have extended the notion of partial moments to develop necessary theoretical framework for lifetime data analysis. In Chapter 1, we have given a brief outline of the work and in Chapter 2 we present the basic concepts and review of literature.

In Chapter 3 we mainly focus on characterizing the bivariate probability models using BUPMs for non-negative continuous random vectors. A new bivariate distribution is proposed by extending the results of Lin (2003) to the bivariate case. It is identified that, that proposed distribution is a modified version of bivariate Gumbel's exponential distribution. Some characterizing relationships between the BUPMs are established to model the proposed distribution. We have also proved characterization theorems that extend the result of Abraham et al. (2007) to characterize bivariate Pareto law using BUPMs. Finally, a data analysis is carried out to illustrate a theoretical result established in the paper using a bivariate failure time data given in Barlow and Proschan (1976).

In Chapter 4 we have introduced the UPMs for the conditionally specified and survival models. Since CUPMs uniquely determines the conditional distributions, some useful characterizations of important bivariate models are studied. The relationships connecting CUPMs and other common reliability measures are obtained. The CUPMs in the context of system reliability (i.e., for minimum and maximum of two component system) are examined. The applications of CUPMs in context of income studies are also investigated. Finally, nonparametric estimators for CUPMs are introduced and validated using simulated and real data sets it is also used for illustrating a characterization result that we proved in the same chapter.

In Chapter 5, we have introduced the LPMs for the conditional models. Since CLPM uniquely determines conditional distributions, some useful characterizations of important bivariate models are studied. The relationships and applications connecting CLPMs with reliability, risk, and income measures are established. Finally, a non-parametric estimator for CLPM is introduced and is validated using simulated and real data sets.

Chapter 6 has studied some reliability aspects of bivariate upper partial moments based on bivariate weighted models. A detailed study on bivariate equilibrium model using BWUPM has been conducted. Alternative definitions to positive (negative) dependence properties in terms of BWUPMs are established. Useful relationships with some important dependence notions viz., expectation dependence, stop-loss distance for bivariate weighted distributions are also derived. The concept of CUPMs are extended to weighted models and studied its various properties.

BUPMs is usually expressed in terms of distribution functions. In Chapter 7, an alternative approach is proposed by considering bivariate copulas instead of bivariate distributions. We define the analogues of BUPMs that are expressed in terms of copulas and study their properties. The proposed copula functions possess several advantages over the usual BUPMs defined in the literature. We can generate new copulas through appropriate choices of the copula-based BUPM and the proposed copula functions satisfy certain properties that are not shared by their distribution-based counterparts. We also illustrate the use of representing partial moments using copulas in the view point of system reliability studies. Applications of the copula based BUPMs and quantile based conditional partial moments in income studies and analysis of risks were also explored.

In the previous chapters we have seen that more results and findings are required to make more advanced study in connection with partial moments of higher orders in higher dimensions. On the basis of the present study it is felt that the following problems are of relevance in the future study.

(i) In order to apply the theoretical results those are establishing through partial moments in practice one must require an estimator. Let $\{X_1, \ldots, X_n\}$ denote n i.i.d set of observations with common survival function $R(t)$. A natural estimator that has been widely using for estimating univariate partial moments is the empirical estimator given by $\hat{p}_r(t) = \frac{1}{n}\sum\limits_{i=1}^{n}(X_i - t)^r I(X_i > t)$ or $\hat{l}_r(t) = \frac{1}{n}\sum\limits_{i=1}^{n}(t - X_i)^r I(X_i \leq t)$, where $I(.)$ is the usual indicator function. The estimators are the r^{th} sample moment of the observations from the events $\{X_i > t, 1 \leq i \leq n\}$ and $\{X_i \leq t, 1 \leq i \leq n\}$. Since these estimator are step functions, it possesses some undesirable properties like high volatile nature. Moreover the partial moments have different interpretations in different contexts. In life length studies the data that are commonly available are subject to censoring. The non-availability of the complete information is reflected in censoring. In most reliability and life testing experiments, due to time constraints or cost considerations the experimenter is forced to terminate the experiment after a specific period of time or after the failure of a specified number of units. Such non-availability of the complete information results the underlying data censored. There are different censoring mechanisms adopted by the experimenters. Also, the independence assumption for the observations is not always adequate in the real life practical situations, for example, sequentially collected economic data. Therefore, in practice it is more adequate to assume that there might exist some sort of dependence in the observed data. Accordingly, as a future work, we plans to derive new nonparametric estimators for the partial moments in different contexts and to validate it using simulated and real life data sets.

(ii) Since most of the real world problems are defined in higher dimensions, we also plan to develop nonparametric estimator's for the partial moments in bivariate and conditional set-up under different schemes such as censoring and dependence in data are present.

(iii) Usually association measures that are widely used to study about the dependence structure of bivariate and multivariate are functions of joint sf, joint CDF or conditional expectation etc. (see Nair and Sankaran (2010a)). Since the partial moments are the more specific measure when the tail events are of interest, we plan to develop an association measure using the target semivariance. Alternatively we are also investigating about the application product moment, a particular case of bivariate partial moment as useful tool as a new association measure.

(iv) In Chapter 7, we developed the concept of copula based BUPMs. The estimation of the proposed measure in the context of life-length and actuarial studies is also a challenging problem. Extension of this copula based measure in bivariate discrete set-up as well as for the bivariate weighted models is also an interesting problem.

We are currently attempting to resolve some of these problems and hopefully work in this direction is expected to be presented in a future work.

List of published/ accepted papers

1. Sunoj, S. M. and Vipin N. (2017). Some properties of conditional partial moments in the context of stochastic modelling. *Statistical Papers*, DOI:10.1007/ s00362 -017-0904-x.

2. Sunoj, S. M. and Vipin N. (2018). On conditional lower partial moments and its applications. *American Journal of Mathematical and Management Sciences*, 37 (1), 14-32.

3. N. Unnikrishnan Nair, Sunoj, S. M. and Vipin N. (2018). On characterizations of some bivariate continuous distributions by properties of higher-degree bivariate stop-loss transforms. *Communications in Statistics–Theory and Methods,* (accepted)

List of communicated papers

1. Vipin N. and Sunoj, S. M. (2018). On some properties and applications of first-order bivariate and conditional partial moments.

2. Vipin N. and Sunoj, S. M. (2018). Bivariate weighted partial moments.

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