A Study on Cross-Connections of Regular Rings

THESIS SUBMITTED TO

COCHIN UNIVERSITY OF SCIENCE AND TECHNOLOGY

FOR THE AWARD OF THE DEGREE OF

DOCTOR OF PHILOSOPHY

UNDER THE FACULTY OF SCIENCE

BY

SREEJAMOL P.R.

Reg.No.4556

DEPARTMENT OF MATHEMATICS

COCHIN UNIVERSITY OF SCIENCE AND TECHNOLOGY

KOCHI - 682 022, KERALA, INDIA

OCTOBER, 2018

A Study on Cross-Connections of Regular Rings

Ph.D. Thesis in the Field of Algebra

Author:
Sreejamol P.R.
Department of Mathematics
Cochin University of Science and Technology
Kochi - 682 022, Kerala, India
Email: *sreejasooraj@gmail.com*

Supervisor:
Dr. P. G. Romeo
Professor and Head
Department of Mathematics
Cochin University of Science and Technology
Kochi - 682 022, Kerala, India.
Email: *romeo_parackal@yahoo.com*

October 2018

Dr. P. G. Romeo
Professor and Head
Department of Mathematics
Cochin University of Science and Technology
Kochi - 682 022, India.

11th October 2018

Certificate

Certified that the work presented in this thesis entitled "**A Study on Cross-Connections of Regular Rings** "is based on the authentic record of research carried out by Mrs. Sreejamol P.R. under my guidance in the Department of Mathematics, Cochin University of Science and Technology, Kochi- 682 022 and has not been included in any other thesis submitted for the award of any degree.

P. G. Romeo
(Supervising Guide)

Phone : 9447663109 Email: *romeo_parckal@yahoo.com*

Department of Mathematics
Cochin University of Science and Technology
Cochin - 682 022

11th October 2018

Certificate

Certified that all the relevant corrections and modifications suggested by the audience during the Pre-synopsis seminar and recommended by the Doctoral Committee of the candidate has been incorporated in the thesis entitled "**A Study on Cross-Connections of Regular Rings**."

Dr. P. G. Romeo
(Supervising Guide)
Professor and Head
Department of Mathematics
Cochin University of Science & Technology
Cochin - 682 022

11th October 2018

Declaration

I, Sreejamol P.R., hereby declare that the work presented in this thesis entitled "**A Study on Cross-Connections of Regular Rings** "is based on the original research work carried out by me under the supervision and guidance of Dr. P. G. Romeo, Professor, Department of Mathematics, Cochin University of Science and Technology, Kochi- 682 022 and has not been included in any other thesis submitted previously for the award of any degree.

Sreejamol P.R.
Research Scholar
Reg. No.: 4556
Department of Mathematics
Cochin University of Science & Technology

To

My Beloved Husband

and

Children

Acknowledgements

"No one can whistle a symphony. It takes a whole orchestra to play it" - H.E.Luccock.

I wholeheartedly thank The Almighty God for giving me mental and physical strength to surpass the hurdles and complete my Ph.D. work.

In all these years, many individuals were instrumental, directly and indirectly, in shaping up my research work. It was hardly impossible for me to thrive in my doctoral work without the precious support of these personalities. Here is a tribute to all those wonderful persons.

First and foremost I would like to express my heartfelt gratitude to my advisor Dr. P.G. Romeo, Professor and Head, Department of Mathematics, CUSAT, for his never-ending patience, motivation, enthusiasm, commitment, immense knowledge, an invaluable continual support and also for providing all the facilities for my research work. I was fortunate enough to have him as my guide.

This thesis could not have been materialized without the generous help of many other important persons. I wish to express my immense thankfulness to Dr. A. Vijayakumar, Retd.Professor, Department of Mathematics, CUSAT, for being my Doctoral committee member and improvising me with valuable suggestions along with a lot of mental support, encouragement and care.

Besides my advisors, a special note of thanks to Dr. T.P. Johnson, Professor, School of Engineering, CUSAT for directing me to Romeo Sir, which became the first stepping stone in my doctoral work and for *his timely help and constant support all the way.*

My sincere thanks to Dr. Krishnamoorthi, Dr. R.S. Chakravarthy, Dr. M.N.N. Namboothiri, Dr. V.B. Kirankumar, Dr. Ambily A.A., Dr. Noufal A., Office staffs, former S.O. Mrs. Jaseela, Librarians and the Authorities of CUSAT for their kind approach and consideration.

Dr. K.S.S. Nambooripad, the legend who has immensely contributed in the area of 'semi-group', gave me his valuable suggestions and guidance, whenever I met him at his home. I was deeply motivated by his greatness and his work, which led me to accomplish my Ph.D. work. I bow my head to him in deep respect.

I acknowledge with sincere gratitude to the team at S.N.M. College, Maliankara, the Management, the Principals - Dr. K.K. Radha, Mrs. M.R. Usha, Mrs. T.H. Haripriya, Dr. K.K. Thampi(Late), Mr. T.S. Rajeev and Dr. Siby Komalan, the Head of Department of Mathematics Mrs. M.S. Asha, my colleagues Mr. Ajan K.A., Dr. Neelima C.A., Mrs. Ninu S. Lal, Mrs. Rejee Ittiyera, FIP substitute teachers Mrs. Renjitha P.R. and Ms. Mary Alphonsa and the office staffs especially Mr. K.V. Ganan and Mr. T.B. Harisarma, for all the supports extended to me in various aspects of my entire Ph.D. programme.

I owe much gratitude to many individuals for their invaluable advices, encouragements, support and co-operation for the completion of my thesis; which include my fellow research scholars Mrs. Lejo J. Manavalan, Mrs. Akhila R., Mrs. Siji Michael, Mr. Bintu Shyam, Mrs. Smisha M.A., Mrs. Smitha Davis, Mrs. Binitha Benny, Mrs. Susan Mathew Panakkal, Mrs. Rasila V.A., Mr. Rahul Rajan, Ms. Aparna Pradeep, Ms. Linet Roslin Anthony, Mrs. Vinitha T., Mrs. Savitha, Mrs. Pinky, Mrs. Divya V., Mrs. Elizabeth Reshma, Ms. Dhanya Shajin, Mr. Manjunath, Mrs. Seethu Varghese, Mrs. Savitha K.S., *Mr. Tijo James, Mrs. Anu Varghese. Their timely help, advices and supports helped me a lot in getting rid of my tensions and making my life at CUSAT a memorable one.*

I also place on record, my thanks to Mr. Azeef, Mr. Asokan and Mrs. Preethy of Kerala University for their co-operation in carrying out my research.

Words are not enough to express my heartfelt thanks to all my teachers, relatives and my friends those who shaped my career. My sincere thanks also goes to the UGC team, for granting me FDP deputation.

Last, first and always, I am deeply indebted to my sweet family which gives me inspiration, encouragement, strength and happiness. My parent-in-laws Mr. V.P. Narayanan (expired in February 2018) and Mrs. Sumithra encouraged me with moral support. Without their encouragement and prayers I could not even imagine about my doctoral work. My parents Mr. P.G. Rajan and Mrs. Drowpathy, brother Mr. Sreekumar and sister-in-law Mrs. Raji supported me a lot. Mere words cannot express my deep gratitude to my husband Mr. V.N. Sooraj and children Master Avinash and Master Karthik for their wholehearted love, devotion, sacrifice, patience and everlasting support, extended to me more than I deserve, throughout my research work.

Sreejamol P.R.

Introduction

A ring $(R, +, \cdot)$ is regular (also called von Neumann regular ring) if the multiplicative semigroup $(R, \cdot)$ is a regular semigroup. A study of the structure of regular ring using the structure of regular semigroup is the theme of this thesis. In this regard we extend the cross-connection theory (categorical approach) used to study the structure of regular semigroup initiated by K.S.S. Nambooripad to the study of the structure of regular rings. Category theory was invented by Samuel Eilenberg and Saunders Mac Lane in the 1940s. Their categorical view point has been widely accepted by working mathematicians. There are many successful attempts to use category theory to study several mathematical structures like semigroups, groups, rings etc. The ESN theorem (Erasmann-Schein-Nambooripad theorem) and inverse categories introduced by Lawson are great achievements in this direction (cf.[19]). There are many approaches to study the structure theory of regular semigroups by W.D. Munn, T.E. Hall, P.A. Grillet, K.S.S. Nambooripad and many others, of which Nambooripad's contribution is remarkable. K.S.S. Nambooripad introduced normal category as a category with subobjects, every morphism has normal factorization and each object is a vertex of an idempotent normal cone (cf.[25]). The principal left (right) ideals of a regular semigroup with suitable translations form normal categories.

A cross-connection is a sort of categorical duality which turns out to be very significant in the study of the structure of the algebraic objects under consideration. The concept of cross-connection was originally introduced by Grillet in 1974 in order to study the structure of regular semigroups using its ideal structure. In [12] he described the cross-connection of regular semigroup S by considering principal left (right) ideals of it as the regular partially ordered sets $\Lambda(S)(I(S))$ and if I is any regular partially ordered set, then the set $N(I)$ of all normal mappings on I is a regular semigroup such that $\Lambda(N(I))$ is order isomorphic to I (cf.[13]). Moreover, given two regular partially ordered sets Λ and I, the relation that should exist between them so that they are respectively order-isomorphic to the partially ordered sets of left and right ideals of a regular semigroup was characterized in terms of a pair of mappings $\Gamma : I \to \Lambda^o$ and $\Delta : \Lambda \to I^o$ where $\Lambda^o[I^o]$ denote the regular partially ordered set of all normal equivalence relations on $\Lambda[I]$ satisfying certain axioms. Grillet calls such a pair (Γ, Δ) of mappings as a *cross-connection* between I and Λ[13]. Any regular semigroup S induces, in a natural fashion, a cross-connection between $I(S)$ and $\Lambda(S)$. Grillet showed that if (Γ, Δ) is a cross-connection between regular partially ordered sets I and Λ, then the set U of all pairs (f, g) of mappings in $N(\Lambda) \times N(I)^{op}$ that respects the given cross-connection is a subsemigroup of $N(\Lambda) \times N(I)^{op}$ and is a fundamental regular semigroup inducing the given cross-connection. $N(I)^{op}$ denotes the left-right dual of the semigroup $N(I)$.

In 1985, K.S.S. Nambooripad and F.J. Pastijn together established the cross-connection (Γ, Δ) of a complemented modular lattice L by replacing the regular partially ordered sets in Grillet's theory with complemented modular lattice L and its dual L^{op}(cf.[27]). They obtained the fundamental regular semigroup $U(L, L^{op}; \Gamma, \Delta)$ of all pairs (f, g) of normal mappings that respecting the cross-connection (see Section 1.5). Later in 1994, K.S.S. Nambooripad extended Grillet's theory to construct arbitrary regular semigroups. He replaced regular partially ordered sets $\Lambda(S)$ and $I(S)$ of left and right ideals of regular semigroup S in Grillet's theory by categories $\mathbb{L}(S)$ and $\mathbb{R}(S)$ of left and right ideals of S with morphisms as appropriate translations and replaced the cross-connection in Grillet's theory by a local isomorphism of $\mathbb{R}(S)$ to the normal dual of $\mathbb{L}(S)$ (see Section 1.7).

In this thesis, we extend the category theoretical approach to the study of the structure theory of arbitrary semigroups, rings, and modules. Also we establish the cross-connection of certain algebraic structures such as regular ring and Boolean lattice.

The thesis is divided into five chapters. The first Chapter is preliminaries in which we include all definitions and basic results needed in the thesis. This Chapter include sections on lattices, semigroups, rings, modules, cross-connection of complemented modular lattice, category theory and cross-connection of normal categories.

In Chapter 2, we describe the cross-connection of Boolean lattice and obtain its representation as a cross-connection ring in which each element is represented as a pair of idempotent normal mappings. The addition is the Boolean addition (symmetric difference) and multiplication meet, its cross-connection determines a Boolean ring (which is a regular ring) and the principal ideals of such a ring again form a Boolean lattice isomorphic to the initial Boolean lattice.

In Chapter 3, we introduce proper categories which are more general than normal categories; in the sense that when restricted to appropriate conditions they reduce to normal categories. Proper categories, preadditive proper categories, abelian proper categories and $RR-$categories are described here and it is shown that the principal left and right ideals of a semigroup are proper categories, that of a ring are $RR-$proper categories. The set of all proper cones in a proper category is a semigroup and set of proper cones in an $RR-$proper category is a ring. In particular if the ring R is regular, then the category of the principal left (right) ideals of R is an $RR-$ normal category (cf.[20]). In Section 3.3, we discuss abelian proper categories. Here it is shown that for an R-module M where R is a commutative ring with unity, the category $S(M)$ whose objects are submodules of M and morphisms R-module homomorphisms is an abelian proper category. In particular when M is a semisimple module, the submodule category $S(M)$ is an abelian normal category and the set of all normal cones form a semisimple $R-$module.

Chapter 4 discusses certain set-valued functors called $H-$functors and using the $H-$functors, the duals of proper categories, preadditive proper categories and $RR-$normal categories are described.

Chapter 5 deals with the cross-connection of $RR-$normal categories. A cross-connection between two $RR-$normal categories $\mathcal{C}$ and $\mathcal{D}$ is a *local isomorphism* $\Gamma : \mathcal{D} \to N^*\mathcal{C}$ where $N^*\mathcal{C}$ is the normal dual of the category $\mathcal{C}$. Local isomorphism $\Gamma^* : \mathcal{C} \to N^*\mathcal{D}$ is the dual cross-connection. Cross connection Γ determines a bifunctor $\Gamma(-,-) : \mathcal{C} \times \mathcal{D} \to \mathbf{Set}$ and cross-connection Γ^* also determines a bifunctor $\Gamma^*(-,-)$. Then there exists a natural isomorphism $\chi_\Gamma : \Gamma(-,-) \to \Gamma^*(-,-)$. Using χ_Γ we get a collection of linked pair of normal cones U which is a regular ring and this ring U is called *cross-connection regular ring* of $RR-$normal categories.

Contents

Chapter 1

Preliminaries

In this chapter we present some basic definitions and results regarding different algebraic structures and categories arising out of these structures used in the sequel. For the concepts in lattice we follow Birkhoff ([4]), Gratzer ([10]), T.S. Blyth ([6]) and Halmos ([8]). Regarding semigroup theory, we follow J.M. Howie ([15]), P.A. Grillet ([12]) and Clifford and Preston ([7]). For rings and modules we follow Musili ([24]), Artin ([2]) and Serge Lang ([18]). For the definitions and results regarding category and cross-connections, we follow S.Mclane ([17]) and K.S.S. Nambooripad ([25]).

1.1 Lattices

Here we recall definitions and basic results regarding partially ordered sets and lattices.

Definition 1.1.1 ([6], page 1)**.** If L is a nonempty set then by a partial order on L we mean a binary relation on L that is reflexive, antisymmetric and transitive. We usually denote a partial order by the symbol $\leq$. Thus $\leq$ is a partial order on L if and only if

(1) $\forall a \in L, a \leq a$ (reflexive);

(2) $\forall a, b \in L$, if $a \leq b$ and $b \leq a$ then $a = b$ (antisymmetric); and

(3) $\forall a, b, c \in L$, if $a \leq b$ and $b \leq c$ then $a \leq c$ (transitive).

Example 1.1.1 (cf.[6], Example 1.3)**.** On the set $\mathbb{N}$ of natural numbers the relation of divisibility is a partial order.

Let $(L, \leq)$ be a poset and $B \subseteq L$.

- $a \in L$ is called an upper bound of $B \Leftrightarrow \forall b \in B : b \leq a$.
- $a \in L$ is called a lower bound of $B \Leftrightarrow \forall b \in B : a \leq b$.
- The greatest amongst the lower bounds, whenever it exists, is called the infimum of B, and is denoted by $infB$.
- The least upper bound of B, whenever it exists, is called the supremum of B, and is denoted by $supB$.

Definition 1.1.2 ([10])**.** A *lattice* is a poset $(L, \leq)$ such that $sup\{a, b\}$ and $inf\{a, b\}$ exist for all $a, b \in L$. A sublattice of L is a nonempty subset K of L such that K is closed under join and meet of L.

Example 1.1.2 (cf.[6], Example 2.8)**.** Let V be a vector space and $SubV$ denotes the set of subspaces of V then $(SubV; \cap, +, \subseteq)$ is a lattice.

A subset I of a lattice L is called an *ideal* if it is a sublattice of L and $x \in I$ and $a \in L$ imply that $x \wedge a \in I$. An ideal I of L is proper if $I \neq L$. The *principal ideal* $L(x)$ of L generated by $x \in L$ is $L(x) = \{y \in L | y \leq x\}$. It is the smallest ideal of L containing x. A lattice in which every subset has meet and join is a *complete lattice* . If $(L, \leq)$ is a lattice, so is its dual $(L, \geq)$.

If a lattice L contains the smallest (greatest) element with respect to $\leq$, then this uniquely determined element is called the *zero element* *(one element)*, denoted by 0 (1). 0 and 1 are called universal bounds.

The *principal ideal* $L(x)$ of L generated by $x \in L$ can also be denoted as the interval $[0, x]$.

Definition 1.1.3 (cf.[6], page 77). A lattice L with 0 and 1 is called *complemented* if for all a in L, there exists at least one element b such that $a \vee b = 1$ and $a \wedge b = 0$. Then b is called the complement of a. A lattice L is called *relatively complemented* if given $a \leq x \leq b$, an element y exists such that $x \wedge y = a$ and $x \vee y = b$. A lattice L is called *modular* if for every $a, b, c \in L, a \leq c \Rightarrow (a \vee b) \wedge c = a \vee (b \wedge c)$. A lattice L called *distributive* if for all $a, b, c \in L$,

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \text{ or}$$

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c).$$

A complemented distributive lattice is called *Boolean lattice*. In a Boolean lattice complement of each element is unique.

Example 1.1.3 (cf.[6], Example 6.2). Lattice of all subspaces of a vector space is a complemented modular lattice, however the complement is not unique.

Example 1.1.4 ([28], page 72). The principal right [left] ideals of a regular ring form relatively complemented modular lattice.

Example 1.1.5 ([8], page 8). The power set of any set X, $(P(X), \cap, \cup, ^c)$ is a Boolean lattice.

Definition 1.1.4 ([8]). An ideal of a Boolean lattice L is a set $I \subseteq L$ such that

1. $0 \in I$,
2. if $a \in I$ and $b \in I$, then $a \vee b \in I$, and
3. if $a \in I$ and $b \in L$, then $a \wedge b \in I$.

The principal ideal of L generated by a in L is $L(a) = [0, a]$.

Definition 1.1.5 (cf.[8], page 202)**.** *Complete ideal* in a Boolean lattice L is an ideal I of L such that if $\{a_i\}$ is a family in I with a supremum a in L, then $a \in I$.

Principal ideals are examples of complete ideals.

Definition 1.1.6 ([8], page 89)**.** Let A and B be Boolean lattices. A (Boolean) homomorphism is a mapping $f : A \to B$ such that, for all $p, q \in A$:

1. $f(p \wedge q) = f(p) \wedge f(q)$,
2. $f(p \vee q) = f(p) \vee f(q)$ and
3. $f(a^c) = f(a)^c$.

Theorem 1.1.1 (cf.[8], Theorem 21)**.** *The class of all complete ideals in a Boolean lattice L is itself a complete Boolean lattice with respect to the distinguished Boolean elements and operations defined by*

(1) $0 = \{0\}$,

(2) $1 = L$,

(3) $M \wedge N = M \cap N$,

(4) $M \vee N = \bigcap\{I : I \text{ is a complete ideal in } L \text{ and } M \cup N \subseteq I\}$,

(5) $M^c = \{p \in L : p \wedge q = 0 \text{ for all } q \in M\}$.

1.2 Semigroups

The formal study of semigroups began in the early twentieth century. A semigroup is a nonempty set S with a binary operation from $S \times S \to S$ as $(x, y) \to xy$ such that $x(yz) = (xy)z$ for all $x, y, z \in S$. A subset T

of a semigroup S is a subsemigroup of S if T is a semigroup with respect to the restriction of the binary operation of S to T. A semigroup with an identity element is called a monoid. A semigroup S is commutative if the product in S is commutative. An element $e \in S$ is said to be an idempotent if $e^2 = e$. The set of idempotents of S is denoted as $E(S)$.

Example 1.2.1 ([15])**.** $(\mathbb{N}, +)$ and $(\mathbb{N}, \cdot)$ are semigroups with respect to addition and multiplication of natural numbers.

Example 1.2.2 (cf.[15], page 6)**.** The set of all maps from a set X into X with the binary operation as composition of maps is a semigroup which is called the *full transformation semigroup* on X.

Example 1.2.3 (cf.[15], page 16)**.** The set of all binary relations on a set X is a semigroup denoted by B_X with the operation '$\circ$' defined as, for all $\rho, \sigma \in B_X$,

$$\rho \circ \sigma = \{(x, y) \in X \times X : (\exists z \in X)(x, z) \in \rho \text{ and } (z, y) \in \sigma\}.$$

An element ϕ of B_X is called a *partial map* of X if $|x\phi| = 1$ for all x in $dom\phi$, that is, if, for all $x, y_1, y_2 \in X$,

$$[(x, y_1) \in \phi \text{ and } (x, y_2) \in \phi] \Rightarrow y_1 = y_2.$$

The set of all partial maps of X denoted as $\mathcal{P}T_X$ is a subsemigroup of B_X with the same operation as in B_X, called partial transformation semigroup. T_X is also a subsemigroup of B_X.

Definition 1.2.1 ([14])**.** An element a of a semigroup S is called *regular* if there exists an element x in S such that $axa = a$. The semigroup S is called *regular semigroup* (von Neumann regular) if all its elements are regular.

Example 1.2.4 ([15])**.** The full transformation semigroup T_X on a set X is regular and $\mathcal{P}T_X$ is a regular subsemigroup of T_X.

Definition 1.2.2 (cf.[25], page 45)**.** A *left translation* in a semigroup S is a mapping $\lambda : S \to S$ such that $\lambda(xy) = \lambda(x)y$ for all $x, y \in S$. If λ and μ are left translations then so is $\lambda\mu$. Dually a right translation in S can be defined.

Ideals and Green's Relations

Green's relations are five equivalence relations that characterise the elements of a semigroup in terms of the principal ideals they generate, are important tools for analyzing the ideals of a semigroup and related notions of structure. The relations are named after James Alexander Green, who introduced them in a paper in 1951. Instead of working directly with a semigroup S, we define Green's relations over the monoid S^1 (see [15]).

Let S be a semigroup. $I \subseteq S$ is called left [right] ideal of a semigroup S if $SI \subseteq I[IS \subseteq I]$. The principal left ideal of a semigroup S generated by a is $S^1a = \{sa|s \in S^1\}$ where S^1 is the semigroup S with an identity adjoined if necessary. That is, S^1a is $Sa \cup \{a\}$. Dually principal right ideal also can be defined.

The Green's relations on a semigroup S written as $\mathcal{L}$, $\mathcal{R}$ and $\mathcal{J}$ are defined as follows: For elements a and b of S,

$$a\mathcal{L}b \Leftrightarrow S^1a = S^1b,$$

$$a\mathcal{R}b \Leftrightarrow aS^1 = bS^1$$

$$a\mathcal{J}b \Leftrightarrow S^1aS^1 = S^1bS^1$$

where S^1a, aS^1 and S^1aS^1 are principal left, right and two sided ideals generated by a respectively. The Green's relations $\mathcal{D}$ and $\mathcal{H}$ are defined as

$$\mathcal{D} = \mathcal{L} \vee \mathcal{R},$$

$$\mathcal{H} = \mathcal{L} \wedge \mathcal{R}.$$

For commutative semigroups all the Green's relations coincide. $\mathcal{L}$-class, $\mathcal{R}$-class, $\mathcal{H}$-class, $\mathcal{D}$-class, $\mathcal{J}$-class containing the element a are denoted by $\mathcal{L}_a, \mathcal{R}_a, \mathcal{H}_a, \mathcal{D}_a, \mathcal{J}_a$ respectively. Partial orders are defined on the quotient sets $S/\mathcal{L}, S/\mathcal{R}, S/\mathcal{J}$ as follows:

$$\mathcal{L}_a \leq \mathcal{L}_b \Leftrightarrow S^1a \subseteq S^1b$$

$$\mathcal{R}_a \leq \mathcal{R}_b \Leftrightarrow aS^1 \subseteq bS^1$$

$$\mathcal{J}_a \leq \mathcal{J}_b \Leftrightarrow S^1aS^1 \subseteq S^1bS^1.$$

We see that $S/\mathcal{L}(S/\mathcal{R}, S/\mathcal{J})$ is isomorphic to the partially ordered set of all principal left (right, two-sided) ideals of S ordered by inclusion.

Proposition 1.2.1 (cf.[15], Proposition 2.1.1)**.** Let a, b be elements of a semigroup S. Then $a\mathcal{L}b$ if and only if there exist $x, y \in S^1$ such that $xa = b, yb = a$. Also $a\mathcal{R}b$ if and only if there exists $u, v \in S^1$ such that $au = b, bv = a$.

Let S be a semigroup. A relation R on the set S is called left compatible (with the operation on S) if

$$(\forall s, t, a \in S),\ (s, t) \in R \Rightarrow (as, at) \in R,$$

and right compatible if

$$(\forall s, t, a \in S),\ (s, t) \in R \Rightarrow (sa, ta) \in R.$$

It is called *compatible* if

$$(\forall s, t, s', t' \in S),\ [(s, t) \in R \text{ and } (s', t') \in R] \Rightarrow (ss', tt') \in R.$$

A left [right] compatible equivalence is called a left [right] congruence and a compatible equivalence relation is called a *congruence* (cf.[14]). Thus it can be seen that $\mathscr{L}$ is a right congruence and $\mathscr{R}$ is a left congruence.

Definition 1.2.3 ([12])**.** A *fundamental semigroup* S is a semigroup in which the equality on S is the only congruence contained in $\mathcal{H}$, that is semigroups having no non-trivial idempotent separating congruences.

The fundamental semigroups were first introduced by Munn in 1966 ([23]).

1.3 Rings

A ring is an algebraic structure with operations that generalize the arithmetic operations of addition and multiplication. A ring is a basic structure in algebra and by a ring we always mean an associative ring with identity.

Definition 1.3.1 (cf.[24], Definition 1.1.1)**.** A nonempty set R together with two binary operations called *addition* $(+)$ and *multiplication* $(\cdot)$ on R is called a ring, if

1. $(R, +)$ is an abelian group,

2. $(R, \cdot)$ is a semigroup and

3. Distributive laws hold.

Thus the theory of rings is a combination of a semigroup and an abelian group structure usually written as $(R, +, \cdot)$. In the ring $(R, +, \cdot)$, if the semigroup $(R, \cdot)$ has an identity, it is unique and is denoted by 1 and is called the *identity element* or the *unity* of R. A ring R is said to be *commutative* if the semigroup$(R, \cdot)$ is commutative. Subring of a ring R is a non-empty subset S of R such that $(S, +)$ is a subgroup of $(R, +)$ and $(S, \cdot)$ is a subsemigroup of $(R, \cdot)$.

Example 1.3.1 (cf.[24], page 5). The set of all integers($\mathbb{Z}$), rational numbers ($\mathbb{Q}$), real numbers ($\mathbb{R}$) and complex numbers ($\mathbb{C}$) are commutative rings with unity under the standard operations of addition and multiplication. The subsets $\mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$ are all subrings of $\mathbb{C}$.

Example 1.3.2 (cf.[24], Definition 1.8.3, page 24). Gaussian integers $\mathbb{Z}[i]$ where $i \in \mathbb{C}$ defined by $\mathbb{Z}[i] = \{a + bi | a, b \in \mathbb{Z}\}$ is a ring.

Example 1.3.3 (cf.[24], page 15). Let $n \in \mathbb{N}$. The set of all $n \times n$ matrices over $\mathbb{R}$ is a ring with respect to usual addition and multiplication of matrices.

Definition 1.3.2 (cf.[24], Definition 3.1.1). A mapping $f : R \to S$ of rings R and S is called a *homomorphism* if $f(a + b) = f(a) + f(b)$ and $f(ab) = f(a)f(b)$ for all $a, b \in R$. An *isomorphism* of rings is a bijective homomorphism.

An integral domain R is a nonzero ring having no zero divisors. That is if $ab = 0$, then $a = 0$ or $b = 0$ and also $1 \neq 0$ in R.

A ring R is a *division ring* if every nonzero element of R has a multiplicative inverse in R. A commutative division ring is called a *field* .

Definition 1.3.3 (cf.[24], Definition 4.1.1, page 107). Let $a, b \in R, a \neq 0$ where R is a ring. We say that *a divides b* or a is a *divisor* of b and written $a|b$ if there exists $c \in R$ such that $b = a \cdot c$.

Definition 1.3.4 (cf.[24], Definition 4.2.1). A commutative integral domain R (with or without unity) is called a *Euclidean domain* if there is a map $d : R^* \to \mathbb{Z}^+$ where $R^* = R \setminus \{0\}$ such that:

1. $\forall a, b \in R^*,\ a|b \Rightarrow d(a) \leq d(b)$
2. Given $a \in R,\ b \in R^*$, there exists $q, r \in R$ (depending on a and b) such that $a = qb + r$ with either $r = 0$ or else $d(r) < d(b)$.

Example 1.3.4 ([24], Example 4.2.3)**.** The ring of integers $\mathbb{Z}$, any field F and ring of Gaussian integers $\mathbb{Z}[i]$ are Euclidean domains.

Definition 1.3.5 ([28])**.** $(R, +, \cdot)$ is a (von Neumann) regular ring if it is a ring with the multiplicative part a regular semigroup (see Definition 1.2.1).

Example 1.3.5 ([28])**.** Every field is von Neumann regular. The ring of $n \times n$ matrices $M_n(F)$ is regular with entries from some field F.

An element $e \in R$ is said to be idempotent if $e.e = e$. $E(R)$ denotes the set of all idempotents of R. The principal ideals of a regular ring are idempotent generated and form a relatively complemented modular lattice ([28]).

Definition 1.3.6 ([24], Definition 1.7.1)**.** A ring R with identity is called *Boolean ring* if every element is an idempotent.

Boolean ring corresponds to Boolean lattice and vice versa ([8]). A subset of a Boolean lattice is a Boolean ideal if and only if it is an ideal in the corresponding Boolean ring [8]. Note that a Boolean ring is von Neumann regular ring, necessarily commutative and has cardinal number a power of 2.

Example 1.3.6 ([24], Example 1.7.2)**.** Let X be any non-empty set and $\mathscr{P}(X)$ be the power set of X with addition and multiplication on $\mathscr{P}(X)$ defined by $A+B = A \oplus B = (A \cup B) \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A)$ and $AB = A \cap B$ is a Boolean ring.

Ideals and Green's relations in rings

Let R be a ring. The *left [right] ideal* I of R is an additive abelian group such that $RI \subseteq I[IR \subseteq I]$. Let $a \in R$. The *principal left (right) ideal* generated by a is $(a)_l = Ra$ and $(a)_r = aR$. If I is simultaneously both left and right ideal of R, we say that I is a *two-sided ideal*. Suppose I

and J are both left or right or two sided ideals of ring R. Their sum $I+J$ is defined as

$$I + J = \{x + y | x \in I, y \in J\}$$

it is the smallest ideal containing both I and J.
Their intersection $I \cap J$ is the usual intersection of sets:

$$I \cap J = \{x | x \in I \text{ and } x \in J\}.$$

Following the Green's relations in semigroups, analogous versions of Green's relations have been defined for rings (cf. [29]).

1.4 Modules

A module is one of the fundamental algebraic structures in abstract algebra. A module over a ring is a generalization of the notion of vector space over a field, wherein the corresponding scalars are the elements of a ring (with identity) and a multiplication (on the left and/or on the right) is defined between elements of the ring and elements of the module. Just as the linear transformations between vector spaces, we have homomorphisms between modules.

Definition 1.4.1 (cf.[24], Definition 5.1.1)**.** Let R be any ring. A left R-module M is an abelian group $(M, +)$ together with a map from $R \times M \to M$ as $(a, x) \mapsto ax$ called the *scalar multiplication* such that

1. $a(x + y) = ax + ay$ for all $a \in R$ and $x, y \in M$
2. $(a + b)x = ax + bx$ for all $a, b \in R$ and $x \in M$
3. $(ab)x = a(bx)$ for all $a, b \in R$ and $x \in M$

A left R-module M is called *unitary left R-module* if $1 \cdot x = x$ for all $x \in M$.

Similarly one can define the *right R-module* as an additive abelian group with scalar multiplication on the right. If R is commutative, the notions of left and right modules coincide.

Let M be an R-module. A nonempty subset N of M is called an *R-submodule* of M if

1. N is an additive subgroup of M, i.e., $x, y \in N \Rightarrow x - y \in N$
2. N is closed for arbitrary scalar multiplication, i.e., $x \in N, a \in R \Rightarrow ax \in N$.

Suppose M is an R-module and P, Q are both submodules of M. Then the *sum* of the submodules $P + Q = \{x + y | x \in P, y \in Q\}$ is the smallest R-submodule containing both P and Q. Their *intersection* $P \cap Q = \{x | x \in P \text{ and } x \in Q\}$ is the intersection of P and Q in the usual sense.

A *homomorphism* of R-modules M and N is a map $f : M \to N$ which is compatible with the laws of composition

$$f(x + y) = f(x) + f(y) \text{ and } f(ax) = af(x),$$

for all $x, y \in M$ and $a \in R$. A bijective homomorphism is called *isomorphism*. The *kernel* of a homomorphism $f : M \to N$ is a submodule of M; denoted by $ker f = \{x \in M | f(x) = 0\}$ and *image* of f is a submodule of N.

Remark 1.4.1 (cf.[24], page 144-145)**.** The *direct product* of two R-modules is again an R-module. For a collection of R-modules $\{M_i\}_{i \in I}$, the *direct product* $\Pi_{i \in I} M_i$ is the product of the underlying sets M_i with R-module structure given by component-wise addition and scalar multiplication.

The direct product $\Pi_{i \in I} M_i$ is equipped with a collection of projection maps $\{\pi_i : \Pi_{i \in I} M_i \to M_i\}_{i \in I}$ given by $\pi_i((m_i)_{i \in I}) = m_i$ for all $i \in I$. Each π_i is an R-module homomorphism.

The *direct sum* $\oplus_{i\in I} M_i$ is a submodule of the direct product $\Pi_{i\in I} M_i$ consisting of elements $(m_i)_{i\in I}$ such that all but a finitely many m_i are zero.

The direct sum $\oplus_{i\in I} M_i$ is equipped with a collection of injection maps $\{p_i : M_i \to \Pi_{i\in I} M_i\}_{i\in I}$ given by $p_i(m) = (m_i)_{i\in I}$ where for all $j \neq i, m_j = 0$ and $m_i = m$, for all $m \in M_i$. Each p_i is an R-module homomorphism.

Example 1.4.1 ([2])**.** If R is a field F, then F-module is an F-vector space. Unitary modules over $\mathbb{Z}$ are simply abelian groups. A ring R can be considered to be both a left R-module and a right R-module.

Definition 1.4.2 (cf.[24], Definition 5.8.3)**.** A nonzero module M is called *simple module* if it has only trivial submodules (0) and M. A field is a simple module viewed as a module over itself. A module is called *semisimple* if it is a direct sum of simple modules.

The module $M_n(D)$ is semisimple for division ring D. Every simple module is semisimple. Note that the ring $\mathbb{Z}$ is not a semisimple module over itself but $\mathbb{Z}_n$ with n, a square free integer is a semisimple module over $\mathbb{Z}$. If M is semisimple R-module, then every submodule and every quotient module of M are semisimple.

Remark 1.4.2. Let M be a semisimple module and $M = \bigoplus_{i\in I} M_i$ where M_i are simple modules and let W be a submodule of M, then $M = W \bigoplus W'$ where $W = \bigoplus_{i:M_i\subset W} M_i$, hence W is semisimple. Also $W' = \bigoplus_{i:M_i\cap W=0} M_i$. As $M/W \cong W'$, it is semisimple and it is the complement of W. The submodules of a semisimple module form *complemented modular lattice* with respect to intersection as meet and sum as join.

Lemma 1.4.1 (cf.[24], Shur's Lemma)**.** Suppose M and N are two simple R-modules. Then any R-module homomorphism (R-linear map) $f : M \to N$ is either 0 or an isomorphism. In particular, the endomorphism ring $End_R(M)$ is a division ring.

The *first isomorphism theorem* for modules states that if $\theta : M \to N$ is an R-module homomorphism between two R-modules M and N then the induced homomorphism $\overline{\theta} : M|_{Ker\theta} \to Im\theta$ is an isomorphism. For semisimple modules $M|_{Ker\theta} \cong (Ker\theta)^c$ and hence $(Ker\theta)^c \cong Im\theta$.

1.5 Cross-connection of complemented modular lattice

Here we describe Grillet's method of cross-connection on regular posets (cf.[12]) and K.S.S. Nambooripad and F.J. Pastijn's method of cross-connection on complemented modular lattices (cf.[27]).

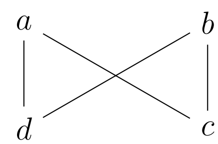
Definition 1.5.1 (cf.[12], page 278). The *ideal* of a partially ordered set X is a subset Y of X such that $x \leq y \in Y$ implies $x \in Y$. The *principal ideal* $X(x)$ of X generated by $x \in X$ is $\{y \in X | y \leq x\}$; it is the smallest ideal of X containing x.

Definition 1.5.2 (cf.[12], page 278). Let X be a partially ordered set, a mapping $f : X \to X$ is a *normal mapping* if it has the following three properties:

1. f is order preserving;
2. the range imf of f is a principal ideal of X;
3. for each $x \in X$ there exists $y \leq x$ such that f maps $X(y)$ isomorphically upon $X(xf)$.

In particular, if f is normal, then there exists at least one element $b \in X$ such that f is an isomorphism of $X(b)$ onto $X(a) = imf$. We denote by $M(f)$ the set of all elements $b \in X$ with this property.

Example 1.5.1. Consider the partially ordered set P below,



The map $f : P \to P$ defined by $f : a \to b,\ b \to b,\ c \to c,\ d \to d$ is a normal mapping with $M(f) = \{a, b\}$.

The set of all normal mappings from X to X, denoted by $N(X)$ is a semigroup under composition. The elements of $N(X)$ will be written as right operators and the elements of its dual $N^{op}(X)$ will be written as left operators. Idempotent normal mappings are called *normal retractions* and a principal ideal $X(a)$ is called *normal retract* if principal ideal $X(a) = ime$ where e is some normal retraction. X is called *regular poset* if every principal ideal of X is a normal retract.

Example 1.5.2 (cf.[12], page 278)**.** If S is a regular semigroup, $\mathcal{L}$ and $\mathcal{R}$ are Green's relations; then $\Lambda = S/\mathcal{L}$ and $I = S/\mathcal{R}$ are regular posets.

Definition 1.5.3 (cf.[12])**.** An equivalence relation ρ on a poset P is said to be *normal* if there exists a normal mapping $f \in N(P)$ such that $kerf = ff^{-1} = \rho$.

The poset (under the reverse of inclusion) of all normal equivalences on P is denoted by P^o such that when P is regular, then so is P^o ([13]). With each $\rho \in P^o$ we may associate the subset $M(\rho)$ defined by $M(\rho) = M(f)$, where f is any normal mapping with $kerf = \rho$ and $a \in M(\rho)$ iff $P(a)$ intersects every ρ-class in exactly one element. Then $P(a) \cap \rho(x)$ contains a single element which is minimal in its ρ-class and the mapping $\varepsilon_p(\rho, a)$ which sends each x in P to the unique element in $P(a) \cap \rho(x)$ is a normal retraction with $ker\varepsilon_p(\rho, a) = \rho$ and

$\mathrm{im}\varepsilon_p(\rho, a) = P(a)$. $\varepsilon_p(\rho, a)$ is called the projection along ρ upon $P(a)$ (cf.[13]).

Proposition 1.5.1 (cf.[27], Proposition 1)**.** Let I and Λ be regular posets and $f : I \to \Lambda$ be a normal mapping. For $\sigma \in \Lambda^o$, define $f^o(\sigma) = ker(f\epsilon_\Lambda(\sigma, u)) = \sigma f^{-1}$ where $u \in M(\sigma)$. Then $f^o : \Lambda^o \to I^o$ is a normal mapping such that $imf^o = I^o(kerf)$ and $M(f^o) = \{\rho \in \Lambda^o | b \in M(\rho)\}$ where $imf = \Lambda(b)$. If P, Q and R are regular partially ordered sets, and if $f : P \to Q$ and $g : Q \to R$ are normal mappings then $(fg)^o = f^o g^o$.

Definition 1.5.4 (cf.[6], page 7)**.** An order preserving mapping $f : P \to Q$ of posets P and Q is said to be *residuated* if there exists an order preserving mapping $f^+ : Q \to P$ such that $f \cdot f^+ \geq id_P$ and $f^+ \cdot f \leq id_Q$. The mapping f^+ is called the *residual* of f.

Example 1.5.3 (cf.[6], Example 1.17)**.** If E is any set and $A \subseteq E$ then for the power set $\mathbb{P}(E)$ of E, $\lambda_A : \mathbb{P}(E) \to \mathbb{P}(E)$ defined by $\lambda_A(X) = A \cap X$ is residuated with residual λ_A^+ given by $\lambda_A^+(Y) = Y \cup A^c$.

Example 1.5.4 (cf.[6], Example 1.20)**.** If S is a semigroup, define a multiplication on the power set $\mathbb{P}(S)$ of S by

$$XY = \begin{cases} \{xy \mid x \in X, y \in Y\} \text{ if } X, Y \neq \phi; \\ \phi, \text{ otherwise.} \end{cases}$$

Then multiplication by a fixed subset of S is a residuated mapping on $\mathbb{P}(S)$.

The set $ResP$ of all residuated maps of P is a semigroup and $f \to f^+$ is a dual isomorphism of $ResP$ onto the semigroup Res^+P of all residuals of elements of $ResP$. An $f \in ResP$ is totally range closed if f maps principal ideals onto principal ideals. Observe that a residuated map that is also normal must be totally range closed. Further $f \in ResP$ is strongly range closed if f and f^+ are totally range closed transformations of P and P^{op} respectively where P^{op} is the dual of P. The set $B(P)$ of all strongly range closed transformations of P is a subsemigroup of $ResP$ and $f \to f^+$ is an isomorphism of $B(P)$ onto $B(P^{op})$. If $f \in ResP$ and if both f and f^+ are normal, then f is *binormal mapping* and $f \in B(P)$.

In [27] it is described that if I and Λ are regular posets and $\Gamma : \Lambda \to I^o$, $\Delta : I \to \Lambda^o$ are order preserving mappings, then $(f, g) \in N(I) \times N(\Lambda)^{op}$ is *compatible* with (Γ, Δ) if the following conditions hold:

(1) $imf = I(x), img = \Lambda(y) \Rightarrow kerf = \Gamma(y), kerg = \Delta(x)$,

(2) the following diagrams commute:

$$\begin{array}{ccc} I & \xrightarrow{\Delta} & \Lambda^o \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle g^o} \\ I & \xrightarrow[\Delta]{} & \Lambda^o \end{array} \qquad \begin{array}{ccc} \Lambda & \xrightarrow{\Gamma} & I^o \\ {\scriptstyle g}\uparrow & & \uparrow{\scriptstyle f^o} \\ \Lambda & \xrightarrow[\Gamma]{} & I^o \end{array}$$

The definition of cross-connection is given in the following theorem:

Theorem 1.5.1 (cf.[27], Theorem 2). *Let I, Λ be regular partially ordered sets and let $\Gamma : \Lambda \to I^o$, $\Delta : I \to \Lambda^o$ be order preserving mappings. Then $[I, \Lambda; \Gamma, \Delta]$ is a cross-connection if and only if the following conditions are satisfied:*

1. *$x \in M(\Gamma(y)) \Leftrightarrow y \in M(\Delta(x))$, $x \in I, y \in \Lambda$,*

2. *if $x \in M(\Gamma(y))$, then the pair*

$$(\varepsilon_I(\Gamma(y), x), \varepsilon_\Lambda(\Delta(x), y))$$

is compatible with (Γ, Δ).

Proposition 1.5.2 ([12], Proposition 2.3). Let $[I, \Lambda; \Gamma, \Delta]$ be a cross-connection between two regular posets I and Λ. Then $U =$

$U(I, \Lambda; \Gamma, \Delta)$ consisting of all the pairs $(f, g) \in N(I) \times N(\Lambda)^{op}$ that are compatible with (Γ, Δ) is a fundamental regular semigroup.

In particular if the regular poset becomes a *complemented modular lattice* L, the cross-connection of complemented modular lattices L and its dual L^{op} is described below (cf.[27]).

Theorem 1.5.2 (cf.[6], Theorem 6.21, page 97). *If L is a lattice then a residuated mapping $f : L \to L$ is totally range closed if and only if*

$$f[f^+(x) \wedge y] = x \wedge f(y), \ (\forall x, y \in L);$$

and is dually totally range closed if and only if

$$f^+[f(x) \vee y] = x \vee f^+(y), \ (\forall x, y \in L).$$

Proposition 1.5.3 (cf.[27], Proposition 3). Let L be a complemented modular lattice, let $a \in L$ and let a^c be a complement of a in L. Then $(a; a^c) : L \to L, x \to (x \vee a) \wedge a^c$ is a binormal idempotent mapping such that $(a; a^c)^+ : L^{op} \to L^{op}, y \to (y \wedge a^c) \vee a$ is the residual of $(a; a^c)$. Further

$$ker(a; a^c) = \Delta(a) = \{(x, y) | x \vee a = y \vee a\}$$

$$ker(a; a^c)^+ = \Gamma(a^c) = \{(x, y) | x \wedge a^c = y \wedge a^c\}$$

and $M(\Gamma(a)) = M(\Delta(a)) = \{a^c | a^c \text{ is a complement of } a \text{ in } L\}$.

Let L be a lattice with 0 and 1. For each $a \in L$ the relation

$$\Gamma(a) = \{(x, y) | x \wedge a = y \wedge a\} \tag{1.1}$$

is an equivalence relation on L, and the mapping $\Gamma : L \to Eq(L)$, $a \mapsto \Gamma(a)$ is an order preserving embedding of L into the poset $Eq(L)$ of all equivalence relations on L ordered under the reverse of inclusion.

Note that $\Gamma(a) = ker f_a$, where $f_a : L \to L, x \mapsto x \wedge a$ is a normal retraction of L. Hence $\Gamma(a) \in L^o$ for all $a \in L$ and

$$\Gamma : L \to L^o, a \mapsto \Gamma(a) \tag{1.2}$$

is an order preserving embedding of L into L^o. Above proposition shows that if L is a complemented modular lattice, Γ is an order preserving embedding of L into $(L^{op})^o$ also. Dually,

$$\Delta(a) = \{(x, y) | x \vee a = y \vee a\} \tag{1.3}$$

is a normal equivalence on L^{op} and

$$\Delta : L^{op} \to (L^{op})^o, a \mapsto \Delta(a) \tag{1.4}$$

is an order preserving embedding of L^{op} into $(L^{op})^o$. Again by the above proposition we see that if L is a complemented modular lattice, then Δ is also an order preserving embedding of L^{op} into L^o.

Theorem 1.5.3 ([27], Theorem 6). *Let L be a lattice with 0 and 1, and define Γ and Δ as defined by equations $1.1, 1.2, 1.3$ and 1.4. Then the following are equivalent:*

(i) L is a complemented modular lattice,

(ii) Δ is an order embedding of L^{op} into L^o,

(iii) Γ is an order embedding of L into $(L^{op})^o$,

(iv) $[L^{op}, L; \Gamma, \Delta]$ is a cross-connection.

If these conditions are satisfied, then the fundamental regular semigroup $U = U(L^{op}, L; \Gamma, \Delta)$ is given by

$$U = \{(f^+, f) | f \in B(L)\}.$$

1.6 Categories

A small category is a category in which the class of objects and class of morphisms are both sets and all categories considered here are small categories. We commence our discussion of the theory of categories with the axiomatic definition of a category and then concentrate on certain types of categories such as preadditive, additive, abelian etc. A detailed survey of the categories with subobjects, factorization etc. and the properties of the ideal categories of a regular semigroup are provided here (cf.[25]). In this thesis all morphisms are written in the order of their composition i.e., from left to right.

Definition 1.6.1 (cf.[17], page 7)**.** A category $\mathcal{C}$ consists of the following data:

1. objects denoted by $a, b, c, ...$ and arrows (morphisms) $f, g, h, ...$
2. for each arrow f there are given objects: $dom(f)$, $cod(f)$ called the domain and codomain of f. We write: $f : a \to b$ to indicate that $a = dom(f)$ and $b = cod(f)$
3. given arrows $f : a \to b$ and $g : b \to c$, that is, with $cod(f) = dom(g)$ then there exists an arrow: $f \cdot g : a \to c$ called the composite of f and g
4. for each object a there is given an arrow: $I_a : a \to a$ called the identity arrow of a.
   These data are required to satisfy the following laws:
5. associativity: $f \cdot (g \cdot h) = (f \cdot g) \cdot h$, $\forall f : a \to b$, $g : b \to c$, $h : c \to d$
6. unit: $f \cdot I_b = f = I_a \cdot f$, $\forall f : a \to b$.

A category is anything that satisfies this definition. For a category $\mathcal{C}$, we denote by $v\mathcal{C}$ the set of objects of $\mathcal{C}$ and for $a, b \in v\mathcal{C}$ the set of morphisms from a to b is denoted by $\mathcal{C}(a, b)$ or $hom(a, b)$ and is called *homset*.

Example 1.6.1 (cf.[17], page 12).

- **Set**: Category of sets with maps,
- $\mathbf{Vct_K}$: Category of vector spaces over a field K with linear mappings,
- **Grp**: Category of groups with group homomorphisms,
- **Ab**: Category of abelian groups with group homomorphisms,
- **Rng**: Category of rings with ring homomorphisms,
- $\mathbf{R-mod}$: Category of left R-modules with module homomorphisms.

A *subcategory* $\mathcal{C}'$ of a category $\mathcal{C}$ is a collection of some of the objects and some of the arrows of $\mathcal{C}$, which includes with each arrow f both the object $domf$ and the object $codf$, with each object s its identity arrow I_s and with each pair of composable arrows $s \to s' \to s''$ their composite. If $\mathcal{C}'(a,b) = \mathcal{C}(a,b)$, then $\mathcal{C}'$ is called full subcategory of $\mathcal{C}$. For example, **Ab** is a full subcategory of **Grp**.

A *functor* is a homomorphism of categories and is defined as follows:

Definition 1.6.2 (cf.[17], page 13). For categories $\mathcal{C}$ and $\mathcal{B}$ a functor $F : \mathcal{C} \to \mathcal{B}$ with domain $\mathcal{C}$ and codomain $\mathcal{B}$ consists of two suitably related functions: The object function F which assigns to each object c of $\mathcal{C}$ an object $F(c)$ of B and the arrow function which assigns to each arrow $f : c \to c'$ of $\mathcal{C}$ an arrow $F(f) : F(c) \to F(c')$ of $\mathcal{B}$, in such a way that

$$F(I_c) = I_{F(c)}, F(f \cdot g) = F(f) \cdot F(g),$$

the later whenever the composite $f \cdot g$ is defined in $\mathcal{C}$. F is called covariant functor.

A simple example is the *powerset functor* $\mathscr{P} : \mathbf{Set} \to \mathbf{Set}$.

A functor $F : \mathcal{C} \to \mathcal{D}$ is called *faithful* if for each $c, c' \in v\mathcal{C}$ the restriction of F to $\mathcal{C}(c, c')$ is injective. F is called *full* if for each $c, c' \in v\mathcal{C}$, F maps $\mathcal{C}(c, c')$ onto $\mathcal{D}(F(c), F(c'))$. An isomorphism of categories is a full and faithful functor F in which vF is a bijection.

Definition 1.6.3 (cf.[25], page 36)**.** Let $\mathcal{C}, \mathcal{D}$ be categories. The *product category* $\mathcal{C} \times \mathcal{D}$ is the category with an object is a pair (c, d) of object c of $\mathcal{C}$ and d of $\mathcal{D}$; a morphism $(c, d) \to (c', d')$ of $\mathcal{C} \times \mathcal{D}$ is a pair (f, g) of arrows $f : c \to c'$ and $g : d \to d'$. The composition of morphisms are defined component wise. A *bifunctor* or a functor in two variables is a functor $F : \mathcal{C} \times \mathcal{D} \to \mathcal{A}$(where $\mathcal{A}$ is another category).

A *natural transformation* is a morphism of functors and is defined as follows:

Definition 1.6.4 (cf.[17], page 40)**.** Given two functors $F, G : \mathcal{C} \to \mathcal{D}$, a *natural transformation* $\tau : F \to G$ is a function which assigns to each object c of $\mathcal{C}$ an arrow $\tau_c : F(c) \to G(c)$ of $\mathcal{D}$ in such a way that every arrow $f : c \to c'$ in $\mathcal{C}$ yields a diagram which is commutative.

$$\begin{tikzcd} F(c) \arrow[r, "\tau_c"] \arrow[d, "F(f)"'] & G(c) \arrow[d, "G(f)"'] \\ F(c') \arrow[r, "\tau_{c'}"'] & G(c') \end{tikzcd}$$

Definition 1.6.5 (cf.[17], page 40)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, then there is an associated category denoted by $[\mathcal{C}, \mathcal{D}]$ in which every functor from $\mathcal{C}$ to $\mathcal{D}$ is an object and every natural transformation between two such functors is a morphism. Any subcategory of $[\mathcal{C}, \mathcal{D}]$ is called a *functor category*.
The category $\mathcal{C}^*$ denote the functor category $[\mathcal{C}, \mathbf{Set}]$ and $\mathcal{C}^*$ is regarded as a dual of $\mathcal{C}$.

A morphism f in a category $\mathcal{C}$ is a monomorphism if for $g, h \in \mathcal{C}$,

$gf = hf$ implies $g = h$; that is f is a monomorphism if it is right cancellable. Dually a morphism $f \in \mathcal{C}$ is an epimorphism if f is left cancellable.

A morphism $f \in \mathcal{C}(c, c')$ is called a *split monomorphism* if there exists a morphism $g \in \mathcal{C}(c', c)$ such that $fg = I_c$. That is f has a right inverse. A morphism $f \in \mathcal{C}(c, c')$ is called a *split epimorphism* if f has a left inverse. Two monomorphisms $f, g \in \mathcal{C}$ are equivalent if there exists $h, k \in \mathcal{C}$ with $f = hg$ and $g = kf$. The fact that f and g are monomorphisms imply that h is an isomorphism and $k = h^{-1}$.

An object a is terminal in $\mathcal{C}$ if for each object b there is exactly one arrow $b \to a$. An object c is initial object if to each object b there is exactly one arrow $c \to b$.

Definition 1.6.6 ([17], page 20)**.** A *zero object* or *null object* z in $\mathcal{C}$ is an object which is both initial and terminal.

For any two objects a and b the unique arrows $a \to z$ and $z \to b$ have a composite $O_a^b : a \to b$ called the *zero morphism* from $a \to b$. The zero object is unique up to isomorphism and the notion of zero arrow is independent of the choice of the zero object.

Example 1.6.2 ([17])**.** Zero module is the zero object in the category $\mathbf{R-mod}$. Trivial group is the zero object in the category **Grp**.

Definition 1.6.7 (cf.[17], page 70)**.** An *equalizer* of $f, g : b \to a$ in $\mathcal{C}$ is an arrow $e : d \to b$ such that $e \cdot f = e \cdot g$ with that to any $h : c \to b$ with $h \cdot f = h \cdot g$ there is a unique $h' : c \to d$ with $h' \cdot e = h$.

Dually *coequalizer* of $f, g : a \to b$ is an arrow $u : b \to d$ such that $f \cdot u = g \cdot u$; and if $h : b \to c$ has $f \cdot h = g \cdot h$, then $h = u \cdot h'$ for a unique arrow $h' : d \to c$.

Let $\mathcal{C}$ has a zero object. A *kernel* of an arrow $f : a \to b$ is defined to be an equalizer of the arrows $f, O : a \rightrightarrows b$. A kernel is necessarily a monomorphism. Dually *cokernel* of $f : a \to b$ is coequalizer of the arrows $f, O : a \rightrightarrows b$. A cokernel is necessarily an epimorphism.

Definition 1.6.8 (cf.[17], page 68). The *product* of two objects a and b of category $\mathcal{C}$ is written $a \times b$ or $a \Pi b$ with two arrows $p_1 : a\Pi b \to a$, $p_2 : a\Pi b \to b$ called the projections of the product $a\Pi b$ such that for any $c \in \mathcal{C}$ with given arrows $f : c \to a$ and $g : c \to b$, there is a unique $h : c \to a\Pi b$ with $h \cdot p_1 = f$ and $h \cdot p_2 = g$.

Dually *coproduct* is written as $a + b$ or $a \amalg b$ with two arrows $q_1 : a \to a \amalg b$, $q_2 : b \to a \amalg b$ called the injections of the coproduct $a \amalg b$ such that for any $d \in \mathcal{C}$ with given arrows $f : a \to d$ and $g : b \to d$, there is a unique $h : a \amalg b \to d$ with $q_1 \cdot h = f$ and $q_2 \cdot h = g$.

Biproduct of a finite collection of objects, in a category with zero objects, is both a product and a coproduct. For example, in the category of $R-$modules $\mathbf{R-mod}$, the direct product of two $R-$modules is a biproduct.

Definition 1.6.9 (cf.[17], page 192). A *preadditive category* (or Ab-category) $\mathcal{A}$ is a category in which each homset $\mathcal{A}(b, c)$ is an additive abelian group and composition of arrows is bilinear relative to this addition and $\mathcal{A}$ has zero object.

A preadditive category with biproduct for each pair of its objects, is called *additive category*.

Definition 1.6.10 (cf.[17], page 198). An *abelian category* $\mathcal{A}$ is a preadditive category satisfying:

1. $\mathcal{A}$ has biproducts,
2. every arrow in $\mathcal{A}$ has a kernel and a cokernel,
3. every monomorphism is a kernel, and every epimorphism is a cokernel.

Example 1.6.3 ([17], page 199). $\mathbf{R-Mod}$ and $\mathbf{Mod-R}$, the categories of left and right R-modules with R-module homomorphisms are abelian categories with the usual kernels and cokernels.

Category with Subobjects

In the following we recall subobject relation in categories and provides some results regarding categories with subobjects from [25].

A *preorder* P is a category such that, for any $p, p' \in vP$; the homset $P(p, p')$ contains at most one morphism. In this case, the relation $\subseteq$ on the class vP defined by $p \subseteq p' \Leftrightarrow P(p, p') \neq \phi$ is a quasi-order on vP . In a preorder, p and p' are isomorphic if and only if $P(p, p') \neq \phi \neq P(p', p)$. Therefore $p \subseteq p'$ is a partial order if and only if P does not contain any nontrivial isomorphisms. Equivalently, the only isomorphisms of P are identity morphisms and in this case P is said to be a *strict preorder*.

Definition 1.6.11 (cf.[25], Definition 1, page 18)**.** Let $\mathcal{C}$ be a category and P be a subcategory of $\mathcal{C}$. Then $(\mathcal{C}, P)$ is called a *category with subobjects* if the following hold:

(1) P is a strict preorder with $vP = v\mathcal{C}$

(2) every $f \in P$ is a monomorphism in $\mathcal{C}$

(3) if $f, g \in P$ and if $f = hg$ for some $h \in \mathcal{C}$, then $h \in P$.

In a category with subobjects if $f : a \to b$ is a morphism in preorder P then f is said to be an *inclusion*, we denote this inclusion by $j(a, b)$.

If there is a morphism $e : b \to a$ such that $j(a, b) \cdot e = I_a$, then e is called a retraction from $b \to a$ and is denoted by $e(b, a)$.

In case a retraction from b to a exists then the inclusion $j(a, b) : a \to b$ is a split inclusion.

Any monomorphism equivalent to an inclusion is called an *embedding*. Clearly every inclusion is an embedding.

Let $\mathcal{C}$ be a category with subobjects. A morphism $f \in \mathcal{C}$ has factorization if

$$f = p \cdot m$$

where p is an epimorphism and m is an embedding (see cf.[25], page 21). A category $\mathcal{C}$ is said to have the *factorization property* if every morphism of $\mathcal{C}$ has a factorization.

Thus, if $\mathcal{C}$ has the factorization property, then any morphism f in $\mathcal{C}$ has at least one factorization of the form $f = qj$, where q is an epimorphism and j is an inclusion. Factorizations of this type are called *canonical factorizations*.

A *normal factorization* of a morphism f in $\mathcal{C}$ is a factorization of the form

$$f = euj$$

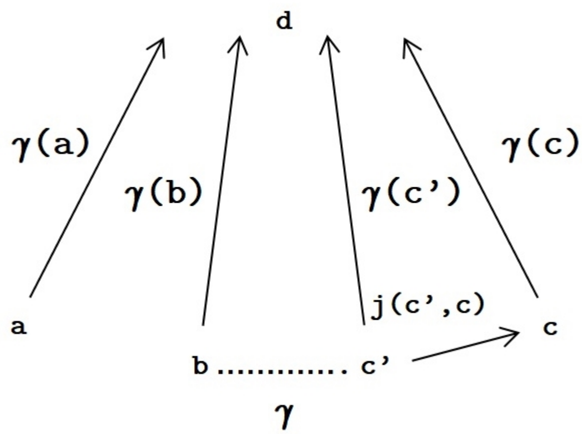
where e is a retraction, u is an isomorphism and j is an inclusion.

A morphism f in a category with subobjects is said to have an *image* if it has a canonical factorization $f = xj$, where x is an epimorphism and j is an inclusion with the property that whenever $f = yj'$ is any other canonical factorization, then there exists an inclusion j'' such that $y = xj''$. A category is said to have *images* if every morphism in $\mathcal{C}$ has an image. In this case, the codomain of x is said to be the *image* of f.

When the morphism f has an image we denote the unique canonical factorization of f by $f = f^o j_f$, where f^o is the *unique epimorphic component* and j_f is the inclusion of f.

Definition 1.6.12 ([25])**.** Let $\mathcal{C}$ be a category with subobjects, images, every morphism in $\mathcal{C}$ has normal factorizations in which the inclusion splits. For $d \in v\mathcal{C}$, a *cone* with vertex d is a collection of maps $\gamma : v\mathcal{C} \to d$ from the base $v\mathcal{C}$ to d satisfying the following:

1. $\gamma(c) \in \mathcal{C}(c, d)$ for all $c \in v\mathcal{C}$,
2. if $c' \subseteq c$ then $j(c', c)\gamma(c) = \gamma(c')$.



Definition 1.6.13 ([25] and [31]). Let $\mathcal{C}$ be a category with subobjects, in which inclusion splits and every morphism has normal [balanced] factorization. Then a normal [balanced] cone in $\mathcal{C}$ is a cone with at least one component isomorphism [balanced morphism]. For a cone $\gamma \in \mathcal{C}$, the M-set [B-set] of γ is defined by

$$M_\gamma = \{c \in v\mathcal{C} : \gamma(c) \text{ is an isomorphism}\} \text{ and}$$

$$B_\gamma = \{c \in v\mathcal{C} : \gamma(c) \text{ is a balanced morphism}\} \text{ respectively.}$$

The vertex d of the cone γ is usually denoted as c_γ.

In [25] it is described that the set of all normal cones in the category $\mathcal{C}$ is a regular semigroup denoted by $\mathcal{TC}$ with respect to the operation; for $\gamma, \beta \in \mathcal{TC}$

$$\gamma \cdot \beta = \gamma \star \beta(c_\gamma)^o. \tag{1.5}$$

That is for every $a \in v\mathcal{C}$,

$$(\gamma \cdot \beta)(a) = \gamma(a) \cdot \beta(c_\gamma)^o.$$

Let $E(\mathcal{TC})$ be the set of all idempotent normal cones in $\mathcal{C}$ and $\mathcal{BC}$ be the concordant semigroup of all balanced cones in category $\mathcal{C}$ (cf.[31]).

Definition 1.6.14 ([25]). A *normal category* is a pair $(\mathcal{C}, P)$ satisfying the following:

1. $(\mathcal{C}, P)$ is a category with subobjects
2. every inclusion in $\mathcal{C}$ splits
3. Any morphism in $\mathcal{C}$ has a normal factorization
4. for each $a \in v\mathcal{C}$ there is a normal cone γ with vertex a and $\gamma(a) = I_a$.

Proposition 1.6.1 ([22], Proposition 1.2.6). Let $\mathcal{C}$ and $\mathcal{D}$ be two isomorphic normal categories, then $\mathcal{TC}$ is isomorphic to $\mathcal{TD}$ as semigroups.

Ideal categories of regular semigroup

Let S be a regular semigroup. The category of principal left ideals $\mathbb{L}(S)$ is defined as $v\mathbb{L}(S) = \{Se : e \in E(S)\}$

$$\mathbb{L}(S)(Se, Sf) = \{\rho : Se \to Sf : (st)\rho = s(t\rho) \text{ for all } s, t \in Se\}$$

Dually, the category of right ideals $\mathbb{R}(S)$ is defined as follows:

$$v\mathbb{R}(S) = \{eS : e \in E(S)\}$$

$$\mathbb{R}(S)(eS, fS) = \{\lambda : eS \to fS : \lambda(st) = (\lambda s)t \text{ for all } s, t \in eS\}.$$

By Lemma 12 of [25], $\mathbb{L}(S)$ is the category whose vertex set is the set of all principal left ideals and whose morphism set is the set of right translations as defined above. Let $\rho(e, u, f) = \rho_u|_{Se}$ where $e, f \in E(S); u \in eSf$. Then we have the following:

1. For every $e, f \in E(S)$ and $u \in eSf, \rho(e, u, f) \in \mathbb{L}(S)(Se, Sf)$. Moreover the map $\rho(e, u, f) \longmapsto u$ is a bijection of $\mathbb{L}(S)(Se, Sf)$ onto eSf.

2. $\rho(e, u, f) = \rho(e', v, f')$ if and only if $e\mathcal{L}e', f\mathcal{L}f', u \in eSf, v \in e'Sf'$ and $v = e'u$.

3. If $\rho(e, u, f)$ and $\rho(g, v, h)$ are composable morphisms in $\mathbb{L}(S)$ (so that $f\mathcal{L}g, u \in eSf$ and $v \in gSh$), then

$$\rho(e, u, f)\rho(g, v, h) = \rho(e, uv, h).$$

In particular $\mathbb{L}(S)$ is a category with subobjects, in which every inclusion splits and every morphism has a normal factorization.

Lemma 1.6.1 (cf.[25], Lemma 15, page 50)**.** Let S be a regular semigroup, $a \in S$ and $f \in E(\mathcal{L}_a)$. Then the map

$$\rho^a(Se) = \rho(e, ea, f)$$

is a normal cone in $\mathbb{L}(S)$ with vertex Sa and such that

$$M\rho^a = \{Se : e \in E(\mathcal{R}_a)\}.$$

Moreover, ρ^a is an idempotent normal cone in $\mathcal{T}\mathbb{L}(S)$ if and only if $a \in E(S)$. $E(\mathcal{L}_a)[E(\mathcal{R}_a)]$ is the set of all idempotents in the $\mathcal{L}[\mathcal{R}]$-class of a.

Thus it is seen that given a regular semigroup S, the category $\mathbb{L}(S)$ described above is a normal category. Further we have the following proposition.

Proposition 1.6.2 (cf.[25], Proposition 13, page 48)**.** Let S be a regular semigroup and the category of principal left ideals of S, $\mathbb{L}(S)$ is a normal category. Let $\rho = \rho(e, u, f) : Se \to Sf$ be a morphism in $\mathbb{L}(S)$. We have the following:

1. The morphism $\rho(e, u, f)$ is a monomorphism iff $\rho(e, u, f)$ is injective and this is true iff $e\mathcal{R}u$.

2. $\rho(e, u, f)$ is an epimorphism if it is surjective and this is true iff $u\mathcal{L}f$.

3. If $Se \subseteq Sf$, then $j(Se, Sf) = \rho(e, e, f)$ and $\rho(f, fe, e) : Sf \to Se$ is a retraction.

Theorem 1.6.1 (cf.[25], Theorem 19, page 53). *Let $\mathcal{C}$ be a normal category. Define F on objects and morphisms of $\mathcal{C}$ as follows. For $c \in v\mathcal{C}$, let*

$$vF(c) = (TC)\epsilon$$

where $\epsilon \in E(\mathcal{TC})$, with $c_\epsilon = c$; and for a morphism $f \in \mathcal{C}(c, d)$, let

$$F : f \mapsto \rho(\epsilon, \epsilon * f^o, \epsilon') : (\mathcal{TC})\epsilon \to (\mathcal{TC})\epsilon'$$

where $\epsilon, \epsilon' \in E(\mathcal{TC})$, with $c_\epsilon = c, c_{\epsilon'} = d$ and $(\epsilon * f^o)(a) = \epsilon(a) \cdot f^o$ for $a \in v\mathcal{C}$. Then $F : \mathcal{C} \to \mathbb{L}(\mathcal{TC})$ is an isomorphism of normal categories.

From this theorem it is clear that for a normal category $\mathcal{C}$ the set of all normal cones $\mathcal{TC}$ is a regular semigroup and its left ideal category $\mathbb{L}(\mathcal{TC})$ is a normal category. Similarly the right ideal category $\mathbb{R}(\mathcal{TC})$ is also a normal category.

Theorem 1.6.2 (cf.[25], Theorem 16, page 51). *Let S be a regular semigroup and S_ρ be the set of all right translations on S. Then $\mathbb{L}(S)$ is a normal category. Moreover there exists a homomorphism $\bar{\rho} : S \to \mathcal{T}\mathbb{L}(S)$ and an injective homomorphism $\phi : S_\rho \to \mathcal{T}\mathbb{L}(S)$ such that the following diagram commutes:*

$$\begin{array}{ccc} S & \xrightarrow{\rho} & S_\rho \\ \downarrow & & \uparrow{\scriptstyle\phi} \\ S & \xrightarrow[\bar{\rho}]{} & \mathcal{T}\mathbb{L}(S) \end{array}$$

Similar results holds for the category $\mathbb{R}(S)$ whose vertex set is the set of all principal right ideals and morphisms set is the set of all left translations.

Normal dual

Let $\mathcal{C}$ be a normal category and $\mathcal{TC}$ be the regular semigroup of normal cones. The poset of right ideals of $\mathcal{TC}$ can be represented as a poset of certain set valued functors, called H-functors. For each $\gamma \in \mathcal{TC}$, define H-functor, $H(\gamma; -)$ on objects and morphisms of $\mathcal{C}$ as follows:

$$H(\gamma; c) = \{\gamma \star f^o : f \in \mathcal{C}(c_\gamma, c)\}$$

$$H(\gamma; g) : \gamma \star f^o \mapsto \gamma \star (fg)^o.$$

Let $\gamma, \gamma' \in \mathcal{TC}$. If $H(\gamma; -) = H(\gamma'; -)$ then $M_\gamma = M_{\gamma'}$ (cf.[25]). In view of this result, we may write $MH(\gamma; -)$ for M_γ.

Theorem 1.6.3 ([25], Theorem 11, page 44)**.** *Let $\mathcal{C}$ be a normal category and $\gamma, \gamma' \in \mathcal{TC}$. Then*

$$\gamma \mathcal{L} \gamma' \Leftrightarrow c_\gamma = c_{\gamma'}.$$

$$\gamma \mathcal{R} \gamma' \Leftrightarrow H(\gamma, -) = H(\gamma', -).$$

$$\gamma \mathcal{D} \gamma' \Leftrightarrow c_\gamma \cong c_{\gamma'}.$$

Definition 1.6.15 (cf.[25], Definition 4, page 55)**.** If $\mathcal{C}$ is a normal category, then the *normal dual* of $\mathcal{C}$, denoted by $N^*\mathcal{C}$ is the full subcategory of $\mathcal{C}^*$ with objects H-functors $H(\varepsilon, -) : \mathcal{C} \to \mathbf{Set}$, where ε is an idempotent normal cone, that is

$$vN^*\mathcal{C} = \{H(\varepsilon; -) : \varepsilon \in E(\mathcal{TC})\}$$

and the morphisms are appropriate natural transformations between such functors.

The following lemma describes morphisms of $N^*\mathcal{C}$ in terms of those of $\mathcal{C}$.

Lemma 1.6.2 ([25], Lemma 21, page 56)**.** To every morphism $\sigma : H(\epsilon; -) \to H(\epsilon'; -)$ in $N^*\mathcal{C}$, there is a unique $\widehat{\sigma} : c_{\epsilon'} \to c_\epsilon$ in $\mathcal{C}$ such that the following diagram commutes.

$$\begin{array}{ccc} H(\epsilon; -) & \xrightarrow{\eta_\epsilon} & \mathcal{C}(c_\epsilon, -) \\ {\scriptstyle\sigma}\big\downarrow & & \big\downarrow{\scriptstyle \mathcal{C}(\widehat{\sigma}, -)} \\ H(\epsilon'; -) & \xrightarrow[\eta_{\epsilon'}]{} & \mathcal{C}(c_{\epsilon'}, -) \end{array}$$

In this case, the component of the natural transformation σ at $c \in v\mathcal{C}$ is the map given by $\sigma(c) : \epsilon * f^o \mapsto \epsilon' * (\widehat{\sigma} f)^o$. In particular, σ is the inclusion $H(\epsilon; -) \subseteq H(\epsilon'; -)$ if and only if $\epsilon = \epsilon' * \widehat{\sigma}$. Moreover, the map $\sigma \mapsto \widehat{\sigma}$ is a bijection of $N^*\mathcal{C}(H(\epsilon; -), H(\epsilon'; -))$ onto $\mathcal{C}(c_{\epsilon'}, c_\epsilon)$.

Lemma 1.6.3 ([25], Lemma 22, page 57)**.** Let $\epsilon, \epsilon' \in E(\mathcal{TC})$. Then the map $\lambda(\epsilon, \gamma, \epsilon') \longmapsto \widetilde{\gamma}$ where $\gamma \in \epsilon'(\mathcal{TC})\epsilon$ and

$$\widetilde{\gamma} = \gamma(c_{\epsilon'}) j(c_\gamma, c_\epsilon) \tag{1.6}$$

is a bijection of $\mathbb{R}(\mathcal{TC})(\epsilon(\mathcal{TC}), \epsilon'(\mathcal{TC}))$ onto $\mathcal{C}(c_{\epsilon'}, c_\epsilon)$.

Lemma 1.6.4 ([25], Lemma 24, page 58)**.** Let $\gamma \in \epsilon'(\mathcal{TC})\epsilon$ and $\gamma' \in \epsilon''(\mathcal{TC})\epsilon'$. Assume that $\widetilde{\gamma}, \widetilde{\gamma'}$ and $\widetilde{\gamma' \cdot \gamma}$ are morphisms defined by Equation 1.6. Then $\widetilde{\gamma' \cdot \gamma} = \widetilde{\gamma'} \cdot \widetilde{\gamma}$.

Theorem 1.6.4 ([25], Theorem 25, page 58)**.** *Let $\mathcal{C}$ be a normal category. Define G on objects and morphisms of $\mathcal{C}$ as follows:*

$$vG(\epsilon(\mathcal{TC})) = H(\epsilon; -)$$

and for $\lambda = \lambda(\epsilon, \gamma, \epsilon') : \epsilon(\mathcal{TC}) \to \epsilon'(\mathcal{TC})$, let $G(\lambda)$ be the natural *transformation making the following diagram commutative.*

$$\begin{array}{ccc} H(\epsilon;-) & \xrightarrow{\eta_\epsilon} & \mathcal{C}(c_\epsilon,-) \\ {\scriptstyle G(\lambda)}\downarrow & & \downarrow{\scriptstyle \mathcal{C}(\tilde{\gamma},-)} \\ H(\epsilon';-) & \xrightarrow[\eta_{\epsilon'}]{} & \mathcal{C}(c_{\epsilon'},-) \end{array}$$

Then $G : \mathbb{R}(\mathcal{TC}) \to N^*\mathcal{C}$ *is an isomorphism of normal categories.*

In the light of the above discussion we can conclude that for a normal category $\mathcal{C}$, its dual $N^*\mathcal{C}$ is also a normal category.

1.7 Cross-connection of normal categories

In this section we describe the cross-connection of normal categories given by K.S.S. Nambooripad in [25].

Let $\mathcal{C}$ be a category with subobjects. Then an *ideal* denoted by $\langle c\rangle$ is the full subcategory of $\mathcal{C}$ whose objects are subobjects of c in $\mathcal{C}$. It is the principal ideal generated by c.

Definition 1.7.1 (cf.[25], Definition 1, page 62)**.** A *local isomorphism* between two normal categories $\mathcal{C}$ and $\mathcal{D}$ is a functor $F : \mathcal{C} \to \mathcal{D}$ which is inclusion preserving, fully faithful and for each $c \in v\mathcal{C}$, $F|_{\langle c\rangle}$ is an isomorphism of the ideal $\langle c\rangle$ onto $\langle F(c)\rangle$.

If $\mathcal{C}$ and $\mathcal{D}$ are normal categories a local isomorphism $\Gamma : \mathcal{D} \to N^*\mathcal{C}$ is called a *connection* of $\mathcal{D}$ with $\mathcal{C}$.

Definition 1.7.2 (cf.[25], Definition 5, page 86)**.** A *cross-connection* between two normal categories $\mathcal{C}$ and $\mathcal{D}$ is a triplet $(\mathcal{D}, \mathcal{C}; \Gamma)$ where $\Gamma : \mathcal{D} \to N^*\mathcal{C}$ is a local isomorphism such that for every $c \in v\mathcal{C}$, there is some $d \in v\mathcal{D}$ such that $c \in M\Gamma(d)$ where $M\Gamma(d)$ is the M-set of normal cone with vertex d.

Proposition 1.7.1 ([25], Proposition 12, page 77)**.** Let $\Gamma : \mathcal{D} \to N^*\mathcal{C}$ be a connection between normal categories $\mathcal{C}$ and $\mathcal{D}$ and let $\mathcal{C}_\Gamma$ be the subcategory of $\mathcal{C}$ such that

$$v\mathcal{C}_\Gamma = \{c \in \mathcal{C} : c \in M\Gamma(d) \text{ for some } d \in v\mathcal{D}\}$$

Then $\mathcal{C}_\Gamma$ is an ideal in $\mathcal{C}$.

Theorem 1.7.1 ([25], Theorem 15, page 81)**.** *Let $\Gamma : \mathcal{D} \to N^*\mathcal{C}$ be a connection between normal categories $\mathcal{C}$. Then there exists a connection $\Gamma^* : \mathcal{C}_\Gamma \to N^*\mathcal{D}$ such that, for $c \in v\mathcal{C}_\Gamma$ and $d \in v\mathcal{D}, c \in M\Gamma(d)$ if and only if $d \in M\Gamma^*(c)$.*

Definition 1.7.3. Let $\Gamma : \mathcal{D} \to N^*\mathcal{C}$ be a connection between normal categories $\mathcal{C}$. The functor $\Gamma^* : \mathcal{C}_\Gamma \to N^*\mathcal{D}$ defined as in the above theorem is called the dual of the connection $\Gamma : \mathcal{D} \to N^*\mathcal{C}$.

Thus given a cross-connection $\Gamma : \mathcal{D} \to N^*\mathcal{C}$, there is a dual cross-connection $\Gamma^* : \mathcal{C} \to N^*\mathcal{D}$ denoted by $(\mathcal{C}, \mathcal{D}; \Gamma^*)$.

Remark 1.7.1 ([25])**.** Given a cross-connection $\Gamma : \mathcal{D} \to N^*\mathcal{C}$, since $N^*\mathcal{C} \subseteq \mathcal{C}^*$ (C^* is the category of all functors from $\mathcal{C}$ to **Set**), by category isomorphisms we get a unique *bifunctor* $\Gamma(-,-) : \mathcal{C} \times \mathcal{D} \to$ **Set** defined by

$$\Gamma(c, d) = \Gamma(d)(c) \text{ and}$$

$$\Gamma(f, g) = (\Gamma(d)(f))(\Gamma(g)(c')) = (\Gamma(g)(c))(\Gamma(d')(f))$$

for all $(c, d) \in v\mathcal{C} \times v\mathcal{D}$ and $(f, g) : (c, d) \to (c', d')$. Similarly corresponding to $\Gamma^* : \mathcal{C} \to N^*\mathcal{D}$, we have $\Gamma^*(-,-) : \mathcal{C} \times \mathcal{D} \to$ **Set** defined by

$$\Gamma^*(c, d) = \Gamma^*(c)(d) \text{ and}$$

$$\Gamma^*(f, g) = (\Gamma^*(c)(g))(\Gamma^*(f)(d')) = (\Gamma^*(f)(d))(\Gamma^*(c')(g))$$

for all $(c, d) \in v\mathcal{C} \times v\mathcal{D}$ and $(f, g) : (c, d) \to (c', d')$.

Theorem 1.7.2 ([25], Theorem 4, page 67). *Given cross-connection $(\mathcal{D}, \mathcal{C}; \Gamma)$, there is a natural isomorphism χ_Γ between the bifunctors $\Gamma(-,-)$ and $\Gamma^*(-,-)$such that $\chi_\Gamma : \Gamma(c,d) \to \Gamma^*(c,d)$ is a bijection. χ_Γ is known as the duality associated with Γ.*

Let $\mathcal{C}$ and $\mathcal{D}$ be $RR-$normal categories and $\Gamma : \mathcal{D} \to N^*\mathcal{C}$ is a cross-connection. $\Gamma^* : \mathcal{C} \to N^*\mathcal{D}$ be its dual cross-connection. Define

$$E_\Gamma = \{(c,d) : c \in v\mathcal{C}_\Gamma, d \in v\mathcal{D} \text{ and } c \in M\Gamma(d)\}.$$

For each $(c,d) \in E_\Gamma, \gamma(c,d)$ denotes the unique cone in $\mathcal{C}$ such that

$$c_{\gamma(c,d)} = c \text{ and } \Gamma(d) = H(\gamma(c,d);-).$$

Similarly for each $(c,d) \in E_\Gamma$, there is a unique cone $\gamma^*(c,d)$ in $\mathcal{D}$ such that

$$c_{\gamma^*(c,d)} = d \text{ and } \Gamma^*(c) = H(\gamma^*(c,d);-).$$

Let $(c,d) \in vC_\Gamma \times v\mathcal{D}$ and choose $c' \in C_\Gamma$ and $d' \in v\mathcal{D}$ such that $(c,d'), (c',d) \in E_\Gamma$. Then every cone in $\Gamma(c,d)$ can be represented as $\gamma(c',d) \star f^o$ with $f \in \mathcal{C}(c',c)$ and every element of $\Gamma^*(c,d)$ can be written as $\gamma^*(c,d') \star g^o$ with $g \in \mathcal{D}(d',d)$. Hence for every $(c,d) \in v\mathcal{C}_\Gamma \times v\mathcal{D}$ and $\gamma(c',d) \star f^o \in \Gamma(c,d)$, we have natural isomorphism

$$\chi_{\Gamma(c,d)}(\gamma(c',d) \star f^o) = \gamma^*(c,d') \star g^o$$

where $(c,d), (c',d') \in E_\Gamma$ and $f \in \mathcal{C}(c,c'), g \in \mathcal{D}(d',d)$ are such that the following diagram commutes.

$$\begin{tikzcd}
\Gamma(d') \arrow[r, "\eta_{\gamma(c,d')}"] \arrow[d, "\Gamma(g)"'] & \mathcal{C}(c,-) \arrow[d, "\mathcal{C}(f,-)"] \\
\Gamma(d) \arrow[r, "\eta_{\gamma(c',d)}"'] & \mathcal{C}(c',-)
\end{tikzcd}$$

Definition 1.7.4 (cf.[25], Section 5.1, page 97). Let Γ be a cross-connection of $\mathcal{D}$ with $\mathcal{C}$. Define

$$U\Gamma = \bigcup\{\Gamma(c,d) : (c,d) \in \upsilon\mathcal{C} \times \upsilon\mathcal{D}\} \tag{1.7}$$

$$U\Gamma^* = \bigcup\{\Gamma^*(c,d) : (c,d) \in \upsilon\mathcal{C} \times \upsilon\mathcal{D}\} \tag{1.8}$$

Proposition 1.7.2 (cf.[25], Proposition 31, page 99). For any cross-connection $\Gamma : \mathcal{D} \to N^*\mathcal{C}$, $U\Gamma$ is a regular subsemigroup of $\mathcal{TC}$ such that $\mathcal{C}$ is isomorphic to $\mathbb{L}(U\Gamma)$. $U\Gamma^*$ is a regular subsemigroup of $\mathcal{TD}$ such that $\mathcal{D}$ is isomorphic to $\mathbb{L}(U\Gamma^*)$.

Definition 1.7.5 (cf.[25], page 100). For a cross-connection Γ : $\mathcal{D} \to N^*\mathcal{C}$, we shall say that $\gamma \in U\Gamma$ is *linked* to $\delta \in U\Gamma^*$ if there is a $(c,d) \in \upsilon\mathcal{C} \times \upsilon\mathcal{D}$ such that $\gamma \in \Gamma(c,d)$ and $\delta = \chi_\Gamma(c,d)(\gamma)$.

Theorem 1.7.3 ([25], Theorem 32, page 101). *Let $\Gamma : \mathcal{D} \to N^*\mathcal{C}$ be a cross-connection. Then*

$$\widetilde{S}\Gamma = \{(\gamma,\delta) \in U\Gamma \times U\Gamma^* : (\gamma,\delta) \text{ is linked }\}$$

is a regular semigroup with the binary operation defined by

$$(\gamma,\delta) \circ (\gamma',\delta') = (\gamma \cdot \gamma', \delta' \cdot \delta)$$

for all $(\gamma,\delta),(\gamma',\delta') \in \widetilde{S}\Gamma$. Then $\widetilde{S}\Gamma$ is a sub direct product of $U\Gamma$ and $U(\Gamma^*)^{op}$ and is called the cross-connection semigroup determined by Γ.

Now consider a regular semigroup S. From the above section it is clear that for a regular semigroup S the categories $\mathbb{L}(S)$ and $\mathbb{R}(S)$ of principal left and right ideals of S are normal categories. The set of all normal cones $\mathcal{T}\mathbb{L}(S)$ and $\mathcal{T}\mathbb{R}(S)$ are regular semigroups.

Proposition 1.7.3 (cf.[25], Proposition 1, page 63). For any regular semigroup S, $FS_\rho : \mathbb{R}(S) \to \mathbb{R}(\mathcal{T}\mathbb{L}(S))$ defined by

$$FS_\rho(eS) = \rho^e(\mathcal{T}\mathbb{L}(S)) \text{ and}$$

$$FS_\rho(\lambda(e,u,f)) = \lambda(\rho^e, \rho^u, \rho^f)$$

is a local isomorphism. Dually, $FS_\lambda : \mathbb{L}(S) \to \mathbb{L}(\mathcal{T}\mathbb{R}(S))$ defined as

$$FS_\lambda(Se) = (\mathcal{T}\mathbb{R}(S))\lambda^e \text{ and}$$

$$FS_\lambda(\rho(e,u,f)) = \rho(\lambda^e, \lambda^u, \lambda^f)$$

is also a local isomorphism.

Theorem 1.7.4 (cf.[25], Theorem 2, page 65)**.** *Let S be a regular semigroup. For $fS \in v\mathbb{R}(S)$ and $\lambda = \lambda(e,u,f)$ in $\mathbb{R}(S)$, let ΓS be defined on objects and morphisms of $\mathbb{R}(S)$ by :*

$$v\Gamma S(fS) = H(\rho^f; -), \Gamma S(\lambda) = \eta_{\rho^e}\mathbb{L}(S)(\rho(f,u,e), -)\eta_{\rho^f}^{-1}.$$

Then ΓS is a local isomorphism from $\mathbb{R}(S)$ to $N^*\mathbb{L}(S)$. Dually, $\Gamma^* S$, defined on objects and morphisms of $\mathbb{L}(S)$ by

$$v\Gamma^* S(Se) = H(\lambda^e; -), \Gamma^* S(\rho) = \eta_{\lambda^f}\mathbb{R}(S)(\lambda(e,u,f), -)\eta_{\lambda^e}^{-1}.$$

for all $Se \in v\mathbb{L}(S)$ and $\rho = \rho(f,u,e) \in \mathbb{L}(S)$, defines a local isomorphism.

Since $\Gamma S : \mathbb{R}(S) \to N^*\mathbb{L}(S) \subseteq \mathbb{L}(S)^*$, by the category isomorphisms, there is a unique bifunctor $\Gamma S(-,-) : \mathbb{L}(S) \times \mathbb{R}(S) \to \boldsymbol{Set}$. $\Gamma S(-,-)$ is defined on objects and morphisms as follows:

$$\Gamma S(Se, fS) = \Gamma S(fS)(Se);$$

$$\Gamma S(\rho, \lambda) = \Gamma S(fS)(\rho)\Gamma S(\lambda)(Se') = \Gamma S(\lambda)(Se)\Gamma S(f'S)(\rho)$$

for all $(Se, fS) \in v\mathbb{L}(S) \times \mathbb{R}(S)$ and $(\rho, \lambda) : (Se, fS) \to (Se', f'S)$.

Theorem 1.7.5 (cf.[25], Theorem 4, page 67)**.** *Let S be a regular semigroup. Then there is a natural isomorphism χ_S from $\Gamma S(-,-)$ to $\Gamma^* S(-,-)$ whose components are defined by*

$$\chi_S(Se, fS) : \rho^f \star \rho(f, u, e)^o \mapsto \lambda^e \star \lambda(e, u, f)^o$$

for each $(Se, fS) \in v\big(\mathbb{L}(S) \times \mathbb{R}(S)\big)$.

Theorem 1.7.6 (cf.[25], Theorem 17, page 87)**.** *Let S be a regular semigroup. Then ΓS is a cross-connection of $\mathbb{R}(S)$ with $\mathbb{L}(S)$. The dual cross-connection is $\Gamma^* S = (\Gamma S)^*$.*

Note that any regular semigroup induces a cross-connection and any cross-connection of normal categories induces a regular semigroup $\widetilde{S}\Gamma$ as in Theorem 1.7.3.

Chapter 2

Cross-connection of Boolean lattice

In this chapter, we discuss the cross-connection of Boolean lattice (complemented distributive lattice) and it is shown that the cross-connection determines a Boolean ring, which is a regular ring. Further it is also shown that the principal ideals of the Boolean ring obtained from cross-connection form a Boolean lattice which is isomorphic to the initial Boolean lattice.

2.1 Cross-connection ring

We have already provided the definition of Boolean lattice and cross-connection of complemented modular lattice in Sections 1.1 and 1.5 respectively. In the following we proceed to describe the cross-connection of Boolean lattice.

Lemma 2.1.1. Let L be a Boolean lattice and L^{op} be the dual of L obtained by reversing the order. For a in L, let a^c be the unique complement of a in L, define

$$f_a : L \to L \text{ as } x \to x \wedge a, \text{ for all } x \in L$$

and

$$f_a^+ : L^{op} \to L^{op}, y \to y \vee a^c \text{ for all } y \in L^{op}.$$

Then f_a and f_a^+ are normal idempotent mappings such that f_a^+ is the residual of f_a.

Proof. It is easy to observe that f_a is order preserving as

$$x \leq y \Rightarrow x \wedge a \leq y \wedge a \Rightarrow xf_a \leq yf_a.$$

Clearly im $f_a = L(a)$ is a principal ideal generated by a. For every $x \in L$, there exists $z = x \wedge a \leq x$ such that $L(z) = L(xf_a)$. Now $f_a : z \to z \wedge a = x \wedge a$ i.e. $f_a : L(z) \to L(xf_a)$ such that f_a acts as an identity morphism. Hence $L(z) \cong L(xf_a)$. Thus f_a and similarly f_a^+ are normal mappings . Obviously $xf_a = x \wedge a, forall x \in L$ is a residuated mapping with residual $f_a^+(y) = y \vee a^c$ (see Definition 1.5.4). □

f_a and f_a^+ defined above are totally range closed mappings and hence f_a is a strongly range closed mapping, $f_a \in B(L)$ and $f_a^+ \in B(L^{op})$.

Let L^o and $(L^{op})^o$ are the Boolean lattices of all normal equivalences on Boolean lattice L and L^{op} respectively. Now define order preserving embeddings $\Gamma : L \to (L^{op})^o$ and $\Delta : L^{op} \to L^o$ as

$$= \Gamma(a) = ker f_a = \{(x, y)/xf_a = yf_a\} = \{(x, y)/x \wedge a = y \wedge a\} \quad (2.1)$$

$$\text{and } \Delta(a) = ker f_{a^c}^+ = \{(x, y)/f_{a^c}^+(x) = f_{a^c}^+(y)\} = \{(x, y)/x \vee a = y \vee a\} \quad (2.2)$$

Obviously $\Gamma(a) = \Delta(a^c)$, for all a in L.

Theorem 2.1.1. *Let L be a Boolean lattice. Γ and Δ are defined as in equations 2.1 and 2.2. Then $[L, L^{op}; \Gamma, \Delta]$ is a cross connection.*

Proof. From the definition of Γ and Δ we get

$$im f_a = L(a) \Rightarrow ker f_a^+ = \Delta(a^c) = \Gamma(a)$$

$$im f_a^+ = L^{op}(a^c) \Rightarrow ker f_a = \Gamma(a) = \Delta(a^c) \text{ for all } a \in L.$$

The following two diagrams commute:

$$\begin{array}{ccc} L & \xrightarrow{\Gamma} & (L^{op})^o \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle (f^+)^o} \\ L & \xrightarrow[\Gamma]{} & (L^{op})^o \end{array} \qquad \begin{array}{ccc} L^{op} & \xrightarrow{\Delta} & L^o \\ {\scriptstyle f^+}\uparrow & & \uparrow{\scriptstyle f^o} \\ L^{op} & \xrightarrow[\Delta]{} & L^o \end{array}$$

we can prove that $\Gamma(xf) = (f^+)^{-1}\Gamma(x)$.

$$\begin{aligned} Let(u,v) \in \Gamma(xf) &\Rightarrow u \wedge xf = v \wedge xf \\ &\Rightarrow f^+(u \wedge xf) = f^+(v \wedge xf) \\ &\text{[since } f^+ \text{ preserves meet operation]} \\ &\Rightarrow f^+u \wedge x = f^+v \wedge x \text{ [by Theorem 1.5.2]} \\ &\Rightarrow (f^+u, f^+v) \in \Gamma(x) \\ &\Rightarrow (u,v) \in (f^+)^{-1}(\Gamma(x)). \end{aligned}$$

$$\Gamma(xf) \subseteq (f^+)^{-1}\Gamma(x).$$

Similarly we can prove that

$$(f^+)^{-1}\Gamma(x) \subseteq \Gamma(xf).$$

Thus we get

$$\Gamma(xf) = (f^+)^o\Gamma(x) = (f^+)^{-1}\Gamma(x) \text{ and similarly}$$

$$\Delta(f^+y) = (\Delta y)f^o = (\Delta y)f^{-1}.$$

Thus (f, f^+) is compatible with (Γ, Δ). For $a \in M(\Gamma(a))$ there exists

f_a such that $ker f_a = \Gamma(a)$ and $a \in M(f_a)$ that is $im f_a = L(a)$, principal ideal generated by a. Dually, $a^c \in M(\Delta(a^c))$ means there exists f_a^+ such that $ker f_a^+ = \Delta(a^c)$ and $a^c \in M(f_a^+)$. That is $im f_a^+ = L^{op}(a^c)$, principal ideal generated by a^c. Thus $a \in M(\Gamma(a)) \Leftrightarrow a^c \in M(\Delta(a^c))$, for every $a \in L$ and being binormal idempotent maps (f, f^+) is compatible with (Γ, Δ). Hence $[L, L^{op}; \Gamma, \Delta]$ is a cross connection. □

For a Boolean lattice L we obtained a cross-connection $[L, L^{op}; \Gamma, \Delta]$. Then

$$U = U([L, L^{op}; \Gamma, \Delta]) = \{(f_a, f_a^+) | f_a \in B(L)\},$$

is the set of all pairs of idempotent normal maps (f_a, f_a^+) with

$$f_a : L \to L, f_a^+ : L^{op} \to L^{op},$$

that are compatible with (Γ, Δ), is a subset of $B(L) \times B(L^{op})$.

Define multiplication and addition in U as

$$(f_a, f_a^+) \cdot (f_b, f_b^+) = (f_{a \wedge b}, f_{a \wedge b}^+)$$

$$\text{and } (f_a, f_a^+) + (f_b, f_b^+) = (f_{a \oplus b}, f_{a \oplus b}^+)$$

where $\oplus$ is the symmetric difference defined by $a \oplus b = (a \wedge b^c) \vee (b \wedge a^c)$.

The following lemmas give the results leading to the ring structure on U. In our discussion let $a, b, c, ...$ denote the elements of Boolean lattice L and $a^c, b^c, c^c, ...$ their unique complements. Note that Boolean lattice L is also a Boolean ring and vice versa.

Lemma 2.1.2. Let L be a Boolean lattice. For $a, b \in L$, $U = \{(f_a, f_a^+) | f_a \in B(L)\}$ is a regular semigroup with respect to multiplication defined by

$$(f_a, f_a^+) \cdot (f_b, f_b^+) = (f_{a \wedge b}, f_{a \wedge b}^+).$$

Proof. For $a, b \in L$, $a \wedge b \in L$ and hence $(f_a, f_a^+) \cdot (f_b, f_b^+) =$

$(f_{a\wedge b}, f^+_{a\wedge b}) \in U$. So U is closed with respect to multiplication. Now

$$\begin{aligned}((f_a, f^+_a)\cdot(f_b, f^+_b))\cdot(f_c, f^+_c) &= (f_{(a\wedge b)\wedge c}, f^+_{(a\wedge b)\wedge c})\\ &= (f_{a\wedge(b\wedge c)}, f^+_{a\wedge(b\wedge c)}))\\ &= (f_a, f^+_a)\cdot((f_b, f^+_b)\cdot(f_c, f^+_c))\end{aligned}$$

U satisfies associativity and hence U is a semigroup with respect to multiplication.

Now for all $(f_a, f^+_a) \in U$,

$$\begin{aligned}(f_a, f^+_a)\cdot(f_a, f^+_a)\cdot(f_a, f^+_a) &= (f_{a\wedge a}, f^+_{a\wedge a})\cdot(f_a, f^+_a)\\ &= (f_{(a\wedge a)\wedge a}, f^+_{(a\wedge a)\wedge a})\\ &= (f_a, f^+_a).\end{aligned}$$

Hence each element (f_a, f^+_a) of U is regular and hence U is a regular semigroup. □

Also note that every element of U is an idempotent as

$$(f_a, f^+_a)\cdot(f_a, f^+_a) = (f_a, f^+_a).$$

Theorem 2.1.2. *Let L be a Boolean lattice. $U = \{(f_a, f^+_a) | f_a \in B(L)\}$ obtained by cross-connection of L, is a regular ring with respect to the addition*

$$(f_a, f^+_a) + (f_b, f^+_b) = (f_{a\oplus b}, f^+_{a\oplus b})$$

where $a\oplus b = (a\wedge b^c)\vee(b\wedge a^c)$ and multiplication

$$(f_a, f^+_a)\cdot(f_b, f^+_b) = (f_{a\wedge b}, f^+_{a\wedge b}).$$

Proof. By the above Lemma U is a regular semigroup. Now we prove that U is an additive abelian group. Since $a\oplus b$ is an element of

Boolean lattice L and $(f_{a\oplus b}, f^+_{a\oplus b})$ is in U, we have

$$\begin{aligned}((f_a, f^+_a) + (f_b, f^+_b)) + (f_c, f^+_c) &= (f_{a\oplus b}, f^+_{a\oplus b}) + (f_c, f^+_c)\\ &= (f_{(a\oplus b)\oplus c}, f^+_{(a\oplus b)\oplus c})\\ &= (f_{a\oplus(b\oplus c)}, f^+_{a\oplus(b\oplus c)})\\ &\quad [\text{ since } L \text{ is a Boolean ring.}]\\ &= (f_a, f^+_a) + (f_{b\oplus c}, f^+_{b\oplus c})\\ &= (f_a, f^+_a) + ((f_b, f^+_b) + (f_c, f^+_c))\end{aligned}$$

$(f_0, f^+_0) \in U$ is the additive identity since L and L^{op} are Boolean rings, with additive identity 0 and

$$\begin{aligned}(f_a, f^+_a) + (f_0, f^+_0) &= (f_{a\oplus 0}, f^+_{a\oplus 0})\\ &= (f_a, f^+_a)\end{aligned}$$

Also every element in U is its own inverse,

$$\begin{aligned}(f_a, f^+_a) + (f_a, f^+_a) &= (f_{a\oplus a}, f^+_{a\oplus a})\\ &= (f_0, f^+_0)\end{aligned}$$

Since L and L^{op} are Boolean rings, addition is commutative in U as:

$$\begin{aligned}(f_a, f^+_a) + (f_b, f^+_b) &= (f_{a\oplus b}, f^+_{a\oplus b})\\ &= (f_{b\oplus a}, f^+_{b\oplus a})\\ &= (f_b, f^+_b) + (f_a, f^+_a)\end{aligned}$$

Hence $U = \{(f_a, f^+_a) | f_a \in B(L)\}$ is an additive abelian group.

Now we prove that multiplication is distributive over addition in U.

For $(f_a, f_a^+), (f_b, f_b^+), (f_c, f_c^+) \in U$,

$$\begin{aligned}(f_a, f_a^+) \cdot [(f_b, f_b^+) + (f_c, f_c^+)] &= (f_a, f_a^+) \cdot (f_{b\oplus c}, f_{b\oplus c}^+)\\ &= (f_{a\wedge(b\oplus c)}, f_{a\wedge(b\oplus c)}^+)\\ &= (f_{(a\wedge b)\oplus(a\wedge c)}, f_{(a\wedge b)\oplus a\wedge c)}^+)\\ &= (f_{a\wedge b}, f_{a\wedge b}^+) + (f_{a\wedge c}, f_{a\wedge c}^+)\\ &= [(f_a, f_a^+) \cdot (f_b, f_b^+)] + [(f_a, f_a^+) \cdot (f_c, f_c^+).\end{aligned}$$

Similarly we get

$$[(f_a, f_a^+) + (f_b, f_b^+)] \cdot (f_c, f_c^+) = [(f_a, f_a^+) \cdot (f_c, f_c^+)] + [(f_b, f_b^+) \cdot (f_c, f_c^+)]$$

Thus multiplication is distributive over addition in U and hence U is a regular ring. $\square$

Remark 2.1.1. The regular ring $U = \{(f_a, f_a^+) | f_a \in B(L)\}$ obtained by cross-connection of L, is a Boolean ring since U is having the identity element (f_1, f_1^+) and each element of U is an idempotent.

U is a Boolean lattice with respect to

$$(f_a, f_a^+) \wedge (f_b, f_b^+) = (f_a, f_a^+) \cdot (f_b, f_b^+) \text{ and} \tag{2.3}$$

$$(f_a, f_a^+) \vee (f_b, f_b^+) = (f_a, f_a^+) + (f_b, f_b^+) + (f_a, f_a^+) \cdot (f_b, f_b^+) \tag{2.4}$$

Lemma 2.1.3. Let L be a Boolean lattice. $U = \{(f_a, f_a^+) | f_a \in B(L)\}$ is the Boolean ring (Boolean lattice) obtained by cross-connection of L. Then for each $(f_a, f_a^+) \in U$, $(f_a, f_a^+)^c = (f_{a^c}, f_{a^c}^+)$.

Proof.

$$(f_a, f_a^+) \wedge (f_{a^c}, f_{a^c}^+) = (f_0, f_0^+) \text{ and}$$

$$\begin{aligned}(f_a, f_a^+) \vee (f_{a^c}, f_{a^c}^+) &= (f_a, f_a^+) + (f_{a^c}, f_{a^c}^+) + (f_a, f_a^+) \cdot (f_{a^c}, f_{a^c}^+) \\ &= (f_{a \oplus a^c}, f_{a \oplus a^c}^+) + (f_0, f_0^+) \\ &= (f_1, f_1^+) + (f_0, f_0^+) \\ &= (f_{1 \oplus 0}, f_{1 \oplus 0}^+) \\ &= (f_1, f_1^+)\end{aligned}$$

hence we get

$$(f_a, f_a^+)^c = (f_{a^c}, f_{a^c}^+).$$

□

Thus we constructed the cross-connection of a Boolean lattice L and obtained a cross-connection Boolean ring U which is a Boolean lattice also.

2.2 Representation of Boolean lattice

Here we prove that the Boolean lattice L is isomorphic to cross-connection Boolean ring U and further the principal ideals of U form a Boolean lattice which is isomorphic to the initial Boolean lattice L.

Definition 2.2.1. A homomorphism from a Boolean lattice L to the ring of pairs of mappings on L is called a *representation* of Boolean lattice.

Theorem 2.2.1. *Let L be a Boolean lattice and U is the cross-connection Boolean ring obtained. Then $\psi : L \to U$ defined by $a \to (f_a, f_a^+)$, for all $a \in L$ is a representation. Further ψ is an isomorphism.*

Proof. The Boolean ring U is a Boolean lattice with respect to the equations 2.3 and 2.4. Also the Boolean lattice L is a Boolean ring with respect to multiplication ‘meet’ and addition ‘symmetric difference’.

Clearly $\psi : a \mapsto (f_a, f_a^+)$ is well defined for each $a \in L$.

$$\begin{aligned}\psi(a \wedge b) &= (f_{a\wedge b}, f^+_{(a\wedge b)})\\ &= (f_a, f_a^+) \wedge (f_b, f_b^+)\\ &= \psi(a) \wedge \psi(b)\end{aligned}$$

$$\begin{aligned}\psi(a \vee b) &= (f_{a\vee b}, f^+_{a\vee b})\\ \psi(a) \vee \psi(b) &= (f_a, f_a^+) \vee (f_b, f_b^+)\\ &= (f_a, f_a^+) + (f_b, f_b^+) + (f_a, f_a^+) \cdot (f_b, f_b^+)\\ &= (f_{a\oplus b}, f^+_{a\oplus b}) + (f_{a\wedge b}, f^+_{a\wedge b})\\ &= (f_{a\oplus b\oplus(a\wedge b)}, f^+_{a\oplus b\oplus(a\wedge b)})\\ &= (f_{a\vee b}, f^+_{a\vee b})\end{aligned}$$

Hence $\psi(a \vee b) = \psi(a) \vee \psi(b)$.

$$\text{Also } \psi(a^c) = ((f_{a^c}, f^+_{a^c}) = (f_a, f_a^+)^c = \psi(a)^c.$$

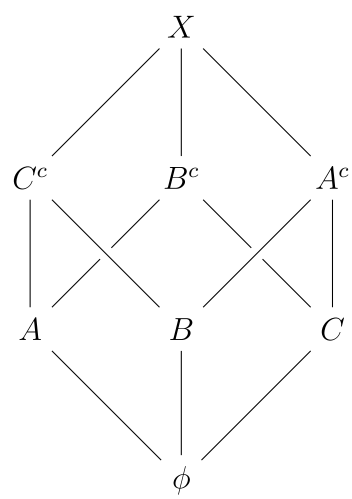
Thus ψ is a Boolean homomorphism. Also ψ is onto as for each $(f_a, f_a^+) \in U$, there exists $a \in L$ such that $\psi(a) = (f_a, f_a^+)$ and ψ is one-one, as for $a \neq b$, $(f_a, f_a^+) \neq (f_b, f_b^+)$ and $\psi(a) \neq \psi(b)$. Hence ψ is an isomorphism from L to U and hence $L \cong U$. □

Consider the Boolean ring U obtained as the cross-connection ring of a Boolean lattice. For any $(f_a, f_a^+) \in U$, the principal ideal generated by (f_a, f_a^+) is a complete ideal and these complete ideals form a complete Boolean lattice L^* (see Theorem 1.1.1). Let $I_a = ((f_a, f_a^+))$, the principal ideal generated by $(f_a, f_a^+) \in U$ and since $(f_a, f_a^+)^c = (f_{a^c}, f^+_{a^c})$ it is obvious that the unique complement of I_a is I_{a^c}.

Similar to the above theorem we can prove that $U \cong L^*$. Thus we get

$$L \cong U \cong L^*.$$

Example 2.2.1. Consider the powerset $(\{a, b, c\}, \subseteq)$ which is a Boolean lattice L with elements $\phi, A = \{a\}, B = \{b\}, C = \{c\}, A^c = \{b, c\}, B^c = \{a, c\}, C^c = \{a, b\}$ and $X = \{a, b, c\}$. For every $A \in L$, define $f_A : L \to L$ as $x \to x \cap A$, for all $x \in L$ and $f_A^+ : L^{op} \to L^{op}$ as $x \to x \cup A^c$, for all $x \in L^{op}$, where L^{op} is the dual lattice of L. The image of f_A, $imf_A = L(A)$, principal ideal generated by A and $imf_A^+ = L^{op}(A^c)$, principal ideal generated by A^c. It is easy to see that f_A and f_A^+ are idempotent normal mappings. f_A is a residuated mapping with residual f_A^+. Clearly f_A and f_A^+ are totally range closed mappings and $\{f_A | \forall A \in L\} \subseteq B(L)$ and $\{f_A^+ | \forall A \in L^{op}\} \subseteq B(L^{op})$. The Boolean lattice L is given in figure,



$\Gamma(A)$ and $\Delta(A)$ denotes the kernels of f_A and $f_{A^c}^+$ which are the normal equivalence relations. Let L^o and $(L^{op})^o$ denotes the corresponding Boolean lattices of these normal equivalences.
For every $A \in L$, $\Gamma(A) = \Delta(A^c)$ and hence

$$imf_A = L(A) \Rightarrow kerf_A^+ = \Delta(A^c) = \Gamma(A)$$

and

$$imf_A^+ = L^{op}(A^c) \Rightarrow kerf_A = \Gamma(A) = \Delta(A^c)$$

and so (f_A, f_A^+) is compatible with (Γ, Δ) and satisfies the conditions of cross connection. Hence $[L, L^{op}; \Gamma, \Delta]$ is a *cross connection.*
$U = \{(f_A, f_A^+) \in B(L) \times B(L^{op})\}$ is the ring with respect to the operations

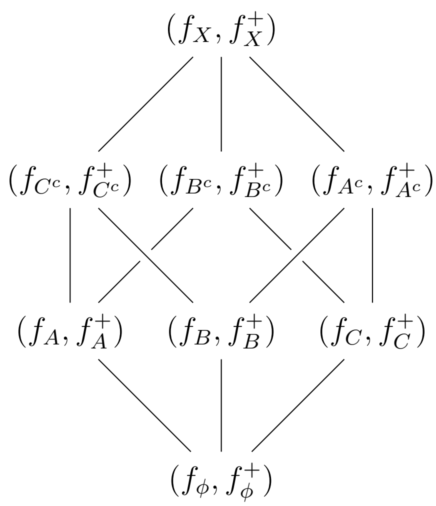
$$(f_A, f_A^+) \cdot (f_B, f_B^+) = (f_{A\wedge B}, f_{A\wedge B}^+) \text{ and}$$

$$(f_A, f_A^+) + (f_B, f_B^+) = (f_{A\oplus B}, f_{A\oplus B}^+).$$

Since every element of U is an idempotent U is a Boolean ring which is given below,

$$\{(f_\phi, f_\phi^+), (f_A, f_A^+), (f_B, f_B^+), (f_C, f_C^+), (f_{A^c}, f_{A^c}^+, (f_{B^c}, f_{B^c}^+), (f_{C^c}, f_{C^c}^+), (f_X, f_X^+)\}.$$

The lattice diagram of U is the following,



The complete ideals of U of the form $I_A = ((f_A, f_A^+))$, for every $A \in L$ are described below.

$$I_\phi = ((f_\phi, f_\phi^+)) = \{(f_\phi, f_\phi^+)\}$$

$$I_A = ((f_A, f_A^+)) = \{(f_\phi, f_\phi^+), (f_A, f_A^+)\}$$

$$I_B = ((f_B, f_B^+)) = \{(f_\phi, f_\phi^+), (f_B, f_B^+)\}$$

$$I_C = ((f_C, f_C^+)) = \{(f_\phi, f_\phi^+), (f_C, f_C^+)\}$$

$$I_{A^c} = ((f_{A^c}, f_{A^c}^+)) = \{(f_\phi, f_\phi^+), (f_{A^c}, f_{A^c}^+), (f_B, f_B^+), (f_C, f_C^+)\}$$

$$I_{B^c} = ((f_{B^c}, f_{B^c}^+)) = \{(f_\phi, f_\phi^+), (f_{B^c}, f_{B^c}^+), (f_A, f_A^+), (f_C, f_C^+)\}$$

$$I_{C^c} = ((f_{C^c}, f_{C^c}^+)) = \{(f_\phi, f_\phi^+), (f_{C^c}, f_{C^c}^+), (f_A, f_A^+)(f_B, f_B^+)\}$$

$$\text{and } I_X = ((f_X, f_X^+)) = U.$$

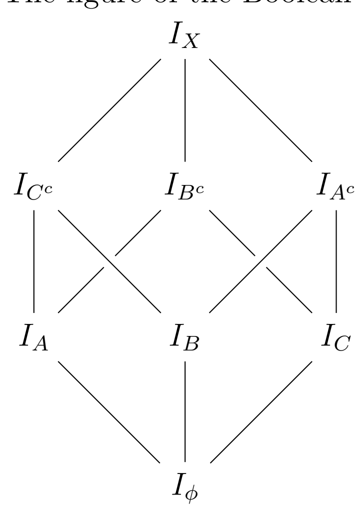
Clearly for every $A \in L$,

$$I_A \cap I_{A^c} = \{(f_\phi, f_\phi^+)\} = I_\phi$$

and

$$I_A \cup I_{A^c} = U = I_X.$$

These complete ideals form a Boolean lattice L^* and obviously it is isomorphic to U which is isomorphic to L. i.e., $L^* \cong U \cong L$.
The figure of the Boolean lattice L^* is given below.



Chapter 3

Categories from semigroups, rings and modules

Here we introduce and study the notion of '*proper category*' and see how this concept is related to the structure of semigroups, rings and modules. The principal left [right] ideals of a semigroup and that of a ring together with suitable translations as morphisms form proper category and preadditive proper category respectively and the submodules of an R-module with R-module homomorphisms form abelian proper category. Corresponding to each of these categories the set of all proper cones is a semigroup, ring and R-module respectively. If we restrict R-module to be semisimple R-module the abelian proper category becomes abelian normal category and the set of all normal cones is a semisimple R-module. Proper cones and proper categories are generalization of normal cones and normal categories introduced by K.S.S. Nambooripad [see Definitions 1.6.13 and 1.6.14]. This chapter is motivated by Nambooripad's work on set-based categories. The definition and example of set-based category (S-category) are provided below.

Let **Set** denotes the set category (see Example 1.6.1). A category $\mathcal{C}$ is *concrete* if there is a faithful functor $U : \mathcal{C} \to \textbf{Set}$ (see Definition

1.6.2). Clearly all small categories are concrete.

Definition 3.0.1. Let $\mathcal{C}$ be a category and let $U : \mathcal{C} \to \mathbf{Set}$ be a functor. $\mathcal{C}$ is *set based*($S-$category for short) with respect to U if the pair $(\mathcal{C}, U)$ satisfy the following:

1. U is an embedding,
2. $U(imf) = imU(f)$ for all $f \in \mathcal{C}$ and
3. the functor U has the property: for $c, c' \in v\mathcal{C}$ and $x \in U(c) \cap U(c')$ there is $d \in v\mathcal{C}$ such that

$$d \subseteq c, d \subseteq c' \text{ and } x \in U(d).$$

Example 3.0.1. Let S be a semigroup. An action of S on the left of a set X is a map

$$\alpha : S \times X \to X$$

such that $s(tx) = (st)x$ for all $s, t \in S$. A left $S-$set(left $S-$module) is a pair (X, α) where X is a set and α an action of S on the left of X. X and Y be left S-sets, a map $\alpha : X \to Y$ satisfying

$$(sx)\alpha = s(x\alpha) \text{ for all } s \in S \text{ and } x \in X$$

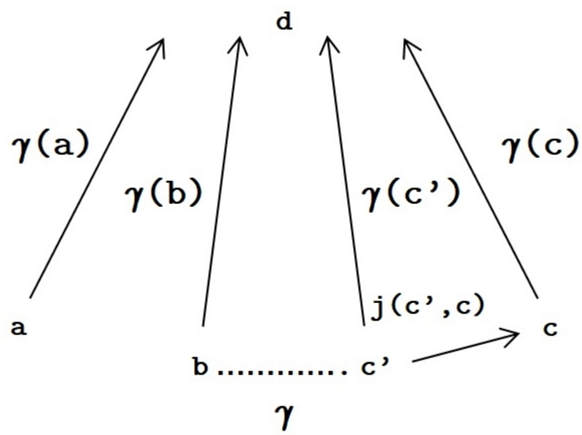
is a left $S-$map. $\mathbf{S-mod}$ the category with $S-$modules as objects and $S-$maps as morphisms is an $S-$category with subobjects and image.

3.1 Proper category

In the following we proceed to describe certain categories which we call proper categories. These are concrete $S-$categories.

Definition 3.1.1. Let $\mathcal{C}$ be a category with subobjects, every inclusion in $\mathcal{C}$ splits and every morphism has canonical factorization.

A proper cone γ in $\mathcal{C}$ is a cone with vertex d such that there exists at least one $c \in v\mathcal{C}$ with $\gamma(c) : c \to d$ is an epimorphism (i.e., $\gamma(c) = \gamma(c)^o$).



The set of all proper cones in category $\mathcal{C}$ is denoted by $\mathcal{PC}$ and for $\gamma \in \mathcal{PC}$, we denote by c_γ the vertex of γ and by N_γ, the $N-$set defined by

$$N_\gamma = \{c \in v\mathcal{C} : \gamma(c) \text{ is epimorphism}\} \tag{3.1}$$

A cone γ in $\mathcal{C}$ is proper, balanced or normal cone according as $N_\gamma \neq \phi$, $B_\gamma \neq \phi$ or $M_\gamma \neq \phi$ respectively and

$$\mathcal{TC} \subseteq \mathcal{BC} \subseteq \mathcal{PC}$$

where $\mathcal{TC}$ and $\mathcal{BC}$ are set of all normal and balanced cones respectively (see Definition 1.6.13).

Definition 3.1.2. A small category $\mathcal{C}$ with subobjects is called *proper category* if it satisfies the following:

1. every inclusion in $\mathcal{C}$ splits,
2. any morphism $f \in \mathcal{C}$ has canonical factorization and
3. each object of $\mathcal{C}$ is a vertex of a proper cone $\gamma \in \mathcal{PC}$.

3.1.1 Semigroup of proper cones

Now we proceed to show that in a proper category $\mathcal{C}$ the set of all proper cones $\mathcal{PC}$ is a semigroup.

Proposition 3.1.1. Let $\mathcal{C}$ be a proper category and γ be a proper cone in $\mathcal{C}$. For $f \in \mathcal{C}(c_\gamma, c)$, the map $\gamma \star f^o : a \mapsto \gamma(a)f^o$ is a proper cone with vertex imf such that for all composable pair of morphisms $f, g \in \mathcal{C}$ with $domf = c_\gamma$

$$\gamma \star (fg)^o = (\gamma \star f^o) \star (j_{c_1}^c g)^o$$

where $c_1 = imf = imf^o$

Proof. Clearly $\gamma \star f^o$ is a cone with vertex imf. To prove that it is proper,

$$\begin{aligned} \text{let } c \in N_\gamma &\Rightarrow \gamma(c) \text{ is an epimorphism} \\ &\Rightarrow \gamma(c) \cdot f^o \text{ is an epimorphism} \\ &\Rightarrow (\gamma \star f^o)(c) \text{ is an epimorphism} \\ &\Rightarrow c \in N_{\gamma \star f^o} \end{aligned}$$

hence $\gamma \star f^o$ is a proper cone.
Now f and g are as in the statement. Let $\eta = \gamma \star f^o$ so that $c_\eta = imf = c_1$. Then we have

$$\begin{aligned} (\gamma \star (fg)^o)(a) &= \gamma(a) f^o (j_{c_1}^c g)^o \\ &= \eta \star (j_{c_1}^c g)^o (a) \\ &= (\gamma \star f^o) \star (j_{c_1}^c g)^o (a) \end{aligned}$$

for all $a \in v\mathcal{C}$. □

Theorem 3.1.1. *Let $\mathcal{C}$ be a proper category. Then $\mathcal{PC}$ the set of all proper cones in $\mathcal{C}$, is a semigroup with respect to the binary* *operation defined by*

$$\gamma \cdot \eta = \gamma \star \eta(c_\gamma)^o$$

for all $\gamma, \eta \in \mathcal{PC}$.

Proof. For $\gamma, \eta \in \mathcal{PC}$, $\gamma \cdot \eta = \gamma \star \eta(c_\gamma)^o$ is a proper cone with vertex $c_{\gamma\cdot\eta} = im\eta(c_\gamma)$ (by Proposition 3.1.1). Now to show that the binary operation defined is associative, let $\alpha, \beta, \gamma \in \mathcal{PC}$ and for $c \in \mathit{v}\mathcal{C}$,

$$\begin{aligned}
(\alpha(\beta\gamma))(c) &= \alpha(c)((\beta\gamma)(c_\alpha))^o \\
&= \alpha(c)(\beta(c_\alpha)((\gamma(c_\beta))^o)^o \\
&= \alpha(c)[(\beta(c_\alpha))^o j^{c_\beta}_{Im\beta(c_\alpha)}(\gamma(c_\beta))^o]^o \\
&= \alpha(c)((\beta(c_\alpha))^o(\gamma(c_{\alpha\beta}))^o \\
((\alpha\beta)\gamma)(c) &= (\alpha\beta)(c)(\gamma(c_{\alpha\beta}))^o \\
&= \alpha(c)((\beta(c_\alpha))^o(\gamma(c_{\alpha\beta}))^o
\end{aligned}$$

Thus $\alpha(\beta\gamma) = (\alpha\beta)\gamma$ and hence $\mathcal{PC}$ is a semigroup. □

Proposition 3.1.2. $\gamma \in \mathcal{PC}$ is an idempotent proper cone if and only if $\gamma(c_\gamma) = I_{c_\gamma}$.

Proof. Suppose γ is an idempotent proper cone and let $c \in N_\gamma$. Then $(\gamma \cdot \gamma)(c) = \gamma(c)$ implies $\gamma(c)(\gamma(c_\gamma))^o = \gamma(c)$. Since $\gamma(c)$ is an epimorphism $(\gamma(c_\gamma))^o = I_{c_\gamma}$. Clearly $\gamma(c_\gamma) \in \mathcal{C}(c_\gamma, c_\gamma)$ and so $\gamma(c_\gamma) = I_{c_\gamma}$. Conversely, if $\gamma(c_\gamma) = I_{c_\gamma}$, then for every $a \in \mathit{v}\mathcal{C}$, $(\gamma \cdot \gamma)(a) = \gamma(a)(\gamma(c_\gamma))^o = \gamma(a)I_{c_\gamma} = \gamma(a)$. Hence γ is an idempotent proper cone. $E(\mathcal{PC})$ denotes the set of all idempotent proper cones in $\mathcal{C}$. □

3.1.2 Ideal categories of semigroups

In the following we proceed to describe the categories of left [right] principal ideals of an arbitrary semigroup S. Let S^1 be the semigroup obtained by adjoining identity to S (if it is necessary). The partially ordered set $S^1/\mathcal{L}$ of all $\mathcal{L}$-classes of semigroup S^1 is order isomorphic with the partially ordered set of all principal left ideals of the semigroup S^1. For $\mathcal{L}_x, \mathcal{L}_y \in S^1/\mathcal{L}$,

$$\mathcal{L}_x \leq \mathcal{L}_y \Leftrightarrow S^1x \subseteq S^1y.$$

Dually the inclusion among the principal right ideals induces a partial order on the set $S^1/\mathcal{R}$ of all $\mathcal{R}$-classes of S^1 and $S^1/\mathcal{R}$ is order isomorphic with the partially ordered set of principal right ideals under inclusion.

Let $\mathbb{L}(S)$ be the set of principal left ideals of S^1 and morphisms between principal left ideals are right translations $\rho : S^1a \to S^1b$ such that $(xy)\rho = x(y\rho)$, for all $x, y \in S^1a$.

Lemma 3.1.1. Let S be a semigroup. Then $\mathbb{L}(S)$, the set of principal left ideals of S^1, is a category whose objects and morphisms are

$$v\mathbb{L}(S) = \{S^1a : a \in S^1\} \text{ and}$$

$$\mathbb{L}(S)(S^1a, S^1b) = \{\rho(a, s, b) : x \to xs \text{ with } x \in S^1a \text{ and } as \in S^1b\}.$$

Proof. For all $a, b \in S^1$ and $x \in S^1a$, $\rho(a, s, b) : x \to xs$ with $as \in S^1b$ is the map $\rho_s|_{S^1a}$ such that $s \in S^1$ and $as \in S^1b$. The composition in $\mathbb{L}(S)$ is given by the rule

$$\rho(a, s, b) \cdot \rho(c, t, d) = \begin{cases} \rho(a, st, d), & \text{if } S^1b = S^1c \\ undefined, & \text{otherwise.} \end{cases}$$

is associative whenever the composition is defined and $\rho(a, 1, a) = I_{S^1a} : S^1a \to S^1a$ for all $a \in S^1$ is the identity morphism and hence $\mathbb{L}(S)$ is a category. □

Dually $\mathbb{R}(S)$ is also a category with objects principal right ideals and morphisms left translations,

$$v\mathbb{R}(S) = \{aS^1 : a \in S^1\} \text{ and}$$

$$\mathbb{R}(S)(aS^1, bS^1) = \{\lambda(a,s,b) : x \to sx \text{ with } x \in aS^1 \text{ and } sa \in bS^1\}.$$

Proposition 3.1.3. Let S be a semigroup and $\mathbb{L}(S)$ be the category of principal left ideals. Let $\rho(a,s,b) : S^1a \to S^1b$ be a morphism in $\mathbb{L}(S)$. Then

1. $\rho(a,s,b)$ is epimorphism if and only if $as\mathcal{L}b$
2. $\rho(a,s,b)$ is a split monomorphism if and only if $a\mathcal{R}as$
3. $\rho(a,s,b)$ is an isomorphism if and only if $a\mathcal{R}as\mathcal{L}b$

Proof. $\rho(a,s,b) : S^1a \to S^1b$ such that for all $x \in S^1a$, $\rho(a,s,b) : x \to xs$ where $as \in S^1b$. So it is easy to observe that $\rho(a,s,b)$ is epimorphism if and only if $S^1as = S^1b$ and so $as\mathcal{L}b$. Now $\rho(a,s,b)$ is a split monomorphism if and only if there is a $\sigma = \rho(b,s',a) : S^1b \to S^1a$ such that $\rho\sigma = I_{S^1a}$ which implies $ass' = a$ and so $a\mathcal{R}as$. (3) follows from (1) and (2). □

Theorem 3.1.2. *The category $\mathbb{L}(S)[\mathbb{R}(S)]$ of all principal left [right] ideals of semigroup S^1 is a proper category.*

Proof. If $\rho(a,s,b) = \rho(a,1,b)$ where $a \cdot 1 \in S^1b$, then $\rho(a,s,b) = j(S^1a, S^1b)$ an inclusion. Identity mapping and set inclusions of principal left ideals are morphisms in the category. So $\mathbb{L}(S)$ is a category with subobjects. $\{\rho(a,1,b) : a \in S^1b\}$ is a choice of subobjects in the category $\mathbb{L}(S)$. If $\rho(a,s,b)$ is a morphism in $\mathbb{L}(S)$, then im $\rho(a,s,b) = S^1as$ and $\rho(a,s,b) = \rho(a,s,as) \cdot \rho(as,1,b)$ gives the canonical factorization of $\rho(a,s,b)$ in $\mathbb{L}(S)$. Let $\rho(a,1,b) : S^1a \subseteq S^1b$ be the inclusion which splits by (2) of Proposition 3.1.3 as $a\mathcal{R}a$. If inclusion splits, canonical factorization is unique([25]). Hence $\mathbb{L}(S)$ is a category with subobjects, every inclusion splits and every morphism has unique canonical factorization.

Now we define $\rho^d : v\mathbb{L}(S) \to S^1d$ as $\rho^d(S^1a) = \rho(a,s,d) : S^1a \to S^1d$, where $S^1a \in v\mathbb{L}(S)$ and $as \in S^1d$. Obviously $\rho^d(S^1a) \in \mathbb{L}(S)(S^1a, S^1d)$

is well defined.
If $S^1a \subseteq S^1b$,

$$\begin{aligned} j(S^1a, S^1b)\rho^d(S^1b) &= \rho(a,1,b) \cdot \rho(b,v,d) \text{ where } bv = qd \in S^1d. \\ &= \rho(a,v,d) \\ &= \rho^d(S^1a) \end{aligned}$$

Since $S^1a \subseteq S^1b$, $a = rb$ for some $r \in S^1$ and so $av = rbv = rqd \in S^1d$. Hence $\rho(a,v,d) = \rho^d(S^1a)$ and $j(S^1a, S^1b) \cdot \rho^d(S^1b) = \rho^d(S^1a)$. Hence ρ^d is a cone in $\mathbb{L}(S)$. Since $\rho^d(S^1a)$ has canonical factorization as $\rho^d(S^1a) = \rho(a,s,d) = \rho(a,s,as) \cdot \rho(as,1,d)$, where $as \in S^1d$. There exists some s such that $S^1as = S^1d$. Thus ρ^d is a proper cone in $\mathbb{L}(S)$ with vertex S^1d and hence $\mathbb{L}(S)$ is a proper category. □

The set of all proper cones in $\mathbb{L}(S)$ is a semigroup with the binary operation

$$(\rho^a \cdot \rho^b)(S^1d) = \rho^a(S^1d) \cdot (\rho^b(S^1a))^o$$

and we denote this semigroup of proper cones by $\mathcal{P}\mathbb{L}(S)$.

Remark 3.1.1. If S^{op} denote the opposite semigroup of S with multiplication given by $a \circ b = b \cdot a$ where the right side is the product in S, then $\mathbb{L}(S^{op}) = \mathbb{R}(S)$ and $\mathbb{R}(S^{op}) = \mathbb{L}(S)$. Thus for any statement which holds for $\mathbb{L}(S)$ (or $\mathbb{R}(S)$) the corresponding dual statement holds for $\mathbb{R}(S)$ (respectively $\mathbb{L}(S)$). Thus for a semigroup S, $\mathbb{R}(S)$ is also a *proper category.* $v\mathbb{R}(S) = \{aS^1 : a \in S^1\}$ and for all $a, b \in S^1$, and for some $s \in S^1$

$$\mathbb{R}(S)(aS^1, bS^1) = \{\lambda(a,s,b) = \lambda_s|_{aS^1} : x \to sx \text{ with } sa \in bS^1\}.$$

Example 3.1.1. Consider the semigroup S given below:

$\cdot$	a	b
a	a	a
b	a	a

then the semigroup S^1 obtained by adjoining 1 to S is

$\cdot$	a	b	1
a	a	a	a
b	a	a	b
1	a	b	1

In S^1, the elements a and 1 are idempotents but b is not an idempotent. As S is commutative, principal left and right ideals coincide. The principal left ideals are $S^1a = \{a\}, S^1b = \{a,b\}, S^1 = \{a,b,1\}$ and the empty set ϕ. So $v\mathbb{L}(S) = \{S^1a, S^1b, S^1, \phi\}$. The proper cone with vertex S^1a is denoted by ρ^a and its components are given by; for all $S^1d \in v\mathbb{L}(S)$, $\rho^a(S^1d) = \rho(d,s,a)$ such that $ds \in S^1a$ for $s \in S^1$. The proper cone ρ^a with $\rho(a,1,a)$ as one component is the idempotent proper cone with vertex S^1a.

The morphisms from S^1a to S^1a are $\rho(a,s,a)$ where $as \in S^1a$, and $s \in S^1$. To find these morphisms,

$$a \cdot a = a \cdot b = a \cdot 1 = a \in S^1a$$

so $\rho(a,a,a) = \rho(a,b,a) = \rho(a,1,a)$. Hence $hom(S^1a, S^1a) = \{\rho(a,1,a)\}$. To find the morphisms from S^1 to S^1a,

$$1 \cdot a = a \in S^1a, 1 \cdot b = b \notin S^1a, 1 \cdot 1 = 1 \notin S^1a$$

so the morphisms from S^1 to S^1a is $\rho(1,a,a)$. In a similar way we can find all homsets

$$hom(S^1a, S^1a) = \{\rho(a,1,a)\},$$

$$hom(S^1, S^1a) = \{\rho(1,a,a)\},$$

$$hom(S^1a, S^1) = \{\rho(a,1,1)\},$$

$$hom(S^1, S^1) = \{\rho(1,a,1), \rho(1,b,1), \rho(1,1,1)\},$$

$$hom(S^1a, S^1b) = \{\rho(a,1,b)\},$$

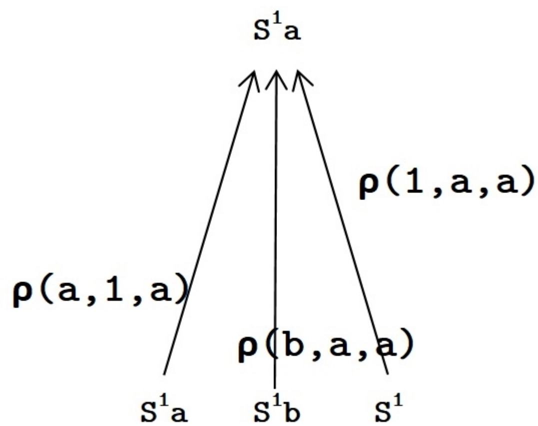
$$hom(S^1b, S^1a) = \{\rho(b, a, a)\},$$

$$hom(S^1b, S^1b) = \{\rho(b, a, b), \rho(b, 1, b)\},$$

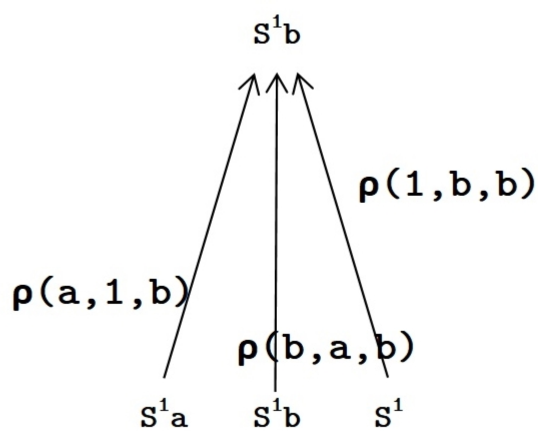
$$hom(S^1b, S^1) = \{\rho(b, a, 1), \rho(b, 1, 1)\},$$

$$hom(S^1, S^1b) = \{\rho(1, a, b), \rho(1, b, b)\}$$

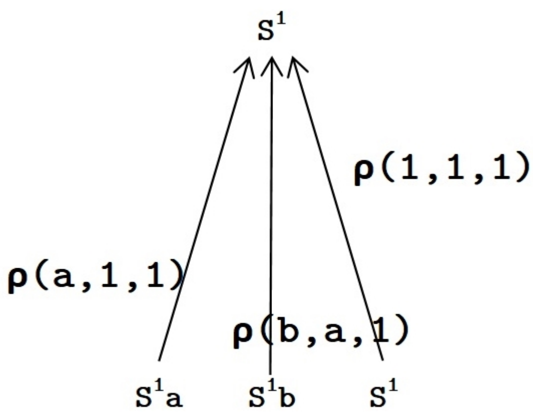
The proper cones are;



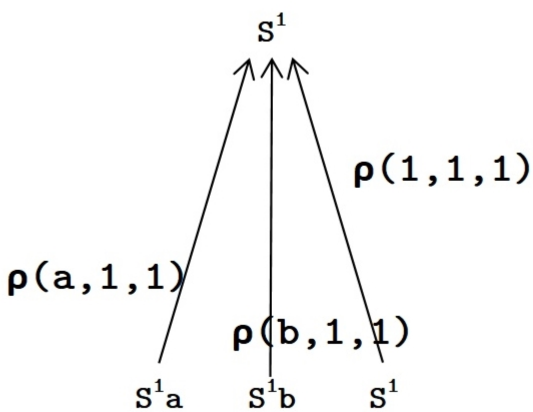
γ_1



γ_2



γ_3



γ_4

$v\mathbb{L}(S) = \{S^1a, S^1b, S^1, \phi\}$ and the set of all proper cones $\mathcal{P}\mathbb{L}(S) = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ is a semigroup. $\gamma_1 \cdot \gamma_1 = \gamma_1, \gamma_1 \cdot \gamma_2 = \gamma_1, \gamma_3 \cdot \gamma_2 = \gamma_2, \gamma_2 \cdot \gamma_3 = \gamma_1$ and so on. γ_1, γ_3 and γ_4 are idempotent proper cones.

The principal right ideal proper category $\mathbb{R}(S)$ is same as category $\mathbb{L}(S)$. The principal left ideals of semigroup $\mathcal{P}\mathbb{L}(S)$ are given by

$$\mathcal{P}\mathbb{L}(S)\gamma_1 = \{\gamma_1\}$$

$$\mathcal{P}\mathbb{L}(S)\gamma_2 = \{\gamma_1, \gamma_2\}$$

$$\mathcal{P}\mathbb{L}(S)\gamma_3 = \{\gamma_1, \gamma_3, \gamma_4\}$$

$$\mathcal{P}\mathbb{L}(S)\gamma_4 = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$$

Hence for the given general semigroup S^1, $\mathcal{P}\mathbb{L}(S)$ is a semigroup and its left ideal category $\mathbb{L}(\mathcal{P}\mathbb{L}(S))$ is a proper category with objects $v\mathbb{L}(\mathcal{P}\mathbb{L}(S)) = \{\mathcal{P}\mathbb{L}(S)\gamma_1, \mathcal{P}\mathbb{L}(S)\gamma_2, \mathcal{P}\mathbb{L}(S)\gamma_3, \mathcal{P}\mathbb{L}(S)\gamma_4\}$ and morphisms right translations.

3.2 Preadditive proper category

Recall the definition of preadditive category (see Definition 1.6.9). A preadditive category which is also a proper category is termed as a *preadditive proper category*. In the following we proceed to discuss certain specific preadditive proper categories and proper cones in such categories.

3.2.1 $RR-$categories and Ring of proper cones

Definition 3.2.1. A preadditive proper category $\mathcal{C}$ satisfying the following conditions:

1. the object set $v\mathcal{C}$ with partial order induced by subobject relation is a relatively complemented lattice and
2. every subset of the object set which has an upper bound contains a maximal element

is called an $RR-$*proper category* and these conditions are termed as $RR-$conditions.

Now we proceed to show that the set of proper cones $\mathcal{PC}$ in an $RR-$proper category $\mathcal{C}$, is a ring.

Proposition 3.2.1. Let γ be a cone with vertex d in an $RR-$proper category $\mathcal{C}$. Let $X = \{im\gamma(a) : a \in v\mathcal{C}\}$ then there exists a unique maximum element $d_0 \leq d$ in X.

Proof. Since $RR-$proper category $\mathcal{C}$ satisfies $RR-$conditions, X has a maximal element say d_0. To prove that it is unique let d_0 and d_1 be two maximal elements of X. Then there exists $b, c \in v\mathcal{C}$ such that $d_0 = im\gamma(b)$ and $d_1 = im\gamma(c)$. By $RR-$conditions $b \vee c \in v\mathcal{C}$. Let $d = b \vee c$. Then $b \subseteq d$ and $c \subseteq d$. Then

$$j(b, d)\gamma(d) = \gamma(b) \text{ and } j(c, d)\gamma(d) = \gamma(c).$$

Obviously

$$imj(b, d)\gamma(d) \subseteq im\gamma(d) \text{ and } imj(c, d)\gamma(d) \subseteq im\gamma(d).$$

Then $im\gamma(b) \subseteq im\gamma(d)$ and $im\gamma(c) \subseteq im\gamma(d)$ so that $d_0 \subseteq im\gamma(d)$ and $d_1 \subseteq im\gamma(d)$. Since d_0 and d_1 are maximal elements in X, $d_0 = im\gamma(d)$ and $d_1 = im\gamma(d)$. So $d_0 = d_1$. Also $im\gamma(a) \leq d_0$ for all $a \in v\mathcal{C}$. $\square$

Lemma 3.2.1. If γ is a cone in an $RR-$proper category $\mathcal{C}$ with vertex d and $d_0 = max\{im\gamma(a) | a \in v\mathcal{C}\}$ then for every retraction $e : d \to d_0$, the cone γ^* with vertex d_0, defined by;

$$\gamma^*(a) = \gamma(a)e(d, d_0), \text{ for all } a \in v\mathcal{C}$$

is a proper cone.

Proof. For all $a \in v\mathcal{C}, \gamma^*(a) : a \to d_0$. Since $e : d \to d_0$ is a retraction, $d_0 \subseteq d$. Let $d_0 = im\gamma(d_1)$ where $d_1 \in v\mathcal{C}$.

$$\begin{aligned} \text{For } a \subseteq b, j_a^b\gamma^*(b) &= j_a^b\gamma(b)e(d, d_0) \\ &= \gamma(a)e(d, d_0) \text{ [since } \gamma \text{ is a cone]} \\ &= \gamma^*(a) \end{aligned}$$

hence γ^* is a cone. Now to prove that γ^* is a proper cone, it is sufficient

to prove that at least one component of γ^* is an epimorphism.
We have $im\gamma(d_1) = d_0$ and $\gamma(d_1) = qj$ is the canonical factorization, where q is the epimorphism and $j : d_0 \subseteq d$, inclusion, then

$$\begin{aligned}\gamma^*(d_1) &= \gamma(d_1)e(d, d_0)\\ &= qj(d_0, d)e(d, d_0)\\ &= q\end{aligned}$$

which is an epimorphism and hence γ^* is a proper cone. □

Remark 3.2.1. If γ is a proper cone, then $\gamma^* = \gamma$.

Lemma 3.2.2. Let γ^* be a proper cone in the $RR-$proper category $\mathcal{C}$ as defined in Lemma 3.2.1, then the epimorphic component of γ^* is γ^* i.e., $(\gamma^*)^o = \gamma^*$.

Proof. $(\gamma^*)^o(a) = [\gamma(a)e(c_\gamma, d_0)]^o$ where $d_0 = max\{im\gamma(a)|a \in \mathit{v}\mathcal{C}\}$. Since $e(c_\gamma, d_0)$ is the retraction, $[\gamma(a)e(c_\gamma, d_0)]^o = \gamma(a)e(c_\gamma, d_0)$. Hence $(\gamma^*)^o(a) = \gamma^*(a)$. □

Lemma 3.2.3. If γ, β are two proper cones in an $RR-$proper category $\mathcal{C}$ and γ^* is defined as in Lemma 3.2.1, then

$$(\gamma \cdot \beta)^* = \gamma \cdot \beta^*$$

Proof. For all $a \in \mathit{v}\mathcal{C}$, and let $d_0 = max\{im(\gamma \cdot \beta)(a)|a \in \mathit{v}\mathcal{C}\}$

$$\begin{aligned}(\gamma \cdot \beta)^*(a) &= (\gamma \cdot \beta)(a) \cdot e(c_\beta, d_0)\\ &= \gamma(a) \cdot [(\beta(c_\gamma))^o \cdot e(c_\beta, d_0)]\\ &= \gamma(a) \cdot [\beta^*(c_\gamma)]^o\\ &= \gamma(a) \cdot [\beta^*(c_\gamma)] \text{ [by Lemma 3.2.2]}\\ &= (\gamma \cdot \beta^*)(a)\end{aligned}$$

Hence the proof. □

Lemma 3.2.4. Let $\mathcal{C}$ be an $RR-$proper category and $\mathcal{PC}$ the semigroup of proper cones in $\mathcal{C}$. For $\gamma, \delta \in \mathcal{PC}$, with vertices $c_\gamma = c$ and $c_\delta = d$ and for all $a \in \upsilon\mathcal{C}$,

$$(\gamma \oplus \delta)(a) = \gamma(a)j(c, c \vee d) + \delta(a)j(d, c \vee d)$$

Then $\gamma \oplus \delta$ is a *cone* with vertex $c \vee d$.

Proof. The result follows from [20]. In [20] Sunny Lukose and A.R.Rajan defined addition of two cones in $RR-$categories as given in the statement. We follow the same addition for proper cones in $RR-$proper categories. $\square$

Corollary 3.2.1. If $c_\gamma = c_\delta$, then the inclusions $j(c_\gamma, c_\gamma \vee c_\delta)$ and $j(c_\delta, c_\gamma \vee c_\delta)$ become identity maps and $(\gamma \oplus \delta)(a) = \gamma(a) + \delta(a)$.

Definition 3.2.2. Let $\mathcal{C}$ be an $RR-$proper category and $\mathcal{PC}$ the semigroup of proper cones in $\mathcal{C}$. For $\gamma, \delta \in \mathcal{PC}$, with vertices $c_\gamma = c$ and $c_\delta = d$, define

$$\gamma + \delta = (\gamma \oplus \delta)^* \tag{3.2}$$

where $(\gamma \oplus \delta)^*(a) = (\gamma \oplus \delta)(a)e(c \vee d, d_0)$ for all $a \in \upsilon\mathcal{C}$. Then $\gamma + \delta$ is a proper cone with vertex d_0, where $d_0 = max\{im(\gamma \oplus \delta)(a), |a \in \upsilon\mathcal{C}\}$.

Lemma 3.2.5. For an $RR-$proper category $\mathcal{C}$, the set of all proper cones $\mathcal{PC}$ is an additive abelian group with respect to the addition defined in equation 3.2.

Proof. For $\gamma, \delta \in \mathcal{PC}$, $\gamma + \delta$ is a proper cone in $\mathcal{PC}$. Since $\mathcal{C}$ is a $RR-$proper category, it is preadditive and hence each homset in $\mathcal{C}$ is an additive abelian group.
For $\gamma, \delta, \rho \in \mathcal{PC}$ and for all $a \in \upsilon\mathcal{C}$, let $\gamma+\delta = \eta$, $\delta+\rho = \tau$, $(c_\gamma \vee c_\delta) \vee c_\rho =$

$c_\gamma \vee (c_\delta \vee c_\rho) = d$, then

$$\begin{aligned}
((\gamma+\delta)+\rho)(a) &= (\eta+\rho)(a)\\
&= (\eta \oplus \rho)^*(a)\\
&= [\eta(a)j(c_\eta, d) + \rho(a)j(c_\beta, d)]^*\\
&= [(\gamma(a)j(c_\gamma, c_\gamma \vee c_\delta)\\
&+ \delta(a)j(c_\delta, c_\gamma \vee c_\delta))e(c_\gamma \vee c_\delta, c_\eta)j(c_\eta, d) + \rho(a)j(c_\rho, d)]^*\\
&= [(\gamma(a)j(c_\gamma, d) + \delta(a)j(c_\delta, d)) + \rho(a)j(c_\rho, d)]^*\\
&= [\gamma(a)j(c_\gamma, d) + (\delta(a)j(c_\delta, d) + \rho(a)j(c_\rho, d))]^*\\
&[\text{since addition is associative in } hom(a,d)]
\end{aligned}$$

and

$$\begin{aligned}
(\gamma+(\delta+\rho))(a) &= (\gamma+\tau)(a)\\
&= (\gamma \oplus \tau)^*(a)\\
&= [\gamma(a)j(c_\gamma, d) + \tau(a)j(c_\tau, d)]^*\\
&= [\gamma(a)j(c_\gamma, d) + (\delta(a)j(c_\delta, c_\delta \vee c_\rho)\\
&+ \rho(a)j(c_\rho, c_\delta \vee c_\rho))e(c_\delta \vee c_\rho, c_\tau)j(c_\tau, d)]^*\\
&= [\gamma(a)j(c_\gamma, d) + (\delta(a)j(c_\delta, d) + \rho(a)j(c_\rho, d))]^*
\end{aligned}$$

hence the addition is associative.
Let 0 be the zero object in $\mathcal{C}$ and γ_0 be the cone with vertex 0, where $\gamma_0(a) = 0$ for all $a \in v\mathcal{C}$, then $\gamma_0(a)$ is the unique morphism from a to 0 and γ_0 is a proper cone in $\mathcal{PC}$. For every $\gamma \in \mathcal{PC}$ and for all $a \in v\mathcal{C}$, let $d = max\{im(\gamma \oplus \gamma_0)(a) | a \in \mathcal{C}\} = c_\gamma$, since γ is a proper cone.

$$\begin{aligned}
(\gamma+\gamma_0)(a) &= (\gamma \oplus \gamma_0)^*(a)\\
&= [\gamma(a)j(c_\gamma, c_\gamma) + \gamma_0(a)j(c_{\gamma_0}, c_\gamma)]e(c_\gamma, d)\\
&= [\gamma(a) + \gamma_0(a)j(0, c_\gamma)]e(c_\gamma, d)\\
&= \gamma(a)
\end{aligned}$$

thus $\gamma + \gamma_0 = \gamma$. Similarly $\gamma_0 + \gamma = \gamma$. Hence γ_0 is the identity element in $\mathcal{PC}$.

For $\gamma \in \mathcal{PC}$, define $-\gamma$ by $-\gamma(a) = -(\gamma(a))$. Clearly $-\gamma$ is a proper cone in $\mathcal{PC}$ and $c_{-\gamma} = c_\gamma$.

$$\begin{aligned}
(\gamma + -(\gamma))(a) &= [\gamma(a)j(c_\gamma, c_\gamma \vee c_{-\gamma}) + (-\gamma)(a)j(c_{-\gamma}, c_\gamma \vee c_{-\gamma})]^* \\
&= [\gamma(a)j(c_\gamma, c_\gamma) + (-\gamma)(a)j(c_\gamma, c_\gamma)]^* \\
&= [\gamma(a)I_{c_\gamma} + (-\gamma)(a)I_{c_\gamma}]^* \\
&= [\gamma(a) + (-\gamma)(a)]^* \\
&= [\gamma_0(a)]^* = \gamma_0(a)
\end{aligned}$$

i.e., $\gamma + (-\gamma) = \gamma_0$. Similarly $(-\gamma) + \gamma = \gamma_0$. Hence $-\gamma$ is the additive inverse of γ.

$$\begin{aligned}
(\gamma + \delta)(a) &= [(\gamma \oplus \delta)(a)]^* \\
&= [\gamma(a)j(c_\gamma, c_\gamma \vee c_\delta) + \delta(a)j(c_\delta, c_\gamma \vee c_\delta)]^* \\
&= [\delta(a)j(c_\delta, c_\gamma \vee c_\delta) + \gamma(a)j(c_\gamma, c_\gamma \vee c_\delta)]^* \\
&= [(\delta \oplus \gamma)(a)]^* \\
&= (\delta + \gamma)(a)
\end{aligned}$$

i.e., $\gamma + \delta = \delta + \gamma$. Thus $\mathcal{PC}$ is an additive abelian group. □

Theorem 3.2.1. *For an $RR-$proper category $\mathcal{C}$, the set of all proper cones $\mathcal{PC}$ is a ring.*

Proof. As $RR-$proper category $\mathcal{C}$ is a proper category it is already known that the set of all proper cones $\mathcal{PC}$ in $\mathcal{C}$ form a semigroup by Theorem 3.1.1 with respect to the multiplication,

$$\gamma \cdot \beta = \gamma \star \beta(c_\gamma)^o$$

where $\gamma, \beta \in \mathcal{PC}$. From the above Lemma 3.2.5, $\mathcal{PC}$ is an additive

abelian group. So it is enough to prove that the multiplication distributes over addition. As $RR-$proper category is preadditive, composition of morphisms is distributive over addition.
For $\gamma, \delta, \rho \in \mathcal{PC}$, and for all $a \in v\mathcal{C}$, let $c_\rho = c$ and $c_\gamma \vee c_\delta = d$

$$\begin{aligned}[\rho \cdot (\gamma + \delta)](a) &= \rho(a) \cdot [(\gamma + \delta)(c)]^o \\ &= \rho(a) \cdot [(\gamma \oplus \delta)^*(c)]^o \\ &= \rho(a) \cdot ([(\gamma(c)j(c_\gamma, d) + \delta(c)j(c_\delta, d))]^*)^o \\ &= \rho(a) \cdot [(\gamma(c)j(c_\gamma, d) + \delta(c)j(c_\delta, d))]^* \text{ [by Lemma 3.2.2]}\end{aligned}$$

$$\begin{aligned}[(\rho \cdot \gamma) + (\rho \cdot \delta)](a) &= [(\rho \cdot \gamma) \oplus (\rho \cdot \delta)]^*(a) \\ &= [\rho(a) \cdot (\gamma(c))^o j(im\gamma(c), d) + \rho(a) \cdot (\delta(c))^o j(im\delta(c), d)]^* \\ &= [\rho(a) \cdot (\gamma(c))^o j(im\gamma(c), c_\gamma) j(c_\gamma, d) \\ &+ \rho(a) \cdot (\delta(c))^o j(im\delta(c), c_\delta) j(c_\delta, d)]^* \\ &= [\rho(a) \cdot (\gamma(c)j(c_\gamma, d) + \delta(c)j(c_\delta, d))]^* \\ &= \rho(a) \cdot [(\gamma(c)j(c_\gamma, d) + \delta(c)j(c_\delta, d))]^* \text{ [by Lemma 3.2.3]}\end{aligned}$$

Hence $\rho \cdot (\gamma + \delta) = (\rho \cdot \gamma) + (\rho \cdot \delta)$. Similarly we can prove that $(\gamma + \delta) \cdot \rho = \gamma \cdot \rho + \delta \cdot \rho$. Thus $\mathcal{PC}$ is a ring. □

3.2.2 Ideal categories of rings

Let R be a ring with unity. Then principal left [right] ideal of a ring R generated by an element a of R is $(a)_l = Ra$ $[(a)_r = aR]$. Since the multiplicative part of a ring is semigroup, it follows that the principal left [right] ideals of a ring R as objects and morphisms right [left] translations form *proper category* $\mathbb{L}(R)[\mathbb{R}(R)]$.

$$v\mathbb{L}(R) = \{Ra : a \in R\} \text{ and for } a, b \in R$$

$$\mathbb{L}(R)(Ra, Rb) = \{\rho(a, s, b) : x \to xs \text{ with } as \in Rb; \text{ for all } x \in Ra\}$$

Let $\rho(a,s,b), \rho(a,t,b) \in \mathbb{L}(R)(Ra,Rb)$, then $as, at \in Rb$. Since $a(s+t) \in Rb$ we have $\rho(a,s,b)+\rho(a,t,b) = \rho(a,s+t,b) \in \mathbb{L}(R)(Ra,Rb)$.

Lemma 3.2.6. The proper category $\mathbb{L}(R)[\mathbb{R}(R)]$ of principal left [right] ideals of a ring R with right [left] translations as morphisms is a preadditive category.

Proof. For $\rho(a,s,b), \rho(a,t,b) \in \mathbb{L}(R)(Ra,Rb)$, the addition is defined by

$$\rho(a,s,b) + \rho(a,t,b) = \rho(a,s+t,b).$$

Under this addition homset $\mathbb{L}(R)(Ra,Rb)$ is an abelian group, further if restricted to $\mathbb{L}(R)(Ra,Ra)$, it is a ring.

For $\rho(a,s,b), \rho(a,t,b) \in \mathbb{L}(R)(Ra,Rb)$ and $\rho(b,u,c), \rho(b,v,c) \in \mathbb{L}(R)(Rb,Rc)$ with $as, at \in Rb$ and $bu, bv \in Rc$,

$$\begin{aligned}\rho(a,s,b)[\rho(b,u,c)+\rho(b,v,c)] &= \rho(a,s,b)[\rho(b,u+v,c)]\\ &= \rho(a,s(u+v),c) = \rho(a,su+sv,c)\\ &= \rho(a,su,c)+\rho(a,sv,c)\\ &= \rho(a,s,b)\rho(b,u,c)+\rho(a,s,b)\rho(b,v,c)\end{aligned}$$

Similarly $[\rho(a,s,b)+\rho(a,t,b)]\rho(b,u,c) = \rho(a,s,b)\rho(b,u,c)+\rho(a,t,b)\rho(b,u,c)$. Clearly the zero ideal is the *zero object* in the category $\mathbb{L}(R)$. Hence for a ring R, the category $\mathbb{L}(R)$ and dually $\mathbb{R}(R)$ are *preadditive proper categories.* □

Let R be a ring with the property that any set of principal ideals which is bounded above has a maximal element then the category $\mathbb{L}(R)$ of principal left ideals of R is preadditive proper category with the property that $v\mathbb{L}(R)$ (principal ideals of ring) form a complete lattice with respect to the partial order induced from strict preorder. The join and meet are defined by; $Ra \vee Rb = Ra + Rb$ and $Ra \wedge Rb = Ra \cap Rb$ where $Ra + Rb$ is the smallest principal ideal containing both Ra and Rb. The trivial ideals R and (0) are the bounds; and every subset of $v\mathbb{L}(R)$ which has an upper bound in $v\mathbb{L}(R)$ contains a maximal element. Then $\mathbb{L}(R)$ is an $RR-$proper category. Examples of such rings are Euclidean domains, ring of integers $\mathbb{Z}$, polynomial ring $F[X]$ for a field F, Gaussian integers $\mathbb{Z}[i]$ etc.

Example 3.2.1. $R = (\mathbb{Z}_4, +, \cdot)$

+	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

As R is a commutative ring, the two sided principal ideals are $(0) = \{0\}$, $(1) = (3) = R$, $(2) = \{0, 2\}$. We get a proper category $\mathbb{L}(R)$ with objects $(0), (1)$ and (2). The morphisms can be found as in Example 3.1.1. The homsets are

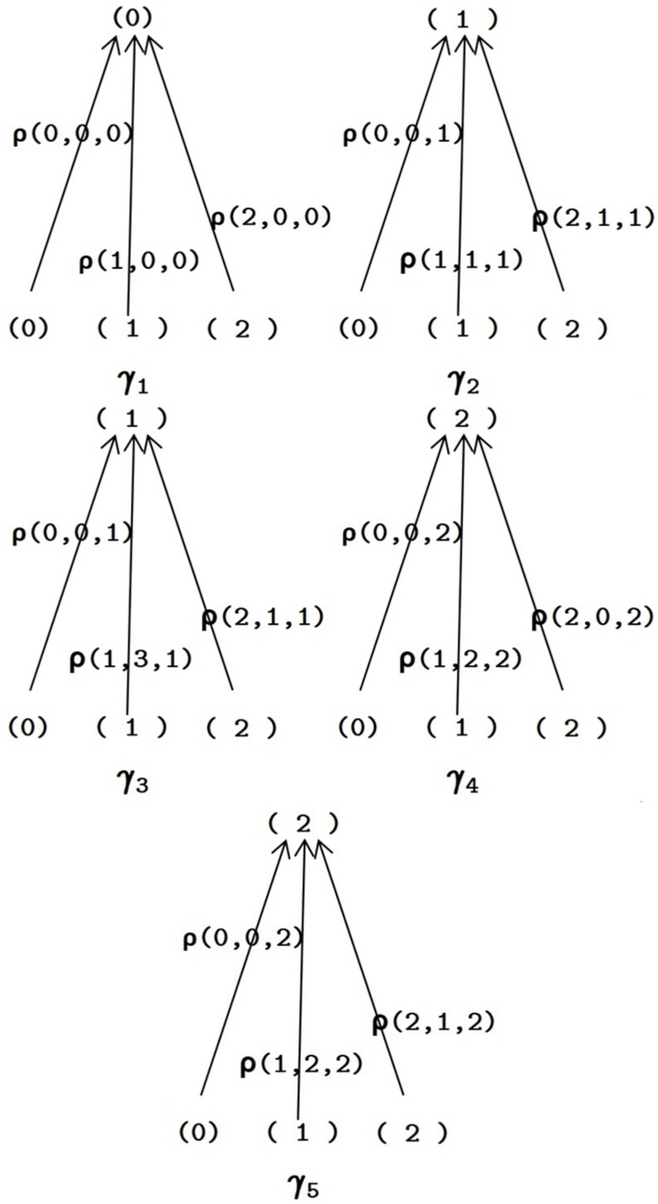
$$hom((1),(2)) = \{\rho(1,0,2), \rho(1,2,2)\},$$

$$hom((1),(1)) = \{\rho(1,0,1), \rho(1,1,1), \rho(1,2,1), \rho(1,3,1)\},$$

$$hom((2),(1)) = \{\rho(2,0,1), \rho(2,1,1)\},$$

$$hom((2),(2)) = \{\rho(2,0,2), \rho(2,1,2)\}.$$

The proper cones are;



γ_1, γ_2 and γ_5 are idempotent proper cones. But γ_3 and γ_4 are not idempotent proper cones.

$\mathcal{P}\mathbb{L}(R) = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5\}$ is a ring. It is easy to see that $\gamma_1 \cdot \gamma_2 = \gamma_1, \gamma_2 \cdot \gamma_3 = \gamma_3, \gamma_3 \cdot \gamma_4 = \gamma_4, \gamma_3 \cdot \gamma_5 = \gamma_4, \gamma_4 \cdot \gamma_5 = \gamma_4$ and so on.

$\gamma_1 + \gamma_2 = \gamma_2, \gamma_1 + \gamma_3 = \gamma_3$ and so on. γ_1 acts as additive identity element. $\gamma_2 + \gamma_2 = \gamma_4$, $\gamma_2 + \gamma_3 = \gamma_0$, $\gamma_3 + \gamma_4 = \gamma_2$ and so on.

3.2.3 $RR-$normal category

A normal category which is preadditive is called *preadditive normal category* (see Definition 1.6.9 and 1.6.14). A preadditive normal category satisfying $RR-$conditions is called *$RR-$normal category* (see Definition 3.2.1). A.R. Rajan and Sunny Lukose proved that the set of all normal cones in an $RR-$normal category is a regular ring (cf.[20]). Let $\mathcal{TC}$ be the set of all normal cones in an $RR-$normal category $\mathcal{C}$ and $E(\mathcal{TC})$ be the set of all idempotent normal cones in $\mathcal{C}$ (see Definition 1.6.13).

Definition 3.2.3. If γ is a cone in an $RR-$normal category $\mathcal{C}$ with vertex d and $d_0 = max\{im\gamma(a)|a \in v\mathcal{C}\}$ then for every retraction $e : d \to d_0$, the cone γ^* with vertex d_0, defined by;

$$\gamma^*(a) = \gamma(a)e(d, d_0), \text{ for all } a \in v\mathcal{C}$$

is a normal cone.

Definition 3.2.4. Let $\mathcal{C}, \mathcal{D}$ be $RR-$normal categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be a functor of $RR-$normal categories if the following hold:

1. if $j : a \to b$ is an inclusion in $\mathcal{C}$ then $F(j) : F(a) \to F(b)$ is an inclusion in $\mathcal{D}$
2. for $a, b \in v\mathcal{C}$ the restriction of F to $hom(a, b)$ is a homomorphism of groups from $hom(a, b)$ to $hom(F(a), F(b))$.

Theorem 3.2.2 (cf.[20], Theorem 2.25)**.** *The set of all normal cones $\mathcal{TC}$ in an RR-normal category $\mathcal{C}$ form a regular ring with respect to the multiplication and addition of normal cones defined by: for*

$\gamma, \delta \in \mathcal{TC}$ with $c_\gamma = c, c_\delta = d$ and for each $a \in v\mathcal{C}$,

$$(\gamma \cdot \delta)(a) = \gamma(a) \cdot \delta(c)^o \text{ and}$$

$$(\gamma + \delta)(a) = (\gamma \oplus \delta)^*(a)$$

where $(\gamma \oplus \delta)^*(a) = [\gamma(a)j(c, c \vee d) + \delta(a)j(d, c \vee d)]e(c \vee d, d_0)$

and $d_0 = max\{im(\gamma \oplus \delta)(a) | a \in v\mathcal{C}\}$ is the vertex of the normal cone $\gamma + \delta$.

Two $RR-$normal categories are isomorphic if there exists an isomorphism which preserves inclusions and addition in homsets. If $\mathcal{C}$ is an $RR-$normal category, the set of all normal cones $\mathcal{TC}$ is a regular ring. The following theorem establishes the isomorphism between $RR-$normal category $\mathcal{C}$ and left ideal category of $\mathcal{TC}$ denoted by $\mathbb{L}(\mathcal{TC})$.

Theorem 3.2.3. *Let $\mathcal{C}$ be an $RR-$normal category. Define F on objects and morphisms of $\mathcal{C}$ as follows: for $c \in v\mathcal{C}$, let*

$$vF(c) = (TC)\epsilon,$$

where $\epsilon \in E(\mathcal{TC})$ with $c_\epsilon = c$. For a morphism $f \in \mathcal{C}(c, b)$, let

$$F(f) = \rho(\epsilon, \epsilon * f^o, \epsilon') : (\mathcal{TC})\epsilon \to (\mathcal{TC})\epsilon'$$

where $\epsilon, \epsilon' \in E(\mathcal{TC})$, with $c_\epsilon = c, c_{\epsilon'} = b$. Then $F : \mathcal{C} \to \mathbb{L}(\mathcal{TC})$ is an isomorphism of $RR-$normal categories.

Proof. Let $\epsilon \in E(\mathcal{TC})$ and $f, g : c_\epsilon \to b$. First we prove that $(\epsilon \star f^o) + (\epsilon \star g^o) = \epsilon \star (f + g)^o$, where $\epsilon \star f^o$ is a normal cone with vertex imf and $\epsilon \star g^o$ is a normal cone with vertex img. Let $v = imf \vee img$ and for $a \in v\mathcal{C}$, let $d_0 = max\{im(\epsilon \star f^o \oplus \epsilon \star g^o)(a) | a \in v\mathcal{C}\}$ then by

Theorem 3.2.2,

$$\begin{aligned}
(\epsilon\star f^o+\epsilon\star g^o)(a) &= [(\epsilon\star f^o)(a)j(imf,v)\\
&\quad +(\epsilon\star g^o)(a)j(img,v)]e(v,d_0)\\
&= [(\epsilon(a)f^o)j(imf,v)+(\epsilon(a)g^o)j(img,v)]e(v,d_0)\\
&= \epsilon(a)[f^oj(imf,v)+g^oj(img,v)]e(v,d_0)\\
&= \epsilon(a)[f^oj(imf,v)+g^oj(img,v)]j(v,b)e(b,v)e(v,d_0)\\
&= \epsilon(a)[f^oj(imf,b)+g^oj(img,b)]e(b,d_0)\\
&= \epsilon(a)[f+g]e(b,d_0)\\
&= \epsilon(a)(f+g)^oj(im(f+g),cod(f+g))e(b,d_0)\\
&= \epsilon(a)(f+g)^oj(im(f+g),b)e(b,d_0)\\
&= \epsilon(a)(f+g)^o[\text{ since } im(f+g)=d_0]\\
&= \epsilon\star(f+g)^o(a)
\end{aligned}$$

By Theorem 1.6.1 $F:\mathcal{C}\to\mathbb{L}(\mathcal{TC})$ is an isomorphism of normal categories. Now it is sufficient to prove that F preserves addition in homsets. Let $f,g:c\to b$. As in the statement of the theorem, $F(f)=\rho(\epsilon,\epsilon*f^o,\epsilon'),F(g)=\rho(\epsilon,\epsilon*g^o,\epsilon'):(\mathcal{TC})\epsilon\to(\mathcal{TC})\epsilon'$. Then $F(f+g)=\rho(\epsilon,\epsilon*(f+g)^o,\epsilon'):(\mathcal{TC})\epsilon\to(\mathcal{TC})\epsilon'$.

$$\begin{aligned}
F(f+g) &= \rho(\epsilon,\epsilon*(f+g)^o,\epsilon')\\
&= \rho(\epsilon,(\epsilon\star f^o+\epsilon\star g^o),\epsilon')\\
&= \rho(\epsilon,\epsilon\star f^o,\epsilon')+\rho(\epsilon,\epsilon\star g^o,\epsilon')\\
&= F(f)+F(g)
\end{aligned}$$

hence $F:\mathcal{C}\to\mathbb{L}(\mathcal{TC})$ is an isomorphism of $RR-$normal categories. □

From the above theorem it is clear that $RR-$normal category $\mathcal{C}$ is isomorphic to $\mathbb{L}(\mathcal{TC})$ and hence $\mathbb{L}(\mathcal{TC})$ is an $RR-$normal category. Dual result holds for $\mathbb{R}(\mathcal{TC})$.

Ideal categories of regular rings

Let R be a regular ring and $\mathbb{L}(R)[\mathbb{R}(R)]$ the category of principal left [right] ideals of R with morphisms right [left] translations is a preadditive proper category by above Section 3.2.2. As the multiplicative part of a regular ring is regular semigroup and ideal categories of regular semigroups are normal categories, the category $\mathbb{L}(R)[\mathbb{R}(R)]$ becomes preadditive normal category .

The principal ideals of a regular ring are generated by idempotents and form complemented modular lattice(cf.[28]). Also $eR \vee fR = gR$ where $e, f, g \in E(R)$. Thus any set of principal ideals which are bounded above has a maximal element. Thus the preadditive normal category $\mathbb{L}(R)$ satisfies the $RR-$conditions and hence it is an $RR-$normal category. Dual result holds for the category $\mathbb{R}(R)$.
For $e, f \in E(R)$,

$$v\mathbb{L}(R) = \{Re|e \in E(R)\}$$

$$\mathbb{L}(R)(Re, Rf) = \{\rho|(st)\rho = s(t\rho); (s+t)\rho = s\rho + t\rho, \text{ for all } s, t \in Re\}$$

$\rho = \rho(e, u, f) = \rho_u|_{Re} : Re \to Rf$ is the translation $x \to xu$ where $u \in eRf$. The multiplication and addition of morphisms in $\mathbb{L}(R)$ are as follows:

$$\rho(e, u, f).\rho(g, v, h) = \rho(e, uv, f) \text{ where } f\mathcal{L}g, u \in eRf \text{ and } v \in gRh$$

$$\text{and } \rho(e, u, f) + \rho(e, v, f) = \rho(e, u+v, f) \text{ where } u, v, u+v \in eRf$$

As each principal ideal of a regular ring R is idempotent generated, each $\mathcal{L}$-class and $\mathcal{R}$-class has at least one idempotent.

Lemma 3.2.7. Let R be a regular ring and $a \in R, f \in E(\mathcal{L}_a)$. Then ρ^a is the normal cone in $\mathbb{L}(R)$ with vertex $Ra = Rf$ whose component at Re is

$$\rho^a(Re) = \rho(e, ea, f)$$

and the $M-$set of ρ^a is,

$$M\rho^a = \{Re : e \in E(\mathcal{R}_a)\}$$

where $E(\mathcal{L}_a)[E(\mathcal{R}_a)]$ is the set of all idempotents in the $\mathcal{L}[\mathcal{R}]$-class of a.

Proof. Since the multiplicative part of R is a regular semigroup, the result follows from Lemma 1.6.1. $\square$

For a regular ring R the left ideal category $\mathbb{L}(R)$ is an $RR-$normal category and by Theorem 3.2.2 the set of all normal cones $\mathcal{T}(\mathbb{L}(R))$ is a regular ring. The following theorem describes a homomorphism of R and $\mathcal{T}(\mathbb{L}(R))$.

Theorem 3.2.4. *Let R be a regular ring and $\mathbb{L}(R)$ the $RR-$normal category of principal left ideals of R. R_ρ is the set of all right translations in R. Then there exists a homomorphism $\bar{\rho} : R \to \mathcal{T}\mathbb{L}(R)$ and an injective homomorphism $\phi : R_\rho \to \mathcal{T}\mathbb{L}(R)$ such that the following diagram commutes.*

$$\begin{tikzcd}
R \arrow[r, "\rho"] \arrow[d] & R_\rho \arrow[d, "\phi"] \\
R \arrow[r, "\bar{\rho}"'] & \mathcal{T}\mathbb{L}(R)
\end{tikzcd}$$

Proof. For $a \in R$, $\phi : R_\rho \to \mathcal{T}\mathbb{L}(R)$ as $\phi(\rho_a) = \rho^a$ and set $\bar{\rho} = \rho\phi$. Then $\bar{\rho}(a) = \rho^a$. By Theorem 1.6.2, $\bar{\rho}$ is a homomorphism for a regular semigroup S, hence to prove that $\bar{\rho}$ is a ring homomorphism it is enough to prove that $\bar{\rho}$ preserves addition.

Given two normal cones ρ^a and ρ^b in $\mathcal{T}\mathbb{L}(R)$, Sunny Lukose and A.R. Rajan proved the existence of the sum $\rho^a + \rho^b$ as the normal cone ρ^{a+b} with vertex $R(a + b)$ in their Ph.D. thesis submitted to University of Kerala in 2003. Hence

$$\begin{aligned}\bar{\rho}(a+b) &= \rho\phi(a+b)\\ &= \phi(\rho(a+b))\\ &= \phi(\rho_{a+b})\\ &= \phi(\rho_a+\rho_b)\\ &= \rho^a+\rho^b\\ &= \bar{\rho}(a)+\bar{\rho}(b).\end{aligned}$$

Hence $\bar{\rho} = \rho\phi : R \to \mathcal{T}\mathbb{L}(R)$ is a ring homomorphism. □

3.3 Abelian proper category

Our main interest in this section is the application of category theory to module theory. A category of modules has a richer structure than an abstract category since the additive structure on modules extends to their homomorphisms.

An abelian category which is also proper [normal] is called *abelian proper [normal] category*. Let M be an R-module, now we proceed to show that the category of submodules of M is an abelian proper category and in particular if we restrict to semisimple R-module, the corresponding category is an *abelian normal category*.

Theorem 3.3.1. *Let M be an R-module. The submodules of M as objects and morphisms the R-module homomorphisms form an abelian proper category $\mathcal{S}(M)$.*

Proof. Let M be an R-module and $N_i, i = 1, 2, ...n$ be submodules of M, then $\mathcal{S}(M)$ with object set $N_i's$ and morphisms are R-module homomorphisms (R-linear maps) is a subcategory of $\mathbf{R-mod}$, the category of $R-$modules with morphisms R-module homomorphisms which is an abelian category. Hence $\mathcal{S}(M)$ is also an abelian category.

Further we prove that $hom(N_i, N_k)$ is an R-module: for $r, r_1, r_2 \in R$, $f, g \in hom(N_i, N_k)$ and $x \in N_i$; $f(x) \in N_k \Rightarrow rf(x) \in N_k$ [since N_k is an $R-$module]. This implies $rf \in hom(N_i, N_k)$. Also

1. $1 \cdot f = f$
2. $(r_1 \cdot r_2)f = r_1(r_2 f)$
3. $(r_1 + r_2)f = r_1 f + r_2 f$
4. $r(f + g) = rf + rg$.

Next we prove that the abelian category $\mathcal{S}(M)$ is a proper category. The submodules of M act as subobjects under usual set inclusion. Hence $\mathcal{S}(M)$ is a category with subobjects. For $N_i, N_k \in v\mathcal{S}(M)$, let $j : N_i \to N_k$ be the inclusion. Then there exists an $R-$module homomorphism $e : N_k \to N_i$ such that for all $x \in N_k$,

$$e(x) = \begin{cases} x, \text{if x} \in N_i \\ 0, \text{if x} \in N_t \end{cases}$$

where $N_t \in v\mathcal{S}(M)$ is such that $N_i \oplus N_k = N_t$. Then $je = I_{N_i}$. Hence every inclusion splits. Every morphism $f : N_i \to N_k$ in $\mathcal{S}(M)$ has canonical factorization as $f = qj$ where $q : N_i \to im\phi$, an epimorphism and $j : im\phi \to N_k$ an inclusion. For $N_i \subseteq M$, let $\gamma : v\mathcal{S}(M) \to N_m$ be the map such that

1. $\gamma(N_i) : N_i \to N_m$,
2. whenever $N_i \subseteq N_k; j(N_i, N_k) \cdot \gamma(N_k) = \gamma(N_i)$,

This collection of morphisms $\{\gamma(N_i) : N_i \in v\mathcal{S}(M)\}$ is a cone with vertex N_m. Since every morphism has canonical factorization there exists a submodule N_l of M such that $\gamma(N_l) : N_l \to N_m$ is an epimorphism. Thus $\gamma : v\mathcal{S}(M) \to N_m$ is a proper cone with vertex N_m. Hence $\mathcal{S}(M)$ is an abelian proper category. □

Next Theorem shows that the set of all proper cones in $\mathcal{S}(M)$ is an $R-$module.

Theorem 3.3.2. *Let $\mathcal{S}(M)$ be the abelian proper category of submodules of an R-module M. Then the set of all proper cones $\mathcal{PS}(M)$ is an R-module.*

Proof. The submodules of M form a relatively complemented modular lattice and any set of submodules bounded above has a maximal element. Hence the abelian proper category $\mathcal{S}(M)$ satisfies $RR-$conditions and hence $\mathcal{S}(M)$ is an $RR-$proper category. By Lemma 3.2.5, for the $RR-$proper category $\mathcal{S}(M)$ the set of all proper cones $\mathcal{PS}(M)$ is an additive abelian group. Now we define a scalar multiplication from $R \times \mathcal{PS}(M) \to \mathcal{PS}(M)$, such that for $r \in R, N_i \in v\mathcal{S}(M)$ and $\gamma \in \mathcal{PS}(M)$, $r\gamma(N_i) = \gamma^*(rN_i)$ where γ^* is as defined in Lemma 3.2.1 which is a proper cone in $\mathcal{PS}(M)$. Since each component of a cone lies in some homset and since each homset is an $R-$module we have

1. $1\gamma = \gamma$,
2. $(r+s)\gamma = r\gamma + s\gamma$;
3. $(rs)\gamma = r(s\gamma)$;
4. $r(\gamma + \delta) = r\gamma + r\delta$.

Hence $\mathcal{PS}(M)$ is an R-module. □

3.3.1 Abelian normal category

Here we consider the category $\mathcal{S}(M)$ of submodules of a semisimple R-module M whose objects are submodules and morphisms R-module homomorphisms. It is already seen that $\mathcal{S}(M)$ is an abelian proper category. In the following we proceed to prove that $\mathcal{S}(M)$ is an abelian normal category. Also we prove that the set of all normal cones in

$\mathcal{S}(M)$ is a *semisimple R-module*. For that we recall addition of normal cones in an RR-normal category (see Theorem 3.2.2).

Theorem 3.3.3. *Let M be a semisimple R-module. The category $\mathcal{S}(M)$ with objects submodules of M with morphisms R-module homomorphism, is an abelian normal category.*

Proof. Let M be a semisimple R-module and $N_i, i = 1, 2, ...n$ be submodules of M. By Theorem 3.3.1, the submodule category $\mathcal{S}(M)$ is an abelian proper category. For $N_i, N_k \in v\mathcal{S}(M)$ every morphism $f : N_i \to N_k$ in $\mathcal{S}(M)$ has normal factorization as $f = quj$ where $q : N_i \to (kerf)^c$, a retraction, $u : (kerf)^c \to imf$, an isomorphism and $j : imf \to N_k$ an inclusion where $N_i = kerf \oplus (kerf)^c$ (see Section 1.4).

Since every morphism has normal factorization, proper cones become normal cones. Normal cone $\gamma : v\mathcal{S}(M) \to \mathcal{S}(M)$ is defined such that

1. for each submodule N_i of M, there is a $\gamma(N_i) : N_i \to N_m$
2. whenever $N_i \subseteq N_k, j(N_i, N_k) \cdot \gamma(N_k) = \gamma(N_i)$ where $j(N_i, N_k)$ is an inclusion from N_i to N_k and
3. there exists at least one $N_l \in v\mathcal{S}(M)$ such that $\gamma(N_l) : N_l \to N_m$ is an isomorphism.

This collection of morphisms $\{\gamma(N_i) : N_i \in v\mathcal{S}(M)\}$ is a normal cone with vertex N_m. Now we prove that each object of $\mathcal{S}(M)$ is a vertex of an idempotent normal cone. Let $f : M \to N_m$ be an R-module homomorphism such that $f(x) = x,$ for all $x \in N_m$. For any $N_i \subseteq M$, define $\gamma(N_i) = f|_{N_i} : N_i \to N_m$. Then $\gamma(N_m) = I_{N_m}$. Thus γ is an idempotent normal cone with vertex $N_m \in v\mathcal{S}(M)$. Hence $\mathcal{S}(M)$ is a normal category. $\square$

Theorem 3.3.4. *Let $\mathcal{S}(M)$ be the abelian normal category of* *submodules of an R-module M. Then the set of all normal cones $\mathcal{TS}(M)$ is a semisimple R-module.*

Proof. Abelian normal category $\mathcal{S}(M)$ is an $RR-$normal category also. By Theorem 3.2.2, for an $RR-$normal category $\mathcal{S}(M)$ the set of all normal cones $\mathcal{TS}(M)$ is an additive abelian group with respect to the addition: for $\gamma, \delta \in \mathcal{TS}(M)$ with $c_\gamma = c$ and $c_\delta = d$,

$$\gamma + \delta = (\gamma \oplus \delta)^*$$

where for each $a \in v\mathcal{S}(M)$,

$$(\gamma \oplus \delta)^*(a) = (\gamma \oplus \delta)(a)e(c \vee d, d_0),$$

$$(\gamma \oplus \delta)(a) = \gamma(a)j(c, c \vee d) + \delta(a)j(d, c \vee d)$$

and

$$d_0 = max\{im(\gamma \oplus \delta)(a) | a \in v\mathcal{C}\}.$$

Now define a scalar multiplication from $R \times \mathcal{TS}(M) \to \mathcal{TS}(M)$ such that for $r \in R$, $N_i \in v\mathcal{S}(M)$ and $\gamma, \delta \in \mathcal{TS}(M)$,

$$r\gamma(N_i) = \gamma^*(rN_i)$$

where $\gamma^*(rN_i) = \gamma(rN_i)e(c_\gamma, d')$; $d' = max\{im\gamma(N_i) | N_i \in v\mathcal{S}(M)\}$ and rN_i is a submodule of M. Then $r\gamma$ is a normal cone in $\mathcal{TS}(M)$ by the definition of γ^* given in Definition 3.2.3. Also,

1. $1\gamma = \gamma$
2. $(r + s)\gamma = r\gamma + s\gamma$
3. $(rs)\gamma = r(s\gamma)$
4. $r(\gamma + \delta) = r\gamma + r\delta$

hence $\mathcal{TS}(M)$ is an R-module.

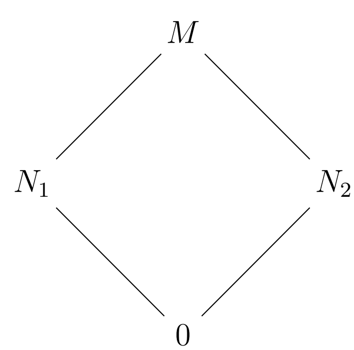
Now we proceed to prove that $\mathcal{TS}(M)$ is semisimple. Let $N_0 = \{0\}, N_1, N_2, ...N_r$ be simple submodules of M such that $M = N_0 \oplus N_1 \oplus N_2 \oplus ... \oplus N_r$ and let A_0 be the set of all normal cones with vertex N_0, A_1 be the set of all normal cones with vertex N_1, A_2 be the set of all normal cones with vertex N_2,... A_r be the set of all normal cones with vertex N_r. Then each A_i is a submodule of $\mathcal{TS}(M)$ and hence each A_i is an $R-$module. For $\gamma, \delta \in A_i$ with vertex N_i, $\gamma + \delta$ is also a normal cone with vertex subobject of N_i that is either N_i or N_0 since N_i is simple. Thus the submodules of A_i are A_i and A_0 and thus $A_i = A_i \oplus A_0$. Hence each A_i is a simple module. Thus $A_0 \oplus A_1 \oplus A_2 \oplus ... \oplus A_r \subseteq \mathcal{TS}(M)$.

The vertex of any normal cone γ in $\mathcal{TS}(M)$ is a submodule of M. So γ is in any submodule of $A_0 \oplus A_1 \oplus A_2 \oplus ... \oplus A_r$. So $\mathcal{TS}(M)) \subseteq A_0 \oplus A_1 \oplus A_2 \oplus ... \oplus A_r$ and hence

$$\mathcal{TS}(M)) = A_1 \oplus A_2 \oplus ... \oplus A_r.$$

Thus $\mathcal{TS}(M)$ is a semisimple R-module. □

Example 3.3.1. Consider the semisimple $\mathbb{Z}$-module $M = \mathbb{Z}_6$. The submodules of M are $\{0\}, N_1 = \{0, 3\}, N_2 = \{0, 2, 4\}$ and M. The submodules form a Boolean lattice whose diagram is,



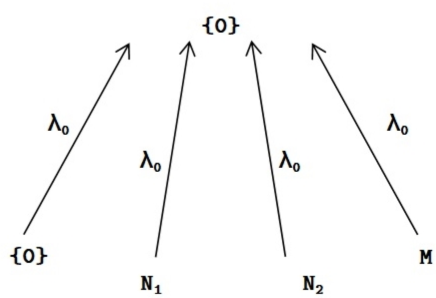
Obviously $M = \{0\} \oplus M$ and $M = N_1 \oplus N_2$.
Consider the abelian normal category $\mathcal{S}(M)$ with objects $v\mathcal{S}(M) = \{\{0\}, N_1, N_2, M\}$ and morphisms $\mathbb{Z}-$module homomorphisms given by $\lambda_n : x \to nx$.
The homsets are

$$hom(N_1, N_1) = \{\lambda_0, \lambda_1\}, hom(N_2, N_2) = \{\lambda_0, \lambda_1, \lambda_2\}, hom(N_1, M) = \{\lambda_0, \lambda_1\},$$

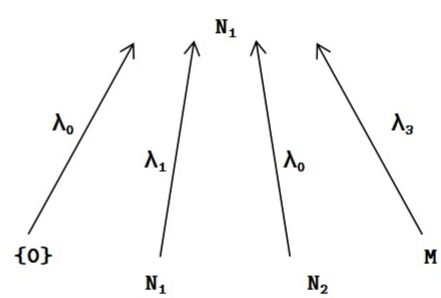
$$hom(M, N_1) = \{\lambda_0, \lambda_3\}, hom(N_2, M) = \{\lambda_0, \lambda_1, \lambda_2\}, hom(M, N_2) = \{\lambda_0, \lambda_2, \lambda_4\},$$

$$hom(N_1, N_2) = \phi, hom(M, M) = \{\lambda_0, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5\} = A(say)$$

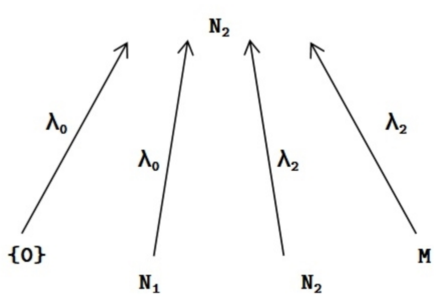
then A is a *von Neumann regular ring* isomorphic to M.
The normal cones are given by



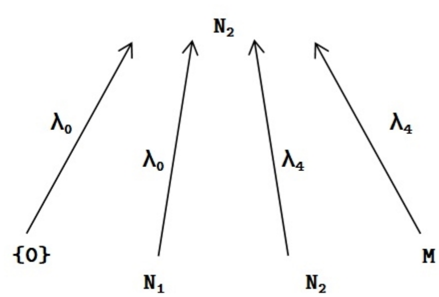
γ_0



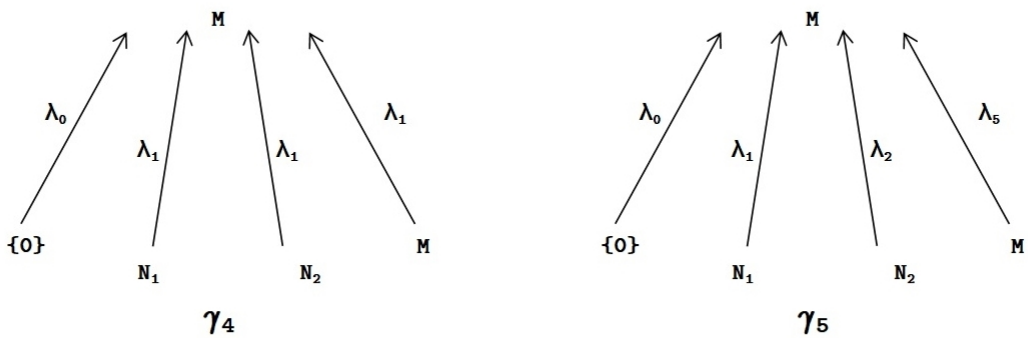
γ_1



γ_2



γ_3



$\mathcal{TS}(M) = \{\gamma_0, \gamma_1, \gamma_2, \gamma_3, \gamma_4, \gamma_5\}$ is isomorphic to M and also isomorphic to $A = hom(M, M)$. The correspondence is

$$\gamma_0 \to \lambda_0, \gamma_1 \to \lambda_3, \gamma_2 \to \lambda_2, \gamma_3 \to \lambda_4, \gamma_4 \to \lambda_1, \gamma_5 \to \lambda_5.$$

It is easy to observe that the set of all normal cones $\mathcal{TS}(M)$ is a *semisimple module* with submodules $B_0 = \{\gamma_0\}, B_1 = \{\gamma_0, \gamma_1\}, B_2 = \{\gamma_0, \gamma_2, \gamma_3\}$ and $\mathcal{TS}(M)$. Obviously

$$B_0 \oplus \mathcal{TS}(M) = \mathcal{TS}(M) \text{ and } B_1 \oplus B_2 = \mathcal{TS}(M).$$

From this example it is clear that the set of all normal cones in the abelian normal category $\mathcal{S}(M)$ of a semisimple module M, is a semisimple module.

Chapter 4

$H-$functors and Dual categories

We have described proper category, preadditive proper category and $RR-$categories in Chapter 3. Now we proceed to describe the dual categories of these categories and it is shown that the dual category of a proper category, preadditive proper category and $RR-$normal category are also proper category, preadditive proper category and $RR-$normal category respectively.

4.1 Dual categories

For a regular semigroup S the category of principal left [right] ideals $\mathbb{L}(S)[\mathbb{R}(S)]$ is a normal category by Proposition 1.6.2. The set of all normal cones in the category $\mathbb{L}(S)$ denoted by $\mathcal{T}\mathbb{L}(S)$ is a regular semigroup and the category of its principal left ideals $\mathbb{L}(\mathcal{T}\mathbb{L}(S))$ is again a normal category isomorphic to $\mathbb{L}(S)$. By Theorem 1.6.4, the normal dual category $N^*\mathbb{L}(S)$ is isomorphic to the category of principal right ideals $\mathbb{R}(S)$. But in an arbitrary semigroup S, all principal ideals are not idempotent generated and ideal categories fails to be normal, however the category of principal left [right] ideals $\mathbb{L}(S)[\mathbb{R}(S)]$ is a proper category (see Theorem 3.1.2). The set of proper cones in $\mathbb{L}(S)$ denoted by $\mathcal{P}\mathbb{L}(S)$ is a semigroup and its left ideal category $\mathbb{L}(\mathcal{P}\mathbb{L}(S))$ is also a proper category.

4.1.1 Dual of proper category

Let $\mathcal{C}$ be a proper category and $\mathcal{PC}$ the semigroup of proper cones in $\mathcal{C}$. Now we proceed to describe certain set valued functors called H-functors on objects and morphisms of the proper category $\mathcal{C}$.

Definition 4.1.1. Let $\mathcal{PC}$ be the semigroup of proper cones in a proper category $\mathcal{C}$. For each $\gamma \in \mathcal{PC}$ we define an *H-functor*, $H(\gamma; -) : \mathcal{C} \to \mathbf{Set}$ defined by; for $c, d \in v\mathcal{C}$ and $g : c \to d$,

$$H(\gamma; c) = \{\gamma * f^o | f : c_\gamma \to c\}$$

$$H(\gamma; g) : H(\gamma; c) \to H(\gamma; d) \text{ maps } \gamma * f^o \mapsto \gamma * (fg)^o.$$

Clearly, $H(\gamma; c)$ is a set for all $c \in v\mathcal{C}$ and by the uniqueness of epimorphic component, $H(\gamma; g)$ is a map of the set $H(\gamma; c)$ to $H(\gamma; d)$.

Lemma 4.1.1. For each $\gamma \in \mathcal{PC}$, $H(\gamma; -) : \mathcal{C} \to \mathbf{Set}$ defined above is an inclusion preserving covariant functor.

Proof. For each $c \in v\mathcal{C}$, $H(\gamma; c)$ is a set and $H(\gamma; g)$ is a map from $H(\gamma; c)$ to $H(\gamma; c')$ where $g : c \to c'$. Now let $h : c' \to c''$

$$\begin{aligned} H(\gamma; g)H(\gamma; h)(\gamma \star f^o) &= H(\gamma; h)(H(\gamma; g)(\gamma \star f^o)) \\ &= H(\gamma; h)(\gamma \star (fg)^o) = \gamma \star ((fg)h)^o \\ &= \gamma \star (fgh)^o = H(\gamma; gh)(\gamma \star f^o) \end{aligned}$$

$$\text{Thus } H(\gamma; gh) = H(\gamma; g)H(\gamma; h).$$

Also $H(\gamma; I_c) = I_{H(\gamma;c)}$ and for $c \subseteq c'$, $H(\gamma; j_c^{c'}) = j_{H(\gamma;c)}^{H(\gamma;c')}$. Hence H is an inclusion preserving covariant functor. ∎

Remark 4.1.1. $H(\gamma;-)$ and $\mathcal{C}(c_\gamma;-)$ are naturally isomorphic functors and the natural isomorphism η_γ is given by $\eta_\gamma(c_\gamma) : H(\gamma;c_\gamma) \to \mathcal{C}(c_\gamma, c_\gamma)$ sending $\gamma \to I_{c_\gamma}$.

Proposition 4.1.1. For $\gamma, \gamma' \in \mathcal{PC}$, $H(\gamma;-) \subseteq H(\gamma';-)$ if and only if there exists a unique epimorphism h from $c_{\gamma'}$ to c_γ such that $\gamma = \gamma' \star h$.

Proof. Let $\gamma = \gamma' \star h$ where $h : c'_\gamma \to c_\gamma$ is an epimorphism and let $c \in v\mathcal{C}$. If $\gamma \star f^o \in H(\gamma;c)$, we have $\gamma \star f^o = \gamma' \star hf^o$. Since h is an epimorphism, by the uniqueness of canonical factorization $(hf)^o = hf^o$ and so $\gamma \star f^o \in H(\gamma';c)$. Thus $H(\gamma;c) \subseteq H(\gamma';c)$ for all $c \in v\mathcal{C}$.

Let $g : c \to c'$ be a morphism. Then

$$H(\gamma;g)(\gamma \star f^o) = \gamma \star fg^o \text{ and}$$

$$\begin{aligned} H(\gamma';g)(\gamma \star f^o) &= H(\gamma';g)(\gamma' \star (hf)^o) \\ &= \gamma' \star (hfg)^o \\ &= \gamma \star (fg)^o \end{aligned}$$

and the following diagram commutes:

$$\begin{tikzcd} H(\gamma';c) \arrow[r, "H(\gamma';g)"] & H(\gamma';c') \\ H(\gamma;c) \arrow[u, "j^{H(\gamma';c)}_{H(\gamma;c)}"] \arrow[r, "H(\gamma;g)"'] & H(\gamma;c') \arrow[u, "j^{H(\gamma';c')}_{H(\gamma;c')}"'] \end{tikzcd}$$

hence $H(\gamma;-) \subseteq H(\gamma';-)$.

Conversely, suppose that $H(\gamma;-) \subseteq H(\gamma';-)$, then $\gamma \in H(\gamma;c_\gamma) \subseteq H(\gamma';c_\gamma)$ and so $\gamma = \gamma' \star f^o$ for some $f \in \mathcal{C}(c_{\gamma'}, c_\gamma)$. Now $c_{\gamma' \star f^o} = imf$ and $\gamma = \gamma' \star f$ implies $imf = c_\gamma = \mathit{codf}$. Hence f is an epimorphism and $\gamma = \gamma' \star f$. If $\gamma' \star h = \gamma' \star k$, then for any $c \in N_{\gamma'}$, we have $\gamma'(c)h = \gamma'(c)k$ and since $\gamma'(c)$ is an epimorphism left cancellation holds and $h = k$.

This proves the uniqueness. □

Corollary 4.1.1. Let $\gamma, \gamma' \in \mathcal{PC}$. If $H(\gamma; -) \subseteq H(\gamma'; -)$ then $N_{\gamma'} \subseteq N_\gamma$.

Proof. Let $c \in N_{\gamma'}$ then $\gamma'(c)$ is an epimorphism. $H(\gamma; -) \subseteq H(\gamma'; -)$ implies $\gamma = \gamma' \star h$ where $h : c_{\gamma'} \to c_\gamma$ is an epimorphism and $\gamma(c) = \gamma'(c)h$ is epimorphism. Hence $c \in N_\gamma$ and thus $N_{\gamma'} \subseteq N_\gamma$. □

Definition 4.1.2. If $\mathcal{C}$ is a proper category then the proper dual of $\mathcal{C}$ denoted by $\mathcal{P}^*\mathcal{C}$ is the full subcategory of $\mathcal{C}^*$ with

$$v\mathcal{P}^*\mathcal{C} = \{H(\gamma; -) | \gamma \in \mathcal{PC}\}$$

where each $H-$functor $H(\gamma; -)$ is as defined in Definition 4.1.1.

Lemma 1.6.2 describes the morphisms of normal dual category in terms of those of normal category $\mathcal{C}$. We can extend the same result to proper category since it differs with normal category only in normal factorization. Since the result is independent of normal factorization we have the following lemma which describes morphisms of $\mathcal{P}^*\mathcal{C}$ in terms of those of $\mathcal{C}$.

Lemma 4.1.2. Let $\mathcal{C}$ be a proper category and its dual $\mathcal{P}^*\mathcal{C}$ as defined above. To every morphism $\sigma : H(\gamma; -) \to H(\gamma'; -)$ in $\mathcal{P}^*\mathcal{C}$, there is a unique $\widehat{\sigma} : c_{\gamma'} \to c_\gamma$ in $\mathcal{C}$ such that the following diagram commutes.

$$\begin{array}{ccc} H(\gamma; -) & \xrightarrow{\eta_\gamma} & \mathcal{C}(c_\gamma, -) \\ \sigma \downarrow & & \downarrow \mathcal{C}(\widehat{\sigma}, -) \\ H(\gamma'; -) & \xrightarrow[\eta_{\gamma'}]{} & \mathcal{C}(c'_\gamma, -) \end{array}$$

In this case, the component of the natural transformation σ at $c \in v\mathcal{C}$ is the map given by $\sigma(c) : \gamma * f^o \mapsto \gamma' * (\widehat{\sigma} f)^o$. In particular, σ is the inclusion $H(\gamma; -) \subseteq H(\gamma'; -)$ if and only if $\gamma = \gamma' * \widehat{\sigma}$. Moreover, the map $\sigma \mapsto \widehat{\sigma}$ is a bijection of $\mathcal{P}^*\mathcal{C}(H(\gamma;-), H(\gamma';-))$ onto $\mathcal{C}(c'_\gamma, c_\gamma)$.

Remark 4.1.2. Let $\mathcal{C}^* = [\mathcal{C}, \mathbf{Set}]$ (see Definition 1.6.5). Since $\mathcal{C}^*$ is a category with subobjects with respect to functorial inclusion, $\mathcal{P}^*\mathcal{C}$ is a category with subobjects, with the subobject relation induced from $\mathcal{C}^*$.

Lemma 4.1.3. Let $\mathcal{C}$ be a proper category. If $f : c \to d$ is an epimorphism in $\mathcal{C}$, then $H(\gamma; f) : H(\gamma; c) \to H(\gamma; d)$ in category $\mathcal{P}^*\mathcal{C}$ is an epimorphism. Similar result hold for monomorphism.

Proof. Let $f : c \to d$ is an epimorphism. Then for $k_1, k_2 : d \to a$, $f \cdot k_1 = f \cdot k_2 \Rightarrow k_1 = k_2$. Consider $H(\gamma; c) = \{\gamma \star h^o | h : c_\gamma \to c\}$, $H(\gamma; f) : H(\gamma; c) \to H(\gamma; d)$ defined as $H(\gamma; f)(\gamma \star h^o) = \gamma \star (hf)^o$. For $H(\gamma; k_1), H(\gamma; k_2) : H(\gamma; d) \to H(\gamma; a)$ and for every $\gamma \star h^o \in H(\gamma; c)$, let

$$H(\gamma; f)H(\gamma; k_1)(\gamma \star h^o) = H(\gamma; f)H(\gamma; k_2)(\gamma \star h^o).$$

$$\begin{aligned} \text{But } H(\gamma; f)H(\gamma; k_1)(\gamma \star h^o) &= H(\gamma; k_1)(H(\gamma; f)(\gamma \star h^o)) \\ &= H(\gamma; k_1)(\gamma \star (hf)^o) \end{aligned}$$

$$\begin{aligned} \text{Now } H(\gamma; k_1)(\gamma \star (hf)^o) &= (\gamma \star (hfk_1)^o) \\ &= (\gamma \star (hfk_2)^o) \text{ [since } k_1 = k_2] \\ &= H(\gamma; k_2)(\gamma \star (hf)^o) \end{aligned}$$

Thus for every $\gamma \star (hf)^o \in H(\gamma; d), H(\gamma; k_1)(\gamma \star (hf)^o) = H(\gamma; k_2)(\gamma \star (hf)^o)$. Hence $H(\gamma; k_1) = H(\gamma; k_2)$ and so $H(\gamma; f)$ is an epimorphism. Dually, if $f : c \to d$ is a monomorphism in proper category $\mathcal{C}$, then $H(\gamma; f) : H(\gamma; c) \to H(\gamma; d)$ in $\mathcal{P}^*\mathcal{C}$ is a monomorphism. $\square$

By Lemma 4.1.2, the homset $\mathcal{P}^*\mathcal{C}(H(\gamma;-), H(\gamma';-))$ is isomorphic to $\mathcal{C}(c'_\gamma, c_\gamma)$ and by Remark 4.1.1, $H(\gamma;-)$ and $\mathcal{C}(c_\gamma;-)$ are naturally isomorphic functors. The next theorem shows that $\mathcal{P}^*\mathcal{C}$ satisfies all the conditions for a proper category.

Theorem 4.1.1. *For a proper category $\mathcal{C}$, its dual $\mathcal{P}^*\mathcal{C}$ is also a proper category.*

Proof. For a proper category $\mathcal{C}$, its dual is defined as in Definition 4.1.2,

$$v\mathcal{P}^*\mathcal{C} = \{H(\gamma; -) : \gamma \in \mathcal{PC}\}$$

such that for $c, d \in v\mathcal{C}$ and for $g : c \to d$,

$$H(\gamma; c) = \{\gamma * f^o | f : c_\gamma \to c\} \text{ and}$$

$$H(\gamma; g) : H(\gamma; c) \to H(\gamma; d) \text{ maps } \gamma * f^o \mapsto \gamma * (fg)^o.$$

By Remark 4.1.2, $\mathcal{P}^*\mathcal{C}$ is a category with subobjects.

Let $H(\gamma; j) : H(\gamma; c) \subseteq H(\gamma; d)$ is an inclusion in $\mathcal{P}^*\mathcal{C}$. Obviously $j : c \subseteq d$ is an inclusion in proper category $\mathcal{C}$ which splits, implies there exists retraction $e : d \to c$, such that $je = I_c$.

$$H(\gamma; c) = \{\gamma \star f^o | f : c_\gamma \to c\} \text{ and } H(\gamma; j)(\gamma \star f^o) = \gamma \star (fj)^o.$$

$H(\gamma; e) : H(\gamma; d) \to H(\gamma; c)$ and

$$H(\gamma; j)H(\gamma; e) = H(\gamma; I_c) = I_{H(\gamma;c)}.$$

Hence $H(\gamma; j)$ splits in $\mathcal{P}^*\mathcal{C}$.

If $\sigma(c) : H(\gamma; c) \to H(\gamma'; c)$ is an inclusion in $\mathcal{P}^*\mathcal{C}$ then by Lemma 4.1.2, $j : c_{\gamma'} \to c_\gamma$ is an inclusion in $\mathcal{C}$ which splits in $\mathcal{C}$. Since the homsets are isomorphic $\sigma(c)$ also splits in $\mathcal{P}^*\mathcal{C}$.

Next we prove that every morphism in $\mathcal{P}^*\mathcal{C}$ has unique canonical factorization. Let $f : c \to d$ be a morphism in $\mathcal{C}$ which has unique canonical factorization $f = qj$ where $q : c \to a$ and $j : a \subseteq d$.
Then $H(\gamma; f) : H(\gamma; c) \to H(\gamma; d)$ is defined as $H(\gamma; f) : \gamma \star h^o \mapsto$

$\gamma \star (hf)^o$ where $h : c_\gamma \to c$

$$\begin{aligned} H(\gamma; q)H(\gamma; j)(\gamma \star h^o) &= H(\gamma; j)(H(\gamma; q)(\gamma \star h^o)) \\ &= H(\gamma; j)(\gamma \star (hq)^o) = (\gamma \star (hqj)^o) \\ &= (\gamma \star (hf)^o) = H(\gamma; f)(\gamma \star h^o) \end{aligned}$$

Thus $H(\gamma; f) = H(\gamma; q)H(\gamma; j)$, where $H(\gamma; q)$ is an epimorphism and $H(\gamma; j)$ is an inclusion by Lemma 4.1.3.

If $\sigma(c) : H(\gamma; c) \to H(\gamma'; c)$, then by Lemma 4.1.2 there exists $\hat{\sigma} : c_{\gamma'} \to c_\gamma$ which has canonical factorization $\hat{\sigma} = qj$ in $\mathcal{C}$. Let $q : c_{\gamma'} \to c''_\gamma$ and $j : c''_\gamma \to c_\gamma$ then $\sigma(c)$ has canonical factorization $q'j'$ where $q' : H(\gamma; c) \to H(\gamma''; c)$ and $j' : H(\gamma''; c) \to H(\gamma'; c)$. Since every inclusion splits, by [25] $\sigma(c)$ has unique canonical factorization.

Let γ be a proper cone in $\mathcal{C}$ with vertex c_γ. For every $a \in v\mathcal{C}$, each component $\gamma(a) : a \to c_\gamma$. Then by Definition 4.1.1, $H(\gamma; \gamma(a)) : H(\gamma; a) \to H(\gamma; c_\gamma)$. Hence there is a proper cone γ' in $\mathcal{P}^*\mathcal{C}$ with vertex $H(\gamma; c_\gamma)$ as follows: for all $H(\gamma; a) \in v\mathcal{P}^*\mathcal{C}$

$$\gamma'(H(\gamma; a)) : H(\gamma; a) \to H(\gamma; c_\gamma)$$

If $H(\gamma; a) \subseteq H(\gamma; b)$ then

$$\begin{aligned} j(H(\gamma; a), H(\gamma; b))H(\gamma; \gamma(b)) &= H(\gamma; j(a, b))H(\gamma; \gamma(b)) \\ &= H(\gamma; j(a, b)\gamma(b)) \\ &= H(\gamma; \gamma(a)). \end{aligned}$$

since γ is a proper cone in $\mathcal{C}$, there exists at least one $c \in v\mathcal{C}$ such that $\gamma(c)$ is an epimorphism. Then by Lemma 4.1.3, $H(\gamma; \gamma(c))$ is an epimorphism which is a component of γ' and hence γ' is a proper cone in $\mathcal{P}^*\mathcal{C}$. In a similar way we get proper cones in $\mathcal{P}^*\mathcal{C}$ with each element of $\mathcal{P}^*\mathcal{C}$ as its vertex. Hence $\mathcal{P}^*\mathcal{C}$ is a proper category. □

Remark 4.1.3. If γ is an idempotent proper cone in $\mathcal{C}$ such that

$\gamma(c_\gamma) = I_{c_\gamma}$. Then $H(\gamma;\gamma(c_\gamma)) = H(\gamma; I_{c_\gamma}) = I_{H(\gamma;c_\gamma)}$ (see Lemma 4.1.1). Hence there exists an idempotent proper cone in $\mathcal{P}^*\mathcal{C}$ with vertex $H(\gamma;-)$.

Example 4.1.1. Let S be the semigroup given in Example 3.1.1, its principal left ideal category $\mathbb{L}(S)$ is a proper category. The dual of this proper category $\mathcal{P}^*\mathbb{L}(S)$ is the following.

$$v\mathcal{P}^*\mathbb{L}(S) = \{H(\gamma_1;-), H(\gamma_2;-), H(\gamma_3;-)\}$$

and morphisms are of the form $H(\gamma_1;\rho(a,a,1)) : H(\gamma_1;S^1a) \to H(\gamma_1;S^1)$. Similarly

$$v\mathcal{P}^*\mathbb{R}(S) = \{H(\gamma_i';-)|i = 1,2,...9\}$$

and morphisms are of the form $H(\gamma_1';\lambda(b,a,a)) : H(\gamma_1';bS^1) \to H(\gamma_1';aS^1)$.

4.1.2 Dual of preadditive proper category

Preadditive proper category $\mathcal{C}$ is described in Section 3.2. Now we proceed to show that its dual $\mathcal{P}^*\mathcal{C}$ is a preadditive proper category.

Theorem 4.1.2. *Let $\mathcal{C}$ be a preadditive proper category. Then its dual category $\mathcal{P}^*\mathcal{C}$ is also a preadditive proper category.*

Proof. Since every preadditive proper category $\mathcal{C}$ is a proper category it is seen that dual $\mathcal{P}^*\mathcal{C}$ is also a proper category (see Theorem 4.1.1). Now it is enough to show that in $\mathcal{P}^*\mathcal{C}$ each homset is an additive abelian group and composition of morphisms is distributive over addition.

$$hom(H(\gamma;c), H(\gamma;d)) = \{H(\gamma;f)|f : c \to d\}$$

since $hom(c,d)$ is an additive abelian group and $H(\gamma;-)$ is a covariant functor; for $f_i, f_j \in hom(c,d)$, $f_i + f_j \in hom(c,d)$. Define

$$H(\gamma; f_i) + H(\gamma; f_j) = H(\gamma; f_i + f_j) \in hom(H(\gamma; c), H(\gamma; d)).$$

Since $hom(c,d)$ is an additive abelian group, under this addition $hom(H(\gamma; c), H(\gamma; d))$ is obviously an additive abelian group. Also composition of morphisms is bilinear in $\mathcal{P}^*\mathcal{C}$.

$$\begin{aligned} H(\gamma; f) \cdot (H(\gamma; g) + H(\gamma; h)) &= H(\gamma; f) \cdot (H(\gamma; g+h) \\ &= H(\gamma; g+h) \cdot H(\gamma; f) = H(\gamma; (g+h)f) \\ &= H(\gamma; gf) + H(\gamma; hf) \\ &= H(\gamma; f) \cdot H(\gamma; g) + H(\gamma; f) \cdot H(\gamma; h). \end{aligned}$$

Now by Remark 4.1.1, $H(\gamma; -)$ and $\mathcal{C}(c_\gamma; -)$ are naturally isomorphic functors. By Lemma 4.1.2, when $\mathcal{C}$ is proper, the homset $\mathcal{P}^*\mathcal{C}(H(\gamma; -), H(\gamma'; -))$ to $\mathcal{C}(c_{\gamma'}, c_\gamma)$ is a bijection. Let

$$\phi : \mathcal{P}^*\mathcal{C}(H(\gamma; -), H(\gamma'; -)) \to \mathcal{C}(c_{\gamma'}, c_\gamma)$$

is defined as $\phi(\sigma) = \widehat{\sigma}$, where $\widehat{\sigma} \in \mathcal{C}(c_{\gamma'}, c_\gamma)$. For $d \in v\mathcal{C}$, since the diagram below commutes, ϕ is an isomorphism.

$$\begin{array}{ccc} H(\gamma; d) & \xrightarrow{\eta_\gamma(d)} & \mathcal{C}(c_\gamma, d) \\ {\scriptstyle \sigma(d)}\downarrow & & \downarrow{\scriptstyle \mathcal{C}(\widehat{\sigma}, d)} \\ H(\gamma'; d) & \xrightarrow[\eta_{\gamma'}(d)]{} & \mathcal{C}(c'_\gamma, d) \end{array}$$

Now to prove the result for preadditive proper categories it is sufficient to prove that addition is preserved between homsets. Let $\sigma_1, \sigma_2 : H(\gamma; -) \to H(\gamma'; -)$ in $\mathcal{P}^*\mathcal{C}$, then there exists $\widehat{\sigma_1}, \widehat{\sigma_2} : c_{\gamma'} \to c_\gamma$ in $\mathcal{C}$. Since $\mathcal{C}$ is preadditive proper category, each homset is an additive abelian group and hence $\widehat{\sigma_1} + \widehat{\sigma_2} \in \mathcal{C}(c'_\gamma, c_\gamma)$. From the diagram

$$C(\widehat{\sigma_1}, -) = \eta_\gamma^{-1}\sigma_1\eta_{\gamma'} : C(c_\gamma, -) \to C(c'_\gamma, -)$$

$$C(\widehat{\sigma_2}, -) = \eta_\gamma^{-1}\sigma_2\eta_{\gamma'} : C(c_\gamma, -) \to C(c'_\gamma, -).$$

$$\begin{aligned} C(\widehat{\sigma_1 + \sigma_2}, -) &= \eta_\gamma^{-1}(\sigma_1 + \sigma_2)\eta_{\gamma'} \\ &= \eta_\gamma^{-1}\sigma_1\eta_{\gamma'} + \eta_\gamma^{-1}\sigma_2\eta_{\gamma'} \\ &[\text{ since } \mathcal{C} \text{ preserves addition in homsets}] \\ &= C(\widehat{\sigma_1}, -) + C(\widehat{\sigma_2}, -). \end{aligned}$$

Thus we get $\widehat{\sigma_1 + \sigma_2} = \widehat{\sigma_1} + \widehat{\sigma_2}$ since $\mathcal{C}$ is preadditive proper category. Hence

$$\begin{aligned} \phi(\sigma_1 + \sigma_2) &= \widehat{\sigma_1 + \sigma_2} \\ &= \widehat{\sigma_1} + \widehat{\sigma_2} \\ &= \phi(\sigma_1) + \phi(\sigma_2). \end{aligned}$$

Thus the homsets $\mathcal{P}^*\mathcal{C}(H(\gamma; -), H(\gamma'; -))$ and $\mathcal{C}(c'_\gamma, c_\gamma)$ are isomorphic.

Thus if each homset in $\mathcal{C}$ an additive abelian group and composition distributes over addition then each homset in $\mathcal{P}^*\mathcal{C}$ also have the same property. If 0 is the zero object of $\mathcal{C}$ then $H(\gamma; 0)$ is the zero object of $\mathcal{P}^*\mathcal{C}$ for each proper cone γ in $\mathcal{C}$. Hence $\mathcal{P}^*\mathcal{C}$ is a preadditive proper category. □

4.1.3 Dual of $RR-$normal category

In Section 3.2.3, the $RR-$normal categories are described and it is shown that for an $RR-$normal category $\mathcal{C}$ the set of all normal cones $\mathcal{TC}$ is a regular ring. Now we proceed to define the dual of $RR-$normal category.

For $\gamma, \gamma' \in \mathcal{TC}$, $\gamma\mathcal{R}\gamma'$ if and only if $H(\gamma; -) = H(\gamma'; -)$. $\mathcal{TC}$ being a regular ring each $\mathcal{R}-$class contains at least one idempotent and so for every $\gamma \in \mathcal{TC}$ we have $H(\gamma;-) = H(\epsilon;-)$ where ϵ is an idempotent normal cone.

Definition 4.1.3. Let $\mathcal{C}$ is an $RR-$normal category, then the dual of $\mathcal{C}$, denoted by $N^*\mathcal{C}$, is the full subcategory of $\mathcal{C}^*$ with $vN^*\mathcal{C} = \{H(\epsilon;-) : \epsilon \in E(\mathcal{TC})\}$ where $H(\epsilon;-)$ is defined on objects and morphisms of $\mathcal{C}$ as for $c, d \in v\mathcal{C}$ and $g : c \to d$;

$$H(\epsilon;c) = \{\epsilon * f^o | f : c_\epsilon \to c\}\text{and}$$

$$H(\epsilon;g) : H(\epsilon;c) \to H(\epsilon;d) \text{ maps } \epsilon * f^o \mapsto \epsilon * (fg)^o.$$

Lemma 4.1.4. Let $\mathcal{C}$ be an $RR-$normal category. To every morphism $\sigma : H(\epsilon;-) \to H(\epsilon';-)$ in $N^*\mathcal{C}$, there is a unique $\widehat{\sigma} : c_{\epsilon'} \to c_\epsilon$ in $\mathcal{C}$ such that the following diagram commutes where $\epsilon, \epsilon' \in E(\mathcal{TC})$.

$$\begin{tikzcd}
H(\epsilon;-) \arrow[r, "\eta_\epsilon"] \arrow[d, "\sigma"'] & \mathcal{C}(c_\epsilon,-) \arrow[d, "\mathcal{C}(\widehat{\sigma},-)"] \\
H(\epsilon';-) \arrow[r, "\eta_{\epsilon'}"'] & \mathcal{C}(c'_\epsilon,-)
\end{tikzcd}$$

In this case, the component of the natural transformation σ at $c \in v\mathcal{C}$ is the map given by

$$\sigma(c) : \epsilon * f^o \mapsto \epsilon' * (\widehat{\sigma} f)^o.$$

In particular, σ is the inclusion $H(\epsilon;-) \subseteq H(\epsilon';-)$ if and only if

$$\epsilon = \epsilon' * \widehat{\sigma}.$$

Moreover, the map $\sigma \mapsto \widehat{\sigma}$ is a bijection of $N^*\mathcal{C}(H(\epsilon;-), H(\epsilon';-))$ onto $\mathcal{C}(c'_\epsilon, c_\epsilon)$.

Proof. By Lemma 1.6.2, $\phi : N^*\mathcal{C}(H(\epsilon;-), H(\epsilon';-)) \to \mathcal{C}(c'_\epsilon, c_\epsilon)$

is an isomorphism between homsets for a normal category $\mathcal{C}$. Now since $RR-$categories are preadditive and since all normal categories are proper categories, $RR-$normal category $\mathcal{C}$ is a preadditive proper category and by Theorem 4.1.2 the homset $N^*\mathcal{C}(H(\epsilon;-), H(\epsilon';-))$ is isomorphic to $\mathcal{C}(c'_\epsilon, c_\epsilon)$. Hence the proof. $\square$

By Theorem 3.2.3, $RR-$normal category $\mathcal{C}$ is isomorphic to $\mathbb{L}(\mathcal{TC})$ so that $\mathbb{L}(\mathcal{TC})$ is an $RR-$normal category. Dually $\mathbb{R}(\mathcal{TC})$ is an $RR-$normal category. To prove that the dual category $N^*\mathcal{C}$ is an $RR-$normal category we show that it is isomorphic to $\mathbb{R}(\mathcal{TC})$. To prove that two $RR-$normal categories are isomorphic, we show that there is an isomorphic functor between the corresponding normal categories which preserves addition in homsets.

Lemma 4.1.5. Let $\mathcal{C}$ be an $RR-$normal category and $\epsilon, \epsilon' \in E(\mathcal{TC})$. Then the map $\lambda(\epsilon, \gamma, \epsilon') \mapsto \widetilde{\gamma}$ where $\gamma \in \epsilon'(\mathcal{TC})\epsilon$ and

$$\widetilde{\gamma} = \gamma(c_{\epsilon'})j(c_\gamma, c_\epsilon)$$

is a bijection of $\mathbb{R}(\mathcal{TC})(\epsilon(\mathcal{TC}), \epsilon'(\mathcal{TC}))$ onto $\mathcal{C}(c'_\epsilon, c_\epsilon)$.

Proof. By Lemma 1.6.3, for a normal category $\mathcal{C}$, there is a bijection between $\mathbb{R}(\mathcal{TC})(\epsilon(\mathcal{TC}), \epsilon'(\mathcal{TC}))$ and $\epsilon'(\mathcal{TC})\epsilon$ where $\mathcal{TC}$ is a regular semigroup. Now for an $RR-$normal category $\mathcal{C}$, $\mathcal{TC}$ is a regular ring. It is sufficient to show that the mapping $\phi : \gamma \to \widetilde{\gamma}$ preserves addition between $\epsilon'(\mathcal{TC})\epsilon$ and $\mathcal{C}(c'_\epsilon, c_\epsilon)$. For $\gamma, \delta \in \epsilon'(\mathcal{TC})\epsilon$, $\gamma + \delta \in \epsilon'(\mathcal{TC})\epsilon$, let

$d_0 = max\{im(\gamma+\delta)(a) | a \in v\mathcal{C}\} = c_{(\gamma+\delta)}$ and let $c_\gamma \vee c_\delta = v$.

$$\begin{aligned}
\phi(\gamma+\delta) &= \widetilde{(\gamma+\delta)} \\
&= (\gamma+\delta)(c_{\epsilon'})j(c_{(\gamma+\delta)}, c_\epsilon) \\
&= \big[\big[\gamma(c_{\epsilon'})j(c_\gamma, v) + \delta(c_{\epsilon'})j(c_\delta, v)\big]e(v, d_0)\big]j(c_{\gamma+\delta}, c_\epsilon) \\
&= \gamma(c_{\epsilon'})j(c_\gamma, v)e(v, d_0)j(d_0, c_\epsilon) + \delta(c_{\epsilon'})j(c_\delta, v)e(v, d_0)j(d_0, c_\epsilon) \\
&\quad [\text{since } c_{\gamma+\delta} = d_0] \\
&= \gamma(c_{\epsilon'})j(c_\gamma, c_\epsilon) + \delta(c_{\epsilon'})j(c_\delta, c_\epsilon) \\
&= \widetilde{\gamma} + \widetilde{\delta} \\
&= \phi(\gamma) + \phi(\delta)
\end{aligned}$$

Hence ϕ preserves addition between $\epsilon'(\mathcal{TC})\epsilon$ and $\mathcal{C}(c'_\epsilon, c_\epsilon)$ and we get

$$\widetilde{(\gamma+\delta)} = \widetilde{\gamma} + \widetilde{\delta} \tag{4.1}$$

□

Theorem 4.1.3. *Let $\mathcal{C}$ be an $RR-$normal category. Define G on objects and morphisms of $\mathbb{R}(\mathcal{TC})$ as*

$$vG(\epsilon(\mathcal{TC})) = H(\epsilon; -)$$

and for $\lambda = \lambda(\epsilon, \gamma, \epsilon') : \epsilon(\mathcal{TC}) \to \epsilon'(\mathcal{TC})$, let $G(\lambda)$ be the natural transformation making the following diagram commutative.

$$\begin{tikzcd}
H(\epsilon; -) \arrow[r, "\eta_\epsilon"] \arrow[d, "G(\lambda)"'] & \mathcal{C}(c_\epsilon, -) \arrow[d, "\mathcal{C}(\widetilde{\gamma}, -)"] \\
H(\epsilon'; -) \arrow[r, "\eta_{\epsilon'}"'] & \mathcal{C}(c'_\epsilon, -)
\end{tikzcd}$$

Then $G : \mathbb{R}(\mathcal{TC}) \to N^*\mathcal{C}$ is an isomorphism of $RR-$normal categories.

Proof. By Theorem 1.6.4, $G : \mathbb{R}(\mathcal{TC}) \to N^*\mathcal{C}$ is an isomorphism of normal categories. Now to prove that G is an isomorphism of $RR-$normal categories it is sufficient to prove that the functor G preserves addition between homsets.

$$\begin{array}{ccccc} \epsilon\mathcal{TC} & \xrightarrow{G} & H(\epsilon;-) & \xrightarrow{\eta_\epsilon} & \mathcal{C}(c_\epsilon,-) \\ {\scriptstyle\lambda}\downarrow & & {\scriptstyle G(\lambda)}\downarrow & & \downarrow{\scriptstyle\mathcal{C}(\widetilde{\gamma},-)} \\ \epsilon'\mathcal{TC} & \xrightarrow{G} & H(\epsilon';-) & \xrightarrow[\eta_{\epsilon'}]{} & \mathcal{C}(c'_\epsilon,-) \end{array}$$

From the diagram the definition of $G(\lambda)$ is equivalent to

$$G(\lambda) = \eta_\epsilon \mathcal{C}(\widetilde{\gamma},-)\eta_{\epsilon'}^{-1}.$$

Let $\lambda_1 = \lambda(\epsilon,\gamma,\epsilon')$ and $\lambda_2 = \lambda(\epsilon,\delta,\epsilon')$ mapping from $\epsilon(\mathcal{TC})$ to $\epsilon'(\mathcal{TC})$. Then $\lambda_1+\lambda_2 = \lambda(\epsilon,\gamma+\delta,\epsilon') : \epsilon(\mathcal{TC}) \to \epsilon'(\mathcal{TC})$.

$$\begin{aligned} G(\lambda_1+\lambda_2) &= \eta_\epsilon\mathcal{C}(\widetilde{(\gamma+\delta)},-)\eta_{\epsilon'}^{-1} \\ &= \eta_\epsilon\mathcal{C}(\widetilde{\gamma}+\widetilde{\delta},-)\eta_{\epsilon'}^{-1} \\ &[\text{ since } \widetilde{(\gamma+\delta)} = \widetilde{\gamma}+\widetilde{\delta} \text{ by Equation 4.1 }] \\ &= \eta_\epsilon\mathcal{C}(\widetilde{\gamma},-)\eta_{\epsilon'}^{-1} + \eta_\epsilon\mathcal{C}(\widetilde{\delta},-)\eta_{\epsilon'}^{-1} \\ &= G(\lambda_1)+G(\lambda_2) \end{aligned}$$

Hence G is an isomorphism of $RR-$normal categories $\mathbb{R}(\mathcal{TC})$ and $N^*\mathcal{C}$. $\square$

Thus for an $RR-$normal category $\mathcal{C}$, $\mathcal{C} \cong \mathbb{L}(\mathcal{TC})$ and $\mathbb{R}(\mathcal{TC}) \cong N^*\mathcal{C}$. Hence the dual category $N^*\mathcal{C}$ is an $RR-$normal category.

Chapter 5

Cross-connection of $RR-$normal categories

The cross-connection of normal categories is described in Section 1.7 and the cross-connection semigroup-which turns out to be a regular semigroup and thus obtained a beautiful structure theorem for regular semigroups. Here we generalize this approach by describing the cross-connections of $RR-$normal categories and obtain a regular ring.

5.1 Cross-connection

In Section 3.2.3, $RR-$normal category is described and now we proceed to describe cross-connection of $RR-$normal categories. In an $RR-$normal category $\mathcal{C}$ the set of all normal cones $\mathcal{TC}$ is a regular ring and $\mathbb{L}(\mathcal{TC})$ is an $RR-$normal category. Theorem 3.2.3 establishes the isomorphism of $\mathcal{C}$ with $\mathbb{L}(\mathcal{TC})$. RR-normal dual category $N^*\mathcal{C}$ is discussed in Section 4.1.3 and by Theorem 4.1.3, $\mathbb{R}(\mathcal{TC})$ is isomorphic to $N^*\mathcal{C}$.

Definition 5.1.1. Let $\mathcal{C}$ and $\mathcal{D}$ be $RR-$normal categories. A functor $F : \mathcal{C} \to \mathcal{D}$ is said to be a *local isomorphism* if F is inclusion preserving, fully-faithful, addition preserving in each homset and for each $c \in \imath\mathcal{C}, F|_{\langle c\rangle}$ is an isomorphism of the ideal $\langle c\rangle$ onto $\langle F(c)\rangle$.

Let R be a (von Neumann) regular ring. The category $\mathbb{L}(R)[\mathbb{R}(R)]$ of principal left [right] ideals of R are $RR-$normal categories (see Section 3.2.3). The set of all normal cones $\mathcal{T}\mathbb{L}(R)[\mathcal{T}\mathbb{R}(R)]$ of $RR-$normal category $\mathbb{L}(R)[\mathbb{R}(R)]$ is a regular ring. By Theorem 3.2.4, it is seen that there is a ring homomorphism $\bar{\rho} : a \mapsto \rho^a$ of R into $\mathcal{T}\mathbb{L}(R)$. The following Proposition shows that the category of principal right ideals $\mathbb{R}(\mathcal{T}\mathbb{L}(R))$ is an $RR-$normal category by showing that it is isomorphic to $\mathbb{R}(R)$.

Proposition 5.1.1. For any regular ring R, $FR_\rho : \mathbb{R}(R) \to \mathbb{R}(\mathcal{T}\mathbb{L}(R))$ defined by $FR_\rho(eR) = \rho^e(\mathcal{T}\mathbb{L}(R))$ and $FR_\rho(\lambda(e,u,f)) = \lambda(\rho^e,\rho^u,\rho^f)$ is a local isomorphism and dually $FR_\lambda : \mathbb{L}(R) \to \mathbb{L}(\mathcal{T}\mathbb{R}(R))$ defined by $FR_\lambda(Re) = (\mathcal{T}\mathbb{R}(R))\lambda^e$ and $FR_\lambda(\rho(e,u,f)) = \rho(\lambda^e,\lambda^u,\lambda^f)$ is also a local isomorphism.

Proof. In Proposition 1.7.3, it is shown that $FS_\rho : \mathbb{R}(S) \to \mathbb{R}(\mathcal{T}\mathbb{L}(S))$ is a local isomorphism where $\mathbb{R}(S), \mathbb{L}(S)$ are ideal categories of a regular semigroup S. To prove the result for a regular ring R and the corresponding $RR-$normal categories $\mathbb{L}(R)$ and $\mathbb{R}(R)$ it is sufficient to prove that the functors defined above preserves addition between homsets.
Let $\lambda_1 = \lambda(e,u,f)$ and $\lambda_2 = \lambda(e,v,f)$ belong to $\mathbb{R}(R)(eR, fR)$

$$\begin{aligned}
FR_\rho(\lambda_1+\lambda_2) &= FR_\rho(\lambda(e,u,f)+\lambda(e,v,f)) \\
&= FR_\rho(\lambda(e,(u+v),f)) \\
&= \lambda(\rho^e,\rho^{u+v},\rho^f) \\
&= \lambda(\rho^e,\rho^u+\rho^v,\rho^f) \\
&\text{[since } \rho^{u+v} = \rho^u+\rho^v] \\
&= \lambda(\rho^e,\rho^u,\rho^f)+\lambda(\rho^e,\rho^v,\rho^f) \\
&= FR_\rho(\lambda_1)+FR_\rho(\lambda_2)
\end{aligned}$$

Hence $FR_\rho : \mathbb{R}(R) \to \mathbb{R}(\mathcal{T}\mathbb{L}(R))$ is a local isomorphism between the $RR-$normal categories $\mathbb{R}(R)$ and $\mathbb{R}(\mathcal{T}\mathbb{L}(R))$. □

Theorem 5.1.1. *Let R be a regular ring and $\mathbb{L}(R)[\mathbb{R}(R)]$ be the $RR-$normal category of principal left [right] ideals of R. For $e, f \in E(R), fR \in v\mathbb{R}(R)$ and $\lambda = \lambda(e, u, f)$ in $\mathbb{R}(R)$, let ΓR be defined on objects and morphisms of $\mathbb{R}(R)$ by :*

$$\Gamma R(fR) = H(\rho^f; -), \Gamma R(\lambda) = \eta_{\rho^e}\mathbb{L}(R)(\rho(f, u, e), -)\eta_{\rho^f}^{-1}.$$

Then ΓR is a local isomorphism from $\mathbb{R}(R)$ to $N^*\mathbb{L}(R)$. Dually, $\Gamma^* R$, defined on objects and morphisms of $\mathbb{L}(R)$ by

$$\Gamma^* R(Re) = H(\lambda^e; -), \Gamma^* R(\rho) = \eta_{\lambda^f}\mathbb{R}(R)(\lambda(e, u, f), -)\eta_{\lambda^e}^{-1}.$$

for all $Re \in v\mathbb{L}(R)$ and $\rho = \rho(f, u, e) \in \mathbb{L}(R)$, defines a local isomorphism.

Proof. We have $\mathbb{L}(R)$ and $\mathbb{R}(R)$ are $RR-$normal categories and $\mathcal{T}\mathbb{L}(R)$ is a regular ring. By Proposition 5.1.1, $FR_\rho : \mathbb{R}(R) \to \mathbb{R}(\mathcal{T}\mathbb{L}(R))$ is a local isomorphism. Also by Theorem 4.1.3, $G : \mathbb{R}(\mathcal{T}\mathbb{L}(R)) \to N^*\mathbb{L}(R)$ is also a local isomorphism. Then obviously $\Gamma R = FR_\rho \cdot G$ is a local isomorphism from $\mathbb{R}(R)$ to $N^*\mathbb{L}(R)$. Dually $\Gamma^* R$ is also a local isomorphism. □

For a regular ring R, the ideal categories $\mathbb{L}(R)$ and $[\mathbb{R}(R)]$ are $RR-$normal categories and the corresponding dual categories $N^*\mathbb{L}(R)$ and $[N^*\mathbb{R}(R)]$ are also $RR-$normal categories. $\Gamma R(-, -) : \mathbb{L}(R) \times \mathbb{R}(R) \to$ **Set** is a bifunctor associated with the local isomorphism $\Gamma R : \mathbb{R}(R) \to N^*\mathbb{L}(R)$ defined on objects and morphisms as follows:

$$\Gamma R(Re, fR) = \Gamma R(fR)(Re);$$

$$\Gamma R(\rho, \lambda) = \Gamma R(fR)(\rho)\Gamma R(\lambda)(Re') = \Gamma R(\lambda)(Re)\Gamma R(f'R)(\rho)$$

for all $(Re, fR) \in v\mathbb{L}(R) \times \mathbb{R}(R)$ and $(\rho, \lambda) : (Re, fR) \to (Re', f'R)$.

Proposition 5.1.2. The bifunctor $\Gamma R(-,-) : \mathbb{L}(R) \times \mathbb{R}(R) \to$ **Set** defined above is addition preserving.

Proof. Let $\rho_1, \rho_2, \rho_1 + \rho_2 : Re \to Re'$ and $\lambda_1, \lambda_2, \lambda_1 + \lambda_2 : fR \to f'R$

$$\begin{aligned}
\Gamma R(\rho_1 + \rho_2, \lambda_1 + \lambda_2) &= \Gamma R(fR)(\rho_1 + \rho_2)\Gamma R(\lambda_1 + \lambda_2)(Re') \\
&= H(\rho^f, \rho_1 + \rho_2)\Gamma R(\lambda_1 + \lambda_2)(Re') \\
&= [H(\rho^f, \rho_1) + H(\rho^f, \rho_2)][\Gamma R(\lambda_1)(Re') + \Gamma R(\lambda_2)(Re')] \\
&= H(\rho^f, \rho_1)\Gamma R(\lambda_1)(Re') + H(\rho^f, \rho_2)\Gamma R(\lambda_2)(Re') \\
&= \Gamma R(\rho_1, \lambda_1) + \Gamma R(\rho_2, \lambda_2)
\end{aligned}$$

Thus $\Gamma R(-,-)$ is an addition preserving bifunctor between the categories $\mathbb{L}(R) \times \mathbb{R}(R)$ and **Set**. $\square$

Similarly $\Gamma^* R(-,-) : \mathbb{L}(R) \times \mathbb{R}(R) \to$ **Set** is also a bifunctor associated with the local isomorphism $\Gamma^* R : \mathbb{L}(R) \to N^*\mathbb{R}(R)$ defined on objects and morphisms as follows:

$$\Gamma^* R(Re, fR) = \Gamma^* R(Re)(fR);$$

$$\Gamma^* R(\rho, \lambda) = \Gamma^* R(Re)(\lambda)\Gamma^* R(\rho)(f'R) = \Gamma^* R(\rho)(fR)\Gamma^* R(Re')(\lambda)$$

for all $(Re, fR) \in v\mathbb{L}(R) \times \mathbb{R}(R)$ and $(\rho, \lambda) : (Re, fR) \to (Re', f'R)$.

Theorem 5.1.2. *Let R be a regular ring. $\Gamma R(-,-)$ and $\Gamma^* R(-,-)$ are two set valued bifunctors defined as above. Then there is a natural isomorphism χ_R from $\Gamma R(-,-)$ to $\Gamma^* R(-,-)$ whose components are defined by*

$$\chi_R(Re, fR) : \rho^f \star \rho(f, u, e)^o \mapsto \lambda^e \star \lambda(e, u, f)^o$$

for each $(Re, fR) \in v\big(\mathbb{L}(R) \times \mathbb{R}(R)\big)$.

Proof. By Theorem 1.7.5, for a regular semigroup S, χ_S is a natural isomorphism between $\Gamma S(-,-)$ and $\Gamma^* S(-,-)$ whose components are defined by

$$\chi_S(Se, fS) : \rho^f \star \rho(f,u,e)^o \mapsto \lambda^e \star \lambda(e,u,f)^o$$

for each $(Se, fS) \in v\big(\mathbb{L}(S) \times \mathbb{R}(S)\big)$. By above Proposition 5.1.2, ΓR and $\Gamma^* R$ are addition preserving bifunctors. So obviously χ_R is also addition preserving. $\square$

$\Gamma R : \mathbb{R}(R) \to N^*\mathbb{L}(R)$ is the *connection* of R and $\Gamma^* R : \mathbb{L}(R) \to N^*\mathbb{R}(R)$ is the *dual connection* of R.

Proposition 5.1.3 (cf.[25])**.** Let $\Gamma : \mathcal{D} \to N^*\mathcal{C}$ be a *connection* between $RR-$normal categories $\mathcal{C}$ and $\mathcal{D}$ and let $\mathcal{C}_\Gamma$ be the subcategory of $\mathcal{C}$ such that

$$v\mathcal{C}_\Gamma = \{c \in \mathcal{C} : c \in M\Gamma(d) \text{ for some } d \in v\mathcal{D}\},$$

where $M\Gamma(d)$ is the $M-$set of normal cone with vertex d. Then $\mathcal{C}_\Gamma$ is an ideal in $\mathcal{C}$.

Definition 5.1.2. Let $\mathcal{C}$ and $\mathcal{D}$ be $RR-$normal categories. A *cross-connection* is a triplet $(\mathcal{D}, \mathcal{C}; \Gamma)$ where $\Gamma : \mathcal{D} \to N^*\mathcal{C}$ is a *local isomorphism* such that for every $c \in v\mathcal{C}$ there is some $d \in v\mathcal{D}$ such that $c \in M\Gamma(d)$.

Theorem 5.1.3. *Let R be a regular ring. Then the connection ΓR of R is a cross-connection of $\mathbb{R}(R)$ and $\mathbb{L}(R)$. Moreover $\Gamma^* R$ is the dual of ΓR.*

Proof. By Theorem 1.7.6, $\Gamma S : \mathbb{R}(S) \to N^*\mathbb{L}(S)$ is a cross-connection for a regular semigroup S and $\Gamma^* S$ is the dual of ΓS. By Theorem 5.1.1, $\Gamma R : \mathbb{R}(R) \to N^*\mathbb{L}(R)$ is a cross-connection for a regular ring R. Dually $\Gamma^* R$ is the dual cross-connection of ΓR. $\square$

5.2 Cross-connection regular ring

Cross-connection of $RR-$normal categories is discussed in Section 5.1. Now we proceed to describe the regular ring corresponding to the cross-connection which we call the *cross-connection regular ring*.

Let $\mathcal{C}$ and $\mathcal{D}$ be $RR-$normal categories. $\Gamma : \mathcal{D} \to N^*\mathcal{C}$ is a cross-connection and $\Gamma^* : \mathcal{C} \to N^*\mathcal{D}$ be its dual cross-connection. Define

$$E_\Gamma = \{(c, d) : c \in v\mathcal{C}_\Gamma, d \in v\mathcal{D} \text{ and } c \in M\Gamma(d)\}.$$

For each $(c, d) \in E_\Gamma, \gamma(c, d)$ denotes the unique cone in $\mathcal{C}$ such that $c_{\gamma(c,d)} = c$ and $\Gamma(d) = H(\gamma(c, d); -)$. Similarly for each $(c, d) \in E_\Gamma$, there is a unique cone $\gamma^*(c, d)$ in $\mathcal{D}$ such that $c_{\gamma^*(c,d)} = d$ and $\Gamma^*(c) = H(\gamma^*(c, d); -)$.

Let $(c, d) \in vC_\Gamma \times v\mathcal{D}$ and choose $c' \in C_\Gamma$ and $d' \in v\mathcal{D}$ such that $(c, d'), (c', d) \in E_\Gamma$. Then every cone in $\Gamma(c, d)$ can be represented as $\gamma(c', d) \star f^o$ with $f \in \mathcal{C}(c', c)$ and every element of $\Gamma^*(c, d)$ can be written as $\gamma^*(c, d') \star g^o$ with $g \in \mathcal{D}(d', d)$.

Hence for every $(c, d) \in v\mathcal{C}_\Gamma \times v\mathcal{D}$ and $\gamma(c', d) \star f^o \in \Gamma(c, d)$, we have natural isomorphism

$$\chi\Gamma(c, d)(\gamma(c', d) \star f^o) = \gamma^*(c, d') \star g^o$$

where $(c, d), (c', d') \in E_\Gamma$ and $f \in \mathcal{C}(c, c'), g \in \mathcal{D}(d', d)$ are such that the following diagram commutes.

$$\begin{tikzcd}
\Gamma(d') \arrow[r, "\eta_{\gamma(c,d')}"] \arrow[d, "\Gamma(g)"'] & \mathcal{C}(c, -) \arrow[d, "\mathcal{C}(f,-)"] \\
\Gamma(d) \arrow[r, "\eta_{\gamma(c',d)}"'] & \mathcal{C}(c', -)
\end{tikzcd}$$

Let Γ be a cross-connection of $\mathcal{D}$ with $\mathcal{C}$. Define

$$U\Gamma = \bigcup\{\gamma(c, d) : (c, d) \in v\mathcal{C} \times v\mathcal{D}\};$$

$$U\Gamma^* = \bigcup\{\gamma^*(c,d) : (c,d) \in v\mathcal{C} \times v\mathcal{D}\}.$$

For any cross-connection $\Gamma : \mathcal{D} \to N^*\mathcal{C}$, $U\Gamma$ is a regular subring of $\mathcal{TC}$ such that

$$E(U\Gamma) = \{\gamma(c,d) : (c,d) \in E_\Gamma\}.$$

Definition 5.2.1. Given a cross-connection $\Gamma : \mathcal{D} \to N^*\mathcal{C}$, between $RR-$normal categories $\mathcal{C}$ and $\mathcal{D}$, a cone $\gamma \in U\Gamma$ is *linked* to $\gamma^* \in U\Gamma^*$ or (γ, γ^*) is *linked* relative to Γ if there is $(c,d) \in v\mathcal{C} \times v\mathcal{D}$ such that

$$\gamma \in \Gamma(c,d) \text{ and } \gamma^* = \chi_{\Gamma(c,d)}(\gamma).$$

Theorem 5.2.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be two $RR-$normal categories and $\Gamma : \mathcal{D} \to N^*\mathcal{C}$ be a cross-connection. Then $\widetilde{R}\Gamma = \{(\gamma, \gamma^*) \in \Gamma \times \Gamma^* :$ (γ, γ^*) is linked $\}$ is a regular ring with binary operation defined by*

$$(\gamma, \gamma^*)(\delta, \delta^*) = (\gamma.\delta, \delta^*.\gamma^*) \text{ and}$$

$$(\gamma, \gamma^*) + (\delta, \delta^*) = (\gamma + \delta, \gamma^* + \delta^*)$$

for all $(\gamma, \gamma^*), (\delta, \delta^*) \in \widetilde{R}\Gamma$.

Proof. By Theorem 3.2.2, for an $RR-$normal category $\mathcal{C}$ the set of all normal cones $\mathcal{TC}$ is a regular ring with respect to the multiplication and addition of normal cones are defined by: for $\gamma, \delta \in \mathcal{TC}$ with $c_\gamma = c, c_\delta = d$ and for each $a \in v\mathcal{C}$, $(\gamma \cdot \delta)(a) = \gamma(a) \cdot \delta(c)^o$ and $(\gamma + \delta)(a) = (\gamma \oplus \delta)^*(a)$ where $(\gamma \oplus \delta)^*(a) = [\gamma(a)j(c, c \vee d) + \delta(a)j(d, c \vee d)]e(c \vee d, d_0)$ and $d_0 = max\{im(\gamma \oplus \delta)(a) | a \in v\mathcal{C}\}$ is the vertex of the normal cone $\gamma + \delta$.

Since $\mathcal{TC}$ and $\mathcal{TD}$ are regular rings for $RR-$normal categories $\mathcal{C}$ and $\mathcal{D}$ respectively, $\mathcal{TC} \times \mathcal{TD}$ is also a regular ring and $\widetilde{R}\Gamma$ is a regular subring of $\mathcal{TC} \times \mathcal{TD}$. □

Example 5.2.1. Cross-connection of the Matrix Ring $M_2(\mathbb{Z}_2)$
Consider the regular ring $R = M_2(\mathbb{Z}_2)$. This ring has 16 elements of

which there are 8 idempotents. The idempotents of R denoted as $E(R)$ are listed below.

$$e_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, e_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, e_3 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$

$$e_4 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, e_5 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, e_6 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, e_7 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

The egg-box picture of the idempotents is the following:

1				
	e_6	e_5		
	e_2		e_4	
		e_1	e_3	
				0

The elements other than idempotents are

$$a_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, a_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, a_3 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

$$a_4 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, a_5 = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, a_6 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, a_7 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, a_8 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$$

$\mathbb{L}(R)$ is the category of idempotent generated principal left ideals of ring R, its object set $v\mathbb{L}(R)$ consists of the following.

$$I_0 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}, R = M_2(\mathbb{Z}_2),$$

$$I_1 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \right\}$$

$$I_2 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$$

$$I_3 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$

Recall that the morphisms of the category $\mathbb{L}(R)$ are defined by $\rho_u = \rho(e,u,f) : Re \to Rf$ is the translation $x \to xu$ where $u \in eRf$. Hence the homsets are

$$hom(I_1, I_1) = \{\rho_{e_0}, \rho_{e_1}\}, hom(I_1, I_2) = \{\rho_{e_0}, \rho_{a_1}\},$$

$$hom(I_2, I_2) = \{\rho_{e_0}, \rho_{e_2}\}, hom(I_1, I_3) = \{\rho_{e_0}, \rho_{e_3}\},$$

$$hom(I_3, I_3) = \{\rho_{e_0}, \rho_{e_3}\}, hom(I_2, I_1) = \{\rho_{e_0}, \rho_{a_2}\},$$

$$hom(I_3, I_1) = \{\rho_{e_0}, \rho_{e_1}\}, hom(I_2, I_3) = \{\rho_{e_0}, \rho_{e_4}\},$$

$$hom(I_3, I_2) = \{\rho_{e_0}, \rho_{a_1}\}, hom(I_1, I_0) = \{\rho_{e_0}\},$$

$$hom(I_2, I_0) = \{\rho_{e_0}\}, hom(I_3, I_0) = \{\rho_{e_0}\},$$

$$hom(R, I_0) = \{\rho_{e_0}\}, hom(R, I_1) = \{\rho_{e_0}, \rho_{e_1}, \rho_{a_2}, \rho_{e_5}\},$$

$$hom(I_1, R) = \{\rho_{e_0}, \rho_{e_1}, \rho_{a_1}, \rho_{e_3}\}, hom(R, I_2) = \{\rho_{e_0}, \rho_{a_1}, \rho_{e_2}, \rho_{e_6}\},$$

$$hom(I_2, R) = \{\rho_{e_0}, \rho_{a_2}, \rho_{e_2}, \rho_{e_4}\}, hom(R, I_3) = \{\rho_{e_0}, \rho_{e_3}, \rho_{e_4}, \rho_{a_3}\},$$

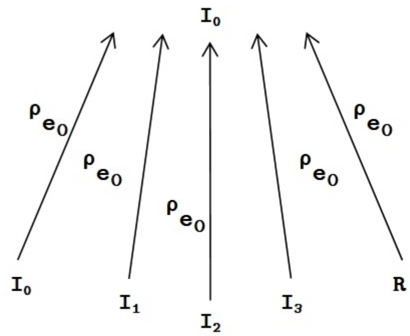
$$hom(I_3, R) = \{\rho_{e_0}, \rho_{e_1}, \rho_{a_1}, \rho_{e_3}\}, hom(R, R) = \{\rho_u | u \in R\}$$

Note that $hom(R,R) = \{\rho_u | u \in R\}$ is also a von Neumann regular ring with respect to the addition and multiplication defined by

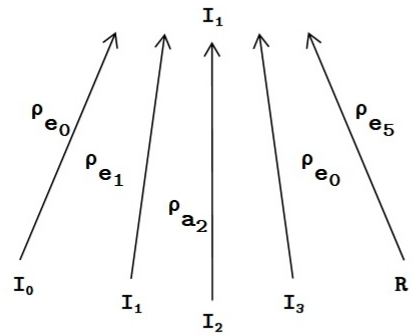
$$\rho_u + \rho_v = \rho_{u+v} \text{ and}$$

$$\rho_u \cdot \rho_v = \rho_{uv}.$$

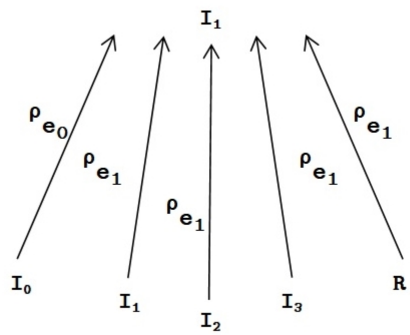
The normal cones in $\mathbb{L}(R)$ are the following:



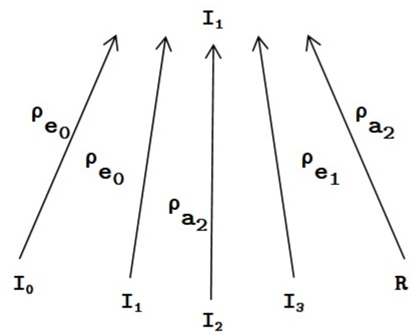
γ_0



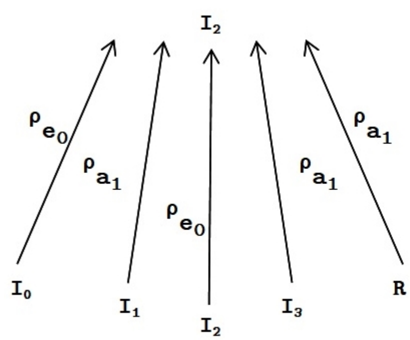
γ_1



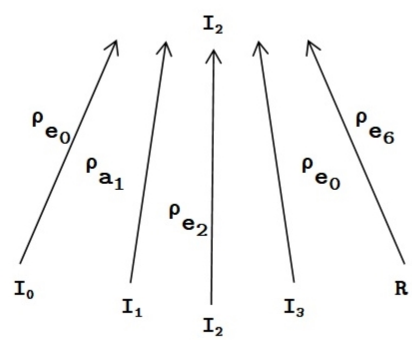
γ_2



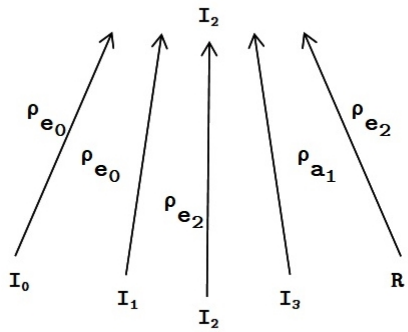
γ_3



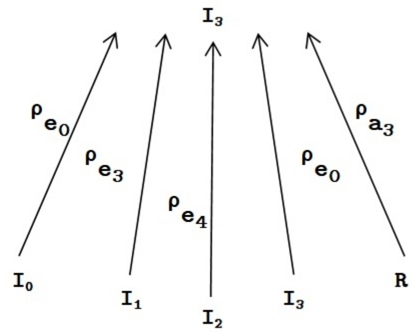
β_1



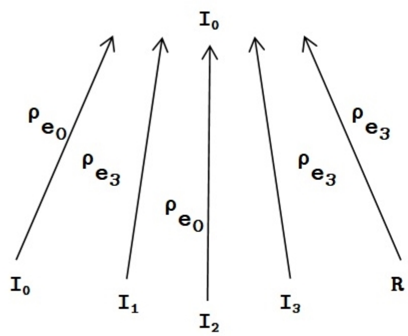
β_2



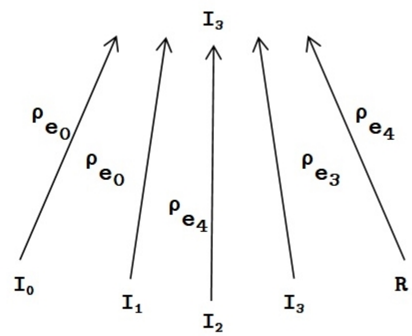
β_3



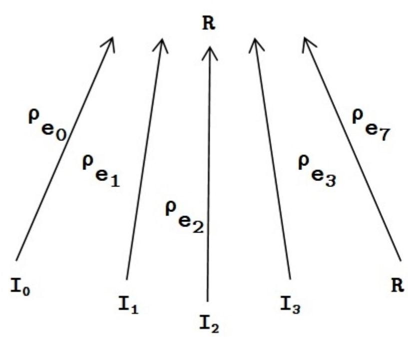
δ_1



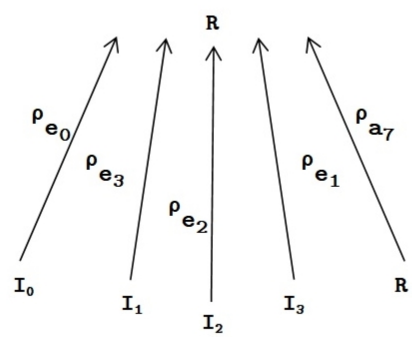
δ_2



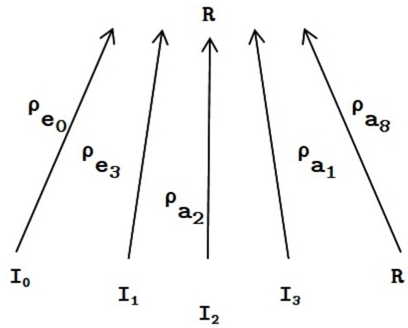
δ_3



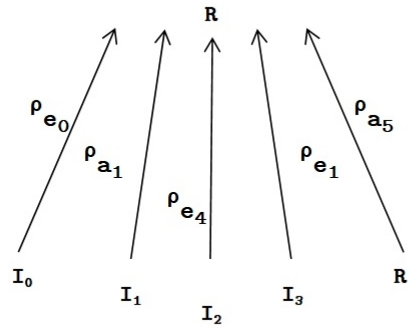
α_1



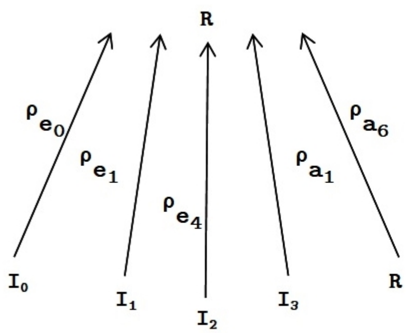
α_2



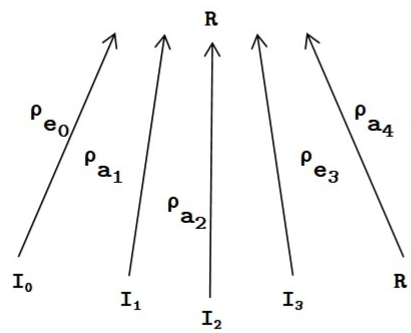
α_3



α_4



α_5



α_6

The set of all normal cones

$$\mathcal{T}\mathbb{L}(\mathcal{R}) = \{\gamma_0, \gamma_1, \gamma_2, \gamma_3, \beta_1, \beta_2, \beta_3, \delta_1, \delta_2, \delta_3, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$$

and it is a regular ring isomorphic to R and $hom(R, R)$. It is easily seen that $\gamma_1 + \gamma_2 = \gamma_3, \beta_1 + \gamma_1 = \alpha_3, \delta_2 + \beta_1 = \gamma_2, \alpha_3 + \delta_2 = \gamma_3$ and so on.

The idempotents in $\mathcal{T}\mathbb{L}(\mathcal{R})$ are

$$E(\mathcal{T}\mathbb{L}(\mathcal{R})) = \{\gamma_0, \gamma_1, \gamma_2, \beta_2, \beta_3, \delta_2, \delta_3, \alpha_1\}.$$

Further $\mathbb{L}(R) \cong \mathbb{L}(\mathcal{T}\mathbb{L}(\mathcal{R}))$ and the correspondence is the following:
$I_0 \to \mathcal{T}\mathbb{L}(\mathcal{R})\gamma_0, I_1 \to \mathcal{T}\mathbb{L}(\mathcal{R})\gamma_1, I_2 \to \mathcal{T}\mathbb{L}(\mathcal{R})\beta_2, I_3 \to \mathcal{T}\mathbb{L}(\mathcal{R})\delta_2, R \to \mathcal{T}\mathbb{L}(\mathcal{R})\alpha_1$.

The dual category $N^*\mathbb{L}(R)$ is the category with

$$vN^*\mathbb{L}(R) = \{H(\gamma_i;-), H(\beta_j;-), H(\delta_j;-), H(\alpha_1;-); i = 0,1,2; j = 2,3\}$$

and morphisms are appropriate natural transformations.
The principal right ideals generated by idempotents are

$$I_0 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}, R = M_2(\mathbb{Z}_2)$$

,

$$J_1 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \right\}$$

$$J_2 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \right\}$$

$$J_3 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$$

The category whose objects are these principal right ideals and morphisms are of the form $\lambda_u = \lambda(e,u,f) : eR \to fR$ is the translation $x \to ux$ where $u \in fRe$ is the normal category $\mathbb{R}(R)$.
The homsets are the following.

$$hom(J_1, J_1) = \{\lambda_{e_0}, \lambda_{e_1}\}, hom(J_1, J_2) = \{\lambda_{e_0}, \lambda_{a_2}\},$$

$$hom(J_2, J_2) = \{\lambda_{e_0}, \lambda_{e_2}\}, hom(J_1, J_3) = \{\lambda_{e_0}, \lambda_{e_5}\},$$

$$hom(J_3, J_3) = \{\lambda_{e_0}, \lambda_{e_5}\}, hom(J_2, J_1) = \{\lambda_{e_0}, \lambda_{a_1}\},$$

$$hom(J_3, J_1) = \{\lambda_{e_0}, \lambda_{e_1}\}, hom(J_2, J_3) = \{\lambda_{e_0}, \lambda_{e_6}\},$$

$$hom(J_3, J_2) = \{\lambda_{e_0}, \lambda_{a_2}\}, hom(J_1, I_0) = \{\lambda_{e_0}\},$$

$$hom(J_2, I_0) = \{\lambda_{e_0}\}, hom(J_3, I_0) = \{\lambda_{e_0}\},$$

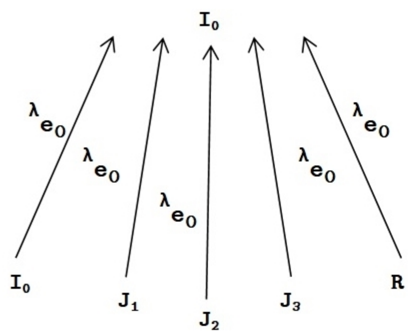
$$hom(R, I_0) = \{\lambda_{e_0}\}, hom(R, J_1) = \{\lambda_{e_0}, \lambda_{e_1}, \lambda_{a_1}, \lambda_{e_3}\},$$

$$hom(J_1, R) = \{\lambda_{e_0}, \lambda_{e_1}, \lambda_{a_2}, \lambda_{e_5}\}, hom(R, J_2) = \{\lambda_{e_0}, \lambda_{e_2}, \lambda_{a_2}, \lambda_{e_4}\},$$

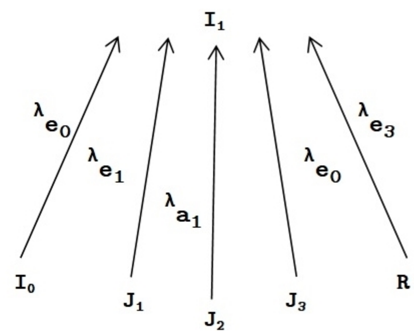
$$hom(J_2, R) = \{\lambda_{e_0}, \lambda_{a_1}, \lambda_{e_2}, \lambda_{a_6}\}, hom(R, J_3) = \{\lambda_{e_0}, \lambda_{e_5}, \lambda_{e_6}, \lambda_{a_3}\},$$

$$hom(J_3, R) = \{\lambda_{e_0}, \lambda_{e_1}, \lambda_{a_2}, \lambda_{e_5}\}, hom(R, R) = \{\lambda_u | u \in R\}$$

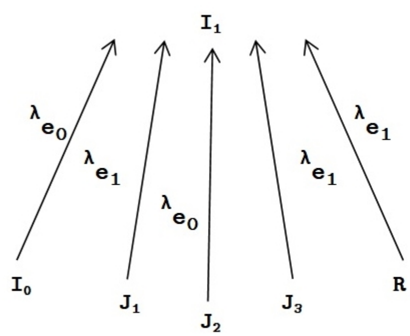
The set of all normal cones $\mathcal{T}\mathbb{R}(\mathcal{R})$ in the category $\mathbb{R}(R)$ are



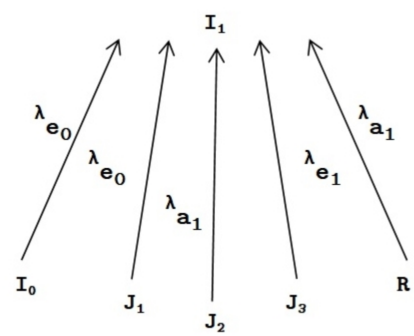
γ_0,



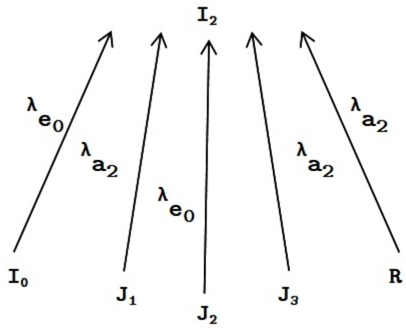
γ_1,



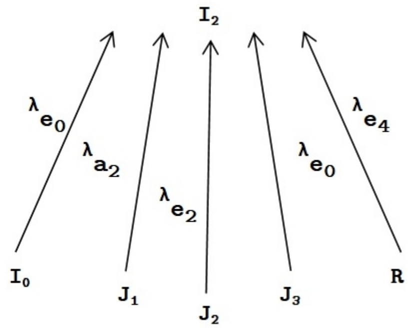
γ_2,



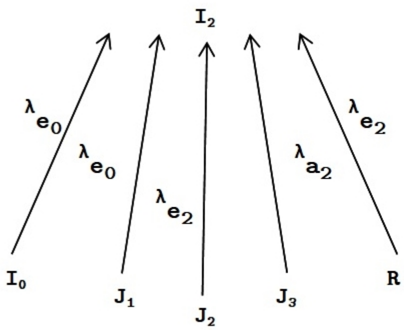
γ_3,



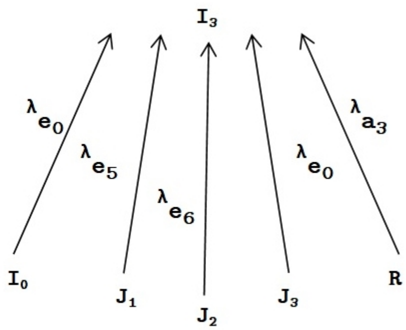
β_1'



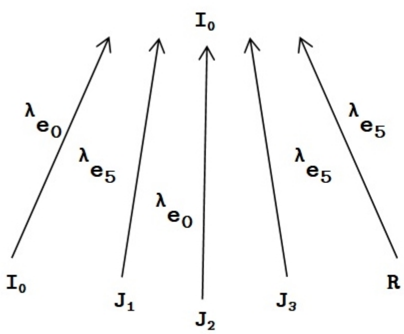
β_2'



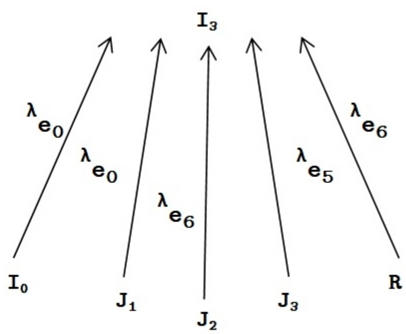
β_3'



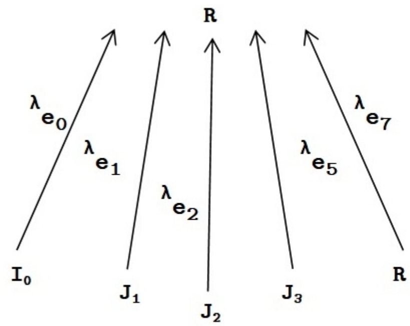
δ_1'



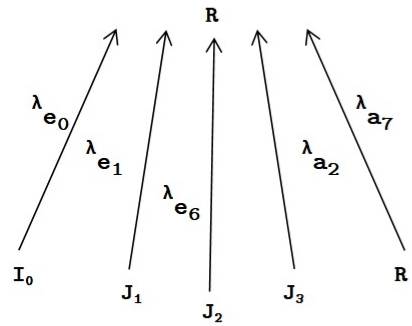
δ_2'



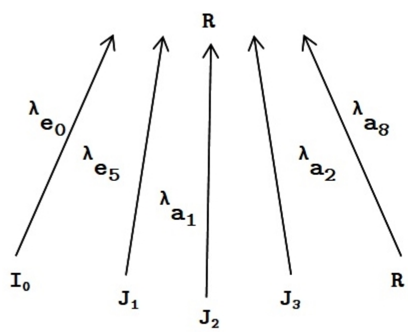
δ_3'



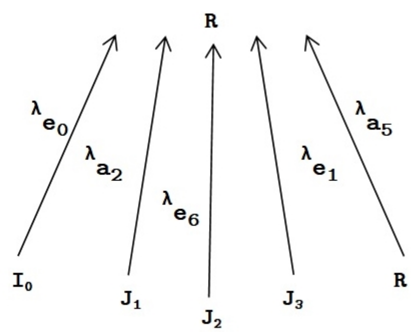
α_1'



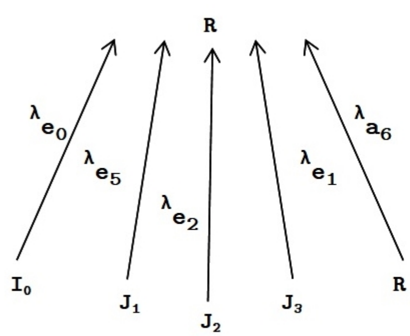
α_2'



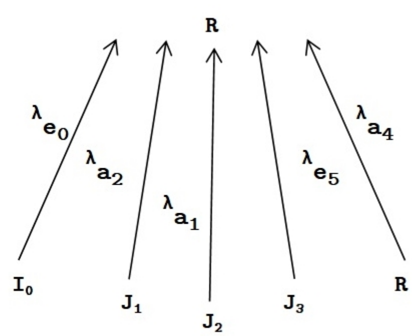
α_3'



α_4'



α_5'



α_6'

$$\mathcal{T}\mathbb{R}(\mathcal{R}) = \{\gamma_0', \gamma_1', \gamma_2', \gamma_3', \beta_1', \beta_2', \beta_3', \delta_1', \delta_2', \delta_3', \alpha_1', \alpha_2', \alpha_3', \alpha_4', \alpha_5', \alpha_6'\}$$

and it is a regular ring isomorphic to R and $hom(R, R)$. The idempotent set $E(\mathcal{T}\mathbb{R}(\mathcal{R})) = \{\gamma_0', \gamma_1', \gamma_2', \beta_2', \beta_3', \delta_2', \delta_3', \alpha_1'\}$.

Further $N^*\mathbb{L}(R) \cong \mathbb{R}(\mathcal{T}\mathbb{L}(\mathcal{R}))$ and the correspondence is given by $H(\gamma_0;-) \to \gamma_0\mathcal{T}\mathbb{L}(\mathcal{R}), H(\gamma_1;-) \to \gamma_1\mathcal{T}\mathbb{L}(\mathcal{R}), H(\gamma_2;-) \to \gamma_2\mathcal{T}\mathbb{L}(\mathcal{R}),$ $H(\beta_2;-) \to \beta_2\mathcal{T}\mathbb{L}(\mathcal{R}), H(\beta_3;-) \to \beta_3\mathcal{T}\mathbb{L}(\mathcal{R}), H(\delta_2;-) \to \delta_2\mathcal{T}\mathbb{L}(\mathcal{R}),$ $H(\delta_3;-) \to \delta_3\mathcal{T}\mathbb{L}(\mathcal{R}), H(\alpha_1;-) \to \alpha_1\mathcal{T}\mathbb{L}(\mathcal{R}).$

Define $\Gamma R : \mathbb{R}(R) \to N^*\mathbb{L}(R)$ on objects and morphisms of $\mathbb{R}(R)$ by

$$\Gamma R(fR) = H(\rho^f;-), \Gamma R(\lambda) = \eta_{\rho^e}\mathbb{L}(R)(\rho(f,u,e),-)\eta_{\rho^f}^{-1}$$

where $\lambda = \lambda(e,u,f)$ in $\mathbb{R}(R)$. ρ^f is the normal cone in $\mathbb{L}(R)$ with vertex Rf. The diagram given below commutes

$$\begin{tikzcd}
eR \arrow[r, "\Gamma R"] \arrow[d, "\lambda"'] & H(\rho^e;-) \arrow[r, "\eta_{\rho^e}"] \arrow[d, "\Gamma R(\lambda)"'] & \mathbb{L}(R)(Re,-) \arrow[d, "\mathbb{L}(R)(\rho(f,u,e),-)"] \\
fR \arrow[r, "\Gamma R"] & H(\rho^f;-) \arrow[r, "\eta_{\rho^f}"'] & \mathbb{L}(R)(Rf,-)
\end{tikzcd}$$

and ΓR is a cross-connection of RR-normal categories $\mathbb{L}(R)$ and $\mathbb{R}(R)$. Now we proceed to find the cross-connection regular ring.
$E_{\Gamma R} = \{(I_0,I_0),(I_1,J_1),(I_1,J_2),(I_1,J_3)(I_2,J_1),(I_2,J_2),(I_2,J_3),(I_3,J_1),$ $(I_3,J_2),(I_3,J_3),(R,R)\}$

For $(I_1,J_1) \in E_{\Gamma R}, \gamma(I_1,J_1)$ denotes the unique cone in $\mathbb{L}(R)$ with vertex I_1. Define an addition preserving bifunctor $\Gamma R(-,-) : \mathbb{L}(R) \times \mathbb{R}(R) \to \mathbf{Set}$ as described below.

$$\begin{aligned}
\Gamma R(I_1,J_1) &= \{\gamma(I_i,J_1) \star (\rho_u)^o | \rho_u \in \mathbb{L}(R)(I_i,I_1); i = 1,2,3\} \\
&= \{\gamma_1 \star (\rho_{e_1})^o, \gamma_2 \star (\rho_{e_1})^o, \beta_2 \star (\rho_{e_2})^o, \beta_3 \star (\rho_{e_2})^o, \delta_2 \star (\rho_{e_1})^o, \delta_3 \star (\rho_{e_1})^o\} \\
&= \{\gamma_1,\gamma_2,\gamma_3\}
\end{aligned}$$

Similarly $\Gamma R(I_1,J_2) = \Gamma R(I_1,J_3) = \{\gamma_1,\gamma_2,\gamma_3\}$,
$\Gamma R(I_2,J_i) = \{\beta_1,\beta_2,\beta_3\}$, $\Gamma R(I_3,J_i) = \{\delta_1,\delta_2,\delta_3\}; i = 1,2,3.$,
$\Gamma R(I_0,J_0) = \{\gamma_0\}$, $\Gamma R(R,R) = \{\alpha_1,\alpha_2,\alpha_3,\alpha_4,\alpha_5,\alpha_6\}$.

Thus we get

$$U\Gamma R = \{\gamma_0, \gamma_1, \gamma_2, \gamma_3, \beta_1, \beta_2, \beta_3, \delta_1, \delta_2, \delta_3, \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6\}$$

Similarly, we can define $\Gamma^* R : \mathbb{L}(R) \to N^*\mathbb{R}(R)$ and obtain

$$U\Gamma^* R = \{\gamma'_0, \gamma'_1, \gamma'_2, \gamma'_3, \beta'_1, \beta'_2, \beta'_3, \delta'_1, \delta'_2, \delta'_3, \alpha'_1, \alpha'_2, \alpha'_3, \alpha'_4, \alpha'_5, \alpha'_6\}.$$

Let $\chi_{\Gamma R}(I_i, J_i)$ is a natural isomorphism from $\Gamma R(I_i, J_i)$ to $\Gamma^* R(I_i, J_i)$ defined by $\gamma(I_i, J_1) \star f^o \mapsto \gamma^*(I_1, J_i) \star g^o$ where $f : I_1 \to I_i$ and $g : J_i \to J_1$. Using $\chi_{\Gamma R}$, the linked pairs of normal cones are
$\widetilde{R}\Gamma = \{(\gamma_0, \gamma'_0), (\gamma_1, \gamma'_1), (\gamma_1, \gamma'_2), (\gamma_1, \gamma'_3), (\gamma_2, \gamma'_1), (\gamma_2, \gamma'_2), (\gamma_2, \gamma'_3), (\gamma_3, \gamma'_1), (\gamma_3, \gamma'_2), (\gamma_3, \gamma'_3), (\beta_1, \beta'_1), (\beta_1, \beta'_2), (\beta_1, \beta'_3), (\beta_2, \beta'_1), (\beta_2, \beta'_2), (\beta_2, \beta'_3), (\beta_3, \beta'_1), (\beta_3, \beta'_2), (\beta_3, \beta'_3), (\delta_1, \delta'_1), (\delta_1, \delta'_2), (\delta_1, \delta'_3), (\delta_2, \delta'_1), (\delta_2, \delta'_2), (\delta_2, \delta'_3), (\delta_3, \delta'_1), (\delta_3, \delta'_2), (\delta_3, \delta'_3), (\alpha_1, \alpha'_1), (\alpha_1, \alpha'_2), (\alpha_1, \alpha'_3), (\alpha_1, \alpha'_4), (\alpha_1, \alpha'_5), (\alpha_1, \alpha'_6), (\alpha_2, \alpha'_1), (\alpha_2, \alpha'_2), (\alpha_2, \alpha'_3), (\alpha_2, \alpha'_4), (\alpha_2, \alpha'_5), (\alpha_2, \alpha'_6), (\alpha_3, \alpha'_1), (\alpha_3, \alpha'_2), (\alpha_3, \alpha'_3), (\alpha_3, \alpha'_4), (\alpha_3, \alpha'_5), (\alpha_3, \alpha'_6), (\alpha_4, \alpha'_1), (\alpha_4, \alpha'_2), (\alpha_4, \alpha'_3), (\alpha_4, \alpha'_4), (\alpha_4, \alpha'_5), (\alpha_4, \alpha'_6), (\alpha_5, \alpha'_1), (\alpha_5, \alpha'_2), (\alpha_5, \alpha'_3), (\alpha_5, \alpha'_4), (\alpha_5, \alpha'_5), (\alpha_5, \alpha'_6), (\alpha_6, \alpha'_1), (\alpha_6, \alpha'_2), (\alpha_6, \alpha'_3), (\alpha_6, \alpha'_4), (\alpha_6, \alpha'_5), (\alpha_6, \alpha'_6)\}$

$\widetilde{R}\Gamma$ is the cross-connection regular ring which is a regular subring of $\mathcal{T}\mathbb{L}(\mathcal{R}) \times \mathcal{T}\mathbb{R}(\mathcal{R})$.

$$(\gamma_i, \gamma'_j) + (\beta_k, \beta'_l) = (\gamma_i + \beta_k, \gamma'_j + \beta'_l)$$

$$\text{and } (\gamma_i, \gamma'_j) \cdot (\beta_k, \beta'_l) = (\gamma_i \cdot \beta_k, \beta'_l \cdot \gamma'_j).$$

The addition and multiplication of two normal cones is as defined in Theorem 3.2.2 and it can be seen that $(\gamma_1, \gamma'_1) + (\delta_1, \delta'_1) = (\beta_2, \beta'_2)$ and so on. The multiplication is distributive over addition as $(\gamma_1, \gamma'_1) \cdot [(\beta_2, \beta'_2) + (\delta_1, \delta'_1)] = (\gamma_1, \gamma'_1) \cdot (\gamma_1, \gamma'_1) = (\gamma_1, \gamma'_1)$ and $(\gamma_1, \gamma'_1) \cdot (\beta_2, \beta'_2) + (\gamma_1, \gamma'_1) \cdot (\delta_1, \delta'_1) = (\beta_2, \beta'_2) + (\delta_1, \delta'_1) = (\gamma_1, \gamma'_1)$ and so on.

Thus the principal left and right ideals of a regular ring are $RR-$normal categories and their cross-connection gives a regular ring.

Scope of Further Study

In this thesis we discussed cross-connections of Boolean lattices and that of regular ring. It is also shown that the ideal categories of arbitrary semigroups, rings and modules as proper category, preadditive proper category and abelian proper category respectively. The duals of proper categories and preadditive proper categories are also discussed.

The cross-connection of proper categories and that of preadditive proper categories are not discussed in this thesis, similarly the dual of abelian proper category is also not discussed, hence these will be a natural choice for any further study.

The submodules of a semisimple R-module form a complemented modular lattice. Moreover it also forms an abelian normal category with object set submodules and morphisms R-module homomorphisms. The cross-connection of the semisimple R-module can be constructed considering it as an abelian normal category and as a complemented modular lattice. These constructions as well as their equivalences will be a part of any future research based on this thesis.

Bibliography

[1] F.W. Anderson and K.R.Fuller, Rings and categories of modules, 2nd edition, Springer-Verlag, New York, ISBN 0-387-97845-3 (1992).

[2] Michael Artin, Algebra, Prentice Hall, New Jersey, ISBN 0-13-004763-5 (1991).

[3] A.J. Berrick and M.E. Keating, Categories and modules with K-theory in view, Cambridge university press, New York, ISBN 0-521-632765 (2000).

[4] Garrett Birkhoff, Lattice theory, American Mathematical Society, ISBN 0 821 810251 (1940).

[5] T.S. Blyth and M.F. Janowitz, Residuation theory, Pergamon, Oxford, ISBN 978-0-08-016408-3 (1972).

[6] T.S. Blyth, Lattices and ordered algebraic structures, Springer Verlag, London, ISBN 1-85233-905-5 (2005).

[7] A.H. Clifford and G.B. Preston, The algebraic theory of semigroups vol 1, Math Surveys No.7, American Mathematical Society, Providence, R.I. (1961).

[8] Steven Givant and Paul Halmos, Introduction to boolean algebras, Springer Verlag, Newyork, ISBN 978-0-387-40293-2 (2009).

[9] K.R. Goodearl, Von Neumann Regular Rings, Pitman publishing limited, London, ISBN 0-273-08400-3 (1979).

[10] George Gratzer, Lattice theory: Foundation, Springer Basel AG, ISBN 978-3-0348-0017-4 (2011).

[11] M.P. Grillet, Green's relations in a semiring, Portugaliae Mathematica, Portugal, 29, 181-195 (1970).

[12] P.A. Grillet, Semigroups, An introduction to the structure theory, Tulane University, New Orleans, Louisiana, ISBN 0-8247-9662-4 (1995).

[13] P. A. Grillet : The structure of regular semigroups. I. A representation, II. Cross-Connections, III. The reduced case, Semigroup Forum, 8, 177-183, 254-259, 260-265 (1974).

[14] J.M. Howie, An introduction to semigroup theory, Academic Press Inc.(London)Ltd., ISBN 75-46333 (1976).

[15] J.M. Howie, Fundamentals of semigroup theory, Clarendon Press, Oxford, ISBN 0-19-851194-9 (1995).

[16] M.E. Keating, A first course in module theory, Imperial college press, London, ISBN 186 094 096 (1998).

[17] S.Mac Lane, Categories for the working mathematician, Springer Verlag, Newyork, ISBN 0-387-98403-8 (1971).

[18] Serge Lang, Algebra, Addison-Wesley publishing company, Newyork, ISBN 0-201-55540-9 (1993).

[19] Mark V Lawson, Inverse semigroups: The theory of partial symmetries, World Scientific, ISBN 978-981-02-3316-7 (1998).

[20] Sunny Lukose and A.R. Rajan, Ring of normal cones, Indian J. Pure. Appl. Math, 41(5):663-681 (2010).

[21] Barry Mitchell, Rings with several objects, Advances in Mathematics, 8, 1-161 (1972).

[22] Azeef Muhammed P.A., Cross connections of linear transformation semigroup, arXiv:1701.06098[math.RA] (2017).

[23] W.D. Munn, Fundamental inverse semigroups, Quart.J.Maths, Oxford, 21, 157-170 (1970).

[24] C. Musili, Introduction to rings and modules, Narosa Publishing House, New Delhi, ISBN 81-85198-64-0 (1992).

[25] K.S.S. Nambooripad, Theory of cross connections, Publication No.28 - Centre for Mathematical Sciences, Trivandrum (1994).

[26] K.S.S. Nambooripad, Relation between biordered sets and cross-connections, Semigroup Forum, 16, 67-81 (1978).

[27] K.S.S. Nambooripad and F.J.Pastijn, The fundamental representation of a strongly regular baer semigroup, J.Algebra, 92,283-302 (1985).

[28] John von Neumann, Continuous geometry, Princeton University Press, London (1960).

[29] Petraq Petro, Green's relations and minimal quasi ideals in rings, Communications in algebra, vol 30, pp.4677-4686 (2002).

[30] K.M. Rangaswamy, Regular and Baer rings, Proceedings of the American Mathematical Society, Vol 42, Number 2, 354-358 (1974).

[31] P.G. Romeo, Concordant semigroups and balanced categories, Southeast Asian Bulletin of Mathematics, 31:949-961 (2007).

[32] Louis Rowen, Ring theory Vol 1, Academic Press, Inc. New York, ISBN 0-12-599841-4 (1988).

[33] Robert Wisbauer, Foundations of module and ring theory, Gordon and Breach Science Publishers, University of Düsseldorf (1991).

Publications

- **On the Category of Ideals of Ring** - Journal of Mathematics and Systems Science, 7, 94-99, March 2017.

- **Cross connection of Boolean lattice** - International Journal of Algebra, Vol. 11, no. 4, 171 - 179, May 2017.

- **Greens Relations and Translations in Rings** - Journal of Statistics and Mathematical Engineering, Vol 3, Issue 3, 1-10, November 2017.

- **Category of partially ordered sets**, Chapter 41, 551-559, Book-Infinitude: Frontiers of Research in Mathematics, Statistics and Computer Science, ISBN: 9788191070627(2016).

- **On the category of R-modules over noncommutative ring**, Bulletin of Kerala Mathematical Association, Vol. 15, no.1, 23-28, June 2018.

- **Proper Category and its dual** - Algebra and its Applications-Proceedings of International Conference on Algebra and Applications, Aligarh Muslim University, Aligarh, Uttar Pradesh, India; de Gruyter Publications, 215-226, ISBN: 978-3-11-054092-5, August 2018.

- **Abelian normal category of semisimple R-modules** - Proceedings of International Conference on Algebra and Discrete

Mathematics, Madurai Kamaraj University, Madurai, Tamilnadu, India;- in association with International Journal of Computer Science, 40-46, January 2018.

Papers presented

- **Representation of Boolean Algebra** - International Conference of Algebra and Applications, Aligarh Muslim University, Aligarh, Uttar Pradesh, India; November 12-14, 2016.

- **Abelian Normal Category of Semisimple R-modules** - International Conference on Algebra and Discrete Mathematics, Madurai Kamaraj University, Madurai, Tamilnadu, India; January 8-10, 2018.

- **On the Translational hull of Rings** - International Seminar on Algebra and Coding theory, St.Berchmans College, Changanasserry, Kerala; February 8-10, 2017.

- **Greens Relations in Rings** - National Seminar on Algebra and Analysis, University College, Thiruvananthapuram, Kerala; February 1-3, 2017.

- **Category of R-modules over non commutative ring** - National Seminar on Mathematics for Computer Science, Amal Jyothi college of Engineering, Kanjirappally, Kerala; November 9-11, 2017.

<u>CURRICULUM VITAE</u>

Name :	Sreejamol P.R.
Present Address :	Department of Mathematics, CUSAT Cochin, Kerala, India – 682 022.
Official Address :	Assistant Professor, Department of Mathematics, SNM College, Maliankara, Ernakulam, Kerala, India – 683 516.
Permanent Address :	Soorya(H), Edayar Road U.C.College P.O, Aluva, Ernakulam, Kerala, India – 683 102.
Email :	sreejasooraj@gmail.com
Qualifications :	**B.Sc.** (Mathematics), 2000, **M.Sc.** (Mathematics), 2002, **B.Ed.** (Mathematics), 2003, M. G. University, Kottayam, Kerala, India.
Research Interest :	Algebra and Category Theory.

Index