STUDIES ON THE NETWORK TOPOLOGICAL PROPERTIES OF SOME GRAPH CLASSES

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By

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To

My Parents and Husband

Department of Mathematics

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Certificate

Certified that the work presented in this thesis entitled **"Studies on the network topological properties of some graph classes"** is based on the authentic record of research carried out by Ms. Savitha K.S under my guidance in the Department of Mathematics, Cochin University of Science and Technology, Cochin-682 022 and has not been included in any other thesis submitted for the award of any degree.

Dr. A. Vijayakumar (Supervisor)

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Certificate

Certified that all the relevant corrections and modifications suggested by the audience during the Pre-synopsis seminar and recommended by the doctoral committee of the candidate has been incorporated in the thesis entitled **"Studies on the network topological properties of some graph classes"**.

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Declaration

I, Savitha K.S, hereby declare that this thesis entitled **"Studies on the network topological properties of some graph classes"** is based on the original work carried out by me under the guidance of Dr. A. Vijayakumar, Professor and Head, Department of Mathematics, Cochin University of Science and Technology, Cochin-22 and contains no material which had been accepted for any other Degree, Diploma or similar titles in any University or Institution. To the best of my knowledge and belief, this thesis contains no material previously published by any person except where due references are made in the text of the thesis.

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STUDIES ON THE NETWORK TOPOLOGICAL PROPERTIES OF SOME GRAPH CLASSES

Contents

Chapter 1

Introduction

Graph theory, a branch of mathematics originated in the 18^{th} century, had its beginning in recreational mathematics problems. But later, it has grown into a significant area of mathematical research with applications in areas like phylogenetics, mathematical chemistry, computer science, economics, environmental conservation, psychology and telecommunications to name a few. Probably the latest and more exciting application of graph theory is Social Network Analysis(SNA). SNA is the study of social networks, their structures and how knowing this structure can lead to a better understanding of the behavior within social networks. The most well known social network is the Facebook. In a typical situation, a problem that arises in a real world is converted to a graph theoretic problem and is solved by applying the existing techniques or by developing the new ones.

The famous Swiss Mathematician Leonard Euler (1707 – 1782) initiated Graph Theory when in 1736 he settled the famous unsolved Königsberg Bridge Problem. Since then it has witnessed an unprecedented growth due to its role as an essential structure in developing modern applied mathematics as well as computer science.

In recent years, graph theory has established itself as an important mathematical tool in a wide variety of subjects, ranging from operational research and chemistry to genetics and linguistics, and from electrical engineering and geography to sociology and architecture. Also, many branches of mathematics such as group theory, matrix theory, probability and topology have close connections with graph theory. At the same time it has also emerged as a worthwhile mathematical discipline in its own right.

An interconnection network (network) depicts the connection pattern of the components of a system in which the components are linked by some means for the transmission of information. Therefore, an interconnection network can be modeled by a graph in a natural way where the vertices represent the components and the edges the links between them. The graph so obtained is usually termed as the topological structure of the interconnection network or network topology[65].

The advent of Very Large Scale Integrated (VLSI) circuit technology has enabled the construction of very complex and large interconnection networks. In such constructions, the designing of an appropriate topological structure is a critical issue and is an area in which many research have been made over the past decade. It is almost impossible to design a network that is optimum in all aspects. So, one has to design a suitable network depending on its properties and requirements. Thus many graphs are proposed as a possible interconnection network topologies. Graph theory is a fundamental and powerful tool for designing and analyzing interconnection networks, since the topological structure of an interconnection network is a graph.

The basic principles to be concentrated in a network design such as small number of connections to a component, small communication delay, high fault tolerance, easy routing algorithm, embeddability of other topologies etc. can be well explained in the language of graph theory. To be more specific, the number of connections to a component is the degree of a vertex, communication delay is measured by the diameter of a graph, fault tolerance can be studied by the vertex as well as edge deletions in a graph, routing in a network specifies the paths between the vertices in the graph and the embeddability of other structures is the existence of subgraphs. Thus, many graph theoretic techniques can be used to measure the reliability and efficiency of a network based on the above mentioned principles. Some of these techniques and other important parameters to analyze an interconnection network can be seen in [33, 65].

Among the many, a few important measures of efficiency of a network that we concentrate are wide diameter, fault diameter, diameter variability, diameter vulnerability, $(l, k)-$ domination, routing, forwarding indices, bisection width, convexity parameters and hamiltonicity. The wide diameter based on the internally disjoint paths between the vertices in a graph is a generalization of the diameter. A small wide diameter is preferred since it enables fast multipath communication. Fault diameter, proposed by Krishnamoorthy and Krishnamurthy [41] estimates the impact on diameter when faults occur. A small fault diameter is also desirable to obtain a small communication delay when a fault occcurs.

The fact that the diameter of a graph can be affected by the addition or deletion of edges, give rise to the concept of diameter variability in graphs. The study of diameter variability in a network becomes important as it determines the communication efficiency when an addition or deletion of a link occurs. Vulnerability [16, 52] is a measure of the ability of the system to withstand vertex or edge faults and $(l, k)-$ domination is a parameter used to characterize the reliability of resources-sharing in a network.

A routing in a network gives an idea about how the message is being transmitted between the components in a network. The forwarding indices measure the quality of a routing in terms of the load (congestion) of a vertex (an edge) which in turn measure the capacity of a network. A 'good' routing should have small vertex-forwarding index and edge-forwarding index. Thus it becomes very significant, to compute the vertex-forwarding index and the edge-forwarding index of a graph, which has received much attention [67].

The bisection width of a graph G, is the least number of edges to be removed to partition G into two components with roughly equal order [33]. The bisection width of interconnection networks has always been important in parallel computing, since it bounds the amount of information that can be moved from one side of a network to another. For some network families, finding the exact value of the bisection width has proven to be challenging.

Another important area of concern in the designing of a network is its security as it involves sharing and transmission of information. As the convex structure in networks allows safe data transmission and also represent locally self sufficient systems, security in network design can be dealt with the study of the convex structure of an interconnection network. It can also be seen as a measure of network redundancy, a concept closely related to robustness and resilience.

Not only is hamiltonicity a fundamental graph-theoretic concept but is also extremely relevant in the context of interconnection networks where the existence of hamiltonian cycles or paths can have a number of applications. In an all-to-all communication pattern, the existence of a hamiltonian cycle enables every node to send its data out so that in a one-port synchronous system what results is an optimal algorithm. The presence of hamiltonian cycles in an interconnection network is also required to meet some specific approaches used in distributed operation systems.

This thesis deals with the study of network topological properties of some graph classes. It mainly discusses the topological properties of an interesting graph transformation called Mycielskian of a graph, its iterations and generalizations. This thesis also analyzes some important properties of two well known networks namely Sierpiński graphs and Fibonacci cubes.

1.1 Definitions

All other notions not mentioned here are from [4].

Definition 1.1.1. A **graph** $G = (V, E)$ consists of a non-empty set V called its **vertices** and a set of unordered pairs of distinct vertices E called its **edges** . The unordered pair of vertices $\{u, v\} \in E$ are called the **end vertices** of the edge $e = \{u, v\}$.

In that case, the vertex u is said to be **adjacent** to the vertex v. Two edges e and e' are said to be **adjacent** if they have a common end vertex. The **neighborhood** of a vertex u is the set $N(u)$ consisting of all vertices v which are adjacent to u. $|V|$ is called the **order** of G and $|E|$ is called the **size** of G.

Definition 1.1.2. The number of vertices adjacent to a vertex v is called the **degree** of the vertex, denoted by $d(v)$. A vertex of degree zero is an **isolated vertex** and of degree one is a **pendant vertex**. The edge incident on a pendant vertex is a **pendant edge**. If G is a graph of order n, then a vertex of degree $n-1$ is called a **universal vertex**. The maximum and the minimum of the degrees of vertices of a graph are denoted by $\Delta(G)$ and $\delta(G)$ respectively. G is regular if $\Delta(G) = \delta(G)$. It is **k-regular**, if $deg(v) = k$ for every vertex $v \in V(G)$.

Definition 1.1.3. The **distance** between two vertices u and v of a connected graph G, denoted by $d(u, v)$, is the length of a shortest u–v path in G. The **eccentricity** of a vertex u is $e(u) = max\ \{d(u, v)/v \in V(G)\}$. The **radius** $r(G)$ and the **diameter** diam(G) are respectively the minimum and the maximum of the vertex eccentricities. For a vertex $u \in V(G)$, if there exists a vertex $v \in V(G)$ such that $d(u, v) =$ diam(G), v is then called a **diametral vertex** of u.

Definition 1.1.4. The **complete graph** K_n is a graph of order n in which each pair of distinct vertices is joined by an edge.

Definition 1.1.5. A graph G is **bipartite** if its vertex set can be partitioned into two non-empty subsets X and Y such that every edge of G has one end vertex in X and the other in Y. The pair (X, Y) is then called a bipartition of G. A bipartite graph in which each vertex of X is adjacent to all the vertices of Y is called a **complete bipartite graph**.

Definition 1.1.6. For a connected graph G, a subset V' of $V(G)$ is a k**-vertex cut** of G if $G - V'$ is disconnected and $|V'| = k$. The **vertex connectivity** of a nontrivial connected graph G, $\kappa(G)$, is the least number of vertices whose deletion from G disconnects G. A graph G is k**-connected**, if $\kappa(G) \geq k$. A vertex v of G is a **cut vertex** of G if $\{v\}$ is a vertex cut of G. The **edge connectivity** of a connected graph G, $\kappa'(G)$, is the least number of edges whose deletion from G disconnects G.

Definition 1.1.7. Let S be the set of all n-tuples in which each position is 0 or 1. The **hypercube** of dimension n, denoted by

Q_n, is the graph whose vertex set is S in which two vertices are adjacent if they differ in exactly one position.

Definition 1.1.8. [33] For every integer w, $1 \leq w \leq \kappa(G)$, a w**-container** between any two distinct vertices u and v of G is a collection of 'w' internally vertex disjoint paths between them. Let $C_w(u, v)$ denote a w-container between u and v. In $C_w(u, v)$, the parameter w is the width of the container. The *length of the container* $l_w(u, v)$ is the length of a longest path among all paths in $C_w(u, v)$. The w**-wide distance** $d_w(u, v)$ between u and v is the minimum $l_w(u, v)$, over all w-containers between u and v. The w**-wide diameter** of G, $D_w(G)$ is the maximum of $d_w(u, v)$ among all pairs of vertices u, v in G, $u \neq v$. $D_{\kappa(G)}(G)$ is called the **wide diameter** of G.
A simple example is shown in the Fig.1.2.

Definition 1.1.9. [41] The **vertex fault diameter** denoted by $f(G)$ is defined as $f(G) = \max\{diam(G \setminus S) : S \subseteq V(G), |S| = \kappa(G) - 1\}$.
Example: $f(Q_n) = n + 1, n \geq 2$.

Definition 1.1.10. The maximum diameter of a graph obtained by deleting t edges from a $(t + 1)-$ edge connected graph with

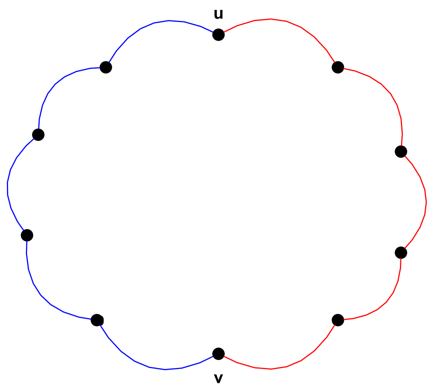

Figure 1.1: $d_2(u,v) = 5$ and $D_2(G) = 9$

diameter d is denoted by $f(t,d)$ and is used to study the **diameter vulnerability** of graphs by edge deletion .

Definition 1.1.11. [65] Let G be a $k-$connected graph ($k \geq 1$), $\phi \neq S \subset V(G)$, and $y \in V(G \setminus S)$. A path from y to some vertex in S is called a $(y,S)-$path. A set of k internally disjoint $(y,S)-$paths is called a $(y,S)-$container, denoted by $C_k(G;y,S)$. The length of a longest path among all paths in $C_k(G;y,S)$ is called the length of $C_k(G;y,S)$. For a given integer $l(\geq 1)$, if there exists a $(y,S)-$container $C_k(G;y,S)$ with length at most l, then we say that S can $(l,k)-$dominate y. S is called an $(l,k)-$*dominating set of* G, if it $(l,k)-$dominates every vertex in $G \setminus S$. The set of all $(l,k)-$dominating sets in G is denoted by $S_{l,k}(G)$. The parameter $\gamma_{l,k}(G) = min\{|S| : S \in S_{l,k}(G)\}$ is called the $(l,k)-$**dominating number of** G and an $(l,k)-$dominating set S of G is called minimum if $|S| = \gamma_{l,k}(G)$.

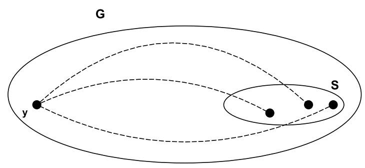

Figure 1.2: A (y, S) container

Definition 1.1.12. [4] For a graph $G = (V, E)$, the **Mycielskian of** G is the graph $\mu(G)$ with the vertex set $V(\mu(G)) = V \cup V' \cup \{w\}$, where $V' = \{u' : u \in V\}$ and the edge set $E(\mu(G)) = E \cup \{uv' : uv \in E\} \cup \{v'w : v' \in V'\}$. The vertex v' is called the twin of the vertex v and vice versa. The vertex w is called the root of $\mu(G)$. For $n \geq 2$, $\mu^n(G)$ is defined iteratively by setting $\mu^n(G) = \mu(\mu^{n-1}(G))$.

$\mu(C_6)$ is shown in the Fig.1.3

Definition 1.1.13. [44] Let G be a graph with vertex set $V^0 = \{v_1^0, v_2^0, \cdots v_n^0\}$ and edge set E^0. Given an integer $m \geq 1$, the m- **Mycielskian of** G, denoted by $\mu_m(G)$, is the graph with vertex set $V^0 \cup V^1 \cup V^2 \cdots V^m \cup \{z\}$, where $V^i = \{v_j^i : v_j^0 \in V^0\}$

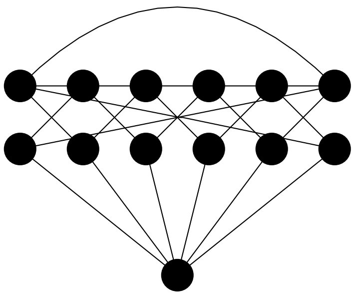

Figure 1.3: $\mu(C_6)$

is the i^{th} distinct copy of V^0 for $i = 1, 2, \cdots, m$ and edge set $E^0 \cup \left(\bigcup_{i=0}^{m-1}\{v_j^i v_{j'}^{i+1} : v_j^0 v_{j'}^0 \in E^0\}\right) \cup \{v_j^m z : v_j^m \in V^m\}$. $\mu_0(G)$ is defined to be the graph obtained from G by adding an universal vertex z and the Mycielskian of G is simply $\mu_1(G)$. We call the vertices of V^i as vertices of level i.

Illustration

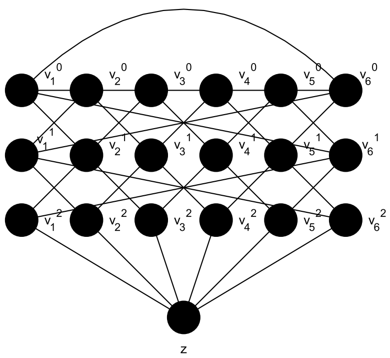

Figure 1.4: $\mu_2(C_6)$

Definition 1.1.14. [62] Let k be an arbitrary integer. The **diameter variability** arising from change of edges of a graph G is defined as follows :

- $D^{-k}(G)$: the least number of edges whose addition to G decreases the diameter by (at least) k;
- $D^{+0}(G)$: the maximum number of edges whose deletion from G does not change the diameter;
- $D^{+k}(G)$: the least number of edges whose deletion from G increases the diameter by (at least) k.

For example, consider the $m-$cycle C_m with vertex set $\{0, 1, 2, \cdots, m-1\}$ and edge set $\{(i, i+1)|0 \leq i \leq m-1\}$, where addition is in integer modulo m. Then $\text{diam}(C_m) = \lfloor \frac{m}{2} \rfloor$. If P_m is the path on m vertices with vertex set $\{0, 1, 2, \cdots, m-1\}$ and edge set $\{(i, i+1)|0 \leq i \leq m-2\}$, then $\text{diam}(P_m) = m-1$. It is easy to see that $D^{-1}(P_m) = D^{-2}(P_m) = \cdots = D^{-(m-1-\lfloor \frac{m}{2} \rfloor)}(P_m) = 1$. $D^{+1}(C_m) = D^{+2}(C_m) = \cdots = D^{+(m-1-\lfloor \frac{m}{2} \rfloor)}(C_m) = 1$.

Definition 1.1.15. Let G be a graph with $|V| = n \geq 2$ and $|E| \geq 1$. A **routing** R in G is a set of $n(n-1)$ paths, one for each ordered pair (x, y) of vertices of G. The path $R(x, y)$

specified by R carries the data transmitted from the source x to the destination y. Let $R(G)$ be the set of routings in a graph G. For a given $R \in R(G)$ and $x \in V(G)$, **the load of** x with respect to R, denoted by $\zeta_x(G;R)$, is defined as the number of paths specified by R going through x. The parameter $\zeta(G;R) = \max\{\zeta_x(G;R) : x \in V(G)\}$ is called the **vertex forwarding index of** G **with respect to** R, and $\zeta(G) = \min\{\zeta(G;R) : R \in R(G)\}$ is called the **vertex forwarding index** of G. The **congestion of an edge** e with respect to R, denoted by $\pi_e(G;R)$, is defined as the number of paths specified by R which go through e. The **edge-forwarding index of** G **with respect to** R, denoted by $\pi(G;R)$, is the maximum number of paths specified by R going through any edge of G, i.e. $\pi(G;R) = \max\{\pi_e(G;R) : e \in E(G)\}$; and the *edge-forwarding index* of G is defined as $\pi(G) = \min\{\pi(G;R) : R \in R(G)\}$.

Definition 1.1.16. [33] The **bisection width** of a graph G, bw(G), is the least number of edges to be removed to partition G into two components with equal number of vertices (or differing by one in the case of an odd number of vertices).
Example: For Q_3, it can be shown that $\zeta(Q_3) = 5, \pi(Q_3) = 2^3$ and $\mathrm{bw}(Q_3) = 2^2$.

Definition 1.1.17. [6] For any two vertices u and v in a graph G, a shortest $u - v$ path is called a $u - v$ **geodesic**. The **closure**(S) of a set S consists of the vertices of S together with all vertices on a geodesic between two vertices of S. A set S is **convex** if $(S) = S$. The process of taking closures can be repeated to obtain a sequence $S^1, S^2, \ldots$ of geodetic closures, where $S^1 = (S)$, $S^2 = (S^1)$, and in general $S^k = (S^{k-1}), k > 1$. Since $V(G)$ is finite, the process must terminate with some smallest n for which $S^n = S^{n-1}$. The resulting set $[[S]]$ is called the **convex hull** of S. This corresponds to the smallest convex set containing S and the value of n is called the **geodetic iteration number**, $\mathrm{gin}(S)$. For a graph G, **gin**(G) is defined as the maximum value of *gin(S)* over all $S \subset V(G)$.
For example, $\mathrm{gin}(K_{2,3}) = 2$.

Definition 1.1.18. [6] A **geodetic cover** of G is a set $S \subset V(G)$ such that every vertex of G is contained in a geodesic joining some pair of vertices in S. The **geodetic number gn(**G**)** of G is the minimum cardinality among its geodetic covers and any geodetic cover of cardinality **gn**(G) is a **geodetic basis**.
It is easy to see that $\mathrm{gn}(K_p) = p$.

Definition 1.1.19. [6] The **hull number** $h(G)$ is the minimum cardinality of a set $S \subset V(G)$ such that its convex hull $[[S]]$ is $V(G)$. Such a set is called a **minimum hull set**.

From this definition of hull number, it is clear that $h(G) \leq \mathrm{gn}(G)$ for a graph G and using this, it can be shown that $h(K_p) = p$.

Definition 1.1.20. [61] An **interval** $I(u, v)$ in G is defined as the closure of the set $\{u, v\}$. A graph G is **interval monotone** if for every pair (u, v), $I(u, v)$ is convex .

Example:Q_n.

Definition 1.1.21. [12] The **convexity number** $c(G)$ of a graph G is the maximum cardinality of a proper convex set of $V(G)$.

For every tree T of order $n \geq 2$, $c(T) = n - 1$.

Definition 1.1.22. [13] A connected graph G is **poly-convex** if for every integer i with $1 \leq i \leq c(G)$ there exists a convex set of cardinality i in G .

Definition 1.1.23. The closure of a subset S of vertices contains every vertex v such that v belongs to some induced path joining two vertices of S is called **monophonic closure**.

Example: In Fig.1.5, monophonic closure of $\{u, x\} = V(G)$.

Based on this definition of closure, we have the parameters

minimal path iteration number(min(G)), **m-convexity number**(mc(G)), **monophonic number**(mn(G)) and **m-hull number** (mh(G)) defined similar to that with respect to a geodesic.

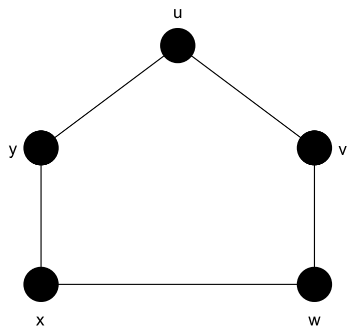

Figure 1.5: Monophonic closure of $\{u, x\} = V(G)$

Basic notations used in this thesis are mostly from [4] and unless otherwise stated, all graphs considered are simple, finite and undirected.

1.2 A survey of previous results

In this section, we provide a brief survey of the previous results on the Mycielskian, its iterates and generalization, forwarding index, bisection width etc.

1.2.1 Mycielskian of a graph

In a search for triangle-free graphs with arbitrarily large chromatic number, Mycielski in 1955 [48] developed an interesting graph transformation known as the Mycielskian of a graph and in recent times, there has been an increasing interest in the study of the Mycielskian. In [23], Fisher et al. studied the hamiltonicity and diameter of the Mycielskian and proved that if G is hamiltonian, then so is $\mu(G)$ and diameter of $\mu(G)$= $\min(\max(2, diam(G)), 4)$. Balakrishnan and Francis Raj determined the vertex connectivity and edge connectivity of Mycielskian in [3]. In [28], L.Guo et al. showed that for a connected graph G with $|V(G)| \geq 2$, $\mu(G)$ is super connected if and only if $\delta(G) < 2\kappa(G)$ and $\mu(G)$ is super edge connected if and only if $G \ncong K_2$. Recently, Chithra M. R studied the diameter variability of Mycielskian in [15]. These results motivated the study of network topological properties of the Mycielskian of a graph.

Generalized Mycielskian of a graph is a natural generalization of the Mycielskian and its several parameters such as circular clique number, total domination number, open packing number and spectrum are determined in [44]. Francis Raj [24] investigated the vertex connectivity and edge connectivity of the generalised mycielskian of digraphs, which turned out to be a generalisation of the results due to Guo and Guo [27].

1.2.2 Some measures of importance in networks

An interconnection network connects the processors of a parallel and distributed system. The topological structure of an interconnection network can be modeled by a graph where the vertices represent components of the network and the edges represent communication links between them. Some graph theoretic techniques that are used to study the efficiency and reliability of a network are discussed in [33, 65]. Network topological notions such as wide diameter, fault diameter, diameter vulnerability and $(l, w)-$domination can be used to study the efficiency and reliability of a network and in this thesis we study these notions in the Mycielskian of a graph and its iterates. Wide diameter and fault diameter of hypercubes studied by Saad and Shultz can be seen in [33]. The study of these two parameters for other interconnection networks and an operator $P_3(G)$ can be seen in

[45, 22]. Vulnerability, a measure of the ability of the system to withstand vertex or edge faults and maximum routing delay is studied in [16, 52]. (l, w)-domination is a parameter used to characterize the reliability of resources-sharing in a network and has been recently studied in [66].

1.2.3 Diameter variability

The study of diameter variability in a network becomes important as it determines the communication efficiency when an addition or deletion of a link occurs. Graham and Harary [25] studied how the diameter of hypercubes $(Q_n, n \geq 1)$ can be affected by adding or deleting edges. They considered changing the diameter with out considering the extent of the change and showed that $D^{-1}(Q_n) = 2, D^{+1}(Q_n) = n - 1$ and $D^{+0}(Q_n) \geq (n-3)2^{n-1} + 2$. Bouabdallah et al. [5] improved the lower bound of $D^{+0}(Q_n)$ and furthermore gave an upper bound. Diameter variability of cycles and tori was determined by Wang et al. [62]. In [64], authors considered the change in the diameter of a hypercube with the addition of edges. In [63], authors studied the changing of the diameter of a diagonal mesh network.

Diameter variability of various graph products was studied in [14].

1.2.4 Forwarding indices and bisection width

The notion of forwarding index is motivated by the problem of maximizing network capacity, which reduces to minimizing the vertex-forwarding index or the edge-forwarding index of a routing. Hence a 'good' routing is expected to have small vertex-forwarding index and edge-forwarding index. Thus it becomes very significant, to compute the vertex-forwarding index and the edge-forwarding index of a graph, which has received much attention. A detailed survey on forwarding indices can be seen in [67].

The bisection width of interconnection networks has always been important in parallel computing, since it bounds the amount of information that can be moved from one side of a network to another. The problem of finding the exact bisection width of the multidimensional torus was posed by Leighton [42]. The exact value of the bisection width of the torus is provided by Aroca and Anta [2]. Optimal bounds of bisection width of De Bruijn and Kautz networks was done by Rolim et al. [53]. The bisection width of cubic graphs was found by Clark and Entringer [17], while the bisection width of crossed cubes, twisted cubes and hypercubes can be found in [9], [43] and [33] respectively.

1.2.5 Sierpiński graphs

S. Klavžar and U. Multinović introduced Sierpiński graphs S_k^n [37] as graphs of a particular variant of the Tower of Hanoi problem. Since then they have been studied extensively because of its interesting properties. In [50], Parisse determined the eccentricity of a vertex, the diameter, the radius and the center of S_k^n. The average eccentricity of Sierpiński graphs is obtained in [32]. Recently, Klavžar and Zemljič introduced the concept of almost-extreme vertex and gave an explicit formula for the distance in S_k^n between an arbitrary vertex and an almost-extreme vertex in [39]. The problem to obtain shortest paths in S_k^n was studied in many papers. In [37], Klavžar and Milutinović proved that there are at most two different shortest paths between any two vertices in S_k^n, and showed that the number of shortest paths between any fixed pair of vertices can be computed in O(n) time. The set $S_u = \{v \in V(S_k^n)) :$ there exist two shortest u, v-paths in S_k^n }, where u is any almost-extreme vertex of S_k^n is determined in [68]. An efficient algorithm to determine *all* shortest paths and their length in Sierpiński graphs was presented by Hinz and Holz auf der Heide [30]. Moreover, these graphs found to be very similar to a class of graphs called WK-recursive networks introduced in computer science. Hence it is studied as a model for interconnection networks also [31].

Due to the fact that the shortest paths in base-3 correspond to optimal solutions in the Tower of Hanoi puzzle, the study of metric properties of Sierpiński graphs has received much attention. In the seminal paper [37], a formula for the distance between vertices in S_p^n was determined. Hinz and Parisse [32] determined the diameter and the eccentricity of these graphs. Klavžar and Zemljič [39] gave explicit formulas for the distance in Sierpiński graphs between an arbitrary vertex and an almost-extreme vertex . They applied this formula to compute the distance of almost-extreme vertices and also to obtain the metric dimension of Sierpiński graphs.

1.2.6 Convexity parameters

It is found interesting to study a few numerical graph invariants (parameters) inspired by the convexity in graphs. Among many convexities geodesic convexity and minimal path convexity are two widely studied notions of convexities in graphs. Therefore we determined the parameters based on both these convexities in the Sierpiński graphs with an intention to explore its convexity nature. The parameters we concentrated are iteration number, convexity number, geodetic number, monophonic number and hull number. We have also studied the interval monotonicity and polyconvexity nature of the Sierpiński graphs.

The geodetic iteration number was studied by Harary and Nieminen [29] who determined the minimum order of a graph G such that $\mathrm{gin}(G) = n$. Buckley, Harary and Quintas [7] characterized those connected graphs G for which $\mathrm{gn}(G) = p, p-1$ or 2. In the same paper, they also determined the $\mathrm{gn}(G)$ for various classes of graphs such as unicyclic graphs, complete multipartite graphs and prisms of an n-cycle. In [11] it was shown that if G is a connected graph of order $n \geq 2$ and diameter d, then $gn(G) \leq n - d + 1$.

The hull number was studied by Everet and Siedman [21] who obtained some bounds for $h(G)$. In [10], it is shown that every two integers a and b with $2 \leq a \leq b$ are realizable as the hull and geodetic numbers, respectively of some graph. Some results in geodetic iteration number and interval monotonicity can also be seen in [51]. A recent study on the convexity of partial cubes can be seen in [1]. In [20] Carathtodory, Helly and Radon type theorems are proved for m-convex sets. The computational complexity of determining the important convexity parameters like m-convexity number, m- hull number, monophonic number etc can be seen in [19] where, it is proved that the decision problems corresponding to the m-convexity and monophonic numbers are NP-complete.

1.2.7 Fibonacci cubes

Fibonacci cubes were introduced as a model for interconnection networks inspired by the famous Fibonacci numbers [34]. These graphs can be embedded in hypercubes, a well studied interconnection network topology. The fact that the order of hypercubes is a power of 2, limits the choice for a network interconnection with a given number of vertices. The initial study of Fibonacci cubes revealed that they contain about $\frac{1}{5}$ fewer edges than the hypercubes for the same number of vertices and admit embedding of basic topologies such as arrays, rings and meshes. Also for a Fibonacci cubes of order n, it was shown [35] that the diameter, the edge connectivity and the node connectivity are $O(\log n)$, which are similar to the hypercubes. Thus, Fibonacci cubes are network topologies having many desirable properties of hypercubes with lesser number of vertices as well as edges. These facts prompted the study of its network topological properties. In spite of its asymmetric and relatively sparse interconnections, the Fibonacci cubes were shown to possess attractive recurrent structures.

It was proved in [18] that every Fibonacci cube has a hamiltonian path and in [49] its independence number is determined. Various enumerative sequences of these graphs are studied in [38, 8, 46]. In mathematical chemistry the concept is related to perfect matchings in hexagonal graphs and the structure of perfect matchings in these graphs is used to determine the stability of benzenoid molecules. In this context it was proved that the resonance graphs of fibonacenes are isomorphic to Fibonacci cubes [40, 70, 71]. Some recent results on the disjoint hypercubes in Fibonacci cubes and q-cube enumerator polynomial can be seen in [26, 47, 59, 60].

1.3 Summary of the thesis

Here, we provide a chapter wise summary of the thesis entitled '**Studies on the Network Topological Properties of some graph classes**' . This thesis is divided into five chapters.
Chapter 1 is an introductory chapter giving a brief history of graph theory and background of our work. Preliminaries, definitions, terminologies and the literature survey are also included in this chapter.
In **Chapter** 2, some network topological properties of the Mycielskian of a graph is studied. We have mainly concentrated on the wide diameter, fault diameter, diameter vulnerability and (l,k)-domination of the Mycielskian which are measures of the reliability and efficiency of a network. The main results obtained are

- If G is a connected graph, then $D_{\kappa(\mu(G))}(\mu(G)) =$

$\max\{D_{\kappa(G)}(G),\ 4\}$.

- $D_{\kappa(\mu^n(G))}(\mu^n(G)) = \max\{D_{\kappa(G)}(G),\ 4\}$, $n \geq 2$.

- Let G be a connected graph with connectivity $\kappa(G) = \delta(G)$. Then $f(\mu(G)) = \max\{f(G),\ 4\}$.

- Let G be a connected graph with connectivity $\kappa(G) = \delta(G)$. Then $f(\mu^n(G)) = \max\{f(G),\ 4\}$.

- Let G be a connected graph with $\delta(G) = \kappa(G) + i$, $i > 0$, then $f(\mu(G)) \leq \max\{$ $f(G)$, 4, diam(G)+2 $\}$.

- Let G be a connected graph with diameter d. If d_μ denote the diameter of the Mycielskian of G, then the diameter vulnerability measure $f(\delta(G), d_\mu) = \max\{d, 3\}$.

- For a connected graph G, $\gamma_{l,\kappa(\mu^n(G))}(\mu^n(G)) = \gamma_{l,\kappa(G)}(G)$, $n \geq 1, l \geq 4$.

The above results show that Mycielskian preserves the desirable properties of networks such as fast multipath communication, high fault tolerance and reliable resource sharing.

Chapter 3 deals with some diameter notions of the generalized Mycielskian of a graph. We estimated the diameter of the generalized Mycielskian and also studied its variability by the addition and deletion of edges. The main results obtained are

- $diam\,(\mu_m(G)) = \min\{\max\{m+1, diam(G)\}, 2(m+1)\}$.
- Bounds for $D^{+0}(\mu_m(G)), D^{+1}(\mu_m(G)), D^{-k}(\mu_m(G))$.
- A necessary and sufficient condition for $D^{+1}(\mu_m(G)) = 1$ and $D^{-1}(\mu_m(G)) = 1$.
- A characterization for the diameter minimality of generalized Mycielskian.

Forwarding indices, bisection width and some convexity parameters of Sierpiński graphs are studied in **Chapter** 4. The main results obtained are

- $\zeta(S_k^n) = 2(k^{n-1} - 1)k^{n-1}$.
- $\pi(S_k^n) = 2 \cdot k^{2(n-1)}$.
- $\text{bw}(S_k^n) = \begin{cases} \frac{k^2}{4} & \text{if } k \text{ is even} \\ n\left\lfloor \frac{k}{2} \right\rfloor^2 + \left\lfloor \frac{k}{2} \right\rfloor & \text{if } k \text{ is odd} \end{cases}$

- The iteration number, convexity number and hull number of Sierpiński graphs with respect to geodesic and minimal path convexity.
- Sierpiński graphs are interval monotone.
- The Sierpiński graphs are poly convex with respect to geodesic convexity where as it is not with respect to the minimal path convexity.

The last chapter **Chapter** 5 deals with some properties of Fibonacci cubes. In this chapter, **We have solved completely a problem posed by S.Klavžar [36] for the Fibonacci cubes of odd order, 'for which vertices v, $\Gamma_n - v$ is hamiltonian'**. We have also studied some diameter notions like wide diameter, fault diameter and diameter variability which are important in networks. The main results obtained are

- Let Γ_n be a Fibonacci cube of odd order. Then $\Gamma_n - v$ is hamiltonian if and only if v lies in the larger bipartition set of Γ_n.
- $D_{\lfloor \frac{n+2}{3} \rfloor}(\Gamma_n) = f(\Gamma_n) = n = diam(\Gamma_n), n \geq 5$

- $D^{-1}(\Gamma_n) = 1, n \geq 2.$
- $D^{-2}(\Gamma_n) = 3, n \geq 4.$
- $D^{+0}(\Gamma_n) = |E(\Gamma_n)| - |V(\Gamma_n)| + 1$
- An upper bound for $D^{-l}(\Gamma_n)$.

Some results of this thesis are included in the papers [54, 55, 56, 57, 58]. The thesis concludes with some suggestions for further study and a list of references.

Chapter 2

Some network topological properties of the Mycielskian of a graph

Efficiency and reliability are two important criteria in the designing of a good interconnection network. Network topological notions such as wide diameter, fault diameter, diameter vulnerability, $(l, k)-$domination etc can be used to study the efficiency

Some results of this chapter are included in the paper.
Savitha K. S, A. Vijayakumar, Some network topological properties of the Mycielskian of a graph, AKCE Int.journal of graphs.combin 13 (1) (2016).

and reliability of a network. In this chapter, we study these notions in the Mycielskian of a graph and its iterates. It is proved that the Mycielskian and its iterates produce large networks and preserve some nice properties of networks such as fast multi-path communication, high fault tolerance and reliable resource sharing.

The following results [3] are used in this chapter.

Lemma 2.0.1. *For a connected graph G, $\kappa(\mu(G)) = \kappa(G) + 1$ if and only if $\delta(G) = \kappa(G)$.*

Lemma 2.0.2. *If G is a connected graph, then $\kappa(\mu(G)) = \kappa(G) + i + 1$ if and only if $\delta(G) = \kappa(G) + i$ for each i, $0 \leq i < \kappa(G)$.*

Lemma 2.0.3. *If G is a connected graph, then $\kappa(\mu(G)) = 2\kappa(G)+1$ if and only if $\delta(G) \geq 2\kappa(G)$ and $\kappa(\mu(G)) = \min\{\delta(G)+1, 2\kappa(G) + 1\}$.*

Lemma 2.0.4. *If G is a connected graph, then $\kappa'(\mu(G)) = \delta(G) + 1$.*

2.1 Wide diameter

To determine the wide diameter of the Mycielskian of a graph, we first study the notion of containers in the Mycielskian.

2.1.1 Containers in the Mycielskian of a graph

In the following five propositions, G is a connected graph with $\delta(G) = \kappa(G) = k$. Then by Lemma 2.0.1, $\kappa(\mu(G)) = \kappa(G)+1 = k+1$.

Proposition 2.1.1. *For every $u, v \in V$, there exists a $k+1-$container in $\mu(G)$ of length $\max\{l_k(u, v)$ in $G, 4\}$ between u and v.*

Proof. Let $\{p_1, p_2, \cdots, p_k\}$ be a $k-$container between u and v in G with length $l_k(u, v)$. Then this will also be a $k-$container between u and v in $\mu(G)$. Let p_{k+1} be the path $uu_1^{'}wu_{n-1}^{'}v$, where $u_1 \in N(u)$ and $u_{n-1} \in N(v)$. Then, $\{p_1, p_2, \cdots, p_k, p_{k+1}\}$ will be a $k+1-$container between u and v in $\mu(G)$ and the length of this container is $\max\{l_k(u, v)$ in $G, 4\}$. □

Proposition 2.1.2. *For $u \in V$ and $v \in V'$, there exists a $k+1-$container between u and v in $\mu(G)$ of length* $\max\{l_k(u,x)$ *in* $G, 3\}$*, where x is the twin of v.*

Proof. Let $v = x', x \in V$ and $C_k(u,x) = \{p_i : uu_1^i u_2^i \cdots u_{n-1}^i x, i = 1, \cdots, k\}$ be a $k-$container between u and x in G. Let $C_{k+1}(u,v) = \{p_i^{'} : uu_1^i u_2^i \ldots u_{n-1}^i x', i = 1, \cdots, k\} \cup \{\ u(u_1^i)'wx'$, for some $i\}$ in $\mu(G)$. Then $C_{k+1}(u,v)$ is a $k+1-$ container between u and v in $\mu(G)$ and the length of this container is $\max\{l_k(u,x)$in $G, 3\}$. □

Proposition 2.1.3. *For every $u, v \in V'$, there exists a $k+1-$container between u and v in $\mu(G)$ of length* $\max\{l_k(x,y)$ *in* $G, 2\}$*, where x and y are the twin of u and v respectively.*

Proof. Let $u = x'$ and $v = y'$, where $x, y \in V$. Consider a container $C_k(x,y)$ in G and replace x and y by u and v to form $C_k(u,v)$. Then $C_k(u,v) \cup \{uwv\}$ is a $k+1-$ container between u and v in $\mu(G)$ and is of length $\max\{l_k(x,y)$ in $G, 2\}$. □

Proposition 2.1.4. *For every $u \in V$, there exists a $k+1-$ container between u and the root vertex w in $\mu(G)$ of length 2 or 3.*

Proof. Since $d(u) \geq k$ in G, every $u \in V \subseteq V(\mu(G))$ has at least k neighbors in V'. Let $C_i(x,y) = uu_i^{'}w$, where $u_i^{'} \in N(u) \cap V'$. Then $C_i(x,y), i \geq k$ is a $u - w$ container of width at least k in $\mu(G)$. If $d(u) > k$, this gives the required $(k+1)-$container by considering any of the $(k+1)$ u_i's. If $d(u) = k$, then $C_k(x,y) \cup \{uu_iu'w\}$ will be the required $(k+1)-$ container. Thus, in this case $l_{k+1}(u,v)$ is either 2 or 3. □

Proposition 2.1.5. *For every $u \in V'$, there exists a $k+1-$ container between u and the root vertex w in $\mu(G)$ of length 4.*

Proof. Let $u = x', x \in V$. Consider the k paths $uu_iu_{i+1}u_i'w$ for $i = 1, 2, 3, \ldots, k$ where $u_i \in N(x) \cap V$ and $u_{i+1} \in N(u_i) \cap V$. For each $v \in V$, $N(v) \cap V \geq k$. Hence we are left with $k, k-1, k-2 \cdots k-(k-1)$ number of choices in the subsequent selection of u_i as well as u_{i+1} and therefore these k paths are vertex disjoint. These k paths, together with the edge uw will then form a $(k+1)-$ container in $\mu(G)$ and the length of this container is 4. □

Proposition 2.1.6. *Let G be a connected graph with $\delta(G) = \kappa(G) + i, i > 0$. Then, between any two vertices $u, v \in V$ there exists a container in $\mu(G)$ of width $\kappa(\mu(G))$ such that*

$l_{\kappa(\mu(G))}(u,v) \leq \max\{l_{\kappa(G)}(u,v)\ in\ G, 4\}$.

Proof. Let $\kappa(G) = k$.

Case 1: $\delta(G) = k + i$, $0 < i < k$.

In this case, $\kappa(\mu(G)) = k+i+1$ by the Lemma 2.0.2. We claim that for every $u, v \in V$, there exists a $u - v$ container of width $k + i + 1$ in $\mu(G)$. For this, consider a $u - v$ path in G, say $uu_1u_2...u_{n-1}v$. Corresponding to this path in G, there are two vertex disjoint paths in $\mu(G)$, namely $uu_1^{'}u_2u_3^{'}...u_{n-2}u_{n-1}^{'}v$ and $uu_1u_2^{'}u_3u_4^{'}...u_{n-1}v$ or $uu_1^{'}u_2u_3^{'}...u_{n-2}^{'}u_{n-1}v$ and $uu_1u_2^{'}u_3u_4^{'}...u_{n-1}^{'}v$ (according as n is even or odd). Thus any $k-$container in G will give a $2k-$container in $\mu(G)$. Any $(k+i+1)$, $i > 0$ paths from this container, will give a $(k+i+1)-$container in $\mu(G)$ of length at most $l_k(u,v)$ in G.

Case 2: $\delta(G) = k + i$, $i \geq k$

Here $\kappa(\mu(G)) = 2k + 1$, by the Lemma 2.0.3. Consider the $2k-$container obtained in Case 1 and include the path $uu_k^{'}wv_k^{'}v$, where $u_k \in N(u) \cap V$ and $v_k \in N(v) \cap V$ to that container(such vertices u_k and v_k exists, as degree of both u and v is at least $2(k+i)$). This is a container in $\mu(G)$ of width $2k+1$ and the length of this container is $\max\{l_k(u,v)$ in $G, 4\}$. $\square$

Proposition 2.1.7. *Let G be a connected graph with $\delta(G) = \kappa(G) + i$, $i > 0$. Then, between any two vertices $u, v \in V(\mu(G))$ there exists a container in $\mu(G)$ of width $\kappa(\mu(G))$ such that $l_{\kappa(\mu(G))}(u, v) \leq \max\{l_{\kappa(G)}(u, v)$ in $G, 4\}$.*

Proof. For vertices in V the result follows from Proposition 4.4.4 and in other cases, the proof is similar to that in Propositions 2.2, 2.3, 2.4 and 2.5. □

2.1.2 Wide diameter

In this section, we determine the wide diameter of the Mycielskian and its iterates.

Theorem 2.1.8. *If G is a connected graph, then*

$$D_{\kappa(\mu(G))}(\mu(G)) = \max\{D_{\kappa(G)}(G), 4\}.$$

Proof. It follows from the propositions in Section 2.1.1, that $D_{\kappa(\mu(G))}(\mu(G)) \leq \max\{D_{\kappa(G)}(G), 4\}$. Now, to prove the reverse inequality, consider a pair of vertices $u, v \in V$ such that $d_{\kappa(G)}(u, v) = D_{\kappa(G)}(G)$. Then, $d_{\kappa(\mu(G))}(u, v) = \max\{D_{\kappa(G)}(G), 4\}$

and hence $D_{\kappa(\mu(G))}(\mu(G)) \geq \max\{D_{\kappa(G)}(G), 4\}$. $\square$

Corollary 2.1.9. *If G is a connected graph, then*

$$D_{\kappa(\mu^n(G))}(\mu^n(G)) = \max\{D_{\kappa(G)}(G), 4\}, \ n \geq 2.$$

2.2 Fault diameter

Fault diameter of the Mycielskian and its iterates is determined in this section. Exact value for the fault diameter is obtained only when $\delta(G) = \kappa(G)$ and in other situations, we could only get a sharp upper bound.

Theorem 2.2.1. *Let G be a connected graph with $\delta(G) = \kappa(G)$. Then $f(\mu(G)) = \max\{f(G), 4\}$.*

Proof. Let $\delta(G) = \kappa(G) = k$. Then $\kappa(\mu(G)) = k + 1$. Take the vertices u, v in G which are at a distance $f(G)$, when $k - 1$ vertices are deleted from G. We claim that when k vertices are deleted from $\mu(G)$, the maximum possible distance between u and v in $\mu(G)$ is $\max\{f(G), 4\}$. For this, let $S \subseteq V(\mu(G))$ with $|S| = k$ be deleted from $V(\mu(G))$. Then,

Case 1: $w \notin S$.

In this case, $S \subseteq V$ or $S \subseteq V'$ or S intersects both V and V'. If $S \subseteq V$, then vertices in V of $\mu(G)$ are connected through their twins and maximum distance between u and v is 4. If $S \subseteq V'$, then maximum distance between u and v is $d(G)$. If S intersects both V and V', then the distance between u and v is maximum when $|S \cap V| = k - 1$ and the corresponding distance is $\max\{f(G), 4\}$. Thus in this case, the maximum possible distance between u and v is $\max\{f(G), 4\}$.

Case 2: $w \in S$.

If $w \in S$, then the distance between u and v is maximum, when the remaining $k - 1$ vertices are deleted from V and the corresponding distance is $f(G)$.

Thus we have a pair of vertices u and v in $\mu(G)$ for which the maximum distance between them is $\max\{f(G), 4\}$, when k vertices are deleted and therefore $f(\mu(G)) \geq \max\{f(G), 4\}$.

Next, we show that $f(\mu(G)) \leq \max\{f(G), 4\}$. For this, consider the following cases.

Case 1: $S \subseteq V$.

In this case, the $< V >$ becomes disconnected and the vertices in V are connected by the path through 'w'. Therefore the maximum distance between them is 4. For all other pairs of vertices, maximum distance possible is less than 4.

Case 2: $S \subseteq V'$.

When all the k vertices are deleted from V', the maximum distance between any pair of vertices in V and that between $u \in V$ and $v' \in V'$ is diam(G). For other pairs of vertices, the maximum distance is 3. Therefore, in this case diam$(G \setminus S)$ is max$\{$diam(G), 3$\}$.

Case 3: S intersects both V and V'.

In this case, the maximum distance occur between the vertices in V and the corresponding distance is $\max\{f(G), 4\}$

Case 4: $w \in S$.

If $w \in S$, then the remaining $k-1$ vertices can be deleted from V, or from V' or from both. But in all these cases maximum distance possible is $f(G)$, which occur when the vertices are deleted entirely from V or from V'.

Thus we have $f(\mu(G)) \leq \max\{f(G), 4\}$ and the result follows. □

Corollary 2.2.2. *Let G be a connected graph with $\delta(G) = \kappa(G)$. Then $f(\mu^n(G)) = \max\{f(G), 4\}$, $n \geq 2$.*

Theorem 2.2.3. *Let G be a connected graph with $\delta(G) = \kappa(G)+$*

$i, 0 < i < \kappa(G)$, then $f(\mu(G)) \leq \max\{f(G), 4, diam(G) + 2\}$.

Proof. Let $\kappa(G) = k$ and S be any set of $\delta(G)$ vertices. By Lemma 2.0.2, $\kappa(\mu(G)) = k + i + 1$, $0 < i < \kappa(G)$. To determine $f(\mu(G))$, consider the following cases.

Case 1: $w \notin S$.

In this case, $S \subseteq V$, $S \subseteq V'$, or S intersects both V and V'. In any situation, the maximum distance possible is $\max\{f(G), 4\}$, which occurs when $k - 1$ vertices from V and the remaining vertices from V' are deleted.

Case 2: $w \in S$.

If $w \in S$, then the remaining $\delta(G) - 1$ vertices are to be deleted from $V \cup V'$. Now, we have the following sub cases.

Case 2a: The remaining vertices are deleted from V.

In this case, the maximum distance occurs between those u' and v', for which all the neighbors except one, belong to $S \subset V$ and the corresponding distance is 2 more than diam(G).

Case 2b: The remaining vertices are deleted from V'.

In this case, the paths in V are unaffected and hence the distance between any pair of vertices in $\mu(G)$ is same as that in G. Thus the maximum possible distance is diam (G).

Case 2c: Some vertices are deleted from V and the remaining from V'.

The maximum distance occurs when $\kappa(G)$ vertices from V and $\delta(G)-\kappa(G)-1$ from V' are deleted. Since any vertex $u \in V$ has at least $\delta(G)$ neighbors in V', there exists at least one neighbor in V' for every $u \in V$. Thus the maximum possible distance between the vertices in this case is $\max\{f(G), \text{diam}(G)+2\}$.
Hence $f(\mu(G)) \leq \max\{f(G), 4, \text{diam}(G)+2\}$. □

Theorem 2.2.4. *Let G be a connected graph with $\delta(G) = \kappa(G) + i$, $i \geq \kappa(G)$. Then $f(\mu(G)) \leq \max\{f(G), 4, \mathit{diam}(G)+2\}$.*

Proof. In this case $\kappa(\mu(G)) = 2\kappa(G)+1$ and the proof is similar to that of Theorem 4.4.5. □

2.3 Diameter vulnerability

It may be noted that $\kappa'(\mu(G)) = \delta(G)+1$ and hence we find $f(\delta(G), \text{diam}\mu(G))$, the maximum diameter of $\mu(G)$ obtained by deleting $\delta(G)$ edges, to study the diameter vulnerability of $\mu(G)$.

Theorem 2.3.1. *Let G be a connected graph with $\mathit{diam}(G) = d$*

and diam $(\mu(G)) = d_\mu$. *Then,* $f(\delta(G), d_\mu) = \max\{d, 3\}$.

Proof. To find $f(\delta(G), d_\mu)$, we consider the following cases of edge deletions.

Case 1: The deleted edges are of the form uv, where $u, v \in V$. If the edges of the form uv are deleted, then the vertices in V are connected by a path through the twins u' and v'. Therefore, they can be at a distance $\min\{d, 4\}$. The other distances are unaffected by this, except possibly that between any vertex u and its twin u'. If the edges are deleted in such a way that the shortest path between u and u' is affected, then the distance $d(u, u')$ is 3 by using the path $u - v' - w - u'$.

Case 2: The deleted edges are of the form uv', where $u \in V$ and $v' \in V'$.

In this case, the distance between u and v is unaffected. If the edges deleted are of the form uv' where $u \in N(v)$, then distance between v and its twin v' is affected and we have to take the path $v - u' - w - v'$ and hence the distance becomes 3. The distance between u and v' is $\min\{d, 4\}$, as we have to consider either the path $u - u_1 - u_2 - \dots u_{n-1} - v'$ where $u - u_1 - u_2 - \dots u_{n-1} - v$ is a path in V or the path $u - u_1 - u' - w - v'$. The maximum possible distance between u and w occurs, when all the edges uv' incident on the vertex u with degree $\delta(G)$ are deleted. Then, we have to take the $u-w$ path as $u-x-u'-w, x \in N(u)$, and hence the distance is 3. Thus if the edges of the form uv' are deleted, then maximum $d(u,v) = \min\{5,d\}$, maximum $d(u,v') = \min\{d,4\}$, maximum $d(u,u') = 3$ and maximum $d(u,w) = 3$.

Case 3: The deleted edges are of the form $v'w$, where $v' \in V'$. In this case maximum $d(u,v) = \min\{d,5\}$. The maximum possible distance between u' and v' is d as we have to take the path $u' - u_1 - \cdots - v_1 - v'$, where $u_1 \in N(u)$ and $v_1 \in N(v)$. Since u has at least $\delta(G)$ neighbors in both V and V' maximum distance between u and w is 3 and that of v' and w is also 3 by considering the $v'-w$ path $v'-u-v_1'-w$, $u \in N(v') \cap V$ and $v_1' \in N(u) \cap V'$(such a u exists as the minimum degree of any v' is $\delta+1$). Thus in this case, maximum $d(u,v) = \min\{d,5\}$, maximum $d(u',v') = d$, maximum $d(u,w) =$ maximum $d(v',w) = 3$. Hence $f(\delta(G), d_\mu) = \max\{d,3\}$. $\square$

Corollary 2.3.2. *If $d \geq 4$, then*

$$f(\delta(G)+n-1, diam(\mu^n(G)) = 4$$

.

2.4 $(l, k)-$dominating number

In order to obtain the $(l, k)-$dominating number of the Mycielskian, we first prove the following lemma.

Lemma 2.4.1. *If $w \notin S$, then S is not a minimum $(l, k)-$ dominating set in $\mu(G)$, where $l \geq 4$.*

Proof. Suppose that S is a minimum $(l, k)-$dominating set in $\mu(G)$ and $w \in S$. Let $S' = S - \{w\}$ and replace the $x - w$ path and the $y' - w$ path in the respective containers by the paths $x - y' - w - z' - s$ and $y' - w - z' - s$ respectively, where $s \in S$ and $z \in N(s) \cap V$. There exist k disjoint paths $w - z' - s$ between w and S also. Thus S' becomes a $(l, k)-$ dominating set of $\mu(G)$, which contradicts the definition of S. □

Theorem 2.4.2. *For a connected graph G, $\gamma_{l,\kappa(\mu(G))}(\mu(G)) = \gamma_{l,\kappa(G)}(G), l \geq 4$.*

Proof. **Case 1:** $\delta(G) = \kappa(G) = k$. Let S be a minimum

$(l,k)-$dominating set in G, $l \geq 4$. Then we claim that S is also a $(l,k+1)-$dominating set in $\mu(G)$.

Clearly $S \subset V$ in $\mu(G)$. Now, consider the following cases.

Case 1a: $x \in V - S$

Consider the $(x,S)-$container in G. To this container include a $(x,S)-$path through w disjoint with the paths already considered. This shows that every $x \in V - S$ is $(l,k+1)-$ dominated by S.

Case 1b: $x \in V'$

Let $x = y', y \in S$.

As $d(y) \geq k$, there exists a (x,S) container of width at least k namely xz_iy, where $z_i \in N(y) \cap V$. If $d(y) > k$, then this will give the required $(x,S)-$ container. Otherwise we have to take one more path through w to get the required container.

If $x = y', y \notin S$, then take the $(y,S)-$container of width k in G, replace y by x and include the path $xwy'_1s,\ s \in S$ and $y_1 \in N(s)$.

Case 1c: $x = w$

In this case, the set of all paths $\mathcal{P} = \{xy'_is, s \in S$ and $y_i \in N(s)\}$will be the required container if $d(s) > k$. If $d(s) = k$, then the required container is $\mathcal{P} \cup \{xy'_js_1 : y_j \neq y_i \in N(s_1), s_1 \neq s \in S\}$.

Thus S is also a $(l,k+1)-$ dominating set in $\mu(G)$, $l \geq 4$ and therefore $\gamma_{l,k+1}(\mu(G)) \leq \gamma_{l,k}(G)$.

Case 2: $\delta(G) = \kappa(G)+i$, $i > 0$. Let S be a minimum $(l,\kappa(G))-$ dominating set in $G, l \geq 4$. Then, corresponding to each path in the (x,S) container in G, we have two vertex disjoint paths in $\mu(G)$ (Proposition 4.4.4) and it can be shown that S is also a $(l,\kappa(\mu(G)))-$dominating set in $\mu(G)$ as in Case 1. Hence it follows that $\gamma_{l,\kappa(\mu(G))}(\mu(G)) \leq \gamma_{l,\kappa(G)}(G)$, $l \geq 4$.

To prove the reverse inequality consider a minimum $(l,\kappa(\mu(G))-$ dominating set S of $\mu(G)$, $l \geq 4$ and define $S' = \{x : x$ or $x' \in S\}$. Then S' is a $(l,\kappa(G))-$dominating set of G and the fact that $w \notin S$ gives $\gamma_{l,\kappa(\mu(G))}(\mu(G)) \geq \gamma_{l,\kappa(G)}(G)$. $\square$

Corollary 2.4.3. *For a connected graph G, $\gamma_{l,\kappa(\mu^n(G))}(\mu^n(G)) = \gamma_{l,\kappa(G)}(G), l \geq 4$.*

Chapter 3

Some diameter notions of the generalized Mycielskian of a graph

Generalized Mycielskian of a graph is the natural generalization of the Mycielskian of a graph which we discussed earlier

Some results of this chapter are included in the following paper.
Savitha K.S, Chithra M.R, A. Vijayakumar, Some diameter notions of the generalised Mycielskian of a graph, LectureNotes in Comput. Sci.(Proceedings of the International Conference on Theoretical Computer Science and Discrete Mathematics(Springer), Kalasalingam University, Krishnankoil,2016).(to appear).

that it preserves some nice properties of a good interconnection network. Thus it is natural to extend the study of the network topological properties to generalised Mycielskian also. It is known that in any interconnection network, diameter is an important basic parameter for communication as it determines maximum communication delay in the network and we could see that even diameter was unknown for generalized Mycielskian. In this chapter, we study the diameter and its variability by the addition and deletion of edges in the generalized Mycielskian of a graph.

3.1 Diameter of $\mu_m(G)$

In[23], the diameter of the Mycielskian of a graph G is found to be $\min\{\max\{2, \mathrm{diam}(G)\}, 4\}$. In this section, we determine the diameter of the generalized Mycielskian of a graph.

Theorem 3.1.1. *Let $diam(G)$ be the diameter of a connected graph G. Then the diameter of $\mu_m(G)$ is given by $diam\,(\mu_m(G)) = \min\{\max\{m+1, diam(G)\}, 2(m+1)\}$.*

Proof. We prove this result by considering the following cases.

Case:1 $\operatorname{diam}(G) \leq m+1$.

In this case, we claim that $\operatorname{diam}(\mu_m(G)) = m+1$. For this, consider the vertices v_i^0 and z and let $v_i^0 v_{i+1}^0 \in E^0$. Then $d(v_i^0, z) = m+1$ by taking the path $v_i^0 - v_{i+1}^1 - v_i^2 - \cdots - v_i^m (\text{or } v_{i+1}^m) - z$ according as m is even (or odd). Therefore $\operatorname{diam}(\mu_m(G)) \geq m+1$. Next, we show that for any pair of vertices u and v, $d_{\mu_m(G)}(u,v) \leq m+1$.

Case:1a $u, v \in V^i$.

Let $u = v_j^i$ and $v = v_k^i$. If $u, v \in V^0$, then the distance between u and v in $\mu_m(G)$ is same as that in G. Hence $d_{\mu_m(G)}(u,v) \leq d(G) \leq m+1$. For $u, v \in V^1$, if v_j^0 and v_k^0 are adjacent, then $d(v_j^1, v_k^1)$ is 3 by taking the path $v_j^1 - v_k^0 - v_j^0 - v_k^1$. If they are non adjacent, we consider the path $v_j^1 - u_1^0 - \cdots - u_{n-1}^0 - v_k^1$, where $v_j^0 - u_1^0 - \cdots - u_{n-1}^0 - v_k^0$ is a shortest $v_j^0 - v_k^0$ path in G. So $d_{\mu_m(G)}(v_j^1, v_k^1) = d_G(v_j^0, v_k^0)$ and hence $d_{\mu_m(G)}(u,v) \leq \operatorname{diam}(G) \leq m+1$.

Now Let $u = v_j^i$ and $v = v_k^i$, $i > 1$, then

$$d_{\mu_m(G)}(u,v) = \begin{cases} d_G(v_j^0, v_k^0) & \text{if } d_G(v_j^0, v_k^0) \text{ is even} \\ \min\{2(m-i)+2, 2i+1\} & \text{if } d_G(v_j^0, v_k^0) \text{ is odd} \end{cases}$$

for, if $d_G(v_j^0, v_k^0)$ is even, we take the path $v_j^i - u_1^{i+1} - u_2^i - \cdots - v_k^i$, where $v_j^0 - u_1^0 - u_2^0 \cdots - v_k^0$ is a shortest $v_j^0 - v_k^0$ path in G and if $d(u,v)$ is odd, we have to take either the path $v_j^i - u_1^{i-1} - \cdots - u_j^0 - u_{j+1}^0 - u_j^1 - \cdots - v_k^i$ or the one which pass through z namely $v_j^i - u_1^{i+1} - u_2^{i+2} - u_3^{i+3} - \cdots - z - u_i^m - u_{i\pm 1}^{m-1} - \cdots - v_k^i$. This shows that $d(v_j^i, v_k^i) \leq m+1, i \geq 1$.

Case:1b $u \in V^i, v = z$.

Let $u = v_j^i$ and $v_j^0 v_{j+1}^0$be an edge in G. Then, there exists the path $v_j^0 - v_{j+1}^1 - v_j^2 - \cdots - v_j^m(v_{j+1}^m) - z$ of length $m+1$ between v_j^0 and z. For all other $v_j^i, i > 1$ there exists the path $v_j^i - v_{j+1}^{i+1} - v_j^{i+2} - \cdots - v_j^m(v_{j+1}^m) - z$ of length less than $m+1$ between v_j^i and z. Thus $d(v_j^i, z) \leq m+1$.

Case:1c $u \in V^i, v \in V^j, i \geq 0, j \geq 1, i < j$.

Case:1c(i) $u = v_k^i$ and $v = v_k^j$.

It is easy to see that $d_{\mu_m(G)}(v_k^0, v_k^1) = 2$. Now, consider $d(v_k^i, v_k^j)$, $i \geq 0, j > 1$. Let v_k^0 be adjacent to v_l^0 in G. If $j - i$ is even, then we have the path $v_k^i - v_l^{i+1} - v_k^{i+2} - \cdots - v_k^j$ between v_k^i and v_k^j of length $j-i$. If $j-i$ is odd, either we can take the path $v_k^i - v_l^{i-1} - v_k^{i-2} \cdots - v_k^0(v_l^0) - v_l^0(v_k^0) - v_k^1(v_l^1) - v_l^2(v_k^2) - \cdots - v_k^j$ or we can take $v_k^j - v_l^{j+1} - \cdots v_k^m(v_l^m) - z - v_l^m(v_k^m) - v_l^{m-1}(v_k^{m-1}) - \cdots - v_k^i$. Hence for $i \geq 0, j > 1, d(v_k^i, v_k^j) \leq \min\{i+j+1, 2(m+1) - (i+j)\}$. Thus, we get $d(v_k^i, v_k^j) \leq m+1$.

Case:1c(ii) $u = v_k^i$ and $v = v_l^j, k \neq l$.

If $v_k^0 - u_1^0 - u_2^0 \cdots - u_{n-1}^0 - v_l^0$ is a path in G, then $v_k^0 - u_1^0 - u_2^0 - \cdots u_{n-1}^0 - v_l^1$ is a path in $\mu_m(G)$ and hence $d(v_k^0, v_l^1) \leq \text{diam}(G)$. Now, consider the pair $(v_k^i, v_l^j), i \geq 0, j \geq 2$. First suppose that v_k^0 and v_l^0 are adjacent. If $j - i$ is odd, then we have the path $v_k^i - v_l^{i+1} - v_k^{i+2} - v_l^{i+3} \cdots - v_l^j$ and hence $d(v_k^i, v_l^j) \leq j - i \leq m + 1$. If $j - i$ is even, either we take $v_k^i - v_l^{i-1} - v_k^{i-2} - \cdots - v_k^0(v_l^0) - v_l^1(v_k^1) - \cdots - v_l^j$ or we take $v_k^i - v_l^{i+1} - v_k^{i+2} - \cdots - v_k^m(v_l^m) - z - v_l^m(v_k^m) - \cdots - v_k^{n-1} - v_l^j$. Hence $d(v_k^i, v_l^j) \leq min\{j + i + 1, 2(m + 1) - (j + i)\}$. If v_k^0 and v_k^l are not adjacent in G, then take a shortest $v_k^0 - v_l^0$ path say $v_k^0 - u_1^0 - u_2^0 - \cdots - u_{n-1}^0 - v_l^0$ in G. Corresponding to this path, we have, the path say $P = v_k^i - u_1^{i+1} - u_2^{i+2} - \cdots - v_l^j$ of length $d(u, v)$ in $\mu_m(G)$ if $d(v_k^0, v_l^0) \leq j - i$. If $j - i < d(v_k^0, v_l^0)$, instead of P we have the path $P' = v_l^j - u_{n-1}^{j-1} - \cdots - v_k^r - u_1^{r-1} - \cdots - u^0 - u_1^1 - u_2^2 - \cdots - v_k^i$, where $r = j - i - d(u, v)$. Therefore $d(v_k^i, v_l^j) \leq m + 1$ and hence $\text{diam}(\mu_m(G)) \leq m + 1$.

Case 2: $m + 1 < \text{diam}(G) < 2(m + 1)$.

In this case, proceeding on similar lines as in case 1, we get $d(u, v) \leq \text{diam}(G)$, $u, v \in V(\mu_m(G))$. If v_i^0 and v_j^0 are the diametral vertices in G, then in $\mu_m(G)$ also, we have v_i^0 and v_j^0 at distance diam(G) and hence it follows that diam$(\mu_m(G)) = d(G)$.

Case 3: diam$(G) \geq 2(m+1)$.

The diametral vertices in G in this case, are at a distance $2(m+1)$ as the shortest path between them is through z in $\mu_m(G)$. For every pair of vertices, we can show that there exists a path of length less than or equal to $2(m+1)$ as in case 1. Hence diam$(\mu_m(G)) = 2(m+1)$ in this case. $\square$

3.2 Diameter variability

Here, we determine $D^{+0}(\mu_m(G)), D^{+1}(\mu_m(G)), D^{-1}(\mu_m(G))$.

Theorem 3.2.1. *Let G be a connected graph such that $G \ncong K_{1,n-1}$ and $m \geq 1$ be an integer. Then*

$$D^{+0}(\mu_m(G)) \geq \begin{cases} 2e+k-(n+1+\sum_{i=1}^{k} e_i), & \\ \qquad\qquad\qquad \text{if } diam(G) \leq m+1, \\ t(2e+k-(n+1+\sum_{i=1}^{k} e_i)), & \\ \qquad\qquad\qquad \text{if } diam(G) > m+t,\ 1 \leq t \leq m, \\ \max\{m(2e+k-(n+1+\sum_{i=1}^{k} e_i)), e\}, & \\ \qquad\qquad\qquad \text{if } diam(G) \geq 2(m+1). \end{cases}$$

where n is the number of vertices in G, e the number of edges in G and e_i's, $i = 1, 2, 3 \cdots, k$ are the number of pendant edges attached to the vertex v_i of G.

Proof. If $\text{diam}(G) \leq m + 1$, then remove all the edges of the form $u_i - u_{i+1}$ from the $\lceil \frac{m-1}{2} \rceil^{th}$ level except the pendant edges, one edge from all but one vertex with $d(v) > 1$ and two from one vertex with $d(v) > 1$. More specifically, let $v_1 - v_2 - \cdots - v_d$ be a diametral path in G where, $v_{e_1}, v_{e_2}, \cdots, v_{e_k}$ are the k vertices v_i with e_i pendant vertices. Then remove all the edges except one from $\{v_1, v_2, \cdots, v_d\} \setminus \{v_{e_1}, v_{e_2}, \cdots, v_{e_k}, v_{d-1}\}$ and remove all the edges except two from v_{d-1} (See Fig:3.2). This removal of edges will not affect the shortest path between the vertices in $\mu_m(G)$ is clear from the discussion of shortest paths in the Section 4.4.4. If $\text{diam}(G) > m+t, 1 \leq t \leq m$, these set of edges can be removed from t levels $m-1, m-2, \cdots, m-t$. If $\text{diam}(G) \geq 2(m+1)$, either the removal of the edges of the above type from m levels or the removal of the edges from the copy of G in $\mu_m(G)$ will not change the diameter of $\mu_m(G)$. $\square$

Illustration of Theorem 3.2.1 is shown in the figure 3.1.

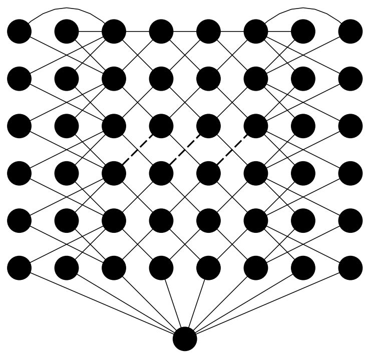

Figure 3.1: Dashed lines are the deleted edges

Theorem 3.2.2. *Let G be any connected graph and $m \geq 2$ be an integer. Then $D^{+1}(\mu_m(G)) = 1$ if and only if diam $(G) \leq m+1$ and G has at least one pendant edge.*

Proof. First, suppose that G has at least one pendant edge and $\mathrm{diam}(G) \leq m+1$. Then $\mathrm{diam}(\mu_m(G)) = m+1$. Let $v_i^0 - v_j^0$ be a pendant edge in G. Consider the pair of vertices (v_j^1, z) in $\mu_m(G)$, which are at distance m. If the edge $v_i^2 - v_j^1$ is deleted, then, $d(v_j^1, z) = m+2$ by the path $v_j^1 - v_i^0 - v_k^1 - v_i^2 - v_j^3 - \cdots - v_i^m - z$ or $v_j^1 - v_j^0 - v_k^1 - v_i^2 - v_j^3 - \cdots - v_j^m - z$ according as m is odd or even respectively (Fig:3). For all other vertices x in $\mu_m(G)$, $d(v_j^1, x) \leq m+2$, since the distance between any other pair is not affected by the removal of this edge. Therefore $D^{+1}(\mu_m(G)) = 1$.

Conversely, assume that $D^{+1}(\mu_m(G)) = 1$. If possible, let

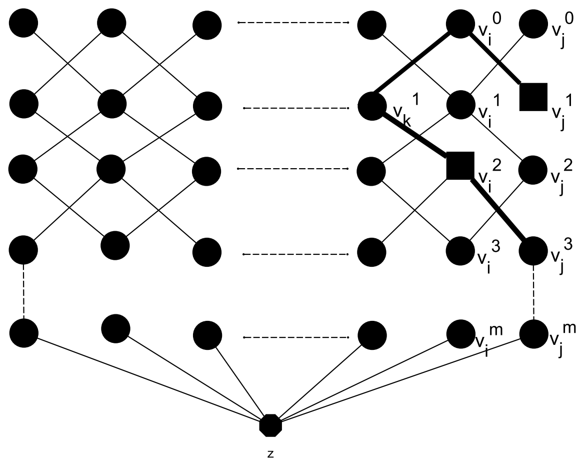

Figure 3.2

diam$(G) \leq m+1$ and G has no pendant edges. Then, diam$(\mu_m(G))$ $= m+1$. Let an edge $v_i^0 - v_{i+1}^0$, be deleted. Then, $d(v_x^0, v_y^0) \leq$ $diam(G)$ by a path, $v_x^0 - v_{(x+1)}^0 - v_{(x+2)}^0 \cdots - v_i^0 - v_{i+1}^1 - v_{i+2}^0 - \cdots -$ v_y^0. Clearly, the distance between any other pair is not affected by the removal of this edge. If an edge $v_k^i - v_l^{i+1}$ is deleted, then $d(v_k^i, v_l^{i+1})$ is affected. Since, $\delta(G) \geq 2$, v_k^i is adjacent to some other vertex v_j^{i+1}, in the $i+1^{th}$ level. Thus, $d(v_k^i, v_l^{i+1}) = 3$, by a path, $v_k^i - v_j^{i+1} - v_k^{i+2} - v_l^{i+1}$. For any other pair (v_k^i, v_x^j), the edge $v_k^i - v_l^{i+1}$ in any $v_k^i - v_x^j$ path can be replaced by the edge $v_k^i - v_j^{i+1}$ for some $v_j^{i+1} \in N(v_k^i)$ and hence $d(v_k^i, v_x^j) \leq m+1$. The removal of an edge $v_i^m - z$ also will not affect the diameter as it changes only the distance between v_i^m and z to 3. Thus we get a contradiction to the fact that $D^{+1}(\mu_m(G)) = 1$ and therefore G has at least one pendant edge.

Next, suppose that diam$(G) > m+1$ and G has pendant edges. Since diam$(G) > m+1$, diam$(\mu_m(G))$ =diam(G). If any edge $v_x^0 - v_y^0$ is removed, the distance is unaffected in $\mu_m(G)$ as alternate paths exist through the duals. Let $v_k^0 - v_l^0$ be a pendant edge in G. If an edge of the form $v_k^{i+1} - v_l^i$ is removed from $\mu_m(G)$, then $d(v_k^{i+1}, v_l^i) = 3$ by the path $v_l^i - v_k^{i-1} - v_x^i - v_k^{i+1}$, where $v_x^0 \in N(v_k^0)$ for $i \neq 0$ and for $i = 0$, $d(v_k^1, v_l^0) = 3$ by the path $v_l^0 - v_k^0 - v_x^0 - v_k^1$. Thus the distance between v_l^0 and z becomes $m+2$ by the path $v_l^0 - v_k^0 - v_l^1 - v_k^2 - \cdots - z$. For $i > 0$, $d(v_l^i, z) < m+2$, by the path $v_l^i - v_k^{i-1} - v_x^i - v_k^{i+1} - v_l^{i+2} - \cdots - v_k^m - z$ or $v_l^i - v_k^{i-1} - v_x^i - v_k^{i+1} - v_l^{i+2} - \cdots - v_l^m - z$ according as m is odd or even respectively, where $d(v_k^{i+1}, v_l^i) = 3$ and $d(v_k^{i+1}, z) < m-1$. The other edge removals will not affect the distance as there are alternate paths. Thus the removal of any single edge does not change the diameter and hence a contradiction. □

Theorem 3.2.3. $D^{+1}(\mu_m(G)) \leq \begin{cases} 2\delta(G)-1 & \\ \qquad\qquad \textit{if } diam(G) \leq m+1 \\ \delta(G) & \\ \qquad \textit{if } m+1 < diam(G) < 2m+1 \\ D^{+1}(G) & \\ \qquad\qquad \textit{if } diam(G) = 2m+1 \\ \Delta(G) & \\ \qquad\qquad \textit{if } diam(G) \geq 2(m+1). \end{cases}$

Proof. To obtain this upper bound, we consider the following cases.

Case 1: $\mathrm{diam}(G) \leq m+1$

In this case, $\mathrm{diam}(\mu_m(G)) = m+1$. Let v_i^0 be a vertex with minimum degree in G. Then $d(v_i^0)$ in $\mu_m(G)$ is $2\delta(G)$. Delete all the edges incident with v_i^0 except one that is adjacent to a vertex in level 0. This deletion will result in a graph with $d(v_i^0, z) = m+2$.

Case 2: $m+1 < \mathrm{diam}(G) < 2m+1$

Since $\mathrm{diam}(\mu_m(G)) = \mathrm{diam}(G)$ in this case, If we delete all the edges incident on a vertex v_i^0 with minimum degree then $d(v_i^0, v_i^1) = 2m+1$. Therefore, $D^{+1}(\mu_m(G)) \leq \delta(G)$.

Case 3: $\operatorname{diam}(G) = 2m + 1$

In this case, delete those edges that are deleted to increase the diameter of G by at least 1 from level 0 of $\mu_m(G)$. This will clearly increase the diameter of $\mu_m(G)$ by at least 1 and hence $D^{+1}(\mu_m(G)) \leq D^{+1}(G)$.

Case 4: $\operatorname{diam}(G) \geq 2(m + 1)$

Here, the shortest paths are through z. Let u^0, v^0 be a pair of diametral vertices in G and let $d(u^0) \leq d(v^0)$. Delete all the edges $u_i^m - z$, $u_i \in N(u^0)$. Then $d_{\mu_m(G)}(u^0, v^0) > 2(m + 1)$. Hence $D^{+1}(\mu_m(G)) \leq d(u^0) \leq \Delta(G)$. □

Note: The above bound can be verified easily. For example if we consider $\mu_3(C_6)$, it is true that $D^{+1}(\mu_3(C_6)) = 2$.

Theorem 3.2.4. *Let G be a connected graph with $D^{-1}(G) = 1, e = v_i^0 - v_j^0$ be an edge in G such that $diam(G+e) = diam(G) - 1$ and $k = \min\{d_G(v_i^0, v_x^0), d_G(v_j^0, v_x^0)\}$, where v_x^0 is an end point of any diametral path in G. Then $D^{-1}(\mu_m(G)) = 1$*

$$\textit{if and only if } m \leq \begin{cases} k + \frac{diam(G)-1}{2} & \textit{if } diam(G) \textit{ is odd} \\ k + \frac{diam(G)-2}{2} & \textit{if } diam(G) \textit{ is even.} \end{cases}$$

Proof. Let diam(G) be odd and consider the edge $e = v_i^0 - v_j^0$ in G such that diam$(G+e)$ = diam$(G) - 1$. Let $k = \min\{d_G(v_i^0, v_x^0), d_G(v_j^0, v_x^0)\}$, where v_x^0 is an end point of any diametral path in G. Let $m \leq k + \frac{\text{diam}(G)-1}{2}$. Then by adding e to $\mu_m(G)$, $d_{\mu_m(G)}(v_i^p, v_j^q) \leq \text{diam}(G) - 1$, for $k+1 \leq p, q \leq m$ by taking the path through z. For $1 \leq p, q \leq k$, the shortest path between the vertices v_i^p and v_j^q will be the path through level 0 which contains e and hence $d_{\mu_m(G)}(v_i^p, v_j^q) \leq \text{diam}(G) - 1$ in this case also. Take a pair of diametral vertices $(v_{i'}^0, v_{j'}^0)$in G. Then $d_{\mu_m(G)}(v_{i'}^0, v_{j'}^0) = \text{diam}(G) - 1$ by the path through level 0. Hence it follows that $D^{-1}(\mu_m(G)) = 1$.

Conversely suppose that $D^{-1}(\mu_m(G)) = 1$. Then clearly diam$\mu_m(G)$ = diam(G). If possible let $m > k + \frac{\text{diam}(G)-1}{2}$. Consider the pair$(v_{i'}^{k+1}, v_{j'}^{k+1})$ which are the dual vertices in the $k+1^{th}$ level of the diametral vertices $v_{i'}$ and $v_{j'}$ in G and let the edge e be added in $\mu_m(G)$. Then, Clearly the shortest path between these vertices is through the level 0 given by $v_{i'}^{k+1} - v_1^k - v_2^{k-1} \cdots - v_{k+1}^0 - v_{k+2}^0 - \cdots - v_{\text{diam}(G)-2(k+1)}^0 - v_{\text{diam}(G)-2k-1}^1 - \cdots - v_{\text{diam}(G)-1}^k - v_j'^{k+1}$, where $v_i^0 - v_1^0 - v_2^0 \cdots - v_{\text{diam}(G)-1}^0 - v_{j'}^0$ is the shortest path between $v_{i'}^0$ and $v_{j'}^0$ in G. Thus by the definition of k, we have $d(v_{i'}^{k+1}, v_{j'}^{k+1})$ =diam(G) in $\mu_m(G)+e$. Now, if we add any other edge in $\mu_m(G)$, then the distance $d(v^0_{i'}, v^0_{j'})$ =diam(G). Thus we get a contradiction to the fact that $D^{-1}(\mu_m(G)) = 1$. Similarly we can prove the case, when diam(G) is even. $\square$

3.3 Diameter minimality of the generalized Mycielskian.

A graph G is **diameter minimal** if diam($G - e$) >diam(G) for any $e \in G$ [6]. In this section, we have obtained a characterization for the generalized Mycielskian of a graph to be diameter minimal. Through out this section, we denote $d_{\mu_m(G)}(v_i, v_j)$ as $d(v_i, v_j)$ for the sake of convenience.

Theorem 3.3.1. *Let G be any connected graph. Then $\mu_m(G), m \geq 1$, is diameter minimal if and only if G is $K_{1,n}$.*

Proof. Let G be $K_{1,n}$, with $d(v_i) = n$. Then, $diam(\mu_m(G)) = m + 1$. We have to prove that $\mu_m(G)$ is diameter minimal. For this, we consider the following possible cases of edge deletions in $\mu_m(G)$.

Case 1: Let an edge $v^0_i - v^0_j$ be deleted.

First suppose that m is even. Consider the pair of vertices (v_j^0, v_i^m) in $\mu_m(G)$. When the edge $v_j^0 - v_i^0$ is deleted, $d(v_j^0, v_i^m) = m + 2$ by the path $v_j^0 - v_i^1 - v_j^2 - v_i^3 - \cdots - v_j^m - z - v_i^m$, where $d(v_j^0, v_j^m) = m$ and $d(v_j^m, v_i^m) = 2$. The distance between any other pair of vertices is not affected by the removal of this edge. When m is odd, we consider the pair of vertices (v_j^0, v_j^m) in $\mu_m(G)$. By deleting the edge $v_i^0 - v_j^0$, $d(v_j^0, v_j^m) = m+2$ by the path $v_j^0 - v_i^1 - v_j^2 - v - i^3 - \cdots - v_i^m - z - v_j^m$, where $d(v_j^0, v_i^m) = m$ and $d(v_i^m, v_j^m) = 2$ and no other distance is affected by this removal.

Case 2a: Let an edge $z - v_j^m$ be deleted.

First, take m is even. Let the edge $z - v_j^m$, be deleted. Then, $d(z, v_j^m) = 3$ by a path $z - v_x^m - v_y^{m-1} - v_j^m$, where $v_x^0 \in N(v_y^0)$ and $v_y^0 \in N(v_j^0)$. Also $d(v_j^m, v_j^1) = m+2$ by the path $v_j^m - v_i^{m-1} - v_j^{m-2} - v_i^{m-3} - \cdots - v_i^1 - v_j^0 - v_i^0 - v_j^1$. The distance between any other vertices is not affected by the removal of this edge. Next, assume that m is odd and an edge $z - v_j^m$ is deleted. Then, $d(z, v_j^m) = 3$ by a path $z - v_x^m - v_i^{m-1} - v_j^m$ and $d(v_j^m, v_i^1) = m+2$ by the path $v_j^m - v_i^{m-1} - v_j^{m-2} - v_i^{m-3} - \cdots - v_j^1 - v_i^0 - v_j^0 - v_i^1$. As before, the distance between any other vertices is not affected by the removal of this edge.

Case 2b: Let an edge $z - v_i^m$ be deleted.

Then, $d(z, v_i^m) = 2m$ by the path $v_i^m - v_j^{m-1} - v_i^{m-2} - \cdots - v_i^1 - v_j^0 - v_i^0 - v_j^1 - v_i^2 - \cdots - v_j^m - z$. The distance between any two other vertices is less than or equal to $2m$.

Case 3: Let an edge $v_i^k - v_j^{k+1}$ be deleted.

Then, $d(v_i^k, v_j^{k+1}) = 3$ by the path $v_i^k - v_x^{k+1} - v_i^{k+2} - v_j^{k+1}$ for $k < m-1$ and by the path $v_i^k - v_x^{k+1} - z - v_j^{k+1}$ for $k = m-1$. If m is even, consider the pair of vertices (v_j^k, v_i^{m-k}) in $\mu_m(G)$. Then $d(v_j^k, v_i^{m-k}) = m+2$ by a path $v_j^k - v_i^{k+1} - v_j^{k+2} - v - i^{k+3} - \cdots - v_i^m - z - v_j^m - v_i^{m-1} - \cdots - v_i^{m-k}$, where $d(v_j^k, z) = (m+1) - k$ and $d(z, v_j^{m-k}) = k+1$. The distance between any two other vertices is at most $m+2$. If m is odd, then $d(v_j^i, v_j^{m-k}) = m+2$ by a path $v_j^k - v_i^{k+1} - v - j^{k+2} - v_i^{k+3} - \cdots - v_j^m - z - v_i^m - v_j^{m-1} - \cdots - v_j^{m-k}$, where $d(v_j^k, z) = (m+1) - k$ and $d(z, v_j^{m-k}) = k+1$. The distance between other pairs of vertices is at most $m+2$.

Case 4: Let an edge $v_i^k - v_j^{k-1}$ be deleted.

In this case, if m is even, $d(v_j^{k-1}, v_j^{m+1-k}) = m+2$ by the path $v_j^{k-1} - v_i^{i-2} - \cdots - v_i^0 - v_j^0 - v_j^1 - v_i^2 - \cdots - v_i^{m-k} - v_j^{m+1-k}$. If m is odd, $d(v_j^{k-1}, v_i^{m+1-k}) = m+2$ by the path $v_j^{k-1} - v_i^{k-2} - \cdots - v_i^0 - v_j^0 - v_j^1 - v_i^2 \cdots - v_i^{m+1-k}$. All the other distances are at most $m+2$. Hence it follows that $\mu_m(G)$ is diameter minimal.

Conversely, assume that $\mu_m(G)$ is diameter minimal and if possible let $\delta(G) \geq 2$.

Consider the following cases.

Case 1: $\text{diam}(\mu_m(G)) = \text{diam}(G)$.

Let an edge $v_i^0 - v_j^0$, be deleted. Then, for any $v_x^0, v_y^0 \in V^0$, $d(v_x^0, v_y^0) \leq diam(G)$ by a path, $v_x^0 - v_{x+1}^0 - v_{x+2}^0 - \cdots - v_i^0 - v_j^1 - \cdots - v_y^0$. Also, the distance between any two other is not affected by the removal of this edge, since $\delta(G) \geq 2$.

Case 2: $\text{diam}(\mu_m(G)) = m + 1$.

When an edge $z - v_i^m$ is deleted, $d(z, v_i^m) = 3$ by a path, $z - v_x^m - v_j^{m-1} - v_i^m$. Since, $\delta(G) \geq 2$, the neighbors of v_i in G will be adjacent to some other vertices. Thus, $d(z, v_a^{m-1}) = 2, \forall v_a^0 \in V^0$, (see Fig:4). Hence, $d(z, v_a^0) = m + 1$, $\forall V_a^0 \in V^0$.

Case 3: $\text{diam}(\mu_m(G)) = 2(m + 1)$.

Let an edge $v_i^0 - v_j^0$ be deleted. Then, $d(v_x^0, v_y^0) \leq 2(m+1)$ by a path, $v_x^0 - v_y^1 - \cdots - z - v_k^m - \cdots - v_y^0$. Also, the distance between any two other is not affected by the removal of this edge, since $\delta(G) \geq 2$.

Thus the above arguments, show that $\mu_m(G)$ can not be diameter minimal. Therefore G must be a connected graph with at least one pendant edge.

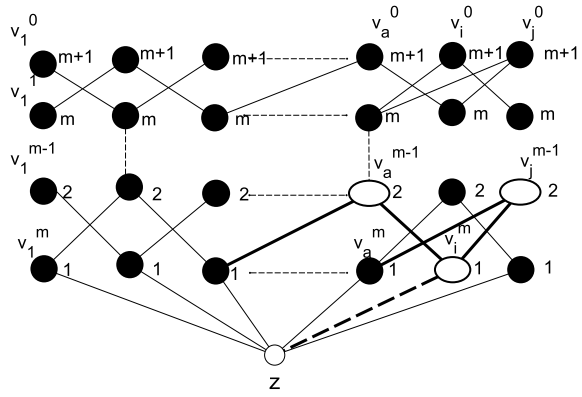

Figure 3.3

Now, from the proof of Theorem 4.4.1, it is clear that deletion of an edge increases the diameter of the generalized Mycielskian if and only if it is a pendant edge and hence G must be $K_{1,n}$. $\square$

3.4 Some bounds for $D^{-k}(\mu_m(G))$

In this section, G is a connected graph of order n and we obtain an upper bound for $D^{-k}(\mu_m(G)), m \geq 1$, depending on the the diameter of G.

Lemma 3.4.1. *If $diam(G) < m+1$, then $D^{-k}(\mu_m(G)) \leq n, 1 \leq k \leq \min\{\frac{m}{2}+1, diam(G)\}$.*

Proof. Since $\mathrm{diam}(G) < m+1$, $\mathrm{diam}\mu_m(G) = m+1$. Now, add the edges $v_i^0 - z \ \forall i = 1, 2, \cdots, n$ in $\mu_m(G)$. Then $d(v_i^j, z) \leq \frac{m}{2} + 1, \forall v_i^j \in V(\mu_m(G))$ and the bound follows. □

Lemma 3.4.2. *Let d' be the diameter of G after adding $D^{-k}(G)$ edges to G and let $m+1 \leq diam(G) < 2(m+1)$. Then $D^{-k}(\mu_m(G)) \leq (m - |E'|)D^{-k}(G)$ where $E' = \{i | 2i \leq d'\}$.*

Proof. Let $v_i^0 - v_{i'}^0, 1 \leq i \leq D^{-k}(G)$, be the edges added in G to reduce the diameter by at least k. Now, add $v_i^j - v_{i'}^j, j = 0, 1, \cdots, |E'|$ in $\mu_m(G)$. This will clearly reduce the diameter of $\mu_m(G)$ by at least k. □

Lemma 3.4.3. *Let $l = D^{-(2(m+1)-k)}(G)$ and $diam(G) \geq 2(m+1)$. Then $D^{-k}(\mu_m(G)) \leq l, k < 2(m+1)$.*

Proof. If $\mathrm{diam}(G) \geq 2(m+1)$, then $\mathrm{diam}(\mu_m(G)) = 2(m+1)$ and hence all shortest paths are the paths through z. Add those l edges which are used to get $D^{-(2(m+1)-k)}(G)$ in the 0^{th} level of $\mu_m(G)$. Then the distance reduces by at least $2(m+1) - k$ in the level 0 and hence the diameter of $\mu_m(G)$ reduces by at least $k, k < 2(m+1)$. □

Theorem 3.4.4.

$$D^{-k}(\mu_m(G)) \leq \begin{cases} n & \textit{if} \;\; diam(G) < m+1, \\ & 1 \leq k \leq \min\{\frac{m}{2}+1, diam(G)\}. \\ (m-|E'|)D^{-k}(G) & \\ & \textit{if} \;\; m+1 \leq diam(G) < 2(m+1). \\ D^{-(2(m+1)-k)}(G) & \textit{if} \;\; diam(G) \geq 2(m+1). \end{cases}$$

where $E' = \{i|2i \leq d'\}, d'$ *the diameter of* G *after adding* $D^{-k}(G)$ *edges to* G.

Proof. The proof follows from the previous lemmas. □

Chapter 4

Some parameters of Sierpiński graphs

In this chapter, our focus is on Sierpiński graphs . Several properties of these graphs are extensively studied earlier. Here, we compute the forwarding indices, bisection width and some convexity parameters of this graph class. The concepts of forwarding index and bisection width are two important measures of

Some results of this chapter are included in the following paper.
Savitha K. S, A. Vijayakumar, Forwarding indices and bisection width of Sierpiński graphs, Bulletin. ICA (76), 107-116, 2016.

efficiency and performance in communication networks. In this chapter, the vertex forwarding index, the edge forwarding index and the bisection width of Sierpiński graphs is determined. Apart from that, some convexity parameters are also determined as convexity plays an important role in analyzing the security of a network.

4.1 Definition and Preliminaries

Definition 4.1.1. For $n \geq 1$ and $k \geq 1$, the vertex set of S_k^n consists of all n-tuples of integers $1, 2, ..., k$, that is, $V(S_k^n) = [1, k]^n$. Two distinct vertices $u = (u_1, u_2, ..., u_n)$ and $v = (v_1, v_2, ..., v_n)$ are adjacent if and only if there exists an $h \in [1, n]$ such that

- $u_t = v_t$ for $t \in [1, h-1]$;
- $u_h \neq v_h$;
- $u_t = v_h$ and $v_t = u_h$ for $t \in [h+1, n]$.

The Sierpiński graph S_4^3 is shown in Fig 5.4.

Now, we discuss some basic properties of Sierpiński graphs S_k^n.

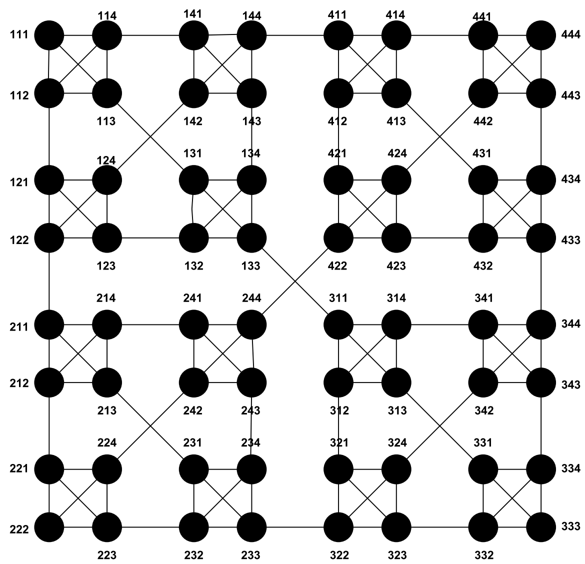

Figure 4.1: Sierpiński graph S_4^3

We denote the vertex

$(u_1, u_2, ..., u_n)$ of S_k^n by $u_1 u_2 \cdots u_n$ for convenience. The vertices $ii \cdots i$, $i \in [1, k]$, are called the *extreme vertices* of S_k^n . For $i \in [1, k]$ and $n \geq 2$, let iS_k^n denote the subgraph of S_k^n induced by the vertices of the form $i \cdots$. Clearly, iS_k^n is isomorphic to S_k^{n-1}. The edges of S_k^n that lie in no induced K_k are called *bridging edges*, where K_k is the complete graph induced by k vertices in S_k^n. Note that iS_k^n and jS_k^n, $i \neq j$, are connected by a single bridging edge between vertices $ijj \cdots j$ and $jii \cdots i$.

For Sierpiński graphs S_k^n, if $n = 1$, then S_k^1 is isomorphic to the complete graph K_k; if $k = 1$, then S_1^n is isomorphic to the trivial graph K_1 consisting of one vertex and no edges and if $k = 2$, S_2^n is isomorphic to the path of length $2^n - 1$.

For the sake of convenience, we use the following notations. For any vertex $u_1u_2\cdots u_n \in V(S_k^n)$ and $i \in [1, n-1]$, if $u_{i+1} = u_{i+2} = \cdots = u_j = l$, then we let $u_1\cdots u_i l^{j-i} u_{j+1}\cdots u_n$ to denote $u_1u_2\cdots u_n$. In particular, if $u_{i+1} = u_{i+2} = \cdots = u_n = l$, then we denote $u_1u_2\cdots u_i l\cdots l$ by $u_1\cdots u_i l^{n-i}$ for $i \in [1, n-1]$. Hence extreme vertices $ii\cdots i$ of S_k^n can be denoted by i^n.

Given a vertex $u_1\cdots u_i l^{j-i} u_{j+1}\cdots u_n$, we consider $u_1\cdots u_0$ and l^0 to be empty strings. Thus, the edge set of S_k^n can be given by $E(S_k^n) = \{(u_1\cdots u_{r-1} j i^{n-r},\ u_1\cdots u_{r-1} i j^{n-r}) :\ u_1\cdots u_{r-1} \in [1, k]^{r-1},\ r \in [1, n-1],\ j \neq i;\ i,\ j \in [1, k]\} \cup \{(u_1\cdots u_n,\ u_1\cdots u_{n-1} l) : u_1\cdots u_{n-1} \in [1, k]^{n-1},\ u_1\cdots u_n \in [1, k]^n,\ l \in [1, k] \setminus \{u_n\}\}$. In fact, the first edge set in the formula above is the set consisting of all bridging edges and the second edge set consists of all edges lying in induced K_k.

Because of the recursive nature of Sierpiński graphs, we can obtain S_k^{n-1} from S_k^n by replacing each K_k of S_k^n by a single vertex and making two such vertices adjacent if their corresponding K_k's are joined by an edge in S_k^n. Here, this process of obtaining S_k^{n-1} from S_k^n is termed as *shrinking* of S_k^n.

4.2 Forwarding index

4.2.1 Routing in Sierpiński graphs to evaluate the forwarding index

The specific routing $\mathbf{R}$ to evaluate the forwarding index of Sierpi$\acute{n}$ski graphs is defined as follows. Let $u = x_1x_2x_3\cdots x_n$ and $v = y_1y_2y_3\cdots y_n$ be any two arbitrary vertices in S_k^n, then the $u-v$ path in $\mathbf{R}$ in the broad sense is defined as $x_1x_2x_3\cdots x_n \rightarrow x_1y_1y_1\cdots y_1 \rightarrow y_1x_1x_1\cdots x_1 \rightarrow y_1y_2y_3\cdots y_n$.

Now, the path from $x_1x_2x_3\cdots x_n$ to $x_1y_1y_1\cdots y_1$ in this routing is taken as $x_1x_2x_3\cdots x_n \rightarrow x_1x_2x_3\cdots x_{n-1}y_1 \rightarrow x_1x_2x_3\cdots y_1x_{n-1} \rightarrow x_1x_2x_3\cdots x_{n-2}y_1y_1 \rightarrow x_1x_2x_3\cdots y_1x_{n-2}x_{n-2} \rightarrow x_1x_2x_3\cdots y_1 x_{n-2}\, y_1 \rightarrow x_1x_2x_3\cdots y_1y_1x_{n-2} \rightarrow x_1x_2x_3\cdots y_1y_1y_1 \cdots \rightarrow x_1y_1y_1 \cdots y_1$.

The path from $y_1x_1x_1\cdots x_1$ to $y_1y_2y_3\cdots y_n$ is given by $y_1x_1x_1\cdots x_1$

$\rightarrow y_1x_1x_1\cdots x_1y_2 \rightarrow y_1x_1x_1\cdots y_2x_1 \rightarrow y_1x_1x_1\cdots y_2y_2 \rightarrow \cdots$ $y_1x_1y_2\cdots y_2y_2 \rightarrow y_1y_2x_1\cdots x_1x_1 \rightarrow \cdots\ y_1y_2y_3\cdots x_1x_1 \rightarrow \cdots$ $y_1y_2y_3\cdots y_n$.

4.2.2 Vertex forwarding index

Using the routing **R** given in the previous section, an expression for the vertex forwarding index of Sierpiński graphs is obtained in the following theorem.

Theorem 4.2.1. *The vertex forwarding index of Sierpiński graph S_k^n is*

$$\zeta(S_k^n) = 2(k^{n-1}-1)k^{n-1}.$$

Proof. Let $w = w_1w_2w_3\cdots w_{n-i-1}xy^i$ be an arbitrary vertex in S_k^n. Under the routing **R**, the vertex w will lie on a $u-v$ path if $u = w_1w_2w_3\cdots w_{n-i-1}xx_1x_2\cdots x_i$ and $v = w_1w_2w_3\cdots w_{n-i-1}$ $yy_1y_2\cdots y_i$ and through $w_1w_2w_3\cdots w_{n-i-1}y^{i+1}$ to S_k^{l+2}, S_k^{l+3} and so on. Also it can lie on a path, where $u = w_1w_2w_3\cdots w_{n-i-1}yx_1x_2$ $\cdots x_i$ and $v = w_1w_2w_3\cdots w_{n-i-1}x^{i+1}$ and through the latter to

S_k^{l+2}, S_k^{l+3} and so on. Therefore the load of w under $\mathbf{R}$ satisfies

$$\zeta_w(S_k^n;\ \mathbf{R}) \leq 2\left[k^i \sum_{j=i+1}^{n-1} k^j + \left(k^i - 1\right) \sum_{j=i+1}^{n-1} k^j + (k^i - 1)k^i\right],\ i < n.$$

Hence the load of w is maximum when $i = n - 1$ and it follows that

$\zeta(S_k^n;\ \mathbf{R}) = 2(k^{n-1} - 1)k^{n-1}$.

Therefore, we have $\zeta(S_k^n) \leq 2(k^{n-1} - 1)k^{n-1}$.

Now, to obtain the lower bound, we find the average load of vertices of the form xy^{n-1} in S_k^n. For this, take the vertices of the form xw and yw', where $x \neq y$ and $w, w' \in \{u_1u_2 \cdots u_{n-1}, u_i \in [1, k]\}$. Then the possible choices of w and w' are the following.

1. $w = y^{n-1}$ and $w' = x^{n-1}$
2. $w = y^{n-1}$ and $w' \neq x^{n-1}$
3. $w \neq y^{n-1}$ and $w' = x^{n-1}$
4. $w \neq y^{n-1}$ and $w' \neq x^{n-1}$.

If we consider all the possible paths from xw to yw', the load of the vertices xy^{n-1} and yx^{n-1} in respective choices of w and

w' are 0, $(k^{n-1}-1)$ for yx^{n-1}, $(k^{n-1}-1)$ for xy^{n-1} and $(k^{n-1}-1)(k^{n-1}-1)$ for both. Thus the average load of the vertices of the form xy^{n-1} is $2k^{n-1}(k^{n-1}-1)$. This shows that $\zeta(S_k^n) \geq 2k^{n-1}(k^{n-1}-1)$ and hence we have $\zeta(S_k^n) = 2k^{n-1}(k^{n-1}-1)$. □

4.2.3 Edge forwarding index

To get the edge forwarding index, the following lemma [33] is used.

Lemma 4.2.2. *Let G be a graph of order n such that the removal of w edges partitions G into two not necessarily connected subgraphs of k vertices and $n-k$ vertices, respectively. Then $\pi(G) \geq \left\lceil \frac{2k(n-k)}{w} \right\rceil$. Moreover, this bound is tight.*

Theorem 4.2.3. *The edge forwarding index of Sierpiński graph S_k^n is $\pi(S_k^n) = 2 \cdot k^{2(n-1)}$.*

Proof. Consider the routing $\mathbf{R}$ given in Section 4.2.1 and take an arbitrary edge e in S_k^n. Then e can be either an edge in the induced K_k or a bridging edge. If it is an edge in the induced K_k, it is of the form $\{(u_1 \cdots u_n, u_1 \cdots u_{n-1}l) : u_1 \cdots u_{n-1} \in [1,k]^{n-1},\ u_1 \cdots u_n \in [1,k]^n,\ l \in [1,k] \setminus u_n\}$. This lies on a path from $u = u_1u_2\cdots u_n$ to lS_k^{n-1}, $lu_{n-1}S_k^{n-2}$ and so on. But the vertex u is of the form $u = w_1w_2\cdots w_{n-i-1}kl^i$ and hence this edge can also lie on a path from $w_1w_2\cdots w_{n-i-1}lk_1k_2\cdots k_i$ to u. If it is a bridging edge, it is of the form $\{(u_1\cdots u_{r-1}ji^{n-r},$ $u_1\cdots u_{r-1}ij^{n-r}) : \ u_1\cdots u_{r-1} \in [1,k]^{r-1},\ r \in [1, n-1],\ j \neq i;\ i,\ j \in [1,k]\}$ and it lies on a path from any vertex of $u_1\cdots u_{r-1}$ jS_k^{n-r} to $u_1\cdots u_{r-1}iS_k^{n-r}$.

Hence it follows that

$$\pi_e(S_k^n;\ \mathbf{R}) \leq 2\left[k^l\left(\sum_{i=l+1}^{n-1} k^i\right) + (k^lk^l)\right],\ l < n.$$

So the congestion of an edge is a maximum when $l = n-1$ and the maximum congestion occurs for the bridging edges.
Thus $\pi(S_k^n;\ \mathbf{R}) = 2\cdot k^{2(n-1)}$ and therefore

$$\pi(S_k^n) \leq 2\cdot k^{2(n-1)}.$$

On the other hand, if we remove the $k-1$ bridging edges from any iS_k^n, $i \in [1,k]$, we will have two components of the graph, containing k^{n-1} and $(k-1)k^{n-1}$ vertices respectively. By Lemma

4.4.5, it follows that

$$\pi(S_k^n) \geq \left\lceil \frac{2k^{n-1}(k-1)k^{n-1}}{k-1} \right\rceil$$

and we get

$$\pi(S_k^n) \geq 2k^{n-1}k^{n-1}.$$

Thus $\pi(S_k^n) = 2 \cdot k^{2(n-1)}$. □

4.3 Bisection width

In this section we use the following corollary of Lemma 4.4.5[33].

Corollary 4.3.1. *Let G be a graph of order n with edge forwarding index $\pi(G)$ and bisection width $\mathrm{bw}(G)$. Then $\pi(G) \geq \left\lceil \frac{n^2}{2\mathrm{bw}(G)} \right\rceil$ if n is even and $\pi(G) \geq \left\lceil \frac{n^2-1}{2\mathrm{bw}(G)} \right\rceil$ if n is odd.*

Theorem 4.3.2. *The bisection width of Sierpiński graphs is* $\mathrm{bw}(S_k^n) = \begin{cases} \frac{k^2}{4} & \text{if } k \text{ is even} \\ n\left\lfloor \frac{k}{2} \right\rfloor^2 + \left\lfloor \frac{k}{2} \right\rfloor & \text{if } k \text{ is odd} \end{cases}$

Proof. **Case 1:** k is even.

To obtain an upper bound for $\mathrm{bw}(S_k^n)$, we remove the bridging edges $\{ij^{n-1}, ji^{n-1}\}$, where $1 \leq i \leq \frac{k}{2}$ and $\frac{k}{2}+1 \leq j \leq k$. Then removal of these $\frac{k^2}{4}$ edges clearly partitions $V(S_k^n)$ into two sets with equal number of vertices and hence we have $\mathrm{bw}(S_k^n) \leq \frac{k^2}{4}$. On the other hand, using Corollary 4.4.6 and the edge forwarding index obtained from Section 4.2.3, we get $\mathrm{bw}(S_k^n) \geq \frac{k^n k^n}{2 \cdot 2 \cdot k^{n-1} k^{n-1}}$ and this shows that $\mathrm{bw}(S_k^n) \geq \frac{k^2}{4}$. Thus we have $\mathrm{bw}(S_k^n) = \frac{k^2}{4}$.

Case 2: k is odd.

In this case we partition the vertex set of S_k^n into two sets with almost equal number of vertices, i.e with a difference of one. For this, as in Case 1, remove the bridging edges $\{ij^{n-1}, ji^{n-1}\}$, where $1 \leq i \leq \left\lfloor \frac{k}{2} \right\rfloor$; $\left\lfloor \frac{k}{2} \right\rfloor + 1 \leq j \leq k$. This will give a partition in which one part contains one S_k^{n-1} more than the other part. For the required bipartition, we have to continue this process until we are left with a K_k for which the bisection width is $\left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right) \left\lfloor \frac{k}{2} \right\rfloor$. This process will terminate with an upper bound for $\mathrm{bw}(S_k^n)$ given by

$$\mathrm{bw}(S_k^n) \leq (n-1) \left\lfloor \frac{k}{2} \right\rfloor \left\lfloor \frac{k}{2} \right\rfloor + \left(\left\lfloor \frac{k}{2} \right\rfloor + 1 \right) \left\lfloor \frac{k}{2} \right\rfloor .$$

To prove the lower bound, we use induction on n. The result is trivially true for $n = 1$. Assume that it is true for S_k^{n-1}. Now, for $n \geq 2$, shrink S_k^n to form S_k^{n-1}, where the result is true by induction assumption. Therefore we require at least $(n-2)\left\lfloor \frac{k}{2} \right\rfloor \left\lfloor \frac{k}{2} \right\rfloor + (\left\lfloor \frac{k}{2} \right\rfloor + 1)\left\lfloor \frac{k}{2} \right\rfloor$ edges to partition this S_k^{n-1} into two with almost equal number of vertices. Take this partition and replace each vertex by K_k, and the corresponding adjacencies, so that we have a partition of S_k^n in which one K_k is more in one part than in the other. Thus to obtain the required bipartition of S_k^n, we have to make the $\left\lfloor \frac{k}{2} \right\rfloor$ vertices of this K_k adjacent to the other component of the previous partition and remove$\left\lfloor \frac{k}{2} \right\rfloor (\left\lfloor \frac{k}{2} \right\rfloor + 1)$ edges within this K_k. Therefore the number of edges to be removed for the required bipartition is at least the sum of the number of edges removed in the two previous steps. Thus we have

$$\mathrm{bw}(S_k^n) \geq (n-1)\left\lfloor \frac{k}{2} \right\rfloor \left\lfloor \frac{k}{2} \right\rfloor + \left(\left\lfloor \frac{k}{2} \right\rfloor + 1\right)\left\lfloor \frac{k}{2} \right\rfloor$$

and the result follows.

$\square$

4.4 Geodesic convexity parameters

In this section, we use the following theorem.

Theorem 4.4.1. *[10] If v is a vertex of a graph G such that $\langle N(v)\rangle$ is complete, then v belongs to every hull set and every geodetic set.*

Here, the vertices which lie on the geodesic between i^n and j^n, $j = [(i+1) mod n]$ are termed as the boundary vertices denoted by $\mathcal{B}(S_k^n)$ and that between the other extreme vertices as the diagonal vertices denoted by $\mathcal{D}(S_k^n)$.

4.4.1 The geodetic iteration number

In this section, we obtain an expression of the geodetic iteration number of the Sierpiński graphs.

Theorem 4.4.2. *The geodetic iteration number of Sierpiński graphs is $gin(S_k^n) = n$.*

Proof. Let $S = \{i^n : 1 \le i \le n\}$. Then $S^1 = (S)$ is the set $S \cup \mathcal{B}(S_k^n) \cup \mathcal{D}(S_k^n)$. Now, $S^2 = S^1 \cup \mathcal{B}(iS_k^n) \cup \mathcal{D}(iS_k^n)$ and so on. Therefore $S^{n-1} = V(S_k^n)$ and hence $S^n = S^{n-1}$. Thus there exists a set S in S_k^n with $\text{gin}(S) = n$. Now, for any other $S \subset V(S_k^n)$ we prove that $\text{gin}(S) \leq n$ by induction on n. It is trivially true for $n = 1$. For $n = 2$, all subsets S other than the extreme vertices have $\text{gin}(S) \leq 2$. Thus we assume that the result is true for $n \leq p$ and suppose that S is any arbitrary subset of S_k^{p+1}. Then, $(S) = \{(S_i), S_i \subset V(iS_k^{p+1}), 1 \leq i \leq k\} \cup I(u_i, u_j), u_i \in S_i, u_j \in S_j, i \neq j$. Therefore $S^m = S_i^m \cup I(u_i, u_j), u_i \in S_i^{m-1}, u_j \in S_j^{m-1}, i \neq j$ and hence $S^i = S^{i-1}$ for $i \leq p+1$ by induction assumption. This shows that $\text{gin}(S) \leq n$. $\square$

4.4.2 The geodetic number

It is easy to see that $\text{gn}(S_k^1) = p$ and $\text{gn}(S_2^n) = 2$ since $S_k^1 = K_p$ and $S_2^n = P_{2^n}$.

Proposition 4.4.3. *$gn(S_k^2) = k$.*

Proof. The extreme vertices of Sierpiński graphs are the vertices with their neighbors inducing a complete subgraph. Therefore extreme vertices are in any geodetic set by the Theorem 4.4.1.

Thus $\mathrm{gn}(S_k^2) \geq k$. Next we claim that the set of extreme vertices in (S_k^n) form a geodetic cover. Let S be the set of vertices other than the extreme vertices in S_k^2. Then $S = \{ij : 1 \leq i \leq k, 1 \leq j \leq k, i \neq j\}$. Take any vertex $ij \in S$. This lies on the geodesic $ii - ij - ji - jj$ between ii and jj. Hence the claim and therefore we have $\mathrm{gn}(S_k^2) \leq k$. □

Proposition 4.4.4. *$gn(S_3^3) = 5$.*

Proof. Let S be any geodetic basis of S_3^3. Then S contains the extreme vertices by the Theorem4.4.1. So the vertices on the boundary are covered by these vertices. Now, to cover the other vertices we include the vertices123 and 322 to S. Thus S becomes a geodetic cover and hence $\mathrm{gn}(S_3^3) \leq 5$. The fact that the extreme vertices present in any geodetic basis shows that $\mathrm{gn}(S_3^3) \geq 3$. Thus, to prove the result it remains to show that $\mathrm{gn}(S_3^3) \neq 4$. If possible let $\mathrm{gn}(S_3^3) = 4$ and let S' be the corresponding geodetic basis. Then $S' = \{111, 222, 333, i'j'k' : i' \neq j' \neq k', 1 \leq i' \leq k, 1 \leq j' \leq k, 1 \leq k' \leq k\}$. The vertex $i'j'k'$ can't be on the boundary of S_3^3 as S' is a geodetic basis. So this can be on the boundary of any of the three S_3^2's present in S_3^3. In this case, the internal vertices of the other two S_3^2's won't be covered. Hence it follows that any geodetic basis of S_3^3 contains more than 4 elements. $\square$

Proposition 4.4.5. $gn(S_4^3) = 12$.

Proof. The vertices not covered by the extreme vertices form an induced C_6 in each S_4^2. To cover these vertices, include any two of them in the geodetic cover. Thus $\mathrm{gn}(S_4^3) \leq 12$. Suppose that there exists a geodetic cover S' with $|S'| < 12$. certainly $\{111, 222, 333, 444\} \subset S'$. We claim that S' contains at least two vertices of each $iS_4^3 \setminus \{iii\}$. If possible S' contains at most one vertex of each $iS_4^3 \setminus \{iii\}$. Then that vertex is one of the end vertex of the bridging edges not in $(\{iii), 1 \leq i \leq 4\})$. But then the remaining vertices won't lie on any $u - v$ geodesic, where $u, v \in S'$, which contradicts the fact that S' is a geodetic cover. Therefore, S' contains at least two vertices of each $iS_4^3 \setminus \{iii\}$. Thus $|S'| \geq 12$ and the result follows. $\square$

Proposition 4.4.6. $gn(s_k^3) = k^2, k \geq 5$.

Proof. As before any geodetic cover contains the k extreme vertices and these cover the boundary and diagonal vertices of S_k^3. The other uncovered vertices induce a subgraph isomorphic to a regular graph that contains $k-1$ mutually adjacent copies of K_{k-2} in each iS_k^2. To cover these vertices we take $k-1$ vertices from each copy. Thus in this cover we have k vertices from each $iS_k^2, 1 \leq i \leq k$. This shows that this cover contains k^2 vertices and therefore $\mathrm{gn}(s_k^3) \leq k^2$.

On the other hand suppose that S is a geodetic cover. Then $\{i^n : 1 \leq i \leq k\} \subset S$. We claim that S contains at least $k-1$ vertices of each $iS_k^3 \setminus \{iii)\}$. If possible, suppose that S contains at most $k-2$ vertices of each $iS_k^3 \setminus \{iii\}$. Since S is a geodetic cover, these $k-2$ vertices lie in different $p-1$ K_k's of iS_k^2. In that case, the vertices on one K_{k-2} will remain uncovered by this S, which is a contradiction. Hence S' contains at least $k-1$ vertices of each $iS_k^3 \setminus \{iii\}$ and therefore $|S| \geq k^2$.

Hence we have $\mathrm{gn}(S_k^3) = k^2, k \geq 5$. $\square$

Theorem 4.4.7. *The geodetic number of Sierpiński graphs is*

$$gn(S_k^n), n \geq 3 = \begin{cases} k^{n-2} + 2k^{n-3}, k = 3 \\ k^{n-1} - k^{n-2}, k = 4 \\ k^{n-1}, k \geq 5. \end{cases}$$

Proof. In S_k^n, there are k^{n-3} S_k^3's and take $S = \bigcup_{i=1}^{k^{n-3}} S_i$, where S_i is a geodetic cover of S_k^3. Then S will be a geodetic cover of S_k^n and the upper bound follows from the Propositions 4.4.4, 4.4.5 and 4.4.6. To prove the lower bound, we use induction on n. The result is true for $n = 3$ follows from the Propositions 4.4.4, 4.4.5 and 4.4.6. Suppose that the result is true for all $n \leq m$ and consider S_k^{m+1}. Let S be a geodetic cover of S_k^{m+1}. Then, $S \cap iS_k^{m+1}$ is a geodetic cover of $iS_k^{m+1}$and hence $\left|S \cap iS_k^{m+1}\right| \geq gn(S_k^m)$. Therefore $|S| \geq \sum_{i=1}^{k} gn(iS_k^{m+1})$ implies $|S| \geq \sum_{i=1}^{k} gn(S_k^m) = k.gn(S_k^m)$. Thus the lower bound follows by induction assumption. □

4.4.3 Hull number

Theorem 4.4.8. *The hull number of Sierpiński graphs is* $h(S_k^n) = k$

Proof. The fact that the extreme vertices present in any hull set implies that $h(S_k^n) \geq k$. Let $S = \{i^n : 1 \leq i \leq k\}$. Then $(S) = \mathcal{B}(S_k^n) \cup \mathcal{D}(S_k^n)$, $S^2 = \mathcal{B}(S_k^{n-1}) \cup \mathcal{D}(S_k^{n-1})$ and so on. Proceeding like this we get finally $[[S]] = V(S_k^n)$. Therefore $h(S_k^n) \leq k$ and the result follows. □

4.4.4 Poly-convexity and Convexity number

Theorem 4.4.9. *The Sierpiński graphs are poly-convex with respect to geodesic convexity.*

Proof. For this, we have to show that for every $i, 1 \leq i \leq k^n - 1$, there exists a convex set of cardinality i in S_k^n. If $i \leq k$, consider the subgraphs of any K_k in S_k^n induced by i vertices. This will be the required convex sets. For $k \leq i \leq 2k$, consider the subgraph induced by the vertices in any K_k and any $i - k$ vertices in other K_k in any S_k^2. Continuing like this we get all the convex sets of cardinality up to k^2 from S_k^2 and proceed likewise with $i^{n-3}S_k^3, i^{n-4}S_k^4, \cdots iS_k^{n-1}$. This will naturally give all the convex sets of cardinality $i, 1 \leq i \leq k^n - 1$. □

Corollary 4.4.10. *The convexity number of Sierpiński graphs is* $c(S_k^n) = k^n - 1$.

4.4.5 Interval monotonicity

Theorem 4.4.11. *Sierpiński graphs are interval monotone with respect to geodesic intervals.*

Proof. In order to show that S_k^n are interval monotone, we have to show that for any pair $u, v \in S_k^n$, $I(u, v)$ is convex. We prove this by induction on n. For $n = 1$, it is trivially true. Let us assume that the result is true for all $n \leq m$ and consider any pair of arbitrary vertices u, v in S_k^{m+1}. If $u, v \in iS_k^m$ for some i, then the result is true by induction. So assume that $u \in iS_k^m$ and $v \in jS_k^m, i \neq j$. Since there are at most two shortest paths between any pair of vertices in S_k^n, we first consider the case when there is a unique shortest path P between u and v. Let $a, b \in I(u, v)$ be such that $I(a, b)$ is not contained in $I(u, v)$. Then there is a $z \in I(a, b)$ such that $z \notin I(u, v)$. If such a z exists, then the path through z between u and v will also be a shortest path between u and v. This contradicts the uniqueness of P. Hence $I(u, v) \supseteq I(a, b)$. Now, suppose that there are two paths P and P' between u and v. Let $u = iu_1u_2u_3 \cdots u_{n-1}$ and $v = jv_1v_2 \cdots v_{n-1}$. Let P be the direct path that contains the bridging edge $ij^{n-1} - ji^{n-1}$ and P' be the one that contains the edges $ih^{n-1} - hi^{n-1}$ and $jh^{n-1} - hj^{n-1}$, $h \neq i, j$. Now, $I(u, v)$ is the disjoint union of $I(u_i, v_i), u_i, v_i \in lS_k^{n-1}, l = i, j, h$ and each $I(u_i, v_i)$ is convex by induction assumption. Therefore $I(u, v)$ must also be convex. $\square$

4.5 Minimal path convexity parameters

In this section, we call a vertex v to be simplicial if $\langle N(v)\rangle$ is complete. Then it is interesting to observe that every monophonic set and m-hull set of any graph contains all the simplicial vertices.

4.5.1 Minimal path iteration number

Theorem 4.5.1. *For any pair of nonadjacent vertices* $u, v \in V(S_k^n), (\{u, v\}) = V(S_k^n) \setminus \{i^n : 1 \leq i \leq k\}$.

Proof. Proof is by induction on n. The result is trivially true for $n = 1$ and it can be easily verified for $n = 2$. So assume that the result is true for $n = m$. Consider S_k^{m+1}. Take two non adjacent vertices u and v. Let $u = iu_1u_2\cdots u_{n-1}$ and $v = jv_1v_2\cdots v_{n-1}$, for $1 \leq i, j \leq k$. Consider all the minimal paths between u and ij^{n-1}. Being non adjacent vertices these paths will cover all the vertices except $ik^{n-1}, k \neq j$ by induction assumption. Now, take all the minimal paths between ji^{n-1} and $jv_1v_2\cdots v_{n-1}$. Union of these paths and paths between u and ij^{n-1} will again give a collection of minimal paths covering all vertices in iS_k^m and jS_k^m except some extreme vertices. Considering the minimal paths between u and $il^{n-1}, l \neq k$ and proceeding as before will cover vertices in the other induced S_k^m except a few almost extreme vertices. These can be covered by taking the minimal paths through the boundary namely $u - \cdots - ul^{n-1} - lu^{n-1} - \cdots - l^{n-2}x - l^{n-2}y - \cdots - l^{n-1}z - zl^{n-1} - \cdots - v$. Thus the closure of $\{u, v\}$ is the entire vertex set with out the extreme vertices. $\square$

Corollary 4.5.2. *For any $S \subseteq V(S_k^n)$, (S) is m-convex.*

Proof. If S is convex (S) is trivially convex. So suppose that S is not convex. Then there exists at least a pair of non adjacent vertices and therefore closure of S will contain all the vertices except the extreme vertices by the previous theorem. Hence the result. $\square$

Corollary 4.5.3. *Minimal path iteration number of Sierpiński graphs is $min(S_k^n) = 2$.*

Proof. Proof follows from the previous corollary. $\square$

4.5.2 Monophonic number

Theorem 4.5.4. *The monophonic number of Sierpiński graphs is $mn(S_k^n) = k$.*

Proof. Since the simplicial vertices belong to any monophonic set we have $\mathrm{mn}(S_k^n) \geq k$. From Theorem 4.5.1, we also know that the closure of all simplicial vertices give the whole vertex set. Thus it follows that $\mathrm{mn}(S_k^n) = k$. □

4.5.3 m- hull number

Theorem 4.5.5. *$m-$hull number of Sierpiński graphs $mh(S_k^n) = k$.*

Proof. Every hull set is also a m- hull set. Therefore $\mathrm{mh}(S_k^n) \leq k$. The k extreme vertices of (S_k^n) are the simplicial vertices and hence belong to any m- hull set . Thus we have $\mathrm{mh}(S_k^n) = k$. □

Observation 4.5.6. *The monophonic number and the $m-$hull number of Sierpiński graphs are both equal to k.*

4.5.4 m-convexity number and poly-convexity

$S = V(S_k^n) \setminus \{i^n.\text{for any} i \in [1,k]\}$ is a convex set of cardinality $k^n - 1$. Therefore $m-$ convexity number of S_k^n is $k^n - 1$. There does not exist convex sets of cardinality i for all $i, 1 \leq i \leq k^n$. For example in S_3^2, there is no convex set of cardinality 4. Thus S_k^n is not poly-convex with respect to minimal path.

Remark 4.5.1. For $S_k^n, mn(S_k^n) = mh(S_k^n)$.

4.5.5 Interval monotonicity

Theorem 4.5.7. *S_k^n is interval monotone with respect to minimal path convexity.*

Proof. Let $S = \{u, v\}, u, v \in V(S_k^n)$. Then $I(u, v) = (S)$ and which is convex by the lemma 4.4.5. Since u and v are arbitrary it follows that S_k^n is interval monotone. □

Observation 4.5.8. *Since minimal path convexity implies geodesic path convexity theorem 4.4.11 directly follows from Theorem 4.5.5 also.*

Chapter 5

Some properties of Fibonacci cubes

The existence of a hamiltonian cycle in Fibonacci cubes is a very important property, especially in the presence of faulty links when a reconfiguration of the network is necessary. It is known that Fibonacci cubes of even order are hamiltonian, whereas that of odd order are not. In this chapter, we solve the problem posed by S.Klavžar [36] for the Fibonacci cubes of odd order, ***for which vertices*** v***,*** $\Gamma_n - v$ ***is hamiltonian?***. We have also studied some diameter notions like wide diameter, fault diameter and diameter variability of Fibonacci cubes in this chapter which are important in networks.

5.1 Definition and Preliminaries

Definition 5.1.1. Let $\mathcal{B} = \{0,1\}$ and for $n \geq 1$ set $\mathcal{B}_n = \{b_1b_2\ldots b_n | b_i \in \mathcal{B}, 1 \leq i \leq n\}$. Let $\mathcal{F}_n = \{b_1b_2\ldots b_n \in \mathcal{B}_n \mid b_i \cdot b_{i+1} = 0, 1 \leq i \leq n-1\}$. The set $\mathcal{F}_n$ thus contains all binary strings of length n that contain no two consecutive 1s. The **Fibonacci cubes** Γ_n, $n \geq 1$, has $\mathcal{F}_n$ as the vertex set, two vertices being adjacent if they differ in exactly one coordinate. In other words, Γ_n is the graph obtained from the hypercubes Q_n by removing all vertices that contain at least two consecutive 1s. Note that $\Gamma_1 = K_2$ and Γ_2 is the path on three vertices. The Fibonacci cubes $\Gamma_n, n = 3,4,5$ are shown in fig 5.1. For convenience we set $\Gamma_0 = K_1$.

Let $n \geq 1$ and consider the partition of $\mathcal{F}_n$ into the sets of strings given by $\mathcal{A}_n = \{b_1b_2\ldots b_n \in \mathcal{F}_n \mid b_1 = 1\}$ and $\mathcal{B}_n = \{b_1b_2\ldots b_n \in \mathcal{F}_n \mid b_1 = 0\}$. Setting $\mathcal{A}_0 = \phi$ and $\mathcal{B}_0 = \{\lambda\}$, where λ is the empty string, $\mathcal{A}_n$ and $\mathcal{B}_n$ can be for any $n \geq 1$ re-

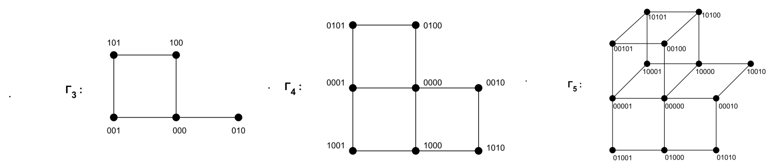

Figure 5.1: Γ_3, Γ_4 and Γ_5

cursively defined by $\mathcal{A}_n = \{1\alpha \mid \alpha \in \mathcal{B}_{n-1}\}$ and $\mathcal{B}_n = \{0\alpha \mid \alpha \in \mathcal{A}_{n-1} \cup \mathcal{B}_{n-1}\}$. Since a string of $\mathcal{A}_n (n \geq 2)$ necessarily starts with 10, the set $\mathcal{A}_n$ induces a subgraph of Γ_n isomorphic to Γ_{n-2}. Similarly, $\mathcal{B}_n$ induces Γ_{n-1} in Γ_n. Moreover, each vertex 1α of $\mathcal{A}_n$ has exactly one neighbor in $\mathcal{B}_n$, the vertex 0α. This recursive structure of Γ_n is called the fundamental decomposition of Γ_n. In this chapter, we denote this fundamental decomposition by $\Gamma_n = 0\Gamma_{n-1}\Psi 10\Gamma_{n-2}$

The fundamental decomposition of Γ_n can be recursively applied to its subgraphs Γ_{n-1} and Γ_{n-2}. To avoid ambiguity with initial conditions, we define the Fibonacci numbers as $F_0 = 0, F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Then $|V(\Gamma_0)| = 1 = F_2$ and $|V(\Gamma_1)| = 2 = F_3$. Hence the fundamental decomposition of Γ_n immediately implies that $|V(\Gamma_n)| = F_{n+2}$.

Cong, Zheng and Sharma [18] showed that Fibonacci cubes contain hamiltonian paths, constructed as follows. The empty sequence $g_0 = \lambda$ and the sequence $g_1 = 0, 1$ are clearly spanning paths of Γ_0 and Γ_1, respectively. For $n \geq 2$, let $g_n = 0\bar{g}_{n-1}, 10\bar{g}_{n-2}$, where $\bar{g}$ denotes the reverse of the sequence g and αg is the sequence obtained from g by appending a fixed string α in front of each of the terms of g. The first few sequences g_n are thus

$$\begin{aligned}
g_0 &= \lambda \\
g_1 &= 0, 1 \\
g_2 &= 01, 00, 10 \\
g_3 &= 010, 000, 001, 101, 100 \\
g_4 &= 0100, 0101, 0001, 0000, 0010, 1010, 1000, 1001
\end{aligned}$$

Using this sequence g_n, a hamiltonian cycle in the Fibonacci cubes of even order is given by $C = \{g_n^0, \bar{g_n^1}\}$, where g_n^0 and g_n^1 respectively denote the subsequence of g_n that have 0 and 1 on the last position of the binary strings [69]. For example, Γ_4 has a hamiltonian cycle $\{0100, 0000, 0010, 1010, 1000, 1001, 00001, 0101, 0100\}$(Fig: 5.1).

5.2 Hamiltonicity of Fibonacci cubes of odd order under vertex deletion

This section contains our main result on Fibonacci cubes, by which we have completely solved the open problem mentioned earlier. We first analyze the bipartiteness of the Fibonacci cubes. A vertex of Γ_n is called **even weighted** if it has even number of 1's and **odd weighted** if it has odd number of 1's. The set of even weighted vertices and odd weighted vertices of Γ_n are denoted by $\mathcal{E}_n$ and $\mathcal{O}_n$ respectively with the vertex $00\cdots0$ included in $\mathcal{E}_n$. With these notions, we have the following.

Proposition 5.2.1. *The Fibonacci cube* Γ_n *of odd order has a bipartition* $\{\mathcal{E}_n, \mathcal{O}_n\}$ *such that* $\big||\mathcal{E}_n| - |\mathcal{O}_n|\big| = 1$.

Proof. The vertices of $\mathcal{E}_n$ and $\mathcal{O}_n$ clearly gives a bipartition of Γ_n. Since Γ_n has a hamiltonian path for every n, $\{\mathcal{E}_n, \mathcal{O}_n\}$ form a bipartition with $\big||\mathcal{E}_n| - |\mathcal{O}_n|\big| = 1$ as Γ_n has odd number of vertices. $\square$

Theorem 5.2.2. *Let* Γ_n *be a Fibonacci cube of odd order. Then*

$\Gamma_n - v$ is hamiltonian if and only if v lies in the larger bipartition set of Γ_n.

Proof. For $n < 3$, the result is trivial. In Γ_3 (Fig: 5.1), the deletion of $010 \in \mathcal{O}_3$ gives the hamiltonian cycle $\{000, 001, 101, 100, 000\}$ and for the other odd weighted vertices v, $\Gamma_3 - v$ is a tree. Now, suppose that $n > 3$. If $\Gamma_n - v$ is hamiltonian, then v must lie in the larger bipartition set of Γ_n as $\Gamma_n - v$ is also bipartite (Proposition 5.2.1).

Conversely, suppose that v lies in the larger bipartition set of Γ_n. We prove that $\Gamma_n - v$ is hamiltonian by induction on n. In Γ_5 (Fig: 5.1), even weighted vertices are more and if we delete any one vertex from $\mathcal{E}_5$, we can find a hamiltonian cycle. For example, if we delete the vertex 10010, then $00000 - 00010 - 01010 - 01000 - 01001 - 00001 - 00101 - 00100 - 10100 - 10101 - 10001 - 10000 - 00000$ is a hamiltonian cycle. If we delete any other vertex of even weight also, we can find the corresponding hamiltonian cycles by combining the hamiltonian path obtained from the hamiltonian cycle in Γ_4 and the hamiltonian path in Γ_3 of Γ_5. In Γ_6, we have a copy of Γ_4 and Γ_5 by the fundamental decomposition. It has more number of even weighted vertices. Therefore deleting an even weighted vertex from Γ_6 is same as that of either deleting an even weighted vertex from Γ_5 or an odd weighted vertex from Γ_4 of Γ_6. If we delete an even weighted vertex from Γ_5, it has a hamiltonian cycle. Γ_4 also has a hamiltonian cycle as it is of even order. Therefore using these two, we can form a hamiltonian cycle in $\Gamma_6 - v, v \in \mathcal{E}_6$. For example, if we delete the vertex 10010, then $\{00000, 00010, 01010, 01000, 01001, 00001, 00101, 00100, 10100, 10101, 10001, 10000, 00000\}$ is the hamiltonian cycle in $\Gamma_5 - 10010$. In the case of any other even weighted vertex also, we can find the corresponding hamiltonian cycles by using the hamiltonian path obtained from the hamiltonian cycle in Γ_4 and the hamiltonian path in Γ_3 of Γ_5.

If we delete an odd weighted vertex from Γ_4, we construct the required cycle as follows. Take the hamiltonian path obtained after deleting the specified vertex from the hamiltonian cycle in Γ_4 given by the sequence g_4^0, g_4^1. Complete this path by starting from one end and enter the Γ_4 of Γ_5 through the other end. Move through the same path in reverse order until we reach an end vertex of the hamiltonian path given by the sequence $0\overline{g_2}, 10\overline{g_1}$ in Γ_3 of Γ_4 in Γ_5 and go to the corresponding vertex in Γ_3 of Γ_5. Complete the hamiltonian path there and come back to Γ_4 in Γ_5, go through the remaining vertices in Γ_4 of Γ_5 and then to the vertex entered to complete the cycle. If we delete for example, 010010 from Γ_6, $\{100000, 100100, 100101, 100001, 101001, 101000, 101010, 100010, 000010, 001010, 001000, 001001, 000001, 000101, 000100, 010100, 010101, 010001, 010000, 000000, 100000\}$ is the hamiltonian cycle.

The hamiltonian cycle obtained after deleting the vertex 101000 in Γ_6 is $\{101001, 100001, 100101, 100000, 100010, 101010, 001010, 000010, 010010, 010000, 010001, 010101, 010100, 000100, 000101, 000001, 000000, 001000, 001001, 101001\}$. Similarly we can find the hamiltonian cycles in other cases also. If Γ_n is of odd order, then $n = 3k - 1$ or $n = 3k$, $k \geq 2$. The basis step for induction follows from the fact that the result is true for Γ_5 and Γ_6. Now, assume that the result is true for all $k \leq m$ and consider $\Gamma_n,\ n = 3(m+1) - 1$ and $n = 3(m+1)$.

Case 1. $n = 3(m+1) - 1 = 3m + 2$.

In this case, Γ_{3m+2} can be decomposed into Γ_{3m+1} and Γ_{3m} by the fundamental decomposition. In this decomposition, Γ_{3m+1}

is hamiltonian as it is of even order and the result is true for Γ_{3m} by induction hypothesis. Thus $|\mathcal{E}_{3m+2}| \gtrless |\mathcal{O}_{3m+2}|$ according as $|\mathcal{O}_{3m}| \gtrless |\mathcal{E}_{3m}|$.

Case 1.1 $|\mathcal{E}_{3m+2}| > |\mathcal{O}_{3m+2}|$

We shall prove that Γ_{3m+2} has a hamiltonian cycle when an even weighted vertex is deleted. This vertex can be either from Γ_{3m+1} or from Γ_{3m}. If it is from Γ_{3m+1}, then the process of deletion is same as the deletion of an even weighted vertex from Γ_{3m+1} and if it is from Γ_{3m}, then it is same as that of an odd weighted vertex from Γ_{3m}.

First consider, $\Gamma_{3m+2}-v, v \in \mathcal{O}_{3m}$. In this case, $\Gamma_{3m}-v$ has a hamiltonian cycle by induction hypothesis. Using this cycle and the hamiltonian cycle in Γ_{3m+1} (which exists being a Fibonacci cube of even order), we can construct the required hamiltonian cycle. Let the cycle in $0\Gamma_{3m+1}$ be taken as $\{00v_1, 00v_2, \cdots, 00v_i,$ $01v_i, \cdots 01v_j, 00v_j, \cdots, 00v_l, 00v_1\}$, $l = F_{3m+3}$ where, $\{01v_i, 01v_{i+1}, \cdots, 01v_j\}$ is a hamiltonian path in $010\Gamma_{3m-1}$ and $\{v_1, v_2, \cdots, v_l, v_1\}$ is a hamiltonian cycle in $\Gamma_{3m} - v$. Then $\{10v_1, 10v_2, \cdots, 10v_l, 00v_l, \cdots, 00v_j, 01v_j, \cdots, 01v_i, 00v_i, \cdots,$

$00v_1, 10v_1\}$ is the required cycle.

Next, consider $\Gamma_{3m+2} - v, v \in \mathcal{E}_{3m+1}$. Here, let P be the hamiltonian path obtained by deleting the specified vertex from the hamiltonian cycle given by $\{(g_{3m+1})^0, \overline{(g_{3m+1})^1)}\}$[69]. Let $g_{n,1}$ and $g_{n,l}$ denote the end vertices of the sequence g_n and $g_{n,i}, 1 < i < l$ the internal vertices of g_n. Since $g_{3m+1} = 0\overline{g_{3m}}, 10\overline{g_{3m-1}}$, the end vertices of P are of the form $00g_{3m}$ or $010g_{3m-1}$(considering them as the vertices in Γ_{3m+2}).

Case 1.1.a One end of P is $00g_{3m}$ and other end is $010g_{3m-1}$.

Let these vertices be denoted as $00g_{3m,i}$ and $010g_{3m-1,j}$ respectively. Then

$$\{00g_{3m,i}, 00g_{3m,i+1}, \cdots, 00g_{3m,l}, 10g_{3m,l}, 10g_{3m,l-1}, \cdots, 10g_{3m,1},$$
$$00g_{3m,1} = 000g_{3m-1,l}(\text{ or } 00g_{3m,1}, \cdots, 000g_{3m-1,l}), 010g_{3m-1,l},$$
$$\cdots, 010g_{3m-1,j}, \cdots, 010g_{3m-1,1}, 000g_{3m-1,1}, 000g_{3m-1,2}, \cdots,$$
$$000g_{3m-1,j}, \cdots, 000g_{3m-1,l-1} = 00g_{3m,2}, 00g_{3m,3}, \cdots, 00g_{3m,i-1},$$
$$00g_{3m,i}\}.$$

OR (if the above situation fails.)

$$\{00g_{3m,i}, 00g_{3m,i-1}, \cdots, 00g_{3m,i} = 000g_{3m-1,1}, 010g_{3m-1,1}, \cdots, 010g_{3m-1,l}, 000g_{3m-1,l} = 00g_{3m,1}, 10g_{3m,1}, 10g_{3m,2}, \cdots, 10g_{3m,l}, 00g_{3m,l}, 00g_{3m,l-1}, \cdots, 00g_{3m,i-1}, \cdots, 00g_{3m,i}\}.$$

is the required cycle.

Case 1.1.b Both ends are in $00g_{3m}$

Let these two ends be denoted as $00g_{3m,i}$ and $00g_{3m,j}$ respectively. Then

$$\{00g_{3m,i}, 00g_{3m,i+1}, \cdots, 00g_{3m,l}, 10g_{3m,l}, \cdots, 10g_{3m,1}, 00g_{3m,1} = 000g_{3m-1,l}(\text{ or } 00g_{3m,1}, \cdots, 000g_{3m-1,l}), 010g_{3m-1,l}, \cdots, 010g_{3m-1,1}, 000g_{3m-1,1}, \cdots, 000g_{3m-1,l-1} = 00g_{3m,2}, \cdots, 00g_{3m,i-1}\}.$$

OR (if the above situation fails.)

$$\{00g_{3m,i}, 00g_{3m,i-1}, \cdots, 00g_{3m,1} = 000g_{3m-1,l}(\text{ or } 00g_{3m,1}, \cdots, 000g_{3m-1,l}), 010g_{3m-1,l}, \cdots, 010g_{3m-1,1}, 000g_{3m-1,1}, \cdots, 00g_{3m,l}, 10g_{3m,l}, \cdots, 10g_{3m,1}, \cdots, 00g_{3m,i}, \cdots, 00g_{3m,i}\}.$$

is the required cycle.

Case 1.1.c Both the ends are in $010g_{3m-1}$

Let these end vertices be denoted as $010g_{3m-1,i}$ and $010g_{3m-1,j}$. Then the required cycle is obtained as

$\{010g_{3m-1,i}, 010g_{3m-1,i+1}, \cdots, 010g_{3m-1,l}, 000g_{3m-1,l} = 00g_{3m,1}$ (or $000g_{3m-1,l}, \cdots, 00g_{3m,1}), 10g_{3m,1}, \cdots - 10g_{3m,l} = 00g_{3m,l}$, $\cdots, 00g_{3m,2} = 000g_{3m-1,l-1}, 010g_{3m-1,l-1}, \cdots, 010g_{3m-1,i-1}$, $010g_{3m-1,i}\}$.

Case 1.2$|\mathcal{E}_{3m+2}| < |\mathcal{O}_{3m+2}|$.

The proof is on similar lines.

Case 2. $n = 3(m+1) = 3m+3$.

In Γ_{3m+3}, we have Γ_{3m+2} and Γ_{3m+1} as induced subgraphs. So $|\mathcal{E}_{3m+3}| \gtrless |\mathcal{O}_{3m+3}|$ according as $|\mathcal{O}_{3m+2}| \gtrless |\mathcal{E}_{3m+2}|$. Now in Γ_{3m+2}, the result holds true by Case 1 and since Γ_{3m+1} is hamiltonian.

Case 2.1 $|\mathcal{E}_{3m+2}| > |\mathcal{O}_{3m+2}|$.

Here also we have two choices for the deletion of an even weighted vertex, either an even weighted vertex from Γ_{3m+2} or an odd weighted vertex from Γ_{3m+1}. If we consider $\Gamma_{3m+3}-v, v \in \mathcal{E}_{3m+2}$, a hamiltonian cycle in $\Gamma_{3m+2} - v$ together with the hamiltonian cycle in Γ_{3m+1} will give the required hamiltonian cycle as in case 1.

Next, we consider $\Gamma_{3m+3} - v, v \in \mathcal{O}_{3m+1}$. Let P' be the hamiltonian path in $\Gamma_{3m+1} - v, v \in \mathcal{O}_{3m+1}$ obtained by deleting the vertex v from the hamiltonian cycle in Γ_{3m+1}. The end vertices of P' are of the form $100g_{3m}$ or $1010g_{3m-1}$. If P' has ends $100g_{3m,i}$ and $1010g_{3m-1,j}$, then

$$\{100g_{3m,i}, 100g_{3m,i-1}, \cdots, 1010g_{3m-1,j}, 0010g_{3m-1,j}, \cdots,$$
$$000g_{3m,k}, \cdots, 000g_{3m,l}, 010g_{3m,l}, \cdots, 010g_{3m,1}, 000g_{3m,1}, \cdots,$$
$$000g_{3m,i}, 100g_{3m,i}\}.$$

is the required cycle. Similarly we can find hamiltonian cycles in other cases also.

Case 2.2 $|\mathcal{E}_{3m+2}| < |\mathcal{O}_{3m+2}|$.

The required cycle is obtained as in case 2.1. $\square$

5.3 Wide diameter and Fault diameter

In this section, we show that the wide diameter and the fault diameter of Fibonacci cube are both equal to its diameter for $n \geq 5$.

Theorem 5.3.1. *The wide diameter of* Γ_n, $D_{\lfloor \frac{n+2}{3} \rfloor}(\Gamma_n) = n, n \geq 5$.

Proof. For $n = 5$, we know that $\kappa(\Gamma_5) = 2$ and for every $u, v \in \Gamma_n$, there exists a container of width 2 and length ≤ 5. Also the vertices $10101, 01010$ are at wide distance shows that, $D_2(\Gamma_5) = 5$. Hence we prove the result by induction on n. Suppose that the result is true for all $n \leq m$ and consider Γ_{m+1}.

Case 1: $m + 1 = 3k + 1$.

In this case, $\kappa(\Gamma_{m+1}) = k + 1$ and $\kappa(\Gamma_m) = \kappa(\Gamma_{m-1}) = k$. Our Claim: for every $u, v \in \Gamma_{m+1}$, there exists a $k+1-$ container of length at most $m + 1$.

Case 1.1 : $u, v \in 0\Gamma_m$.

If $u, v \in 0\Gamma_m$, by induction assumption there exists a container of width k and length at most m. One more path vertex disjoint with this container between u and v is obtained by going through Γ_{m-1} if it is not there in $0\Gamma_m$. then length of this path is still at most $m - 1 + 2 = m + 1$.

Case 1.2 : $u, v \in 10\Gamma_{m-1}$.

Container is obtained in a similar fashion as in the previous case with the last container is being taken through Γ_m.

Case 1.3 : $u \in 0\Gamma_m$and $v \in 10\Gamma_{m-1}$.

Let $u = 0u_1u_2 \cdots u_m$ and $v = 10v_1v_2 \cdots v_{m-1}$. Then take the $k-$ containers $P_i, 1 \leq i \leq k$ between u and $00v_1v_2 \cdots v_{m-1}$ given by $P_i = u - w_1 - w_2 - \cdots - 00w_i - \cdots - 00v_1v_2 \cdots v_{m-1}, w_i \in \Gamma_{m-1}$ and set $P_{i'} = u - w_1 - w_2 - \cdots - 10w_i - \cdots - 10v_1v_2 \cdots v_{m-1} = v$. This will give a $k-$ container between u and v and the $k + 1^{th}$ path given by $P_{k+1} = u - 00x - 10x - v,$ for some$v \in \Gamma_m$. Length of this container is clearly at most $m + 1$. See Fig:.

Case 2: $m + 1 = 3k$ or $3k + 2$.

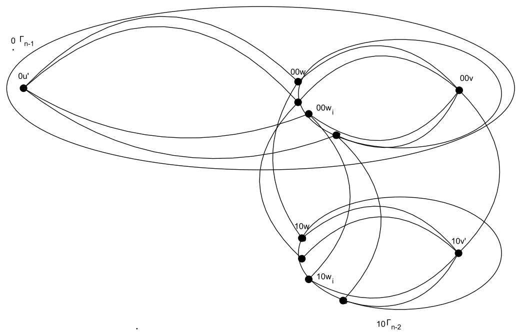

Figure 5.2: Containers in Γ_n

In this case $\kappa(\Gamma_i) = k$ or $k+1$ $\forall i = m+1, m, m-1$. So the required container can be obtained in a similar manner.

Considering all the possible cases as above it follows that $D_{\lfloor \frac{n+2}{3} \rfloor}(\Gamma_{m+1}) \leq m+1$. Now the fact that the vertices $u_1 = 1010\cdots10(01)$ and $v_1 = 0101\cdots01(10)$ are at wide distance $m+1$ shows that $D_{\lfloor \frac{n+2}{3} \rfloor}(\Gamma_{m+1}) = m+1$. □

Theorem 5.3.2. *The fault diameter of* Γ_n $f(\Gamma_n) = n$.

Proof. Let $u = 1010\cdots10(01)$ and $v = 0101\cdots01(10)$ and let $F = \{u^i/0 \leq i \leq \lceil \frac{n+2}{3} \rceil - 1\}$, where u^i denotes the vertex where the i^{th} one becomes zero. Then there will be only one fault free node $10001\cdots10(01)$. Then distance between this vertex and v in $\Gamma_n - F$ is at least $n-1$ and hence $d(u,v) \geq n$. Since wide diameter is always an upper bound to fault diameter it follows that $f(\Gamma_n) = n$. $\square$

Corollary 5.3.3. $D_{\lfloor \frac{n+2}{3} \rfloor}(\Gamma_n) = f(\Gamma_n) = n = diam(\Gamma_n), n \geq 5$.

5.4 Diameter variability

First recall that $D^{-k}(G)$ is the least number of edges whose addition to G decreases the diameter by (at least) k, $D^{+0}(G)$ is the maximum number of edges whose deletion from G does not change the diameter and $D^{+k}(G)$ is the least number of edges whose deletion from G increases the diameter by (at least) k. Here, we determine $D^{-1}(\Gamma_n)$, $D^{-2}(\Gamma_n)$, $D^{+0}(\Gamma_n)$ and also an upper bound for $D^{-l}(\Gamma_n)$.

Theorem 5.4.1. $D^{-1}(\Gamma_n) = 1, n \geq 2$.

Proof. To prove this, it is enough to find an edge that must be added to Γ_n to change the diameter to $n-1$. Let $u = 00\cdots01$ and $v = 00\cdots10$ and let $e = u{-}v$. Claim $D(\Gamma_n){+}e = n{-}1$. We prove this by induction. Consider Γ_2. Adding the edge $01-10$, Γ_2 becomes K_3 and the result becomes true. So suppose that the result is true for Γ_k and consider Γ_{k+1}. By fundamental decomposition, we can write $\Gamma_{k+1} = 0\Gamma_k \Psi 10\Gamma_{k-1}$. Add the edge $e = 00\cdots01 - 00\cdots10$. Then this reduces the diameter of $0\Gamma_k$ by $k-1$ by induction assumption. Now, we prove that $\mathrm{diam}(\Gamma_{k+1}) + e = k$. For this, consider the following cases.

Case 3. $u, v \in 10\Gamma_{k-1}$.

In this case $d(u,v) \leq k-1$ as these vertices can be considered as two vertices in Γ_{k-1}.

Case 4. $u, v \in 00\Gamma_k + e$.

Because of the induction assumption it follows that $d(u,v) \leq k-1$.

Case 5. $u \in 00\Gamma_k + e, v \in 10\Gamma_{k-1}$.

Let $u = 00u_1u_2\cdots u_{k-1}$ and $v = 10v_1v_2\cdots v_{k-1}$. Consider the path $P = u - \cdots 00v_1v_2\cdots v_{k-1} - v$. Therefore $d(u,v) \leq$ length of $P \leq (k-2)+1 = k-1$. If $u = 01u_1u_2\cdots u_{k-1}$, then there is a $u-v$ path defined by $P^{'} = u - 00u_1u_2\cdots u_{k-1} - \cdots 00v_1v_2\cdots v_{k-1} - v$. Then $d(u,v) \leq$ length of $P^{'} \leq 1 + (k-2) + 1 = k$.

Thus $u, v \in \Gamma_{k+1} + e, d(u,v) \leq k$ and hence $\mathrm{diam}\Gamma_{k+1} + e \leq k$.

Now consider the vertices $u^{'} = 1010\cdots10$ and $v^{'} = 0101\cdots01$ if k is odd and $u^{'} = 1010\cdots101$ and $v^{'} = 0101\cdots010$ if k is even. Then clearly $d(u^{'}, v^{'}) = k$ in $\Gamma_{k+1} + e$ and hence the result. $\square$

Lemma 5.4.2. *In $\Gamma_n, n \geq 4$ there exists at least three pairs of vertices at distance at least $n-1$ with the corresponding shortest paths have at most one edge in common.*

Proof. Consider Γ_4 and the vertex pairs $(0101, 1010), (0010, 1001), (0100, 1001)$. Let the corresponding shortest paths be $P_1 = 0101 - 0100 - 0000 - 1000 - 1010$, $P_2 = 0010 - 0000 - 0001 - 1001$, and $P_3 = 0100 - 0101 - 0001 - 1001$. These paths satisfy the required condition. Using this, we can find the required three paths in Γ_5 by adding to each $0P_i$, the vertices $10101, 10100, 10010$ suitably at one end. This process can be continued to get the required paths in any $\Gamma_n, n \geq 4$. $\square$

Theorem 5.4.3. $D^{-2}(\Gamma_n) = 3, n \geq 4$.

Proof. Consider Γ_4. Add the edges $e_1 = 0001 - 0010, e_2 = 0101 - 1010, e_3 = 0100 - 1000$. Then it is easy to see that $\operatorname{diam}(\Gamma_4) + \{e_1, e_2, e_3\} = 2$ and this the minimum. Next we claim that $D^{-2}(\Gamma_n) = 3, n > 4$. For this, add the edges $e_1 =$

$00\cdots0001{-}00\cdots0010, e_2 = 00\cdots0101{-}00\cdots1010, e_3 = 00\cdots 0100 - 00\cdots1000$ in Γ_n. Then by induction and fundamental decomposition it is easy to prove that $\text{diam}(\Gamma_n + \{e_1, e_2, e_3\}) = n-2, n > 4$. Thus it remains to prove that addition of any two non adjacent edges will not decrease the diameter by two. If we add any two non adjacent vertices, then there exists at least one pair of vertices which are at distance at least $n-1$ and hence the diameter remains to be at least $n-1$ by the previous lemma. □

Lemma 5.4.4. *For all Γ_n, there exists a diameter preserving spanning tree.*

Proof. Let the required spanning tree for Γ_n be denoted by T_n. Then, for $n \leq 2$, $T_n = \Gamma_n$. Now T_3 for Γ_3 is shown in the figure Fig:2. Using T_2 and T_3 we can find T_4 by just joining the edge

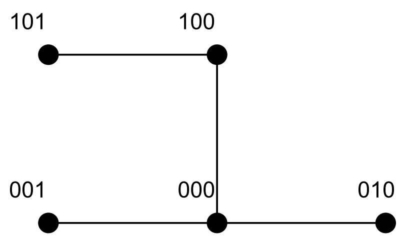

Figure 5.3: T_3 : Spanning tree in Γ_3

$0000 - 1000$(see Fig : 3). Now, define $T_n = 0T_{n-1} \cup 10T_{n-2} \cup \{e =$

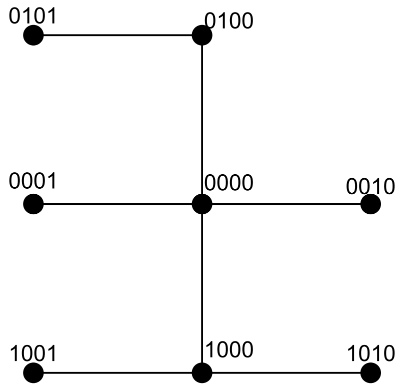

Figure 5.4: T_4 : Spanning tree in Γ_4

$00 \cdots 00 - 10 \cdots 00\}$. This T_n will be the required spanning tree which can be proved easily by induction. □

Theorem 5.4.5. $D^{+0}(\Gamma_n) = |E(\Gamma_n)| - |V(\Gamma_n)| + 1.$

Proof. By the previous lemma, there exists a spanning tree T_n with diameter n. $|E(T_n)| = |T_{n-1}| + |T_{n-2}| + 1 = F_{n+2} - 1$. Therefore the number of edges removed to get T_n is $|E(\Gamma_n)| - (F_{n+2} - 1)$. Hence the lower bound and since its a tree, the equality holds. □

Theorem 5.4.6. $D^{-l}(\Gamma_n) \leq D^{-l}(\Gamma_{n-1}) + D^{-(l-2)}(\Gamma_{n-2}), l > 2, n > 4.$

Proof. Let e_1 and e_2 be the number of edges added in Γ_{n-1} and Γ_{n-2} respectively to reduce the diameter by at least $l-1$ and l respectively. Then $\mathrm{diam}(\Gamma_{n-1})$ changes from $n-1$ to $n-1-l$ and that of Γ_{n-2} from $n-2$ to $n-l$ by the addition of these edges. The fundamental decomposition of Γ_n and these addition of edges in $0\Gamma_{n-1}$ and in $10\Gamma_{n-2}$, then give the distance between any pair of vertices in Γ_n to be at most $n-l$. Hence the bound. $\square$

Concluding Remarks

In this thesis, we have made an attempt to study the network topological properties of some graph classes. Through this study, we observed that Mycielskian is a graph operator which can produce large networks which preserves some nice properties. We could also study the various diameter aspects and other measures of efficiency in two well known class of graphs namely Sierpinski graphs and Fibonacci cubes. However, we list below some problems which we found interesting, but could not be attempted for various reasons.

- Wide diameter for edge variation, $(l, w)-$ independence number, Restricted connectivity and diameter of Mycielskian.

- Since we have studied only diameter and its variabilty of generalized Mycielskian, many problems like wide diameter, fault diameter, diameter vulnerability, $(l, w)-$ domination and independence number are all still open.

- The properties like spanning connectivity, spanning diameter, mutually independent hamiltonian paths and cycles can be studied in both Sierpinski graphs and Fibonacci cubes.

- Forwarding indices and bisection width of Sierpinski-like graphs can be studied.

- All the measures of reliability and efficiency of networks discussed here can be extended to other networks.

We conclude the thesis with an optimistic note that some of the problems mentioned above will be solved soon.

List of symbols

$G = (V, E)$	-	Graph with vertex set V and edge set E.
$\Delta(G)$	-	The maximum degree of G.
$\delta(G)$	-	The minimum degree of G.
$\kappa(G)$	-	The vertex connectivity of G.
$\kappa'(G)$	-	The edge connectivity of G.
$\mu(G)$	-	The Mycielskian of G.
$\mu_m(G)$	-	The m-Mycielskian of G.
$\pi(G)$	-	The edge-forwarding index of G.
$\zeta(G)$	-	The vertex forwarding index of G.
Γ_n	-	The n-dimensional Fibonacci cube.

bw(G)	-	The bisection width of G.
$c(G)$	-	The convexity number of G.
C_n	-	The cycle of length n.
$d(u,v)$	-	The distance between u and v.
$d_G(u,v)$	-	The distance between u and v in G.
$\mathrm{diam}(G)$	-	The diameter of G.
$d(v)$	-	The degree of the vertex v.
$D_\kappa(G)$	-	Wide diameter of G.
$D^{-k}(G)$	-	The least number of edges whose addition to G decreases the diameter by (at least) k.
$D^{+0}(G)$	-	The maximum number of edges whose deletion from G does not change the diameter.
$D^{+k}(G)$	-	The least number of edges whose deletion from G increases the diameter by (at least) k.
$e(v)$	-	The eccentricity of vertex v.
$f(G)$	-	The fault diameter of G.
$\mathrm{gin}(G)$	-	The geodetic iteration number of G.
$\mathrm{gn}(G)$	-	The geodetic number of G.

$G + uv$	-	The supergraph obtained by adding the new edge uv.
$G - e$	-	The subgraph of G obtained by deleting the edge e.
$G - E'$	-	The subgraph of G obtained by the deletion of the edges in $E' \subset E$.
$G - v$	-	The subgraph of G obtained by deleting the vertex v.
$G - S$	-	The subgraph of G obtained by the deletion of the vertices in $S \subset V$.
$h(G)$	-	The hull number of G.
$I(u, v)$	-	The interval in G.
K_n	-	The complete graph on n vertices.
$K_{1,q}$	-	The star of size q.
$K_{p,q}$	-	The complete bipartite graph with part sizes p and q.
$\text{mc}(G)$	-	The m-convexity number of G.
$\min(G)$	-	The minimal path iteration number of G.
$\text{mh}(G)$	-	The m-hull number of G.
$\text{mn}(G)$	-	The monophonic number of G.

$N(v)$	-	The neighborhood of the vertex v.
P_k	-	The path of length k.
Q_n	-	n-dimensional hypercube.
$r(G)$	-	The radius of G.
$< S >$	-	The subgraph of G induced by the subset S of V.
$[[S]]$	-	The convex hull of G.
S_k^n	-	The Sierpiński graphs.

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List of Publications

Papers published /communicated

1. Savitha K.S, A. Vijayakumar, Some network topological notions of the Mycielskian of a graph, *AKCE Int. J. Graphs Comb.* **13** (2016), 31–37.

2. Savitha K.S, Chithra M.R, A. Vijayakumar, Some diameter notions of the generalized Mycielskian of a graph, *Lecture Notes in Comput. Sci.*(Springer)(To appear).

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4. Savitha K.S, A. Vijayakumar, Hamiltonicity of Fibonacci cubes of odd order under vertex deletion (communicated).

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Papers presented

1. Some network topological properties of the Mycielskian of a graph, 23rd international conference of forum for interdisciplinary Mathematics, NIT Karnataka, Suratkal, $18-20$, December 2014.

2. $(l, k)-$ domination of the Mycielskian of a graph National seminar on Discrete mathematics, St.Paul's College, Kalamassery, $15-16$, December 2015.

3. Some diameter notions of the generalized Mycielskian of a graph, International Conference on Theoretical Computer Science and Discrete Mathematics, Kalasalingam University, Krishnankoil, Tamilnadu, $19-21$, December 2016.

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