STUDIES ON THE POWER DOMINATION PROBLEM IN GRAPHS

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under the **Faculty of Science**

By

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To

My Parents

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Certificate

Certified that the work presented in this thesis entitled **"Studies on the Power Domination Problem in Graphs"** is based on the authentic record of research carried out by Ms. Seethu Varghese under my guidance in the Department of Mathematics, Cochin University of Science and Technology, Cochin- 682 022 and has not been included in any other thesis submitted for the award of any degree.

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Certified that all the relevant corrections and modifications suggested by the audience during the Pre-synopsis seminar and recommended by the doctoral committee of the candidate has been incorporated in the thesis entitled **"Studies on the Power Domination Problem in Graphs"**.

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Declaration

I, Seethu Varghese, hereby declare that this thesis entitled **"Studies on the Power Domination Problem in Graphs"** is based on the original work carried out by me under the guidance of Dr. A. Vijayakumar, Professor and Head, Department of Mathematics, Cochin University of Science and Technology, Cochin-22 and contains no material which had been accepted for any other Degree, Diploma or similar titles in any University or Institution. To the best of my knowledge and belief, this thesis contains no material previously published by any person except where due references are made in the text of the thesis.

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Chapter 1

Introduction

In 1735, the renowned Swiss mathematician Leonhard Euler settled the famous 'Königsberg bridge problem', which had perplexed scholars for many years, and his method of solution to the problem initiated the study of an entire new branch of mathematics, called Graph Theory. The origin of graph theory is well recorded in [13]. After Euler's discovery, graph theory boomed with major contributions made by great mathematicians like Kirkman, Hamilton, Cayley, Kirchhoff, Sylvester and Polya. This branch of mathematics has developed into a substantial body of knowledge with a variety of applications in diverse fields such as physics, chemistry, economics, psychology, business, sociology, anthropology, linguistics and geography. Volumes have been written on the rich theory and the very many applications of graphs in [4, 11, 33, 71], including the pioneering works of C. Berge [10], F. Harary [36] and O. Ore [58].

In the past decade, graph theory has gone through a remarkable shift and a profound transformation. Graph theory is now emerging as a central part of the information revolution. In contrast to its origin in recreational mathematics, graph theory today uses sophisticated combinatorial, probabilistic, and spectral methods with deep connections with a variety of areas in mathematics and computer science. The area of network science calls for a solid scientific foundation and rigorous analysis for which graph theory is ideally suited. Graph theory provides a fundamental tool for designing and analyzing large scale networks [47]. Real world networks such as the World Wide Web, collaboration networks, citation networks and social networks lead to new directions for research in graph theory [5, 19].

This thesis aims at making a humble effort to contribute to the innumerous set of results in graph theory.

Domination is a well studied graph parameter, and a classical topic in graph theory. The historical roots of domination dates back to 1862 when C. F. De Jaenisch studied the problem of determining the minimum number of queens necessary to cover $n \times n$ chessboard [22]. The evolution and the subsequent development of this fertile area of domination theory from the chessboard problems is very well surveyed by Watkins in [70]. The mathematical study of domination theory in graphs started around 1960 by Berge [10] and Ore [58]. Domination has applications in facility location problems, coding theory, computer communication networks, biological networks and social networks.

One of the reasons that stimulate much research into domination is the multitude of variations of domination. Each type of domination meets a specific purpose in real life applications. Various types of domination are obtained by imposing an additional condition on the method of dominating. Domination parameters are in general discussed and characterized for their properties and bounds on various graphs. A detailed study on the motivation and applications of graph domination, and comprehensive treatment of variety of domination parameters can be found in [20, 40, 41].

Power domination, introduced by Baldwin et al. [6] in 1993, is a variation of domination which arises in the context of monitoring electric power networks. A power network contains a set of buses and a set of branches connecting the buses. It also contains a set of generators, which supply power, and a set of loads, where the power is directed to. The important task for electric power companies is to monitor their power system continuously. In order to monitor a power network, we need to measure all the state variables of the network by placing measurement devices. A Phasor Measurement Unit (PMU) is a measurement device placed on a bus that has the ability to measure the voltage at the bus and current phase of the branches connected to the bus. PMUs are used to monitor large system stability and to give warnings of system-wide failures. The ability to measure the current phasors as well as the voltage gives the PMU an advantage over other measurement units, some of which require one measurement device per bus. As PMUs are expensive devices, the goal of power system monitoring problem is to install the minimum number of PMUs such that the whole system is monitored. This problem has been formulated as a graph theoretic problem by Haynes et. al in [39] in 2002.

Two fundamental laws of physics that can be used to reduce the number of PMUs required to achieve complete monitoring of a power system are as follows:

1. **Ohm's law:** The current through a conductor between two points is directly proportional to the voltage across the two buses, and is inversely proportional to the resistance between them.
2. **Kirchhoff's current law:** At any bus in a circuit, the sum of currents flowing into it is equal to the sum of currents flowing out of it.

Consider an 18-bus system depicted in Figure 1.1. We place the PMUs at bus 5 and 14 and the associated phasor measurements are assigned to the branches 5-4, 5-6, 5-10, 14-8, 14-12 and 14-15. From Ohm's law, the voltage phasors at buses 4, 6, 8, 10,12 and 15 can be obtained from the branch current (Figure 1.2). From the known voltage phasors, the currents of the branches 6-8 and 10-12 can be calculated using Ohm's law (Figure 1.3). From the known currents of branches 5-6 and 6-8, the current of the branch 6-7 can be inferred by using Kirchhoff's

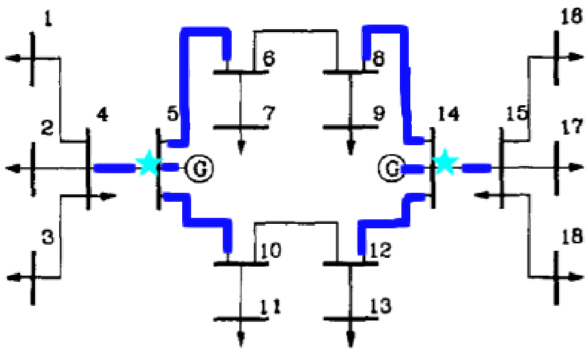


Figure 1.1: 18-Bus system (From Baldwin et al. [6]).

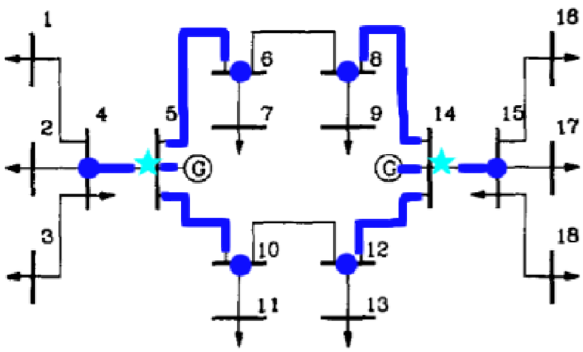


Figure 1.2: 18-Bus system (From Baldwin et al. [6]).

law. Using Ohm's law, the voltage phasor at bus 7 can be deduced. Similarly, the current of the branch 8-9 and the voltage phasor at bus 9 can be calculated (Figure 1.4). Since the coverage of the two PMUs cannot be expanded any further, consider the placement of PMUs at buses 4, 6 and 15. This allows us to assign current measurements to the branches 4-1, 4-2, 4-3, 4-5, 6-5, 6-7, 6-8, 15-14, 15-16, 15-17 and 15-18, and to calculate

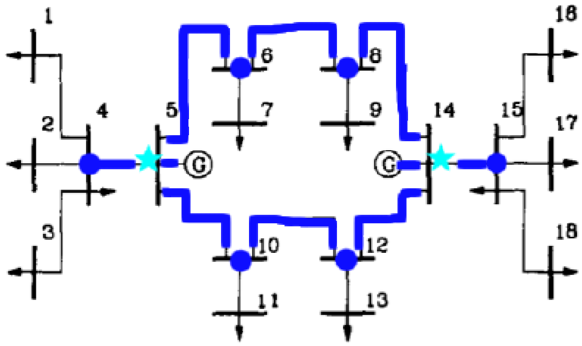


Figure 1.3: 18-Bus system (From Baldwin et al. [6]).

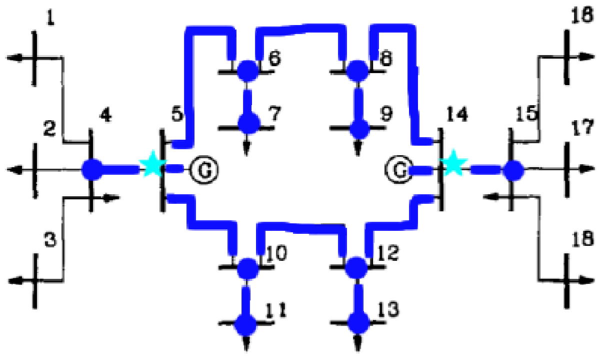


Figure 1.4: 18-Bus system (From Baldwin et al. [6]).

the voltage phasors at buses 1, 2, 3, 5, 7, 8, 14, 16, 17 and 18. From the voltages at buses 5 and 14, the current measurements can be assigned to the branches 5-10 and 14-12 by Kirchhoff's law. Again by applying Ohm's law, voltages at buses 10 and 12, and current of the branch 10-12 can be obtained. Finally by applying Kirchhoff's law, currents at branches 10-11 and 12-13 can be calculated, which will then give the voltage phasors at buses 11 and 13 by Ohm's law. However, this placement set of 3 PMUs is an optimal one (Figure 1.5).

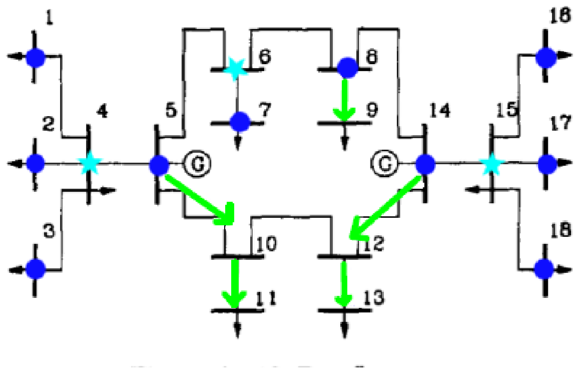


Figure 1.5: 18-Bus system (From Baldwin et al. [6]).

To see the power system monitoring problem and its graph theoretic formulation [1, 2] in more detail consider a graph $G = (V(G), E(G))$ that represents a power network. Here, vertices represent buses and edges are associated with the branches joining two buses. The resistance of the branches in the power network is a property of the material with which it is made and hence it can be assumed to be known. For simplicity, assume that there are no generators and loads. Our goal is to measure the voltages at all vertices and electrical currents at the edges. By placing a PMU at a vertex v we can measure the voltage of v and the electrical current on each edge incident to v. Next, by using Ohm's law we can compute the voltage of any vertex in the neighbourhood of v. Now, assume that the voltage on v and all of its neighbours except w is known. By applying Ohm's law, we can compute the current on the edges incident to v except the edge vw. Next by using Kirchhoff's current law, we compute the current on the edge vw. Finally, applying Ohm's law on the edge vw gives us the voltage of w. Once we get the value of voltage at a vertex and electrical current on the edge incident to it, we say that the corresponding vertex and the edge are '**monitored**'. Using Ohm and Kirchhoff laws, it is then possible to infer from initial knowledge of the status of some part of the network the status of new edges or vertices. The graphical representation of the monitoring of 18-bus system discussed earlier is shown in Figure 1.6.

In terms of graphs, the monitoring of vertices and edges by a PMU can now be described by the following rules [39].

1. Any vertex where a PMU is placed and its incident edges are monitored.

2. If one end vertex of a monitored edge is monitored, then the other end vertex is monitored (*by Ohm's law*).

3. Any edge joining two monitored vertices is monitored (*by

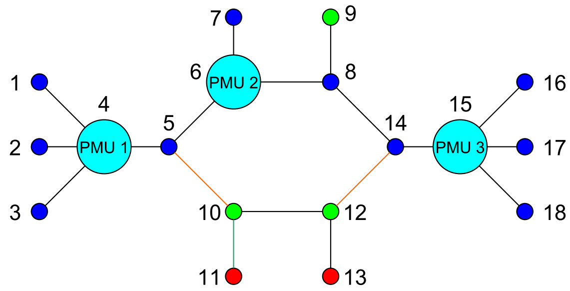


Figure 1.6

Ohm's law).

4. If a vertex is of degree $r > 1$, and $r - 1$ of its incident edges are monitored, then all r incident edges are monitored (*by Kirchhoff's law*).

The power system monitoring problem was considered in [39] in a slightly more complicated way by treating both vertices and edges of a given graph. It was later noticed in [2, 17, 28, 34, 52] that the problem can be studied considering only vertices. However, it was easily shown that both approaches are equivalent, as observed in [28]. The above four monitoring rules of a PMU can now be simplified into following two monitoring rules.

• (**Rule 1**) A PMU on a vertex v monitors v and all its neighbours.

• (**Rule 2**) If a monitored vertex u has all its neighbours monitored except one neighbour w, then w becomes monitored as well.

The monitoring of vertices is governed by the above two rules. The first rule is called the **domination rule** and the second one is called the **propagation rule**. From this propagation, a vertex can end up to be monitored even though it is at a large distance from any vertex selected to carry a PMU.

A graph is '**monitored**' if all its vertices are monitored according to the domination and propagation rules. The **power domination problem in graphs** consists of finding a set of vertices of minimum cardinality that monitors the entire graph, by applying the two monitoring rules- the domination rule and the propagation rule. In terms of physical network, those vertices will provide the locations where the PMUs should be placed in order to monitor the entire network at the minimum cost.

Power domination is generalized to k-power domination by Chang et al. [18] in 2012 by adding the possibility of propagating up to k vertices, k a non-negative integer. The k-power domination reduces to the usual power domination when $k = 1$

and to the domination when $k = 0$.

This thesis is centered around the power domination problem in graphs.

The basic notations, terminology and definitions used in this thesis are from [4, 35, 36, 40, 71].

1.1 Notations

When $G = (V(G), E(G))$ is a graph,

$V(G)$	:	the vertex set of G
$E(G)$	:	the edge set of G, a collection of 2-element subsets of $V(G)$
$\lvert V(G)\rvert$	:	the order of G
$\lvert E(G)\rvert$	:	the size of G
$d_G(v)$ or $d(v)$	:	the degree of v in G
$\delta(G)$	:	the minimum degree of G
$\Delta(G)$	:	the maximum degree of G
$d_G(u, v)$ or $d(u, v)$	:	the shortest distance between u and v in G

u, v-path	:	path from a vertex u to a vertex v
$r(G)$	:	the radius of G
$N_G(v)$ or $N(v)$	:	the open neighbourhood of v in G
$N_G[v]$ or $N[v]$	:	the closed neighbourhood of v in G
$N_G(S) = \cup_{v \in S} N_G(v)$ or $N(S)$	:	the open neighbourhood of a subset S of $V(G)$
$N_G[S] = \cup_{v \in S} N_G[v]$ or $N[S]$	:	the closed neighbourhood of a subset S of $V(G)$
$\lambda(G)$	:	the edge connectivity of G
$G \cong H$	:	G is isomorphic to H
$G \cup H$	:	union of G and H
$< X >$	:	the subgraph induced by a subset X of $V(G)$
P_n	:	the path on n vertices
C_n	:	the cycle on n vertices
K_n	:	the complete graph on n vertices
$K_{1,n}$	:	the star of size n

$K_{m,n}$	:	the complete bipartite graph where m and n are the cardinalities of the partitions
$G - v$	:	the subgraph of G obtained by deleting the vertex v
$G - e$	:	the subgraph of G obtained by deleting the edge e
$G - A$	:	the subgraph of G obtained by the deletion of the vertices in A
$G - B$	:	the subgraph of G obtained by the deletion of the edges in B

1.2 Definitions

Definition 1.2.1. [4] A graph G is **trivial** or **empty** if its vertex set is a singleton and it contains no edges, and **nontrivial** or **nonempty** if it has at least one edge. A vertex of degree zero is an **isolated vertex** and of degree one is a **pendant vertex**. The edge incident on a pendant vertex is a **pendant edge**. If G is a graph of order n, then a vertex of degree $n - 1$ is called a **universal vertex**. A spanning 1-regular graph is called a **perfect matching**. A maximal complete subgraph of order p

is called a ***p*-clique**.

Definition 1.2.2. [4] A **subdivision of an edge** $e = xy$ of a graph G is obtained by introducing a new vertex z in e, that is, by replacing the edge $e = xy$ of G by the path xzy of length two so that the new vertex z is of degree two in the resulting graph. The graph obtained by **contraction of an edge** $e = xy$, denoted by G/e, is obtained from $G - e$ by replacing x and y by a new vertex v_{xy} (contracted vertex) which is adjacent to all vertices in $N_{G-e}(x) \cup N_{G-e}(y)$.

Definition 1.2.3. [39] A **spider** is the tree formed from a star by subdividing any number of its edges any number of times.

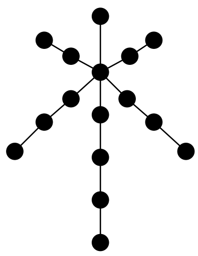


Figure 1.7: A spider graph obtained by subdividing the edges of the star $K_{1,6}$.

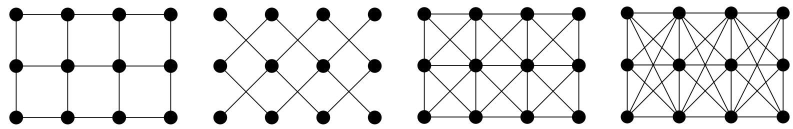
Definition 1.2.4. [4] The **join** of two graphs G and H, denoted by $G \vee H$, is the graph with vertex set $V(G) \cup V(H)$ and the edge set $E(G) \cup E(H) \cup \{gh\colon g \in V(G), h \in V(H)\}$. The graph $K_1 \vee C_{n-1}$ is called the **wheel**, W_n. The graph $K_1 \vee P_{n-1}$ is called the **fan**, F_n.

Definition 1.2.5. [35] The **Cartesian product** of two graphs G and H, denoted by $G \square H$, is the graph with vertex set $V(G) \times V(H)$ and two vertices (g, h) and (g', h') in $V(G) \times V(H)$ are adjacent in $G \square H$ if either $g = g'$ and $hh' \in E(H)$, or $h = h'$ and $gg' \in E(G)$.

Definition 1.2.6. [35] The **direct product** of two graphs G and H, denoted by $G \times H$, is the graph with vertex set $V(G) \times V(H)$ and two vertices (g, h) and (g', h') in $V(G) \times V(H)$ are adjacent in $G \times H$ if $gg' \in E(G)$ and $hh' \in E(H)$.

Definition 1.2.7. [35] The **strong product** of two graphs G and H, denoted by $G \boxtimes H$, is the graph with vertex set $V(G) \times V(H)$ and the edge set $E(G \boxtimes H) = E(G \square H) \cup E(G \times H)$.

Definition 1.2.8. [35] The **lexicographic product** of two graphs G and H, denoted by $G \circ H$, is the graph with vertex set $V(G) \times V(H)$ and two vertices (g, h) and (g', h') in $V(G) \times V(H)$ are adjacent in $G \circ H$ if either $gg' \in E(G)$, or $g = g'$ and $hh' \in E(H)$.



(a) $P_4 \square P_3$ (b) $P_4 \times P_3$ (c) $P_4 \boxtimes P_3$ (d) $P_4 \circ P_3$

Figure 1.8: The four graph products.

Remark 1.2.1. The four product graphs, except for the lexicographic product, are commutative [35].

Definition 1.2.9. [35] Let $G * H$ be any of the graph products. For any vertex $g \in V(G)$, the subgraph of $G * H$ induced by $\{g\} \times V(H)$ is called the H-**fiber** at g and denoted by ${}^gH$. For any vertex $h \in V(H)$, the subgraph of $G * H$ induced by $V(G) \times \{h\}$ is called the G-**fiber** at h and denoted by G^h. The graphs G and H are called the **factor graphs** of $G * H$.

Definition 1.2.10. [40] A subset $S \subseteq V(G)$ of vertices in a graph G is a **dominating** set if every vertex $v \in V(G)\backslash S$ is adjacent to at least one vertex in S. If S is a dominating set then $N_G[S] = V(G)$. A dominating set of minimum cardinality in G is called a **minimum dominating set**, and its cardinality, the **domination number** of G, denoted by $\gamma(G)$. A $\gamma(G)$-set is a dominating set in G of cardinality $\gamma(G)$.

Remark 1.2.2. A vertex v in a graph G is said to **dominate**

its closed neighbourhood $N_G[v]$. For two nonempty subsets S and X of $V(G)$, the set S **dominates** X if each vertex in $X \setminus S$ is adjacent to a vertex in S.

Definition 1.2.11. [40] A subset $S \subseteq V(G)$ of vertices in a graph G is a **total dominating** set if every vertex $v \in V(G)$ is adjacent to at least one vertex in S. The minimum cardinality of a total dominating set in a graph G is called its **total domination number**, denoted by $\gamma_t(G)$. A $\gamma_t(G)$-set is a total dominating set in G of cardinality $\gamma_t(G)$.

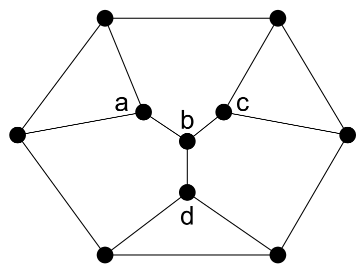


Figure 1.9: The graph G.

For the graph G in Figure 1.9, $\gamma(G) = 3$ ($\{a, c, d\}$ is a $\gamma(G)$-set) and $\gamma_t(G) = 4$ ($\{a, b, c, d\}$ is a $\gamma_t(G)$-set).

Definition 1.2.12. [26] Let G be a graph and $S \subseteq V(G)$. The **set monitored by S** , denoted by $M(S)$ is defined recursively as follows:

• (*domination step*) $M(S) \leftarrow S \cup N_G(S)$,

• (*propagation step*) as long as there exists a vertex $v \in M(S)$ such that $N_G(v) \cap (V(G) \setminus M(S)) = \{w\}$, set $M(S) \leftarrow M(S) \cup \{w\}$.

Remark 1.2.3. The set $M(S)$ initially consists of all vertices in $N_G[S]$. This set is then iteratively extended by including all vertices $w \in V(G)$ that have a neighbour v in $M(S)$ such that all the other neighbours of v, except w, are already in $M(S)$. In that case, we say '*v monitors w by propagation*'. This is continued until no such vertex w exists, at which stage the set monitored by S has been constructed. For example, consider Figure 1.10. For the set $S = \{u, v\}$, the vertices of $M(S)$ are coloured black in the figure.

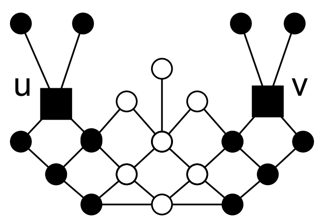


Figure 1.10: $M(S)$ where $S = \{u, v\}$.

Definition 1.2.13. [26] A set $S \subseteq V(G)$ is called a **power dominating set** (**PDS**) of a graph G if $M(S) = V(G)$ and the **power domination number** of G, denoted by $\gamma_{\mathrm{P}}(G)$, is the minimum cardinality of a power dominating set of G. A $\gamma_{\mathrm{P}}(G)$-set is a power dominating set in G of cardinality $\gamma_{\mathrm{P}}(G)$.

For the graph G in Figure 1.9, the set $\{a, c\}$ is a $\gamma_{\mathrm{P}}(G)$-set and therefore $\gamma_{\mathrm{P}}(G) = 2$.

Definition 1.2.14. [18] Let G be a graph, $S \subseteq V(G)$ and $k \geq 0$. The sets $\left(\mathcal{P}^{i}_{G,k}(S)\right)_{i \geq 0}$ of **vertices monitored by $\boldsymbol{S}$ at step $\boldsymbol{i}$** are defined as follows:

- (*domination step*) $\mathcal{P}^{0}_{G,k}(S) = N_G[S]$, and
- (*propagation steps*) $\mathcal{P}^{i+1}_{G,k}(S) = \bigcup\{N_G[v] \colon v \in \mathcal{P}^{i}_{G,k}(S)$ such that $\left|N_G[v] \setminus \mathcal{P}^{i}_{G,k}(S)\right| \leq k\}$.

For $i \geq 0$ we have $\mathcal{P}^{i}_{G,k}(S) \subseteq \mathcal{P}^{i+1}_{G,k}(S)$. This is easy to check by induction, using the fact that whenever $N_G[v]$ has been included in $\mathcal{P}^{i}_{G,k}(S)$, it is included in $\mathcal{P}^{i+1}_{G,k}(S)$. This also implies that $\mathcal{P}^{i}_{G,k}(S)$ is always a union of neighbourhoods. Furthermore, if a vertex v in the set $\mathcal{P}^{i}_{G,k}(S)$ has at most k neighbours outside the set, then the set $\mathcal{P}^{i+1}_{G,k}(S)$ contains $N_G[v]$. We say, 'v *monitors its* k *unmonitored neighbours by propagation at step* $i+1$'. If $\mathcal{P}^{i_0}_{G,k}(S) = \mathcal{P}^{i_0+1}_{G,k}(S)$ for some integer i_0, then $\mathcal{P}^{j}_{G,k}(S) = \mathcal{P}^{i_0}_{G,k}(S)$ for every $j \geq i_0$. We thus denote this set $\mathcal{P}^{i_0}_{G,k}(S)$ by $\mathcal{P}^{\infty}_{G,k}(S)$.

Remark 1.2.4. When the graph G is clear from the context, we simplify the notation to $\mathcal{P}^{i}_{k}(S)$ and $\mathcal{P}^{\infty}_{k}(S)$. The definition for the monitored set $M(S)$ is obtained by replacing k by 1 in the definition of $\left(\mathcal{P}^{i}_{G,k}(S)\right)_{i\geq 0}$.

Definition 1.2.15. [18] A set $S \subseteq V(G)$ is a ***k*-power dominating set** (***k*-PDS**) of a graph G if $\mathcal{P}^{\infty}_{G,k}(S) = V(G)$. A k-PDS of minimum cardinality in G is called a **minimum *k*-PDS** and its cardinality is called the ***k*-power domination number** of G, denoted by $\gamma_{\mathrm{P},k}(G)$. A $\gamma_{\mathrm{P},k}(G)$-set is a k-PDS in G of cardinality $\gamma_{\mathrm{P},k}(G)$.

Remark 1.2.5. Observe that $\gamma_{\mathrm{P},0}(G) = \gamma(G)$ and $\gamma_{\mathrm{P},1}(G) = \gamma_{\mathrm{P}}(G)$.

Consider the graph G' in Figure 1.11.

Case (i) : $k = 0$.

For $S_0 = \{e, g, k, s\}$, $\mathcal{P}^{0}_{G',0}(S_0) = V(G')$.

Case (ii) : $k = 1$.

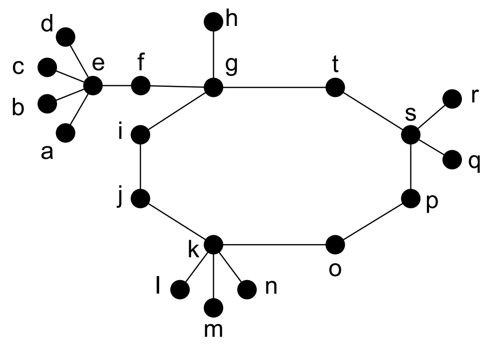


Figure 1.11: The graph G'.

For $S_1 = \{e, k, s\}$,

$\mathcal{P}^0_{G',1}(S_1) = \{e, k, s, a, b, c, d, f, j, l, m, n, o, p, q, r, t\}$.

$\mathcal{P}^1_{G',1}(S_1) = \{g, i\} \cup \mathcal{P}^0_{G',1}(S_1)$.

$\mathcal{P}^2_{G',1}(S_1) = \{h\} \cup \mathcal{P}^1_{G',1}(S_1) = V(G')$.

Case $(iii) : k = 2$.

For $S_2 = \{e, k\}$,

$\mathcal{P}^0_{G',2}(S_2) = \{e, k, a, b, c, d, f, j, l, m, n, o\}$.

$\mathcal{P}^1_{G',2}(S_2) = \{g, i, p\} \cup \mathcal{P}^0_{G',2}(S_2)$.

$\mathcal{P}^2_{G',2}(S_2) = \{h, s, t\} \cup \mathcal{P}^1_{G',2}(S_2)$.

$\mathcal{P}^3_{G',2}(S_2) = \{r, q\} \cup \mathcal{P}^2_{G',2}(S_2) = V(G')$.

Case $(iv) : k = 3$.

For $S_3 = \{e\}$,

$\mathcal{P}^0_{G',3}(S_3) = \{e, a, b, c, d, f\}$.

$\mathcal{P}^1_{G',3}(S_3) = \{g\} \cup \mathcal{P}^0_{G',3}(S_3)$.

$\mathcal{P}^2_{G',3}(S_3) = \{h, i, t\} \cup \mathcal{P}^1_{G',3}(S_3)$.

$\mathcal{P}^3_{G',3}(S_3) = \{j, s\} \cup \mathcal{P}^2_{G',3}(S_3)$.

$\mathcal{P}^4_{G',3}(S_3) = \{k, p, q, r\} \cup \mathcal{P}^3_{G',3}(S_3)$.

$\mathcal{P}^5_{G',3}(S_3) = \{o\} \cup \mathcal{P}^4_{G',3}(S_3)$.

$\mathcal{P}^6_{G',k}(S_3) = \{l, m, n\} \cup \mathcal{P}^5_{G',3}(S_3) = V(G')$.

For $k \geq 4$, we get that $\mathcal{P}^5_{G',k}(S_k) = V(G')$, for $S_k = \{e\}$. Thus, for each k, we can observe that S_k is a unique k-PDS of G'. Therefore $\gamma_{\mathrm{P},0}(G') = \gamma(G') = 4$, $\gamma_{\mathrm{P},1}(G') = \gamma_{\mathrm{P}}(G') = 3$, $\gamma_{\mathrm{P},2}(G') = 2$ and for $k \geq 3$, $\gamma_{\mathrm{P},k}(G') = 1$.

Definition 1.2.16. [25] The radius of a k-PDS S of a graph G is defined by $\mathrm{rad}_{\mathrm{P},k}(G, S) = 1 + \min\{i : \mathcal{P}^i_{G,k}(S) = V(G)\}$. The ***k*-propagation radius** of a graph G, denoted by $\mathrm{rad}_{\mathrm{P},k}(G)$, is $\mathrm{rad}_{\mathrm{P},k}(G) = \min\{\mathrm{rad}_{\mathrm{P},k}(G, S),\ S$ is a k-PDS of G, $|S| = \gamma_{\mathrm{P},k}(G)\}$.

For example, in Figure 1.11, $\mathrm{rad}_{\mathrm{P},0}(G') = 1$, $\mathrm{rad}_{\mathrm{P},1}(G') = 3$, $\mathrm{rad}_{\mathrm{P},2}(G') = 4$, $\mathrm{rad}_{\mathrm{P},3}(G') = 7$ and for $k \geq 4$, $\mathrm{rad}_{\mathrm{P},k}(G') = 6$.

Definition 1.2.17. [31] The **bondage number**, denoted by $b(G)$, of a nonempty graph G is the minimum cardinality among all sets of edges B for which $\gamma(G - B) > \gamma(G)$.

In Figure 1.11, $\gamma(G' - ae) = 5 > 4 = \gamma(G')$. Therefore $b(G') = 1$.

Definition 1.2.18. [3] **Color-change rule**: If G is a graph with each vertex coloured either white or black, v is a black vertex of G, and exactly one neighbour w of v is white, then change the colour of w to black. Given a colouring of G, the **derived coloring** is the result of applying the color-change rule until no more changes are possible.

Definition 1.2.19. [3] A **zero forcing set** of a graph G is a set $Z \subseteq V(G)$ such that if initially the vertices in Z are coloured black and the remaining vertices are coloured white, the entire graph G may be coloured black by repeatedly applying the colour-change rule. The **zero forcing number** of G, denoted by $Z(G)$, is the minimum cardinality of a zero forcing set.

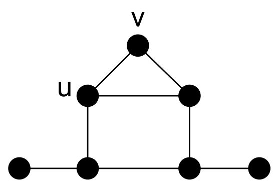


Figure 1.12: The graph H.

In Figure 1.12, $\{u, v\}$ is a minimum zero forcing set of H and therefore $Z(H) = 2$

We will use the notations $\mathbb{N}_0 \colon = \{0, 1, \ldots\}$ and $\mathbb{N}_t := \{t, t+1, \ldots\} \subseteq \mathbb{N}_0$, $t \in \mathbb{N}_0$. For $t \in \mathbb{N}_1$, $[t] := \{1, \ldots, t\} \subseteq \mathbb{N}_1$, $[t]_0 := \{0, \ldots, t-1\} \subseteq \mathbb{N}_0$ and for $t, s \in \mathbb{N}_3$, $[t]_0^{s-2} = \{a_{s-2}a_{s-3} \ldots a_1 \colon a_i \in [t]_0$ for all $i\}$. Note that $|[t]_0| = t = |[t]|$.

Definition 1.2.20. [30] The **Knödel graph** on 2ν vertices, where $\nu \in \mathbb{N}_1$, and of maximum degree $\Delta \in [1 + \lfloor \log_2(\nu) \rfloor]$ is denoted by $W_{\Delta,2\nu}$. The vertices of $W_{\Delta,2\nu}$ are the pairs (i, j) with $i = 1, 2$ and $j \in [\nu]_0$. For every such j, there is an edge between vertex $(1, j)$ and any vertex $(2, j + 2^\ell - 1 \,(\mathrm{mod}\, \nu))$ with $\ell \in [\Delta]_0$.

Remark 1.2.6. An edge of $W_{\Delta,2\nu}$ which connects a vertex $(1, j)$ with the vertex $(2, j + 2^\ell - 1 \,(\mathrm{mod}\, \nu))$ is called an **edge in dimension $\boldsymbol{\ell}$**.

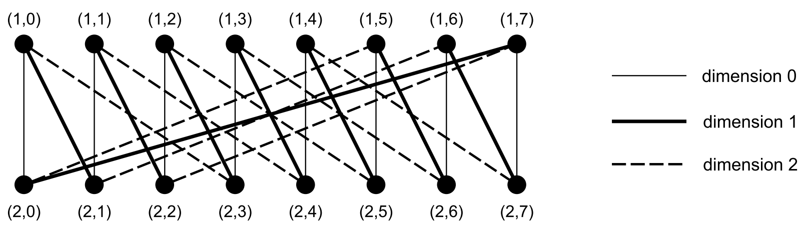


Figure 1.13: The graph $W_{3,16}$.

Definition 1.2.21. [43] The **Hanoi graph**, denoted by H_p^n, for base $p \in \mathbb{N}_3$ and exponent $n \in \mathbb{N}_0$ are defined as follows.
$V(H_p^n) = \{s_n \dots s_1 \colon s_d \in [p]_0 \text{ for } d \in [n]\}$,
$E(H_p^n) = \big\{\{\underline{s}i\overline{s}, \underline{s}j\overline{s}\} \colon i, j \in [p]_0,\ i \neq j,\ \underline{s} \in [p]_0^{n-d}$,
$\overline{s} \in ([p]_0 \backslash \{i, j\})^{d-1}, d \in [n]\big\}$.

Remark 1.2.7. The edge sets of Hanoi graphs can also be expressed in a recursive definition:

$$E(H_p^0) = \emptyset,$$

$$\forall n \in \mathbb{N}_0 \colon E(H_p^{1+n}) = \big\{\{ir, is\} \colon i \in [p]_0,\ \{r, s\} \in E(H_p^n)\big\} \cup \big\{\{ir, jr\} \colon i, j \in [p]_0,\ i \neq j,\ r \in ([p]_0 \backslash \{i, j\})^n\big\}.$$

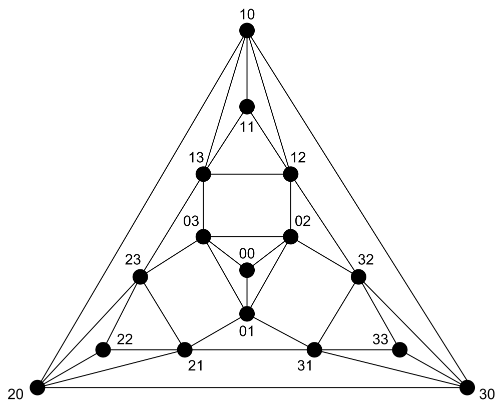


Figure 1.14: The graph H_4^2.

Definition 1.2.22. [45] For $C, L \in \mathbb{N}_1$, an $L-$level **WK-Recursive**

mesh, denoted by $WK_{(C,L)}$, consists of a set of vertices $V(WK_{(C,L)}) = \{(a_La_{L-1}\ldots a_1)\colon a_i \in [C]_0 \text{ for } i \in [L]\}$. The vertex with address $(a_La_{L-1}\ldots a_1)$ is adjacent

1. to all the vertices with addresses $(a_La_{L-1}\ldots a_2a_j)$ such that $a_j \in [C]_0$, $a_j \neq a_1$ and

2. to a vertex $(a_La_{L-1}\ldots a_{j+1}a_{j-1}(a_j)^{j-1})$, if there exists one j such that $2 \leq j \leq L, a_{j-1} = a_{j-2} = \ldots = a_1$ and $a_j \neq a_{j-1}$.

Remark 1.2.8. The notation $(a_j)^{j-1}$ denotes that the term a_j is repeated $j-1$ times. The WK-Recursive mesh is isomorphic to the Sierpiński graph, denoted by S_p^n, defined in [25]. Here the parameters p and n of S_p^n correspond to C and L of $WK_{(C,L)}$, respectively.

Definition 1.2.23. [45] For $C, L \in \mathbb{N}_1$, a **WK-Pyramid network**, denoted by $WKP_{(C,L)}$, consists of a set of vertices $V(WKP_{(C,L)}) = \{(r,(a_ra_{r-1}\ldots a_1))\colon r \in [L], a_i \in [C]_0 \text{ for } i \in [r]\} \cup \{(0,(1))\}$. The vertex $(0,(1))$ is adjacent to every vertex in level 1. A vertex with address $(r,(a_ra_{r-1}\ldots a_1))$ at level $r > 0$ is adjacent

1. to vertices $(r, (a_r a_{r-1} \dots a_2 a_j)) \in V(WKP_{(C,L)})$, for $a_j \in [C]_0$, $a_j \neq a_1$,

2. to a vertex $(r, (a_r a_{r-1} \dots a_{j+1} a_{j-1} (a_j)^{j-1}))$, if there exists one j such that $2 \leq j \leq L, a_{j-1} = a_{j-2} = \dots = a_1$ and $a_j \neq a_{j-1}$,

3. to vertices $(r+1, (a_r a_{r-1} \dots a_2 a_1 a_j))$, for $a_j \in [C]_0$, in level $r+1$ and

4. to a vertex $(r-1, (a_r a_{r-1} \dots a_2))$, in level $r-1$.

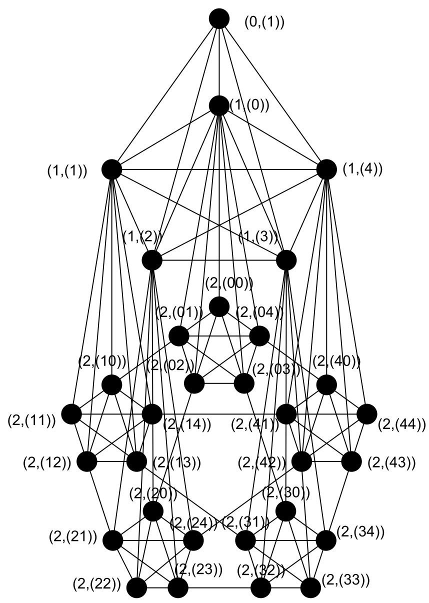


Figure 1.15: The graph $WKP_{(5,2)}$.

1.3 New definitions

Definition 1.3.1. [66] Let $k \geq 0$. The ***k*-power bondage number**, denoted by $b_{\mathrm{P},k}(G)$, of a nonempty graph G is $b_{\mathrm{P},k}(G) = \min\{|B|\colon B \subseteq E(G), \gamma_{\mathrm{P},k}(G-B) > \gamma_{\mathrm{P},k}(G)\}$.

Remark 1.3.1. An edge set B with $\gamma_{\mathrm{P},k}(G-B) > \gamma_{\mathrm{P},k}(G)$ is the k**-power bondage set** of G. Clearly, $b_{\mathrm{P},0}(G) = b(G)$.

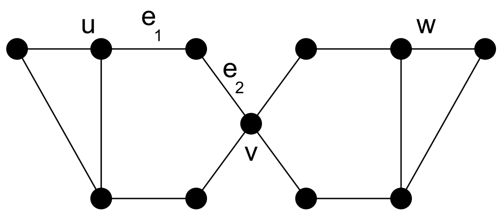


Figure 1.16: The graph H'.

In Figure 1.16, $\{u, v, w\}$ is a $\gamma(H')$- set and for $k \geq 1$, $\{v\}$ is a $\gamma_{\mathrm{P},k}(H')$- set. Therefore $\gamma(H') = 3$ and $\gamma_{\mathrm{P},k}(H') = 1$ for $k \geq 1$. For the set $B = \{e_1, e_2\}$, $\gamma(H' - B) = 4$ and for the set $B' = \{e_1\}$, $\gamma_{\mathrm{P}}(H' - B') = 2$. Hence $b(H') = b_{\mathrm{P},0}(H') = |B| = 2$ and $b_{\mathrm{P},1}(H') = |B'| = 1$. For $k \geq 2$, $\gamma_{\mathrm{P},k}(H' - B) = 2$ and therefore $b_{\mathrm{P},k}(H') = |B| = 2$.

Definition 1.3.2. [27] For $k \geq 1$, a ***k*-generalized spider** is a tree with at most one vertex of degree $k + 2$ or more.

Remark 1.3.2. The *spider* in Definition 1.2.3 is a 1-generalized spider and thus the k-generalized spider is the generalization of spider. Figure 1.17 depicts a k-generalized spider for $k \geq 2$.

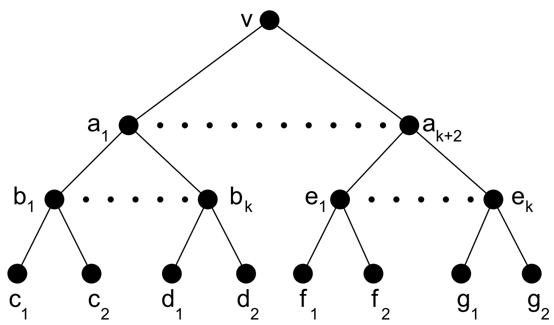


Figure 1.17: A k-generalized spider, T.

Definition 1.3.3. A **tailed star** is a tree obtained from a star by subdividing at most one edge any number of times.

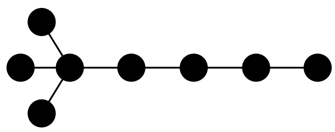


Figure 1.18: A tailed star obtained by subdividing an edge of $K_{1,4}$.

1.4 A survey of previous results

In this section, we shall provide a brief survey of literature on the power domination problem, which we will require in subsequent chapters of the thesis.

The problem of finding a dominating set of minimum cardinality is an important problem that has been extensively studied [14, 20, 35, 40, 41]. The following theorem gives a useful property of dominating sets.

Theorem 1.4.1. *[14] Let G be a graph without isolated vertices. Then G has a $\gamma(G)$-set D such that for every $u \in D$, there exists a vertex $v \in V(G) \setminus D$ that is adjacent to u but to no other vertex of D.*

Our focus is on a variation of domination called the power dominating set problem. The computational complexity of the power domination problem is considered in various papers [1, 2, 34, 39], in which it is proved to be NP-complete on bipartite and chordal graphs as well as for bounded propagation variants. On the other hand, the efficient solutions for the problem are known for trees [39], interval graphs [56] and block graphs [74]. Upper bounds for $\gamma_{\mathrm{P}}(G)$ for an arbitrary graph G are given in [76]. The power domination problem in planar graphs is studied in [75]. It is proved in [75] that if G is an outerplanar graph with diameter 2 or a 2-connected outerplanar graph with diameter 3, then $\gamma_{\mathrm{P}}(G) = 1$ and if G is a planar graph with diameter 2, then $\gamma_{\mathrm{P}}(G) \leq 2$.

The problem of characterizing the power domination number of a graph is non trivial for simple families of graphs. Early studies try to characterize it for products of paths and cycles [7, 26, 28]. The problem of finding the power domination number is also studied in generalized Petersen graphs [73], hypercubes [21, 59] and hexagonal grids [29]. The following results from [7, 26, 28, 39, 62] are of interest to us.

Observation 1.4.2. *[39] For any graph G, $1 \leq \gamma_{\mathrm{P}}(G) \leq \gamma(G)$.*

Theorem 1.4.3. *[39] For any tree T,*
(a) $\gamma_{\mathrm{P}}(T) = 1$ if and only if T is a spider.
(b) $\gamma_{\mathrm{P}}(T) = sp(T)$, where $sp(T)$ is the minimum number of subsets into which $V(T)$ can be partitioned so that each subset induces a spider.

Theorem 1.4.4. *[28] Let $G = P_n \square P_m$. For $m \geq n \geq 1$,*

$$\gamma_{\mathrm{P}}(G) = \begin{cases} \left\lceil \frac{n+1}{4} \right\rceil, & n \equiv 4\,(mod\ 8); \\ \left\lceil \frac{n}{4} \right\rceil, & otherwise. \end{cases}$$

Theorem 1.4.5. *[7] Let $n \geq 2$, $m \geq 3$ and $G = P_n \Box C_m$. Then $\gamma_{\mathrm{P}}(G) \leq min\{\left\lceil \frac{m+1}{4} \right\rceil, \left\lceil \frac{n+1}{2} \right\rceil\}$.*

Theorem 1.4.6. *[7] Let $n, m \geq 3$ and $G = C_n \Box C_m$. For* $n \leq m$, $\gamma_{\mathrm{P}}(G) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil, & n \equiv 2\,(mod\ 4); \\ \left\lceil \frac{n+1}{2} \right\rceil, & otherwise. \end{cases}$

Theorem 1.4.7. *[62] We have,*

$(a)\gamma_{\mathrm{P}}(K_m \Box P_n) = 1, m, n \geq 1.$

$(b)\gamma_{\mathrm{P}}(K_m \Box C_n) = 2, m > 2, n \geq 4.$

$(c)\gamma_{\mathrm{P}}(K_m \Box F_n) = 2, m, n \geq 3.$

$(d)\gamma_{\mathrm{P}}(K_m \Box W_n) = 3, m, n \geq 4.$

$(e)\gamma_{\mathrm{P}}(K_m \times C_n) = \gamma_t(C_n), m \geq 3, n \geq 4.$

Theorem 1.4.8. *[26] For any nontrivial graphs G and H,*

$$\gamma_{\mathrm{P}}(G \circ H) = \begin{cases} \gamma(G), & \gamma_{\mathrm{P}}(H) = 1; \\ \gamma_t(G), & \gamma_{\mathrm{P}}(H) > 1. \end{cases}$$

Linear-time algorithms are known for computing minimum k-power dominating sets in trees [18] and in block graphs [69]. In [18], the authors showed along with some early results about

k-power domination that some bounds, extremal graphs and properties can be expressed for any k, including the case of domination. In [24], a bound from [76] on regular graphs is also generalized to any k. Closed formulae for the k-power domination number of Sierpiński graphs [25] are also known.

Observation 1.4.9. *[18] Let G be a graph and $k \geq 1$. Then*
(a) $1 \leq \gamma_{\mathrm{P},k+1}(G) \leq \gamma_{\mathrm{P},k}(G) \leq \gamma_{\mathrm{P},k-1}(G) \leq \gamma(G)$.
(b) If G is connected and $\Delta(G) \leq k+1$, then $\gamma_{\mathrm{P},k}(G) = 1$.
(c) If G contains a vertex v which is adjacent to at least $k+2$ pendant vertices, then v is in every $\gamma_{\mathrm{P},k}(G)$-set.

Theorem 1.4.10. *[18] For $k \geq 1$, if G is a connected graph with $\Delta(G) \geq k+2$, then there exists a $\gamma_{\mathrm{P},k}(G)$-set containing only vertices of degree at least $k+2$.*

Theorem 1.4.11. *[18] Let G be a connected graph of order n and $k \geq 1$.*
(a) If $n \geq k+2$, then $\gamma_{\mathrm{P},k}(G) \leq \frac{n}{k+2}$.
(b) If G is $(k+2)$-regular and $G \neq K_{k+2,k+2}$, then $\gamma_{\mathrm{P},k}(G) \leq \frac{n}{k+3}$ and this bound is tight.

Theorem 1.4.12. *[25] Let $k \geq 1$. The k-power domination number of Sierpiński graph, S_p^n, $p, n \geq 1$, is*

$$\gamma_{\mathrm{P},k}(S_p^n) = \begin{cases} 1, & p = 1 \ \mathit{or}\ p = 2 \ \mathit{or}\ n = 1 \ \mathit{or}\ p \leq k+1; \\ p-k, & n = 2, p \geq k+2; \\ (p-k-1)p^{n-2}, & p, n \geq 3, p \geq k+2. \end{cases}$$

Another recent but natural question about power domination is related to the propagation radius. In a graph, a vertex that is arbitrarily far apart from any vertex in the set S may eventually get monitored by S. In practice, besides the minimum cardinality of a k-PDS, the information in how many propagation steps the graph is monitored from a given k-PDS could also be important. Therefore, it is natural to consider power domination with bounded time constraints, as was first studied in [1] and then in [57]. Inspired by this study, the k-propagation radius of a graph G, $\mathrm{rad}_{\mathrm{P},k}(G)$, was introduced in [25] as a way to measure the efficiency of a minimum k-PDS. It gives the minimum number of propagation steps required to monitor the entire graph over all minimum k-PDS.

From the electric power network modelling perspective, if there are $|S|$ Phasor Measurement Units capable of monitoring the entire network, then the propagation radius of S may represent the number of times that Kirchhoff's law has to be applied to achieve the monitoring of whole network. However, the repeated application of Kirchhoff's laws would induce an unreasonable cumulated margin of error and therefore the study of propagation radius is important.

The following is an interesting observation from [25].

Proposition 1.4.13. *[25] Let $k \geq 1$ and let G be a graph. Then $\gamma_{\mathrm{P},k}(G) = \gamma(G)$ if and only if $\mathrm{rad}_{\mathrm{P},k}(G) = 1$.*

The k-propagation radius of Sierpiński graphs, S_p^n, is obtained in [25] and the result is as follows.

Theorem 1.4.14. *[25] For $k, p \geq 1$,*

$$\mathrm{rad}_{\mathrm{P},k}(S_p^n) = \begin{cases} 1, & n = 1 \text{ or } p = 1; \\ 2, & n = 2 \text{ and } (k = 1 \text{ or } p = 2); \\ 3, & n = 2, k \geq 2, p \geq 3. \end{cases}$$

Theorem 1.4.15. *[25] For $k, p \geq 1$ and $n \geq 3$,*

$$\mathrm{rad}_{\mathrm{P},k}(S_p^n) = \begin{cases} 3, & p \geq 2k+3; \\ 4 \text{ or } 5, & 2k+2 \geq p \geq k+1+\sqrt{k+1}; \\ 5, & k+1+\sqrt{k+1} > p \geq k+2; \\ r(S_p^n), & p \leq k+1. \end{cases}$$

Some of the recent works related to the power domination problem are in [9, 15, 32].

An important consideration in the topological design of the network is fault tolerance, that is the ability of the network to provide service even when it contains a faulty component or components. The behaviour of the network in the presence of a fault can be analyzed by determining the effect that removing an edge (link failure) or a vertex (processor failure) from its underlying graph G has on the fault tolerance criterion.

In 1990, Fink et al. [31] introduced the concept of bondage number as a measure of the vulnerability of the interconnection network under link failure. In [31], the exact values of $b(G)$ for some classes of graphs and sharp upper bounds for $b(G)$ in terms of its order and degree were obtained. In [38], Hartnell and Rall gave other bounds for $b(G)$ and disproved a conjecture proposed in [31]. The bondage number was then studied in trees [68] and planar graphs [50]. The bondage problem is proved to be NP-complete in [48]. Variations of this notion such as total bondage number [54], paired bondage number [61] and distance k-bondage number [37] are also available in the literature. For a survey about the bondage number, see [72].

Knödel graphs, $W_{\Delta,2\nu}$, have been introduced by W. Knödel in [53] as the network topology underlying an optimal-time algorithm for gossiping among n nodes. They have been widely studied as interconnection networks mainly because of their favourable properties in terms of broadcasting and gossiping [12]. $W_{r,2^r}$ is one of the three nonisomorphic infinite graph families known to be minimum broadcast and gossip graphs. The other two families are the hypercube of dimension r, H_r [55] and the recursive circulant graph $G(2^r, 4)$ [60]. Vertex transitivity, high vertex and edge connectivity, dimensionality and embedding properties make the Knödel graph a suitable candidate for a network topology and an appropriate architecture for parallel computing. For a survey about the Knödel graphs, see [30].

The Tower of Hanoi (TH) problem, invented by the French number theorist É. Lucas in 1883, has presented a challenge in mathematics as well as in computer science and psychology. The classical problem consists of three pegs and is thoroughly studied in [42]. On the other hand, as soon as there are at least four pegs, the problem turned into a notorious open question. The general

TH problem has $p \in \mathbb{N}_3$ pegs and $n \in \mathbb{N}_0$ discs of mutually different size. A **legal move** is a transfer of the topmost disc from one peg to another peg, no disc being placed onto a smaller one. Initially, all discs lie on one peg in small-on-large ordering, that is, in a **perfect state**. The objective is to transfer all the discs from one perfect state to another in the minimum number of legal moves. A state (= distribution of discs on pegs) is called **regular** if on every peg the discs lie in the small-on-large ordering. The Hanoi graphs, H_p^n, form a natural mathematical model for the TH problem. The graph is constructed with all regular states as vertices, and two states are adjacent whenever one is obtained from the other by a legal move. Many properties of Hanoi graphs have been studied in [43].

The WK-Pyramid network, $WKP_{(C,L)}$, an interconnection network based on the WK-recursive mesh [23], was introduced in [45] for massively parallel computers. It eliminates some drawbacks of the conventional pyramid network, stemming from the fact that the connections within the layers of this network form a WK-recursive mesh. It is of much less network cost than the hypercube, k-ary n-cube and WK-Recursive networks. It also has small average distance and diameter, large connectivity and high degree of scalability and expandability. Because of the desirable properties of this network, it is suitable for medium or large sized networks and also a better alternative for mesh and traditional pyramid interconnection topologies.

WK-recursive networks [23] are very similar to the family of Sierpiński graphs [25, 51] and they can be obtained from Sierpiński graphs by adding a link to each of its extreme vertices. Also, one can observe that the subgraph induced by the vertices of each layer of a WK-Pyramid network form a Sierpiński graph. The k-power domination number and k-propagation radius of Sierpiński graphs are studied in [25]. This motivates the study of generalized power domination and propagation radius in $WKP_{(C,L)}$. Routing in WK-Pyramid network was studied in [45] and the Hamiltonian properties of the network were studied in [44, 45, 46].

1.5 Summary of the thesis

This thesis entitled **'Studies on the Power Domination Problem in Graphs'** deals with the power domination problem in graphs, motivated by the research contributions mentioned earlier. It mainly discusses the heredity property of power domination, the problem of finding the power domination number of certain classes of graphs such as product graphs, Knödel graphs, Hanoi graphs and WK-Pyramid networks etc.

This thesis is divided into five chapters. The first chapter is an introduction and contains the literature on the power domination problem and various graph classes studied in this thesis. It also includes the basic definitions and terminology.

The second chapter deals with the heredity property of the generalized power domination. We study the behaviour of k-power domination number of a graph by small changes on the graph such as removing a vertex or an edge, or contracting an edge. The behaviour of the k-propagation radius of graphs by similar modifications is also studied. We prove that though the behaviour of the k-power domination is similar to the domination in the case of the removal of a vertex, the removal of an edge can decrease the k-power domination number and the contraction of an edge can increase the k-power domination number, both phenomena that are impossible in usual domination. A technical lemma is given that proves useful while considering the different cases of vertex removal, edge removal and edge contraction. The main results in this chapter are listed below.

- ► For any graph G, $v \in V(G)$ and $k \geq 1$, $\gamma_{\mathrm{P},k}(G - v) \geq \gamma_{\mathrm{P},k}(G) - 1$ and there is no upper bound for $\gamma_{\mathrm{P},k}(G - v)$ in terms of $\gamma_{\mathrm{P},k}(G)$.
- ► Let G be a graph and e be an edge in G. Then, for $k \geq 1$,

  $\gamma_{\mathrm{P},k}(G) - 1 \leq \gamma_{\mathrm{P},k}(G - e) \leq \gamma_{\mathrm{P},k}(G) + 1$ and

  $\gamma_{\mathrm{P},k}(G) - 1 \leq \gamma_{\mathrm{P},k}(G/e) \leq \gamma_{\mathrm{P},k}(G) + 1$.
- ► Several examples of graphs for which the bounds are sharp.
- ► Characterization of graphs for which the removal of any edge increases the k-power domination number.

In the third chapter, we extend the study on the k-power domination number of a graph when several edges are deleted. We introduce the notion of bondage number in power domination. The following are some of the results obtained.

- ⧫ For any connected nonempty graph G with $\Delta(G) \leq k+1$, where $k \geq 1$, $b_{\mathrm{P},k}(G) = \lambda(G)$.

- ♦ For $k \geq 1$ and any connected nonempty graph G, $b_{\mathrm{P},k}(G) \leq \Delta(G) + \delta(G) - 1$.

- ♦ For $k \geq 1$ and $n \geq 2$, $b_{\mathrm{P},k}(K_n) = n - 1$.

- ♦ For $k, n \geq 1$, $b_{\mathrm{P},k}(K_{n,n}) = \begin{cases} n, & n \leq k+1 \text{ or } n \geq k+3; \\ 2n-1, & n = k+2. \end{cases}$

- ♦ For any nonempty tree T and $k \geq 1$, $b_{\mathrm{P},k}(T) \leq 2$.

- ♦ If the nonempty tree T has a vertex v that belongs to every minimum k-PDS of T, then $b_{\mathrm{P},k}(T) = 1$.

In the fourth chapter, the power domination problem in Cartesian product, direct product and lexicographic product is studied. The main results are,

- ✠ For any two nontrivial graphs G and H, $\gamma_{\mathrm{P}}(G \,\square\, H) \leq \min\{\gamma_{\mathrm{P}}(G)|V(H)|, \gamma_{\mathrm{P}}(H)|V(G)|\}$.

- ✠ For any nontrivial graph G, sharp upper bounds for $\gamma_{\mathrm{P}}(G \,\square\, P_n)$ and $\gamma_{\mathrm{P}}(G \,\square\, H)$, where H is a nontrivial graph with a universal vertex.

✠ Let G and H be two graphs of order at least four. Then $\gamma_{\mathrm{P}}(G \,\Box\, H) = 1$ if and only if one of the graphs has a universal vertex and the other is isomorphic to a path.

✠ Sharp upper bounds for the power domination number of direct products when one of the factor graphs has a universal vertex.

✠ For each positive integer ℓ, let $\mathcal{A}_\ell$ be the family of all nontrivial graphs F such that $\gamma_{\mathrm{P},\ell}(F) = 1$. And, let $\mathcal{B}_\ell$ be the family of all disconnected graphs F such that $F = F_1 \cup \ldots \cup F_r, 2 \leq r \leq \ell+1$, where each F_i is a component of F, having the property that F_1 is a connected nontrivial graph with $\gamma_{\mathrm{P},\ell}(F_1) = 1$ and sum of the order of the remaining components of F is at most ℓ, i.e. $1 \leq |V(F_2)| + \ldots + |V(F_r)| \leq \ell$. Let $\mathcal{F}_\ell = \mathcal{A}_\ell \cup \mathcal{B}_\ell$. Let G be a nontrivial graph without isolated vertices. For any nontrivial graph H and $1 \leq k \leq |V(H)| - 1$, $\gamma_{\mathrm{P},k}(G \circ H) = \begin{cases} \gamma(G), & H \in \mathcal{F}_k; \\ \gamma_t(G), & H \notin \mathcal{F}_k. \end{cases}$

✠ Let G be a nontrivial graph without isolated vertices and H be a connected nontrivial graph. If $k \geq |V(H)|$, then $\gamma_{\mathrm{P},k}(G \circ H) = \gamma_{\mathrm{P},\lfloor \frac{\mathrm{k}}{|\mathrm{V(H)}|} \rfloor}(G)$.

The power domination problem in more classes of graphs such as Knödel graphs, Hanoi graphs and WK-Pyramid networks are discussed in the fifth chapter and we have the following results.

★ Knödel graphs

∗ For $\nu \in \mathbb{N}_4$, $\gamma_{\mathrm{P}}(W_{3,2\nu}) = 2$.

∗ For $r \in \mathbb{N}_4, \gamma_{\mathrm{P}}(W_{r,2^r}) \leq 2^{r-3}$ and the bound is sharp.

★ Hanoi graphs

∗ For $k \in \mathbb{N}_1$ and $p \in \mathbb{N}_4$, $\gamma_{\mathrm{P},k}(H_p^2) = \begin{cases} 1, & k \in \mathbb{N}_{p-2}; \\ p-k-1, & k \in [p-3]. \end{cases}$

∗ For $k \in \mathbb{N}_1$ and $p \in \mathbb{N}_4$, $\mathrm{rad}_{\mathrm{P},k}(H_p^2) = 3$.

★ WK-Pyramid networks

∗ Let $k, C, L \in \mathbb{N}_1$. Then

$$\gamma_{\mathrm{P},k}(WKP_{(C,L)}) = \begin{cases} 1, & C = 1 \text{ or } L = 1 \text{ or } k \in \mathbb{N}_C; \\ C-k, & L = 2, C \in \mathbb{N}_2, k \in [C-1]; \\ (C-k-1)C^{L-2}, & C, L \in \mathbb{N}_3, k \in [C-2]. \end{cases}$$

∗ For $k \in \mathbb{N}_1$ and $C \in \mathbb{N}_2$, $\mathrm{rad}_{\mathrm{P},k}(WKP_{(C,2)}) = \begin{cases} 2, & k \in \mathbb{N}_C; \\ 3, & k \in [C-1]. \end{cases}$

All the graphs considered in this thesis are finite, undirected and simple. Unless otherwise stated, the symbol k denotes a positive integer, in the subsequent chapters.

Some results of this thesis are included in the papers [27, 63, 64, 65, 66]. The thesis concludes with some suggestions for further study and a bibliography.

Chapter 2

Heredity for generalized power domination

In general, it remains difficult to establish lower bounds for the power domination number of a graph. One of the reasons, why it is so, is that power domination does not behave well when we consider subgraphs. In this chapter, we explore in detail the behaviour of the k-power domination number of a graph by small changes on the graph, namely edge or vertex deletion,

Some results of this chapter are included in the following paper.
Paul Dorbec, Seethu Varghese, A. Vijayakumar, Heredity for generalized power domination, Discrete Math. Theor. Comput. Sci. 18 (3) (2016).

or edge contraction. We also consider the behaviour of the k-propagation radius of graphs by similar modifications.

2.1 Variations on the k-power domination number

Before considering the different cases of vertex removal, edge removal and edge contraction, we propose the following technical lemma which should prove useful. It states that if two graphs differ only on parts that are already monitored, then propagation in the not yet monitored parts behave the same. For a graph $G = (V(G), E(G))$ and two subsets X and Y of $V(G)$, we denote by $E_G(X, Y)$ the set of edges $uv \in E(G)$ such that $u \in X$ and $v \in Y$. Note that if $X \subseteq Y$, $E_G(X, Y)$ in particular contains all edges of the induced subgraph $< X >$ of G on X.

Lemma 2.1.1. *Let $G = (V_G, E_G)$ and $H = (V_H, E_H)$ be two graphs, S a subset of vertices of G and i a nonnegative integer. Define $X = V_G \setminus \mathcal{P}^i_{G,k}(S)$ and the subgraph G' with vertex set $N_G[X]$ and edge set $E_G(X, N_G[X])$.*

Suppose there exists a subset $Y \subseteq V_H$ such that the subgraph $H' = (N_H[Y], E_H(Y, N_H[Y]))$ is isomorphic to G' with a mapping $\varphi : N_G[X] \to N_H[Y]$ that maps X precisely to Y. Then, if for some k-power dominating set $T \subseteq V_H$ and some nonnegative integer j, $Y \subseteq V_H \setminus \mathcal{P}^{j}_{H,k}(T)$, then S is a k-PDS of G and $\mathrm{rad}_{\mathrm{P},k}(G, S) \leq i - j + \mathrm{rad}_{\mathrm{P},k}(H, T)$.

Proof. For $\ell \geq 0$, denote the sets $X \cap \mathcal{P}^{i+\ell}_{G,k}(S)$ and $Y \cap \mathcal{P}^{j+\ell}_{H,k}(T)$ by X^ℓ and Y^ℓ, respectively. We prove by induction that for all ℓ, $Y^\ell \subseteq \varphi(X^\ell)$.

By hypothesis, $X^0 = \emptyset$ and so $\varphi(X^0) = \emptyset = Y^0$, so it holds for $\ell = 0$. Now assume that the property is true for some $\ell \geq 0$. Suppose that some vertex $v = \varphi(u) \in N_H[Y]$ satisfies the conditions for propagation in H at step $j + \ell$, i.e. $v \in \mathcal{P}^{j+\ell}_{H,k}(T)$ and $|N_H[v] \setminus \mathcal{P}^{j+\ell}_{H,k}(T)| \leq k$. We show that u also satisfies the conditions for propagation in G. First, remark that u is monitored at step $i+\ell$: indeed, if $u \notin X$, then by definition of X, $u \in \mathcal{P}^{i+\ell}_{G,k}(S)$, otherwise if $u \in X$, then $v \in Y \cap \mathcal{P}^{j+\ell}_{H,k}(T) = Y^\ell$, and thus by the induction hypothesis, $u \in X^\ell \subseteq \mathcal{P}^{i+\ell}_{G,k}(S)$. Now consider any neighbour u' of u not yet dominated. Then $u' \in X \setminus X^\ell$ and $\varphi(u') \in Y \setminus Y^\ell$. Moreover, by the isomorphism between

G' and H', $\varphi(u')$ is also adjacent to v, and was among the at most k unmonitored neighbours of v in H. Therefore, u has at most k unmonitored neighbours in G, and also propagates in G. Applying this statement to all vertices in G', we infer that $Y^{\ell+1} \subseteq \varphi(X^{\ell+1})$. By induction, this is also true for $\ell = \mathrm{rad}_{\mathrm{P},k}(H,T) - j - 1$, and we deduce that

$$\begin{aligned} X = \varphi^{-1}(Y) \subseteq & \varphi^{-1}\Big(Y^\ell = \big(\mathcal{P}_{H,k}^{\mathrm{rad}_{\mathrm{P},k}(H,T)-1}(T) \cap Y\big)\Big) \\ \subseteq & \Big(X^\ell = \big(\mathcal{P}_{G,k}^{\mathrm{rad}_{\mathrm{P},k}(H,T)-j+i-1}(S) \cap X\big)\Big), \end{aligned}$$

and thus that S is a k-power dominating set of G and $\mathrm{rad}_{\mathrm{P},k}(G,S) \leq i - j + \mathrm{rad}_{\mathrm{P},k}(H,T)$. $\square$

We now use this lemma to state how the k-power domination number of a graph may change with atomic variations of the graph, such as the vertex or edge removal and edge contraction.

2.1.1 Vertex removal

Similar to what happens for domination [40], we have the following:

Theorem 2.1.2. *Let G be a graph and v be a vertex in G. There is no upper bound for $\gamma_{\mathrm{P},k}(G-v)$ in terms of $\gamma_{\mathrm{P},k}(G)$. On the other hand, we have $\gamma_{\mathrm{P},k}(G-v) \geq \gamma_{\mathrm{P},k}(G) - 1$. Moreover, if $\gamma_{\mathrm{P},k}(G-v) = \gamma_{\mathrm{P},k}(G) - 1$, then $\mathrm{rad}_{\mathrm{P},k}(G) \leq \mathrm{rad}_{\mathrm{P},k}(G-v)$.*

Proof. We first prove the lower bound, using Lemma 2.1.1. We define $H = G - v$ with the obvious mapping φ from $V(G) \setminus v$ to $V(H)$. Let T be a k-PDS of $H = G - v$, that induces the minimum propagation radius. Then for the set $S = T \cup \{v\}$, the conditions of Lemma 2.1.1 hold already from $i = 0$ and $j = 0$ and the bound follows. Moreover, we also get that $\mathrm{rad}_{\mathrm{P},k}(G, S) \leq j - i + \mathrm{rad}_{\mathrm{P},k}(H, T) = \mathrm{rad}_{\mathrm{P},k}(G-v)$. For proving there is no upper bound for $\gamma_{\mathrm{P},k}(G-v)$ in terms of $\gamma_{\mathrm{P},k}(G)$, we can consider the star with n pendant vertices $K_{1,n}$, for which the removal of the central vertex increases the k-power domination number from 1 to n. $\square$

We now describe examples that tighten the lower bound of the above theorem or illustrate better the absence of upper bound (in particular for graphs that remain connected). A first example for which the tightness of the lower bound can be observed is the 4×4 grid $P_4 \,\square\, P_4$, for which we get $\gamma_{\mathrm{P}}(P_4 \,\square\, P_4) = 2$ (see [28]) and $\gamma_{\mathrm{P}}((P_4 \,\square\, P_4) - v) = 1$ for any vertex v. Simple examples for larger k are the graphs $K_{k+2,k+2}$, for which the removal of any vertex drops the k-power domination number from 2 to 1 (those were the only exceptions in [24]), as well as the complete bipartite graph $K_{k+3,k+3}$ minus a perfect matching.

We now describe infinite families of graphs to illustrate these bounds. The family of graphs $D_{k,n}$ was defined in [18]. It is made of n copies of $k+3$-cliques minus an edge, organized into a cycle, and where the end vertices of the missing edges are linked to the corresponding vertices in the adjacent cliques in the cycle (see Figure 2.1). Note that $\gamma_{\mathrm{P},k}(D_{k,n}) = n$, as each copy of $K_{k+3} - e$ must contain a vertex of a k-PDS. Its k-propagation radius is 1 since $D_{k,n}$ has a dominating set of cardinality n. The removal of an end vertex of the edges linking two cliques (e.g. u in Figure 2.1) does not change its k-power domination number, but the removal of any other vertex (e.g. v in Figure 2.1) decreases it by one, and increases the k-propagation radius from 1 to 2. So this forms an infinite family tightening the lower bound for any value of k and $\gamma_{\mathrm{P},k}(G)$.

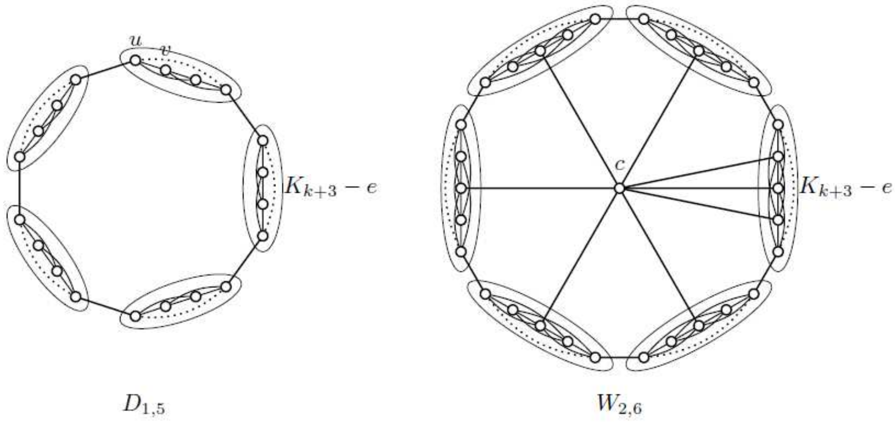


Figure 2.1: The graphs $D_{k,n}$ and $W_{k,n}$ obtained by the addition of vertex c.

Now, an infinite family of graphs proving the absence of an upper bound is a generalization $W_{k,n}$ of the wheel (depicted in Figure 2.1). It is made of $D_{k,n}$ together with a vertex c adjacent to three vertices of degree $k+2$ in one particular clique and to one vertex of degree $k+2$ in all the other cliques. Observe that for $n \geq k+2$, $\{c\}$ is the only k-PDS of $W_{k,n}$ of cardinality 1, and thus we get $\mathrm{rad}_{\mathrm{P},k}(W_{k,n}) = \mathrm{rad}_{\mathrm{P},k}(W_{k,n}, \{c\}) = 2+3\lfloor \frac{n-1}{2} \rfloor + 2(n-1 \pmod 2))$. The removal of c induces the graph $D_{k,n}$, increasing the k-power domination number from 1 to n, and dropping the k-propagation radius from roughly $\frac{3n}{2}$ to 1.

More constructions could be proposed to show that the propagation radius of a graph can evolve quite freely when a vertex is removed, and there is little hope for other bounds on this parameter when a vertex is removed. The most unlikely example is that the removal of a vertex increase both the k-power domination number and the k-propagation radius by unbounded value. This is possible with the following variation on $W_{k,pn}$.

Consider pn subgraphs $(H_i)_{0\leq i<pn}$, all isomorphic to a clique minus an edge, on $k+3$ vertices when $i \equiv 0 \pmod{p}$ and on $k+1$ vertices otherwise. We again connect the end vertices of the missing edges in the clique into a cycle joining H_i to $H_{i+1 \pmod{pn}}$, and add a vertex c adjacent to three vertices of degree $k+2$ in all copies H_i when $i \equiv 0 \pmod{p}$, and to one vertex of degree k in all the other copies. Let G be the graph thus obtained.

We get that $\{c\}$ is a k-PDS of G inducing a k-propagation radius of 2. On the other hand, $\gamma_{\mathrm{P},k}(G-c)=n$ (one vertex is needed in each $H_i, i \equiv 0 \pmod{p}$). The k-propagation radius of G is $1+3\lfloor\frac{p-1}{2}\rfloor+2(p-1 \pmod{2})$.

2.1.2 Edge removal

In a graph G, removing an edge e can never decrease the domination number. More generally, we have that for any graph G, $\gamma(G) \leq \gamma(G-e) \leq \gamma(G)+1$. On the contrary, the removal of an edge can decrease the k-power domination number as stated in the following result. Indeed, it may happen that the removal of one edge allows the propagation through another edge incident to a common vertex, and thus decreases the k-power domination number.

Theorem 2.1.3. *Let G be a graph and e be an edge in G. Then*

$$\gamma_{\mathrm{P},k}(G) - 1 \leq \gamma_{\mathrm{P},k}(G-e) \leq \gamma_{\mathrm{P},k}(G) + 1\,.$$

Moreover,

$$\begin{cases} \textit{if } \gamma_{\mathrm{P},k}(G) - 1 = \gamma_{\mathrm{P},k}(G-e), \textit{ then } \mathrm{rad}_{\mathrm{P},k}(G) \leq \mathrm{rad}_{\mathrm{P},k}(G-e). \\ \textit{if } \gamma_{\mathrm{P},k}(G-e) = \gamma_{\mathrm{P},k}(G) + 1, \textit{ then } \mathrm{rad}_{\mathrm{P},k}(G-e) \leq \mathrm{rad}_{\mathrm{P},k}(G). \end{cases}$$

Proof. We first prove that $\gamma_{\mathrm{P},k}(G-e) \leq \gamma_{\mathrm{P},k}(G)+1$. Let T be a $\gamma_{\mathrm{P},k}(G)$-set. If T is also a k-PDS of $G-e$, then we are done, so assume T is not. Let j_0 be the smallest integer j such that $\mathcal{P}^{j}_{G,k}(T) \supsetneq \mathcal{P}^{j}_{G-e,k}(T)$, and let v be a vertex in $\mathcal{P}^{j_0}_{G,k}(T) \setminus \mathcal{P}^{j_0}_{G-e,k}(T)$. Since $v \in \mathcal{P}^{j_0}_{G,k}(T)$, there exists some neighbour u of v in $\mathcal{P}^{j_0-1}_{G,k}(T)$ such that $|N_G[u] \setminus \mathcal{P}^{j_0-1}_{G,k}(T)| \leq k$. Since $N_{G-e}[u] \subseteq N_G[u]$, $N_{G-e}[u]$ is also included in $\mathcal{P}^{j_0}_{G-e,k}(T)$, and v cannot be a neighbour of u any more, so $e = uv$. Thus we choose $S = T \cup \{v\}$ and using Lemma 2.1.1 (with the obvious mapping from $G - e$ to G, and $i = j = j_0$), we get that S is a k-PDS of $G - e$ of cardinality $\gamma_{\mathrm{P},k}(G) + 1$. We also get that if $\gamma_{\mathrm{P},k}(G - e) = \gamma_{\mathrm{P},k}(G) + 1$, then $\mathrm{rad}_{\mathrm{P},k}(G - e) \leq \mathrm{rad}_{\mathrm{P},k}(G)$.

We now prove that $\gamma_{\mathrm{P},k}(G) - 1 \leq \gamma_{\mathrm{P},k}(G - e)$. Let T be a minimum k-PDS of $H = G - e$ and u be an end vertex of e. We apply Lemma 2.1.1, for $S = T \cup \{u\}$ and $i = j = 0$. We get that S is a k-PDS of G and $\mathrm{rad}_{\mathrm{P},k}(G, S) = \mathrm{rad}_{\mathrm{P},k}(G - e, T)$. We infer that if S is minimal (that is $\gamma_{\mathrm{P},k}(G) = \gamma_{\mathrm{P},k}(G - e) + 1$), then $\mathrm{rad}_{\mathrm{P},k}(G) \leq \mathrm{rad}_{\mathrm{P},k}(G - e)$. $\square$

As a first illustration of these possibilities, in the graph G drawn in Figure 2.2, the removal of the edge e_1 decreases the k-power domination number, the removal of the edge e_3 increases it, and the removal of the edge e_2 does not have any consequence.

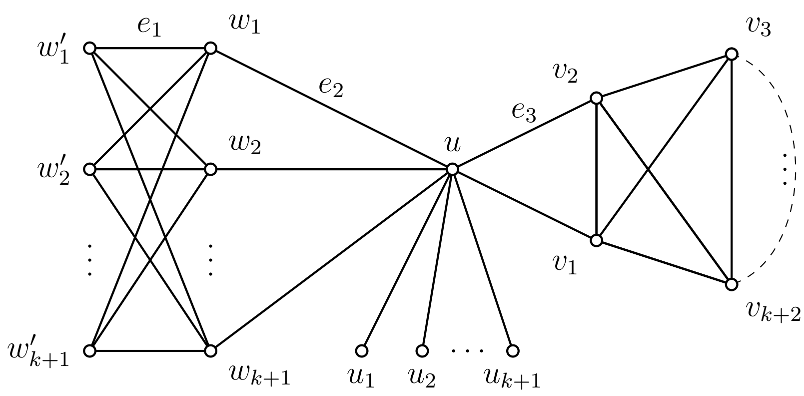


Figure 2.2: A graph G where $\gamma_{\mathrm{P},k}(G) = 2 = \gamma_{\mathrm{P},k}(G - e_2), \gamma_{\mathrm{P},k}(G - e_1) = 1, \gamma_{\mathrm{P},k}(G - e_3) = 3$.

We now propose a graph family where the removal of an edge decreases the k-power domination number but increases its k-propagation radius arbitrarily. The graph $G_{k,r,a}$ represented in Figure 2.3 satisfies $\gamma_{\mathrm{P},k}(G) = 2$ and $\mathrm{rad}_{\mathrm{P},k}(G) = a + 2$ (which is reached with the initial set $\{u, v\}$). If the edge e is removed, we get a new graph whose k-power domination number is 1 and which has k-propagation radius $(r + 3)(a + 1) + 2$. So no upper bound can be found for $\mathrm{rad}_{\mathrm{P},k}(G - e)$ (in terms of $\mathrm{rad}_{\mathrm{P},k}(G)$) when the removal of an edge decreases the k-power domination number.

Similar graphs where the edge removal increases the k-power domination number can also be found. For example, in the graph $G_{k,r,a}$ (Figure 2.3), if we remove the topmost path of length $a+2$ from w to v, except for the vertex adjacent to v, we get another graph G' such that $\{u\}$ is the only $\gamma_{\mathrm{P},k}(G')$-set of cardinality 1, and with $\mathrm{rad}_{\mathrm{P},k}(G') = (r+2)(a+1)+3$. Removing the same edge e, now $\{u, v\}$ is a $\gamma_{\mathrm{P},k}(G'-e)$-set and $\mathrm{rad}_{\mathrm{P},k}(G'-e) = a+2$. This illustrates the fact that no lower bound can be found for $\mathrm{rad}_{\mathrm{P},k}(G-e)$ (in terms of $\mathrm{rad}_{\mathrm{P},k}(G)$) when the removal of an edge increases the k-power domination number.

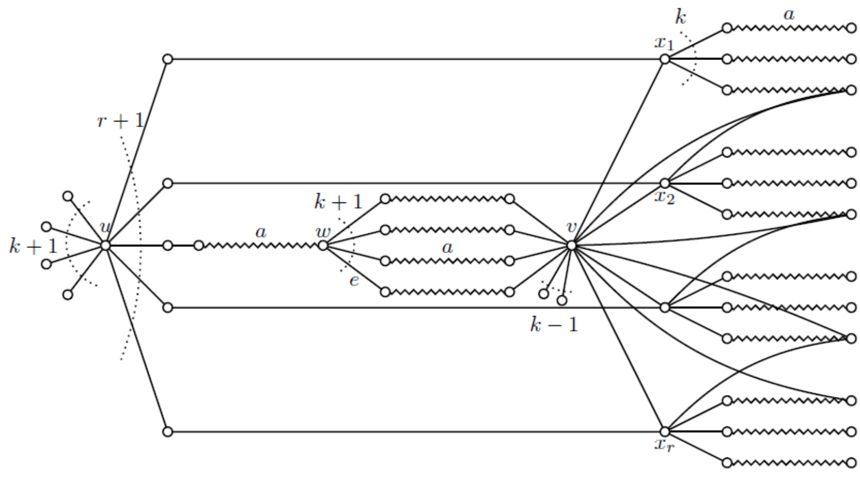


Figure 2.3: The graph $G_{k,r,a}$ for $k = 3$ and $r = 4$ (zigzag edges represent paths of length a).

We now characterize the graphs for which the removal of any edge increases the k-power domination number. Recall that, a ***k*-generalized spider** is a tree with at most one vertex of degree $k+2$ or more (Figure 1.17).

Theorem 2.1.4. *Let G be a graph. For each edge e in G, $\gamma_{\mathrm{P},k}(G-e) > \gamma_{\mathrm{P},k}(G)$ if and only if G is a disjoint union of k-generalized spiders.*

Proof. First observe that if G is a disjoint union of k-generalized spiders, then its k-power domination number is exactly its number of components, and clearly $\gamma_{\mathrm{P},k}(G-e) > \gamma_{\mathrm{P},k}(G)$ for any edge e in G.

Let G be a graph and let S be a $\gamma_{\mathrm{P},k}(G)$-set. We label the vertices of G with integers from 1 to n and consider the subsequent natural ordering on the vertices. For $i \geq 0$, we define $E'_i \subseteq E(G)$ as follows:

$$\begin{cases} E'_0 & = \{uv \in E(G) \mid v \in N(S) \setminus S, u = \min\{x \in N(v) \cap S\}\} \\ E'_{i+1} & = \{uv \in E(G) \mid v \in \mathcal{P}_k^{i+1}(S) \setminus \mathcal{P}_k^{i}(S), \\ & \qquad u = \min\{x \in \mathcal{P}_k^{i}(S) \cap N(v), |N[x] \setminus \mathcal{P}_k^{i}(S)| \leq k\}\} \end{cases}$$

where the minima are taken according to the ordering of the vertices. Let E' be the union of all E'_i for $i \geq 0$. If we consider the edges of E' as defined above oriented from u to v, then the in-degree of each vertex not in S is 1, of vertices in S is 0. Also the graph is acyclic, and each vertex not in S has out-degree at most k. Thus the graph induced by E' is a forest of k-generalized spiders. Observe that each k-generalized spider contains exactly one vertex of S, which forms its k-PDS . Hence S is a k-PDS of this forest of k-generalized spiders. We now assume that for any edge $e \in E(G)$, $\gamma_{\mathrm{P},k}(G-e) > \gamma_{\mathrm{P},k}(G)$, and we then prove that $E' = E(G)$.

By way of contradiction, suppose there exists an edge e in $E(G)$ and not in E'. We prove that S is a k-PDS of $G-e$. For that, we prove by induction that for all $i, \mathcal{P}^i_{G,k}(S) \subseteq \mathcal{P}^i_{G-e,k}(S)$. First observe that $\mathcal{P}^0_{G-e,k}(S) = \mathcal{P}^0_{G,k}(S)$. Indeed, suppose there exists a vertex x in $\mathcal{P}^0_{G,k}(S)$ but not in $\mathcal{P}^0_{G-e,k}(S)$, then e has to be of the form xv with $v \in S$. But since $e \notin E'_0$, there exists another vertex $u < v$ in S such that $ux \in E'_0$, and $x \in \mathcal{P}^0_{G-e,k}(S)$.

Assume now $\mathcal{P}^i_{G,k}(S) \subseteq \mathcal{P}^i_{G-e,k}(S)$ for some $i \geq 0$, and let us prove that $\mathcal{P}^{i+1}_{G,k}(S) \subseteq \mathcal{P}^{i+1}_{G-e,k}(S)$. Let x be a vertex in $\mathcal{P}^{i+1}_{G,k}(S)$. If $x \in \mathcal{P}^i_{G,k}(S)$, then by the induction hypothesis, $x \in \mathcal{P}^{i+1}_{G-e,k}(S)$. If $x \notin \mathcal{P}^i_{G,k}(S)$, then there exists a vertex $v \in \mathcal{P}^i_{G,k}(S), x \in N_G[v]$

such that $|N_G[v] \setminus \mathcal{P}^i_{G,k}(S)| \leq k$. Suppose $e \neq xv$. Then, since $N_{G-e}[v] \subseteq N_G[v]$ and by the induction hypothesis, $v \in \mathcal{P}^i_{G-e,k}(S), x \in N_{G-e}[v]$ and $|N_{G-e}[v] \setminus \mathcal{P}^i_{G-e,k}(S)| \leq k$, which implies $x \in \mathcal{P}^{i+1}_{G-e,k}(S)$. If $e = xv$ then by the choice of $E^{'}_{i+1}$, there exists a vertex $w \in \mathcal{P}^i_{G,k}(S), w < v, wx \in E^{'}_{i+1}$ such that $|N_G[w] \setminus \mathcal{P}^i_{G,k}(S)| \leq k$ and $x \in N_G[w] \setminus \mathcal{P}^i_{G,k}(S)$. Then by the induction hypothesis, $w \in \mathcal{P}^i_{G-e,k}(S), x \in N_{G-e}[w]$ and $|N_{G-e}[w] \setminus \mathcal{P}^i_{G-e,k}(S)| \leq k$, which implies $x \in \mathcal{P}^{i+1}_{G-e,k}(S)$. Therefore $E(G) = E'$ and G is indeed a union of k-generalized spiders. $\square$

Observe that there also exist graphs for which the removal of any edge decreases the k-power domination number, though we did not manage to characterize them. The simplest example is the complete bipartite graph $K_{k+2,k+2}$, in which the removal of any edge decreases the k-power domination number from 2 to 1. Another example is the graph $K_{k+3,k+3} - M$, where M is a perfect matching, in which we have $\gamma_{\mathrm{P},k}(K_{k+3,k+3} - M) = 2$ and $\gamma_{\mathrm{P},k}((K_{k+3,k+3} - M) - e) = 1$ for any edge e.

More complex examples are the Cartesian product of K_4 and W_5, where the k-power domination number decreases from

3 to 2. A general family of graphs having this property is the Cartesian product of two complete graphs of the same order $K_a \square K_a$, which shall be described in subsection 2.1.4.

2.1.3 Edge contraction

Contracting an edge in a graph may decrease its domination number by one, but can never increase it [49]. We have that $\gamma(G) - 1 \leq \gamma(G/e) \leq \gamma(G)$. As we prove in the following, increase of the k-power domination number may occur.

Theorem 2.1.5. *Let G be a graph and e be an edge in G. Then*

$$\gamma_{\mathrm{P},k}(G) - 1 \leq \gamma_{\mathrm{P},k}(G/e) \leq \gamma_{\mathrm{P},k}(G) + 1\,.$$

Moreover,

$$\begin{cases} \textit{if } \gamma_{\mathrm{P},k}(G) - 1 = \gamma_{\mathrm{P},k}(G/e), \textit{ then } \mathrm{rad}_{\mathrm{P},k}(G) \leq \mathrm{rad}_{\mathrm{P},k}(G/e). \\ \textit{if } \gamma_{\mathrm{P},k}(G/e) = \gamma_{\mathrm{P},k}(G) + 1, \textit{ then } \mathrm{rad}_{\mathrm{P},k}(G/e) \leq \mathrm{rad}_{\mathrm{P},k}(G). \end{cases}$$

Proof. Let $e = xy$ be an arbitrary edge in G, we denote by v_{xy}, the vertex obtained by contraction of e in G/e. We first prove that $\gamma_{\mathrm{P},k}(G/e) \geq \gamma_{\mathrm{P},k}(G) - 1$.

Let T be a minimum k-PDS of $H = G/e$. Suppose first that the vertex $v_{xy} \in T$, then by taking $S = T \setminus \{v_{xy}\} \cup \{x, y\}$, the conditions of Lemma 2.1.1 hold from $i = j = 0$ with the natural mapping from $G - \{x, y\}$ to $H - v_{xy}$. We infer that S is a k-PDS of G and $\mathrm{rad}_{\mathrm{P},k}(G, S) = \mathrm{rad}_{\mathrm{P},k}(G/e, T)$. We now consider the case when $v_{xy} \notin T$. Let j_0 be the smallest j such that $v_{xy} \in \mathcal{P}^{j}_{G/e,k}(T)$. Let w be a neighbour of v_{xy} that brought v_{xy} into $\mathcal{P}^{j}_{G/e,k}(T)$, i.e. if $j_0 = 0$, w is a neighbour of v_{xy} in T, otherwise when $j_0 > 0$, w is a neighbour of v_{xy} in $\mathcal{P}^{j_0-1}_{G/e,k}(T)$ such that $|N_{G/e}[w] \setminus \mathcal{P}^{j_0-1}_{G/e,k}(T)| \leq k$. By definition of edge contraction, the edge wv_{xy} corresponds to an edge wx or wy in $E(G)$. If $wx \in E(G)$, then take $S = T \cup \{y\}$, otherwise take $S = T \cup \{x\}$. Then, by applying Lemma 2.1.1 (with the natural mapping from $G - \{x, y\}$ to $H - v_{xy}$ and $i = j = j_0$), we get that S is a k-PDS of G and $\mathrm{rad}_{\mathrm{P},k}(G, S) = \mathrm{rad}_{\mathrm{P},k}(G/e, T)$. This implies that if $\gamma_{\mathrm{P},k}(G) = \gamma_{\mathrm{P},k}(G/e) + 1$, then $\mathrm{rad}_{\mathrm{P},k}(G) \leq \mathrm{rad}_{\mathrm{P},k}(G/e)$.

We now prove that $\gamma_{\mathrm{P},k}(G/e) \leq \gamma_{\mathrm{P},k}(G) + 1$. Let T be a minimum k-PDS of G and let $S = T \setminus \{x, y\} \cup \{v_{xy}\}$. Let j_0 be the smallest j such that $N_G[x] \cup N_G[y] \subseteq \mathcal{P}^{j}_{G,k}(T)$. Here also, we can use Lemma 2.1.1 (with the natural mapping from $(G/e) - v_{xy}$ to $G - \{x, y\}$ and $i = j = j_0$), and get that S is k-PDS of G/e. We also get that if $\gamma_{\mathrm{P},k}(G/e) = \gamma_{\mathrm{P},k}(G)+1$, then $\mathrm{rad}_{\mathrm{P},k}(G/e) \leq \mathrm{rad}_{\mathrm{P},k}(G)$. $\square$

The bounds in Theorem 2.1.5 are tight. For example, the lower bound holds for the graphs $K_{k+2,k+2}$ and $K_{k+3,k+3} - M$, where M is a perfect matching, but also for the Cartesian product of two complete graphs of same order $K_a \,\square\, K_a$, as is described in the next subsection. The upper bound is attained for example for the k-generalized spider T in Figure 1.17, which satisfy $\gamma_{\mathrm{P},k}(T) = 1$ and $\gamma_{\mathrm{P},k}(T/a_1b_1) = 2$ for $k \geq 2$. For the 1-generalized spider (Figure 1.7), the contraction of any edge in the graph does not have any consequence.

2.1.4 On the Cartesian product of twin complete graphs

The Cartesian product of two complete graphs of same (large enough) order is such that removing a vertex, removing an edge or contracting an edge decrease its k-power domination number.

Observation 2.1.6. *Let $a \geq 1$ and $G = K_a \,\square\, K_a$. Then*

$$\gamma_{\mathrm{P},k}(G) = \begin{cases} a-k, & a \geq k+2\,; \\ 1, & a \leq k+1. \end{cases}$$

Proof. Let the vertices of K_a be denoted by $\{v_1, \ldots, v_a\}$. If $a < k+2$, then any vertex in $G = K_a \,\square\, K_a$ is a minimum k-PDS. Now, assume $a \geq k+2$. Let $S = \{(v_i, v_i) \colon 1 \leq i \leq a-k\}$. Then $\mathcal{P}_k^0(S) = \{(v_i, v_j) \colon i \leq a-k \text{ or } j \leq a-k\}$ and the set of vertices $A = \{(v_i, v_j) \colon a-k+1 \leq i, j \leq a\}$ is yet to be monitored. Since any vertex in $\mathcal{P}_k^0(S) \setminus A$ has either 0 or k neighbours in A and each vertex in A is adjacent to some vertex in $\mathcal{P}_k^0(S)$, $\mathcal{P}_k^1(S)$ covers the whole graph. Thus S is a k-PDS of G. Therefore, $\gamma_{\mathrm{P},k}(G) \leq a-k$.

We now prove that $\gamma_{\mathrm{P},k}(G) \geq a-k$. By way of contradiction, suppose S is a k-PDS of G such that $|S| \leq a-k-1$. Without loss of generality, assume that the elements of S belong to the first $a-k-1$ columns and rows of G. Then the vertices in the set $A' = \{(v_i, v_j) \colon a-k \leq i, j \leq a\}$ are not adjacent to any vertex in S, and $\mathcal{P}_k^0(S) \cap A' = \emptyset$. Since any vertex in $G - A'$ has either 0 or $k+1$ neighbours in A', no vertices from this set may get monitored later on, a contradiction. $\square$

Observation 2.1.7. *Let $a \geq k+2$ and $G = K_a \,\square\, K_a$. Then $\gamma_{\mathrm{P},k}(G - v) = a - k - 1$ for any vertex v in G.*

Proof. Let the vertices of K_a be denoted by $\{v_1, \ldots, v_a\}$. We prove the result for $v = (v_1, v_1)$ which implies the result for any v by vertex transitivity. First observe that $S = \{(v_i, v_i) \colon 2 \leq i \leq a - k\}$ is a k-PDS of $G - v$. Indeed $\mathcal{P}_k^0(S) = \{(v_i, v_j) \colon 2 \leq i \leq a - k \text{ or } 2 \leq j \leq a - k\}$ then vertices (v_i, v_1) (resp. (v_1, v_i)) with $2 \leq i \leq a - k$ have only vertices (v_j, v_1) (resp. (v_1, v_j)) with $a - k + 1 \leq j \leq a$ as unmonitored neighbours, which are thus all in $\mathcal{P}_k^1(S)$. The next propagation step covers the graph. Thus S is a k-PDS of $G - v$ and $\gamma_{\mathrm{P},k}(G - v) \leq a - k - 1$. Now by Theorem 2.1.2 and Observation 2.1.6, $\gamma_{\mathrm{P},k}(G - v) \geq a - k - 1$. $\square$

Observation 2.1.8. *Let $a \geq k+2$ and $G = K_a \,\square\, K_a$. Then $\gamma_{\mathrm{P},k}(G - e) = a - k - 1$ for any edge e in G.*

Proof. Let the vertices of K_a be denoted by $\{v_1, \ldots, v_a\}$. By edge transitivity of G, we can assume that $e = (v_1, v_1)(v_2, v_1)$. Let $S = \{(v_i, v_i) \colon 2 \leq i \leq a - k\}$. Then $\mathcal{P}_k^0(S) = \{(v_i, v_j) \colon 2 \leq i \leq a - k \text{ or } 2 \leq j \leq a - k\}$. Now the vertex (v_2, v_1) has only k unmonitored neighbours, namely the vertices (v_j, v_1) for

$a-k < j \leq a$, and they all are in $\mathcal{P}_k^1(S)$. Then all vertices (v_j, v_2) for $a-k < j \leq a$ have only k unmonitored neighbours and thus $\mathcal{P}_k^2(S)$ contains all vertices (v_i, v_j) for $i \geq 2$. Then $\mathcal{P}_k^3(S)$ contains the whole graph and $\gamma_{\mathrm{P},k}(G-e) \leq a-k-1$. The lower bound follows from Theorem 2.1.3 and Observation 2.1.6. $\square$

Observation 2.1.9. *Let $a \geq k+2$ and $G = K_a \,\square\, K_a$. Then $\gamma_{\mathrm{P},k}(G/e) = a-k-1$ for any edge e in G.*

Proof. Let the vertices of K_a be denoted by $\{v_1, \ldots, v_a\}$. By edge transitivity of G, we can assume that $e = (v_1, v_1)(v_2, v_1)$ and we denote by v_e the vertex in G/e obtained by contracting (v_1, v_1) and (v_2, v_1). Let $S = \{v_e\} \cup \{(v_i, v_i) \colon 3 \leq i \leq a-k\}$. Then $\mathcal{P}_k^0(S)$ contains all vertices (v_i, v_j) with $1 \leq i \leq a-k$ and $1 \leq j \leq a$. After one propagation step, the whole graph is monitored so $\gamma_{\mathrm{P},k}(G/e) \leq a-k-1$. The lower bound follows from Theorem 2.1.5 and Observation 2.1.6. $\square$

Note 2.1.1. We are not aware of any general results relating $\gamma_{\mathrm{P},k}(G-e)$ and $\gamma_{\mathrm{P},k}(G/e)$ for any graph G.

Chapter 3

The k-power bondage number of a graph

In the previous chapter, we studied the behaviour of the k-power domination number of a graph when a single edge is deleted. Then it is natural to extend the study on the power domination number of a graph when several edges are deleted. Also, in the monitoring of electric power networks, the links of the network

Some results of this chapter are included in the following paper.
Seethu Varghese, A. Vijayakumar, The k-power bondage number of a graph, Discrete Math. Algorithms Appl. 8 (4) (2016) 1650064 pp. 13. (DOI: 10.1142/S1793830916500646)

through which the propagation occurs play a significant role. Any sort of failures to such links may lead to an increase in the number of PMUs required to monitor the network, thereby increasing the cost for monitoring the entire network. So we must consider whether its function remains good when the network is attacked and thereby any link failures have occurred. Motivated by these observations, we initiate the study of the k-power bondage number of a graph G, denoted by $b_{\mathrm{P},k}(G)$. It gives the number of edges to be deleted from G which is just enough to increase its k-power domination number. $b_{\mathrm{P},k}(G)$ is a generalization of the bondage number, $b(G)$.

In this chapter, we consider all graphs to be nonempty.

3.1 k-power bondage number in general graphs

In this section, we establish an upper bound for $b_{\mathrm{P},k}(G)$.

Theorem 3.1.1. *If G is a connected graph and $\Delta(G) \leq k+1$, then $b_{\mathrm{P},k}(G) = \lambda(G)$, where $\lambda(G)$ is the edge connectivity of G.*

Proof. By the hypothesis, $\gamma_{\mathrm{P},k}(G) = 1$ (see Observation 1.4.9 (b)). Let $B \subseteq E(G)$ be a set of cardinality $\lambda(G)$ such that the subgraph $G - B$ is disconnected. Then $\gamma_{\mathrm{P},k}(G - B) \geq 2 > 1 = \gamma_{\mathrm{P},k}(G)$. Now, for any $B \subseteq E(G)$ with $|B| < \lambda(G)$, the subgraph $G - B$ is still connected with $\Delta(G - B) \leq k + 1$, which implies that $\gamma_{\mathrm{P},k}(G - B) = 1$. $\square$

On the other hand, $\gamma_{\mathrm{P},k}(G) = 1$ in general does not imply that $b_{\mathrm{P},k}(G) = \lambda(G)$. For example, in Figure 3.1, for $k = 1$ we get $\gamma_{\mathrm{P},1}(P_4 \,\Box\, K_2) = 1, \lambda(P_4 \,\Box\, K_2) = 2$ but for the edge uv, $\gamma_{\mathrm{P},1}((P_4 \,\Box\, K_2) - uv) = 2$, hence $b_{\mathrm{P},1}(P_4 \,\Box\, K_2) = 1$.

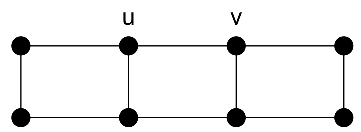


Figure 3.1: The graph $P_4 \,\Box\, K_2$.

Theorem 3.1.2. *Let G be a graph. Then*
$b_{\mathrm{P},k}(G) \leq \min\{d(u) + d(v) - 1 \colon uv \in E(G)\}$.

Proof. Let α denote the right side of the above inequality and let u and v be adjacent vertices of G such that $d(u) + d(v) - 1 = \alpha$. Assume that $b_{\mathrm{P},k}(G) > \alpha$. If E' denotes the set of edges that are incident with at least one of u and v, then $|E'| = \alpha$ and therefore $\gamma_{\mathrm{P},k}(G - E') \leq \gamma_{\mathrm{P},k}(G)$. Since u and v are isolated vertices in $G - E'$, we get $\gamma_{\mathrm{P},k}(G - u - v) \leq \gamma_{\mathrm{P},k}(G) - 2$. Let S' be a $\gamma_{\mathrm{P},k}(G - u - v)$-set.

We claim that the set $S = S' \cup \{u\}$ is a k-PDS of G. Clearly, $\mathcal{P}^0_{G-u-v,k}(S') \subsetneq \mathcal{P}^0_{G,k}(S)$. We prove by induction that for all $\ell, \mathcal{P}^{\ell}_{G-u-v,k}(S') \subsetneq \mathcal{P}^{\ell}_{G,k}(S)$. The result is true for $\ell = 0$. Suppose that $\mathcal{P}^{\ell}_{G-u-v,k}(S') \subsetneq \mathcal{P}^{\ell}_{G,k}(S)$ for some $\ell \geq 0$. Let x be a vertex in $\mathcal{P}^{\ell+1}_{G-u-v,k}(S')$. If x is an isolated vertex in $G - u - v$, then clearly x is in S' and thereby in S. Otherwise, there exists a vertex $y \in \mathcal{P}^{\ell}_{G-u-v,k}(S'), y \in N_{G-u-v}[x]$ such that $\left|N_{G-u-v}[y] \setminus \mathcal{P}^{\ell}_{G-u-v,k}(S')\right| \leq k$. If $y \notin N_G(u) \cup N_G(v)$, then $N_G[y] = N_{G-u-v}[y]$. Therefore, by the induction hypothesis, $y \in \mathcal{P}^{\ell}_{G,k}(S)$ and $\left|N_G[y] \setminus \mathcal{P}^{\ell}_{G,k}(S)\right| \leq k$, which implies that $x \in \mathcal{P}^{\ell+1}_{G,k}(S)$. If $y \in N_G(u) \cup N_G(v)$, then $N_G[y] \subseteq N_{G-u-v}[y] \cup \{u, v\}$. But, since $u, v \in \mathcal{P}^0_{G,k}(S)$ and by the induction hypothesis, $y \in \mathcal{P}^{\ell}_{G,k}(S)$ and $\left|N_G[y] \setminus \mathcal{P}^{\ell}_{G,k}(S)\right| \leq k$, which implies that $x \in \mathcal{P}^{\ell+1}_{G,k}(S)$. Since S' is a $\gamma_{\mathrm{P},k}(G - u - v)$-set, $V(G - u - v) = \mathcal{P}^{\ell'}_{G-u-v,k}(S') \subsetneq \mathcal{P}^{\ell'}_{G,k}(S)$ for some $\ell' \geq 0$. Since $u \in S, V(G) = \mathcal{P}^{\ell'}_{G,k}(S)$. Thus S is a k-PDS of G of cardinality at most $\gamma_{\mathrm{P},k}(G) - 1$, which is a contradiction. Therefore,

$b_{\mathrm{P},k}(G) \leq \alpha$. $\square$

Corollary 3.1.3. *Let G be a connected graph. Then* $b_{\mathrm{P},k}(G) \leq \Delta(G) + \delta(G) - 1$.

Proof. Let u be a vertex such that $d(u) = \delta(G)$ and let v be any neighbour of u. Then, by Theorem 3.1.2,

$$\begin{aligned} b_{\mathrm{P},k}(G) &\leq d(u) + d(v) - 1 = \delta(G) + d(v) - 1 \\ &\leq \delta(G) + \Delta(G) - 1. \end{aligned}$$

$\square$

Sharpness of Theorem 3.1.2 and Corollary 3.1.3 are given in Remark 3.2.1.

3.2 k-power bondage number of some graph classes

In this section, we shall compute the value of $b_{\mathrm{P},k}(G)$ for some well known classes of graphs. We first determine $b_{\mathrm{P},k}(K_n)$. It is known that $b(K_n) = \lceil \frac{n}{2} \rceil$, for $n \geq 2$ [31].

Theorem 3.2.1. *Let $G = K_n, n \geq 2$. Then $b_{\mathrm{P},k}(G) = n - 1$.*

Proof. The result is true for $n = 2, 3$. Let $n \geq 4$. The removal of $n - 1$ edges incident to a vertex $v \in V(G)$ yields a disconnected graph H with $\gamma_{\mathrm{P},k}(H) = 2$. Hence $b_{\mathrm{P},k}(G) \leq n - 1$.

If $B \subseteq E(G)$ is a set of edges with cardinality less than $n-1$, then the spanning subgraph $G-B$ has a vertex of degree at least $n - 2$. Otherwise, if every vertex in $G - B$ has degree at most $n - 3$, then $|B|$ is at least n, which is a contradiction. If $G - B$ has a vertex of degree $n - 1$, then clearly B is not a k-power bondage set of G. Now, suppose $G - B$ has no vertex of degree $n - 1$. Then $G - B$ has a vertex, say v, of degree $n - 2$. Take $S = \{v\}$. Then the $n - 2$ neighbours of v belong to $\mathcal{P}^0_{G-B,k}(S)$. Since $\lambda(G) = n - 1$, the graph $G - B$ is connected. Thus the remaining one unmonitored vertex, say u, which is adjacent to some vertex in $N_{G-B}(v)$, gets monitored by propagation, i.e. $u \in \mathcal{P}^1_{G-B,k}(S)$. Thus $\mathcal{P}^1_{G-B,k}(S) = V(G - B)$, which implies that S is a k-PDS of $G - B$. Thus B is not a k-power bondage set of G. Hence $b_{\mathrm{P},k}(G) \geq n - 1$. $\square$

We now determine the k-power bondage number of the complete bipartite graph $K_{n,n}, n \geq 1$. The graph $K_{k+2,k+2}$ already played a noticeable role among the $k+2$-regular graphs, as observed in [24]. Observe that for any $n \geq 1$,

$$\gamma_{\mathrm{P},k}(K_{n,n}) = \begin{cases} 1, & n \leq k+1; \\ 2, & n \geq k+2. \end{cases}$$

Lemma 3.2.2. *Let* $G = K_{k+2,k+2}$. *Then* $b_{\mathrm{P},k}(G) = 2k+3$.

Proof. Let $V(G) = V_1 \cup V_2$, where $V_1 = \{u_1, \ldots, u_{k+2}\}$, $V_2 = \{v_1, \ldots, v_{k+2}\}$ is the bipartition. We know $\gamma_{\mathrm{P},k}(G) = 2$. Let the edge set $B = \{u_1v_j \colon 1 \leq j \leq k+2\} \cup \{u_jv_1 \colon 1 \leq j \leq k+2\}$. Then the spanning subgraph $G - B = K_{k+1,k+1} \cup \{u_1, v_1\}$. Since $\Delta(K_{k+1,k+1}) = k+1, \gamma_{\mathrm{P},k}(K_{k+1,k+1}) = 1$ (by Observation 1.4.9 (b)). Hence $\gamma_{\mathrm{P},k}(G-B) = 3 > 2 = \gamma_{\mathrm{P},k}(G)$, which implies that B is a k-power bondage set of G. Thus $b_{\mathrm{P},k}(G) \leq |B| = 2k+3$.

We now prove that $b_{\mathrm{P},k}(G) \geq 2k+3$. Let $B \subseteq E(G)$ be a set of edges with cardinality at most $2k+2$. If every vertex in $G-B$ has degree at most k, then the spanning subgraph $G-B$ has at most $k(k+2)$ edges. Thus $|B|$ is at least $2k+4$, which is a contradiction. Hence $G - B$ has a vertex of degree at least $k+1$. Suppose $G-B$ has a vertex of degree $k+2$. Without loss of generality, assume $d_{G-B}(u_1) = k+2$. Since $|B| \leq 2k+2$, the subgraph $G - B$ has at most one isolated vertex. Hence $G - B$ can either be connected or of the form $G' \cup \{u_j\}$ for some j, where G' is a connected component in G.

Case (a): $G - B$ is connected.

If $G - B$ contains at least one vertex v_i of degree $k + 2$, say v_1, then the set $\{u_1, v_1\}$ forms a k-PDS of $G - B$, which implies that $\gamma_{\mathrm{P},k}(G - B) \leq 2$. If not, $1 \leq d_{G-B}(v_i) \leq k + 1$ for all i such that $1 \leq i \leq k + 2$. Let $S = \{u_1\}$. Then $\mathcal{P}^0_{G-B,k}(S) = \{u_1, v_1, \ldots, v_{k+2}\}$. Now, since $\left|N_{G-B}[v_i] \setminus \mathcal{P}^0_{G-B,k}(S)\right| \leq k$ for all i, each v_i can monitor their neighbours in V_1. Since $G - B$ is connected, $\bigcup_{i=1}^{k+2} N_{G-B}(v_i) = V_1$. Hence $\mathcal{P}^1_{G-B,k}(S) = V(G-B)$, which implies that S is a k-PDS of $G - B$.

Case (b): $G - B = G' \cup \{u_j\}$, for some j.

Let $S = \{u_1, u_j\}$. Then $\mathcal{P}^0_{G-B,k}(S) = \{u_1, u_j, v_1, \ldots, v_{k+2}\}$. Since u_j is an isolated vertex in $G-B$, $\left|N_{G-B}[v_i] \setminus \mathcal{P}^0_{G-B,k}(S)\right| \leq k$ for all i. Hence the vertices $v_i, i \in \{1, \ldots, k+2\}$, can monitor their neighbours in V_1 by propagation. Since G' is connected, $\bigcup_{i=1}^{k+2} N_{G'}(v_i) = V_1 \setminus \{u_j\}$. Hence $\mathcal{P}^1_{G-B,k}(S) = V(G - B)$, which implies that S is a k-PDS of $G - B$.

Hence in both the cases, we get $\gamma_{\mathrm{P},k}(G - B) \leq 2$. Now suppose $G - B$ has no vertices of degree $k + 2$. Therefore, $d_{G-B}(v) \leq k + 1$ for every $v \in V(G - B)$. Then we have the following three cases.

Case 1: $G - B$ is connected.

Case 2: $G - B = G' \cup \{e\}$, where the edge $e = u_i v_j$ for some i and j, and G' is a connected component in G.

Case 3: $G - B = G' \cup \{v\}$, where v is an isolated vertex in $G - B$ and G' is a connected component in G.

In Case 1, since $G - B$ is connected with $\Delta(G - B) \leq k + 1, \gamma_{\mathrm{P},k}(G - B) = 1$. In Case 2 and Case 3, $\gamma_{\mathrm{P},k}(G - B) = 2$. Hence an edge set B with $|B| \leq 2k + 2$ cannot be a k-power bondage set of B. □

Remark 3.2.1. It follows from Lemma 3.2.2 that the bounds in Theorem 3.1.2 and Corollary 3.1.3 are sharp.

Theorem 3.2.3. *For $n \geq 1$,*

$$b_{\mathrm{P},k}(K_{n,n}) = \begin{cases} n, & n \leq k+1 \text{ or } n \geq k+3; \\ 2n-1, & n = k+2. \end{cases}$$

Proof. Let $G = K_{n,n}$ and $V(G) = V_1 \cup V_2$, where $V_1 = \{u_1, \ldots, u_n\}$, $V_2 = \{v_1, \ldots, v_n\}$ be the bipartite sets. When $n \leq k+1$, by Theorem 3.1.1, we have $b_{\mathrm{P},k}(G) = \lambda(G) = n$. And, by Lemma 3.2.2, the result holds for $n = k+2$. One can easily verify the result for $n \in \{1, 2, 3\}$. So, let $n \geq 4$.

Assume now that $n \geq k+3$. Then we have $\gamma_{\mathrm{P},k}(G) = 2$. The removal of n edges incident to any vertex, say u_1, in G results in a disconnected graph $H = K_{n-1,n} \cup \{u_1\}$. Any single vertex in $K_{n-1,n}$ cannot itself monitor the entire graph as each of the monitored vertices has at least $n-2$ unmonitored neighours after the domination step, thereby prevents the propagation step. Thus $\gamma_{\mathrm{P},k}(K_{n-1,n}) = 2$ and hence we get $\gamma_{\mathrm{P},k}(H) = 3 > 2 = \gamma_{\mathrm{P},k}(G)$. Let $B \subseteq E(G)$ be an edge set with cardinality at most $n-1$. Then the spanning subgraph $G - B$ is a connected bipartite graph having at least one vertex with degree n in $G - B$. Without loss of generality, assume $d_{G-B}(u_1) = n$. Then there exists at least one vertex v_i that has degree $n-1$ in $G - B$. Otherwise, $1 \leq d_{G-B}(v_i) \leq n-2$ for all i, which implies that B has at least $2n$ edges, a contradiction. We may assume that $d_{G-B}(v_1) = n-1$. Then the set $\{u_1, v_1\}$ clearly forms a k-PDS of $G - B$ and thus B is not a k-power bondage set of G. Hence

$b_{\mathrm{P},k}(G) = n$. $\square$

Observation 3.2.4. *For $n \geq 3$, $b_{\mathrm{P},k}(C_n) = 2$ and for $n \geq 2$, $b_{\mathrm{P},k}(P_n) = 1$.*

Observation 3.2.5. *In general, the values of $b(G)$ and $b_{\mathrm{P},k}(G)$ are unrelated.*

For example, let G be a graph obtained from the star $K_{1,n}$, $n \geq 1$ by subdividing each of its edges exactly twice. Then $\gamma(G) = n+1, \gamma_{\mathrm{P},k}(G) = 1$ and $b(G) = 2 > 1 = b_{\mathrm{P},k}(G)$, whereas in the graph $H = K_{k+2,k+2}$, $\gamma(H) = 2 = \gamma_{\mathrm{P},k}(H)$ and $b(H) = k+2 < 2k+3 = b_{\mathrm{P},k}(H)$. Also $b_{\mathrm{P},k+1}(H) = k+2 = b_{\mathrm{P},k-1}(H)$. For complete graphs, we have $b(K_n) < b_{\mathrm{P},k}(K_n)$ for $n \geq 4$. On the other hand, $b(G') = b_{\mathrm{P},k}(G')$ when G' is a star.

3.3 k-power bondage number of trees

Lemma 3.3.1. *Let G be a graph and $e = uv$ be an edge in G. Let S' be a k-PDS of $G-e$. Suppose there is a set S and integers i and j such that $\mathcal{P}^i_{G-e,k}(S') \subseteq \mathcal{P}^j_{G,k}(S)$ and $u, v \in \mathcal{P}^j_{G,k}(S)$. Then S is a k-PDS of G.*

Proof. We prove by induction that for all ℓ, $\mathcal{P}^{i+\ell}_{G-e,k}(S') \subseteq \mathcal{P}^{j+\ell}_{G,k}(S)$. By the hypothesis, it holds for $\ell = 0$. Now assume that the property is true for some $\ell \geq 0$. Let x be a vertex that satisfies the conditions for propagation at step $i+\ell$, i.e. $x \in \mathcal{P}^{i+\ell}_{G-e,k}(S')$ and $|N_{G-e}[x] \setminus \mathcal{P}^{i+\ell}_{G-e,k}(S')| \leq k$. Then we have $N_{G-e}[x] \subseteq \mathcal{P}^{i+\ell+1}_{G-e,k}(S')$. By the induction hypothesis, $x \in \mathcal{P}^{j+\ell}_{G,k}(S)$ and $|N_{G-e}[x] \setminus \mathcal{P}^{j+\ell}_{G,k}(S)| \leq k$. Now possibly x is u or v and has one more neighbour in G than in $G-e$, namely v or u. In that case though, by the initial hypothesis, both u and v are in $\mathcal{P}^{j+\ell}_{G,k}(S)$ and so $|N_G[x] \setminus \mathcal{P}^{j+\ell}_{G,k}(S)| \leq k$. Thus again $N_G[x] \subseteq \mathcal{P}^{j+\ell+1}_{G,k}(S)$ and applying this to all vertices satisfying the propagation properties, we get $\mathcal{P}^{i+\ell+1}_{G-e,k}(S') \subseteq \mathcal{P}^{j+\ell+1}_{G,k}(S)$. □

It was proved in Chapter 2 that $\gamma_{\mathrm{P},k}(G-v) \geq \gamma_{\mathrm{P},k}(G) - 1$ and $\gamma_{\mathrm{P},k}(G) - 1 \leq \gamma_{\mathrm{P},k}(G-e) \leq \gamma_{\mathrm{P},k}(G) + 1$. However for a tree T we have,

Theorem 3.3.2. *Let T be a tree and e be an edge in T. Then $\gamma_{\mathrm{P},k}(T-e) \geq \gamma_{\mathrm{P},k}(T)$.*

Proof. Let $e = uv$ be an edge in T. Let S be a k-PDS of $T-e$. If either u or v belong to S, then S is also a k-PDS of

T. Hence assume neither u nor v belong to S. Let T_1 and T_2 be the components in $T-e$ containing u and v, respectively. Also, let $S = S_1 \cup S_2$, where S_1 and S_2 are k-PDS of T_1 and T_2, respectively. Let i, j be the smallest integers such that $u \in \mathcal{P}^i_{T-e,k}(S)$ and $v \in \mathcal{P}^j_{T-e,k}(S)$, where $i, j \geq 0$. This means that $u \in \mathcal{P}^i_{T-e,k}(S_1)$ and $v \in \mathcal{P}^j_{T-e,k}(S_2)$. Without loss of generality, assume that $i \leq j$.

Claim 1: $\mathcal{P}^\ell_{T-e,k}(S_1) \subset \mathcal{P}^\ell_{T,k}(S), 0 \leq \ell \leq i.$

If $\ell = 0$, then $\mathcal{P}^0_{T-e,k}(S) = \mathcal{P}^0_{T,k}(S)$, since $u, v \notin S$. Hence $\mathcal{P}^0_{T-e,k}(S_1) \subset \mathcal{P}^0_{T,k}(S)$. Thus, for $\ell > 0$, assume that $\mathcal{P}^{\ell-1}_{T-e,k}(S_1) \subset \mathcal{P}^{\ell-1}_{T,k}(S)$. Let x be a vertex in $\mathcal{P}^\ell_{T-e,k}(S_1)$. Then there exists a vertex $w \in \mathcal{P}^{\ell-1}_{T-e,k}(S_1), w \in N_{T-e}[x]$ such that $\left|N_{T-e}[w] \setminus \mathcal{P}^{\ell-1}_{T-e,k}(S_1)\right| \leq k$. Since $\ell \leq i$ and by minimality of i, $w \neq u$. Thus $N_{T-e}[w] = N_T[w]$. Now, by the induction hypothesis, $w \in \mathcal{P}^{\ell-1}_{T,k}(S)$ and $\left|N_T[w] \setminus \mathcal{P}^{\ell-1}_{T,k}(S)\right| \leq k$. Thus $x \in \mathcal{P}^\ell_{T,k}(S)$.

Claim 2: $\mathcal{P}^\ell_{T-e,k}(S_2) \subset \mathcal{P}^\ell_{T,k}(S), 0 \leq \ell \leq j.$

The proof of Claim 2 is similar to Claim 1.

We now prove that $\mathcal{P}^{i+1}_{T-e,k}(S) \subseteq \mathcal{P}^{j+1}_{T,k}(S)$. Let x be a vertex in $\mathcal{P}^{i+1}_{T-e,k}(S), x \neq u, v$. Then there exists a vertex $w \in$

$\mathcal{P}^i_{T-e,k}(S), w \in N_{T-e}[x]$ such that

$$\left|N_{T-e}[w] \setminus \mathcal{P}^i_{T-e,k}(S)\right| \leq k. \tag{3.1}$$

By Claim 1, $\mathcal{P}^i_{T-e,k}(S_1) \subset \mathcal{P}^i_{T,k}(S)$ and by Claim 2, $\mathcal{P}^i_{T-e,k}(S_2) \subset \mathcal{P}^i_{T,k}(S)$. Therefore,

$$\mathcal{P}^i_{T-e,k}(S) = \mathcal{P}^i_{T-e,k}(S_1) \cup \mathcal{P}^i_{T-e,k}(S_2) \subseteq \mathcal{P}^i_{T,k}(S). \tag{3.2}$$

Case 1: $w \notin \{u, v\}$.

Clearly, $N_{T-e}[w] = N_T[w]$. Hence the equations (3.1) and (3.2) imply that $w \in \mathcal{P}^i_{T,k}(S)$ and $\left|N_T[w] \setminus \mathcal{P}^i_{T,k}(S)\right| \leq k$. Thus $x \in \mathcal{P}^{i+1}_{T,k}(S) \subseteq \mathcal{P}^{j+1}_{T,k}(S)$.

Case 2: $w = u$.

The equations (3.1) and (3.2) imply that $u \in \mathcal{P}^i_{T,k}(S)$ and $\left|N_{T-e}[u] \setminus \mathcal{P}^i_{T,k}(S)\right| \leq k$. Since $i \leq j, \left|N_{T-e}[u] \setminus \mathcal{P}^j_{T,k}(S)\right| \leq k$. Now, u has one more neighbour, v, in T than in $T - e$. But, $v \in \mathcal{P}^j_{T-e,k}(S_2) \subset \mathcal{P}^j_{T,k}(S)$, thus $\left|N_T[u] \setminus \mathcal{P}^j_{T,k}(S)\right| \leq k$. Hence $x \in \mathcal{P}^{j+1}_{T,k}(S)$.

Case 3: $w = v$.

This is the case when $i = j$. Again, the equations (3.1) and (3.2) imply that $v \in \mathcal{P}^i_{T,k}(S)$ and $\left|N_{T-e}[v] \setminus \mathcal{P}^i_{T,k}(S)\right| \leq k$. But, by equation (3.2), $u \in \mathcal{P}^i_{T,k}(S)$ and hence $\left|N_T[v] \setminus \mathcal{P}^i_{T,k}(S)\right| \leq k$. Thus $x \in \mathcal{P}^{i+1}_{T,k}(S)$.

Hence there exist integers i and j such that $\mathcal{P}^{i+1}_{T-e,k}(S) \subseteq \mathcal{P}^{j+1}_{T,k}(S)$. Now, $u \in \mathcal{P}^i_{T,k}(S_1) \subset \mathcal{P}^i_{T,k}(S) \subseteq \mathcal{P}^{j+1}_{T,k}(S)$ (by Claim 1). And $v \in \mathcal{P}^j_{T,k}(S_2) \subset \mathcal{P}^j_{T,k}(S) \subseteq \mathcal{P}^{j+1}_{T,k}(S)$ (by Claim 2). Thus, by Lemma 3.3.1, S is a k-PDS of T. Therefore, $\gamma_{\mathrm{P},k}(T) \leq \gamma_{\mathrm{P},k}(T-e)$. □

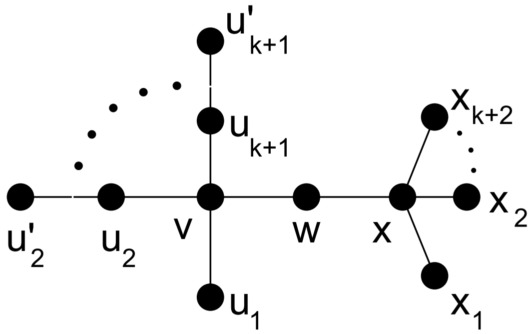


Figure 3.2: The tree T.

For the tree T in Figure 3.2, $\{v, x\}$ is a $\gamma_{\mathrm{P},k}(T)$-set and $\gamma_{\mathrm{P},k}(T) = 2$. For each $i, j, 1 \leq i \leq k+2$ and $2 \leq j \leq k+1$, $\gamma_{\mathrm{P},k}(T - xx_i) = \gamma_{\mathrm{P},k}(T - uu'_j) = 3 > \gamma_{\mathrm{P},k}(T)$. For edges vw and wx, $\gamma_{\mathrm{P},k}(T - vw) = \gamma_{\mathrm{P},k}(T) = \gamma_{\mathrm{P},k}(T - wx)$. Also, for each $j, 1 \leq j \leq k+1$, $\{x, u_j\}$ is a $\gamma_{\mathrm{P},k}(T - vu_j)$-set and therefore $\gamma_{\mathrm{P},k}(T - vu_j) = \gamma_{\mathrm{P},k}(T)$. Hence for any edge e in T, $\gamma_{\mathrm{P},k}(T - e) \geq \gamma_{\mathrm{P},k}(T)$.

Proposition 3.3.3. *Let u be a pendant vertex and v its neighbour in a tree T. Then $\gamma_{\mathrm{P},k}(T - uv) = \gamma_{\mathrm{P},k}(T)$ if and only if $\gamma_{\mathrm{P},k}(T - u) = \gamma_{\mathrm{P},k}(T) - 1$.*

Proof. $\gamma_{\mathrm{P},k}(T-uv) = \gamma_{\mathrm{P},k}(T) \Longleftrightarrow \gamma_{\mathrm{P},k}(T-u)+1 = \gamma_{\mathrm{P},k}(T) \Longleftrightarrow \gamma_{\mathrm{P},k}(T - u) < \gamma_{\mathrm{P},k}(T)$. □

In Figure 3.2, for the pendant vertex u_1 in T, $\gamma_{\mathrm{P},k}(T-u_1v) = 2 = \gamma_{\mathrm{P},k}(T)$ and $\gamma_{\mathrm{P},k}(T - u_1) = 1 < \gamma_{\mathrm{P},k}(T)$.

We now investigate the k-power bondage number of trees. For this purpose, we shall need the following terms. If T is a tree rooted at the vertex x and v is a vertex of T, then the **level number** of v, which we denote by $\ell(v)$, is the length of the unique x,v-path in T. If a vertex v of T is adjacent to u and $\ell(u) > \ell(v)$, then u is called a **child** of v and v is the **parent** of u. A vertex w is a **descendant** of v (and v is an **ancestor** of w) if the level numbers of the vertices on the v,w-path are monotonically increasing.

In [8], Bauer et al. proved that $b_{\mathrm{P},k}(T) \leq 2$ for $k = 0$. We generalize this result to any integer $k \geq 1$.

Theorem 3.3.4. *Let T be a tree. Then $b_{\mathrm{P},k}(T) \leq 2$.*

Proof. If $\Delta(T) \leq k+1$, then by Theorem 3.1.1, $b_{\mathrm{P},k}(T) = 1$. Now assume that $\Delta(T) \geq k+2$. If T contains a vertex w which is adjacent to at least $k+2$ pendant vertices, $w_1, \ldots, w_{k+2}$, then w is in every $\gamma_{\mathrm{P},k}(T)$-set (by Observation 1.4.9 (c)). Let $e = w_1 w$. However, both w_1 and either w or any of the pendant vertices $w_2, \ldots, w_{k+2}$ will be in every k-PDS of $T - e$. Hence $\gamma_{\mathrm{P},k}(T-e) > \gamma_{\mathrm{P},k}(T)$, which implies that $b_{\mathrm{P},k}(T) = 1$.

Now, assume that no vertex of T is adjacent to $k+2$ or more pendant vertices. We have the following two cases.

Case 1: T contains a vertex v which is adjacent to $k+1$ pendant vertices.

Let $u_1, \ldots, u_{k+1}$ be the pendant vertices adjacent to v. For every $\gamma_{\mathrm{P},k}(T)$-set, S of T, $|S \cap \{v, u_1, \ldots, u_{k+1}\}| = 1$. If $\gamma_{\mathrm{P},k}(T - u_i v) > \gamma_{\mathrm{P},k}(T)$ for some i, then we are done. Otherwise, for some pendant vertex, say u_1, $\gamma_{\mathrm{P},k}(T - u_1 v) = \gamma_{\mathrm{P},k}(T)$, by Theorem 3.3.2. Thus, by Proposition 3.3.3, $\gamma_{\mathrm{P},k}(T - u_1) = \gamma_{\mathrm{P},k}(T) - 1$. Let S' be a $\gamma_{\mathrm{P},k}(T - u_1)$-set. Then $v \notin S'$. If $v \in S'$, then S'

is a k-PDS of T of cardinality less than $\gamma_{\mathrm{P},k}(T)$, which is a contradiction. Clearly, S' is a k-PDS of $T - u_1 - u_2$. Thus, $\gamma_{\mathrm{P},k}(T - u_1 - u_2) \leq \gamma_{\mathrm{P},k}(T - u_1)$. Hence $\gamma_{\mathrm{P},k}(T - u_1 - u_2) = \gamma_{\mathrm{P},k}(T-u_1)$. Otherwise, $\gamma_{\mathrm{P},k}(T-u_1-u_2) = \gamma_{\mathrm{P},k}(T-u_1)-1$. But, then, $S'' \cup \{v\}$ is a k-PDS of T of cardinality less than $\gamma_{\mathrm{P},k}(T)$, where S'' is a $\gamma_{\mathrm{P},k}(T - u_1 - u_2)$-set. Hence $\gamma_{\mathrm{P},k}(T - u_1v - u_2v) = \gamma_{\mathrm{P},k}(T - u_1 - u_2) + 2 > \gamma_{\mathrm{P},k}(T)$, which implies that $b_{\mathrm{P},k}(T) \leq 2$.

Case 2: Every vertex of T is adjacent to at most k pendant vertices.

We may assume that T is rooted at a vertex x. Let u be a vertex at the maximum distance from x in T. Clearly, u is a pendant vertex and let v be the vertex adjacent to u (v is the parent of u). Then, by definition of u, all descendants of v are pendant vertices and thus v has at most k descendants. Hence v has degree at most $k + 1$ in T. Let S' be a k-PDS of $T - u$ containing only vertices of degree at least $k + 2$ in $T - u$ (S' exists by Theorem 1.4.10). Then S' does not contain v or any of its descendants. Now, since S' is a k-PDS, v gets monitored by its parent, say w, at some propagation step i in $T - u$. Consequently, v can monitor all its descendants in the next propagation step as it has at most $k - 1$ unmonitored neighbours in

$T-u$ after the i^{th} propagation step. Next, when we consider the monitoring of S' in T, v will get monitored by w at step i and since v has at most k descendants in T, it can again monitor all its descendants in the $(i+1)^{\text{th}}$ propagation step in T. Thus S' is also a k-PDS of T and we get $\gamma_{\mathrm{P},k}(T) \leq \gamma_{\mathrm{P},k}(T-u)$. Also, since u is a pendant vertex of T, $\gamma_{\mathrm{P},k}(T-u) \leq \gamma_{\mathrm{P},k}(T)$. Hence $\gamma_{\mathrm{P},k}(T-uv) = \gamma_{\mathrm{P},k}(T-u)+1 = \gamma_{\mathrm{P},k}(T)+1$, which implies $b_{\mathrm{P},k}(T) = 1$. □

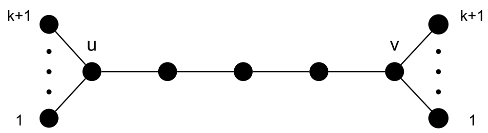


Figure 3.3: The tree T'.

For the tree T' in Figure 3.3, where both vertices u and v are adjacent to $k+1$ pendant vertices, we get that $b_{\mathrm{P},k}(T') = 2$.

Corollary 3.3.5. *If some vertex of a tree T is adjacent to $k+2$ or more vertices of degree one, then $b_{\mathrm{P},k}(T) = 1$.*

We now provide a condition on T for which $b_{\mathrm{P},k}(T) = 1$.

A vertex v of a graph G is **$\boldsymbol{\gamma_{\mathrm{P},k}}$-universal** if it belongs to every $\gamma_{\mathrm{P},k}(G)$-set of G.

Theorem 3.3.6. *Let T be a tree. If T has a $\gamma_{\mathrm{P},k}$-universal vertex, then $b_{\mathrm{P},k}(T) = 1$.*

Proof. Let S be a $\gamma_{\mathrm{P},k}(T)$-set of T and v be a $\gamma_{\mathrm{P},k}$-universal vertex of T. Let $N(v) = \{v_1, \ldots, v_r\}$ and T_i be the component of $T - v$ containing v_i. Since v is $\gamma_{\mathrm{P},k}$-universal, we have $S = \{v\} \cup \big(\cup_{i=1}^{r} S \cap V(T_i)\big)$.

If $S \cap V(T_i)$ is a k-PDS of T_i for all i, then we can extend this k-PDS of T_i to a k-PDS of T that avoids v and has cardinality at most $\gamma_{\mathrm{P},k}(T)$, which is a contradiction. Thus there exists at least one i such that $S \cap V(T_i)$ is not a k-PDS of T_i. Hence assume that $S \cap V(T_i)$ is not a k-PDS of T_i and let $e = vv_i$ for some i.

Claim: $\gamma_{\mathrm{P},k}(T - e) > \gamma_{\mathrm{P},k}(T)$.

Assume not. Then, by Theorem 3.3.2, $\gamma_{\mathrm{P},k}(T - e) = \gamma_{\mathrm{P},k}(T)$. Let S' be a $\gamma_{\mathrm{P},k}(T - e)$-set of $T - e$. Then $S' = S_1' \cup S_2'$, where S_1' is a $\gamma_{\mathrm{P},k}(T_i)$-set of T_i and S_2' is a $\gamma_{\mathrm{P},k}(T - T_i)$-set of $T - T_i$. Clearly, $|S_1'| > |S \cap V(T_i)|$. Otherwise, if $|S_1'| \leq |S \cap V(T_i)|$, then we can replace $S \cap V(T_i)$ by S_1'. Moreover, $|S_1'| = |S \cap V(T_i)| + 1$. (Observe that $(S \cap V(T_i)) \cup \{v_i\}$ is a k-PDS of T_i.)

Hence,

$$\begin{aligned}
\gamma_{\mathrm{P},k}(T-e) &= \gamma_{\mathrm{P},k}(T) \\
\Rightarrow |S_1'| + |S_2'| &= 1 + \sum_{j\neq i} |S\cap V(T_j)| + |S\cap V(T_i)| \\
\Rightarrow |S\cap V(T_i)| + 1 + |S_2'| &= 1 + \sum_{j\neq i} |S\cap V(T_j)| + |S\cap V(T_i)| \\
\Rightarrow |S_2'| &= \sum_{j\neq i} |S\cap V(T_j)|. \qquad (3.3)
\end{aligned}$$

Suppose $S_2' \cap \{v\} = \emptyset$. Then, since any k-PDS of $T-e$ is a k-PDS of T (as seen in the proof of Theorem 3.3.2), $S_1' \cup S_2'$ is a k-PDS of T of cardinality $\gamma_{\mathrm{P},k}(T)$ that does not contain v, which is a contradiction. Thus the vertex v belongs to S_2'. Now, since S_2' is a k-PDS of $T - T_i$ and $(S\cap V(T_i)) \cup \{v\}$ is a k-PDS of $T_i \cup \{v\}$, $S_2' \cup (S \cap V(T_i))$ is a k-PDS of T of cardinality $\gamma_{\mathrm{P},k}(T) - 1$ (follows from (3.3)), which is again a contradiction. Hence the claim. Thus $b_{\mathrm{P},k}(T) = 1$. $\square$

Remark 3.3.1. Theorem 3.3.6 need not hold in general. Let G_k be the graph constructed from a complete graph K_{k+2} such that K_{k+1} is attached to exactly one vertex, say u, of K_{k+2}. Then $\gamma_{\mathrm{P},k}(G_k) = 1$ and u is the unique $\gamma_{\mathrm{P},k}$-universal vertex of G_k. But, $\gamma_{\mathrm{P},k}(G_k - e) = \gamma_{\mathrm{P},k}(G_k)$ for any edge e and hence

$b_{\mathrm{P},k}(G_k) \neq 1$ (See Figure 3.4).

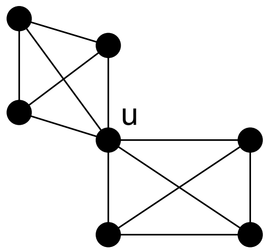


Figure 3.4: The graph G_k for $k = 2$.

Remark 3.3.2. The converse of Theorem 3.3.6 is obviously not true as in the case of paths.

Chapter 4

Power domination in graph products

Many large networks can be efficiently modelled by graph products. When designing large scale networks, the product graphs serve a base for easy and economical control of large scale systems. Any graphical invariant can be studied on product graphs.

Some results of this chapter are included in the following paper.
Seethu Varghese, A. Vijayakumar, On the Power Domination Number of Graph Products, In: S. Govindarajan, A. Maheshwari (Eds.), CALDAM 2016, Lecture Notes in Comput. Sci, Vol. 9602, pp: 357–367, Springer (2016).

A standard question that arise here is the relationship between the invariant of the product and the invariant of the factors.

In this chapter, we discuss the power domination problem in Cartesian product, direct product and lexicographic product. For Cartesian and direct products, we study $\gamma_{\mathrm{P},k}(G \,\square\, H)$ and $\gamma_{\mathrm{P},k}(G \times H)$ for the value $k = 1$. We determine a general upper bound for $\gamma_{\mathrm{P},1}(G \,\square\, H)$ in terms of $\gamma_{\mathrm{P},1}(G)$ and $\gamma_{\mathrm{P},1}(H)$. We establish some sharp upper bounds for $\gamma_{\mathrm{P},1}(G \,\square\, H)$ and $\gamma_{\mathrm{P},1}(G \times H)$, where the graph H has a universal vertex. Characterization of the graphs G and H of order at least four for which $\gamma_{\mathrm{P},1}(G \,\square\, H) = 1$ is obtained. We consider the generalized version of power domination in lexicographic products and obtain the k-power domination number of $G \circ H$.

4.1 The Cartesian Product

We first give a general upper bound for the power domination number of Cartesian product of two graphs.

Theorem 4.1.1. *For any two nontrivial graphs G and H, $\gamma_{\mathrm{P}}(G \,\square\, H) \leq \min\{\gamma_{\mathrm{P}}(G)|V(H)|, \gamma_{\mathrm{P}}(H)|V(G)|\}$.*

Proof. Let S be a PDS of the graph G and let S' be the set $\{(g,h)\colon g \in S, h \in V(H)\}$. Then $\mathcal{P}^0_{G\,\square\, H,1}(S') = \{V({}^gH)\colon g \in \mathcal{P}^0_{G,1}(S)\}$. In order to prove that S' is a PDS of $G \,\square\, H$, it is enough to prove that for a vertex g in G, if $g \in \mathcal{P}^i_{G,1}(S)$, then $V({}^gH) \in \mathcal{P}^i_{G\,\square\, H,1}(S')$ for all $i \geq 0$.

The proof is by induction. The property holds for $i = 0$ and so suppose that it is true for some $i \geq 0$. Let g be a vertex in $\mathcal{P}^{i+1}_{G,1}(S)$. If $g \in \mathcal{P}^i_{G,1}(S)$, then by the induction hypothesis $V({}^gH) \in \mathcal{P}^i_{G\,\square\, H,1}(S')$. Otherwise, there exists a neighbour g' of g in $\mathcal{P}^i_{G,1}(S)$ such that $\left|N_G[g'] \setminus \mathcal{P}^i_{G,1}(S)\right| \leq 1$. By the induction hypothesis, $V({}^{g'}H) \in \mathcal{P}^i_{G\,\square\, H,1}(S')$ and therefore, for $h \in V(H)$, $\left|N_{G\,\square\, H}[(g',h)] \setminus \mathcal{P}^i_{G\,\square\, H,1}(S')\right| = \left|\{(v,h)\colon g'v \in E(G), (v,h) \notin \mathcal{P}^i_{G\,\square\, H,1}(S')\}\right| = \left|\{v\colon g'v \in E(G), v \notin \mathcal{P}^i_{G,1}(S)\}\right| = \left|N_G[g'] \setminus \mathcal{P}^i_{G,1}(S)\right| \leq 1$. Therefore, $N_{G\,\square\, H}[(g',h)] \subseteq \mathcal{P}^{i+1}_{G\,\square\, H,1}(S')$ which implies that $(g,h) \in \mathcal{P}^{i+1}_{G\,\square\, H,1}(S')$. Since this is true for any h, $V({}^gH) \in \mathcal{P}^{i+1}_{G\,\square\, H,1}(S')$. Therefore S' is a PDS of $G \,\square\, H$ and hence $\gamma_{\mathrm{P}}(G \,\square\, H) \leq |S'| \leq \gamma_{\mathrm{P}}(G)|V(H)|$. Similarly we can prove that $\gamma_{\mathrm{P}}(G \,\square\, H) \leq \gamma_{\mathrm{P}}(H)|V(G)|$. $\square$

A close examination of the power domination definition leads naturally to the study of **zero forcing**. The zero forcing number was introduced in [3] to aid in the study of minimum rank/ maximum nullity problems. The minimum rank problem for a (simple) graph asks for the determination of the minimum rank among all real symmetric matrices with the zero-nonzero pattern of off-diagonal entries described by a given graph (the diagonal of the matrix is free); the maximum nullity of the graph is the maximum nullity over the same set of matrices.

One can observe that the colour change rule in zero forcing and the propagation rule in power domination are closely related. The monitoring rules in power domination on a graph G can be described as choosing a set $S \subseteq V(G)$ and applying the zero forcing process to $N[S]$. Also, it is easy to observe that $S \subseteq V(G)$ is a PDS of a graph G if and only if $N[S]$ is a zero forcing set of G. Therefore, it is interesting to compare the two parameters, particularly when these parameters have been studied with different motivations.

Given a zero forcing set, we denote the set of vertices that get coloured black at each stage of applying the color change rule, in the following manner.

Let Z be a zero forcing set of a graph G. The sets $\left(\mathcal{B}^i_G(Z)\right)_{i\geq 0}$ of **vertices that are coloured black by $\boldsymbol{Z}$ at step $\boldsymbol{i}$** are denoted as follows:

$\mathcal{B}^0_G(Z) \quad = Z$, and

$\mathcal{B}^{i+1}_G(Z) = \left\{v\colon vu \in E(G), u \in \mathcal{B}^i_G(Z) \text{ such that } N_G[u]\backslash\mathcal{B}^i_G(Z) = \{v\}\right\} \cup \mathcal{B}^i_G(Z)$.

The following theorem shows that we can construct a power dominating set for $G \,\square\, H$ from a zero forcing set of one of the factor graphs when the other factor has a universal vertex, and thereby obtaining a bound on the power domination number using the zero forcing number.

Theorem 4.1.2. *Let G and H be two nontrivial graphs. If H has a universal vertex, then $\gamma_{\mathrm{P}}(G \,\square\, H) \leq Z(G)$.*

Proof. Let Z be a zero forcing set of G and x be a universal vertex of H. Let $Z' = Z \times \{x\}$. Clearly, $\mathcal{P}^0_{G \,\square\, H,1}(Z')$ contains all vertices of ${}^gH, g \in Z$. We now prove by induction that for a vertex g in G, if $g \in \mathcal{B}^i_G(Z)$, then $V({}^gH) \in \mathcal{P}^i_{G \,\square\, H,1}(Z')$ for all $i \geq 0$. Clearly it holds for $i = 0$. Suppose that the property holds for some $i \geq 0$. Let g be a vertex in $\mathcal{B}_G^{i+1}(Z)$. If g is not in $\mathcal{B}_G^i(Z)$, then there exists a neighbour g' of g in $\mathcal{B}_G^i(Z)$ such that $\left|N_G[g'] \setminus \mathcal{B}_G^i(Z)\right| = 1$. By the induction hypothesis, $V({}^{g'}H) \in \mathcal{P}^i_{G\,\square\,H,1}(Z')$ and therefore, for $h \in V(H)$, $\left|N_{G\,\square\,H}[(g',h)] \setminus \mathcal{P}^i_{G\,\square\,H,1}(Z')\right| = \left|\{(v,h)\colon g'v \in E(G), (v,h) \notin \mathcal{P}^i_{G\,\square\,H,1}(Z')\}\right| = \left|\{v\colon g'v \in E(G), v \notin \mathcal{B}_G^i(Z)\}\right| = \left|N_G[g'] \setminus \mathcal{B}_G^i(Z)\right| = 1$. Hence the vertex (g',h) has only one neighbour yet to be monitored, which implies that $N_{G\,\square\,H}[(g',h)] \subseteq \mathcal{P}^{i+1}_{G\,\square\,H,1}(Z')$. Therefore $(g,h) \in \mathcal{P}^{i+1}_{G\,\square\,H,1}(Z')$. Since h is arbitrary, $V({}^{g}H) \in \mathcal{P}^{i+1}_{G\,\square\,H,1}(Z')$. Now, since Z is a zero forcing set of G, there exists some nonnegative integer j such that $\mathcal{B}_G^j(Z) = V(G)$ and hence $V(G \,\square\, H) = \mathcal{P}^j_{G\,\square\,H,1}(Z')$. Therefore Z' is a PDS of $G \,\square\, H$. $\square$

The bound in Theorem 4.1.2 is sharp for $G = P_m, C_m, W_m$ or F_m and $H = K_n, m, n \geq 4$ by Theorem 1.4.7 $(a) - (d)$.

Theorem 4.1.3. *For any nontrivial graph G and $n \geq 2$,*
$\gamma_{\mathrm{P}}(G \,\square\, P_n) \leq \gamma(G)$.

Proof. Let D be a dominating set of G. Let x be a pendant vertex of P_n. Take $D' = D \times \{x\}$. Since D is a dominating set

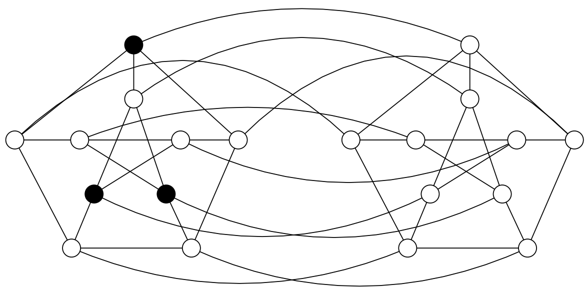


Figure 4.1: Cartesian product of the Petersen graph and P_2.

of G, $V(G^x) \in \mathcal{P}^0_{G\,\square\,P_n,1}(D')$ and therefore the next propagation step covers all the vertices of G^y-fiber, where $xy \in E(P_n)$. The propagation continues in a similar fashion till the last G-fiber and thus D' is a PDS of $G\,\square\,P_n$. $\square$

The bound in Theorem 4.1.3 is sharp for graphs G with $\gamma(G) = 1$. For $n = 2$, the bound is attained for the Petersen graph (Figure 4.1). The black coloured vertices in the figure form a minimum PDS of the graph and we get $\gamma_{\mathrm{P}}(P\,\square\,P_2) = 3 = \gamma(P)$, where P is the Petersen graph.

From Theorems 4.1.2 and 4.1.3, we obtain the following corollary.

Corollary 4.1.4. *For any nontrivial graph G,*
$\gamma_{\mathrm{P}}(G\,\square\,K_2) \leq \min\{\gamma(G), Z(G)\}$.

Theorem 4.1.5. *For any nontrivial graph G,*
$\gamma_{\mathrm{P}}(G \,\square\, K_2) \geq \gamma_{\mathrm{P}}(G)$.

Proof. Let S be a PDS of $G \,\square\, K_2$. Define $S_G = \{g \colon (g,h) \in S$ for some $h \in V(K_2)\}$. We prove that S_G is PDS of G. For that, it is enough to prove that if $(u,v) \in \mathcal{P}^i_{G \,\square\, K_2,1}(S)$, then $u \in \mathcal{P}^i_{G,1}(S_G)$ for all $i \geq 0$.

Let (u,v) be a vertex in $\mathcal{P}^0_{G \,\square\, K_2,1}(S)$. If $(u,v) \in S$, then by definition of S_G, $u \in S_G$. Otherwise, (u,v) is adjacent to some vertex (g,h) in S. Then, either $g = u$ or $h = v$. If $g = u$, then by definition of S_G, $u \in S_G$. If $h = v$, then u is adjacent to g in G. Clearly, g is in S_G. Therefore $u \in \mathcal{P}^0_{G,1}(S_G)$. Hence the property is true for $i = 0$. Assume that the property holds for some $i = \ell \geq 0$. Suppose that $(u,v) \in \mathcal{P}^{\ell+1}_{G \,\square\, K_2,1}(S)$. If $(u,v) \in \mathcal{P}^{\ell}_{G \,\square\, K_2,1}(S)$, then by the induction hypothesis $u \in \mathcal{P}^{\ell}_{G,1}(S_G)$. Otherwise, there exists a vertex $(g,h) \in \mathcal{P}^{\ell}_{G \,\square\, K_2,1}(S)$ such that (u,v) is the only unmonitored neighbour of (g,h) in $G \,\square\, K_2$ after the ℓ^{th} propagation step. Since $(g,h) \in \mathcal{P}^{\ell}_{G \,\square\, K_2,1}(S)$, by the induction hypothesis $g \in \mathcal{P}^{\ell}_{G,1}(S_G)$. We have the following two cases.

Case 1: $g = u$.

Since g is in $\mathcal{P}^{\ell}_{G,1}(S_G)$, $u \in \mathcal{P}^{\ell}_{G,1}(S_G) \subseteq \mathcal{P}^{\ell+1}_{G,1}(S_G)$.

Case 2: $h = v$.

This implies that u is a neighbour of g in G. If possible, assume that the vertex g in $\mathcal{P}^{\ell}_{G,1}(S_G)$ has at least two unmonitored neighbours, say g_1 and g_2, in G after the ℓ^{th} propagation step, i.e. $g_1, g_2 \notin \mathcal{P}^{\ell}_{G,1}(S_G)$. This implies that the vertices $(g_1, h), (g_2, h), (g_1, h'), (g_2, h')$, where $h' \in V(K_2)$, cannot be in $\mathcal{P}^{\ell}_{G\,\square\,K_2,1}(S)$. (Otherwise, g_1 and g_2 would be in $\mathcal{P}^{\ell}_{G,1}(S_G)$, by the induction hypothesis). This means that (g, h) has two unmonitored neighbours (g_1, h) and (g_2, h) in $G \,\square\, K_2$ after the step ℓ, which is a contradiction. Therefore g has at most one unmonitored neighbour in G after the step ℓ, which implies that $N_G[g] \subseteq \mathcal{P}^{\ell+1}_{G,1}(S_G)$ and hence $u \in \mathcal{P}^{\ell+1}_{G,1}(S_G)$. Thus S_G is PDS of G and we get $\gamma_{\mathrm{P}}(G) \leq |S_G| \leq |S| \leq \gamma_{\mathrm{P}}(G \,\square\, K_2)$. □

From Theorems 4.1.1 and 4.1.5, we have the following corollary.

Corollary 4.1.6. *For any nontrivial graph G,*
$\gamma_{\mathrm{P}}(G) \leq \gamma_{\mathrm{P}}(G \,\square\, K_2) \leq 2\gamma_{\mathrm{P}}(G)$.

The above bounds are sharp as depicted in Figure 4.2

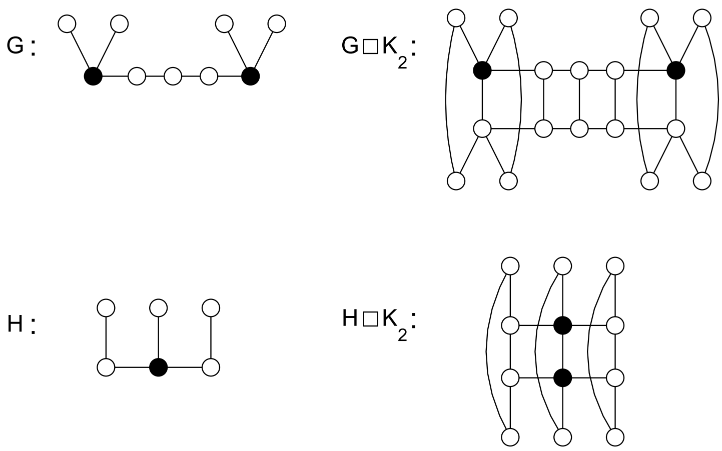


Figure 4.2: $\gamma_{\mathrm{P}}(G \,\square\, K_2) = 2 = \gamma_{\mathrm{P}}(G)$ and $\gamma_{\mathrm{P}}(H \,\square\, K_2) = 2 = 2\gamma_{\mathrm{P}}(H)$.

In general, it remains difficult to identify the graphs G for which $\gamma_{\mathrm{P}}(G) = 1$. Such graphs are identified only in the case of trees (refer Theorem 1.4.3 (a)). We here characterize the graphs G and H of order at least four for which $\gamma_{\mathrm{P}}(G \,\square\, H) = 1$. This condition clearly implies that the factor graphs G and H are connected.

Theorem 4.1.7. *Let G and H be two graphs of order at least four. Then $\gamma_{\mathrm{P}}(G \,\square\, H) = 1$ if and only if one of the graphs has a universal vertex and the other is isomorphic to a path.*

Proof. Suppose that $\gamma_{\mathrm{P}}(G \,\square\, H) = 1$. Let $S = \{(g, h)\}$ be a PDS of $G \,\square\, H$ for some $g \in V(G)$ and $h \in V(H)$. Then, since G and H are connected graphs of order greater than two, at least one of the vertices g or h has degree greater than one. But, if both g and h have degree at least two, then no more vertices get monitored after the domination step. Therefore, assume that g has degree one in G and h has degree at least two in H. Let $d_H(h) = r, r \geq 2$ and $A = \{h' \in N_H(h) \colon N_H[h'] \subseteq N_H[h]\}$.

Claim: $|A| = r$.

If possible, assume that $|A| \leq r-1$. Then the set $B = N_H(h) \setminus A$ is nonempty. Let g' be the neighbour of g. Since $r \geq 2$, the dominated vertex (g', h) has at least two neighbours in its H-fiber and therefore the first propagation step is possible only from the dominated vertices in the ${}^{g}H$-fiber. Since g has degree one, the vertices in the set $\{(g, h') \colon h' \in A\}$ can monitor their corresponding neighbour in the ${}^{g'}H$-fiber. The remaining dominated vertices in the ${}^{g}H$-fiber given by $\{(g, h') \colon h' \in B\}$ have unmonitored neighbours both in ${}^{g}H$- and ${}^{g'}H$-fibers. Since G is a connected graph of order at least four, g' has degree at least two in G. But as no more propagation is possible from any of the dominated vertices in the ${}^{g}H$-fiber, the next step of propagation occurs from the monitored vertices in the ${}^{g'}H$-

fiber, which in turn implies that $d_G(g') = 2$. Let $C = \{h' \in N_H(h)\colon h' \in A, h' \notin N_H(v)$ for every v in $B\}$, i.e. $C \subseteq A$ is the set of vertices in A that are adjacent to none of the vertices in B. Assume that C is nonempty. If g'' is the other neighbour of g', then the vertices in the set $\{(g', h')\colon h' \in C\}$ can hence monitor their neighbour in the ${}^{g''}\!H$-fiber. Since $|B| \geq 1$, the vertex (g', h) and the other monitored vertices ${}^{g'}\!H$-fiber given by $\{(g', h')\colon h' \in A \setminus C\}$ have unmonitored neighbours in their corresponding G- and H-fibers. Again, since $|V(G)| \geq 4$ and G is connected, g'' has at least one neighbour other than g', which in turn prevents any more propagation from the monitored vertices in the ${}^{g''}\!H$-fiber as each of the monitored vertex $(g'', h'), h' \in C$ in the ${}^{g'}\!H$-fiber has unmonitored neighbours in their corresponding G- and H-fibers. Hence the claim.

Suppose now that there exists a vertex x in H which is not adjacent to h. Let P be a path in H connecting h and x. Then there exist adjacent vertices p, q in P such that $p \in N(h)$ and $q \notin N(h)$, which is a contradiction to the claim proved above. Hence h is a universal vertex of H. Therefore the propagation occurs from every vertex in the ${}^{g}\!H$-fiber to their neighbouring H-fiber after the domination step of S. Further propagation is possible only if the neighbour of g in G has degree two. Continuing the same, we get that every vertex of G has degree at most two. Thus G is isomorphic to a path of order at least four with g as one of the pendant vertices.

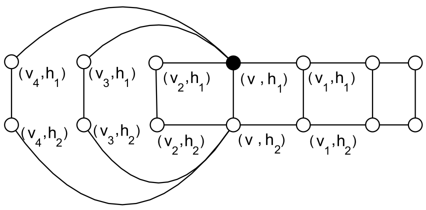
To prove the sufficiency part, assume that G is a path and h is a universal vertex of H. Then it is easy to observe that $\{(g, h)\}$ is a PDS of $G \,\square\, H$, where g is a pendant vertex of G. $\square$

In the previous theorem, we considered any general graphs of order at least four. We now consider the Cartesian products whose one of the factor graphs is a tree T and the other is K_2. We characterize graphs for which $\gamma_{\mathrm{P}}(T \,\square\, K_2) = 1$.

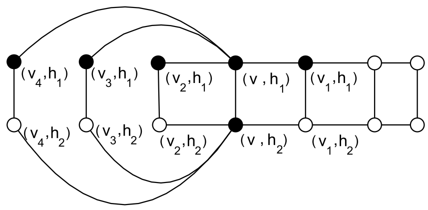
Lemma 4.1.8. *If a tree T is a tailed star, then $\gamma_{\mathrm{P}}(T \,\square\, K_2) = 1$.*

Proof. Let $V(K_2) = \{h_1, h_2\}$. If T is a star, then its universal vertex, say u, is the dominating set of T. Then, from the proof of Theorem 4.1.3, we get the set $\{(u, h_1)\}$ is a PDS of $T \,\square\, K_2$. Suppose now that T is not a star. Then, either T is a path of order at least four or T has a unique vertex having degree at least three. In the former case, by Theorem 4.1.3, $\gamma_{\mathrm{P}}(T \,\square\, K_2) = 1$ and in the latter case, let v be the unique vertex of T of degree three or more. Let $v_1, \ldots, v_m$ be the neighbours of v. Then exactly one neighbour, say v_1, has degree two and all the other neighbours of v have degree one in T. Take $S = \{(v, h_1)\}$. Then $\mathcal{P}^0_{T \,\square\, K_2,1}(S)$ contains the set of vertices $\{(v, h_1), (v, h_2)\} \cup \{(v_i, h_1) \colon 1 \le i \le m\}$. For all $i, i \neq 1$, since v_i is a pendant vertex, the vertex (v_i, h_1) can monitor their only unmonitored neighbour (v_i, h_2) in the $K_2^{v_i}$-fiber by propagation. Consequently, the vertex (v, h_2) has exactly one unmonitored neighbour (v_1, h_2) which gets monitored in the second propagation step. Let u be the other neighbour of v_1. Since $d_T(v_1) = 2$, the vertices (v_1, h_1) and (v_1, h_2) can now monitor their only unmonitored neighbour in their corresponding T-fiber, namely (u, h_1) and (u, h_2), respectively. By definition of a tailed star, the subtree induced by the vertex set $\{t \in V(T) \colon d_T(t, v_1) < d_T(t, v)\}$ is a path. Hence, in a similar fashion, the propagation continues through the path in T-fibers and thus the set S monitors the entire graph. $\square$

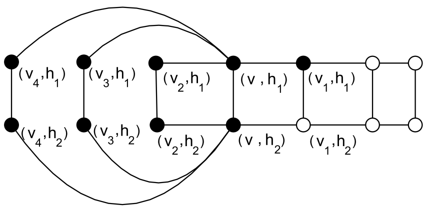
In Figure 4.3, the monitored vertices at each step of propagation are drawn as black vertices. Starting with the initial set $S = \{(v, h_1)\}$, we can observe that all the vertices of the graph



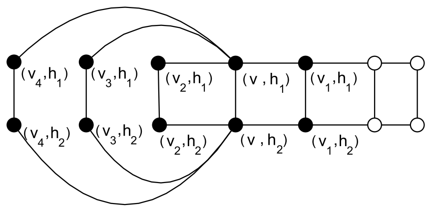
(a) $S = \{(v, h_1)\}$



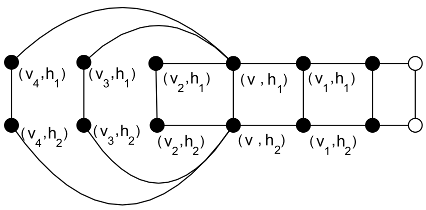
(b) $\mathcal{P}_1^0(S)$



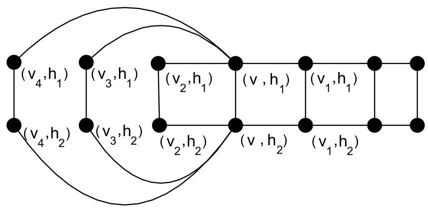
(c) $\mathcal{P}_1^1(S)$



(d) $\mathcal{P}_1^2(S)$



(e) $\mathcal{P}_1^3(S)$



(f) $\mathcal{P}_1^4(S)$

Figure 4.3: Monitoring of vertices in the graph $T \,\square\, K_2$.

$T \,\square\, K_2$ get monitored by step 4 and hence the lemma holds.

Theorem 4.1.9. *For any tree T of order at least two,* $\gamma_{\mathrm{P}}(T \,\square\, K_2) = 1$ *if and only if T is a tailed star.*

Proof. Let $V(K_2) = \{h_1, h_2\}$. Suppose that $\gamma_{\mathrm{P}}(T \,\square\, K_2) = 1$. Let $S = \{(v, h_1)\}$ be a PDS of $T \,\square\, K_2$ for some $v \in V(T)$.

If $|V((T)| \leq 3$, then clearly the result holds. Therefore, let $|V((T)| \geq 4$.

Assume first that $d_T(v) = 1$. Let v_1 be the neighbour of v. Then the vertices dominated by S are $(v, h_1), (v, h_2)$ and (v_1, h_1). Since $d_T(v) = 1$, (v, h_2) monitors (v_1, h_2) in the first propagation step. If $d_T(v_1) \geq 3$, then the vertices (v_1, h_1) and (v_1, h_2) have at least two unmonitored neighbours in their corresponding T-fiber, which prevents the further monitoring of vertices. Therefore $d_T(v_1) = 2$. Let v_2 be the other neighbour of v_1. Arguing the same as in the case of v_1, we can prove that $d_T(v_2) = 2$. Proceeding with the same argument, we finally get that every vertex of T has degree at most two. This implies that T is a path, which is a tailed star.

Assume now that $d_T(v) > 1$. Let $v_1, \ldots, v_m$ be the neighbours of $v, m \geq 2$. If all the neighbours of v are pendant vertices, then we get that T is isomorphic to a star with the universal vertex v and clearly the result holds. Therefore we can assume that v has at least one neighbour which is not a pendant vertex. Let r be the number of neighbours of v that are not pendant vertices. If possible, assume that $r \geq 2$. Let $v_{j_1}, \ldots, v_{j_r}$ be such neighbours of v. We have $\mathcal{P}^0_{T\,\Box\,K_2,1}(S) = \{(v, h_1), (v, h_2)\} \cup \{(v_i, h_1)\colon 1 \le i \le m\}$. Each vertex $u \in N_T(v)\backslash\{v_{j_1}, \ldots, v_{j_r}\}$, has degree one in T and therefore the (u, h_1) can monitor its single unmonitored neighbour (u, h_2) in the first propagation step. For each $i, 1 \le i \le r$, we have $d_T(v_{j_i}) \ge 2$ and therefore the dominated vertex (v_{j_i}, h_1) has at least one unmonitored neighbour in its T-fiber and an unmonitored neighbour (v_{j_i}, h_2) in its K_2-fiber. This prevents the monitoring of vertices from any such dominated vertices. This implies that the dominated vertex (v, h_2) has the set of vertices $\{(v_{j_i}, h_2)\colon 1 \le i \le r\}$ as its unmonitored neighbours in its T-fiber. Since $r \ge 2$, (v, h_2) has at least two unmonitored neighbours in its T-fiber and therefore no possible propagation from any of the monitored vertices. This is a contradiction, since S is a PDS of $T\,\Box\,K_2$. Hence $r = 1$, which means that v has exactly one neighbour, say v_1, which is not a pendant vertex of T. Let us assume that v_1 has at least two neighbours, w_1 and w_2, other than v in T. For each $i \neq 1$,$1 \le i \le m$, since the vertex v_i is pendant, the vertex (v_i, h_1) can monitor (v_i, h_2) by propagation. Consequently, the vertex (v, h_2) can monitor (v_1, h_2) in the second propagation step. But, again the vertex (v_1, h_2) has (w_1, h_2) and (w_2, h_2) as unmonitored neighbours. Similarly, the case with (v_1, h_1). This blocks the further propagation and therefore v_1 has exactly one neighbour, w_1, other than v. By arguing the same as above for the vertex w_1, we get that degree of w_1 is at most two. Continuing with this argument if needed, we can conclude that the subgraph induced by the vertices $\{t \in V(T)\colon d_T(t, v_1) < d_T(t, v)\}$ is a path with v_1 as one of its pendant vertices. Thus we proved that T is indeed a tailed star. The sufficiency part follows from Lemma 4.1.8. $\square$

From Theorems 1.4.3 (a) and 4.1.9, we get the following corollary.

Corollary 4.1.10. *For any spider T, $\gamma_{\mathrm{P}}(T \,\square\, K_2) = \gamma_{\mathrm{P}}(T)$ if and only if T is a tailed star.*

4.2 The Direct Product

Upper bounds for the domination number of the direct products are studied in [16]. We obtain some sharp upper bounds for the power domination number of direct products under the condition that one of the factor graphs has a universal vertex.

Let D be a total dominating set of a graph G. As any total dominating set is a PDS, let $\gamma_{\mathrm{P_D}}(G)$ denote the least cardinality of a subset S of D such that S is a PDS of G. Note that $\gamma_{\mathrm{P_D}}(G) \leq |D|$ and hence $\gamma_{\mathrm{P_D}}(G)$ is well-defined.

Theorem 4.2.1. *Let G be a graph without isolated vertices and H be a nontrivial graph with a universal vertex h. Then $\gamma_{\mathrm{P}}(G \times H) \leq \min\{|D| + \gamma_{\mathrm{P_D}}(G)\}$, where the minimum is taken over all total dominating sets D of G. If G has a $\gamma_t(G)$-set D' which is also its zero forcing set, then $D' \times \{h\}$ is a PDS of $G \times H$ and $\gamma_{\mathrm{P}}(G \times H) \leq \gamma_t(G)$.*

Proof. Let S be a PDS of G with cardinality $\gamma_{\mathrm{P_D}}(G)$ such that $S \subseteq D$. We prove that the set S' given by $S' = (D \times \{h\}) \cup (S \times \{h'\})$, for some $h' \in V(H), h' \neq h$, is a PDS of $G \times H$. Since h is a universal vertex of H and D is a total dominating set of graph G with no isolated vertices, $\mathcal{P}^0_{G\times H,1}(S')$ contains all the vertices of $V(G^v), v \neq h$. Therefore, only those vertices in the G^h-fiber that are not in $(D \times \{h\}) \cup N_{G\times H}(S')$ are yet to be monitored.

We now prove that for a vertex g in G, if $g \in \mathcal{P}^i_{G,1}(S)$, then

$(g,h) \in \mathcal{P}^i_{G\times H,1}(S')$ for all $i \geq 0$. The proof is by induction. Let g be a vertex in $\mathcal{P}^0_{G,1}(S)$. If $g \in S$, then, since $S \subseteq D$, $(g,h) \in S'$. Otherwise, g is adjacent to some g' in S. Then, by definition of S', the vertex (g',h') in S' dominates the vertex (g,h) and therefore $(g,h) \in \mathcal{P}^0_{G\times H,1}(S')$. The property holds for $i = 0$. Suppose that it is true for some $i \geq 0$. Let g be a vertex in $\mathcal{P}^{i+1}_{G,1}(S)$. If $g \in \mathcal{P}^i_{G,1}(S)$, then by the induction hypothesis, $(g,h) \in \mathcal{P}^i_{G\times H,1}(S')$. Otherwise, there exists a neighbour g' of g in $\mathcal{P}^i_{G,1}(S)$ such that $|N_G[g'] \setminus \mathcal{P}^i_{G,1}(S)| \leq 1$. For any $h'' \neq h$, we have $(g',h'') \in \mathcal{P}^0_{G\times H,1}(S')$. Therefore,

$$
\begin{aligned}
&|N_{G\times H}[(g',h'')] \setminus \mathcal{P}^i_{G\times H,1}(S')|\\
&= |\{(u,v)\colon u \in N_G(g'), v \in N_H(h''), (u,v) \notin \mathcal{P}^i_{G\times H,1}(S')\}|\\
&= |\{(u,h)\colon u \in N_G(g'), (u,h) \notin \mathcal{P}^i_{G\times H,1}(S')\}|\\
&= |\{u\colon u \in N_G(g'), u \notin \mathcal{P}^i_{G,1}(S)\}| \text{(by the induction hypothesis)}\\
&= |N_G[g'] \setminus \mathcal{P}^i_{G,1}(S)|\\
&\leq 1.
\end{aligned}
$$

Hence $N_{G\times H}[(g',h'')] \subseteq \mathcal{P}^{i+1}_{G\times H,1}(S')$ and $(g,h) \in \mathcal{P}^{i+1}_{G\times H,1}(S')$. Then, since S is a PDS of G, we get that S' is a PDS of $G \times H$.

If G has a $\gamma_t(G)$-set D' which is also its zero forcing set, then take $S' = D' \times \{h\}$. Then the dominated set, $\mathcal{P}^0_{G\times H,1}(S')$ in $G\times H$ is given by $V(G\times H)\setminus\{(g,h)\colon g \notin D'\}$. Therefore, only

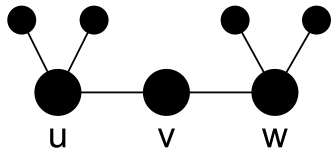


Figure 4.4: The graph G.

those vertices in the G^h-fiber that are not in S' are yet to be monitored. For a vertex g in G, if $g \in \mathcal{B}_G^0(D')$, then $(g,h) \in S'$. Let g be a vertex in $\mathcal{B}_G^1(D)$. If $g \notin D'$, then, since D' is a zero forcing set of G, there exists a neighbour g' of g in D' such that all the neighbours of g' except g are in D'. Therefore for any $h' \neq h$ in H, the vertex (g',h') in $\mathcal{P}^0_{G\times H,1}(S')$ has (g,h) as the single unmonitored neighbour and hence (g,h) belongs to $\mathcal{P}^1_{G\times H,1}(S')$. Now one can prove by induction that if $g \in \mathcal{B}_G^i(D')$, then $(g,h) \in \mathcal{P}^i_{G\times H,1}(S')$ for all $i \geq 0$. Since D' is a zero forcing set of G, this property implies that S' is a PDS of $G \times H$ and $\gamma_{\mathrm{P}}(G \times H) \leq |S'| = \gamma_t(G)$. $\square$

We now give examples of graphs G for which the bounds in Theorem 4.2.1 is sharp. For the graph G in Figure 4.4, $\gamma_t(G) = 3, \gamma_{\mathrm{P_D}}(G) = 2$ for the total dominating set $D = \{u, v, w\}$ and for $H = K_{1,4}$, we get that $\gamma_{\mathrm{P}}(G \times H) = 5 = \gamma_t(G) + \gamma_{\mathrm{P_D}}(G)$.

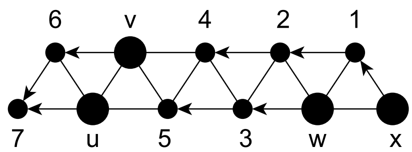


Figure 4.5: The graph G'.

For cycles, C_n, any $\gamma_t(C_n)$-set is its zero forcing set. It is obtained in Theorem 1.4.7 (e) that $\gamma_{\mathrm{P}}(K_m \times C_n) = \gamma_t(C_n)$ for $m \geq 3, n \geq 4$. Thus the bound is sharp. The graph G' given in Figure 4.5 has a $\gamma_t(G')$-set $D = \{u, v, w, x\}$ which is also its zero forcing set. Also, one can observe that if we colour the vertices of D black and the remaining vertices of G' white, then by applying the colour-change rule, the vertices that are marked i will receive the colour black in the i^{th} step for all $i, 1 \leq i \leq 7$ and if H is a graph of order at least three and with a universal vertex h, then the H-fiber ${}^iH$ is monitored by the set $D \times \{h\}$ in the i^{th} propagation step in $G' \times H$. As D is a zero forcing set of G', we get that $D \times \{h\}$ is a PDS of $G' \times H$ (as explained in the proof of Theorem 4.2.1). (The arrow mark in Figure 4.5 indicates the direction in which the propagation occurs.)

In Theorem 4.2.1, we have assumed that G has no isolated vertices. If G contains p isolated vertices, then let G' be the subgraph of G induced by the nonisolated vertices. Then $G = G' \cup p\, K_1$ and $G \times H$ is the disjoint union of $G' \times H$ and $p.|V(H)|$ isolated vertices. Consequently, $\gamma_{\mathrm{P}}(G \times H) = \gamma_{\mathrm{P}}(G' \times H) + p\,|V(H)|$.

We now give an upper bound for $\gamma_{\mathrm{P}}(G \times K_2)$ in terms of $\gamma_{\mathrm{P}}(G)$.

Theorem 4.2.2. *Let G be a nontrivial graph. Then $\gamma_{\mathrm{P}}(G \times K_2) \leq 2\gamma_{\mathrm{P}}(G)$. The equality holds if G is a bipartite graph.*

Proof. Let S be a PDS of G. Let $S' = (S \times \{h_1\}) \cup (S \times \{h_2\})$, where $h_1, h_2 \in V(K_2)$. We prove that S' is a PDS of $G \times K_2$. For that we prove that for all $i \geq 0$ and a vertex g in G, if $g \in \mathcal{P}^i_{G,1}(S)$, then both (g, h_1) and (g, h_2) belong to $\mathcal{P}^i_{G\times K_2,1}(S')$. If $g \in S$, then clearly $(g, h_1), (g, h_2) \in S'$. If g is adjacent to some g' in S, then the vertices (g', h_1) and (g', h_2) dominate (g, h_2) and (g, h_1), respectively. Assume now that the property holds for some $i \geq 0$. Let g be a vertex in $\mathcal{P}^{i+1}_{G,1}(S)$. If g is not in $\mathcal{P}^i_{G,1}(S)$, then there exists a neighbour g' of g in $\mathcal{P}^i_{G,1}(S)$ such that $|N_G[g'] \setminus \mathcal{P}^i_{G,1}(S)| \leq 1$. Hence (g', h_1) and (g', h_2) are in $\mathcal{P}^i_{G\times K_2,1}(S')$ and $|N_{G\times H}[(g', h_1)] \setminus \mathcal{P}^i_{G\times K_2,1}(S')| = |\{(u, h_2)\colon u \in N_G(g'), (u, h_2) \notin$

$\mathcal{P}^i_{G\times K_2,1}(S')\}| = |N_G[g'] \setminus \mathcal{P}^i_{G,1}(S)| \leq 1$. Therefore, $(g,h_2) \in \mathcal{P}^{i+1}_{G\times K_2,1}(S')$. Similarly, we get that $(g,h_1) \in \mathcal{P}^{i+1}_{G\times K_2,1}(S')$.

If G is a bipartite graph, then $G \times K_2$ consists of two copies of G, hence the equality clearly holds. □

4.3 The Lexicographic Product

The power domination number of the lexicographic product is determined in [26]. In this section, we compute the k-power domination number of the lexicographic product.

For each positive integer ℓ, let $\mathcal{A}_\ell$ be the family of all nontrivial graphs F such that $\gamma_{\mathrm{P},\ell}(F) = 1$. And, let $\mathcal{B}_\ell$ be the family of all disconnected graphs F such that $F = F_1 \cup \ldots \cup F_r, 2 \leq r \leq \ell+1$, where each F_i is a component of F, having the property that F_1 is a connected nontrivial graph with $\gamma_{\mathrm{P},\ell}(F_1) = 1$ and sum of the order of the remaining components of F is at most ℓ, i.e. $1 \leq |V(F_2)| + \ldots + |V(F_r)| \leq \ell$. Denote $\mathcal{F}_\ell$ by $\mathcal{A}_\ell \cup \mathcal{B}_\ell$.

Theorem 4.3.1. *Let G be a nontrivial graph without isolated vertices. For any nontrivial graph H and $1 \leq k \leq |V(H)| - 1$,*

$$\gamma_{\mathrm{P},k}(G \circ H) = \begin{cases} \gamma(G), & H \in \mathcal{F}_k; \\ \gamma_t(G), & H \notin \mathcal{F}_k. \end{cases}$$

Proof. We first consider the case when H is in $\mathcal{F}_k$. Suppose that $H \in \mathcal{A}_k$. Then, by definition of $\mathcal{A}_k$, $\gamma_{\mathrm{P},k}(H) = 1$. Let $\{h\}$ be a k-PDS of H and D be a dominating set of G. Then we prove that $D \times \{h\}$ is a k-PDS of $G \circ H$. For a vertex g of G, if $g \notin D$, then any vertex of ${}^gH$ is in the neighbourhood of (g', h), for some $g' \in D$ with $gg' \in E(G)$. If $g \in D$, then any neighbour of a vertex of ${}^gH$ not in ${}^gH$ is dominated and also the set $\{(g, h') \colon h' \in N_H[h]\}$ is dominated. Therefore, since $\{h\}$ is a k-PDS of H, the fiber ${}^gH$ is monitored. Suppose now that $H \in \mathcal{B}_k$. Then, $H = H_1 \cup \ldots \cup H_r$, $2 \leq r \leq k+1$, $H_1, \ldots, H_r$ being the components of H such that $\gamma_{\mathrm{P},k}(H_1) = 1$ and $1 \leq |V(H_2 \cup \ldots \cup H_r)| \leq k$. By Theorem 1.4.1, there exists a $\gamma(G)$-set D of G such that every vertex $u \in D$ has a neighbour $v \in V(G) \setminus D$ such that $N[v] \cap D = \{u\}$. We can call v as the *private neighbour* of u with respect to D. We prove that $D \times \{h\}$ is a k-PDS of $G \circ H$, for a k-PDS $\{h\}$ of H_1. For a vertex g of G, the fiber ${}^gH$ is dominated if $g \notin D$. Assume that $g \in D$. Clearly, every neighbouring H-fibers of ${}^gH$ is dominated.

Therefore, since $\{h\}$ is a k-PDS of H_1, the vertices in the set $\{g\} \times V(H_1)$ is monitored. Let g' be a private neighbour of g in G with respect to D. Then for any $u \in N_G[g'], u \neq g$, we get u is not in D and hence the fiber ${}^{u}H$ is dominated. Therefore the set of unmonitored neighbours of any vertex of ${}^{g'}H$ is given by $\{(g, h')\colon h' \in V(H_2 \cup \ldots \cup H_r)\}$. Since $|V(H_2 \cup \ldots \cup H_r)| \leq k$, the fiber ${}^{g}H$ is monitored. Hence $\gamma_{\mathrm{P},k}(G \circ H) \leq \gamma(G)$.

Assume that $G \circ H$ has a k-PDS S with $|S| < \gamma(G)$. Then there exists an H-fiber ${}^{g}H$ that contains no vertex of $N[S]$. Therefore the vertices of ${}^{g}H$ are monitored by propagation. But every vertex in $V(G \circ H) \setminus V({}^{g}H)$ has either 0 or $|V(H)|$ neighbours in ${}^{g}H$-fiber and therefore, since $|V(H)| \geq 2$ and $1 \leq k \leq |V(H)| - 1$, there can be no propagation in ${}^{g}H$. Hence $\gamma_{\mathrm{P},k}(G \circ H) \geq \gamma(G)$.

Suppose now that H is not in $\mathcal{F}_k$. Let D be total dominating set of G. Then for any h of H, $D \times \{h\}$ is a dominating set of $G \circ H$ and hence a k-PDS of $G \circ H$. Thus $\gamma_{\mathrm{P},k}(G \circ H) \leq \gamma_t(G)$.

Let S be a $\gamma_{\mathrm{P},k}(G \circ H)$-set of $G \circ H$. Suppose that there is an H-fiber ${}^{g}H$ that contains at least two vertices of S. Let S' be obtained by removing from S all vertices of ${}^{g}H$ but one and adding an arbitrary vertex of a neighbouring H-fiber (if there is none yet). Then $N[S] \subseteq N[S']$ and hence S' is a k-PDS of $G \circ H$ with $|S'| \leq |S|$. Repeating this process if necessary, we may now assume that every H-fiber of $G \circ H$ contains at most one vertex in S.

Suppose that there exists an H-fiber ${}^gH$ such that for any neighbour g' of g in G, $V({}^{g'}H) \cap S$ is empty. We know that each vertex in ${}^{g'}H$ is adjacent to all the vertices of ${}^gH$. But, since $1 \leq k \leq |V(H)| - 1$, at least $|V(H)| - k$ vertices of ${}^gH$ have to be monitored by the vertices in $V({}^gH) \cap S$ so that the remaining at most $k(\leq |V(H)| - 1)$ unmonitored vertices of ${}^gH$ can be monitored by propagation from any monitored vertex of its neighbouring H-fiber. Also, ${}^gH$ contains at most one vertex in S. Therefore, S contains a vertex (g, h) in ${}^gH$ such that (g, h) monitors at least $|V(H)| - k$ vertices of ${}^gH$. If H is connected, then this implies that $\{h\}$ is a k-PDS of H and hence $H \in \mathcal{A}_k$. This is a contradiction since $H \notin \mathcal{F}_k$. Therefore assume that H is not connected. Let $H_1, \ldots, H_r, r \geq 2$ be the components of H. Let H_1 be the component that contains the vertex h. Therefore we get that (g, h) monitors at least $|V(H)| - k$ vertices of the fiber ${}^gH_1$ and $|V(H_1)| \geq |V(H)| - k$. Since H is disconnected, we have $|V(H_1)| \leq |V(H)|-1$. Therefore the fiber ${}^{g}H_1$ has at most $k-1$ more unmonitored neighbours, which will eventually get monitored as H_1 is connected. This implies that every vertex of H_1 is monitored by the vertex h and thus $\gamma_{\mathrm{P},k}(H_1) = 1$. Also, we get $1 \leq |V(H_2 \cup \ldots \cup H_r)| \leq k$. Hence H is in $\mathcal{B}_k$, which is again a contradiction. Therefore the set $\{g' \in V(G)\colon V({}^{g'}H) \cap S \neq \emptyset\}$ is a total dominating set of G and we conclude that $\gamma_{\mathrm{P},k}(G \circ H) = |S| \geq \gamma_t(G)$. □

Theorem 4.3.2. *Let G be a nontrivial graph without isolated vertices and H be a connected nontrivial graph. If $k \geq |V(H)|$, then $\gamma_{\mathrm{P},k}(G \circ H) = \gamma_{\mathrm{P},\left\lfloor \frac{\mathrm{k}}{|\mathrm{V(H)}|} \right\rfloor}(G)$.*

Proof. Let $\ell = \left\lfloor \frac{k}{|V(H)|} \right\rfloor$. We first prove that $\gamma_{\mathrm{P},k}(G \circ H) \leq \gamma_{\mathrm{P},\ell}(G)$. Let S be a minimum ℓ-PDS of G. Take $S' = S \times \{h\}$ for some $h \in V(H)$. Let u be a vertex in S. Then any neighbour of a vertex of ${}^{u}H$ not in ${}^{u}H$ is dominated and also the vertex (u,h) dominates all its neighbours in its H-fiber. Since H is connected and $k \geq |V(H)|$, $\{h\}$ is a k-PDS of H and therefore once all the neighbouring H-fibers of ${}^{u}H$ are dominated, the fiber ${}^{u}H$ is monitored by propagation. For a vertex u in S, let j be the smallest integer such that $V({}^{u}H) \in \mathcal{P}^{j}_{G \circ H,k}(S')$.

We now prove that if g is a vertex in $\mathcal{P}^{i}_{G,\ell}(S)$, then $V({}^{g}H) \in \mathcal{P}^{j+i}_{G\circ H,k}(S')$ for all $i \geq 0$. Let g be a vertex in $\mathcal{P}^{0}_{G,\ell}(S)$. If g is in S, then by definition of j, $V({}^{g}H)$ is contained in $\mathcal{P}^{j}_{G\circ H,k}(S')$. If g is not in S, then the vertices of the fiber ${}^{g}H$ are in the neighbourhood of (g', h) for some vertex g' in S with $gg' \in E(G)$ and any vertex of ${}^{g}H$ is dominated. Hence the property is true for $i = 0$. Let g be a vertex in $\mathcal{P}^{1}_{G,\ell}(S)$. If g is not in $\mathcal{P}^{0}_{G,\ell}(S)$, then there exists some neighbour g' of g in $\mathcal{P}^{0}_{G,\ell}(S)$ such that $|N_G[g'] \setminus \mathcal{P}^{0}_{G,\ell}(S)| \leq \ell$. Since the property is true for $i = 0$, we get that $V({}^{g'}H) \in \mathcal{P}^{j}_{G\circ H,k}(S')$. Then for any $h' \in V(H)$, the vertex (g', h') is in $\mathcal{P}^{j}_{G\circ H,k}(S')$ and it has at most $|V(H)|.\ell$ $(\leq k)$ unmonitored neighbours in $G \circ H$ after the step j. Therefore all the neighbouring H-fibers of (g', h') are monitored by propagation in the $(j+1)^{\text{th}}$ step and $N_{G\circ H}[(g', h')] \subseteq \mathcal{P}^{j+1}_{G\circ H,k}(S')$. Hence $V({}^{g}H) \in \mathcal{P}^{j+1}_{G\circ H,k}(S')$. Therefore the property holds for $i = 1$. Similarly the property can be proved for $i \geq 2$. Thus the propagation in $G \circ H$ continues in a similar manner and since S is a ℓ-PDS of G, S' is a k-PDS of $G \circ H$.

To prove the lower bound, let S' be a k-PDS of $G \circ H$. Let $S'_G = \{g\colon (g, v) \in S' \text{ for some } v \in V(H)\}$. For any vertex (g, h) in $\mathcal{P}^{0}_{G\circ H,k}(S')$, clearly $g \in \mathcal{P}^{0}_{G,\ell}(S'_G)$. Let (g, h) be a vertex in $\mathcal{P}^1_{G\circ H,k}(S')$. If $(g,h) \notin \mathcal{P}^0_{G\circ H,k}(S')$, then there exists some neighbour (g',h') of (g,h) in $\mathcal{P}^0_{G\circ H,k}(S')$ such that $|N_{G\circ H}[(g',h')] \setminus \mathcal{P}^0_{G\circ H,k}(S')| \leq k$. We know that for any vertex (u,v) in $G\circ H$, if $uu' \in E(G)$, then (u,v) has $|V(H)|$ neighbours in ${}^{u'}H$. Let r be the number of neighbours (u,v) of (g',h') with $V({}^{u}H) \cap \mathcal{P}^0_{G\circ H,k}(S') = \emptyset$ and that get monitored by (g',h') at the first propagation step of S'. Then $r = m.|V(H)|$ for some nonnegative integer m. Indeed, m is the number of unmonitored neighbours of g' after the domination step of S'_G in G. Also $r \leq k$ and thus $m \leq \ell$. Therefore the vertex g' in $\mathcal{P}^0_{G,\ell}(S'_G)$ monitors its m unmonitored neighbours at the first propagation step of S'_G and hence $N_G[g'] \subseteq \mathcal{P}^1_{G,\ell}(S'_G)$ and $g \in \mathcal{P}^1_{G,\ell}(S'_G)$. In a similar fashion, the propagation occurs in G and we get that if $(g,h) \in \mathcal{P}^i_{G\circ H,k}(S')$, then $g \in \mathcal{P}^i_{G,\ell}(S'_G)$ for all $i \geq 0$. Hence S'_G is a $\ell-$PDS of G. $\square$

Remark 4.3.1. If G contains p isolated vertices, then $G\circ H$ is the disjoint union of $G'\circ H$ and p copies of H, where G' is the subgraph of G induced by the nonisolated vertices of G. Hence, $\gamma_{\mathrm{P},k}(G\circ H) = \gamma_{\mathrm{P},k}(G'\circ H) + p\,\gamma_{\mathrm{P},k}(H)$. Also, from the definitions of strong and lexicographic products, it follows that $G \boxtimes K_m \cong G\circ K_m$ and therefore $\gamma_{\mathrm{P},k}(G\boxtimes K_m) = \gamma_{\mathrm{P},k}(G\circ K_m), m \geq 1$.

Chapter 5

Power domination in some classes of graphs

The power domination number of various classes of graphs has been determined using a two-step process: Finding an upper bound and a lower bound. The upper bound is usually obtained by providing a pattern to construct a set, together with a proof

Some results of this chapter are included in the following papers.
1. Seethu Varghese, Andreas M. Hinz, A. Vijayakumar, Power domination in Knödel graphs and Hanoi graphs, Discuss. Math. Graph Theory (to appear).
2. Seethu Varghese, A. Vijayakumar, Generalized power domination in WK-Pyramid Networks, Bull. Inst. Combin. Appl. (to appear).

that constructed set is a PDS. The lower bound is usually found by exploiting the structural properties of the particular class of graphs. Not many exact values of $\gamma_{\mathrm{P},k}$ for special graph classes are known. In this chapter, we determine the power domination number of 3-regular Knödel graphs and provide an upper bound for $\gamma_{\mathrm{P}}(W_{r+1,2^{r+1}}), r \geq 3$. We compute $\gamma_{\mathrm{P},k}$ and $\mathrm{rad}_{\mathrm{P},k}$ of H_p^2. We also study $\gamma_{\mathrm{P},k}$ of $WKP_{(C,L)}$ and this is the first network class with the pyramid structure for which the k-power domination number is studied.

5.1 Knödel graphs

In this section, we study the power domination number of Knödel graphs.

It is clear from Definition 1.2.20 that $W_{\Delta,2\nu}$ is bipartite. Also, $W_{\Delta,2\nu}$ is connected if and only if $\Delta \geq 2$, since in that case it suffices to alternate edges in dimension 0 and 1 to get a Hamiltonian cycle.

From Observation 1.4.9 (b), we get that $\gamma_{\mathrm{P},k}(W_{\Delta,2\nu}) = 1$ for

$\Delta \geq 2$ and $k \geq \Delta - 1$. Therefore it is interesting to study k-power domination number of $W_{\Delta,2\nu}$ for $k \leq \Delta - 2$.

For $\Delta = 1, W_{1,2\nu}$ consists of ν disjoint copies of K_2 and therefore $\gamma_{\mathrm{P}}(W_{1,2\nu}) = \nu$. For $\nu \in \mathbb{N}_2$ and $\Delta = 2, W_{2,2\nu}$ is a cycle on 2ν vertices and clearly $\gamma_{\mathrm{P}}(W_{2,2\nu}) = 1$. We have the following theorem for the case $\Delta = 3$, if $\nu \in \mathbb{N}_4$.

Theorem 5.1.1. *For $\nu \in \mathbb{N}_4$, $\gamma_{\mathrm{P}}(W_{3,2\nu}) = 2$.*

Proof. We prove that the set $S = \{(1,0),(2,2)\}$ is a PDS of $W_{3,2\nu}$. Then the set of dominated vertices is given by $\mathcal{P}_1^0(S) = \{(i,j)\colon i \in [2], j \in [3]_0\} \cup \{(1,\nu-1),(2,3)\}$. For $\nu = 4$, S is a dominating set of $W_{3,8}$ and for $\nu = 5,6$, we can easily observe that all vertices of $W_{3,2\nu}$ get monitored after the first propagation step and therefore S is a PDS. Let $\nu \in \mathbb{N}_7$. Depending on whether ν is odd or even, we write $\nu = 2m-1$ or $\nu = 2m$, $m \in \mathbb{N}_4$, respectively. Then for $i \in [m-3]$,

$$\begin{aligned}\mathcal{P}_1^i(S) =& \big(\{(1,j)\colon j \in [i+3]_0\} \cup \{(1,\nu-j)\colon j \in [i+2]\}\big)\\ &\cup \big(\{(2,j)\colon j \in [i+5]_0\} \cup \{(2,\nu-j)\colon j \in [i]\}\big)\,.\end{aligned}$$

We get that $\mathcal{P}_1^{m-3}(S) = V(W_{3,2\nu})$, if ν is odd, and $\mathcal{P}_1^{m-2}(S) =$

$\mathcal{P}_1^{m-3}(S) \cup \{(1,m),(2,m+2)\} = V(W_{3,2\nu})$, if ν is even. Hence, in both cases we see that every vertex of $W_{3,2\nu}$ gets monitored after step $\left\lfloor \frac{\nu}{2} \right\rfloor - 2$ and therefore S is a PDS of $W_{3,2\nu}$.

To prove that $\gamma_{\mathrm{P}}(W_{3,2\nu}) \geq 2$, let us assume that $\{v\}$ is a PDS of $W_{3,2\nu}$. Then, since $W_{3,2\nu}$ is bipartite, after the domination step, each of the neighbours of v has exactly two unmonitored neighbours which prevents the further propagation. Hence $\gamma_{\mathrm{P}}(W_{3,2\nu}) = 2$. $\square$

We now focus on the family of Knödel graphs $W_{r+1,2^{r+1}}$. In the next theorem, we prove that the power domination number of $W_{r+1,2^{r+1}}$ is at most 2^{r-2}. For that, we construct a PDS of cardinality 2^{r-2} in $W_{r+1,2^{r+1}}$. One can easily check that $S' = \{(1,1),(2,6)\}$ is a PDS of $W_{4,16}$. It is proved in [30] that $W_{r+1,2^{r+1}}$ can be constructed by taking two copies of $W_{r,2^r}$ and linking the vertices of each copy by a certain perfect matching. Therefore, in order to construct a PDS for $W_{5,32}$, we take two copies of the set S', each from a copy of $W_{4,16}$ that lies in $W_{5,32}$ and then prove that the new set is a PDS of $W_{5,32}$. We now extend the same idea to construct a PDS of $W_{r+1,2^{r+1}}$ for larger values of r. In the proof of the following theorem, we first produce a set S and then give the set of vertices that are dominated given by $\mathcal{P}_1^0(S)$. After that we give the elements in the set $\mathcal{P}_1^1(S)$and $\mathcal{P}_1^2(S)$, the sets of vertices that get monitored at the first and second propagation step, respectively. We obtain that the entire graph get monitored in two propagation steps and thus S is a PDS of $W_{r+1,2^{r+1}}$.

Theorem 5.1.2. *For* $r \in \mathbb{N}_3, \gamma_{\mathrm{P}}(W_{r+1,2^{r+1}}) \leq 2^{r-2}$.

Proof. Let $\nu = 2^r$ and $S = \{(1, 2^{r-3} + j), (2, 7 \cdot 2^{r-3} - 1 + j) \colon j \in [2^{r-3}]_0\}$. Then

$$\begin{aligned} \mathcal{P}_1^0(S) = S \cup \{&(1, 7 \cdot 2^{r-3} + j - 2^{\ell} \, (\mathrm{mod}\, \nu)), \\ &(2, 2^{r-3} + j + 2^{\ell} - 1 \, (\mathrm{mod}\, \nu)) \ : \\ &j \in [2^{r-3}]_0, \ \ell = r-3, r-2, r-1, r\}. \end{aligned}$$

For $r = 3$, the vertex $(1, 2j + 1)$ monitors $(2, 2j + 1)$ for every $j \in [3]$ and the vertex $(2, 2j)$ monitors $(1, 2j)$ for every $j \in [3]_0$. Thus we get $\mathcal{P}_1^1(S) = V(W_{4,16})$. Assume now that $r \in \mathbb{N}_4$. Then, for each j and ℓ, where $j \in [2^{r-4}]_0$, $\ell = r-2, r-1, r$, the vertices in the set $\{(1, 7 \cdot 2^{r-3} + j - 2^{\ell} \, (\mathrm{mod}\, \nu))\}$ monitor the vertices in the set $\{(2, 8 \cdot 2^{r-3} + j - 2^{\ell} - 1 \, (\mathrm{mod}\, \nu))\}$ by propagation. Also, for each j and ℓ, where $2^{r-4} \leq j \leq 2^{r-3}-1$, $\ell = r-2, r-1, r$, the vertices in the set $\{(2, 2^{r-3}+j+2^{\ell}-1 \pmod{\nu})\}$ monitor the vertices in the set $\{(1, j+2^{\ell} \pmod{\nu})\}$ by propagation.

Hence the set of vertices monitored at step 1 is given by

$$\begin{aligned}\mathcal{P}_1^1(S) = &\{(1, j+2^{\ell} \pmod{\nu}) \colon 2^{r-4} \leq j \leq 2^{r-3}-1,\ \ell = r-2, r-1, r\} \\ &\cup \{(2, 8\cdot 2^{r-3}+j-2^{\ell}-1 \pmod{\nu}) \colon \\ &\quad j \in [2^{r-4}]_0,\ \ell = r-2, r-1, r\} \\ &\cup\ \mathcal{P}_1^0(S).\end{aligned}$$

Again following the propagation rule, for each j and ℓ, where $2^{r-4} \leq j \leq 2^{r-3}-1$, $\ell = r-2, r-1, r$, the vertices in the set $\{(1, 7\cdot 2^{r-3}+j-2^{\ell} \pmod{\nu})\}$ monitor the vertices in the set $\{(2, 8\cdot 2^{r-3}+j-2^{\ell}-1 \pmod{\nu})\}$. And, for each j and ℓ, where $j \in [2^{r-4}]_0$, $\ell = r-2, r-1, r$, the vertices in the set $\{(2, 2^{r-3}+j+2^{\ell}-1 \pmod{\nu})\}$ monitor the vertices in the set $\{(1, j+2^{\ell} \pmod{\nu})\}$ by propagation. Hence the set of vertices monitored at step 2 is given by

$$\begin{aligned}
\mathcal{P}_1^2(S) =& \{(1, j + 2^{\ell} \,(\mathrm{mod}\, \nu))\colon j \in [2^{r-4}]_0,\ \ell = r-2, r-1, r\} \\
& \cup \{(2, 8 \cdot 2^{r-3} + j - 2^{\ell} - 1 \,(\mathrm{mod}\, \nu))\colon \\
& \qquad 2^{r-4} \leq j \leq 2^{r-3} - 1,\ \ell = r-2, r-1, r\} \\
& \cup\ \mathcal{P}_1^1(S) \\
=& V(W_{r+1,2^{r+1}}).
\end{aligned}$$

Therefore every vertex of $W_{r+1,2^{r+1}}$ gets monitored after step 2 and hence S is a PDS of $W_{r+1,2^{r+1}}$ and $\gamma_{\mathrm{P}}(W_{r+1,2^{r+1}}) \leq |S| = 2^{r-2}$. □

For $r = 3$, any singleton set $\{v\}, v \in W_{4,16}$ cannot itself power dominate the entire graph, as each of the neighbours of v will have exactly three unmonitored neighbours after the domination step. Hence the bound in Theorem 5.1.2 is sharp for $r = 3$. We further illustrate Theorem 5.1.2 for the graph $W_{5,32}$. The vertices of the set S as defined in the theorem are coloured black in Figure 5.1. In Figure 5.2, $\mathcal{P}_1^0(S)$, the set of dominated vertices, are coloured black and the remaining vertices white. The black vertices in Figure 5.3 and Figure 5.4 represent the

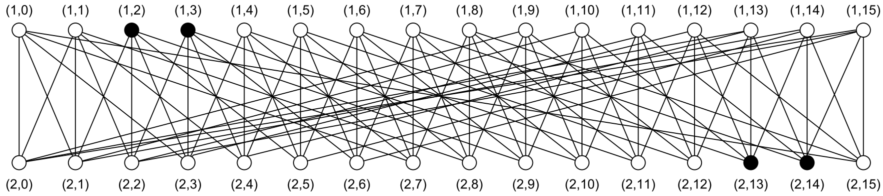


Figure 5.1: A power dominating set in the graph $W_{5,32}$.

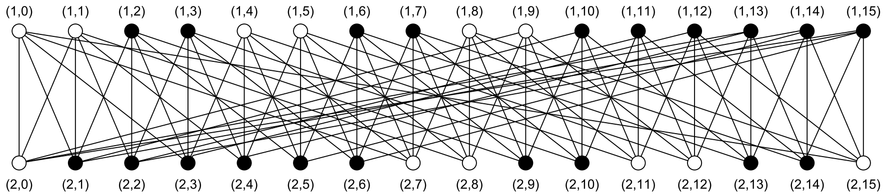


Figure 5.2: Neighbourhood is monitored.

vertices in the set $\mathcal{P}_1^1(S)$ and $\mathcal{P}_1^2(S)$, respectively. The directed edges in the figures indicate the direction in which the propagation occurs at each step. For instance, the directed edge $[(2,2),(1,1)]$ in Figure 5.3 indicates that $(2,2)$ monitors $(1,1)$ in the first propagation step. We observe that all the vertices get monitored by step 2 and therefore S is a PDS of $W_{5,32}$.

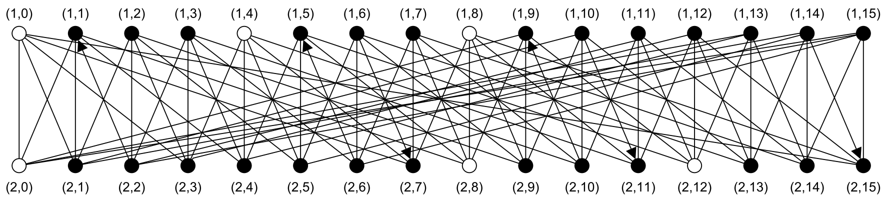


Figure 5.3: Propagation occurs.

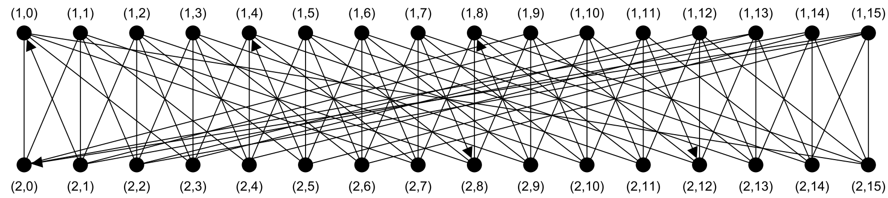


Figure 5.4: End of propagation.

5.2 Hanoi graphs

We get from Definition 1.2.21 that H_1^n is the graph K_1 for any $n \in \mathbb{N}_0$. For $n \in \mathbb{N}_1$, H_2^n is the disjoint union of 2^{n-1} copies of K_2, i.e. $H_2^n \cong W_{1,2^n}$.

In this section, we study the behaviour of power domination in H_p^2. The cases $p \in [2]$ are trivial with $\gamma_{\mathrm{P},k}(H_1^2) = \gamma_{\mathrm{P},k}(K_1) = 1$ and $\gamma_{\mathrm{P},k}(H_2^2) = 2 = \gamma_{\mathrm{P},k}(W_{1,4})$, respectively, for all k.

Recall that for $p \in \mathbb{N}_3$ and $n = 2$,
$V(H_p^2) = \{s_2 s_1 \colon s_1, s_2 \in [p]_0\}$ and
$E(H_p^2) = \big\{\{ri, rj\}, \{i\ell, j\ell\} \colon r, i, j \in [p]_0,\ i \neq j,\ \ell \in [p]_0 \backslash \{i, j\}\big\}$.

Vertices of the form ss are called the **extreme vertices** of H_p^2. Note that the extreme vertices are of degree $p-1$ and all the other vertices are of degree $2p-3$ in H_p^2. It is easy to observe that $\gamma(H_p^2) = p$. Indeed, any set containing a vertex from each of the p cliques in H_p^2 forms a dominating set of H_p^2. Since each of the p cliques contains an extreme vertex, any dominating set of H_p^2 must contain at least p vertices and hence $\gamma(H_p^2) = p$.

For $p = 3$, H_3^n is isomorphic to the Sierpiński graph, S_3^n, see [43, p.143 ff]. It is proved (refer Theorem 1.4.12) that

$$\gamma_{\mathrm{P},k}(S_3^n) = \begin{cases} 1, & n = 1 \text{ or } k \in \mathbb{N}_2; \\ 2, & n = 2 \text{ and } k = 1; \\ 3^{n-2}, & n \in \mathbb{N}_3 \text{ and } k = 1. \end{cases}$$

Therefore $\gamma_{\mathrm{P},1}(H_3^2) = 2$ and $\gamma_{\mathrm{P},k}(H_3^2) = 1$ for $k \in \mathbb{N}_2$.

There are perfect codes for all Hanoi graphs isomorphic to Sierpiński graphs and also for H_p^2 [43]. But, for $p \in \mathbb{N}_4$, the Hanoi graphs do not contain perfect codes for $n \in \mathbb{N}_3$, as found out by Q. Stierstorfer [67]. The domination number of these graphs is not known. Therefore we concentrate on $n = 2$. (For $n = 1$, $H_p^1 \cong K_p \cong S_p^1$.)

Theorem 5.2.1. *Let $p \in \mathbb{N}_4$. Then*

$$\gamma_{\mathrm{P},k}(H_p^2) = \begin{cases} 1, & k \in \mathbb{N}_{p-2}; \\ p - k - 1, & k \in [p-3]. \end{cases}$$

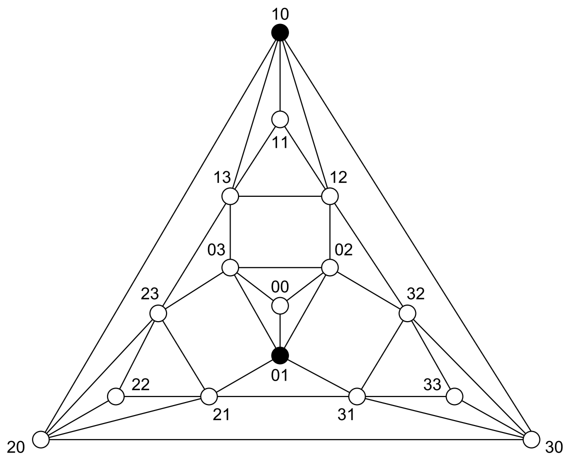


Figure 5.5: The graph H_4^2.

Proof. **Case 1:** $k \in \mathbb{N}_{p-2}$.

Let v be an arbitrary vertex in H_p^2. Let K_p^i denote the subgraph induced by the vertices $\{ij\colon j \in [p]_0\}$. Assume that $v \in K_p^i$ for some i. Let $S = \{v\}$. Then $V(K_p^i) \subseteq \mathcal{P}_k^0(S)$. Since each vertex in K_p^i other than the vertex ii has $p-2$ neighbours outside K_p^i, for any $j \neq i$, $V(K_p^j)\backslash\{jj, ji\} \subseteq \mathcal{P}_k^1(S)$. Hence any vertex $j\ell$ in K_p^j, $\ell \neq i, j$, will have two unmonitored neighbours, namely jj and ji. Since $k \geq p-2 \geq 2$, these vertices will get monitored by propagation, i.e. $V(K_p^j) \subseteq \mathcal{P}_k^2(S)$. Since this is true for any $j \neq i$, S is a k-PDS of H_p^2.

Case 2: $k \in [p-3]$.

We first prove that $\gamma_{\mathrm{P},k}(H_p^2) \leq p-k-1$. Let S be the set of vertices $\{i(i-1)\colon i \in [p-k-2]\} \cup \{0(p-k-2)\}$ (For $k=1$ and $p=4$, the vertices of S are coloured black in Figure 5.5.) Then $\mathcal{P}_k^0(S) = \{V(K_p^i)\colon i \in [p-k-1]_0\} \cup \{ij\colon p-k-1 \leq i \leq p-1,\, j \in [p-k-2]_0\} \cup \{i(p-k-2)\colon p-k-1 \leq i \leq p-1\}$. Let Y be the set of vertices $\{ij\colon i \in [p-k-1]_0,\, p-k-1 \leq j \leq p-1\}$. Then any vertex $v = i'j'$ in Y has exactly k unmonitored neighbours given by $\{\ell j'\colon p-k-1 \leq \ell \leq p-1,\, \ell \neq j'\}$ which will get monitored by propagation. Therefore, the remaining set of unmonitored vertices is given by $\{jj\colon V(K_p^j) \cap S = \emptyset\}$, which will then get monitored by propagation by its neighbours in K_p^j. Thus S is a k-PDS of H_p^2, which implies $\gamma_{\mathrm{P},k}(H_p^2) \leq p-k-1$.

We next prove that $\gamma_{\mathrm{P},k}(H_p^2) \geq p-k-1$. Let S be a k-PDS of H_p^2. Suppose on the contrary that $\gamma_{\mathrm{P},k}(H_p^2) \leq p-k-2$. Assume first that S has exactly one vertex in p-cliques K_p^i for $i \in \{i_1, \ldots, i_{p-k-2}\}$. Let $\{i_1 j_1, \ldots, i_{p-k-2} j_{p-k-2}\}$ be the set of $p-k-2$ vertices in S. Then $S \cap V(K_p^{i'}) = \emptyset$ for any $i' \in I' = [p]_0 \backslash \{i_1, \ldots, i_{p-k-2}\}$. Let $X = \{i'j_1, \ldots, i'j_{p-k-2}\}$. Then $\mathcal{P}_k^0(S) \cap V(K_p^{i'}) \subseteq X$. This holds for any $i' \in I'$. Let $J' = [p]_0 \backslash \{j_1, \ldots, j_{p-k-2}\}$. Then the set of vertices $\{i'j'\colon i' \in I',\, j' \in J'\}$ has an empty intersection with $\mathcal{P}_k^0(S)$. Since every vertex in H_p^2 has either no or more than k neighbours in this set, no vertex from this set can get monitored later on, a contradiction. Assume next that $|S| < p-k-2$ or that S intersects some K_p^i in more than one vertex. Then we can conclude analogously that not all vertices of $K_p^{i'}$ will be monitored and hence $\gamma_{\mathrm{P},k}(H_p^2) \geq p-k-1$. $\square$

It is obtained in Theorem 1.4.12 that for $p \in \mathbb{N}_4$,

$$\gamma_{\mathrm{P},k}(S_p^2) = \begin{cases} 1, & k \in \mathbb{N}_{p-1}; \\ p-k, & k \in [p-2]. \end{cases}$$

We can observe that for $p \in \mathbb{N}_4$, $\gamma_{\mathrm{P},k}(S_p^2) - \gamma_{\mathrm{P},k}(H_p^2) = 1$ if and only if $k \in [p-2]$ and for $k \in \mathbb{N}_{p-1}$, the two values coincide.

We now compute the k-propagation radius of H_p^2. For $p = 3$, it is proved that $\mathrm{rad}_{\mathrm{P},1}(H_3^2) = 2$ and $\mathrm{rad}_{\mathrm{P},k}(H_3^2) = 3$ for $k \in \mathbb{N}_2$ (refer Theorem 1.4.14). The following theorem indicates that the graph H_p^2 can be monitored in 3 steps.

Theorem 5.2.2. *For $p \in \mathbb{N}_4$, $\mathrm{rad}_{\mathrm{P},k}(H_p^2) = 3$.*

Proof. For $k \in \mathbb{N}_{p-2}$, $\gamma_{\mathrm{P},k}(H_p^2) = 1$ and let $S = \{ij\}$ be a k-PDS of H_p^2. If $i \neq j$, we prove that the the vertices ji and jj do not belong to $\mathcal{P}_k^1(S)$. Clearly, $ji, jj \notin \mathcal{P}_k^0(S)$. Also none of the neighbours of ji and jj belongs to $\mathcal{P}_k^0(S)$. Therefore, ji and jj cannot be monitored in step 1. For $i = j$, we can similarly prove that the vertices ℓi and $\ell\ell$, for $\ell \neq i$, do not belong to $\mathcal{P}_k^1(S)$ and hence $\mathrm{rad}_{\mathrm{P},k}(H_p^2) \geq 3$. To prove the upper bound, consider the set $S = \{ii\}$. Then,

$$
\begin{aligned}
\mathcal{P}_k^0(S) &= V(K_p^i)\,,\\
\mathcal{P}_k^1(S) &= \mathcal{P}_k^0(S) \cup \bigcup \left\{V(K_p^\ell)\setminus\{\ell i, \ell\ell\} \colon \ell \in [p]_0 \setminus \{i\}\right\}\,,\\
\mathcal{P}_k^2(S) &= \mathcal{P}_k^1(S) \cup \{\ell i, \ell\ell \colon \ell \in [p]_0 \setminus \{i\}\} = V(H_p^2).
\end{aligned}
$$

Hence $\mathrm{rad}_{\mathrm{P},k}(H_p^2) \leq \mathrm{rad}_{\mathrm{P},k}(G, S) = 3$.

Suppose that $k \in [p-3]$ and let S be a minimum k-PDS of H_p^2. Then $\gamma_{\mathrm{P},k}(H_p^2) = p-k-1$ and thus there exist at least $k+1$ p-cliques K_p^i not containing any vertex of S. Let $K_p^{i'}$ be an arbitrary such clique. We prove that the vertex $i'i'$ is not in $\mathcal{P}_k^1(S)$. Clearly, the vertex $i'i'$ does not belong to $\mathcal{P}_k^0(S)$. Moreover, $|V(K_p^{i'}) \cap \mathcal{P}_k^0(S)| \leq p-k-1$ and therefore $|V(K_p^{i'})\setminus\mathcal{P}_k^0(S)| \geq k+1$. Hence any neighbour of $i'i'$ has more than k unmonitored vertices preventing any propagation to this vertex on that step.

Thus $i'i'$ is not in $\mathcal{P}_k^1(S)$. To prove the upper bound, consider the set $S = \{i(i-1) \colon i \in [p-k-2]\} \cup \{0(p-k-2)\}$. Then,

$$\begin{aligned}
\mathcal{P}_k^0(S) =& \{V(K_p^i) \colon i \in [p-k-1]_0\} \\
& \cup \{ij \colon p-k-1 \le i \le p-1,\, j \in [p-k-1]_0\}\,, \\
\mathcal{P}_k^1(S) =& \mathcal{P}_k^0(S) \cup \{ij \colon p-k-1 \le i, j \le p-1,\, i \ne j\}\,, \\
\mathcal{P}_k^2(S) =& \mathcal{P}_k^1(S) \cup \{ii \colon p-k-1 \le i \le p-1\} = V(H_p^2).
\end{aligned}$$

□

5.3 WK-Pyramid networks

In this section, we determine the k-power domination number of $WKP_{(C,L)}$. We also obtain the k-propagation radius of $WKP_{(C,L)}$ in some cases.

Observe from Definition 1.2.22 that WK-Recursive mesh, $WK_{(C,L)}$, has C^L vertices and $\frac{C}{2}(C^L - 1)$ edges. Vertices in $WK_{(C,L)}$ which are of the form $(\overbrace{a \ldots a}^{L \text{ times}})$ are called extreme vertices of $WK_{(C,L)}$. Clearly, $WK_{(C,L)}$ contains C extreme vertices of degree $C-1$ and all the other vertices are of degree C. We have $WK_{(1,L)} \cong K_1$ ($L \geq 1$), $WK_{(2,L)} \cong P_{2^L}$ ($L \geq 1$) and $WK_{(C,1)} \cong K_C$ ($C \geq 1$).

A vertex of $WKP_{(C,L)}$ with the addressing scheme $(r, (a_r a_{r-1} \dots a_1))$ is called a **vertex at level $\boldsymbol{r}$**. The part $(a_r a_{r-1} \dots a_1)$ of the address determines the address of a vertex within the WK-recursive mesh at level r. All vertices in level $r > 0$ of $WKP_{(C,L)}$ induce a WK-recursive mesh $WK_{(C,r)}$. Hence $|V(WKP_{(C,L)})| = \sum_{i=0}^{L} C^i = \frac{C^{L+1}-1}{C-1}$. Note that $WKP_{(C,1)} \cong K_{C+1}$ ($C \geq 1$), $WKP_{(1,L)} \cong P_{L+1}$ ($L \geq 1$). Vertices of the form $(r, (\overbrace{a \dots a}^{r \text{ times}}))$ are called the **extreme vertices** of $WKP_{(C,L)}$. The vertex $(0, (1))$ has degree C and at any level except the L^{th} level, the extreme vertices are of degree $2C$ and the other vertices are of degree $2C + 1$. In the L^{th} level, the extreme vertices have degree C and the other vertices have degree $C + 1$.

We shall use the following notations in the rest of the chapter.

Let V^1 and V^2 denote the set of vertices of $WKP_{(C,2)}$ in levels 1 and 2, respectively. Let Q_i denote a C-clique induced by the set of vertices $\{(2, (ij)) \colon j \in [C]_0\}$ for some i.

For $C, L \in \mathbb{N}_3$, let $w \in [C]_0^{L-2}$. Denote $V_w^{C,L} = \{(L, (wij)) \in$

$WKP_{(C,L)}$: $i, j \in [C]_0\}$ and $G_w^{C,L} = < V_w^{C,L} >$, i.e. $G_w^{C,L}$ is the induced subgraph in level L of $WKP_{(C,L)}$. In fact, $G_w^{C,L}$ is isomorphic to $WK_{(C,2)}$ for any $w \in [C]_0^{L-2}$ and any $L \in \mathbb{N}_3$ (Figure 5.6). We first consider the easier case as stated in the

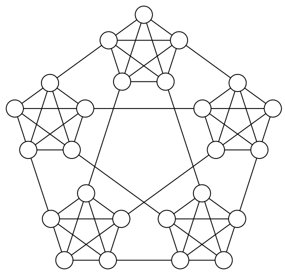


Figure 5.6: The induced subgraph $G_w^{5,L}$ of $WKP_{(5,L)}$.

following theorem.

Theorem 5.3.1. *Let $C, L \in \mathbb{N}_1$. If $C = 1$ or $L = 1$ or $k \geq C$, then $\gamma_{\mathrm{P},k}(WKP_{(C,L)}) = 1$.*

Proof. Recall that $WKP_{(C,1)} \cong K_{C+1}$ $(C \geq 1)$ and that $WKP_{(1,L)}$ $\cong P_{L+1}$ $(L \geq 1)$. Hence $\gamma_{\mathrm{P},k}(G) = 1$ for these graphs G.

If $k \geq C$, then take $S = \{(0, (1))\}$. It monitors the vertices in level 1. Since each vertex in level r has exactly C neighbours in its successive level $r+1$, once the level r is monitored, the vertices in level $r + 1$ get monitored by propagation. This propagation goes on till level L and hence S is a k-PDS of $WKP_{(C,L)}$. $\square$

We have determined the value of $\gamma_{\mathrm{P},k}(WKP_{(C,L)})$ when $k \geq C$. Now, we consider the remaining case $k \leq C-1$. We begin with the computation of $\gamma_{\mathrm{P},k}$ for $L=2$ and will prove in Theorem 5.3.4 that $\gamma_{\mathrm{P},k}(WKP_{(C,2)}) = C-k$ for $C \geq 2, k \leq C-1$.

We first obtain the following upper bound. For that, we produce a set S of cardinality $C-k$ and prove that S monitors the whole graph in two propagation steps.

Lemma 5.3.2. *For $C \in \mathbb{N}_3$ and $k \in [C-1]$,*
$\gamma_{\mathrm{P},k}(WKP_{(C,2)}) \leq C-k$.

Proof. Let $S = \{(1,(i))\colon k \leq i \leq C-1\}$. (For $k=1$ and $C=5$, the vertices in S are coloured black in Figure 5.7.)
Then $\mathcal{P}_k^0(S) = \{(1,(j))\colon j \in [C]_0\} \cup \{(2,(ij))\colon k \leq i \leq C-1, j \in [C]_0\} \cup \{(0,(1))\}$,
$\mathcal{P}_k^1(S) = \mathcal{P}_k^0(S) \cup \{(2,(ij))\colon i \in [k]_0, k \leq j \leq C-1\}$ and
$\mathcal{P}_k^2(S) = \mathcal{P}^1(S) \cup \{(2,(ij))\colon i,j \in [k]_0\} = V(WKP_{(C,2)})$.
Hence S is a k-PDS, which implies $\gamma_{\mathrm{P},k}(WKP_{(C,2)}) \leq |S| = C-k$. □

Lemma 5.3.3. *For $C \in \mathbb{N}_3$ and $k \in [C-2]$,*
$\gamma_{\mathrm{P},k}(WKP_{(C,2)}) \geq C-k$.

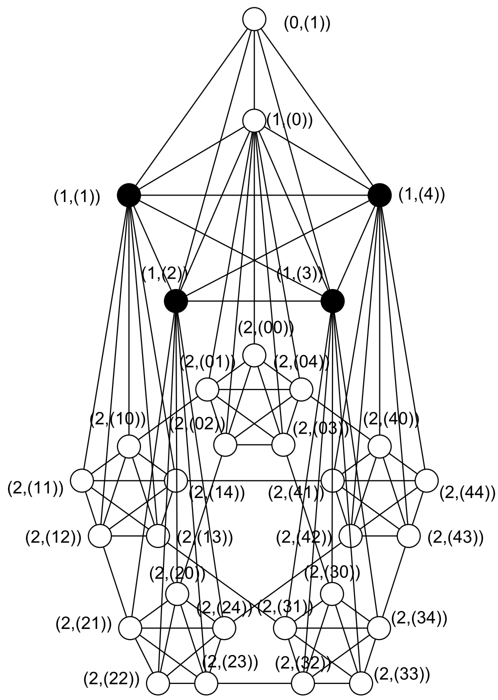


Figure 5.7: The graph $WKP_{(5,2)}$.

Proof. Let S be a minimum k-PDS of $WKP_{(C,2)}$. We may assume that $S \subseteq V^1 \cup V^2$.

Claim: $|S \cap (V^1 \cup V^2)| \geq C - k$.

Suppose on the contrary that $|S \cap (V^1 \cup V^2)| \leq C - k - 1$. We consider the case when S contains vertices from both V^1 and V^2. Assume first that $|S \cap (V^1 \cup V^2)| = C - k - 1$ and that S contains a vertex $(1, (i')) \in V^1$ and the remaining $C - k - 2$

vertices from the C-cliques $Q_{i_1}, \ldots, Q_{i_{C-k-2}}$, where $i' \neq i_\ell, \ell \in [C-k-2]$ such that each of these C-cliques contains exactly one vertex in S. Let Q_ℓ be an arbitrary clique that does not contain any vertex of S, where $\ell \neq i'$. Let $X = \{(2,(\ell i'))\} \cup \{(2,(\ell i_1)), \ldots, (2,(\ell i_{C-k-2}))\}$. Then $\mathcal{P}_k^1(S) \cap V(Q_\ell) = X$. This holds for every $l \in I = [C]_0 \setminus \{i', i_1, \ldots, i_{C-k-2}\}$. Thus the set of vertices $J = \{(2,(\ell\ell'))\colon \ell \in I, \ell' \in I\}$ has an empty intersection with $\mathcal{P}_k^1(S)$. Since every vertex in $WKP_{(C,L)} - J$ has either 0 or $k+1$ neighbours in J, no vertex from this set J may get monitored later on, which is a contradiction. Assume next that $|S \cap (V^1 \cup V^2)| < C-k-1$ or that S intersects some C-clique Q_i in more than one vertex. Then we can analogously conclude that not all vertices of Q_ℓ will be monitored. Now, the case when $S \cap V^1 = \emptyset$ or $S \cap V^2 = \emptyset$ can be proved in a similar manner. Hence the claim.

Therefore $\gamma_{\mathrm{P},k}(WKP_{(C,2)}) = |S| = |S \cap (V^1 \cup V^2)| \geq C - k$. □

From Lemmas 5.3.2 and 5.3.3, we can easily deduce the following theorem.

Theorem 5.3.4. *For $C \in \mathbb{N}_2$ and $k \in [C-1]$,*

$\gamma_{\mathrm{P},k}(WKP_{(C,2)}) = C - k$.

Proof. Clearly, $\gamma_{\mathrm{P},1}(WKP_{(2,2)}) = 1$. Let $C \geq 3$. For $k = C - 1$, any vertex in level 1 forms a k-PDS of $WKP_{(C,2)}$. For $k \in [C-2]$, the result follows from Lemmas 5.3.2 and 5.3.3. □

Thus we compute $\gamma_{\mathrm{P},k}(WKP_{(C,2)})$ for all values of k and C. We now consider the case $C \in \mathbb{N}_3, L \in \mathbb{N}_3$ and $k \in [C - 2]$ and prove an upper bound in the following lemma. We construct a set $S \subseteq V(WKP_{(C,L)})$ that monitors the whole graph. The idea is to construct S in such a way that it initially monitors all the vertices of level L and $L - 1$. For that, we use the hamiltonian property of its subgraphs. Since the graph possesses a pyramid structure, each vertex in a level has exactly one neighbour in its preceding level. Therefore once the levels L and $L-1$ get monitored, the preceding levels can be monitored by propagation.

Lemma 5.3.5. *For $C \in \mathbb{N}_3, L \in \mathbb{N}_3$ and $k \in [C - 2]$,*
$\gamma_{\mathrm{P},k}(WKP_{(C,L)}) \leq (C - k - 1)C^{L-2}$.

Proof. In $WKP_{(C,L)}$, the vertices in the L^{th} level induce $WK_{(C,L)}$ which is hamiltonian [45, 51]. Also, by contracting each of the subgraphs $G_w^{C,L}$ into a single vertex, the graph induced by the vertices in level L is isomorphic to $WK_{(C,L-2)}$. Hence, in level L of $WKP_{(C,L)}$, we can arrange the subgraphs of the form $G_w^{C,L}$ into a cycle such that there exists exactly one edge between the consecutive subgraphs. We now construct a set S in such a way that corresponding to each subgraph $G_w^{C,L}$ in level L, the set S contains one vertex from the neighbour set of $G_w^{C,L}$ in level $L-1$ (which induces a clique) and $C-k-2$ additional vertices from $G_w^{C,L}$.

Let $w', w'' \in [C]_0^{L-2}$. Let $G_w^{C,L}, G_{w'}^{C,L}$ and $G_{w''}^{C,L}$ be consecutive subgraphs in the selected hamiltonian order. Let xx' be the edge between $G_w^{C,L}$ and $G_{w'}^{C,L}$, where $x \in G_w^{C,L}, x' \in G_{w'}^{C,L}$ and let $y'y''$ be the edge between $G_{w'}^{C,L}$ and $G_{w''}^{C,L}$, where $y' \in G_{w'}^{C,L}, y'' \in G_{w''}^{C,L}$. Let H and Q be the C-cliques in $G_{w'}^{C,L}$ that contain the vertices x' and y', respectively. Denote $x = (L, (wii))$ and $y' = (L, (w'jj))$ for some i and j, $i \neq j$. We now construct a set S as explained above. We first choose the elements of S corresponding to the subgraph $G_{w'}^{C,L}$. Let S contain the vertex $(L-1, (w'j))$, which is the neighbour of y' in the $(L-1)^{\text{th}}$ level. Then $C-k-2$ additional vertices from $G_{w'}^{C,L}$ are added to S in such a way that no two vertices lying in the same C-clique in $G_{w'}^{C,L}$ and no one lying in the C-cliques, H and Q (i.e. $S \cap V(H) = \emptyset$ and $S \cap V(Q) = \emptyset$). Now, do this in parallel for all the corresponding subgraphs. In particular, the vertex $(L-1,(wi))$ in the $(L-1)^{\text{th}}$ level corresponding to the vertex x is put into S, when considering $G_w^{C,L}$. Thus $C-k$ vertices of H lie in $\mathcal{P}_k^1(S)$: one of these vertices is x', the other $C-k-1$ are those vertices of H that have a neighbour in the C-cliques in $G_{w'}^{C,L}$ that contain $C-k-2$ vertices of S and that have a neighbour in the C-clique Q in $G_{w'}^{C,L}$. Also, the neighbour of H in the $(L-1)^{\text{th}}$ level belongs to $\mathcal{P}_k^0(S)$, since $(L-1,(wi)) \in S$. Hence the remaining k vertices of H lie in $\mathcal{P}_k^2(S)$ and it is straightforward to check that all the vertices of $G_{w'}^{C,L}$ lie in $\mathcal{P}_k^\infty(S)$. In a similar way, every vertex in the L^{th} level is monitored. We know that, for any w, the neighbours of $G_w^{C,L}$ in the $(L-1)^{\text{th}}$ level induce a C-clique. By the construction of S, each C-clique in the $(L-1)^{\text{th}}$ level contains a vertex in S. Thus we get that all the vertices in levels $L-1$ and $L-2$ belong to $\mathcal{P}_k^0(S)$. Now, since each vertex in level $L-2$ has exactly one neighbour in its preceding level, vertices in the $(L-3)^{\text{rd}}$ level are monitored by propagation. This propagation continues to the preceding levels and hence the whole graph gets monitored. Thus we conclude that S is a

k-PDS. Since each subgraph $G_w^{C,L}$ contains $C-k-1$ vertices of S, $|S| \leq (C-k-1)C^{L-2}$. $\square$

An illustration of Lemma 5.3.5 is included in the last section of this chapter.

Lemma 5.3.6. *For $C \in \mathbb{N}_3, L \in \mathbb{N}_3$ and $k \in [C-2]$,* $\gamma_{\mathrm{P},k}(WKP_{(C,L)}) \geq (C-k-1)C^{L-2}$.

Proof. Let S be a minimum k-PDS of $WKP_{(C,L)}$ and $w \in [C]_0^{L-2}$. Denote $V_w^{C,L-1} = \{(L-1,(wi)) \in WKP_{(C,L)} \colon i \in [C]_0\}$.

Claim: $\left|S \cap (V_w^{C,L} \cup V_w^{C,L-1})\right| \geq C-k-1$.

Suppose on the contrary that $\left|S \cap (V_w^{C,L} \cup V_w^{C,L-1})\right| \leq C-k-2$. Consider the case when $S \cap V_w^{C,L-1} = \emptyset$. Then $|S \cap V_w^{C,L}| \leq C-k-2$. Assume first that $\left|S \cap V_w^{C,L}\right| = C-k-2$. Let H_i be a C-clique in $G_w^{C,L}$, i.e. H_i is induced by the set of vertices $\{(L,(wij)) \in WKP_{(C,L)} \colon j \in [C]_0\}$ for some i. Assume that S has exactly one vertex in C-cliques H_i for $i \in \{i_1,\ldots,i_{C-k-2}\}$. Then $S \cap V(H_{i'}) = \emptyset$ holds for other $k+2$ coordinates i'. Let H_ℓ be an arbitrary such clique in $G_w^{C,L}$ that does not contain any vertex of S. Let $X = \{(L,(w\ell i_1)),\ldots,(L,(w\ell i_{C-k-2}))\} \cup$

$\{(L,(w\ell\ell))\}$. Then $\mathcal{P}_k^1(S) \cap V(H_\ell) \subseteq X$. This holds for every $\ell \in I = [C]_0 \setminus \{i_1, \ldots, i_{C-k-2}\}$. Thus the set of vertices $\{(L,(w\ell\ell'))\colon \ell \in I, \ell' \in I, \ell \neq \ell'\}$ has an empty intersection with $\mathcal{P}_k^1(S)$. Since every vertex in $WKP_{(C,L)}$ has either 0 or $k+1$ neighbours in this set, no vertex from this set may get monitored later on, a contradiction. Assume next that $\left|S \cap V_w^{C,L}\right| < C-k-2$ or that S intersects some C-clique H_i in more than one vertex. Then we can analogously conclude that not all vertices of H_ℓ will be monitored. Thus the case that $S \cap V_w^{C,L-1} = \emptyset$ is not possible.

Now suppose that $S \cap V_w^{C,L-1} \neq \emptyset$. Assume first that $\left|S \cap (V_w^{C,L} \cup V_w^{C,L-1})\right| = C-k-2$ and that S contains a vertex $(L-1,(wi')) \in V_w^{C,L-1}$ and the remaining $C-k-3$ vertices from the C-cliques $H_{i_1}, \ldots, H_{i_{C-k-3}}$, where $i' \neq i_\ell, \ell \in [C-k-3]$ such that each of these C-cliques contains exactly one vertex in S. Let H_ℓ be an arbitrary clique in $G_w^{C,L}$ that does not contain any vertex of S, where $\ell \neq i'$. Let $X = \{(L,(w\ell i'))\} \cup \{(L,(w\ell\ell))\} \cup \{(L,(w\ell i_1)), \ldots, (L,(w\ell i_{C-k-3}))\}$. Then $\mathcal{P}_k^1(S) \cap V(H_\ell) \subseteq X$. This holds for every $l \in I' = [C]_0 \setminus \{i', i_1, \ldots, i_{C-k-3}\}$. Thus the set of vertices $\{(L,(w\ell\ell'))\colon \ell \in I', \ell' \in I', \ell \neq \ell'\}$ has an empty intersection with $\mathcal{P}_k^1(S)$. Since every vertex in $WKP_{(C,L)}$

has either 0 or $k+1$ neighbours in this set, no vertex from this set may get monitored later on, which is a contradiction. Assume next that $\left|S \cap (V_w^{C,L} \cup V_w^{C,L-1})\right| < C-k-2$ or that S intersects some C-clique H_i in more than one vertex. Then we can analogously conclude that not all vertices of H_ℓ will be monitored. Hence the claim $\left|S \cap (V_w^{C,L} \cup V_w^{C,L-1})\right| \geq C-k-1$ is proved. Therefore, $\left|S \cap \left(V(G_w^{C,L}) \cup N_{L-1}(G_w^{C,L})\right)\right| \geq C-k-1$, where $N_{L-1}(G_w^{C,L})$ is the set of neighbours of $G_w^{C,L}$ in the $(L-1)^{\text{th}}$ level. Hence corresponding to each $G_w^{C,L}$ in the L^{th} level, we get at least $C-k-1$ vertices in S.

Hence $|S| \geq \sum_{w \in [C]_0^{L-2}} (C-k-1) = (C-k-1)C^{L-2}$. □

The following theorem gives the exact value of $\gamma_{\mathrm{P},k}(WKP_{(C,L)})$ for $C \in \mathbb{N}_3, L \in \mathbb{N}_3$ and $k \in [C-2]$.

Theorem 5.3.7. *For $C \in \mathbb{N}_3, L \in \mathbb{N}_3$ and $k \in [C-2]$,*
$\gamma_{\mathrm{P},k}(WKP_{(C,L)}) = (C-k-1)C^{L-2}$.

Proof. Follows from Lemmas 5.3.5 and 5.3.6. □

Thus we have the following consolidated result:

Let $C, L \in \mathbb{N}_1$. Then

$$\gamma_{\mathrm{P},k}(WKP_{(C,L)}) = \begin{cases} 1, \;\; C = 1 \text{ or } L = 1 \text{ or } k \in \mathbb{N}_C; \\ C - k, \;\; L = 2, C \in \mathbb{N}_2, k \in [C-1]; \\ (C-k-1)C^{L-2}, \;\; C, L \in \mathbb{N}_3, k \in [C-2]. \end{cases}$$

For $k = C - 1$, $C \in \mathbb{N}_2$ and $L \in \mathbb{N}_3$, we prove the following upper bound.

Theorem 5.3.8. *For $C \in \mathbb{N}_2$ and $L \in \mathbb{N}_3$,*
$\gamma_{\mathrm{P,C-1}}(WKP_{(C,L)}) \leq \lceil \frac{L+1}{3} \rceil$.

Proof. We consider three cases.

Case 1: $L = 3m$, $m \in \mathbb{N}_1$.

$S = \{\, \bigcup_{i=1}^{m} (3i-1, (0)^{3i-1})\} \cup \{(0,(1))\}$.

Here, $|S| = m + 1$. Also, $\lceil \frac{L+1}{3} \rceil = \left\lceil \frac{(3m)+1}{3} \right\rceil = m + 1$.

Case 2: $L = 3m + 1$, $m \in \mathbb{N}_1$.

$S = \{\, \bigcup_{i=1}^{m} (3i, (0)^{3i})\} \cup \{(1,(0))\}$.

Here, $|S| = m + 1$. Also, $\lceil \frac{L+1}{3} \rceil = \left\lceil \frac{(3m+1)+1}{3} \right\rceil = m + 1$.

Case 3: $L = 3m + 2$, $m \in \mathbb{N}_1$.

$S = \{ \bigcup_{i=1}^{m+1} (3i - 2, (0)^{3i-2})\}$.

Here, $|S| = m + 1$. Also, $\lceil \frac{L+1}{3} \rceil = \left\lceil \frac{(3m+2)+1}{3} \right\rceil = m + 1$.

In each case, $\mathcal{P}^{\infty}_{C-1}(S) = V(WKP_{(C,L)})$ and thus S is a k-PDS of order $\lceil \frac{L+1}{3} \rceil$. Hence $\gamma_{\mathrm{P,C-1}}(WKP_{(C,L)}) \leq \lceil \frac{L+1}{3} \rceil$. $\square$

We now determine the k-propagation radius of $WKP_{(C,L)}$ for $C \in \mathbb{N}_1$ and $L = 1, 2$. If $L = 1$, the graph is a complete graph and its k-propagation radius is 1. If $C = 1, \mathrm{rad}_{\mathrm{P},k}(WKP_{(1,L)}) = \mathrm{rad}_{\mathrm{P},k}(P_{L+1}) = \lfloor \frac{L+1}{2} \rfloor$.

Lemma 5.3.9. *Let $C \in \mathbb{N}_3$, $k \in [C - 1]$ and S be a minimum k-PDS of $WKP_{(C,2)}$. Then $S \cap V^1 \neq \emptyset$.*

Proof. Suppose that $S \cap V^1 = \emptyset$. Consider the case when $(0, (1)) \notin S$. Then by Theorem 5.3.4, $|S \cap V^2| = C - k$. Assume first that S has exactly one vertex in C-cliques, Q_i, for $i \in \{i_1, \ldots, i_{C-k}\}$. Then $S \cap V(Q_{i'}) = \emptyset$ for k coordinates i'. Let Q_ℓ be an arbitrary such subgraph. Let $X = \{(2, (\ell i_1)), \ldots, (2, (\ell i_{C-k}))\}$. Then $\mathcal{P}^1_k(S) \cap V(Q_\ell) = X$ and $\mathcal{P}^1_k(S) \cap V^1 = \{(1, i_1), \ldots, (1, i_{C-k})\}$. This holds for any $\ell \in$

$J = [C]_0 \setminus \{i_1, \ldots, i_{C-k}\}$. Therefore the set of vertices $K = \{(2,(ij))\colon i,j \in J\} \cup \{(1,(i))\colon i \in J\} \cup \{(0,(1))\}$ has an empty intersection with $\mathcal{P}^1_k(S)$. Since every vertex of $WKP_{(C,2)} - K$ has either 0 or $k+1$ neighbours in K, no vertex from this set may get monitored later on, a contradiction. The case when $(0,(1)) \in S$ or that S intersects some Q_i in more than one vertex can be proved analogously. $\square$

We can now determine the k-propagation radius of $WKP_{(C,2)}$ using the previous lemma.

Theorem 5.3.10. *Let $C \in \mathbb{N}_2$. Then*

$$\mathrm{rad}_{\mathrm{P},k}(WKP_{(C,2)}) = \begin{cases} 2, & k \geq C; \\ 3, & k \in [C-1]. \end{cases}$$

Proof. For $k \geq C$, $\gamma_{\mathrm{P},k}(WKP_{(C,2)}) = 1$, by Theorem 5.3.1 and observe that $\gamma(WKP_{(C,2)}) > 1$. Therefore, $\mathrm{rad}_{\mathrm{P},k}(WKP_{(C,2)}) \geq 2$ (by Proposition 1.4.13). And, for the set $S = \{(0,(1))\}$, we get that $\mathcal{P}^0_k(S) = S \cup V^1$ and $\mathcal{P}^1_k(S) = V(WKP_{(C,2)})$. Now let $k \in [C-1]$. For $C = 2$, the result easily follows. Let $C \geq 3$. By Theorem 5.3.4, $\gamma_{\mathrm{P},k}(WKP_{(C,2)}) = C-k$ and therefore by Lemma 5.3.9, $|S \cap V^2| \leq C - k - 1$ for every minimum

k-PDS S. Then there exist at least $k+1$ C-cliques, Q_i, not containing any vertex of S. Let $Q_{i'}$ be an arbitrary clique such that $S \cap V(Q_{i'}) = \emptyset$ and $(1,(i')) \notin S$. We prove that the vertex $(2,(i'i'))$ is not in $\mathcal{P}_k^1(S)$. Clearly, $(2,(i'i')) \notin \mathcal{P}_k^0(S)$. Moreover, $|V(Q_{i'}) \cap \mathcal{P}_k^0(S)| \leq C-k-1$ and $|V(Q_{i'}) \setminus \mathcal{P}_k^0(S)| \geq k+1$. Therefore any neighbour of $(2,(i'i'))$ in $Q_{i'}$ is adjacent to more than k unmonitored vertices preventing any propagation to this vertex at this step. Also, since $(1,(i'))$ has more than k unmonitored vertices as its neighbours, $(2,(i'i'))$ cannot be monitored by $(1,(i'))$ at this step. Hence $\mathrm{rad}_{\mathrm{P},k}(WKP_{(C,2)}) \geq 3$. Also, by Lemma 5.3.2, $\mathrm{rad}_{\mathrm{P},k}(WKP_{(C,2)}) \leq 3$. □

Remark 5.3.1. For $C, L \in \mathbb{N}_3$, by observing the propagation behaviour described in the proof of Theorem 5.3.1 and Lemma 5.3.5, one can obtain that $\mathrm{rad}_{\mathrm{P},k}(WKP_{(C,L)}) \leq L$ if $k \geq C$ and $\mathrm{rad}_{\mathrm{P},k}(WKP_{(C,L)}) \leq \max\{5, L-1\}$ if $k \in [C-2]$.

Illustration of Lemma 5.3.5

We illustrate Lemma 5.3.5 for the case $k=1, C=5$ and $L=3$. Figure 5.8 depicts the graph $WKP_{(5,3)}$. We know the vertices in the third level of $WKP_{(5,3)}$induce the subgraph $WK_{(5,3)}$ which is hamiltonian. And, the cycle in the subgraph $WK_{(5,3)}$, as defined in Lemma 5.3.5, are drawn as bold edges in the figure. The consecutive subgraphs $G^{5,3}_{w}$, $G^{5,3}_{w'}$ and $G^{5,3}_{w''}$ in the hamiltonian cycle of $WK_{(5,3)}$ and the vertices x, x', y' and y'' as chosen in the lemma are also marked in the figure. The vertices of the set S as constructed in the lemma are coloured black in Figure 5.8. In Figure 5.9, the vertices in the set $\mathcal{P}^0_1(S)$, are coloured black and the remaining vertices white. The black vertices in Figure 5.10 and Figure 5.11 represent the vertices in the set $\mathcal{P}^1_1(S)$ and $\mathcal{P}^2_1(S)$, respectively. The directed edges in the figures indicate the direction in which the propagation occurs at each step. We can observe that all vertices of $WKP_{(5,3)}$ get monitored by step 2 and $\mathcal{P}^2_1(S) = V(WKP_{(5,3)})$. Therefore, S is a 1-PDS of $WKP_{(5,3)}$ and $|S| = 15 = (5 - 1 - 1) \cdot 5^{3-2}$.

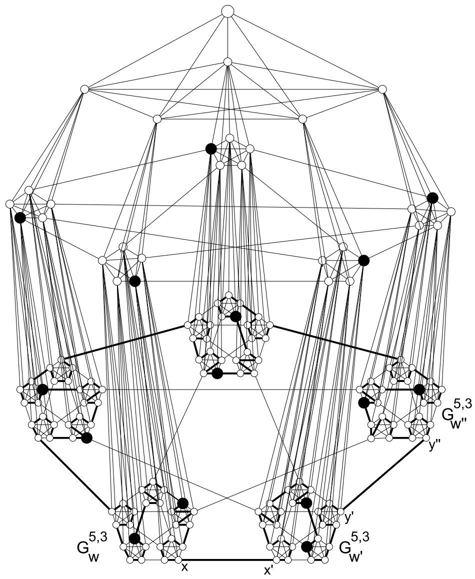


Figure 5.8: The graph $WKP_{(5,3)}$.

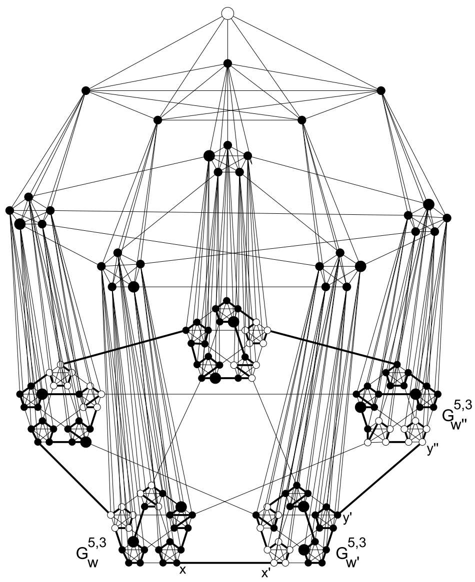


Figure 5.9: $\mathcal{P}_1^0(S)$.

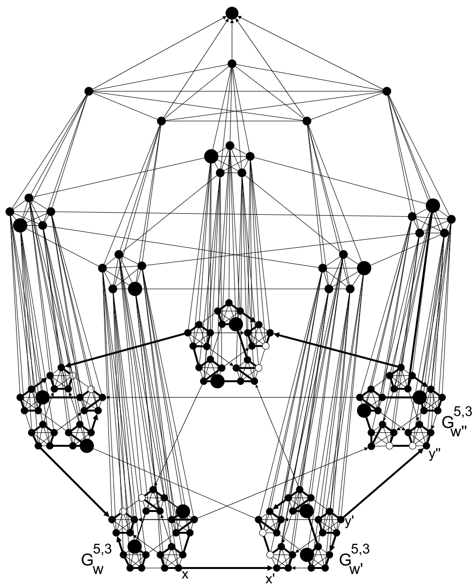


Figure 5.10: $\mathcal{P}_1^1(S)$.

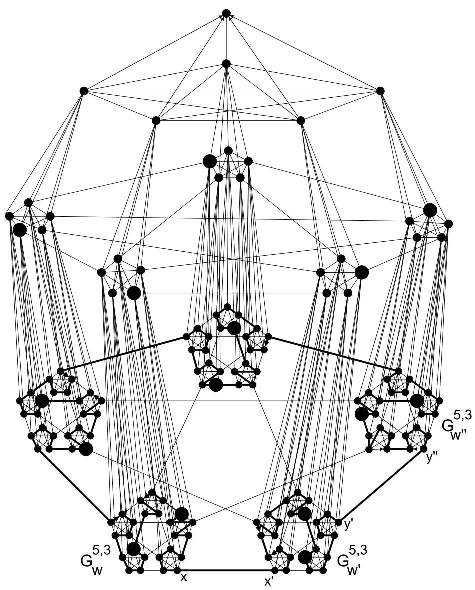


Figure 5.11: $\mathcal{P}_1^2(S)$.

Concluding Remarks

We list below some problems which we found are interesting.

- For each $e \in E(G)$, characterize the graphs G for which

  (*i*) $\gamma_{\mathrm{P},k}(G - e) = \gamma_{\mathrm{P},k}(G)$.

  (*ii*) $\gamma_{\mathrm{P},k}(G - e) = \gamma_{\mathrm{P},k}(G) - 1$.

  (*iii*) $\gamma_{\mathrm{P},k}(G/e) = \gamma_{\mathrm{P},k}(G)$.

  (*iv*) $\gamma_{\mathrm{P},k}(G/e) = \gamma_{\mathrm{P},k}(G) - 1$.

  (*v*) $\gamma_{\mathrm{P},k}(G/e) = \gamma_{\mathrm{P},k}(G) + 1$.

- Estimate some sharp upper and lower bounds for $b_{\mathrm{P},k}(G)$ in terms of other graph parameters.

- Characterize the trees T for which $b_{\mathrm{P},k}(T) = 1$ or $b_{\mathrm{P},k}(T) = 2$.

- For some product $* \in \{\,\Box\,, \times, \boxtimes\}$, can we relate $\gamma_{\mathrm{P},k}(G)$, $\gamma_{\mathrm{P},\ell}(H)$ and some $\gamma_{\mathrm{P},f(k,\ell)}(G * H)$?

- Explore more on the relationship between $Z(G)$ and $\gamma_{\mathrm{P}}(G)$ to get new results in power domination.

- Find $\mathrm{rad}_{\mathrm{P},k}$ of any product $G * H$ for which $\gamma_{\mathrm{P},k}(G * H)$ is known. For example, the power domination number of $P_n \,\Box\, P_m$ is known [28]. So, it would be interesting to determine $\mathrm{rad}_{\mathrm{P},k}(P_n \,\Box\, P_m)$.

As power domination is recently introduced, many interesting questions can be posed on this topic. Characterizing graphs G for which $\gamma_{\mathrm{P},k}(G) = \gamma_{\mathrm{P},k+1}(G)$ for some k is particularly interesting. One can attempt to study the power domination number of line graphs and some other graph operators. Finding lower bounds for $\gamma_{\mathrm{P},k}$ seems to be a difficult task. By introducing the k-propagation radius, one can bring some good lower bounds for $\gamma_{\mathrm{P},k}$ and thereby determine the k-power domination number of some families of graphs.

We conclude the thesis with an optimistic note that some of the problems mentioned above will be solved soon.

List of symbols

$b(G)$	-	the bondage number of G
$b_{\mathrm{P},k}(G)$	-	the k-power bondage number of G
$\mathcal{B}_G^i(Z)$	-	the set of vertices of G that are coloured black by Z at step i
C_n	-	the cycle on n vertices
$d_G(v)$ or $d(v)$	-	the degree of v in G
$d_G(u,v)$ or $d(u,v)$	-	the distance between u and v in G
F_n	-	the fan on n vertices
$E(G)$	-	the edge set of G
$G \cong H$	-	G is isomorphic to H
$G \,\square\, H$	-	Cartesian product of G and H
$G \times H$	-	Direct product of G and H
$G \boxtimes H$	-	Strong product of G and H

$G \circ H$	-	Lexicographic product of G and H
$G \cup H$	-	Union of G and H
$G \vee H$	-	Join of G and H
$G - v$	-	the subgraph of G obtained by deleting the vertex v
$G - e$	-	the subgraph of G obtained by deleting the edge e
$G - A$	-	the subgraph of G obtained by the deletion of the vertices in A
$G - B$	-	the subgraph of G obtained by the deletion of the edges in B
G/e	-	the graph obtained from G by contracting the edge e
H_p^n	-	Hanoi graph for base p and exponent n
K_n	-	the complete graph on n vertices
$K_{1,n}$	-	the star of size n
$K_{m,n}$	-	the complete bipartite graph where m and n are the cardinalities of the partitions
$M(S)$	-	the set monitored by S
$N_G(v)$ or $N(v)$	-	the open neighbourhood of v in G
$N_G[v]$ or $N[v]$	-	the closed neighbourhood of v in G

$N_G(S)$ or $N(S)$	-	the open neighbourhood of a subset S of $V(G)$
$N_G[S]$ or $N[S]$	-	the closed neighbourhood of a subset S of $V(G)$
$\mathcal{P}^i_{G,k}(S)$	-	the set monitored by S at step i in k-power domination
P_n	-	the path on n vertices
$r(G)$	-	the radius of G
S^n_p	-	Sierpiński graph on p^n vertices
$V(G)$	-	the vertex set of G
W_n	-	the wheel on n vertices
$W_{\Delta,2\nu}$	-	Knödel graph of order 2ν and degree Δ
$WK_{(C,L)}$	-	WK-Recursive mesh on C^L vertices
$WKP_{(C,L)}$	-	WK-Pyramid network on $\frac{C^{L+1}-1}{C-1}$ vertices
$Z(G)$	-	the zero forcing number of G
$\lceil x \rceil$	-	Smallest integer $\geq x$
$\lfloor x \rfloor$	-	Greatest integer $\leq x$
$< X >$	-	the subgraph induced by a subset X of $V(G)$

$\lambda(G)$	-	the edge connectivity of G
$\Delta(G)$	-	the maximum degree of G
$\delta(G)$	-	the minimum degree of G
$\gamma(G)$	-	the domination number of G
$\gamma_t(G)$	-	the total domination number of G
$\gamma_{\mathrm{P}}(G)$	-	the power domination number of G
$\gamma_{\mathrm{P},k}(G)$	-	the k-power domination number of G

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List of Publications

Papers published / accepted

1. P. Dorbec, Seethu Varghese, A. Vijayakumar, Heredity for generalized power domination, Discrete Math. Theor. Comput. Sci. 18 (3) (2016).

2. Seethu Varghese, A. Vijayakumar, On the Power Domination Number of Graph Products, In: S. Govindarajan, A. Maheshwari (Eds.), CALDAM 2016, Lecture Notes in Comput. Sci, Vol. 9602, pp: 357–367, Springer (2016).

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