Stochastic Modelling: Analysis and Applications

ON QUEUEING-INVENTORY MODELS – PRODUCT FORM SOLUTION; RESERVATION, CANCELLATION AND COMMON LIFE TIME

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ON QUEUEING-INVENTORY MODELS – PRODUCT FORM SOLUTION; RESERVATION, CANCELLATION AND COMMON LIFE TIME

Ph.D. thesis in the field of Stochastic Modelling & Analysis

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Certificate

Certified that the work presented in this thesis entitled "**ON QUEUEING-INVENTORY MODELS – PRODUCT FORM SOLUTION; RESERVATION, CANCELLATION AND COMMON LIFE TIME** "is based on the authentic record of research carried out by Miss. Dhanya Shajin under my guidance in the Department of Mathematics, Cochin University of Science and Technology, Kochi- 682 022 and has not been included in any other thesis submitted for the award of any degree.

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Certified that all the relevant corrections and modifications suggested by the audience during the Pre-synopsis seminar and recommended by the Doctoral Committee of the candidate has been incorporated in the thesis entitled "**ON QUEUEING-INVENTORY MODELS – PRODUCT FORM SOLUTION; RESERVATION, CANCELLATION AND COMMON LIFE TIME.** "

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Declaration

I, DHANYA SHAJIN, hereby declare that the work presented in this thesis entitled "**ON QUEUEING-INVENTORY MODELS – PRODUCT FORM SOLUTION; RESERVATION, CANCELLATION AND COMMON LIFE TIME** "is based on the original research work carried out by me under the supervision and guidance of Dr. B. Lakshmy, Associate Professor, Department of Mathematics, Cochin University of Science and Technology, Kochi- 682 022 and has not been included in any other thesis submitted previously for the award of any degree.

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To

My loving

Appa, Amma

and

Teachers

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Dhanya Shajin

Contents

Notations and Abbreviations used

$\boldsymbol{e}$:	column vector of 1's with appropriate dimension
$\mathbf{0}$:	vector consisting of 0's with appropriate dimension
$\mathbf{O}$:	zero matrix with appropriate dimension
$\boldsymbol{e}_j$:	column vector of appropriate dimension with 1 in the j^{th} position and 0 elsewhere
$\boldsymbol{e}'_j$:	row vector of appropriate dimension with 1 in the j^{th} position and 0 elsewhere
I :	identity matrix of appropriate dimension
I_r :	identity matrix of dimension r
PH:	Phase type
$CTMC$:	Continuous Time Markov Chain
QBD:	Quasi-birth-and-death
$LIQBD$:	Level Independent Quasi-Birth-and-Death process
$LDQBD$:	Level Dependent Quasi-Birth-and-Death process
CLT:	Common Life Time

Chapter 1

Introduction

In any service station where it takes a positive amount of time to serve customers, queue of customers get formed. In classical queues, service process goes on if there is at least one customer and the server is ready to serve – availability of resources is not taken into consideration. This is not the case with inventory. Absence of inventory (no item) results in no service even when there are customers and the server is ready to serve. In classical inventory, queue of customers get formed only when there is no item in inventory, provided customers are allowed to join in the absence of inventory. This is a consequence of negligible service time. In practice it takes sometime to serve an item and so inventory with positive service time presents a more realistic situation than that with negligible service time. Such models are referred to as inventory with positive service time, or often called queueing-inventory problems by several researchers. In this thesis one encounters models providing explicit solution for system state distribution and also those that need algorithmic analysis.

Queueing phenomena can be found in almost all walks of life. For instance, in airport check-in systems, traffic intersections, supermarket check-out counters, telecommunication systems, manufacturing systems, bank branches.

Queueing theory is the mathematical study of ‘queue’ or ‘waiting lines’ where an item from inventory is provided to the customer on completion of service. A typical queueing system consists of a queue and a server. Customers arrive in the system from outside and join the queue in a certain way. The server picks up customers and serves them according to certain service discipline. Customers leave the system immediately after their service is completed.

For queueing systems, *queue length, waiting time and busy period* are of primary interest to applications. The theory permits the derivation and calculation of several performance measures including the average waiting time in the queue or the system, mean queue length, traffic intensity, the expected number waiting or receiving service, mean busy period, distribution of queue length, and the probability of encountering the system in certain states, such as empty, full, having an available server or having to wait a certain time to be served.

The simplest form of queuing models are based on the birth and death process, where the birth process describes the inter-arrival time (time between two arrivals) to the queue and the death process describes the service or holding time in the queue. Birth-death processes have many applications in demography, queueing theory and in biology, for example to study the evolution of bacteria. The state i, of the process represents the current size of the population. The transitions are limited to births and deaths. When a birth occurs, the process goes from state i to $i+1$ and with the occurrence of a death, the process goes from state i to state $i-1$.

When analyzing several stochastic system, block-structured stochastic models are found to be a useful and effective mathematical tool. The block-structured stochastic models began with studying the matrix-geometric stationary probability of a quasi-birth-and-death process (QBD) process. The initial attention was directed toward performance computation.

1.1 Quasi-birth-and-death process

A continuous time ***quasi-birth-and-death*** (QBD) process $\Omega = \{(X_k, J_k), k \geq 0\}$ is a continuous time Markov chain with state space $\{(0, j); 1 \leq j \leq a\} \bigcup \{(n, j); n \geq 1, 1 \leq j \leq b\}$ where a and b are positive integers. We call X_k the ***level variable*** and J_k the ***phase variable***. The Markov chain is called a quasi-birth-and-death process if the level variable X_k increases or decreases its value by at most one at each transition: it is possible to move in one step from $(n, j) \to (m, j')$ only if $m = n, n+1$ or $n-1$ (provided in the last case that $n \geq 1$). If $n = 0$, then $m = 0$ or 1. If the transition rates are level independent, then the QBD process is called ***level independent quasi-birth-and-death process*** (LIQBD); else it is called ***level dependent quasi-birth-and-death process*** (LDQBD). An LIQBD is a Markov process with state space $\{(0, j); 1 \leq j \leq a\} \bigcup \{(n, j); n \geq 1; 1 \leq j \leq b\}$ and its infinitesimal generator is of the form

$$\boldsymbol{\mathcal{Q}} = \begin{bmatrix} A_{00} & A_{01} & & & \\ A_{10} & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix} \tag{1.1}$$

where A_{00} is an $a \times a$ matrix, A_{01} is an $a \times b$ matrix, A_{10} is a $b \times a$ matrix and A_0, A_1, A_2 are square matrices of order b. A_0 represents the arrival of a customer to the system; that is transition from level $n \to n+1$. A_2 represents departure of a customer after service completion: $n \to n-1$, A_1 describes all transitions in which the level does not change (transitions within levels). Also matrices $A_{00}, A_{01}, A_{10}, A_0, A_1, A_2$ satisfy $(A_{00} + A_{01})\mathbf{e} = (A_{10} + A_1 + A_0)\mathbf{e} = (A_2 + A_1 + A_0)\mathbf{e} = \mathbf{0}$. Essentially we have a quasi-Toeplitz structure for $\boldsymbol{\mathcal{Q}}$ in the LIQBD process.

Let $A = A_0 + A_1 + A_2$. Then A is a generator matrix of order b, which governs the transitions of the phase variable, given that the level variable is 2 or greater. The generator matrix A and its invariant probability vector play an important role in the study of structured Markov chain.

1.2 Matrix Analytic Method

Marcel F Neuts pioneered ***Matrix Analytic Methods*** in the study of queueing models in the 1970's. It is a tool to construct and analyze a wide class of stochastic models, particularly telecommunication networks, transportation systems, supply chain systems, manufacturing systems and inventory systems, using a matrix formalism to develop algorithmically tractable solution. For a detailed description of this method see Latouche and Ramaswami [28], Neuts [33], Qi-Ming He [37] and for specialized subjects see Alfa [2] for matrix analytic methods in discrete time queues, Artalejo and Gomez-Corral [4] for matrix analytic methods applied to retrial queues, Bini et al. [9] for matrix analytic methods and numerical computation, Breuer and Baum [11] for matrix analytic methods and queueing theory and Tian and Zhang [50] for matrix analytic methods on vacation queues. The matrix geometric method is for quasi-birth-and-death process (QBD) whereas matrix analytic method is for GI/M/1 type structures.

From Neuts [33] we have the following theorem:

Theorem 1.2.1. *The process $\boldsymbol{Q}$ in (1.1) is positive recurrent if and only if the minimal non-negative solution R to the matrix quadratic equation*

$$R^2 A_2 + RA_1 + A_0 = \boldsymbol{O} \tag{1.2}$$

has all its eigenvalues lie inside the unit disk and the finite system of equations

$$\begin{aligned} \mathbf{x}_0 A_{00} + \mathbf{x}_1 A_{10} &= \boldsymbol{0}, \\ \mathbf{x}_0 A_{01} + \mathbf{x}_1 [A_1 + RA_2] &= \boldsymbol{0}, \\ \mathbf{x}_0 \boldsymbol{e} + \mathbf{x}_1 (I - R)^{-1} \boldsymbol{e} &= 1 \end{aligned} \tag{1.3}$$

has a unique positive solution for $\mathbf{x}_0$ *and* $\mathbf{x}_1$.
If the matrix $A = A_0 + A_l + A_2$ *is irreducible, then* $sp(R) < 1$ *if and only if*

$$\boldsymbol{\pi} A_0 \boldsymbol{e} < \boldsymbol{\pi} A_2 \boldsymbol{e} \tag{1.4}$$

where $\boldsymbol{\pi}$ *is the stationary probability vector of the matrix* A. *The stationary probability vector* $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_l, \mathbf{x}_2, ...)$ *of* $\boldsymbol{Q}$ *is given by*

$$\mathbf{x}_i = \mathbf{x}_1 R^{i-1}, \quad i \geq 2. \tag{1.5}$$

We can use an iterative algorithm or logarithmic reduction algorithm (see Latouche and Ramaswami ([28], [29])) for computing rate matrix R of the equation (1.2).

1.2.1 GI/M/1 type Markov chain

A continuous time $GI/M/1$ ***type Markov chain*** (see Neuts ([33], [34])) $\{(X_k, J_k), k \geq 0\}$ is a continuous time Markov chain with state space $\{((n, j); n \geq 0; 1 \leq j \leq m\}$ where m is a positive integer and the infinitesimal generator is of the form

$$\tilde{\boldsymbol{Q}} = \begin{bmatrix} B_0 & A_0 & & & \\ B_1 & A_1 & A_0 & & \\ B_2 & A_2 & A_1 & A_0 & \\ \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}. \tag{1.6}$$

The block matrices satisfy the following condition: $B_n\mathbf{e} + (A_0 + A_1 + ... + A_n)\mathbf{e} = \mathbf{0}$, for $n \geq 0$. The matrix $A = \sum_{k=0}^{\infty} A_k$ has negative diagonal and non-negative off-diagonal elements. Its row sums are non-positive.

Theorem 1.2.2. *(see Neuts [33]) The irreducible Markov process* $\tilde{\boldsymbol{Q}}$ *in (1.6) is positive recurrent if and only if the minimal non-negative solution* R *of the equation*

$$\sum_{k=0}^{\infty} R^k A_k = \boldsymbol{O} \tag{1.7}$$

has $sp(R) < 1$ *and if there exists a positive vector* $\mathbf{x}_0$ *such that*

$$\mathbf{x}_0 B[R] = 0. \tag{1.8}$$

The matrix $B[R] = \sum_{k=0}^{\infty} R^k B_k$ *is a generator.*

The stationary probability vector $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, ...)$ *of* $\tilde{\boldsymbol{Q}}$ *is given by*

$$\mathbf{x}_k = \mathbf{x}_0 R^k, \text{ for } k \geq 0 \tag{1.9}$$

together with the normalizing condition

$$\mathbf{x}_0 (I - R)^{-1} \boldsymbol{e} = 1. \tag{1.10}$$

The matrix R *has a positive maximal eigenvalue* θ. *If the generator* A *is irreducible, the left eigenvector* u^* *of* R, *corresponding to* θ, *is determined up to a multiplicative constant and may be chosen to be positive. Then the matrix* R *satisfies* $sp(R) < 1$ *if and only if*

$$\boldsymbol{\pi} A_0 \boldsymbol{e} < \sum_{k=2}^{\infty} (k-1) \boldsymbol{\pi} A_k \boldsymbol{e} \tag{1.11}$$

where $\boldsymbol{\pi}$ *satisfies* $\boldsymbol{\pi} A = 0$ *and* $\boldsymbol{\pi} \boldsymbol{e} = 1$.

Whenever $\theta = sp(R) < 1$, *the equality*

$$A_0 \boldsymbol{e} = \sum_{k=1}^{\infty} R^k \sum_{v=k+1}^{\infty} A_v \boldsymbol{e} \tag{1.12}$$

holds.

Phase type distribution (continuous time)

Phase type distributions (PH-distributions) were introduced by Neuts (1975) as a generalization of the exponential distribution.

The PH-distribution is characterized by an absorbing Markov chain with a finite set of states, which is measured by the time that the underlying Markov chain spends in all the transient states until the first absorption.

Consider a continuous time Markov chain with state space $\{1, 2, ..., m, m+1\}$ whose infinitesimal generator is given by

$$\mathcal{Q} = \begin{bmatrix} T & \mathbf{T^0} \\ \mathbf{0} & 0 \end{bmatrix} \tag{1.13}$$

where $\mathbf{T^0} \geq 0$ and $T\mathbf{e} + \mathbf{T^0} = \mathbf{0}$. The state $m+1$ is an absorbing state and all the other are transient. Let $(\boldsymbol{\alpha}, \alpha_{m+1})$ be the initial probability vector of the Markov chain, where $\boldsymbol{\alpha}\mathbf{e} + \alpha_{m+1} = 1$.

A nonnegative random variable X has a ***phase type distribution*** if its distribution function is given by

$$F(t) = P\{X \leq t\} = 1 - \boldsymbol{\alpha} \exp(Tt)\mathbf{e} = 1 - \boldsymbol{\alpha} \left(\sum_{n=0}^{\infty} \frac{t^n}{n!} T^n \right) \mathbf{e}, \ t \geq 0. \tag{1.14}$$

For the Markov chain given in (1.13), T is called a PH-generator. The 2-tuple $(\boldsymbol{\alpha}, T)$ is called a ***phase type representation*** (PH-representation) of order m for the PH-distribution.

- Density function $f(t) = \boldsymbol{\alpha} \exp(Tt)\mathbf{T^0}$.
- Moments $\mu'_i = (-1)^i i! \boldsymbol{\alpha} T^{-1}\mathbf{e}, \ i \geq 0$.
- Laplace-Stieltjes transform $\tilde{f}(s) = \alpha_{m+1} + \boldsymbol{\alpha}(sI - T)^{-1}\mathbf{T^0}, \quad \mathrm{Re}\ s \geq 0$.

For further information about the PH distribution (see Latouche and Ramaswami [28], Neuts [33]).

1.3 Inventory system

Inventory management is one of the most important tasks in business. ***Inventory*** is any stored resource that is used to satisfy current as well as future needs. Raw materials, work-in-process, and finished goods are examples of inventory. Inventory models have a wide range of application in hospitals, educational institutions, banks, agriculture, industries etc.

The fundamental problem in inventory management can be described by the following two questions: (for more details about the inventory system see Beyer et al. [6], Hadley and Whitin [16], Naddor [32], Sahin [40], Sivazlian and Stanfel [48])

(1) When should an order be placed?

(2) How much should be ordered?

The reorder point (when) and order quantity (how much) are normally determined by minimizing the total inventory cost that can be expressed as a function of these two variables. The total inventory cost is generally composed of the following components:

- Setup cost: The ***setup cost (ordering cost)*** represents the fixed charge incurred when an order is placed. Thus, frequent smaller orders will result in a higher setup cost than less frequent larger orders. The latter, in turn, results in increasing holding cost.

- Purchase cost: The ***purchase cost*** becomes an important factor when the commodity unit price becomes dependent on the size of the order. This situation is normally expressed in terms of a quantity discount, where the unit price of the item decreases with the increase in the quantity ordered.

- Holding cost: The ***holding cost*** which represents the costs of carrying inventory in stock (for example; interest on invested capital, storage, handling, depreciation and maintenance) normally increases with the increasing level of inventory.

- Shortage cost: The ***shortage cost*** is a penalty incurred when we run out of stock of a commodity that is asked for. It generally includes costs due to loss of customer's goodwill as well as potential loss in income.

There are two basic trade-offs in an inventory problem. One is the trade-off between setup costs and inventory holding costs. By placing orders frequently, the size of each order can be made relatively small. Therefore, the holding costs can be reduced. However, the total setup costs will go up. Conversely, less frequent orders will save setup costs but incur higher holding costs. The other trade-off is between holding costs and stockout costs. Holding more inventory reduces the likelihood of stockout, and vice versa. These trade-offs give rise to an optimization problem of finding the optimal ordering policy that minimizes the overall cost.

While dealing with inventory systems, there are several factors which are to be taken into consideration.

1. Demand Rate

   The number of items required per unit time is called ***demand rate***. The demand pattern of an item may be either deterministic or probabilistic.

2. Lead Time

   When an order is placed, it may take some time before delivery is effected. The time lag from the point at which an order is placed until the order is delivered is called the lead time. If the replenishment is instantaneous, then the ***lead time*** is zero, otherwise lead time is positive.

3. Ordering Policy

   - (s, S)- policy: In (s, S)-*policy*, the order is such that number of items needed at the time of replenishment is that many units to bring the level back to S, where s is the reorder level and S is the maximum inventory level.

- (s, Q)- policy: In (s, Q)- *policy*, the number of items ordered is fixed and is equal to $Q = S - s$. Here we take s as the reorder level and Q as the fixed ordering quantity.
- $(S-1, S)$- policy: In $(S-1, S)$- policy, an order is placed for exactly one unit at each epoch of occurrence of a demand. That is, the one-for-one $(S-1, S)$ inventory policy calls for a replenishment order after each demand, equal in magnitude to the size of the demand. This is often advocated for controlling the stock levels of expensive, slow-moving items.

There seems to be one more policy discussed in Schwarz et al. [43].

Queueing-inventory systems

ln queueing-inventory models the availability of items are also to be considered in addition to the features in queueing theory. If the time required to serve the items to the customers is taken to be positive, then a queue is formed. In inventory models with negligible service time, queue of customers is formed only when the system is out of stock and unsatisfied customers are permitted to wait. In the case of inventory with positive service time, queue is formed even when inventoried items are available because new customers can join while a service is going on. If either lead time or service time or both are taken to be positive, then also a queue is formed.

1.4 Review of related work

Given below is a review of work related to the theme of this thesis. Most of the real life situations need positive amount of time to serve the inventory. Sigman and Levi [46] were the first to introduce inventory models with positive service time. They assume that the processing of inventory require an arbitrarily distributed positive amount of time, thus leading to the formation of queue. Since then numerous studies on inventory models with positive service time are reported. Nevertheless product form solution could be arrived at in substantially limited number of investigations. We refer to the survey paper by Krishnamoorthy et al. [24] for details on queueing-inventory models with positive service time.

Schwarz et al. [43] discuss M/M/1 queueing system with inventory where the lead time is exponentially distributed. They analyze the problem for $(s, Q), (s, S), (S-1, S)$ and random order inventory policies and produce product form solution for these models by assuming that no customer joins the queue when inventory level is zero. Schwarz et al. [45] consider queueing networks with attached inventory where again product form solution for the system state is established. In this paper replenishment lead times are taken to be non-zero and random which depend on the load of the system. At each service station an order for replenishment is made when the inventory level at that station drops to its reorder level. When the inventory level depletes to zero, the server with zero attached inventory does not accept new customers; however, the lost sales are not lost to the system, instead rerouted to nodes with positive inventory. They derive stationary distributions of joint queue lengths and inventory processes in explicit product form.

Saffari et al. [39] consider a queueing-inventory system under the (s, Q) policy with lost sales in which demands occur according to a Poisson process. Service time duration follows exponential distribution. Replenishment lead time is arbitrarily distributed, independent of the on-hand inventory and number of customers in the system. In addition to these assumptions the condition that no customer joins when inventory level is zero, leads them to a product form solution for the long run system state distribution, thereby subsuming Schwarz et al. [43]. Krenzler and Daduna ([20], [21]) analyze a single server queueing inventory system with positive service time in a random environment. The service system and the environment interact in both directions. Whenever the environment enters a specific subset of its state space, the service process is completely blocked and new arrivals are lost. They obtain a necessary and sufficient condition for a product form steady state distribution of the joint queueing-environment process.

Krishnamoorthy and Viswanath [26] is the first reported work on production inventory system with positive service time; in that paper the time for producing each item is assumed to follow a Markovian production scheme. The customer arrival process follows a Markovian arrival process and the service time to each customer has a phase-type distribution. The analysis is purely algorithmic in that study. This is followed by Krishnamoorthy and Viswanath [27] analyzing an (s, S) production inventory system where again the processing of inventory requires some positive amount of time. With demand process assumed to be Poisson, service time exponentially distributed and no customer joining the queue when the inventory level is zero, they obtain an explicit product form solution. Recently Baek and Moon [5] discuss a single server production inventory system with lost sales in which demands are assumed to occur according to a Poisson process, service time is exponentially distributed. The stocks are replenished by an external order under (r, Q)-policy or an internal production. The internal production process is assumed to be a Poisson process. They derive the stationary joint distribution of the queue length and the on-hand inventory in product form.

An $(S - 1, S)$ inventory system with multiple classes of customers was first introduced by Ha [15] who analyzed a make-to-stock production system with lost sales and showed that for a certain class of problems the $(S - 1, S)$ policy with rationing is optimal. For the $(S - 1, S)$- policy Otten et al. [36] extensively analyze a multiple waiting line problem. With one server each assigned to each line they establish product form solution for the system state distribution.

In discrete time set up Lian et al. [30] consider inventory with common life time, namely discrete phase type distribution. The lead time is taken as zero; demand process geometric. However service time is assumed to be negligible. The concept of rationing of inventoried item is introduced in Sapna Isotupa [42]. She analyzes an $(S-1, S)$ inventory system with two demand classes and showed that under certain conditions there is a sub-optimal rationing policy which yields a lower cost for the supplier and higher service levels for both the high priority and low priority customers than the optimal policy where the two customers are treated alike. Once the on-hand inventory goes down to a prescribed level, demands of high priority customers alone are entertained until the next replenishment. She also assumes that the service time is negligible.

Queues with postponed work is introduced by Deepak et al. [13]. Customers arrive to a single server system and joins a buffer of finite capacity K. An arrival, encountering the buffer full, joins a pool of postponed work of infinite capacity with certain probability and with complementary probability leaves the system forever. Customers from the pool are transferred to the buffer, one at a time, with probability p, whenever the number in the buffer at a service completion epoch is less than $L(< K)$. Further this transfer is with probability one if no customer left in the buffer at a service completion epoch. This system is analyzed in the stationary case and a number of performance measures are obtained. This notion of postponement of work has been introduced into inventory by a few researchers (see Arivarignan et al. [3], Krishnamoorthy and Islam [23], Paul Manuel et al. [35], Sivakumar and Arivarignan [48]).

1.5 Summary of the thesis

In this thesis a few queueing-inventory models are discussed by identifying continuous time Markov chains. The resulting LIQBD and GI/M/1 Type Markov chains are analyzed algorithmically where closed form expressions could not be arrived at. In some cases we have obtained product form solutions for the system state. Numerical examples are done using MATLAB Program.

Now we turn to the content of the thesis. This thesis entitled "**On queueing-inventory models – product form solution; reservation, cancellation and common life time**" is divided into 7 chapters.

Chapter 1 provides an introduction to the theme of this thesis, which includes description of stochastic process, queueing theory, inventory system and matrix analytic method. It contains a survey on related work and also provides a brief of the work done in this thesis.

In Chapter 2, we analyze and compare three different single server queueing-inventory models with infinite capacity. Demand process is assumed to be Poisson, service time and lead time are independent exponentially distributed random variables. In model I, whenever the inventory level is less than or equal to the reorder level, new arrivals join only if the number of customers in the system is less than the on-hand inventory. On the other hand in model II, whenever the inventory level is less than or equal to the reorder level new arrivals do not join. In the third model, we consider the case where, when the inventory level enters a specific subset of its state space new arrivals do not join the system. Inventory cycle time distribution is obtained. Optimization problems associated with the models are investigated. Finally we discuss a special case where, whenever the inventory level enters a complete blocking set, new arrivals do not join and service process is also completely blocked. In this case we obtain the system state distribution as the product of their marginals. Further we investigate how large the blocking set can be in terms of inventory level. Numerical illustrations of the system behavior are also provided.

Chapter 3 is on a single server supply chain model in which stocks are kept in both the manufacturer warehouse (production centre) and the retail shop (distribution centre). Arrival of customers to the retail shop form a Poisson process and their service time follows exponential distribution. The maximum stock of the distribution centre is limited to $s + Q(= S)$. When the inventory level depletes to s due to services, it demands Q units at a time from the production centre. The lead time follows an exponential distribution. If the production centre has the required stock on-hand, the items are shipped on receiving the order; else it takes additional time to have Q items and then the shipping time. The production centre adopts a (rQ, KQ)-policy where the processing of inventory requires a positive random amount of time. Production time for unit item is exponentially distributed. Also we assume that no customer joins the queue when the inventory level in the distribution centre is zero. This assumption leads to an explicit product form solution for the steady state probability vector. *(Published in OPSEARCH (Springer) under the title: **Product form solution for some queueing-inventory supply chain problem**).*

In chapter 4 we analyze two single server, lost sales $(S - 1, S)$ queueing-inventory systems with two demand classes – high priority and low priority. The service of non-priority customers are preempted with arrival of high priority customers. We compare two different models – one in which low priority customers do not join the system only when the on-hand inventory is zero and in the other case when there is no high priority customer present but there is positive inventory an arriving low priority customer join is assumed to the system. In the second model we obtain stochastic decomposition of the system. On the contrary this property is absent in model I. We investigate the behavior of both of these queuing-inventory systems. Several performance measures are evaluated. Numerical illustrations of the system behavior are also provided. An optimization problem of interest of both models is discussed through an example. *(Presented in 27^{th} European Conference on Operational Research (EURO), Glasgow, July 2015 under the title: **Product form solution in***

two priority queueing-inventory systems).

In Chapter 5, a queueing-inventory system that has applications in railway / airline / bus reservation systems is discussed. Maximum items in the inventory is S which have a random common life time; this includes those that are sold in a particular cycle. A customer, on arrival to an idle server with at least one item in inventory, is immediately taken for service; or else he joins the buffer of maximum size S depending on number of items in the inventory (the buffer capacity varies and is, at any time, equal to the number of items in the inventory). The arrival of customers constitutes a Poisson process, demanding exactly one item each from the inventory. If there is no item in the inventory, the arriving customer first queue up in a finite waiting space of capacity K. When it overflows an arrival goes to an orbit of infinite capacity with probability p or is lost forever with probability $1 - p$. From the orbit he retries for service according to an exponentially distributed inter-occurrence time. The service time follows an exponential distribution. Cancellation of sold items before its expiry is permitted. Inventory gets added through cancellation of purchased items, until the expiry time. Cancellation time is assumed to be negligible. We analyze this system. Several performance characteristics are computed; expected sojourn time of the system in a cycle with "no inventory" and also "maximum inventory" are computed. Some illustrative numerical examples are presented. An optimization problem is numerically analyzed. (*Presented in* 10^{th} *International Workshop on Retrial Queues (WRQ), Tokyo, July 2014 & Published in Special issue of Annals of Operation Research (Springer) under the title:* ***On a queueing-inventory with reservation, cancellation, common life time and retrial***).

In Chapter 6 we consider two single server queueing-inventory systems in which items in the inventory have a random common life time. On realization of common life time, all customers in the system are flushed out. Subsequently the inventory reaches its maximum level S through a (positive lead time) replenishment for the next cycle which follows an exponential distribution. Through cancellation of purchases, inventory gets added until their expiry time; where cancellation time follows exponential distribution. Customers arrive according to a Poisson process and service time is exponentially distributed. On arrival if a customer finds the server busy, then he joins a buffer of varying size. If there is no inventory, the arriving customer first try to queue up in a finite waiting room of capacity K. Finding that full, he joins a pool of infinite capacity with probability γ $(0 < \gamma < 1)$; else it is lost to the system forever. We discuss two models based on 'transfer' of customers from the pool to the waiting room / buffer. In Model 1, if at a service completion epoch the waiting room size drops to a preassigned level $L - 1$ $(1 < L < K)$ or below, a customer is transferred from pool to waiting room with probability p $(0 < p < 1)$ and positioned as the last among the waiting customers. If at a departure epoch the waiting room turns out to be empty and there is at least one customer in the pool, then the one ahead of all waiting in the pool gets transferred to the waiting room with probability one. We introduce a totally different transfer mechanism in Model 2: when at a service completion epoch, the server turns idle with at least one item in the inventory, the head of the pooled customers is immediately taken for service. At the time of a cancellation, if the server is idle with none, one or more customers in the waiting room, then the head of the pooled customers goes to the buffer directly for service. Also we assume that no customer joins the system when there is no item in the inventory. Several system performance measures are obtained. A cost function is discussed for each model and some numerical illustrations are presented. Finally a comparison of the two models are made.*(Invited paper to the special issue "Stochastic Models" of Indian Journal of Pure and Applied Mathematics, Guest Editor: Professor M. K. Ghosh paper title:* ***GI/M/1 type queueing-inventory systems with postponed work, reservation, cancellation and common life time****).*

The focus in chapter 7 is on a single server queueing-inventory system in which items in the inventory have a random common life time. On realization of common life time, customers are flushed out from the finite buffer and waiting room. Subsequently the inventory reaches its maximum S through an instantaneous (zero lead time) replenishment for the next cycle. Also pooled customers, if any, immediately get transferred to the buffer and waiting room, subject to a maximum of $S + K$. Through cancellation of purchases, inventory gets added until their expiry time, where inter cancellation time follows exponential distribution. Customers arrive according to a Poisson process and service time is exponentially distributed. On arrival, if a customer finds the server busy, then he joins a buffer of varying size provided there is "space". If there is no inventory, the arriving customer first go to a finite waiting room of capacity K. If that is also full, he joins a pool of infinite capacity with probability γ $(0 < \gamma < 1)$; else it is lost to the system forever. When, at a service completion epoch the waiting room size drops to a preassigned level $L - 1$ $(1 < L < K)$ or below, a customer is transferred from pool to waiting room with probability p $(0 < p < 1)$ and positioned as the last among the waiting customers. If at a departure epoch the waiting room turns out to be empty and there is at least one customer in the pool, then the one ahead of all waiting in the pool gets transferred to the waiting room with probability one. Several system performance measures are obtained. A cost function is discussed and some numerical illustrations are presented.

Finally a section giving a few concluding remarks and some further possible investigations is provided.

Chapter 2

On partial and complete blocking set of states in queueing-inventory models

This chapter focuses on partial and complete blocking sets. Whereas both service process and new arrivals are completely blocked in the completely blocking set, in the partial blocking set service is provided as long as inventory is available. The only action that is not prevented while in complete blocking set is the replenishment of inventory. Through replenishment the system gets freed temporarily from blocking set.

Three queueing-inventory models are investigated in this chapter:
(1) When inventory level is less than or equal to the reorder level s, new arrivals join only when the number of customers in the system is less than the on-hand inventory level.

Some results in this chapter are included in the paper:
A. Krishnamoorthy, Dhanya Shajin and B. Lakshmy : **On partial and complete blocking set of states in queueing-inventory models** (communicated).

(2) Newly arriving customers do not join the system when the inventory level is less than or equal to the reorder level s.
(3) When the inventory level enters a specific subset $\{0, 1, 2, ..., s_2\}$ of $\{0, 1, 2, ..., s_2, s_2 + 1, ..., s_1, ..., S\}$ new arrivals do not join the system irrespective of the number of customers in the system.

In these three models partial blocking is considered, that is, service is provided as long as at least an item is available in the inventory subject to customer availability. As a particular case of Model 3 we examine a system with complete blocking set to produce product form solution.

Krenzler and Daduna ([20], [21]) have analyzed a single server queueing inventory system with positive service time in a random environment. The service system and the environment interact in both directions. Whenever the environment enters a specific subset of its state space, the service process is completely blocked and new arrivals are lost (blocking set of states with positive inventory). They obtain a necessary and sufficient condition for the stability of the system and then derive a product form steady state distribution of the joint queueing-environment process.

This chapter is motivated by Krenzler and Daduna [20] mentioned above where the authors assume that whenever the environment enters a specific subset (blocking state), the service process will be interrupted and no new arrivals are admitted to the system, rather they are lost to the system forever. Under this assumption a product form solution for the steady state distribution is obtained by them. However, they are silent as to how large this blocking set could be in terms of the number of items in the inventory. For example, if there is inventory and customers are waiting, blocking service proves costly.

In the present work, we assume that new arrivals join the system depending on the inventory level as well as the number of customers waiting. When the inventory level enters a partially blocking set, new arrivals are restricted, though service is continued to be provided. The decomposition property evades; worse still is that we do not have a closed form expression for system state distribution. Both (s, S) and (s, Q) ordering policies are considered.

Further we consider a special case of Model 3 in which, when the inventory level enters a complete blocking subset of the set of states, new arrivals do not join the system as well as no service is given until the system exits from this blocking set of states (here 'exit from the blocking set' is by way of replenishment against the order placed). In this case we obtain the joint distribution of the number of customers and the number of items in the inventory as the product of their marginals, which is similar to the condition imposed in the paper of Krenzler and Daduna [20]. Further we investigate the optimal size of the blocking set. We prove that the cost turns out to be minimum when the blocking set has the inventory coordinate equal to zero (see Table 2.15).

2.1 Mathematical formulation of Model 1

We consider an inventory system with positive service time. Arrival process is assumed to be Poisson with rate λ. Each customer demands one unit of the item, having a random duration of service time which is exponentially distributed with parameter μ. The maximum capacity of the storage system is fixed as S. The lead time is exponentially distributed with parameter β which is independent of the service and arrival processes. We consider two distinct replenishment policies: (a) (s, Q) policy (b) (s, S) policy.

Assumptions

- If the inventory level is greater than the reorder level s, then newly arriving customers join.
- If the inventory level is less than or equal to s and the number of customers in the system is less than the inventory level then also an arriving customer joins.

- If the inventory level is less than or equal to s and also less than or equal to the number of customers in the system, then new arrivals do not join until the next replenishment.

The model is studied as a quasi-birth-and-death process and a matrix geometric solution is obtained (see Latouche and Ramaswami [28], Neuts [33]). To obtain the state space of the QBD in the sequel we use the following notations.

$\boldsymbol{\mathcal{N}}(t)$	Number of customers in the system at time t
$\boldsymbol{\mathcal{I}}(t)$	Number of items in the inventory at time t

2.1.1 Analysis: (s, Q) policy

In this model when the inventoried items reach the level $s \geq 0$, an order for replenishment by fixed quatity Q, where $Q = S - s$, is placed. Then $\Omega = \{(\boldsymbol{\mathcal{N}}(t), \boldsymbol{\mathcal{I}}(t)); t \geq 0\}$ is a continuous time Markov chain with state space $E = \{0, 1, 2, ...\} \times \{0, 1, 2, ..., S\}$ and the infinitesimal generator matrix $\boldsymbol{\mathcal{Q}_1}$ is given by

$$\boldsymbol{\mathcal{Q}_1} = \begin{bmatrix} A_{00} & A_{01} & & & & & & \\ A_2 & A_{11} & A_{12} & & & & & \\ & A_2 & A_{22} & A_{23} & & & & \\ & & \ddots & \ddots & \ddots & & & \\ & & & A_2 & A_{s-1\ s-1} & A_{s-1\ s} & & \\ & & & & A_2 & A_1 & A_0 & \\ & & & & & \ddots & \ddots & \ddots \end{bmatrix}. \tag{2.1}$$

All submatrices in the above are square matrices of order $S+1$. The non-diagonal elements of $\boldsymbol{\mathcal{Q}_1} = (q_1((n,i),(m,j)); (n,i),(m,j) \in E)$ are:

- if the inventory level is less than or equal to s and the number of customers in the system is less than the inventory level then also an arriving customer joins. The transition rates are

$(n,i) \to (n+1,i) = \quad \lambda; \quad 0 \le n \le s-1, \quad i = n+1, n+2, ..., S$

- if the inventory level is greater than the reorder level s, then newly arriving customers join. Here transition rates are

  $(n,i) \to (n+1,i) = \quad \lambda; \quad n \ge s, \qquad\qquad i = s+1, s+2, ..., S$

- $(n,i) \to (n-1,i-1) = \quad \mu; \quad n \ge 1, \qquad\qquad i = 1, 2, ..., S$
- $(n,i) \to (n,i+Q) = \quad \beta; \quad n \ge 0, \qquad\qquad i = 0, 1, ..., s$
- $(n,i) \to (m,j) = \quad 0; \quad$ otherwise.

The diagonal entries are such that each row sum is zero.

Steady-state analysis

For determining the stability condition for the original system, consider $A = A_0 + A_1 + A_2$:

$$A = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ s \\ s+1 \\ \vdots \\ Q \\ \vdots \\ S-1 \\ S \end{array} \begin{array}{c} \begin{array}{cccccccccc} 0 & 1 & \cdots & s & s+1 & \cdots & Q & \cdots & S-1 & S \end{array} \\ \left(\begin{array}{cccccccccc} -\beta & & & & & & \beta & & & \\ \mu & -(\mu+\beta) & & & & & & \beta & & \\ & \ddots & \ddots & & & & & & \ddots & \\ & & & -(\mu+\beta) & & & & & & \beta \\ & & & \mu & -\mu & & & & & \\ & & & & \ddots & \ddots & & & & \\ & & & & & \mu & -\mu & & & \\ & & & & & & \ddots & \ddots & & \\ & & & & & & & \mu & -\mu & \\ & & & & & & & & \mu & -\mu \end{array} \right) \end{array}$$

Lemma 2.1.1. The steady-state probability distribution $\Pi = (\pi_0, \pi_1, ..., \pi_S)$ of the Markov chain corresponding to the generator A is given by

$$\pi_i = \begin{cases} \frac{\beta}{\mu}(\frac{\beta+\mu}{\mu})^{i-1}\pi_0, & i = 1, 2, ..., s \\ \frac{\beta}{\mu}(\frac{\beta+\mu}{\mu})^{s}\pi_0, & i = s+1, s+2, ..., Q \end{cases}$$

$$\pi_{Q+i} = \frac{\beta}{\mu}\left(\frac{\beta+\mu}{\mu}\right)^{i-1}\left[\left(\frac{\beta+\mu}{\mu}\right)^{s-(i-1)} - 1\right]\pi_0,\ i = 1, 2, ..., s$$

and π_0 can be obtained from the normalizing equation $\Pi\mathbf{e} = 1$.

Proof. We have $\Pi A = 0$ and $\Pi\mathbf{e} = 1$. The first equation of the above yields the following set of equations:

$$\begin{aligned}
-\beta\pi_0 + \mu\pi_1 &= 0,\\
-(\beta+\mu)\pi_i + \mu\pi_{i+1} &= 0,\ i = 1, 2, ..., s,\\
-\mu\pi_i + \mu\pi_{i+1} &= 0,\ i = s+1, s+2, ..., Q-1,\\
\beta\pi_i - \mu\pi_{Q+i} + \mu\pi_{Q+i+1} &= 0,\ i = 0, 1, ..., s-1,\\
\beta\pi_s - \mu\pi_S &= 0.
\end{aligned}$$

These equations can be recursively solved to get

$$\pi_i = \begin{cases} \frac{\beta}{\mu}(\frac{\beta+\mu}{\mu})^{i-1}\pi_0,\ i = 1, 2, ..., s \\ \frac{\beta}{\mu}(\frac{\beta+\mu}{\mu})^{s}\pi_0,\ i = s+1, s+2, ..., Q \end{cases}$$

$$\pi_{Q+i} = \frac{\beta}{\mu}\left(\frac{\beta+\mu}{\mu}\right)^{i-1}\left[\left(\frac{\beta+\mu}{\mu}\right)^{s-(i-1)} - 1\right]\pi_0,\ i = 1, 2, ..., s.$$

Using the normalizing condition $\Pi\mathbf{e} = 1$, we get

$$\pi_0 = \left[1 + (S-s)\frac{\beta}{\mu}\left(\frac{\beta+\mu}{\mu}\right)^s\right]^{-1}.$$

□

Lemma 2.1.2. The stability condition of the system under study is given by

$$\lambda < \frac{1 - \left[1 + (S-s)\frac{\beta}{\mu}\left(\frac{\beta+\mu}{\mu}\right)^s\right]^{-1}}{1 - \left(\frac{\beta+\mu}{\mu}\right)^s\left[1 + (S-s)\frac{\beta}{\mu}\left(\frac{\beta+\mu}{\mu}\right)^s\right]^{-1}}\mu \tag{2.2}$$

Proof. We look at the left as well as right drift rates of the Markov chain. These are respectively $\Pi A_2\mathbf{e}$ and $\Pi A_0\mathbf{e}$. When the former exceeds the latter the system is stable. Computation yields

$$\Pi A_0\mathbf{e} = \lambda\left[1 - \left(\frac{\beta+\mu}{\mu}\right)^s\left[1 + (S-s)\frac{\beta}{\mu}\left(\frac{\beta+\mu}{\mu}\right)^s\right]^{-1}\right],$$

$$\Pi A_2\mathbf{e} = \mu\left(1 - \left[1 + (S-s)\frac{\beta}{\mu}\left(\frac{\beta+\mu}{\mu}\right)^s\right]^{-1}\right).$$

$\square$

2.1.2 Analysis: (s, S) policy

In (s, S) policy, when the inventory level falls to s, order for replenishment is placed to bring the level back to S at the time of replenishment. The lead time for replenishment is exponentially distributed with parameter β. Under the above assumptions we get the infinitesimal generator matrix $\boldsymbol{\mathcal{Q}_1}$ given in (2.1). The transition rates $(q_1((n,i),(m,j)); (n,i),(m,j) \in E)$ are given by

$$\begin{array}{llll}
(n,i) \to (n+1,i) & = \lambda; & 0 \le n \le s-1, & i = n+1, n+2, ..., S, \\
(n,i) \to (n+1,i) & = \lambda; & n \ge s, & i = s+1, s+2, ..., S, \\
(n,i) \to (n-1,i-1) & = \mu; & n \ge 1, & i = 1, 2, ..., S, \\
(n,i) \to (n,S) & = \beta; & n \ge 0, & i = 0, 1, ..., s, \\
(n,i) \to (m,j) & = 0; & \text{otherwise.} &
\end{array}$$

The diagonal entries are such that each row sum is zero.

Now for determining the stability condition for the system, we define $\tilde{A} = A_0 + A_1 + A_2$.

Let $\mathbf{\Phi} = (\phi_0, \phi_1, ..., \phi_S)$ be the steady-state probability vector of the matrix $\tilde{A}$. Then $\mathbf{\Phi}$ satisfies the equation $\mathbf{\Phi}\tilde{A} = 0$ and $\mathbf{\Phi}\mathbf{e} = 1$.

Equation $\mathbf{\Phi}\tilde{A} = 0 \Rightarrow$

$$\begin{aligned} -\beta\phi_0 + \mu\phi_1 &= 0, \\ -(\beta+\mu)\phi_i + \mu\phi_{i+1} &= 0,\ i = 1, 2, ..., s, \\ -\mu\phi_i + \mu\phi_{i+1} &= 0,\ i = s+1, s+2, ..., S-1, \\ \beta\left[\phi_0 + \phi_1 + ... + \phi_s\right] - \mu\phi_S &= 0. \end{aligned}$$

Solving these equations recursively we get,

$$\phi_i = \begin{cases} \frac{\beta}{\mu}(\frac{\beta+\mu}{\mu})^{i-1}\phi_0, & i = 1, 2, ..., s, \\ \frac{\beta}{\mu}(\frac{\beta+\mu}{\mu})^{s}\phi_0, & i = s+1, s+2, ..., S. \end{cases}$$

Using the normalizing condition $\mathbf{\Phi e} = 1$, we further get

$$\phi_0 = \left(\frac{\beta+\mu}{\mu}\right)^{-s}\left[1 + \frac{\beta}{\mu}(S-s)\right]^{-1}.$$

Lemma 2.1.3. The stability condition of the original system under study is given by

$$\lambda < \frac{1 - \left(\frac{\beta+\mu}{\mu}\right)^{-s}\left[1 + \frac{\beta}{\mu}(S-s)\right]^{-1}}{1 - \left[1 + \frac{\beta}{\mu}(S-s)\right]^{-1}}\mu. \tag{2.3}$$

Proof. This is on the same lines as that for Lemma 2.1.2 and hence is omitted. □

2.1.3 Steady-state probability vector of $\mathcal{Q}_1$

We calculate the steady-state probability vector of $\mathcal{Q}_1$ under the stability condition. Let the steady-state probability vector $\mathbf{x}$ of $\mathcal{Q}_1$ be partitioned according to the number of customers in the system as $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, ...)$ where the sub-vectors $\mathbf{x}_i,\ \ i \geq 0$, contains $S+1$ elements $x_i(0), x_i(1), ..., x_i(S)$.

The sub-vectors satisfy the equations:

$$\begin{aligned}
\mathbf{x}_0 A_{00} + \mathbf{x}_1 A_2 &= 0, \\
\mathbf{x}_{i-1} A_{i-1\ i} + \mathbf{x}_i A_{i\ i} + \mathbf{x}_{i+1} A_2 &= 0, \quad 1 \le i \le s-1 \\
\mathbf{x}_{s-1} A_{s-1\ s} + \mathbf{x}_s A_1 + \mathbf{x}_{s+1} A_2 &= 0, \\
\mathbf{x}_{i-1} A_0 + \mathbf{x}_i A_1 + \mathbf{x}_{i+1} A_2 &= 0, \quad i \ge s+1.
\end{aligned}$$

From the matrix- geometric method (see Neuts [33]), we have

$$\mathbf{x}_{i+1} = \mathbf{x}_i R, \ i \ge s$$

where R is the minimal non-negative solution to the matrix quadratic equation $R^2 A_2 + RA_1 + A_0 = \mathbf{O}$ and the vectors $\mathbf{x}_0, \mathbf{x}_1, ..., \mathbf{x}_{s-1}, \mathbf{x}_s$ are obtained from the boundary equations:

$$\begin{aligned}
\mathbf{x}_0 A_{00} + \mathbf{x}_1 A_2 &= 0, \\
\mathbf{x}_{i-1} A_{i-1\ i} + \mathbf{x}_i A_{i\ i} + \mathbf{x}_{i+1} A_2 &= 0, \quad 1 \le i \le s-1, \\
\mathbf{x}_{s-1} A_{s-1\ s} + \mathbf{x}_s [A_1 + RA_2] &= 0.
\end{aligned}$$

Once R is obtained, from the boundary equation we obtain $\mathbf{x}_s = \mathbf{x}_{s-1} R_{s-1}$ and $\mathbf{x}_i = \mathbf{x}_{i-1} R_{i-1}, 1 \le i \le s-1$ which gives

$$\mathbf{x}_s = \mathbf{x}_0 \prod_{i=0}^{s-1} R_i$$

where (see Neuts [33])

$$\begin{aligned}
R_{s-1} &= A_{s-1\ s} \left[- \left(A_1 + RA_2 \right) \right]^{-1}, \\
R_{i-1} &= A_{i-1\ i} \left[- \left(A_{i\ i} + R_i A_2 \right) \right]^{-1}, \quad 1 \le i \le s-1.
\end{aligned}$$

$\mathbf{x}_0$ is obtained from $\mathbf{xe} = 1$, as

$$\mathbf{x}_0 \left[I + \sum_{k=0}^{s-2} \prod_{j=0}^{k} R_j + \prod_{j=0}^{s-1} R_j (I-R)^{-1} \right] \mathbf{e} = 1.$$

Performance Measures

1. Mean number of customers in the system $E_C = \sum_{i=0}^{\infty} i\mathbf{x}_i\mathbf{e}$.

2. Expected inventory level in the system $E_I = \sum_{i=0}^{\infty}\sum_{j=0}^{S} jx_i(j)$.

3. Expected reorder rate $E_R = \mu \sum_{i=1}^{\infty} x_i(s+1)$.

4. Expected loss rate of customers

$$E_L = \lambda \left[\sum_{i=0}^{s}\sum_{j=0}^{i} x_i(j) + \sum_{i=s+1}^{\infty}\sum_{j=0}^{s} x_i(j) \right].$$

2.2 Analysis of inventory cycle time

In the (s, S) policy with positive lead time L, an order is placed so as to bring its level to S at the replenishment epoch whenever the inventory level reaches s. Let T be the time elapsed, starting with s items in the inventory triggering an order for replenishment, until the inventory level returns to s for the first time (with exactly one replenishment in between). The inventory cycle time is a random variable whose distribution depends both on the number of customers in the system when the replenishment order is placed and length of the lead time. Let j be the number of customers in the system when the inventory level is s (that is, j is the number of customers in the starting state). The analysis of the inventory cycle time T using the formula $\sum_{i=1}^{4} P(T|\boldsymbol{\mathcal{A}}_i)P(\boldsymbol{\mathcal{A}}_i)$ where each $\boldsymbol{\mathcal{A}}_i,\ \ i = 1, 2, 3, 4$, is described below for **Case 1** in detail.

- t_0 is some arbitrary but fixed instant of time (inventory cycle time starting at time t_0).

- τ is a stopping time when the first replenishment occurs ($\tau > t_0$).
- ν is a stopping time when the inventory reaches state s again.
- $[\boldsymbol{\mathcal{I}}(t) = \ell]$ = "A new replenishment occurs, when the inventory level is ℓ".
- $\boldsymbol{\mathcal{A}}$ is any measurable set. It is used as a place holder for arguments of probability measures.

Case 1: The number of customers $j \geq S - \ell,\ 1 \leq \ell \leq s$

When the number of customers $j \geq S - \ell$, future arrivals need not be considered. The state space of $\boldsymbol{\mathcal{X}}(t) = (\boldsymbol{\mathcal{N}}(t), \boldsymbol{\mathcal{I}}(t))$ is $\{(n,m)\} \bigcup \{\Delta\}$ where $\{\Delta\}$ is the absorbing state namely, $\{(j - s + \ell, S)\}$. Here $s \geq m \geq \ell$, and $n = j$. During lead time the inventory level decreases from s to ℓ ($1 \leq \ell \leq s$), consequent to service completions. That is, the time before replenishment

$$P(\nu \in \boldsymbol{\mathcal{A}} / \boldsymbol{\mathcal{I}}(\tau) = \ell \cap \boldsymbol{\mathcal{N}}(t_0) \geq S - \ell \cap \boldsymbol{\mathcal{I}}(t_0) = s)$$

is Erlang distributed with parameter μ and $s - \ell$ stages. The probability that the replenishment will occur when the inventory level is $\ell,\ 1 \leq \ell \leq s$ is

$$P(\boldsymbol{\mathcal{I}}(\tau = \ell / \boldsymbol{\mathcal{N}}(t_0) \geq S - \ell \cap \boldsymbol{\mathcal{I}}(t_0) = s) = \left(\frac{\mu}{\beta + \mu}\right)^{s-\ell} \left(\frac{\beta}{\beta + \mu}\right).$$

Then from S, the inventory level drops to s, due to $S - s$ service completions with parameter μ each, which therefore follows Erlang distribution with $S - s$ stages. That is, $P(\nu \in \boldsymbol{\mathcal{A}} / \boldsymbol{\mathcal{I}}(\tau) = \ell \cap \boldsymbol{\mathcal{N}}(t_0) \geq S - \ell) = E_{\mu, S-s}(\boldsymbol{\mathcal{A}})$.

Thus the conditional distribution of the cycle time T is

$$\sum_{\ell=1}^{s} P(\nu \in \boldsymbol{\mathcal{A}} \cap \boldsymbol{\mathcal{I}}(\tau) = \ell / \boldsymbol{\mathcal{N}}(t_0) \geq S - \ell \cap \boldsymbol{\mathcal{I}}(t_0) = s)$$

$$= \sum_{\ell=1}^{s} E_{\mu, s-\ell}(\boldsymbol{\mathcal{A}}) P[\text{time taken for } s - \ell \text{ services} < L$$

$$< \text{time taken for } s-\ell+1 \text{ service}] * E_{\mu,S-s}(\boldsymbol{\mathcal{A}})$$
$$= \sum_{\ell=1}^{s} \frac{\beta}{\beta+\mu}\left(\frac{\mu}{\beta+\mu}\right)^{s-\ell} E_{\mu,s-\ell}(\boldsymbol{\mathcal{A}}) * E_{\mu,S-s}(\boldsymbol{\mathcal{A}}).$$

In the above and in what to follow, * denotes convolution.

Case 2: $j \geq S$ and $\ell = 0$; inventory level reaches zero before replenishment

In this case, if the number of customers is $j(\geq S)$ at the epoch of order placement, future arrivals need not be considered. Here we assume that the lead time realization occurs only after inventory level drops to zero. Thus the inventory cycle time T consists of the time taken for s service completions each following exponential distribution with parameter μ, followed by an idle time of the system having random duration following exponential distribution with parameter β plus service time of $S-s$ customers each of which has exponential distribution with parameter μ. The probability that the inventory will be empty before replenishment is

$$P(\boldsymbol{\mathcal{I}}(\tau) = 0/\boldsymbol{\mathcal{N}}(t_0) \geq S \cap \boldsymbol{\mathcal{I}}(t_0) = s) = \left(\frac{\mu}{\beta+\mu}\right)^{s}.$$

Thus the conditional inventory cycle time distribution is

$$P(\nu \in \boldsymbol{\mathcal{A}}/\boldsymbol{\mathcal{I}}(\tau) = 0 \cap \boldsymbol{\mathcal{N}}(t_0) \geq S \cap \boldsymbol{\mathcal{I}}(t_0) = s)$$
$$= E_{\mu,s}(\boldsymbol{\mathcal{A}}) P[L > \text{time taken for } s \text{ services}] * \exp(\beta)(\boldsymbol{\mathcal{A}}) * E_{\mu,S-s}(\boldsymbol{\mathcal{A}})$$
$$= \left(\frac{\mu}{\beta+\mu}\right)^{s} E_{\mu,s}(\boldsymbol{\mathcal{A}}) \; * \exp(\beta)(\boldsymbol{\mathcal{A}}) * E_{\mu,S-s}(\boldsymbol{\mathcal{A}}).$$

Case 3: The number of customers $j < s$

Here we have to consider the future arrivals until the number of customers and the inventory level become equal. Assume that the replenishment takes place only after the inventory level falls down below j. The state space of the process $\{\boldsymbol{\mathcal{X}}(t)\}$ is $\{(n,m)\}\bigcup\{\Delta\}$ where $0 \leq n, m \leq s$. All possible transitions of the finite Markov chain $\{\boldsymbol{\mathcal{X}}(t)\}$ and the corresponding instantaneous rates are given in Table 2.1. Thus the infinitesimal generator $\boldsymbol{\mathcal{Q}}_{j_1}$ of the Markov

from	To	Rate			
(n,m)	$(n+1,m)$	λ	$m = s-k$, $1 \leq m \leq s-(j+1)$,	$0 \leq k \leq j$,	$j-k \leq n \leq m-1$ $0 \leq n \leq m-1$
(n,m)	$(n-1,m-1)$	μ	$m = s-k$, $1 \leq m \leq s-j$,	$0 \leq k \leq j-1$,	$j-k \leq n \leq m$ $1 \leq n \leq m$
(n,m)	Δ	β	$0 \leq m < j-1$,		$0 \leq n \leq m$

Table 2.1: Transition rates in $\boldsymbol{\mathcal{Q}}_{j_1}$

chain $\{\boldsymbol{\mathcal{X}}(t)\}$ is of the form $\boldsymbol{\mathcal{Q}}_{j_1} = \begin{pmatrix} \boldsymbol{\mathcal{T}}_{j_1} & \boldsymbol{\mathcal{T}}^0_{j_1} \\ \mathbf{0} & 0 \end{pmatrix}$, with initial probability vector $\boldsymbol{\alpha}_{j_1} = (0,0,...,1,0,...,0)$ where 1 is at the $[(j+1)(s-j+1)]^{th}$ position. $\boldsymbol{\mathcal{T}}_{j_1}$ is a square matrix of order $(s-j+1)[(j+1)+(s-j)/2]$ and $\boldsymbol{\alpha}_{j_1}$ has $(s-j+1)[(j+1)+(s-j)/2]$ elements. Therefore, when $j < s$, the time till absorption to $\{\Delta\} = \{(n,S); 0 \leq n < j-1\}$, denoted by τ_{j_1}, follows Phase-type distribution having representation $(\boldsymbol{\alpha}_{j_1}, \boldsymbol{\mathcal{T}}_{j_1})$ and expected value $E(\tau_{j_1})(\boldsymbol{\mathcal{A}}) = -\boldsymbol{\alpha}_{j_1}(\boldsymbol{\mathcal{T}}_{j_1})^{-1}\mathbf{e}$.

Now we consider the time till absorption to s, starting from $\{\Delta\}$. Consider the Markov chain $\{\boldsymbol{\mathcal{X}}'(t)\} = \{(\boldsymbol{\mathcal{N}}(t), \boldsymbol{\mathcal{I}}(t)), t \geq 0\}$. The state space of $\boldsymbol{\mathcal{X}}'(t)$ is $\{(n,m)\}\bigcup\{\Delta'\}$ where $\{\Delta'\} = \{(n,s); 0 \leq n \leq s\}$ denotes the absorbing state.

If the system contains at least $S-s$ customers at replenishment epoch, future arrivals need not have to be considered. Then from S, the inventory level goes down to s after $S-s$ service completions. Therefore the time τ'_{j_1} till absorption to s, starting from $\{\Delta\}$, follows Erlang distribution $E_{\mu,S-s}(\boldsymbol{\mathcal{A}})$. Thus the inventory cycle time T follows PH-distribution, which is the convolution of PH-distribution $(\boldsymbol{\alpha}_{j_1}, \boldsymbol{\mathcal{T}}_{j_1})$ and Erlang distribution $E_{\mu,S-s}(\boldsymbol{\mathcal{A}})$.

If the system has $j'(< S-s)$ customers only, then we have to consider future arrivals. Clearly, $\{\boldsymbol{\mathcal{X}}'(t)\}$ is a finite state space Markov chain, the possible transitions and the corresponding instantaneous rates are given in Table 2.2. Hence the infinitesimal generator $\boldsymbol{\mathcal{Q}}'_{j_1}$ of this Markov chain is of the form

from	To	Rate			
(n,m)	$(n+1,m)$	λ	$m = S-k$	$0 \leq k \leq j'$	$j'-k \leq n \leq m-1$
			$s+1 \leq m \leq S-(j'+1)$		$0 \leq n \leq m$
(n,m)	$(n-1,m-1)$	μ	$m = S-k$	$0 \leq k \leq j'-1$	$j'-k \leq n \leq m$
			$s+2 \leq m \leq S-j'$		$1 \leq n \leq m$
$(n,s+1)$	Δ'	μ			$1 \leq n \leq s+1$

Table 2.2: Transition rates in $\boldsymbol{\mathcal{Q}}'_{j_1}$

$\boldsymbol{\mathcal{Q}}'_{j_1} = \begin{pmatrix} \boldsymbol{\mathcal{T}}'_{j_1} & \boldsymbol{\mathcal{T}}'^{0}_{j_1} \\ \mathbf{0} & 0 \end{pmatrix}$, with initial probability vector $\boldsymbol{\alpha}'_{j_1} = (0, 0, ..., 1, 0, ..., 0)$ of order $(S-j'+1)[(j'+1)+(S-j')/2]-(s+1)(s+2)/2$ having 1 at the $[(j'+1)(S-j'+1)]$ position. Therefore, the time till absorption to s, starting from $\{\Delta\}$, denoted by τ'_{j_1}, follows phase-type distribution whose expected value is $E(\tau'_{j_1})(\boldsymbol{\mathcal{A}}) = -\boldsymbol{\alpha}'_{j_1}(\boldsymbol{\mathcal{T}}'_{j_1})^{-1}\mathsf{e}$.

Thus the inventory cycle time T follows PH-distribution, which is the convolution of two phase-type distributions with representations $(\boldsymbol{\alpha}_{j_1}, \boldsymbol{\mathcal{T}}_{j_1})$ and $(\boldsymbol{\alpha}'_{j_1}, \boldsymbol{\mathcal{T}}'_{j_1})$ respectively.

Case 4: The number of customers $j \geq s$

When the number of customers $j(\geq s)$, future arrivals need not be considered. The state space of $\boldsymbol{\mathcal{X}}(t)$ is $\{(n,m)\} \bigcup \{\Delta\}$ where $0 \leq m \leq s,\ n = j$. In this case the inventory level decreases from level s to some level ℓ $(0 \leq \ell \leq s)$, due to service of $s-\ell$ customers. At the inventory level ℓ, the lead time realization occurs and it is restocked to S. Thus the time τ till absorption to $\{\Delta\}$, follows Erlang distribution $E_{\mu,s-\ell}(\boldsymbol{\mathcal{A}})$.

The time till absorption to s, starting from $\{\Delta\}$, is as discussed above. Thus the inventory cycle time T follows the Erlang distribution $E_{\mu,s-\ell}(\boldsymbol{\mathcal{A}}) * E_{\mu,S-s}(\boldsymbol{\mathcal{A}})$ when the system contains at least $S-s$ customers after replenishment, otherwise inventory cycle time T follows the convolution of Erlang distribution $E_{\mu,s-\ell}(\boldsymbol{\mathcal{A}})$ and phase-type distributions $(\boldsymbol{\alpha}'_{j_1}, \boldsymbol{\mathcal{T}}'_{j_1})$.

2.3 Mathematical formulation of Model 2

In this model also arrival process is assumed to be Poisson with rate λ. Each customer demands exactly one item from the inventory, having random duration of service which follows exponential distribution with parameter μ. The maximum capacity of the inventory level is fixed as S. The lead time is exponentially distributed with parameter β which is independent of the service time distribution as well as the arrival process of customers. As in the case of model 1, here we discuss two distinct replenishment policies: (a) (s, Q) policy (b) (s, S) policy.

Assumptions

- As long as the inventory level is greater than the reorder level s, newly arriving customers join.

- If the inventory level is less than or equal to reorder level s, then new arrivals do not join, irrespective of the number of customers already present in the system.

2.3.1 Analysis: (s,Q) policy

The process $\Omega' = \{(\boldsymbol{\mathcal{N}}(t), \boldsymbol{\mathcal{I}}(t)); t \geq 0\}$ is a $CTMC$ with state space $E' = \{(n, i); n \geq 0, 0 \leq i \leq S\}$. The infinitesimal generator $\boldsymbol{\mathcal{Q}}_2$ of this $CTMC$ is an $LIQBD$ of the form:

$$\boldsymbol{\mathcal{Q}}_2 = \begin{bmatrix} B_{00} & B_0 & & & \\ B_2 & B_1 & B_0 & & \\ & B_2 & B_1 & B_0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix} \tag{2.4}$$

where the sub-matrices B_{00}, B_0, B_1 and B_2 are square matrices of order $S+1$.

Define $\boldsymbol{\mathcal{Q}}_2((n,i);(m,j))$ as the transition rates from the state $(n,i) \to (m,j)$ where n (m) represents number of customers in the system and i (j) is the number of items in the inventory. Then the non-diagonal entries are given by

$$\boldsymbol{\mathcal{Q}}_2((n,i),(m,j)) = \begin{cases} \lambda & \text{for } m = n+1,\ j = i; \qquad n \geq 0,\ i = s+1, s+2, ..., S. \\ \mu & \text{for } m = n-1,\ j = i-1;\ \ n \geq 1,\ i = 1, 2, ..., S. \\ \beta & \text{for } m = n,\ j = i+Q; \qquad n \geq 0,\ i = 0, 1, ..., s. \\ 0 & \text{otherwise.} \end{cases}$$

The diagonal entries are such that each row sum is zero. The stability condition of this system is same as that of Model 1 given in (2.2). This is despite the fact that we imposed strong conditions on the admission of customers in this model.

2.3.2 Analysis: (s,S) policy

Under the (s,S) policy, the infinitesimal generator $\boldsymbol{\mathcal{Q}}_2$ in block partitioned form is as given in (2.4).

The non-diagonal elements of $\boldsymbol{\mathcal{Q}}_2 = (q_2((n,i),(m,j)); (n,i),(m,j) \in E)$ are

$$q_2((n,i),(m,j)) = \begin{cases} \lambda & \text{for } m = n+1,\ j = i; \qquad n \geq 0,\ i = s+1, s+2, ..., S. \\ \mu & \text{for } m = n-1,\ j = i-1;\ \ n \geq 1,\ i = 1, 2, ..., S. \\ \beta & \text{for } m = n,\ j = S; \qquad n \geq 0,\ i = 0, 1, ..., s. \\ 0 & \text{otherwise.} \end{cases}$$

The diagonal entries are such that each row sum is zero. The stability condition of this system is the same as the one given in (2.3).

2.3.3 Steady-state probability vector of $\boldsymbol{\mathcal{Q}}_2$

Let the steady-state probability vector $\tilde{\mathbf{x}}$ of $\boldsymbol{\mathcal{Q}}_2$ be partitioned by the levels into sub-vectors $\tilde{\mathbf{x}_i}$ for $i \geq 0$, which contains $S+1$ elements $\tilde{x}_i(0), \tilde{x}_i(1), ..., \tilde{x}_i(S)$.

Theorem 2.3.1. *When the stability condition holds, the steady-state probability vector* $\tilde{\mathbf{x}} = (\tilde{\mathbf{x}_0}, \tilde{\mathbf{x}_1}, \tilde{\mathbf{x}_2}, ...)$ *is given by*

$$\tilde{\mathbf{x}_i} = \tilde{\mathbf{x}_0} R'^i, \;\; i \geq 1$$

where the matrix R' satisfies the matrix quadratic equation

$$R'^2 B_2 + R' B_1 + B_0 = \boldsymbol{O}$$

and the vector $\tilde{x_0}$ is obtained by solving

$$\tilde{\mathbf{x}_0} \left[B_{00} + R' B_2 \right] = 0$$

subject to the normalizing condition

$$\tilde{\mathbf{x}_0} \left[I - R' \right]^{-1} \mathbf{e} = 1.$$

Proof. We have $\tilde{\mathbf{x}} \boldsymbol{\mathcal{Q}}_2 = 0$ and $\tilde{\mathbf{x}} \mathbf{e} = 1$.

The first equation of the above yields the set of equations

$$\tilde{\mathbf{x}_0} B_{00} + \tilde{\mathbf{x}_1} B_2 = 0, \tag{2.5}$$

$$\tilde{\mathbf{x}}_i B_0 + \tilde{\mathbf{x}}_{i+1} B_1 + \tilde{\mathbf{x}}_{i+2} B_2 = 0, \;\; i \geq 0. \tag{2.6}$$

In order to express the solution in a recursive form, we assume

$$\tilde{\mathbf{x}_i} = \tilde{\mathbf{x}_0} R'^i, \;\; i \geq 1$$

where the spectral radius of R' is less than 1, which is ensured by the stability condition. From equation (2.6) we get

$$\tilde{\mathbf{x}}_i \left[R'^2 B_2 + R' B_1 + B_0\right] = 0, \; i \geq 0.$$

Since the above equation is true for $i \geq 0$, we get

$$R'^2 B_2 + R' B_1 + B_0 = 0.$$

Thus R' is a solution of above matrix quadratic equation. Post multiplying the above equation by $\mathbf{e}$ we get

$$\left[I - R'\right] \; \left[\underline{\boldsymbol{\lambda}} - R'\underline{\boldsymbol{\mu}}\right] = 0,$$

where $\underline{\boldsymbol{\mu}} = (0, \mu, \mu, ..., \mu)^T$ and $\underline{\boldsymbol{\lambda}} = (0, 0, ..., 0, \lambda, \lambda, ..., \lambda)^T$ are column vectors of order $S + 1$. (In $\underline{\boldsymbol{\mu}}$, first element is 0, remaining S entries are μ and in $\underline{\boldsymbol{\lambda}}$ first $s + 1$ entries are 0, remaining are λ).

Since spectral radius of R' is less than one, $I - R'$ is non-singular. Hence we have $R'\underline{\boldsymbol{\mu}} = \underline{\boldsymbol{\lambda}}$.

From (2.5) we get $\tilde{\mathbf{x}_0}\,[B_{00} + R'B_2] = 0$. Also $\tilde{\mathbf{x}}\mathbf{e} = 1$, which implies

$$\tilde{\mathbf{x}_0}\left[I + \sum_{i=0}^{\infty} R'^{i+1}\right]\mathbf{e} = 1 \Rightarrow \tilde{\mathbf{x}_0}\left[I - R'\right]^{-1}\mathbf{e} = 1.$$

Thus $\tilde{\mathbf{x}_0}$ is obtained by solving

$$\tilde{\mathbf{x}_0}\left[B_{00} + R'B_2\right] = 0$$

subject to the normalizing condition

$$\tilde{\mathbf{x}_0}\left[I - R'\right]^{-1}\mathbf{e} = 1.$$

This completes the proof of the theorem. $\square$

Performance Measures

1. Mean number of customers in the system $E_C = \sum_{i=1}^{\infty} i\tilde{\mathbf{x}}_i\mathbf{e}$.

2. Expected inventory level in the system $E_I = \sum_{i=0}^{\infty}\sum_{j=0}^{S} j\tilde{x}_i(j)$.

3. Expected reorder rate $E_R = \mu\sum_{i=1}^{\infty}\tilde{x}_i(s+1)$.

4. Expected loss rate of customers $E_L = \lambda\sum_{i=0}^{\infty}\sum_{j=0}^{s}\tilde{x}_i(j)$.

2.4 Mathematical formulation of Model 3

Consider an infinite capacity queueing-inventory system with a single server to which customers arrive according to a Poisson process with rate λ. The service time is assumed to follow an exponential distribution with parameter μ. The replenishment rule is based on (s_1, Q) policy. That is, when the on-hand inventory reaches $s_1 > 0$, a replenishment order is placed for Q units and fix $S = Q + s_1$ as the maximum capacity of the inventory level. We introduce a partially blocking set (partially blocking, since complete blocking involves a bit more stronger assumption). In a complete blocking set all activities other than replenishment, are blocked. Assume that the new arrivals do not join the system whenever the inventory level falls in the blocking set $\{0, 1, 2, ..., s_2\}$, where $s_2 < s_1$, but the service process is continued as long as customers are available. This model involves positive lead time which follows an exponential distribution with parameter β. Then the process $\Omega = \{(\boldsymbol{\mathcal{N}}(t), \boldsymbol{\mathcal{I}}(t)) : t \geq 0\}$ is a $CTMC$ whose state space is $E = \{(n, i); n \geq 0, 0 \leq i \leq S\}$ and the infinitesimal generator $\boldsymbol{\mathcal{Q}}_3$ given by

$$\boldsymbol{\mathcal{Q}}_3 = \begin{bmatrix} C_{00} & C_0 & & & \\ C_2 & C_1 & C_0 & & \\ & C_2 & C_1 & C_0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix} \tag{2.7}$$

where the sub-matrices C_{00}, C_0, C_1 and C_2 are square matrices of order $S+1$.

Entries in the sub-matrices are given by

$$\begin{array}{lcll} (n,i) \to (n+1,i) & = & \lambda; & n \geq 0, \quad i = s_2+1, s_2+2, ..., S, \\ (n,i) \to (n-1,i-1) & = & \mu; & n \geq 1, \qquad i = 1,2,...,S, \\ (n,i) \to (n,i+Q) & = & \beta; & n \geq 0, \qquad i = 0,1,...,s_1, \\ (n,i) \to (m,j) & = & 0; & \text{otherwise.} \end{array}$$

The diagonal entries are determined by the fact that each row sum is zero.

Let $\boldsymbol{\xi} = (\xi_0, \xi_1, ..., \xi_S)$ be the steady-state probability vector of the generator $C = C_0 + C_1 + C_2$. That is, $\boldsymbol{\xi} C = 0$ and $\boldsymbol{\xi}\mathbf{e} = 1$.

Using the above equations, the vector $\boldsymbol{\xi}$ can be obtained explicitly in terms of the parameters of the model as:

$$\xi_i = \begin{cases} \frac{\beta}{\mu}\left(\frac{\beta+\mu}{\mu}\right)^{i-1}\xi_0, & 1 \leq i \leq s_1, \\ \frac{\beta}{\mu}\left(\frac{\beta+\mu}{\mu}\right)^{s_1}\xi_0, & s_1+1 \leq i \leq Q, \\ \frac{\beta}{\mu}\left(\frac{\beta+\mu}{\mu}\right)^{i-Q-1}\left[\left(\frac{\beta+\mu}{\mu}\right)^{s_1-(i-Q-1)} - 1\right]\xi_0, & Q+1 \leq i \leq S \end{cases}$$

and the unknown probability ξ_0 is found using the normalizing condition $\boldsymbol{\xi}\mathbf{e} = 1$ as

$$\xi_0 = \left[1 + Q\frac{\beta}{\mu}\left(\frac{\beta+\mu}{\mu}\right)^{s_1}\right]^{-1}$$

where ξ_i is the probability that the inventory level is $i,\ 0 \leq i \leq S$. Because of the QBD structure of the model, the queueing system is stable if and only if (see Neuts [33])

$$\boldsymbol{\xi} C_0 \mathbf{e} < \boldsymbol{\xi} C_2 \mathbf{e}.$$

This reduces to

$$\frac{\lambda}{\mu}\left[1-\left(\frac{\beta+\mu}{\mu}\right)^{s_2}\left[1+Q\frac{\beta}{\mu}\left(\frac{\beta+\mu}{\mu}\right)^{s_1}\right]^{-1}\right] < \left[1-\left[1+Q\frac{\beta}{\mu}\left(\frac{\beta+\mu}{\mu}\right)^{s_1}\right]^{-1}\right].$$

2.4.1 Steady-state distribution

Assume that the stability condition holds. Let $\mathbf{x}'$ denote the steady-state probability vector of the generator $\boldsymbol{\mathcal{Q}}_3$ given in (2.7). That is,

$$\mathbf{x}'\boldsymbol{\mathcal{Q}}_3 = 0 \quad \text{and} \quad \mathbf{x}'\mathbf{e} = 1.$$

Partition the steady-state probability vector $\mathbf{x}'$ according to the number of customers in the system as $\mathbf{x}' = (\mathbf{x}'_0, \mathbf{x}'_1, \mathbf{x}'_2, ...)$ where the sub-vectors $\mathbf{x}'_n,\ n \geq 0$, contains $S+1$ elements. The sub-vectors satisfy the equations

$$\begin{aligned} \mathbf{x}'_0 C_{00} + \mathbf{x}'_1 C_2 &= 0, \\ \mathbf{x}'_{n-1} C_0 + \mathbf{x}'_n C_1 + \mathbf{x}'_{n+1} C_2 &= 0, \quad n \geq 1. \end{aligned}$$

From the matrix-geometric method (see Neuts [33]),we have

$$\mathbf{x}'_n = \mathbf{x}'_0 \tilde{\Re}^n, \quad n \geq 1$$

where $\tilde{\Re}$ is the minimal non-negative solution to the matrix quadratic equation

$$\tilde{\Re}^2 C_2 + \tilde{\Re} C_1 + C_0 = \mathbf{O}.$$

The subvector $\mathbf{x}'_0$ is obtained by solving the boundary equation

$$\mathbf{x}'_0[C_{00} + \tilde{\Re} C_2] = 0$$

and the normalizing condition

$$\mathbf{x}'_0(I - \tilde{\Re})^{-1}\mathbf{e} = 1.$$

System Performance Measures

1. Expected number of customers in the system $E_C = \sum_{i=1}^{\infty} i\mathbf{x}'_i\mathbf{e}$.

2. Expected inventory level in the system $E_I = \sum_{i=0}^{\infty}\sum_{j=1}^{S} jx'_i(j)$.

3. Expected re-order rate $E_R = \mu\sum_{i=1}^{\infty} x'_i(s_1+1)$.

4. Expected loss rate of customers $E_L = \lambda\sum_{i=0}^{\infty}\sum_{j=0}^{s_2} x'_i(j)$.

2.4.2 Expected time to emptiness of the inventory

We consider $M/M/1/s_2+1$ queueing-inventory system. Starting from a state with inventory level S, we need to find the time till absorption to the state with zero inventory. Let n be the number of customers in the system when the inventory level is S, $0 \leq n \leq s_2+1$ (i.e. n is the number of customers in the starting state).

Consider a finite state Markov chain $\Omega' = \{(\boldsymbol{\mathcal{N}}(t), \boldsymbol{\mathcal{I}}(t)), t \geq 0\}$. The state space of Ω' is $\{(i,j)\}\bigcup\{\Delta\}$ where $0 \leq i \leq s_2+1,\ 1 \leq j \leq S$ and $\{\Delta\}$ is the absorbing state which denotes zero inventory. The possible transitions and corresponding instantaneous rates are given in Table 2.3.

	From	To	Rate		
Transition	(i,j)	$(i+1,j)$	λ	$0 \leq i \leq s_2$	$1 \leq j \leq S$
rates of	(i,j)	$(i-1,j-1)$	μ	$1 \leq i \leq s_2+1$	$2 \leq j \leq S$
$\tilde{\boldsymbol{\mathcal{T}}}_n$	(i,j)	$(i,j+Q)$	β	$0 \leq i \leq s_2+1$	$1 \leq j \leq s_1,$
Transition rates of $\tilde{\boldsymbol{\mathcal{T}}}_n^0$	$(i,1)$	$\{\Delta\}$	μ	$1 \leq i \leq s_2+1$	

Table 2.3: Transition rates in $\tilde{\boldsymbol{\mathcal{Q}}}$

Hence the infinitesimal generator $\tilde{\boldsymbol{\mathcal{Q}}}$, of the Markov chain Ω' is of the form

$\begin{pmatrix} \tilde{\boldsymbol{\mathcal{T}}}_n & \tilde{\boldsymbol{\mathcal{T}}}^0_n \\ 0 & \mathbf{0} \end{pmatrix}$ with initial probability vector $\boldsymbol{\alpha}_n == (0, ..., 0, 1, 0, ..., 0)$ where 1 is in the $S(n+1)^{th}$ position, $0 \leq n \leq s_2 + 1$. $\tilde{\boldsymbol{\mathcal{T}}}_n$ is a square matrix of order $S(s_2+2)$ and $\boldsymbol{\alpha}_n$ has $S(s_2+2)$ elements and $\tilde{\boldsymbol{\mathcal{T}}}_n\mathbf{e} + \tilde{\boldsymbol{\mathcal{T}}}^0_n = \mathbf{0}$. Therefore, the expected time τ until absorption to $\{\Delta\}$ follows PH-distribution with representation $(\boldsymbol{\alpha}_n, \tilde{\boldsymbol{\mathcal{T}}}_n)$ and $E(\tau) = -\boldsymbol{\alpha}_n(\tilde{\boldsymbol{\mathcal{T}}}_n)^{-1}\mathbf{e}$.

2.5 Case of complete blocking

Here we strengthen the assumption in the model described earlier (see also Krenzler and Daduna [20]): Whenever the inventory level reaches the blocking set, newly arriving customers are lost and service process is completely blocked. All other assumptions remain same as in Section 2.4. Thus the infinitesimal generator $\tilde{\boldsymbol{\mathcal{Q}}}_3$ has a form as given in (2.7). However the entries in the block matrices are as described below,

$$\begin{array}{lcll} (n,i) \rightarrow (n+1,i) & = & \lambda; \quad n \geq 0, & i = s_2+1, s_2+2, ..., S, \\ (n,i) \rightarrow (n-1,i-1) & = & \mu; \quad n \geq 1, & i = s_2+1, s_2+2, ..., S, \\ (n,i) \rightarrow (n,i+Q) & = & \beta; \quad n \geq 0, & i = 0, 1, ..., s_1, \\ (n,i) \rightarrow (m,j) & = & 0; \quad \text{otherwise.} & \end{array}$$

The diagonal entries are determined by the fact that each row sum is zero.

The matrix $C = C_0 + C_1 + C_2$ is a finite dimensional irreducible generator. Therefore there exists a unique positive stochastic solution $\tilde{\boldsymbol{\zeta}}$ of the steady state equation

$$\tilde{\boldsymbol{\zeta}}C = 0 \text{ and } \tilde{\boldsymbol{\zeta}}\mathbf{e} = 1.$$

The system is stable if and only if

$$\tilde{\boldsymbol{\zeta}}C_0\mathbf{e} < \tilde{\boldsymbol{\zeta}}C_2\mathbf{e} \Leftrightarrow \left(\sum_{i=s_2+1}^{S} \zeta_i\right)\lambda < \left(\sum_{i=s_2+1}^{S} \zeta_i\right)\mu \Leftrightarrow \lambda < \mu.$$

Thus we have the following lemma.

Lemma 2.5.1. The system with complete blocking of activities, other than replenishment, is stable if and only if $\lambda < \mu$.

For finding the steady state probability vector, we first consider an inventory system where the serving of the inventory is instantaneous, that is, the service time is negligible. Its infinitesimal generator $\boldsymbol{\mathcal{W}}$ is

$$
\begin{array}{c}
\\ 0 \\ \vdots \\ s_2 \\ s_2+1 \\ \vdots \\ s_1 \\ s_1+1 \\ \vdots \\ Q \\ \vdots \\ Q+s_2 \\ \vdots \\ S
\end{array}
\begin{array}{c}
\begin{array}{ccccccccccccc}
0 & \cdots & s_2 & s_2+1 & \cdots & s_1 & s_1+1 & \cdots & Q & \cdots & Q+s_2 & \cdots & S
\end{array}\\
\begin{pmatrix}
-\beta & & & & & & & & \beta & & & & \\
& \ddots & & & & & & & & \ddots & & & \\
& & -\beta & & & & & & & & \beta & & \\
& & \lambda & -(\lambda+\beta) & & & & & & & \ddots & & \\
& & & \ddots & \ddots & & & & & & & \ddots & \\
& & & & \lambda & -(\lambda+\beta) & & & & & & & \beta \\
& & & & & \mu & -\mu & & & & & & \\
& & & & & & \ddots & \ddots & & & & & \\
& & & & & & & \mu & -\mu & & & & \\
& & & & & & & & \ddots & \ddots & & & \\
& & & & & & & & & \mu & -\mu & & \\
& & & & & & & & & & \ddots & \ddots & \\
& & & & & & & & & & & \mu & -\mu
\end{pmatrix}
\end{array}.
$$

Let $\boldsymbol{\xi}' = (\xi_0', \xi_1', ..., \xi_S')$ be the steady state vector of the generator $\boldsymbol{\mathcal{W}}$. Then $\boldsymbol{\xi}'$ satisfies the equations $\boldsymbol{\xi}'\boldsymbol{\mathcal{W}} = 0$ and $\boldsymbol{\xi}'\mathbf{e} = 1$ and can be obtained as

$$
\xi_i' = \begin{cases}
0, & 0 \le i \le s_2 - 1, \\
\frac{\beta}{\lambda}\left(\frac{\beta+\lambda}{\lambda}\right)^{i-(s_2+1)} \xi_{s_2}', & s_2 + 1 \le i \le s_1, \\
\frac{\beta}{\lambda}\left(\frac{\beta+\lambda}{\lambda}\right)^{s_1-s_2} \xi_{s_2}', & s_1 + 1 \le i \le Q + s_2, \\
\frac{\beta}{\lambda}\left(\frac{\beta+\lambda}{\lambda}\right)^{i-(Q+s_2+1)} \left[\left(\frac{\beta+\lambda}{\lambda}\right)^{(Q+s_1+1)-i} - 1\right] \xi_{s_2}', & Q + s_2 + 1 \le i \le Q + s_1.
\end{cases}
$$

The unknown probability ξ_{s_2}' can be found from the normalizing condition $\boldsymbol{\xi}'\mathbf{e} = 1$ as

$$
\xi_{s_2}' = \left[1 + Q\frac{\beta}{\lambda}\left(\frac{\beta+\lambda}{\lambda}\right)^{s_1-s_2}\right]^{-1}.
$$

Now using the vector $\boldsymbol{\xi}'$ we can find the steady-state vector of $\boldsymbol{\mathcal{Q}}_3$. Let $\boldsymbol{\eta} = (\boldsymbol{\eta}_0, \boldsymbol{\eta}_1, \boldsymbol{\eta}_2, ...)$ be the steady-state probability vector of the original system.

Then $\boldsymbol{\eta}\boldsymbol{Q}_3 = 0$ and $\boldsymbol{\eta}\mathbf{e} = 1$. These equations reduce to

$$\boldsymbol{\eta}_0 C_{00} + \boldsymbol{\eta}_1 C_2 = 0,$$

$$\boldsymbol{\eta}_{i-1} C_0 + \boldsymbol{\eta}_i C_1 + \boldsymbol{\eta}_{i+1} C_2 = 0.$$

Assume the $\boldsymbol{\eta}_i$ to have a form:

$$\boldsymbol{\eta}_i = k \left(\frac{\lambda}{\mu}\right)^i \boldsymbol{\xi}' \quad \text{for} \quad i \geq 0, \tag{2.8}$$

where k is an unknown constant to be evaluated. Under the above assumption, we have

$$\boldsymbol{\eta}_0 C_{00} + \boldsymbol{\eta}_1 C_2 = k\boldsymbol{\xi}' \left[C_{00} + \frac{\lambda}{\mu} C_2\right],$$

$$\boldsymbol{\eta}_{i-1} C_0 + \boldsymbol{\eta}_i C_1 + \boldsymbol{\eta}_{i+1} C_2 = k \left(\frac{\lambda}{\mu}\right)^i \boldsymbol{\xi}' \left[C_{00} + \frac{\lambda}{\mu} C_2\right]$$

where $C_{00} + \frac{\lambda}{\mu} C_2 = \boldsymbol{\mathcal{W}}$. Also we have $\boldsymbol{\xi}'\boldsymbol{\mathcal{W}} = 0$.

Now applying the normalizing condition $\boldsymbol{\eta}\mathbf{e} = 1$, we get

$$k \left(1 + \left(\frac{\lambda}{\mu}\right) + \left(\frac{\lambda}{\mu}\right)^2 + \ldots\right) = 1.$$

Hence under the condition that $\lambda < \mu$, we have $k = 1 - \left(\frac{\lambda}{\mu}\right)$. *That is, we get a stochastic decomposition of the state space under the assumption that service process is interrupted in blocked set and further we are able to write the joint probability of the system state as product of their marginals.* Thus we have proved the main result for the special case discussed in this section which we summarize in

Theorem 2.5.1. *The system under consideration has stochastic decomposition with the joint distribution of the system state equal to the product of their marginal distributions.*

System performance measures

1. Expected number of customers in the system $E_C = \dfrac{\lambda}{\mu - \lambda}$.

2. Expected inventory level in the system

$$E_I = \left\{ s_2 + Q + \left[\frac{\beta}{2\lambda} (Q + 2s_1 + 1) - 1 \right] Q \left(\frac{\beta + \lambda}{\lambda} \right)^{s_1 - s_2} \right\} \xi'_{s_2}.$$

3. Expected re-order rate $E_R = \beta \ \xi'_{s_2}$.

4. Expected loss rate of customers $E_L = \lambda \xi'_{s_2}$.

2.6 Numerical illustration

In this section we provide numerical illustration of the system performance measures with variation in values of underlying parameters so as to compare the relative superiority of Models 1 and 2. The conclusions drawn in what to follow are specific to values assigned to parameters.

Effect of λ on various performance measures of Models 1 and 2

Table 2.4 indicates the variation in the system performance measures with arrival rate λ. The increase in the values of the performance measures like expected number of customers in the system, expected reorder rate and expected loss rate are on predictable lines in Models 1 and 2.

Note that the expected number of customers in Model 1 remains larger whereas that of inventory is higher in Model 2. This is a consequence of the assumption in Model 2 that customers do not join once inventory level drops to s until next replenishment. As a consequence, in Model 1 the expected reorder rate is higher and expected loss rate of customers is lesser.

	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2
λ	E_C	E_C	E_I	E_I	E_R	E_R	E_L	E_L
1	0.2441	0.2239	7.0342	7.4739	0.1091	0.1000	0.0179	0.1000
1.5	0.4014	0.3625	6.6087	7.2212	0.1590	0.1428	0.0694	0.2142
2	0.5873	0.5303	6.2327	6.9667	0.2037	0.1818	0.1670	0.3632
2.5	0.8122	0.7396	5.9054	6.7099	0.2429	0.2171	0.3135	0.5408
3	1.0948	1.0073	5.6231	6.4562	0.2771	0.2493	0.5065	0.7404
3.5	1.4676	1.3543	5.3805	6.2136	0.3066	0.2784	0.7410	0.9549

Table 2.4: Effect of λ with $\beta = 1, \mu = 5, s = 3, Q = 9$.

	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2
μ	E_C	E_C	E_I	E_I	E_R	E_R	E_L	E_L
3	1.5068	1.3703	6.2769	6.8383	0.1976	0.1802	0.2220	0.3545
4	0.8473	0.7721	6.2475	6.9172	0.2016	0.1815	0.1854	0.3617
5	0.5873	0.5303	6.2327	6.9667	0.2037	0.1818	0.1670	0.3632
6	0.4487	0.4023	6.2235	6.9966	0.2048	0.1818	0.1564	0.3635
7	0.3627	0.3236	6.2173	7.0157	0.2056	0.1818	0.1497	0.3636
8	0.3043	0.2705	6.2127	7.0285	0.2061	0.1818	0.1452	0.3636

Table 2.5: Effect of μ: Take $\beta = 1, \lambda = 2, s = 3, Q = 9$.

Effect of μ on various performance measures of Models 1 and 2

From Table 2.5 we can make the following observations. As the service rate increases, expected inventory level and expected loss rate increase in Model 2. But in Model 1 expected inventory level and expected loss rate decrease with increase in value of the parameter μ. This shows that expected loss rate is higher in Model 2.

Effect of β on various performance measures of Models 1 and 2

	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2	Model 1	Model 2
β	E_C	E_C	E_I	E_I	E_R	E_R	E_L	E_L
1	0.5873	0.5303	6.2327	6.9667	0.2037	0.1818	0.1670	0.3632
2	0.6326	0.5838	7.0419	7.4270	0.2163	0.1999	0.0530	0.1997
3	0.6461	0.6058	7.3473	7.6026	0.2193	0.2067	0.0260	0.1377
4	0.6522	0.6181	7.5061	7.6955	0.2205	0.2103	0.0158	0.1051
5	0.6556	0.6260	7.6031	7.7531	0.2210	0.2126	0.0110	0.0849

Table 2.6: Effect of β: Take $\mu = 5, \lambda = 2, s = 3, Q = 9$.

Table 2.6 shows that an increase in the value of parameter β makes an increase in the values of measures like expected number of customers, expected inventory level and expected reorder rate, whereas the expected loss rate decreases in Models 1 and 2.

Effect of the arrival rate in Model 3

λ	E_C	E_I	E_R	E_L
2	0.3665	11.0594	0.1900	0.1379
3	0.5852	9.5899	0.2594	0.5117
4	0.8207	8.3900	0.3060	1.0705
5	1.0872	7.5151	0.3359	1.7228
6	1.3994	6.9009	0.3548	2.4112

Table 2.7: Effect of λ: Fix $\beta = 1, \mu = 7, s_2 = 3, s_1 = 8, Q = 10$.

Table 2.7 shows that an increase in the arrival rate λ results in an increase in measures like expected number of customers in the system, expected reorder rate and expected loss rate, whereas the expected inventory level decreases.

Effect of the service rate in Model 3

μ	E_C	E_I	E_R	E_L
4	1.3193	9.7352	0.2133	0.4782
5	0.9401	9.5833	0.2344	0.5119
6	0.7236	**9.5603**	0.2490	**0.5178**
7	0.5852	9.5899	0.2594	0.5117
8	0.4896	9.6377	0.2670	0.5013
9	0.4200	9.6887	0.2726	0.4899

Table 2.8: Effect of μ: Take $\beta = 1, \lambda = 3, s_2 = 3, s_1 = 8, Q = 10$.

From Table 2.8, we observe that as service rate increases, expected number of customers decreases and expected reorder rate increases.

Effect of the replenishment rate in Model 3

The behavior of measures like the expected number of customers, expected inventory level and expected loss rate, as β increases, are on expected lines; where as the first two increase, the third one decreases. Another interesting observation that we get from Table 2.9 is that an optimal value could be obtained for expected reorder rate as the replenishment rate increases.

β	E_C	E_I	E_R	E_L
1	0.5852	9.5899	0.2594	0.5117
2	0.6350	11.7151	**0.2627**	0.1017
3	0.6418	12.3439	0.2613	0.0329
4	0.6439	12.6172	0.2608	0.0161
5	0.6439	12.7713	0.2606	0.0100

Table 2.9: Effect of β: Fix $\mu = 7, \lambda = 3, s_2 = 3, s_1 = 8, Q = 10$.

2.6.1 Optimization problem

(s,Q) policy

In this section we provide the optimal values of the inventory level s and the fixed order quantity Q. For computing the minimal costs (s, Q) of the given queueing-inventory models we introduce the cost function $K(s, Q)$ as

$$K(s, Q) = [\boldsymbol{\mathcal{C}}_0 + Q\boldsymbol{\mathcal{C}}_1]E_R + \boldsymbol{\mathcal{C}}_2 E_I + \boldsymbol{\mathcal{C}}_3 E_C + \boldsymbol{\mathcal{C}}_4 E_L$$

where
$\boldsymbol{\mathcal{C}}_0$: Fixed cost for placing an order
$\boldsymbol{\mathcal{C}}_1$: Procurement cost / unit
$\boldsymbol{\mathcal{C}}_2$: Holding cost of inventory / unit / unit time
$\boldsymbol{\mathcal{C}}_3$: Holding cost of customers / unit / unit time
$\boldsymbol{\mathcal{C}}_4$: Cost due to the loss of customers / unit / unit time
Due to the complexity of the cost function we are unable to compute the optimal pair (s, Q) explicitly, so we arrive at these by using numerical procedures.

We fix the values of the parameters: $\lambda = 3, \mu = 5, \beta = 1, \boldsymbol{\mathcal{C}}_0 = \$100, \boldsymbol{\mathcal{C}}_1 = \$10, \boldsymbol{\mathcal{C}}_2 = \$1, \boldsymbol{\mathcal{C}}_3 = \$0.5, \boldsymbol{\mathcal{C}}_4 = \$20$ and vary the values of s and Q.

Optimal (s, Q) pair	**(1,25)**	(2,25)	(3,25)	(4,25)	(5,24)	(6,24)	(7,24)	(8,24)
Minimum cost	**56.758**	57.2	57.684	58.242	58.881	59.596	60.376	61.21

Table 2.10: Optimal (s, Q) pair and corresponding minimum cost of Model 1

Optimal(s, Q) pair	**(1,25)**	(2,25)	(3,25)	(4,25)	(5,25)	(6,26)	(7,26)	(8,26)
Minimum cost	**56.568**	57.52	58.496	59.486	60.481	61.479	62.479	63.479

Table 2.11: Optimal (s, Q) pair and corresponding minimum cost of Model 2

From Table 2.10 we observe that $(s, Q) = (1, 25)$ is the optimal pair and the corresponding cost (minimum) is \$ 56.758. This obviously depends on the input parameter values. From Table 2.11 we get the optimal pair of (s, Q) as (1,25) and the corresponding minimum cost is \$ 56.568.

(s,S) policy

Now for computing the minimal costs of (s, S) models we introduce the cost function $K(s, S)$ as

$$K(s, S) = \left[\boldsymbol{\mathcal{C}}_0 + \sum_{i=0}^{s}(S - i)\boldsymbol{\mathcal{C}}_1\right] E_R + \boldsymbol{\mathcal{C}}_2 E_I + \boldsymbol{\mathcal{C}}_3 E_C + \boldsymbol{\mathcal{C}}_4 E_L$$

For numerical comparison we assign the following values to the parameters: $\lambda = 3, \mu = 5, \beta = 1, \mathcal{C}_0 = \$100, \mathcal{C}_1 = \$10, \mathcal{C}_2 = \$1, \mathcal{C}_3 = \$0.5, \mathcal{C}_4 = \$20$ and vary the values of s and S.

Optimal (s, S) pair	**(1,25)**	(2,24)	(3,24)	(4,26)	(5,30)	(6,34)	(7,40)	(8,45)
Minimum cost	**85.418**	114.24	144.6	176.74	210.4	245.1	280.46	316.21

Table 2.12: Optimal (s, S) pair and corresponding minimum cost of Model 1

Optimal (s, S) pair	**(1,26)**	(2,26)	(3,28)	(4,31)	(5,35)	(6,40)	(7,45)	(8,49)
Minimum cost	**85.157**	114.68	145.35	177.13	209.72	242.96	276.67	310.72

Table 2.13: Optimal (s, S) pair and corresponding minimum cost of Model 2

From Table 2.12 we observe that $(s, S) = (1, 25)$ is the optimal pair and the corresponding cost is \$ 85.418. Table 2.13 shows \$ 85.157 as the minimum cost corresponding to the optimal pair $(s, S) =$ (1,26).

Cost analysis of Model 3

We discuss the optimal values of the inventory levels s_2, s_1 and the fixed order quantity Q. For computing the minimal costs of the given queueing-inventory model we introduce the cost function $K(s_2, s_1, Q)$ as

$$K(s_2, s_1, Q) = [\boldsymbol{\mathcal{C}}_0 + Q\boldsymbol{\mathcal{C}}_1]E_R + \boldsymbol{\mathcal{C}}_2 E_I + \boldsymbol{\mathcal{C}}_3 E_C + \boldsymbol{\mathcal{C}}_4 E_L$$

where $\boldsymbol{\mathcal{C}}_0, \boldsymbol{\mathcal{C}}_1, \boldsymbol{\mathcal{C}}_2, \boldsymbol{\mathcal{C}}_3$ and $\boldsymbol{\mathcal{C}}_4$ are defined earlier.

We fix the values of the parameters: $\lambda = 6, \mu = 7, \beta = 1, \boldsymbol{\mathcal{C}}_0 = \$100, \boldsymbol{\mathcal{C}}_1 = \$10, \boldsymbol{\mathcal{C}}_2 = \$1, \boldsymbol{\mathcal{C}}_3 = \$0.5, \boldsymbol{\mathcal{C}}_4 = \$20$ and vary the values of s_2, s_1 and Q. We obtain the following Table which provide the optimal pairs (s_2, s_1, Q) and the corresponding minimum cost (in \$). Table 2.14 indicate that optimal

Optimal values	(0,11,26)	(1,10,29)	**(2,9,30)**	(3,10,30)	(4,11,30)	(5,12,30)	(6,13,30)
Minimum cost	69.182	68.946	**66.737**	67.739	68.74	69.741	70.742

Table 2.14: Optimal values (s_2, s_1, Q) and minimum cost

pair (s_1, Q) is obtained when $s_2 = 2$ and the corresponding minimum cost is \$66.737. In this model it is difficult to prove analytically that the cost function is convex in s_2 because of high non-linearity of the function. Nevertheless, all numerical experiments we have performed indicate that this cost function first decreases in s_2, attains a minimum and then starts going up.

Analysis of cost function in the case of complete blocking

Based on the above performance measures, we construct a cost function for checking the optimality of the inventory level s_2.

$$K(s_2, s_1, Q) = \xi'_{s_2}\left\{\beta(\boldsymbol{\mathcal{C}}_0 + Q\boldsymbol{\mathcal{C}}_1) + \lambda\boldsymbol{\mathcal{C}}_4 + \left[s_2 + Q + Q\left(\frac{\beta+\lambda}{\lambda}\right)^{s_1-s_2}\left(\frac{\beta}{2\lambda}[Q + 2s_1 + 1] - 1\right)\right]\boldsymbol{\mathcal{C}}_2\right\} + \left(\frac{\lambda}{\mu-\lambda}\right)\boldsymbol{\mathcal{C}}_3.$$

where $\boldsymbol{\mathcal{C}}_0, \boldsymbol{\mathcal{C}}_1, \boldsymbol{\mathcal{C}}_2, \boldsymbol{\mathcal{C}}_3$ and $\boldsymbol{\mathcal{C}}_4$ are defined earlier.

Fixing the values of the parameters: $\lambda = 5, \mu = 7, \beta = 1, \boldsymbol{\mathcal{C}}_0 = \$100, \boldsymbol{\mathcal{C}}_1 = \$10, \boldsymbol{\mathcal{C}}_2 = \$1, \boldsymbol{\mathcal{C}}_3 = \$0.5, \boldsymbol{\mathcal{C}}_4 = \$20$ and vary the values of s_2, s_1 and Q we obtain the following Table:

Optimal triplet	(0,3,32)	(1,4,32)	(2,5,32)	(3,6,32)	(4,7,32)	(5,8,32)
Minimum cost	**82.362**	83.362	84.363	85.363	86.363	87.363

Table 2.15: Optimal (s_2, s_1, Q) and minimum cost

From Table 2.15 optimal pair (s_1, Q) is $(3, 32)$, obtained when $s_2 = 0$ and the corresponding minimum cost is \$82.362.

Chapter 3

Product form solution for a queueing-inventory supply chain problem

In the previous chapter we established product form solution for the (s_1, Q) inventory model, under heavy restrictions (Special case of Model 3 discussed there). In the present chapter we analyze a single server supply chain model in which stocks are kept in both the production centre and the distribution centre. Arrival of customers to the retail shop forms a Poisson process and their service time are exponentially distributed. The maximum stock level at the distribution centre is limited to $s + Q(= S)$. When the inventory level depletes to s due to services, it demands Q units of items from the production centre. The lead time follows exponential distribution. If the production centre has the required stock on-hand, the items are despatched immediately.

Some results in this chapter are appeared in Journal of OPSEARCH.

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Supply of items from the production centre to the distribution centre is done only in packets of Q units at a time. So if a packet of size Q is not available the distribution centre has to wait till Q units accumulate in the production centre. The production plant adopts an (rQ, KQ) policy where the processing of inventory requires a positive random amount of time. Production time for unit item is exponentially distributed. We impose the condition that no customer joins the queue when the inventory level in the distribution centre is zero.

The motivation behind this work comes from Krishnamoorthy and Viswanath [27]. As in that paper, in the present case also we consider the condition that no customer joins the system when the inventory level is zero at the distribution centre. Those who are already in the queue do not leave the system. This assumption has lead to a product form solution for the system state distribution in the problem under discussion here.

This chapter has theoretical significance in that it provides product form solution despite strong dependence between the number of customers joining during lead time (as long as inventory level is positive) and the duration of the lead time.

Its practical application lies in the following: Customers who join the system would like to minimize their waiting time in the system. As a consequence an arriving customer prefers not to join the system if the inventory level is found to be zero. Under this condition we get a simple expression for the system stability which is independent of the lead time distribution. However, if customers join the system (with positive probability) when inventory level is zero then the expression for system stability will involve the lead time distribution parameter. Further the system state distribution will not have the closed form solution that we could arrive at in the present chapter.

3.1 Mathematical formulation and analysis

A supply chain model with one production centre and one distribution centre is considered in this chapter. Stocks are kept in the retail shop for meeting customer demands whereas the items produced at the plant (which is also considered as the warehouse) are stocked there for meeting demands from the distribution centre. The distribution centre has one server to serve customers when on-hand inventory is positive. Customers arrive to the distribution centre according to a Poisson process of rate λ. Service time follows exponential distribution with parameter μ. The maximum stock at the distribution centre is limited to $s + Q(= S)$.

When the stock of the distribution centre depletes to s, it demands Q items from the production centre. At that instant if the production centre has the required stock on-hand, the items are supplied to the distribution centre. It takes an exponentially distributed amount of time (lead time) with parameter β for the item to reach the distribution centre. At the time of receiving the order if the production centre has less than Q items on stock then an additional time for inventory to accumulate to Q by production is needed. Production time for each unit is exponentially distributed with parameter η.

The production centre adopts an (rQ, KQ) policy where $0 < r < K < \infty$ and are positive integers. When the inventory level depletes to rQ the production process is immediately switched on. The production process is kept in the on mode until the inventory level becomes KQ. We assume that no customer is allowed to join the queue when the inventory level in the distribution centre is zero; such demands are considered as lost. These are referred to as lost sales.

To describe the state space of the QBD we use the following notations in the sequel.

$\mathcal{N}(t)$	number of customers in the system
$\mathcal{I}_P(t)$	number of items present in the production centre
$\mathcal{I}_D(t)$	inventory level in the distribution centre
$\mathcal{M}(t)$	status of the production process: on or off mode

The production process is always in on mode if $0 \leq \mathcal{I}_P(t) \leq rQ$ and it is in off mode if $\mathcal{I}_P(t) = KQ$; but when the inventory level lies between $rQ+1$ and $KQ-1$, $\mathcal{M}(t)$ is either 1 or 0 according as the production is in on or off mode, respectively.

Under the above assumptions $\Omega = \{(\mathcal{N}(t), \mathcal{I}_D(t), \mathcal{I}_P(t), \mathcal{M}(t)), t \geq 0\}$ forms a $CTMC$ with state space $\bigcup_{n=0}^{\infty} \ell(n)$ where

$$\begin{aligned}\ell(n) = \quad & \{(n, i_D, i_P, 1); 0 \leq i_D \leq S, 0 \leq i_P \leq rQ\} \\ & \bigcup \{(n, i_D, i_P, m); 0 \leq i_D \leq S, rQ+1 \leq i_P \leq KQ-1, m = 1, 0\} \\ & \bigcup \{(n, i_D, KQ, 0); 0 \leq i_D \leq S\}.\end{aligned}$$

We now describe the infinitesimal generator matrix $\mathcal{Q}$ of this $CTMC$. Note that by the assumptions made above the $CTMC$ Ω is a $LIQBD$.

We have

$$\mathcal{Q} = \begin{bmatrix} B_0 & A_0 & & & \\ A_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix} \tag{3.1}$$

Each matrix B_0, A_0, A_1, A_2 is a square of order $(S+1)(2K-r)Q$ where

$$A_0 = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \lambda I_{S(2K-r)Q} \end{bmatrix}, A_2 = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mu I_{S(2K-r)Q} & \mathbf{0} \end{bmatrix}$$

$$B_0 = \begin{bmatrix} F & & M & & & \\ & F_1 & & \ddots & & \\ & & \ddots & & \ddots & \\ & & & F_1 & & M \\ & & & & F_2 & \\ & & & & & \ddots \\ & & & & & F_2 \end{bmatrix}, A_1 = \begin{bmatrix} F & & M & & & \\ & F_3 & & \ddots & & \\ & & \ddots & & \ddots & \\ & & & F_3 & & M \\ & & & & F_4 & \\ & & & & & \ddots \\ & & & & & F_4 \end{bmatrix}$$

with F, F_1, F_2, F_3, F_4 and M are square matrices of order $(2K-r)Q$.

Define $\boldsymbol{\mathcal{H}}_{(i_P,k_1)}^{(j_P,k_2)}$ for $\boldsymbol{\mathcal{H}} = F, F_1, F_2, F_3, F_4, M$ as the transition rates from the state $(i_P, k_1) \to (j_P, k_2)$ where i_P, j_P represent the number of items in the production centre and k_1, k_2 are the mode of production process.

$$F_{(i_P,k_1)}^{(j_P,k_2)} = \begin{cases} \eta & j_P = i_P + 1, 0 \le i_P \le KQ - 2, k_2 = k_1 = 1 \\ \eta & j_P = i_P + 1, i_P = KQ - 1, k_2 = 0, k_1 = 1 \\ -\eta & j_P = i_P, 0 \le i_P \le Q - 1, k_2 = k_1 = 1 \\ -(\eta + \beta) & j_P = i_P, Q \le i_P \le KQ - 1, k_2 = k_1 = 1 \\ -\beta & j_P = i_P, rQ + 1 \le i_P \le KQ, k_2 = k_1 = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$F_{1(i_P,k_1)}^{(j_P,k_2)} = \begin{cases} \eta & j_P = i_P + 1, 0 \le i_P \le KQ - 2, k_2 = k_1 = 1 \\ \eta & j_P = i_P + 1, i_P = KQ - 1, k_2 = 0, k_1 = 1 \\ -(\lambda + \eta) & j_P = i_P, 0 \le i_P \le Q - 1, k_2 = k_1 = 1 \\ -(\lambda + \eta + \beta) & j_P = i_P, Q \le i_P \le KQ - 1, k_2 = k_1 = 1 \\ -(\lambda + \beta) & j_P = i_P, rQ + 1 \le i_P \le KQ, k_2 = k_1 = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$M_{(i_P,k_1)}^{(j_P,k_2)} = \begin{cases} \beta & j_P = i_P - Q, Q \le i_P \le KQ - 1, k_2 = k_1 = 1 \\ \beta & j_P = i_P - Q, rQ + 1 \le i_P \le (r+1)Q, k_2 = 1, k_1 = 0 \\ \beta & j_P = i_P - Q, (r+1)Q + 1 \le i_P \le KQ, k_2 = k_1 = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$F_{2(i_P,k_1)}^{(j_P,k_2)} = \begin{cases} \eta & j_P = i_P + 1, 0 \le i_P \le KQ - 2, k_2 = k_1 = 1 \\ \eta & j_P = i_P + 1, i_P = KQ - 1, k_2 = 0, k_1 = 1 \\ -(\lambda + \eta) & j_P = i_P, 0 \le i_P \le KQ - 1, k_2 = k_1 = 1 \\ -\lambda & j_P = i_P, rQ + 1 \le i_P \le KQ, k_2 = k_1 = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$F_{3(i_P,k_1)}^{(j_P,k_2)} = \begin{cases} \eta & j_P = i_P + 1, 0 \le i_P \le KQ - 2, k_2 = k_1 = 1 \\ \eta & j_P = i_P + 1, i_P = KQ - 1, k_2 = 0, k_1 = 1 \\ -(\lambda + \mu + \eta) & j_P = i_P, 0 \le i_P \le Q - 1, k_2 = k_1 = 1 \\ -(\lambda + \mu + \eta + \beta) & j_P = i_P, Q \le i_P \le KQ - 1, k_2 = k_1 = 1 \\ -(\lambda + \mu + \beta) & j_P = i_P, rQ + 1 \le i_P \le KQ, k_2 = k_1 = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$F_{4(i_P,k_1)}^{(j_P,k_2)} = \begin{cases} \eta & j_P = i_P + 1, 0 \le i_P \le KQ - 2, k_2 = k_1 = 1 \\ \eta & j_P = i_P + 1, i_P = KQ - 1, k_2 = 0, k_1 = 1 \\ -(\lambda + \mu + \eta) & j_P = i_P, 0 \le i_P \le KQ - 1, k_2 = k_1 = 1 \\ -(\lambda + \mu) & j_P = i_P, rQ + 1 \le i_P \le KQ, k_2 = k_1 = 0 \\ 0 & \text{otherwise.} \end{cases}$$

3.1.1 System stability

Next we examine the system stability. Define $A = A_0 + A_1 + A_2$. This is the infinitesimal generator of the finite state $CTMC$ $\Omega' = \{(\boldsymbol{\mathcal{I}}_D(t), \boldsymbol{\mathcal{I}}_P(t), \boldsymbol{\mathcal{M}}(t)),$ $t \ge 0\}$ corresponding to the inventory level $\{0, 1, ..., S\}$ in the distribution centre. Let $\boldsymbol{\pi} = (\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, ..., \boldsymbol{\pi}_S)$ be the steady-state probability vector of A where each

$$\boldsymbol{\pi}_i = \Big(\pi_i(0,1), ..., \pi_i(rQ,1), \pi_i(rQ+1,1), \pi_i(rQ+1,0), ...,$$
$$\pi_i(KQ-1,1), \pi_i(KQ-1,0), \pi_i(KQ,0)\Big), \quad \text{for} \quad 0 \le i \le S,$$

is of order $(2K - r)Q$. Then $\boldsymbol{\pi}A = 0, \quad \boldsymbol{\pi}\mathbf{e} = 1.$

This Markov chain is stable if and only if the left drift rate exceeds the right drift rate which amounts to

$$\boldsymbol{\pi}A_0\mathbf{e} < \boldsymbol{\pi}A_2\mathbf{e} \tag{3.2}$$

From the matrices A_0, A_2 we have $\boldsymbol{\pi}A_0\mathbf{e} = \lambda\sum_{i=1}^{S}\boldsymbol{\pi}_i\mathbf{e}$ and $\boldsymbol{\pi}A_2\mathbf{e} = \mu\sum_{i=1}^{S}\boldsymbol{\pi}_i\mathbf{e}$. Using relation (3.2) we obtain the stability condition as $\lambda < \mu$. Thus we have the following result.

Theorem 3.1.1. *The system under study is stable if and only if* $\lambda < \mu$.

3.2 Steady-state analysis

For finding the steady-state probability vector of the $CTMC$ Ω, we first consider the distribution centre of the system where the serving of the inventory is instantaneous. The corresponding Markov chain is defined as $\{(\boldsymbol{\mathcal{I}}_D(t), \boldsymbol{\mathcal{I}}_P(t), \boldsymbol{\mathcal{M}}(t)), t \geq 0\}$ where $\boldsymbol{\mathcal{I}}_D(t), \boldsymbol{\mathcal{I}}_P(t)$ and $\boldsymbol{\mathcal{M}}(t)$ represent the entities described earlier. The infinitesimal generator of this $CTMC$ is given by

$$\tilde{A} = \begin{bmatrix} F & & & M & & & \\ L & F_1 & & & \ddots & & \\ & \ddots & \ddots & & & \ddots & \\ & & L & F_1 & & & M \\ & & & L & F_2 & & \\ & & & & \ddots & \ddots & \\ & & & & & L & F_2 \end{bmatrix}$$

where $L = \lambda I_{(2K-r)Q}$ and all other sub matrices are as defined previously in matrix B_0.

Let $\boldsymbol{\xi} = (\boldsymbol{\xi}_0, \boldsymbol{\xi}_1, ..., \boldsymbol{\xi}_S)$ be the steady-state vector of $\tilde{A}$. Then $\boldsymbol{\xi}$ satisfies the equations $\boldsymbol{\xi}\tilde{A} = 0,\ \boldsymbol{\xi}\mathbf{e} = 1$.

Each $\boldsymbol{\xi}_i$ can be obtained as

$$\boldsymbol{\xi}_i = \begin{cases} \boldsymbol{\xi}_0 \boldsymbol{\mathcal{V}}(0)[\boldsymbol{\mathcal{V}}(1)]^{(i-1)}, & 1 \leq i \leq s+1 \\ \boldsymbol{\xi}_0 \boldsymbol{\mathcal{V}}(0)[\boldsymbol{\mathcal{V}}(1)]^{s}[\boldsymbol{\mathcal{V}}(2)]^{i-(s+1)}, & s+2 \leq i \leq Q \\ \boldsymbol{\xi}_0 \boldsymbol{\mathcal{V}}(i-1), & Q+1 \leq i \leq S. \end{cases}$$

The unknown probability $\boldsymbol{\xi}_0$ can be found from the normalizing condition

$$\boldsymbol{\xi}_0 \left[I + \boldsymbol{\mathcal{V}}(0) \sum_{i=0}^{s-1} (\boldsymbol{\mathcal{V}}(1))^i + \boldsymbol{\mathcal{V}}(0)(\boldsymbol{\mathcal{V}}(1))^s \sum_{i=0}^{Q-(s+1)} (\boldsymbol{\mathcal{V}}(2))^i + \sum_{i=Q}^{S-1} \boldsymbol{\mathcal{V}}(i) \right] \mathbf{e} = 1$$

where $\boldsymbol{\mathcal{V}}(0) = -FL^{-1}, \quad \boldsymbol{\mathcal{V}}(1) = -F_1L^{-1}, \quad \boldsymbol{\mathcal{V}}(2) = -F_2L^{-1}$,
$\boldsymbol{\mathcal{V}}(Q) = -\left[M + \boldsymbol{\mathcal{V}}(0)\ (\boldsymbol{\mathcal{V}}(1))^s\ (\boldsymbol{\mathcal{V}}(2))^{Q-(s+1)}\ F_2\right]\ L^{-1}$,
$\boldsymbol{\mathcal{V}}(i) = -[\boldsymbol{\mathcal{V}}(0)\ (\boldsymbol{\mathcal{V}}(1))^{i-(Q+1)}\ M + \boldsymbol{\mathcal{V}}(i-1)\ F_2]\ L^{-1}, \quad Q+1 \leq i \leq S-1.$

Now using the vector $\boldsymbol{\xi}$, we can find the steady-state vector of the given system. Let $\mathbf{x}$ be the steady-state vector of the generator $\boldsymbol{\mathcal{Q}}$. Then $\mathbf{x}$ must satisfy the set of equations $\mathbf{x}\boldsymbol{\mathcal{Q}} = 0,\ \mathbf{x}\mathbf{e} = 1$.

Partition $\mathbf{x}$ as $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, ...)$. Then the above system of equations reduces to:

$$\mathbf{x}_0 B_0 + \mathbf{x}_1 A_2 = 0, \tag{3.3}$$

$$\mathbf{x}_{i-1} A_0 + \mathbf{x}_i A_1 + \mathbf{x}_{i+1} A_2 = 0, \quad i \geq 1. \tag{3.4}$$

Now assume a solution of the form

$$\mathbf{x}_i = \kappa \left(\frac{\lambda}{\mu}\right)^i \boldsymbol{\xi} \ \text{ for } \ i \geq 0 \tag{3.5}$$

where κ is a constant.

We verify that equation (3.5) satisfies (3.3) and (3.4). We have

$$\mathbf{x}_0 B_0 + \mathbf{x}_1 A_2 = \kappa\boldsymbol{\xi}\left(B_0 + \frac{\lambda}{\mu} A_2\right) = \kappa\boldsymbol{\xi}\tilde{A} = 0,$$

$$\mathbf{x}_{i-1} A_0 + \mathbf{x}_i A_1 + \mathbf{x}_{i+1} A_2 = \left(\frac{\lambda}{\mu}\right)^i \kappa\boldsymbol{\xi}\left(B_0 + \frac{\lambda}{\mu} A_2\right) = \left(\frac{\lambda}{\mu}\right)^i \kappa\boldsymbol{\xi}\tilde{A} = 0.$$

Hence it follows that if we take the vector $\mathbf{x}$ as given by (3.5), Equations (3.3) and (3.4) are satisfied. Now applying the normalizing condition $\mathbf{x}\mathbf{e} = 1$, we get

$$\kappa\left[1 + \left(\frac{\lambda}{\mu}\right) + \left(\frac{\lambda}{\mu}\right)^2 + ...\right]\ \boldsymbol{\xi}\mathbf{e} = 1.$$

Hence under the condition that $\lambda < \mu$, we have $\kappa = 1 - \frac{\lambda}{\mu}$. Thus we have proved the main result of this paper:

Theorem 3.2.1. *The system under consideration has stochastic decomposition with the joint distribution of the system state equal to the product of their marginal distributions.*

3.2.1 Distribution of the replenishment time

Now we derive the distribution of the time until the replenishment is realized after placing an order. There are two cases to be considered:

a. The number of items in the inventory at the production centre $< Q$ at the time of receiving the replenishment order.

b. The inventory level at the production centre $\geq Q$.

The distribution centre demands Q items from the production centre when the inventory level there depletes to s. Here the expected time required to deliver an order depends on the number of items present in the production centre. Thus we consider the following cases:

Let i be the number of items present in the production centre when an order is placed from the distribution centre.

Case a: $0 \leq i < Q$

Production centre delivers in packets of Q items. So at a time when an order is placed if the number of items present in the production centre is $i(< Q)$, the despatch of the packet takes place only when the inventory level at the production centre reaches Q. Thus we have to consider a Markov chain on the states $\{i, i+1, i+2, ..., Q\} \bigcup \{Q*\}$ where $\{Q*\}$ is an absorbing state which represents a packet of Q items delivered to the distribution centre. The infinitesimal generator has of the form

$$\mathcal{W} = \begin{bmatrix} \mathcal{T}_i & \mathcal{T}_i^0 \\ \mathbf{0} & 0 \end{bmatrix}$$

where $$\boldsymbol{\mathcal{T}}_i = \begin{array}{cc} & \begin{array}{cccccc} i & i+1 & i+2 & \dots & Q-1 & Q \end{array} \\ \begin{array}{c} i \\ i+1 \\ i+2 \\ \vdots \\ Q-1 \\ Q \end{array} & \begin{pmatrix} -\eta & \eta & & & & \\ & -\eta & \eta & & & \\ & & -\eta & \ddots & & \\ & & & \ddots & \eta & \\ & & & & -\eta & \eta \\ & & & & & -\beta \end{pmatrix} \end{array}, \quad \boldsymbol{\mathcal{T}}_i^0 = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \beta \end{pmatrix}.$$

Also $\boldsymbol{\mathcal{T}}_i\mathbf{e} + \boldsymbol{\mathcal{T}}_i^0 = 0$ and the initial probability vector $\boldsymbol{\alpha}_i = (1, 0, 0, ...)$ of order $(Q - i + 1)$, to compute the time to despatching the order quantity.

That is, the waiting time at the distribution centre for despatch follows PH-distribution with representation $PH(\boldsymbol{\alpha}_i, \boldsymbol{\mathcal{T}}_i)$. The unconditional distribution for time to replenishment is $\sum_{i=0}^{Q-1} PH(\boldsymbol{\alpha}_i, \boldsymbol{\mathcal{T}}_i)\ P[\boldsymbol{\mathcal{I}}_P = i]$ and the corresponding expected time is $E(\hat{\mathbf{T}}) = \sum_{i=0}^{Q-1} -\boldsymbol{\alpha}_i \boldsymbol{\mathcal{T}}_i^{-1}\mathbf{e}\ P[\boldsymbol{\mathcal{I}}_P = i]$.

Case b: $i \geq Q$

In this case the moment the order for replenishment is received, Q units are despatched to the distribution centre. It takes an exponentially distributed amount of time with parameter β to receive the item at the distribution centre. Then the expected time until replenishment after placing the order is $E(\breve{\mathbf{T}}) = \sum_{i \geq Q} \frac{1}{\beta}\ P[\boldsymbol{\mathcal{I}}_P = i]$ and the unconditional distribution for replenishment is $\sum_{i=Q}^{KQ} \exp(\beta)\ P[\boldsymbol{\mathcal{I}}_P = i]$.

Thus from these two cases we obtain the unconditional distribution for realization of the order at the distribution centre is $\sum_{i=0}^{Q-1} PH(\alpha_i, \boldsymbol{\mathcal{T}}_i)\ P[\boldsymbol{\mathcal{I}}_P =$

$i] + \sum_{i=Q}^{KQ} \exp(\beta)\ P[\boldsymbol{\mathcal{I}}_P = i]$ and expected total waiting time for replenishment as $E_T = E(\hat{\mathbf{T}}) + E(\breve{\mathbf{T}})$.

3.2.2 Analysis of production on-time

The production process is switched on when the inventory level at the production centre depletes to rQ (that is r packets, each of size Q), starting with KQ at the beginning of the cycle. We analyze the length of the production on-time as the time until absorption in a Markov chain $\psi = \{(\boldsymbol{\mathcal{I}}_P(t), \boldsymbol{\mathcal{N}}(t), \boldsymbol{\mathcal{I}}_D(t)), t \geq 0\}$ where $\boldsymbol{\mathcal{N}}(t)$ denotes the number of customers in the system where variation is from 0 to $\mathbf{B}$ (finite but sufficiently large to ensure that with only very small probability ϵ customers are lost when inventory level is positive.) The state space of ψ is given by $\{(i_P, n, i_D), 0 \leq i_P \leq KQ-1, 0 \leq n \leq \mathbf{B}, 0 \leq i_D \leq S\} \bigcup \{*\}$ where $\{*\}$ denotes the absorbing state, which represents switching off the production process. The infinitesimal generator $\boldsymbol{\mathcal{W}}'$ of the process ψ has the form

$$\boldsymbol{\mathcal{W}}' = \begin{bmatrix} \boldsymbol{\mathcal{T}} & \boldsymbol{\mathcal{T}}^0 \\ \mathbf{0} & 0 \end{bmatrix}$$

where $\boldsymbol{\mathcal{T}}$ is of order $KQ(S+1)(\mathbf{B}+1)$ and is given by

$$\boldsymbol{\mathcal{T}} = \begin{bmatrix} E & E_0 & & & & & & & \\ & \ddots & \ddots & & & & & & \\ & & \ddots & \ddots & & & & & \\ & & & E & E_0 & & & & \\ E_2 & & & & E_1 & E_0 & & & \\ & \ddots & & & & \ddots & \ddots & & \\ & & E_2 & & & & E_1 & E_0 \\ & & & E_2 & & & & E_1 \end{bmatrix}, \quad \boldsymbol{\mathcal{T}}^0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ E_0' \end{bmatrix}$$

with $E_0 = \eta I_{(S+1)(\mathbf{B}+1)}, E_0' = \eta \mathbf{e}$.

The matrix E_0 in $\boldsymbol{\mathcal{T}}$ represents an item added to inventory by production and E_2 represents the case of Q items from production centre despatched to the distribution centre.

Define $\boldsymbol{\mathcal{E}}^{(n,i_D)}_{(m,j_D)}$ as the transition rates from $(n, i_D) \rightarrow (m, j_D)$ where $\boldsymbol{\mathcal{E}} = E, E_1, E_2$. These transitions are

$$E^{(n,i_D)}_{2(m,j_D)} = \begin{cases} \beta & j_D = i_D + Q, 0 \leq i_D \leq s, m = n, 0 \leq n \leq B \\ 0 & \text{otherwise.} \end{cases}$$

The non-diagonal elements of E and E_1 are

$$E^{(n,i_D)}_{1(m,j_D)} = \begin{cases} \lambda & j_D = i_D, 1 \leq i_D \leq S, m = n+1, 0 \leq n \leq B-1 \\ \mu & j_D = i_D - 1, 1 \leq i_D \leq S, m = n-1, 1 \leq n \leq B \\ 0 & \text{otherwise,} \end{cases}$$

$$E^{(n,i_D)}_{(m,j_D)} = \begin{cases} \lambda & j_D = i_D, 1 \leq i_D \leq S, m = n+1, 0 \leq n \leq B-1 \\ \mu & j_D = i_D - 1, 1 \leq i_D \leq S, m = n-1, 1 \leq n \leq B \\ 0 & \text{otherwise.} \end{cases}$$

The diagonal entries of E and E_1 are such that each row sum is zero.

Thus the production on-time τ follows phase type distribution and the expected duration of production on time is given by $E_{on}(\tau) = -\boldsymbol{\gamma}\boldsymbol{\mathcal{T}}^{-1}\mathbf{e}$ where $\boldsymbol{\gamma} = (0, .., 0, 1, 0, .., 0)$ is the initial probability vector of order $KQ(\mathbf{B}+1)(S+1)$.

3.2.3 Analysis of production off time

Next we consider the switching off of the production process. When the inventory level at the production centre reaches KQ, the process is switched off to restart at the epoch when the inventory depletes to rQ.

Case 1

When the number of customers at the distribution centre is $i \geq (K - r)Q$, future arrivals need not be considered. In this case we analyze the length of the production off period as the time until absorption (to rQ) in the finite state Markov chain $\varpi = \{(\boldsymbol{\mathcal{I}}_P(t), \boldsymbol{\mathcal{N}}(t), \boldsymbol{\mathcal{I}}_D(t)), t \geq 0\}$. The state space of $\varpi = \{(i_P, n, i_D),\ i_P = (r+1)Q, (r+2)Q..., KQ,\ 0 \leq n \leq i,\ 0 \leq i_D \leq S\} \bigcup \{\Delta\}$, where $\{\Delta\}$ denotes the absorbing state, which represents switching on of the production process. The infinitesimal generator $\tilde{\boldsymbol{\mathcal{W}}}_i$ is of the form

$$\tilde{\boldsymbol{\mathcal{W}}}_i = \begin{bmatrix} \tilde{\boldsymbol{\mathcal{T}}}_i & \tilde{\boldsymbol{\mathcal{T}}}_i^0 \\ \mathbf{0} & 0 \end{bmatrix}$$

with initial probability vector $\boldsymbol{\gamma}_i = (1, 0, 0, ...)$ of order $(K-r)(i+1)(S+1)$.

Here

$$\tilde{\boldsymbol{\mathcal{T}}}_i = \begin{bmatrix} D_1 & & & & \\ D_2 & D_1 & & & \\ & \ddots & \ddots & & \\ & & D_2 & D_1 & \\ & & & D_2 & D_1 \end{bmatrix}, \tilde{\boldsymbol{\mathcal{T}}}_i^0 = \begin{bmatrix} D_2' \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

with

$$D_1 = \begin{bmatrix} U & U_1 & & & \\ & U & U_1 & & \\ & & \ddots & \ddots & \\ & & & U & U_1 \\ & & & & U' \end{bmatrix}, D_2 = \begin{bmatrix} U_2 & & & & \\ & U_2 & & & \\ & & \ddots & & \\ & & & U_2 & \\ & & & & U_2 \end{bmatrix},$$

$D_2' = U_2'\mathbf{e}$ where $U' = diag(-\beta, ..., -\beta, 0, ..., 0)$, $U_2' = [\beta, ..., \beta, 0, ..., 0]^T$, $U = diag(-\beta, -(\mu+\beta), ..., -(\mu+\beta), -\mu, ..., -\mu)$,

$$U_1 = \begin{bmatrix} & & & & \\ \mu & & & & \\ & \mu & & & \\ & & \ddots & & \\ & & & \mu & \end{bmatrix}, U_2 = \begin{bmatrix} & & \beta & & \\ & & & \ddots & \\ & & & & \beta \\ & & & & \\ & & & & \end{bmatrix}.$$

The production off time τ_i (when $i \geq (K-r)Q$) follows phase type distribution. Therefore the expected duration of production off time when there are i customers in the system is given by $E_{off}(\tau_i) = -\gamma_i \tilde{\boldsymbol{\mathcal{T}}}_i^{-1}\mathbf{e}$.

Case 2

When the number i of customers at the distribution centre is less than $(K-r)Q$, new arrivals are to be considered until the epoch at which the number of customers equals the number of items in the production exceeding rQ.

Thus for computing the production off time we have to consider a Markov chain $\varpi' = \{\boldsymbol{\mathcal{N}}_1(t), \boldsymbol{\mathcal{N}}_2(t), \boldsymbol{\mathcal{I}}_P(t), \boldsymbol{\mathcal{I}}_D(t)), t \geq 0\}$ where $\boldsymbol{\mathcal{N}}_1(t)$ is the number of new arrivals to the system and $\boldsymbol{\mathcal{N}}_2(t)$ is the number of customers present in the distribution centre at time t. The state space is $\{(n_1, n_2, i_P, i_D),\ 0 \leq n_1 \leq (K-r)Q - i,\ n_2 = n_1 - (K-r)Q,\ i_P = (r+1)Q, (r+2)Q..., KQ,\ 0 \leq i_D \leq S\} \bigcup \{\Delta'\}$ where $\{\Delta'\}$ denotes the absorbing state, which represents switching on of the production process. The infinitesimal generator $\tilde{\tilde{\boldsymbol{\mathcal{W}}}}_i$ of the process ϖ' is of the form

$$\tilde{\tilde{\boldsymbol{\mathcal{W}}}}_i = \begin{bmatrix} \tilde{\tilde{\boldsymbol{\mathcal{T}}}}_i & \tilde{\tilde{\boldsymbol{\mathcal{T}}}}_i^0 \\ \mathbf{0} & 0 \end{bmatrix}$$

with initial probability vector $\tilde{\gamma}_i = (1, 0, ..., 0)$ of order $(K-r)(S+1)[(K-r)Q-(i-1)]$. Here

$$\tilde{\tilde{\boldsymbol{\mathcal{T}}}}_i = \begin{bmatrix} D & D_5 & & & \\ D_3 & D_4 & D_5 & & \\ & \ddots & \ddots & \ddots & \\ & & D_3 & D_4 & D_5 \\ & & & D_3 & D_4' \end{bmatrix}, \tilde{\tilde{\boldsymbol{\mathcal{T}}}}_i^0 = \begin{bmatrix} D_6 \\ \vdots \\ D_6 \end{bmatrix}$$

where

$$D = \begin{bmatrix} U_3' & U_2 & & \\ & \ddots & \ddots & \\ & & U_3' & U_2 \\ & & & U_3' \end{bmatrix}, D_4' = \begin{bmatrix} U & U_2 & & \\ & \ddots & \ddots & \\ & & U & U_2 \\ & & & U \end{bmatrix}, D_4 = \begin{bmatrix} U_3 & U_2 & & \\ & \ddots & \ddots & \\ & & U_3 & U_2 \\ & & & U_3 \end{bmatrix},$$

$D_3 = diag(U_1, ..., U_1)$, $D_5 = diag(U_0, ..., U_0)$, $D_6 = [0, ..., 0, U_2']^T$ with

$$(U_0)_{ij} = \begin{cases} \lambda & j = i, 2 \leq i \leq S+1 \\ 0 & \text{otherwise,} \end{cases}$$

$$(U_3)_{ij} = \begin{cases} -\beta & j = i, i = 1 \\ -(\lambda+\beta+\mu) & j = i, 2 \leq i \leq s+1 \\ -(\lambda+\mu) & j = i, s+2 \leq i \leq S+1 \\ 0 & \text{otherwise,} \end{cases}$$

$$(U_3')_{ij} = \begin{cases} -\beta & j = i, i = 1 \\ -(\lambda+\beta) & j = i, 2 \leq i \leq s+1 \\ -\lambda & j = i, s+2 \leq i \leq S+1 \\ 0 & \text{otherwise.} \end{cases}$$

Matrices U, U_1, U_2 and U_2' are as given in Case 1. The production off time τ_i' (when $i < (K-r)Q$) follows phase type distribution. These lead to the expression for the expected duration of production off time as $E_{off}(\tau_i') = -\tilde{\gamma}_i \tilde{\tilde{\boldsymbol{\mathcal{T}}}}_i^{-1} \mathbf{e}$.

Performance Measures

1. Expected number of customers in the system

$$E_C = \frac{\lambda}{\mu - \lambda}.$$

This expression indicates that as λ increases, subject to the condition $\lambda < \mu$, E_C increases.

2. Expected inventory level in the distribution centre

$$E_{ID} = \sum_{j=1}^{S} \left[\sum_{k=0}^{rQ} j\xi_j(k,1) + \sum_{k=rQ+1}^{KQ-1} j[\xi_j(k,0) + \xi_j(k,1)] + j\xi_j(KQ,0) \right].$$

Though the production rate η does not appear explicitly in the above expression, we note that E_{ID} increases with η increasing. The reason for this is easy to trace – low values of η would mean that finished products in batches of size Q can not be expected at the time when replenishment order is placed, whereas with increasing values of η the waiting time for packets of size Q turns out to be smaller.

3. Expected inventory level in the production centre

$$E_{IP} = \sum_{j=0}^{S} \left[\sum_{k=0}^{rQ} k\xi_j(k,1) + \sum_{k=rQ+1}^{KQ-1} k[\xi_j(k,0) + \xi_j(k,1)] + KQ\xi_j(KQ,0) \right].$$

Here again E_{IP} can be seen to increase with increase in production rate η.

4. Expected loss rate of customers when the inventory level in the distribution centre is zero

$$E_L = \lambda \left[\sum_{k=0}^{rQ} \xi_0(k,1) + \sum_{k=rQ+1}^{KQ-1} [\xi_0(k,0) + \xi_0(k,1)] + \xi_0(KQ,0) \right].$$

We notice from the above that E_L increases (decreases) with increase (decrease) in the value of λ.

5. Expected production rate

$$E_P = \eta \sum_{j=0}^{S} \sum_{k=0}^{KQ-1} \xi_j(k,1).$$

This tells that E_P increases / decreases as η increases (decreases). It is also seen that for fixed η, E_P increases with λ increasing, reaches a maximum and then starts decreasing. Beyond a certain value, large number of customers are lost.

6. Expected reorder rate

$$E_R = \mu \sum_{i=1}^{\infty} \left(1 - \frac{\lambda}{\mu}\right) \left(\frac{\lambda}{\mu}\right)^i \left[\sum_{k=0}^{KQ-1} \xi_{s+1}(k,1) + \sum_{k=rQ+1}^{KQ} \xi_{s+1}(k,0)\right].$$

E_R is seen to increase / decrease with μ increasing / decreasing.

7. Expected rate at which production process is switched on

$$E_{PR} = \beta \sum_{j=0}^{s} \xi_j((r+1)Q, 0).$$

E_{PR} is increasing / decreasing linear function of β.

8. Fraction of time the production process is on

$$T_{on} = \sum_{j=0}^{S} \sum_{k=0}^{KQ-1} \xi_j(k,1).$$

Using Renewal reward theorem (see Ross [38]) we get the above expression.

A simple proof for the above expression runs as follows: consider the indicator random variables I_{ON} and I_{OFF} representing production 'on' and 'off' respectively. The expectations of these random variables are their corresponding probabilities. We give a reward of one unit when production is in on mode. In one unit of time, total time spent in 'on' and 'off' modes put together is also one unit. Thus the denominator has value one. The numerator is the expectation $E(I_{ON})$.

3.3 Optimization problem

Based on the above performance measures we construct a cost function for checking the optimality of the reorder level s and the fixed order quantity Q

in the distribution centre. Also we check the effect of r and K on the cost function.

Consider a cost function $F(s, Q, r, K)$ defined as

$$\begin{aligned} F(s,Q,r,K) = \;& \mathcal{C}_1 E_{PR} + (\mathcal{C}_2 + \mathcal{C}_3 Q)E_R + \mathcal{C}_4 E_L + \mathcal{C}_5 E_P + \mathcal{C}_6 E_{ID} \\ & + \mathcal{C}_7 E_{IP} + \mathcal{C}_8 E_C + \mathcal{C}_9 E_T + \mathcal{C}_{10} T_{on} \end{aligned}$$

where

$\mathcal{C}_1$: Fixed cost for starting the production
$\mathcal{C}_2$: Fixed cost for placing an order
$\mathcal{C}_3$: Procurement cost per unit
$\mathcal{C}_4$: Cost incurred due to loss of customers
$\mathcal{C}_5$: Running cost per unit time of production process
$\mathcal{C}_6$: Holding cost per unit time per inventory in the distribution centre
$\mathcal{C}_7$: Holding cost per unit time per inventory in the production centre
$\mathcal{C}_8$: Holding cost of customer per unit time
$\mathcal{C}_9$: Penalty cost per unit time delay in replenishment
$\mathcal{C}_{10}$: Cost per unit when production is on (Running cost)

The problem of minimizing the cost for various parameter values are carried out. Below we provide local minimum costs and local optimal pairs.

Effect of variation in s and Q

We assign the following values to the parameters: $\mathcal{C}_1 = \$20000, \mathcal{C}_2 = \$2000, \mathcal{C}_3 = \$200, \mathcal{C}_4 = \$100, \mathcal{C}_5 = \$500, \mathcal{C}_6 = \$25, \mathcal{C}_7 = \$20, \mathcal{C}_8 = \$15, \mathcal{C}_9 = \$100, \mathcal{C}_{10} = \$150, \lambda = 5, \mu = 7, \beta = 2, \eta = 3, r = 2$ and $K = 4$. We obtain the following table (Table 3.1) which provide the optimal (s, Q) pair and the corresponding minimum cost (in \$). Here (3,9) is the local optimal (s, Q) pair and the corresponding minimum local cost is \$4595.7.

s \ Q	6	7	8	9	10
1	5354	4862.4	4854.5	4650.3	4660.1
2	5033.6	4723.8	4714.5	4597.5	4613.6
3	4916.2	**4695.1**	**4674.4**	**4595.7**	**4611.6**
4	**4895.4**	4713.9	4681.9	4618.7	4630.8
5	4918.2	4752.9	4710.6	4691.3	4683.3

Table 3.1: Effect of s and Q on expected total cost

Effect of variation in r and K

In order to study the variation in r and K on expected total cost we fix $\lambda = 5, \mu = 7, \beta = 2, \eta = 3, s = 1, \mathcal{C}_1 = \$20000, \mathcal{C}_2 = \$2000, \mathcal{C}_3 = \$200, \mathcal{C}_4 = \$100, \mathcal{C}_5 = \$500, \mathcal{C}_6 = \$25, \mathcal{C}_7 = \$20, \mathcal{C}_8 = \$15, \mathcal{C}_9 = \$100, \mathcal{C}_{10} = \$150$ and for different values of r and K, the expected total costs are calculated, as they are presented in Tables 3.2 and 3.3. These tables show that the expected total cost is minimum when $r = 3$ and $K = 10$.

r \ K	8	9	10	11	12
1	6618.3	6491.9	6396	6322	6264.3
2	5953.8	5859.7	5798.4	5757	5729.8
3	5589.4	5309.5	**5094.6**	**5101.2**	**5163.3**
4	**5199.3**	**5157.2**	5264.7	5166	5193.2
5	5234.3	5217	5343.9	5225.4	5226.3

Table 3.2: Effect of r and K on expected total cost for $Q = 2$

r \ K	8	9	10	11	12
1	6751.1	6608	6499.1	6414.8	6348.7
2	6123	6010.8	5934.5	5880.6	5842.8
3	5693.6	5449.7	**5229.7**	**5234**	5296
4	**5271.4**	**5256.6**	5364.2	5264	**5291.3**
5	5294.3	5289.7	5417.7	5297.7	5298.4

Table 3.3: Effect of r and K on expected total cost for $Q = 3$

Further numerical experiments were conducted and the above findings are seen to hold.

Chapter 4

Product form solution in two priority queueing-inventory system

In chapters 2 and 3 we assumed the demand to arise from a single class of customers. Literature on multi class customers as source for inventory demand is just a hand full (see for example Isotupa [17], Sapna Isotupa [42], Zhao and Lian [51]).

In this chapter we analyze single server, lost sales $(S-1, S)$ queueing-inventory system with two demand classes – high priority and low priority. The service of non-priority customers are preempted with arrival of high priority customers. We compare two different models – one in which, low priority customers do not join the system when the on-hand inventory is zero and in the other case when there is no high priority customer present but there

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Some results in this chapter are included in the following paper.

A. Krishnamoorthy, Dhanya Shajin and B. Lakshmy : **Product form solution in two priority queueing-inventory system**, Presented in 27^{th} European Conference on Operational Research (EURO), Glasgow, July 2015.

is positive inventory, an arriving low priority customer joins the system. In the second model we obtain stochastic decomposition of the system. On the contrary this property is absent in model I.

Appropriate blocking sets can be constructed to come up with product form solution in certain queueing-inventory systems. For example, in the $M/M/1$ queueing-inventory problem, the smallest blocking set is the the set $\{(n, 0)/n \geq 0\}$ where n is the number of customers in the system and the second coordinate stands for zero inventory. One can enlarge the blocking set by allowing no activity other than replenishment when the system state is $\{(n, i)/n \geq 0, 0 \leq i \leq l < S\}$. This means that no customer is permitted to join the queue even when inventory level is positive and no service takes place while the system is in that blocking set. However, this turns out to be prohibitively expensive to the system. Thus the optimal blocking set turns out to be $\{(n, 0)/n \geq 0\}$. When extended to priority system, the above statement is not valid. For example, with $\{(n_1, n_2, 0)/n_1, n_2 \geq 0\}$ where $n_1(n_2)$ is the number of high (low) priority customers and no inventory (last coordinate) in the system we are not able to produce the system state distribution as product of the marginals! However, a mild relaxation in the blocking set resulted in even the stochastic decomposition ruled out (see model I).

4.1 Mathematical formulation of model I

We consider a single server queueing-inventory system with two types of customers - high priority (HP) and low priority (LP), each of which arrives according to Poisson process of rates λ_1 and λ_2, respectively. HP-customers receive priority over LP-customers; arrival of an HP-customer preempts the service of an LP-customer currently, if any in service. The inventory is controlled by $(S - 1, S)$-policy, where S is the maximum inventory level in the system. No customer (both HP and LP) joins the system when the inventory level is zero. The lead time for replenishment is exponentially distributed with parameter β. The HP-customers join a finite queue of maximum size S. Size of this finite queue varies according to the number of items available in the inventory. In other words at any time this finite queue cannot have more HP-customers than the number of items in the inventory. This ensures that all HP-customers in the system are assured of the item even in the absence of replenishment. LP-customers join an infinite capacity queue. The service time for HP and LP-customers are independent and exponentially distributed with parameters μ_1 and μ_2, respectively.

In the sequel we use the following notations:

$\boldsymbol{\mathcal{N}}_1(t)$: Number of HP customers in the system at time t

$\boldsymbol{\mathcal{N}}_2(t)$: Number of LP customers in the system at time t

$\boldsymbol{\mathcal{I}}(t)$: Number of items in the inventory at time t

The process $\{(\boldsymbol{\mathcal{N}}_2(t), \boldsymbol{\mathcal{I}}(t), \boldsymbol{\mathcal{N}}_1(t)), t \geq 0\}$ is a $CTMC$ whose state space

$$\Omega = \{(n_2, i, n_1); n_2 \geq 0, 0 \leq i \leq S, 0 \leq n_1 \leq i\}.$$

Thus the infinitesimal generator $\boldsymbol{\mathcal{Q}}_1$ of this $CTMC$ is $LIQBD$ with

$$\boldsymbol{\mathcal{Q}}_1 = \begin{bmatrix} A_{00} & A_0 & & & \\ A_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}. \tag{4.1}$$

The matrices A_{00}, A_0, A_1 and A_2 are square matrices of the same order $(S+1)(S+2)/2$ with A_{00} containing transition rates within level 0, A_0 represents transition from level n_2 to $n_2+1, n_2 \geq 0$, A_1 represents the transitions within level $n_2, n_2 \geq 1$, and A_2 represents transitions from level n_2 to level $n_2-1, n_2 \geq 1$. Define $A_{k(n_1,m_1)}^{(i,j)}$, $k = 00, 0, 1, 2$ as the transition rates from $(i, n_1) \rightarrow (j, m_1)$ where i represents the number of items in the inventory and n_1 represents the number of HP-customers. These transition rates are

$$A_{00(n_1,m_1)}^{(i,j)} = \begin{cases} \beta & j = i+1, 0 \le i \le S-1; m_1 = n_1, 0 \le n_1 \le i \\ \lambda_1 & j = i, 1 \le i \le S; m_1 = n_1+1, 0 \le n_1 \le i-1, \\ \mu_1 & j = i-1, 1 \le i \le S; m_1 = n_1-1, 1 \le n_1 \le i, \\ -\beta & j = i = 0; m_1 = n_1 = 0, \\ -(\lambda_1+\lambda_2+\beta) & j = i, 1 \le i \le S-1; m_1 = n_1 = 0, \\ -(\lambda_1+\lambda_2+\mu_1+\beta) & j = i, 2 \le i \le S-1; m_1 = n_1, 1 \le n_1 \le i-1, \\ -(\lambda_2+\mu_1+\beta) & j = i, 1 \le i \le S-1; m_1 = n_1 = i, \\ -(\lambda_1+\lambda_2) & j = i = S; m_1 = n_1 = 0, \\ -(\lambda_2+\mu_1) & j = i = S; m_1 = n_1 = S, \\ -(\lambda_1+\lambda_2+\mu_1) & j = i = S; m_1 = n_1, 1 \le n_1 \le S-1, \\ 0 & \text{otherwise,} \end{cases}$$

$$A_{1(n_1,m_1)}^{(i,j)} = \begin{cases} \beta & j = i+1, 0 \le i \le S-1; m_1 = n_1, 0 \le n_1 \le i \\ \lambda_1 & j = i, 1 \le i \le S; m_1 = n_1+1, 0 \le n_1 \le i-1, \\ \mu_1 & j = i-1, 1 \le i \le S; m_1 = n_1-1, 1 \le n_1 \le i, \\ -\beta & j = i = 0; m_1 = n_1 = 0, \\ -(\lambda_1+\lambda_2+\mu_2+\beta) & j = i, 1 \le i \le S-1; m_1 = n_1 = 0, \\ -(\lambda_1+\lambda_2+\mu_1+\beta) & j = i, 2 \le i \le S-1; m_1 = n_1, 1 \le n_1 \le i-1, \\ -(\lambda_2+\mu_1+\beta) & j = i, 1 \le i \le S-1; m_1 = n_1 = i, \\ -(\lambda_1+\lambda_2+\mu_2) & j = i = S; m_1 = n_1 = 0, \\ -(\lambda_2+\mu_1) & j = i = S; m_1 = n_1 = S, \\ -(\lambda_1+\lambda_2+\mu_1) & j = i = S; m_1 = n_1, 1 \le n_1 \le S-1, \\ 0 & \text{otherwise,} \end{cases}$$

$$A_{0(n_1,m_1)}^{(i,j)} = \begin{cases} \lambda_2 & j = i, 1 \le i \le S; m_1 = n_1, 0 \le n_1 \le i, \\ 0 & \text{otherwise,} \end{cases}$$

$$A_{2(n_1,m_1)}^{(i,j)} = \begin{cases} \mu_2, & j = i-1, 1 \le i \le S; m_1 = n_1 = 0, \\ 0, & \text{otherwise.} \end{cases}$$

4.2 Steady-state analysis

We proceed with the steady-state analysis of the queueing-inventory system under study. The first stage in this direction is to look for the condition for stability.

4.2.1 Stability condition

Let $\boldsymbol{\pi} = (\boldsymbol{\pi}_0, \boldsymbol{\pi}_1, ..., \boldsymbol{\pi}_S)$ denote the steady-state probability vector of the generator

$$A = A_0 + A_1 + A_2 = \begin{bmatrix} F_0 & B_0 & & & \\ M_1 & F_1 & B_1 & & \\ & \ddots & \ddots & \ddots & \\ & & M_{S-1} & F_{S-1} & B_{S-1} \\ & & & M_S & F_S \end{bmatrix} \tag{4.2}$$

where

$$M_i(j,k) = \begin{cases} \mu_1 & k = j-1,\ 1 \le j \le i,\ 1 \le i \le S \\ 0 & \text{otherwise,} \end{cases}$$

$$B_i(j,k) = \begin{cases} \beta & k = j,\ 0 \le j \le i,\ 0 \le i \le S-1 \\ 0 & \text{otherwise,} \end{cases}$$

$$F_i(j,k) = \begin{cases} -\beta & k = j = 1,\ i = 0 \\ -(\lambda_1 + \mu_2 + \beta) & k = j = 1,\ 1 \le i \le S \\ -(\lambda_1 + \mu_1 + \beta) & k = j, 2 \le j \le i,\ 2 \le i \le S \\ -(\mu_1 + \beta) & k = j, j = i+1,\ 1 \le i \le S \\ \lambda_1 & k = j+1, 1 \le j \le i,\ 1 \le i \le S \\ 0 & \text{otherwise} \end{cases}$$

with dimension of M_i, B_i are $(i+1) \times i, i \times (i+1)$ respectively and F_i is square matrix of order $(i+1)$. That is,

$$\boldsymbol{\pi} A = 0, \quad \boldsymbol{\pi}\mathbf{e} = 1. \tag{4.3}$$

The $LIQBD$ description of the model indicates that the queueing system is stable (see Neuts [33]) if and only if

$$\boldsymbol{\pi} A_0 \mathbf{e} < \boldsymbol{\pi} A_2 \mathbf{e}. \tag{4.4}$$

The vector $\boldsymbol{\pi}$ cannot be obtained explicitly in terms of the parameters of the model, and hence the stability condition is known only implicitly as given in (4.7). From expressions in (4.3) we get

$$\boldsymbol{\pi}_i = \boldsymbol{\pi}_{i-1} \boldsymbol{\mathcal{U}}_{i-1}, \quad 1 \leq i \leq S \tag{4.5}$$

where

$$\begin{aligned} \boldsymbol{\mathcal{U}}_{S-1} = & \; -B_{S-1} F_S^{-1} \\ \boldsymbol{\mathcal{U}}_i = & \; -B_i [F_{i+1} + \boldsymbol{\mathcal{U}}_{i+1} M_{i+2}]^{-1}, \quad 0 \leq i \leq S-2. \end{aligned}$$

From the normalizing condition $\boldsymbol{\pi}\mathbf{e} = 1$ we have

$$\boldsymbol{\pi}_0 \left[\sum_{j=0}^{S-1} \prod_{i=0}^{j} \boldsymbol{\mathcal{U}}_i + I \right] \mathbf{e} = 1. \tag{4.6}$$

Inequality (4.4) gives the stability condition as

$$\boldsymbol{\pi}_0 \left[\sum_{j=1}^{S} \prod_{i=0}^{j-1} \boldsymbol{\mathcal{U}}_i L_j \right] \mathbf{e} < \boldsymbol{\pi}_0 \left[\sum_{j=1}^{S} \prod_{i=0}^{j-1} \boldsymbol{\mathcal{U}}_i M_j \right] \mathbf{e} \tag{4.7}$$

where L_i is a square matrix of order $(i+1)$ with

$$L_i(j,k) = \begin{cases} \lambda_2, & k = j = 0, \ 1 \leq i \leq S \\ 0, & \text{otherwise.} \end{cases}$$

4.2.2 Steady-state probability vector

Assuming that (4.7) is satisfied, we briefly outline the computation of the steady-state probability of the system state. Let $\mathbf{x}$ denote the steady-state probability vector of the generator $\boldsymbol{\mathcal{Q}}_1$. Then

$$\mathbf{x}\boldsymbol{\mathcal{Q}}_1 = 0, \quad \mathbf{x}\mathbf{e} = 1. \tag{4.8}$$

Partitioning $\mathbf{x}$ as $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, ...)$ we see that $\mathbf{x}$, under the assumption that the stability condition (4.7) holds, is obtained as (see Neuts [33])

$$\mathbf{x}_n = \mathbf{x}_0 R^n, \quad n \geq 1. \tag{4.9}$$

where R is the minimal nonnegative solution to the matrix quadratic equation:

$$A_0 + RA_1 + R^2 A_2 = \mathbf{O} \tag{4.10}$$

and the boundary equation is given by

$$\mathbf{x}_0 [A_{00} + RA_2] = 0. \tag{4.11}$$

The normalizing condition (4.8) gives

$$\mathbf{x}_0 (I - R)^{-1} \mathbf{e} = 1. \tag{4.12}$$

Now we look at a few of the system performance measures.

Performance Measures

1. Expected number of low priority customers in the system

$$E_{LP}(N) = \sum_{n_2=1}^{\infty} n_2 \mathbf{x}_{n_2} \mathbf{e}$$

2. Expected number of high priority customers in the system

$$E_{HP}(N) = \sum_{n_2=0}^{\infty} \sum_{i=1}^{S} \sum_{n_1=1}^{i} n_1 \mathbf{x}_{n_2}(i, n_1)$$

3. Expected number of items in the inventory

$$E(I) = \sum_{n_2=0}^{\infty} \sum_{i=0}^{S} \sum_{n_1=0}^{i} i \mathbf{x}_{n_2}(i, n_1)$$

4. Expected loss rate of low priority customers

$$E_{LP}(L) = \lambda_2 \sum_{n_2=0}^{\infty} \mathbf{x}_{n_2}(0,0)$$

5. Expected loss rate of high priority customers

$$E_{HP}(L) = \lambda_1 \sum_{n_2=0}^{\infty} \sum_{i=0}^{S} \mathbf{x}_{n_2}(i,i)$$

6. Expected reorder rate

$$E(R) = \mu_1 \sum_{n_2=0}^{\infty} \sum_{i=1}^{S} \sum_{n_1=1}^{i} \mathbf{x}_{n_2}(i,n_1) + \mu_2 \sum_{n_2=1}^{\infty} \sum_{i=1}^{S} \mathbf{x}_{n_2}(i,0)$$

4.2.3 Distribution of waiting time of a HP-customer

For computing the expected waiting time of an HP-customer who joins as the r^{th} customer ($r > 0$) in the queue at the time he joins (joining time taken as time origin, of course provided he is able to join the system). We consider the Markov process $W_1(t) = \{(\boldsymbol{\mathcal{N}}(t), \boldsymbol{\mathcal{I}}(t)), t \geq 0\}$ where $\boldsymbol{\mathcal{N}}(t)$ is the rank of the customer at time t. The rank $\boldsymbol{\mathcal{N}}(t)$ of the customer is r if he is the r^{th} customer in the queue at time t. His rank decreases to 1 as the customers ahead of him leave the system after completing service. Thus the state space of the process is $\{(n,i), 1 \leq n \leq r, n \leq i \leq S\} \bigcup \{\mathbf{0}\}$ where $\{\mathbf{0}\}$ is the absorbing state indicating that the tagged customer is selected for service. The infinitesimal generator $\boldsymbol{\mathcal{W}}_1$ of $W_1(t)$ has the form

$$\boldsymbol{\mathcal{W}}_1 = \begin{bmatrix} \boldsymbol{\mathcal{T}} & \boldsymbol{\mathcal{T}}^0 \\ \mathbf{0} & 0 \end{bmatrix}$$

where

$$\boldsymbol{\mathcal{T}} = \begin{bmatrix} D_r^{(1)} & D_r^{(2)} & & & & \\ & D_{r-1}^{(1)} & D_{r-1}^{(2)} & & & \\ & & \ddots & \ddots & & \\ & & & D_2^{(1)} & D_2^{(2)} \\ & & & & D_1^{(1)} \end{bmatrix}, \boldsymbol{\mathcal{T}}^0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ D_1 \end{bmatrix}$$

with

$$D_j^{(1)} = \begin{bmatrix} -(\mu_1+\beta) & \beta & & & \\ & \ddots & \ddots & & \\ & & -(\mu_1+\beta) & \beta \\ & & & -\mu_1 \end{bmatrix}_{S-j+1\times S-j+1}, 1 \le j \le r,$$

$$D_1 = \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_1 \end{bmatrix}_{s\times 1}, D_j^{(2)} = \begin{bmatrix} \mu_1 & & \\ & \ddots & \\ & & \mu_1 \end{bmatrix}_{S-j+1\times S-j+2}, 2 \le j \le r.$$

Now, the waiting time of HP-customer, who joins the queue as the r^{th} customer is the time until absorption of the Markov chain $\{W_1(t), t \ge 0\}$. The distribution of $\boldsymbol{\mathcal{W}}_1$ is Phase type. Thus the vector of expected waiting time of this particular customer is given by the column vector,

$$E_T^r = -\boldsymbol{\mathcal{T}}^{-1}\mathbf{e}.$$

Hence, the expected waiting time of a general HP-customer in the system is,

$$E_{HP}(W) = \sum_{n_1=0}^{\infty}\sum_{r=1}^{S}\mathbf{y}_{n_1}^r E_T^r$$

where $\mathbf{y}_i^r = (\mathbf{x}_i(r,r), \mathbf{x}_i(r+1,r), ..., \mathbf{x}_i(S,r), ..., \mathbf{x}_i(1,1), ..., \mathbf{x}_i(S,1))$ is a row vector of dimension $r\left[S - \frac{(r-1)}{2}\right]$.

4.2.4 Expected waiting time of a low priority customer

We compute the expected waiting time of a LP-customer in the same way we computed that for the HP-customer. However, the preemption of LP by HP-customer has to be considered, which infact, can be arbitrarily large for any sample LP-customer. So first we give a bound on this and proceed. Here we consider the Markov process $W_2(t) = \{(\mathcal{N}'(t), \mathcal{N}(t), \mathcal{I}(t), \mathcal{N}_1(t)), t \geq 0\}$ where $\mathcal{N}'(t)$ is the number of preemptions $(0 \leq l < \infty)$, $\mathcal{N}(t)$ is the rank of the tagged low priority customer at time t. As defined earlier $\mathcal{I}(t)$ and $\mathcal{N}_1(t)$ respectively denote the inventory level and number of HP-customers at time t in the system. Its state space is $\{(n', n, i, n_1), 0 \leq n' \leq l, 1 \leq n \leq r, 0 \leq i \leq S, 0 \leq n_1 \leq i\} \bigcup \{\mathbf{0}\}$ where the absorbing state $\{\mathbf{0}\}$ indicates that the tagged customer is either selected for service or number of preemption is maximum. Thus the infinitesimal generator of $\{W_2(t), t \geq 0\}$ is of the form

$$\mathcal{W}_2 = \begin{bmatrix} \hat{\mathcal{T}} & \hat{\mathcal{T}}^0 \\ \mathbf{0} & 0 \end{bmatrix}$$

where

$$\hat{\mathcal{T}} = \begin{bmatrix} \mathcal{D}_1 & \mathcal{D}_0 & & \\ & \ddots & \ddots & \\ & & \mathcal{D}_1 & \mathcal{D}_0 \\ & & & \mathcal{D}_1 \end{bmatrix}, \hat{\mathcal{T}}^0 = \begin{bmatrix} \mathcal{D} \\ \vdots \\ \mathcal{D} \\ \mathcal{D}' \end{bmatrix}$$

with

$$\mathcal{D} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ G_2' \end{bmatrix}, \mathcal{D}_1 = \begin{bmatrix} G_1 & G_2 & & \\ & \ddots & \ddots & \\ & & G_1 & G_2 \\ & & & G_1 \end{bmatrix}, \mathcal{D}' = \begin{bmatrix} G_0' \\ \vdots \\ G_0' \\ G \end{bmatrix}, \mathcal{D}_0 = \begin{bmatrix} G_0 & & \\ & \ddots & \\ & & G_0 \end{bmatrix},$$

$$G_1 = \begin{bmatrix} -\beta & B_0 & & & \\ M_1 & F_1' & B_1 & & \\ & \ddots & \ddots & \ddots & \\ & & M_{S-1} & F_{S-1}' & B_{S-1} \\ & & & M_S & F_S' \end{bmatrix}, G_0 = \begin{bmatrix} 0 & & & & \\ & L_1^{(1)} & & & \\ & & \ddots & & \\ & & & L_{S-1}^{(1)} & \\ & & & & L_S^{(1)} \end{bmatrix},$$

$$G_2 = \begin{bmatrix} 0 & & & & \\ & L_1 & & & \\ & & \ddots & & \\ & & & L_{S-1} & \\ & & & & L_S \end{bmatrix}, G_2' = G_2\mathbf{e}, G_0' = G_0\mathbf{e}, G = G_2' + G_0',$$

$$L_i^{(1)}(j,k) = \begin{cases} \lambda_1 & j = 0, k = 1,\ 1 \leq i \leq S \\ 0 & \text{otherwise,} \end{cases}$$

$$F_i'(j,k) = \begin{cases} -(\lambda_1 + \mu_2 + \beta) & k = j = 0,\ 1 \leq i \leq S-1 \\ -(\lambda_1 + \mu_1 + \beta) & k = j, 1 \leq j \leq i-1,\ 2 \leq i \leq S-1 \\ -(\mu_1 + \beta) & k = j, j = i,\ 1 \leq i \leq S-1 \\ \lambda_1 & k = j+1, 1 \leq j \leq i-1,\ 1 \leq i \leq S-1 \\ 0 & \text{otherwise,} \end{cases}$$

$$F_S'(j,k) = \begin{cases} -(\lambda_1 + \mu_2) & k = j = 0, \\ -(\lambda_1 + \mu_1) & k = j, 1 \leq j \leq S-1, \\ -\mu_1 & k = j, j = S, \\ \lambda_1 & k = j+1, 1 \leq j \leq S-1, \\ 0 & \text{otherwise} \end{cases}$$

and matrices $B_i, 0 \leq i \leq S-1$ and $M_i, 1 \leq i \leq S$ are as given in Section 4.2.1.

The expected waiting time of the tagged customer according to the position of the customer at the time of his arrival, is a column vector

$$E_{\hat{T}}^r = -\hat{\boldsymbol{\mathcal{T}}}^{-1}\mathbf{e}$$

of order $(l+1)r(S+1)(s+2)/2$.

Hence the expected waiting time of an LP-customer in the system is

$$E_{LP}(W) = \sum_{l=0}^{\infty}\sum_{r=1}^{\infty} \mathbf{z}_r^l E_{\hat{T}}^r$$

where $\mathbf{z}_r^l = \big(\mathbf{e}_{l+1}' \otimes (\mathbf{x}_r, \mathbf{x}_{r-1}, ..., \mathbf{x}_1)\big)$.

We now proceed to the analysis of the second model.

4.3 Mathematical formulation of model II

In this model we introduce some additional assumptions: low priority customers do not join the system when HP-customers are present. This is too strong a restriction on the system. However, our purpose is to produce a product form solution for this highly dependent system. Nothing less than this works for the intented purpose. Despite this we are able to separate the system into two independent components only, the first of which is the number of priority customer together with number of items in the inventory and the second is the number of low priority customers. All other assumptions remain the same as in model I. Thus the process $\{(\boldsymbol{\mathcal{N}}_2(t), \boldsymbol{\mathcal{I}}(t), \boldsymbol{\mathcal{N}}_1(t)), t \geq 0\}$ is a $CTMC$ whose state space $\{(n_2, i, n_1); n_2 \geq 0, 0 \leq i \leq S, 0 \leq n_1 \leq i\}$ and its infinitesimal generator $\boldsymbol{\mathcal{Q}}_2$ has the form

$$\boldsymbol{\mathcal{Q}}_2 = \begin{bmatrix} A'_{00} & A'_0 & & & \\ A'_2 & A'_1 & A'_0 & & \\ & A'_2 & A'_1 & A'_0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}. \tag{4.13}$$

Define $\boldsymbol{\mathcal{Q}}_{2(n_2,i,n_1)}^{(m_2,j,m_1)}$ as the transition rates from the state $(n_2, i, n_1) \to (m_2, j, m_1)$. Thus the transition rates in this $CTMC$ are

$$\boldsymbol{\mathcal{Q}}_{2(n_2,i,n_1)}^{(m_2,j,m_1)} = \begin{cases} \lambda_2 & m_2 = n_2 + 1, n_2 \geq 0; \qquad j = i, 1 \leq i \leq S; \\ & m_1 = n_1 = 0, \\ \mu_2 & m_2 = n_2 - 1, n_2 \geq 1; \qquad j = i - 1, 1 \leq i \leq S; \\ & m_1 = n_1 = 0, \\ \lambda_1 & m_2 = n_2, n_2 \geq 0; \qquad j = i, 1 \leq i \leq S; \\ & m_1 = n_1 + 1, 0 \leq n_1 \leq i - 1, \\ \mu_2 & m_2 = n_2, n_2 \geq 0; \qquad j = i - 1, 1 \leq i \leq S; \\ & m_1 = n_1 - 1, 1 \leq n_1 \leq i, \\ \beta & m_2 = n_2, n_2 \geq 0; \qquad j = i + 1, 0 \leq i \leq S - 1; \\ & m_1 = n_1, 0 \leq n_1 \leq i, \\ 0 & \text{otherwise}, \end{cases}$$

and its diagonal entries are such that each row sum is zero.

4.3.1 Stability condition

Now look at the finite state space $CTMC$ $\{(\boldsymbol{\mathcal{I}}(t), \boldsymbol{\mathcal{N}}_1(t)), t \geq 0\}$ defined on the phases $\{(i, n_1)/0 \leq i \leq S, 0 \leq n_1 \leq i\}$. Denote its infinitesimal generator by $A' = A'_0 + A'_1 + A'_2$ where A'_0, A'_1, A'_2 are matrices indicating transition rates, to one higher level, within the same level and to the immediate lower level in the repeating part, respectively (that is, the transition at the boundary are excluded).
Let $\boldsymbol{\pi}'$ be the steady state probability vector of A'. Then $\boldsymbol{\pi}' A' = 0, \quad \boldsymbol{\pi}'\mathbf{e} = 1.$

The Markov chain is stable if and only if

$$\boldsymbol{\pi}' A'_0 \mathbf{e} < \boldsymbol{\pi}' A'_2 \mathbf{e}. \tag{4.14}$$

This simplifies to $\lambda_2 < \mu_2$. Thus we have the following lemma.

Lemma 4.3.1. The system under study is stable if and only if $\lambda_2 < \mu_2$.

4.4 Steady-state analysis

For the stable system (that is under the condition $\lambda_2 < \mu_2$), we will prove that the system state can be decomposed. We establish that the joint distribution of the system state equals the product of their marginals. For computing the steady-state probability vector of the system, we first consider an inventory system with negligible service time for LP-customers alone. The rest of the assumptions are the same as given earlier. The corresponding Markov chain is denoted by $\Omega' = \{(\boldsymbol{\mathcal{I}}(t), \boldsymbol{\mathcal{N}}_1(t)), t \geq 0\}$. The state space of this finite $CTMC$

is $\{(i, n_1)/0 \leq i \leq S, 0 \ \leq n_1 \leq i\}$ and its infinitesimal generator is given by

$$\tilde{A} = \begin{bmatrix} -\beta & B_0 & & & \\ M_1' & F_1^{'(0)} & B_1 & & \\ & \ddots & \ddots & \ddots & \\ & & M_{S-1}^{'} & F_{S-1}^{'(0)} & B_{S-1} \\ & & & M_S^{'} & F_S^{'(0)} \end{bmatrix} \tag{4.15}$$

where $F_i^{'(0)}(j,k), B_i(j,k), M_i^{'}(j,k)$ denote the transition rates of HP-customers from j to k with

$$F_i^{'(0)}(j,k) = \begin{cases} -(\lambda_1 + \lambda_2 + \beta) & k = j = 0;\ 1 \leq i \leq S-1 \\ -(\lambda_1 + \mu_1 + \beta) & k = j, 1 \leq j \leq i-1;\ 2 \leq i \leq S-1 \\ -(\mu_1 + \beta) & k = j, j = i;\ 1 \leq i \leq S-1 \\ \lambda_1 & k = j+1, 0 \leq j \leq i-1;\ 1 \leq i \leq S-1 \\ 0 & \text{otherwise,} \end{cases}$$

$$F_S^{'(0)}(j,k) = \begin{cases} -(\lambda_1 + \lambda_2) & k = j = 0, \\ -(\lambda_1 + \mu_1) & k = j, 1 \leq j \leq S-1, \\ \mu_1 & k = j, j = S, \\ \lambda_1 & k = j+1, 0 \leq j \leq S-1, \\ 0 & \text{otherwise,} \end{cases}$$

$$B_i(j,k) = \begin{cases} \beta & k = j,\ 0 \leq j \leq i;\ 0 \leq i \leq S-1 \\ 0 & \text{otherwise,} \end{cases}$$

$$M_i^{'}(j,k) = \begin{cases} \lambda_2 & k = j = 0;\ 1 \leq i \leq S \\ \mu_1 & k = j-1,\ 1 \leq j \leq i;\ 1 \leq i \leq S \\ 0 & \text{otherwise.} \end{cases}$$

Let $\boldsymbol{\xi} = (\boldsymbol{\xi}_0, \boldsymbol{\xi}_1, ..., \boldsymbol{\xi}_S)$ be the steady-state vector of $\tilde{A}$. Then $\boldsymbol{\xi}$ satisfies the equations

$$\boldsymbol{\xi}\tilde{A} = 0, \quad \boldsymbol{\xi}\mathbf{e} = 1. \tag{4.16}$$

Sub-vectors of $\boldsymbol{\xi}$ are further partitioned as

$$\boldsymbol{\xi}_i = (\xi_i(0), \xi_i(1), ..., \xi_i(i)), \quad 0 \le i \le S$$

and can be obtained as

$$\boldsymbol{\xi}_i = \boldsymbol{\xi}_{i-1} \boldsymbol{\mathcal{V}}_{i-1}, \quad 1 \le i \le S$$

where

$$\begin{aligned} \boldsymbol{\mathcal{V}}_{S-1} = & \ -B_{S-1}[F_S^{'(0)}]^{-1} \\ \boldsymbol{\mathcal{V}}_i = & \ -B_i[F_{i+1}^{'(0)} + \boldsymbol{\mathcal{V}}_{i+1} M'_{i+2}]^{-1}, \quad 0 \le i \le S-2. \end{aligned}$$

The unknown probability $\boldsymbol{\xi}_0$ can be found from the normalizing condition

$$\boldsymbol{\xi}_0 \left[I + \sum_{j=0}^{S-1} \prod_{i=0}^{j} \boldsymbol{\mathcal{V}}_i \right] \mathbf{e} = 1.$$

Using the components of the probability vector $\boldsymbol{\xi}$ we shall find the steady-state probability vector to the original system. Let $\mathbf{x}'$ be the steady-state vector of the generator $\boldsymbol{\mathcal{Q}}_2$. Then $\mathbf{x}'$ must satisfy the set of equations

$$\mathbf{x}' \boldsymbol{\mathcal{Q}}_2 = 0, \quad \mathbf{x}' \mathbf{e} = 1. \tag{4.17}$$

Partition $\mathbf{x}'$ as $\mathbf{x}' = (\mathbf{x}'_0, \mathbf{x}'_1, \mathbf{x}'_2, ...)$. Then the above system of equations reduces to:

$$\mathbf{x}'_0 A'_{00} + \mathbf{x}'_1 A'_2 = 0, \tag{4.18}$$

$$\mathbf{x}'_{i-1} A_0 + \mathbf{x}'_i A'_1 + \mathbf{x}'_{i+1} A'_2 = 0, \quad i \ge 1. \tag{4.19}$$

We take as a candidate for the solution

$$\mathbf{x}'_i = \gamma \rho_2^i \boldsymbol{\xi}, \quad i \ge 0 \tag{4.20}$$

where $\rho_2 = \frac{\lambda_2}{\mu_2}$ and γ is a constant. That (4.18) and (4.19) are satisfied by (4.20) can be easily verified: from (4.18), we have

$$\mathbf{x}'_0 A'_{00} + \mathbf{x}'_1 A'_2 = \gamma \boldsymbol{\xi} \left[A'_{00} + \rho_2 A'_2 \right] \tag{4.21}$$

and from (4.19),

$$\mathbf{x}'_{i-1}A_0 + \mathbf{x}'_iA'_1 + \mathbf{x}'_{i+1}A'_2 = \gamma\rho_2^i\boldsymbol{\xi}\left[A'_{00} + \rho_2A'_2\right], \quad i \geq 1. \tag{4.22}$$

We have $\tilde{A} = [A'_{00} + \rho_2A'_2]$ and from (4.16), the right hand side of equations (4.21), (4.22) are zero. Hence it follows that if we take the expression of the vector $\mathbf{x}'$ as given by (4.20), equations (4.18) and (4.19) are satisfied. Now applying the normalizing condition $\mathbf{x}'\mathbf{e} = 1$, we get

$$\gamma\left[1 + \rho_2 + \rho_2^2 + ...\right] = 1.$$

Hence under the condition that $\lambda_2 < \mu_2$, we have $\gamma = 1 - \rho_2$. Thus we have proved the main result of model II:

Theorem 4.4.1. *The system under consideration has stochastic decomposition with the joint distribution of the system state equal to the product of their marginal distributions.*

We now turn to compute a few of the important system characteristics.

Performance Measures

1. Expected number of low priority customers in the system

$$E_{LP}(N) = \frac{\lambda_2}{\mu_2 - \lambda_2}.$$

2. Expected number of high priority customers in the system

$$E_{HP}(N) = \sum_{i=1}^{S}\sum_{n_1=1}^{i} n_1\xi_i(n_1).$$

3. Expected number of items in the inventory

$$E(I) = \sum_{i=1}^{S}\sum_{n_1=0}^{i} i\xi_i(n_1).$$

4. Expected loss rate of low priority customers

$$E_{LP}(L) = \lambda_2 \left[\xi_0(0) + \sum_{i=1}^{S} \sum_{n_1=1}^{i} \xi_i(n_1) \right].$$

5. Expected loss rate of high priority customers

$$E_{HP}(L) = \lambda_1 \sum_{i=0}^{S} \xi_i(i).$$

6. Expected reorder rate

$$E(R) = \sum_{i=1}^{S} \left[\mu_1 \sum_{n_1=1}^{i} \xi_i(n_1) + \lambda_2 \xi_i(0) \right].$$

7. Expected waiting time of low priority customer

$$E_{LP}(W) = \sum_{l=0}^{\infty} \sum_{r=1}^{\infty} \mathbf{z}_r^{\prime l} E_{\hat{T}}^r$$

   where $\mathbf{z}_r^{\prime l} = \left(\mathbf{e}'_{l+1} \otimes (\mathbf{x}'_r \mathbf{x}'_{r-1} ... \mathbf{x}'_1) \right)$ and $E_{\hat{T}}^r$ given in Section 4.2.4.

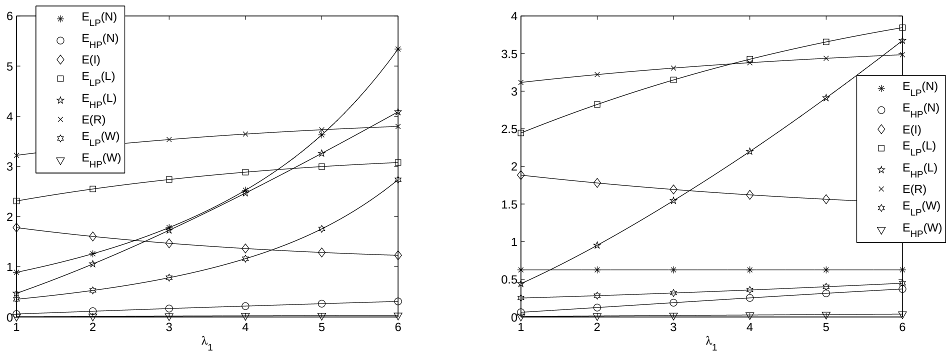
8. Expected waiting time of high priority customer

$$E_{HP}(W) = \sum_{r=1}^{S} \mathbf{y}'_r E_T^r$$

   where $\mathbf{y}'_r = (\xi_r(r), ..., \xi_S(r), \xi_{r-1}(r-1), ..., \xi_S(r-1), ..., \xi_1(1), ..., \xi_S(1))$ and E_T^r given in Section 4.2.3.

4.5 Numerical illustration

In this section we provide numerical illustration of the system performance measures with variation in the values of the underlying parameters.



(a) model I (b) model II

Figure 4.1: Effect of λ_1 on various performance measures with $\lambda_2 = 5, \mu_1 = 10, \mu_2 = 13, \beta = 1, S = 5$ in models I & II

Effect of λ_1 on various performance measures

From Figure 4.1 we can make the following observations. As the arrival rate of high priority customer increases, expected reorder rate, expected waiting time of HP-customers, expected waiting time of LP-customers, expected loss rate for LP-customers and expected loss rate for HP-customers increase. However, the expected number of items in the inventory decreases.

Effect of μ_1 on various performance measures

Table 4.1 indicates the variation in the system performance measures with a high priority customer's service rate μ_1. As μ_1 increases, the behavior of measures like expected number of HP/LP-customers in the system, expected number of items in the inventory, expected reorder rate, expected waiting time of HP/LP-customer are on expected lines. The decrease in the expected number of LP-customers in model II, with increase in μ_1 is attributed to the increase in number HP-customers getting into the system and consequently resulting

Model I								
μ_1	$E_{LP}(N)$	$E_{HP}(N)$	$E(I)$	$E_{LP}(L)$	$E_{HP}(L)$	$E(R)$	$E_{LP}(W)$	$E_{HP}(W)$
5	7.5961	0.4161	1.5688	2.6115	1.7932	3.4634	3.2240	0.0832
6	4.6498	0.3213	1.5179	2.6755	1.7764	3.5036	1.9645	0.0536
7	3.2674	0.2606	1.4935	2.7059	1.7599	3.5196	1.3871	0.0372
8	2.5206	0.2189	1.4806	2.7220	1.7461	3.5272	1.0811	0.0274
9	2.0721	0.1887	1.4729	2.7317	1.7352	3.5318	0.8992	0.0210
10	1.7802	0.1667	1.4678	2.7382	1.7265	3.5351	0.7813	0.0166
Model II								
μ_1	$E_{LP}(N)$	$E_{HP}(N)$	$E(I)$	$E_{LP}(L)$	$E_{HP}(L)$	$E(R)$	$E_{LP}(W)$	$E_{HP}(W)$
5	0.6250	0.5073	2.0094	3.5417	1.4677	2.9906	0.4015	0.1015
6	0.6250	0.3868	1.8982	3.4117	1.4865	3.1018	0.3666	0.0645
7	0.6250	0.3092	1.8221	3.3177	1.5044	3.1779	0.3451	0.0442
8	0.6250	0.2561	1.7671	3.2471	1.5200	3.2329	0.3321	0.0320
9	0.6250	0.2177	1.7256	3.1922	1.5334	3.2744	0.3237	0.0242
10	0.6250	0.1889	1.6933	3.1486	1.5447	3.3067	0.3178	0.0189

Table 4.1: Effect of μ_1 on various performance measures with $\lambda_2 = 5, \lambda_1 = 3, \mu_2 = 13, \beta = 1, S = 5$ in models I & II

in the reduction of LP-customers (while HP-customer is getting served). However this is reversed in model I since LP-customers join the system even when an HP-customer is in service. This is a consequence of presence of the larger number of LP-customers resulting in the inventory becoming empty more often. We stress that HP-customers do not join the system while inventory is zero in both models.

Effect of λ_2 on various performance measures

Table 4.2 shows that an increase in the arrival rate of low priority customers results in an increase in measures like the expected number of LP-customers, expected loss rate of both LP/HP-customers, expected waiting time of LP/HP-customers and expected reorder rate, whereas the expected number of HP-customers and expected number of items show a decreasing trend. Both low priority and high priority customers loss rate increase due to the increase of η_0 in model I and in model II due to increase in the values of $\mathbf{x}_{n_2}(0,0),\ n_2 \geq 0$.

Model I								
λ_2	$E_{LP}(N)$	$E_{HP}(N)$	$E(I)$	$E_{LP}(L)$	$E_{HP}(L)$	$E(R)$	$E_{LP}(W)$	$E_{HP}(W)$
3	0.7753	0.2009	1.8159	1.3604	1.4555	3.1841	0.3896	0.0201
4	1.1909	0.1813	1.6233	2.0194	1.6035	3.3772	0.5588	0.0181
5	1.7802	0.1657	1.4678	2.7382	1.7265	3.5351	0.7813	0.0166
6	2.6793	0.1543	1.3424	3.4980	1.8286	3.6680	1.1017	0.0154
7	4.1414	0.1470	1.2438	4.2792	1.9113	3.7797	1.6024	0.0147
8	6.6268	0.1426	1.1702	5.0619	1.9744	3.8676	2.4288	0.0143
Model II								
λ_2	$E_{LP}(N)$	$E_{HP}(N)$	$E(I)$	$E_{LP}(L)$	$E_{HP}(L)$	$E(R)$	$E_{LP}(W)$	$E_{HP}(W)$
3	0.3000	0.2212	2.0133	1.7105	1.3027	2.9867	0.1557	0.0221
4	0.4444	0.2039	1.8398	2.4076	1.4322	3.1602	0.2303	0.0204
5	0.6250	0.1889	1.6933	3.1486	1.5447	3.3067	0.3178	0.0189
6	0.8571	0.1759	1.5682	3.9250	1.6431	3.4319	0.4223	0.0176
7	1.1667	0.1645	1.4602	4.7300	1.7294	3.5399	0.5509	0.0164
8	1.6000	0.1545	1.3667	5.5575	1.8053	3.6340	0.7171	0.0154

Table 4.2: Effect of λ_2 on various performance measures with $\lambda_1 = 3, \mu_1 = 10, \mu_2 = 13, \beta = 1, S = 5$ in models I & II

Model I								
μ_2	$E_{LP}(N)$	$E_{HP}(N)$	$E(I)$	$E_{LP}(L)$	$E_{HP}(L)$	$E(R)$	$E_{LP}(W)$	$E_{HP}(W)$
9	4.2276	0.1721	1.4802	2.7178	1.7180	3.5423	1.8547	0.0172
10	3.1029	0.1691	1.4729	2.7289	1.7230	3.5407	1.3548	0.0169
11	2.4621	0.1673	1.4697	2.7341	1.7251	3.5383	1.0749	0.0167
12	2.0569	0.1663	1.4684	2.7368	1.7261	3.5364	0.8999	0.0166
13	1.7802	0.1657	1.4678	2.7382	1.7265	3.5351	0.7813	0.0166
14	1.5803	0.1654	1.4677	2.7390	1.7268	3.5342	0.6961	0.0165
Model II								
μ_2	$E_{LP}(N)$	$E_{HP}(N)$	$E(I)$	$E_{LP}(L)$	$E_{HP}(L)$	$E(R)$	$E_{LP}(W)$	$E_{HP}(W)$
9	1.25	0.1889	1.6936	3.1482	1.5445	3.3065	0.6318	0.0189
10	1.0000	0.1889	1.6934	3.1485	1.5447	3.3066	0.5055	0.0189
11	0.8333	0.1889	1.6933	3.1485	1.5447	3.3067	0.4218	0.0189
12	0.7143	0.1889	1.6933	3.1486	1.5447	3.3067	0.3623	0.0189
13	0.6250	0.1889	1.6933	3.1486	1.5447	3.3067	0.3128	0.0189
14	0.5556	0.1889	1.6933	3.1486	1.5447	3.3067	0.2833	0.0189

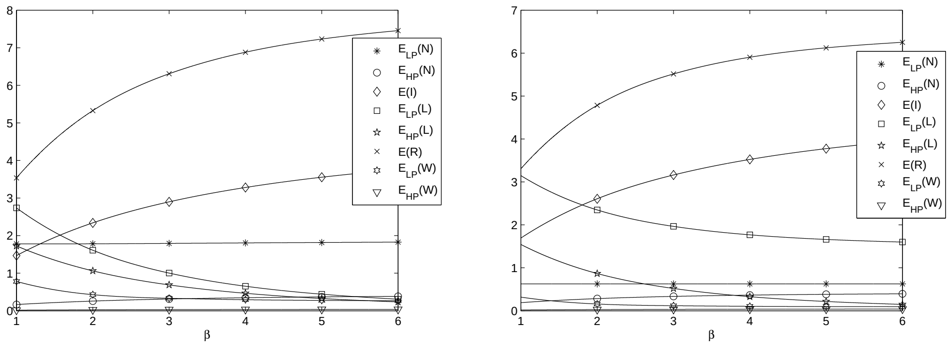
Table 4.3: Effect of μ_2 on various performance measures with $\lambda_1 = 3, \mu_1 = 10, \lambda_2 = 5, \beta = 1, S = 5$ in models I & II

Effect of μ_2 on various performance measures

Table 4.3 shows that in model II, with increasing value of μ_2, the expected inventory level decreases and consequently the reorder rate increases. This behaviour looks quite rational whereas it shows a decreasing trend in model I. We do not have an explanation for the latter.

Effect of β on various performance measures

Figure 4.2 indicates the variation in the system performance measures with replenishment rate β. This shows that the behavior of the system performance measures with increase in β is similar to that with increase in S, which is on expected lines.



(a) model I (b) model II

Figure 4.2: Effect of β on various performance measures with $\lambda_1 = 3, \mu_1 = 10, \lambda_2 = 5, \mu_2 = 13, S = 5$ in models I & II

4.5.1 Optimization problem

Based on the above performance measures we construct a cost function for checking the optimality of the maximum inventory S in the system.

Let $\mathcal{C}^{(L)}_{LP}(\mathcal{C}^{(L)}_{HP})$ be the loss due to a single LP (HP) customer being lost to the system, $\mathcal{C}^{(R)}$ be the purchase price/unit, $\mathcal{C}^{(I)}$ be the inventory carrying cost/unit/unit time, $\mathcal{C}^{(W)}_{LP}$ be the waiting cost in the system per LP-customer and $\mathcal{C}^{(W)}_{HP}$ be that for the HP-customer.

The total expected cost can be expressed as

$$\begin{aligned}\mathbf{F}(S) = \; & \boldsymbol{\mathcal{C}}_{LP}^{(L)} E_{LP}(L) + \boldsymbol{\mathcal{C}}_{HP}^{(L)} E_{HP}(L) + \boldsymbol{\mathcal{C}}^{(R)} E(R) + \boldsymbol{\mathcal{C}}^{(I)} E(I) \\ & + \boldsymbol{\mathcal{C}}_{LP}^{(W)} E_{LP}(W) + \boldsymbol{\mathcal{C}}_{HP}^{(W)} E_{HP}(W).\end{aligned}$$

This is a function of S. Hence we compute the effect of S on $\mathbf{F}(S)$. We assign the following values to the parameters: $\lambda_1 = 2, \mu_1 = 7, \lambda_2 = 3, \mu_2 = 5, \beta = 1, \boldsymbol{\mathcal{C}}_{LP}^{(L)} = \$10, \boldsymbol{\mathcal{C}}_{HP}^{(L)} = \$20, \boldsymbol{\mathcal{C}}_{LP}^{(W)} = \$5, \boldsymbol{\mathcal{C}}_{HP}^{(W)} = \$10, \boldsymbol{\mathcal{C}}^{(R)} = \$50$ and $\boldsymbol{\mathcal{C}}^{(I)} = \$5$. $\mathbf{F}(S)$ value for different values of S are tabulated in Table 4.4.

S	Model I $\mathbf{F}(S)$	Model II $\mathbf{F}(S)$
2	113.5168	99.6193
3	104.1112	92.7812
4	97.9858	88.5918
5	94.0210	86.2826
6	91.6896	**85.4251**
7	**90.6554**	85.7196
8	90.6728	86.9370
9	91.5511	87.2153

Table 4.4: Effect of variation in S

Chapter 5

On a queueing-inventory with reservation, cancellation, common life time and retrial

In chapter 2, 3 and 4 we analyzed inventory systems, for most of which product form solution could be arrived at. From now on we consider real life situations which do not yield product form solution for the system state probability. The reason for the non availability of that may be the ‘dimension’ problem in modelling.

Advance reservation / purchase of inventory for future use is a common phenomena. Sometimes items reserved are subject to cancellation. A typical example is flight / train / bus seats for travel. The seats are considered as inventory. In this context once the flight / train / bus departs, the one holding inventory, but not using it, will lose the inventory as well. This is so since the

Part of this chapter appeared in Annals of Operations Research, under the title: **On a queueing-inventory with reservation, cancellation, common life time and retrial**, *A. Krishnamoorthy, Dhanya Shajin and B. Lakshmy*, Annals of Operations Research (Springer), DOI 10.1007/s10479-015-1849-x.

life time of the inventory has expired. Thus in our cited example, all items in the inventory have a common life time.

In this chapter we consider maximum items in the inventory to be S which have a random common life time; this includes those that are sold in a particular cycle. A customer, on arrival to an idle server, with at least one item in inventory, is immediately taken for service; else he joins the buffer of maximum size S depending on number of items in the inventory (the buffer capacity varies and is, at any time, equal to the number of items in the inventory). The arrival of customers constitutes a Poisson process, demanding exactly one item each from the inventory. If there is no item in the inventory, the arriving customer first queues up in a finite waiting space of capacity K. When it overflows an arrival goes to an orbit of infinite capacity with probability p or is lost forever with probability $1-p$. From the orbit he retries for service according to an exponentially distributed inter-occurrence time. The service time follows an exponential distribution. Cancellation of reservation before its expiry is permitted. Inventory gets added through cancellation of purchased items until the expiry time. Cancellation time is assumed to be negligible.

It is interesting to note that in a recent investigation, with much reduced dimension, we arrived at product form solution. However, the findings are not reported in this thesis.

5.1 Mathematical formulation

We consider an infinite capacity queueing-inventory system with positive service time to which customers arrive according to a Poisson process with rate λ demanding one item each. At the beginning of each cycle there are S new items in the inventory which have a common life time, exponentially distributed with parameter α. Here cycle is the time duration from the epoch at which we start with S items at a replenishment epoch, to the moment when the common life time is realized. The service time for each customer follows an exponential distribution with parameter μ. If on arrival a customer finds the server busy, it joins a buffer of varying size (which depends on the number of items in the inventory). In the absence of vacant position in the buffer, the customer joins a finite waiting room of capacity K, provided vacant position is available there. If the waiting space has reached the maximum capacity, then an arriving customer joins an orbit of infinite capacity with probability p or is lost forever with probability $1 - p$. From the orbit he retries for service at a constant rate η. Cancellation of item sold (that is, returning of a sold item) before its expiry is permitted. This takes place according to an exponentially distributed inter-occurrence time with parameter $i\beta$, when $(S-i)$ items are in the inventory. Through cancellation of purchases, inventory gets added until their expiry time. On expiry of common life time, the inventory reaches its maximum level S through an instantaneous (zero lead time) replenishment for the next cycle. Note that through cancellation inventory level will not go above S since exactly i items are 'in sold list' (which is the maximum possible number that could appear for cancellation) when $(S - i)$ items are held in the inventory. Cancellation time is assumed to be negligible. We start with the case of customers being flushed out from the finite buffer and the waiting room, but not from the orbit, at the epoch of occurrence of the common life time is realized.

When the buffer is full (that is, the number of items in inventory equal to number of customers in the buffer) a new arrival has to go to the waiting room. From waiting room customers go to buffer in the order of arrival as and when cancellation of purchased inventory occurs. Overflow of waiting room results in new arrivals going to an orbit of infinite capacity; the probability of such a customer joining being $p, 0 < p < 1$. With complementary probability the customer leave the system forever. From orbit, customers try to access the waiting room through retrial, if buffer is full; else go to buffer. Failed retrials get back to orbit. Note that $p = 0$ provides a finite state space system and $0 < p \leq 1$, an infinite system; $p = 1$ provides a system with no balking.

In the sequel we use the following notations:

$\boldsymbol{\mathcal{N}}_1(t)$ Number of customers in the orbit at time t
$\boldsymbol{\mathcal{N}}_2(t)$ Number of customers in the waiting room at time t
$\boldsymbol{\mathcal{N}}_3(t)$ Number of customers in the buffer at time t
$\boldsymbol{\mathcal{I}}(t)$ Number of items in the inventory at time t
$\boldsymbol{\mathcal{N}}_4(t)$ Number of revisits to S upto time t (within the same cycle).

$$\boldsymbol{\mathcal{S}}(t) = \begin{cases} 0; & \text{if server is idle at time } t, \\ 1; & \text{if server is busy at time } t. \end{cases}$$

$U_1 = (S+1)(S+2)/2$
$U_2 = (S+1)S/2 + 2$

The process $\{(\boldsymbol{\mathcal{N}}_1(t), \boldsymbol{\mathcal{N}}_2(t), \boldsymbol{\mathcal{I}}(t), \boldsymbol{\mathcal{N}}_3(t), \boldsymbol{\mathcal{S}}(t)), t \geq 0\}$ is a $CTMC$ which is $LIQBD$, with state space

$$\Omega = \{(n_1, 0, i, 0, 0); n_1 \geq 0; 0 \leq i \leq S\} \bigcup \{(n_1, n_2, 0, 0, 0); n_1 \geq 0; 1 \leq n_2 \leq K\}$$
$$\bigcup \{(n_1, n_2, i, n_3, 1); n_1 \geq 0; 1 \leq n_2 \leq K; 1 \leq i \leq S; n_3 = i\} \bigcup$$
$$\{(n_1, 0, i, n_3, 1); n_1 \geq 0; 1 \leq i \leq S; 1 \leq n_3 \leq i\} \bigcup \{(n_1, 0, S, 0); n_1 \geq 0\}.$$

Once the CLT is realized the inventory level reaches its maximum by way of fresh replenishment with no customer in the finite buffer and waiting room. The subset of the state space constituting such states is given by $\{(n_1, 0, S, 0); n_1 \geq 0\}$. Thus the infinitesimal generator $\boldsymbol{\mathcal{Q}}$ of this $CTMC$ is

$$\boldsymbol{\mathcal{Q}} = \begin{bmatrix} B_{00} & B_0 & & & \\ B_2 & B_1 & B_0 & & \\ & B_2 & B_1 & B_0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}.$$

The matrices B_{00}, B_0, B_1 and B_2 are square matrices of order $U_1+K(S+1)+1$.

$$B_{00} = \begin{bmatrix} A_{00} & \bar{\alpha} \\ \bar{\lambda} & -\lambda \end{bmatrix}, B_0 = \begin{bmatrix} A_0 & \mathbf{0} \\ 0 & 0 \end{bmatrix}, B_1 = \begin{bmatrix} A_1 & \bar{\alpha} \\ \bar{\lambda} & -(\eta+\lambda) \end{bmatrix}, B_2 = \begin{bmatrix} A_2 & \mathbf{0} \\ \bar{\eta} & 0 \end{bmatrix}$$

where $\bar{\lambda} = (0, ..., 0, \lambda, 0, ..., 0)$ and $\bar{\eta} = (0, ..., 0, \eta, 0, ..., 0)$ with λ and η at the U_2^{th} position and $\bar{\alpha} = \alpha\mathbf{e}$. A_{00}, A_0, A_1, A_2 are square matrices of order $U_1 + K(S+1)$ which are represented by

$$A_{00} = \begin{bmatrix} H_0 & L_0 & & & & \\ M_0 & H & L & & & \\ & M & H & L & & \\ & & \ddots & \ddots & \ddots & \\ & & & M & H & L \\ & & & & M & H_1 \end{bmatrix}, A_1 = \begin{bmatrix} H_0' & L_0 & & & & \\ M_0 & H & L & & & \\ & M & H & L & & \\ & & \ddots & \ddots & \ddots & \\ & & & M & H & L \\ & & & & M & H_1 \end{bmatrix},$$

$A_0 = diag(0, ..., 0, L_1)$ and $A_2 = diag(N, 0, ..., 0)$.

The sub-matrices H, H_1, L, L_1, M are square matrices of order $(S+1)$; H_0, H_0', N are square matrices of order U_1; dimension of L_0 and M_0 are $U_1 \times (S+1)$, $(S+1) \times U_1$ respectively. These sub-matrices give transition rates from the state $(i, n_3, k_1) \to (j, m_3, k_2)$ where $i(j)$ represents the number of items in the inventory; $n_3(m_3)$, the number of customers in the buffer and k_ℓ, for $\ell = 1, 2$, are status of the server.

$$N_{(i,n_3,k_1;j,m_3,k_2)} = \begin{cases} \eta & j = i, m_3 = n_3 + 1, k_2 = 1 \\ & 1 \leq i \leq S, n_3 = 0, k_1 = 0 \\ 0 & \text{otherwise,} \end{cases}$$

$$L_{0(i,n_3,k_1;j,m_3,k_2)} = \begin{cases} \lambda & j = i, m_3 = n_3, k_2 = 0 \\ & i = 0, n_3 = 0, k_1 = 0 \\ \lambda & j = i, m_3 = n_3, k_2 = k_1 \\ & 1 \leq i \leq S, n_3 = i, k_1 = 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$L_{(i,n_3,k_1;j,m_3,k_2)} = \begin{cases} \lambda & j=i, m_3=n_3, k_2=0 \\ & i=0, n_3=0, k_1=0 \\ \lambda & j=i, m_3=n_3, k_2=k_1 \\ & 1 \le i \le S, n_3=i, k_1=1 \\ 0 & \text{otherwise,} \end{cases}$$

$$H_{(i,n_3,k_1;j,m_3,k_2)} = \begin{cases} -(\lambda+S\beta+\alpha) & j=i, m_3=n_3, k_2=0 \\ & i=0, n_3=0, k_1=0 \\ -(\lambda+(S-i)\beta+\mu+\alpha) & j=i, m_3=n_3, k_2=k_1 \\ & 1 \le i \le S, n_3=i, k_1=1 \\ \mu & j=i-1, m_3=n_3-1, k_2=k_1 \\ & 2 \le i \le S, n_3=i, k_1=1 \\ \mu & j=i-1, m_3=n_3-1, k_2=0 \\ & i=1, n_3=i, k_1=1 \\ 0 & \text{otherwise,} \end{cases}$$

$$H_{0(i,n_3,k_1;j,m_3,k_2)} = \begin{cases} -(\lambda+(S-i)\beta+\alpha) & j=i, m_3=n_3, k_2=k_1 \\ & 0 \le i \le S, n_3=0, k_1=0 \\ -(\lambda+(S-i)\beta+\mu+\alpha) & j=i, m_3=n_3, k_2=k_1 \\ & 1 \le i \le S, 1 \le n_3 \le i, k_1=1 \\ (S-i)\beta & j=i+1, m_3=n_3, k_2=k_1 \\ & 0 \le i \le S-1, n_3=0, k_1=0 \\ (S-i)\beta & j=i+1, m_3=n_3, k_2=k_1 \\ & 0 \le i \le S-1, 1 \le n_3 \le i, k_1=1 \\ \mu & j=i-1, m_3=n_3-1, k_2=0 \\ & 1 \le i \le S, n_3=1, k_1=1 \\ \mu & j=i-1, m_3=n_3-1, k_2=1 \\ & 2 \le i \le S, 2 \le n_3 \le i, k_1=1 \\ \lambda & j=i, m_3=n_3+1, k_2=1 \\ & 1 \le i \le S, n_3=0, k_1=0 \\ \lambda & j=i, m_3=n_3+1, k_2=1 \\ & 2 \le i \le S, 1 \le n_3 \le i-1, k_1=1 \\ 0 & \text{otherwise,} \end{cases}$$

$$H'_{0(i,n_3,k_1;j,m_3,k_2)} = \begin{cases} -(\lambda+\eta+(S-i)\beta+\alpha) & j=i, m_3=n_3, k_2=k_1 \\ & 1\le i\le S, n_3=0, k_1=0 \\ -(\lambda+(S-i)\beta+\alpha) & j=i, m_3=n_3, k_2=k_1 \\ & i=0, n_3=0, k_1=0 \\ -(\lambda+(S-i)\beta+\mu+\alpha) & j=i, m_3=n_3, k_2=k_1 \\ & 1\le i\le S, 1\le n_3\le i, k_1=1 \\ (S-i)\beta & j=i+1, m_3=n_3, k_2=k_1 \\ & 0\le i\le S-1, n_3=0, k_1=0 \\ (S-i)\beta & j=i+1, m_3=n_3, k_2=k_1 \\ & 0\le i\le S-1, 1\le n_3\le i, k_1=1 \\ \mu & j=i-1, m_3=n_3-1, k_2=0 \\ & 1\le i\le S, n_3=1, k_1=1 \\ \mu & j=i-1, m_3=n_3-1, k_2=1 \\ & 2\le i\le S, 2\le n_3\le i, k_1=1 \\ \lambda & j=i, m_3=n_3+1, k_2=1 \\ & 1\le i\le S, n_3=0, k_1=0 \\ \lambda & j=i, m_3=n_3+1, k_2=1 \\ & 2\le i\le S, 1\le n_3\le i-1, k_1=1 \\ 0 & \text{otherwise,} \end{cases}$$

$$L_{1(i,n_3,k_1;j,m_3,k_2)} = \begin{cases} p\lambda & j=i, m_3=n_3, k_2=0 \\ & i=0, n_3=0, k_1=0 \\ p\lambda & j=i, m_3=n_3, k_2=k_1 \\ & 1\le i\le S, n_3=i, k_1=1 \\ 0 & \text{otherwise,} \end{cases}$$

$$H_{1(i,n_3,k_1;j,m_3,k_2)} = \begin{cases} -(p\lambda+S\beta+\alpha) & j=i, m_3=n_3, k_2=0 \\ & i=0, n_3=0, k_1=0 \\ -(p\lambda+(S-i)\beta+\mu+\alpha) & j=i, m_3=n_3, k_2=k_1 \\ & 1\le i\le S, n_3=i, k_1=1 \\ \mu & j=i-1, m_3=n_3-1, k_2=k_1 \\ & 2\le i\le S, n_3=i, k_1=1 \\ \mu & j=i-1, m_3=n_3-1, k_2=0 \\ & i=1, n_3=i, k_1=1 \\ 0 & \text{otherwise,} \end{cases}$$

$$M_{0(i,n_3,k_1;j,m_3,k_2)} = \begin{cases} S\beta & j=i+1, m_3=n_3+1, k_2=1 \\ & i=0, n_3=0, k_1=0 \\ (S-i)\beta & j=i+1, m_3=n_3+1, k_2=k_1 \\ & 1 \le i \le S-1, n_3=i, k_1=1 \\ 0 & \text{otherwise}, \end{cases}$$

$$M_{(i,n_3,k_1;j,m_3,k_2)} = \begin{cases} S\beta & j=i+1, m_3=n_3+1, k_2=1 \\ & i=0, n_3=0, k_1=0 \\ (S-i)\beta & j=i+1, m_3=n_3+1, k_2=k_1 \\ & 1 \le i \le S-1, n_3=i, k_1=1 \\ 0 & \text{otherwise}. \end{cases}$$

5.2 Steady-state analysis

In this section, we perform the steady-state analysis of the queueing-inventory model under study by first establishing the stability condition of the system.

5.2.1 Stability condition

To establish the stability condition, we consider the Markov chain $\{(\boldsymbol{\mathcal{N}}_2(t), \boldsymbol{\mathcal{I}}(t), \boldsymbol{\mathcal{N}}_3(t), \boldsymbol{\mathcal{S}}(t)), t \ge 0\}$ on the finite state space $\{(0,i,0,0), 0 \le i \le S\} \bigcup \{(n_2,0,0,0), 1 \le n_2 \le K\} \bigcup \{(0,i,n_3,1), 1 \le i \le S, 1 \le n_3 \le i\} \bigcup \{(n_2,i,n_3,1), 1 \le n_2 \le K, 1 \le i \le S, n_3 = i\} \bigcup \{(0,S,0)\}$.

Let $\boldsymbol{\phi} = (\boldsymbol{\phi}_0, ..., \boldsymbol{\phi}_K, \phi_S^*)$ denote the steady-state probability vector of this Markov chain. Its infinitesimal generator is

$$B(= B_0 + B_1 + B_2) = \begin{bmatrix} H_0 & L_0 & & & & & \bar{\alpha} \\ M_0 & H & L & & & & \bar{\alpha} \\ & M & H & L & & & \bar{\alpha} \\ & & \ddots & \ddots & \ddots & & \vdots \\ & & & M & H & L & \bar{\alpha} \\ & & & & M & H_1' & \bar{\alpha} \\ L' & & & & & & -(\eta+\lambda) \end{bmatrix}$$

which is of order $U_1+(S+1)K+1$ where

$$L' = (\mathbf{0}, \mathbf{0}, ..., (0, \eta+\lambda, 0, ..., 0)),$$

$$H'_{1(i,n_3,k_1;j,m_3,k_2)} = \begin{cases} -(S\beta+\alpha) & j=i, m_3=n_3, k_2=0 \\ & i=0, n_3=0, k_1=0 \\ -((S-i)\beta+\mu+\alpha) & j=i, m_3=n_3, k_2=k_1 \\ & 1 \le i \le S, n_3=i, k_1=1 \\ \mu & j=i-1, m_3=n_3-1, k_2=k_1 \\ & 2 \le i \le S, n_3=i, k_1=1 \\ \mu & j=i-1, m_3=n_3-1, k_2=0 \\ & i=1, n_3=i, k_1=1 \\ 0 & \text{otherwise.} \end{cases}$$

We have

$$\boldsymbol{\phi} B = 0 \text{ and } \boldsymbol{\phi}\mathbf{e} = 1. \tag{5.1}$$

The *LIQBD* description of the model indicates that the queueing system is stable (see Neuts [33]) if and only if

$$\boldsymbol{\phi} B_0 \mathbf{e} < \boldsymbol{\phi} B_2 \mathbf{e}. \tag{5.2}$$

Partition $\boldsymbol{\phi}_i$ as

$$\boldsymbol{\phi}_0 = \{\phi_0(j,0,0); 0 \le j \le S\} \bigcup \{\phi_0(j,k,1); 1 \le j \le S, 1 \le k \le j\},$$

$$\boldsymbol{\phi}_i = \{\phi_i(0,0,0)\} \bigcup \{\phi_i(j,k,1); 1 \le j \le S, k=j\}, 1 \le i \le K.$$

From equation (5.1) we get

$$\boldsymbol{\phi}_i = \boldsymbol{\phi}_{i-1} \boldsymbol{\mathcal{V}}_{i-1}, \quad 1 \le i \le K$$

where

$$\boldsymbol{\mathcal{V}}_i = \begin{cases} -L\,(H'_1)^{-1} & \text{if } \ i=K-1, \\ -L\,[H+\boldsymbol{\mathcal{V}}_{i+1}M]^{-1} & \text{if } \ 1 \le i \le K-2, \\ -L_0\,[H+\boldsymbol{\mathcal{V}}_1 M]^{-1} & \text{if } \ i=0 \end{cases}$$

and

$$\boldsymbol{\phi}_0 = \phi_S^* \boldsymbol{\mathcal{V}}_S^* \tag{5.3}$$

where $\boldsymbol{\mathcal{V}}_S^* = -L'(H_0 + \boldsymbol{\mathcal{V}}_0 M_0)^{-1}$.

From the normalizing condition we have $(\boldsymbol{\phi}_0 + \boldsymbol{\phi}_1 + ... + \boldsymbol{\phi}_K)\mathbf{e} = 1 - \phi_S^*$. Thus the equation (5.1) implies that $\phi_S^* = \frac{\alpha}{\alpha+\lambda+\eta}$ and

$$\frac{\alpha}{\alpha+\lambda+\eta}\left(\boldsymbol{\mathcal{V}}_S^*\left[I + \sum_{j=0}^{K-1}\prod_{i=0}^{j}\boldsymbol{\mathcal{V}}_i\right]\mathbf{e} + 1\right) = 1. \tag{5.4}$$

Relation (5.2) gives the stability condition explicitly as

$$\boldsymbol{\mathcal{V}}_S^* \prod_{i=0}^{K-1} \boldsymbol{\mathcal{V}}_i L_1 \mathbf{e} < \boldsymbol{\mathcal{V}}_S^* N \mathbf{e} + \eta. \tag{5.5}$$

Thus we have

Lemma 5.2.1. The system described in Section 5.1 is stable if and only if

$$L'(H_0 + \boldsymbol{\mathcal{V}}_0 M_0)^{-1} \prod_{i=0}^{K-1} \boldsymbol{\mathcal{V}}_i L_1 \mathbf{e} > L'(H_0 + \boldsymbol{\mathcal{V}}_0 M_0)^{-1} N \mathbf{e} - \eta. \tag{5.6}$$

where N is given in Section 5.1.

5.2.2 Steady-state probability vector

Assuming that (5.6) is satisfied, we briefly outline the computation of the long run system state probability.

Let $\mathbf{x}$ denote the steady-state probability vector of the generator $\boldsymbol{\mathcal{Q}}$. Then we have

$$\mathbf{x}\ \boldsymbol{\mathcal{Q}} = 0 \text{ with } \mathbf{x}\ \mathbf{e} = 1. \tag{5.7}$$

Partitioning $\mathbf{x}$ as $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, ...)$ and then each of the sub-vectors as

$$\mathbf{x}_{n_1} = \{\mathbf{x}_{n_1}(0, S, 0)\} \bigcup \{\mathbf{x}_{n_1}(0, i, 0, 0); 0 \le i \le S\} \bigcup$$

$$\{\mathbf{x}_{n_1}(n_2,0,0,0); 1 \le n_2 \le K\} \bigcup \{\mathbf{x}_{n_1}(0,i,n_3,1); 1 \le i \le S, 1 \le n_3 \le i\} \bigcup$$
$$\{\mathbf{x}_{n_1}(n_2,i,i,1); 1 \le n_2 \le K, 1 \le i \le S\}\,, \text{ for } n_1 \ge 0,$$

we see that $\mathbf{x}$, under the assumption that the stability condition (5.6) holds, is obtained as (see Neuts [33])

$$\mathbf{x}_{n_1} = \mathbf{x}_0\ R^{n_1}, \quad n_1 \ge 1 \tag{5.8}$$

where R is the minimal nonnegative solution to the matrix quadratic equation:

$$B_0 + R\ B_1 + R^2\ B_2 = \mathbf{O} \tag{5.9}$$

and the boundary equation is given by

$$\mathbf{x}_0[B_{00} + R\ B_2] = 0.$$

The normalizing condition (5.7) gives

$$\mathbf{x}_0(I-R)^{-1}\mathbf{e} = 1. \tag{5.10}$$

5.2.3 Expected sojourn time in a cycle in maximum inventory level S before realization of common life time

In order to compute the sojourn time of the system in a cycle, with inventory at the maximum S, we consider the case of a finite orbit. For numerical procedure the truncation level K_1 (size of the orbit) is taken such that the probability of the number of customers in the orbit going above the truncated size is of the order less than ϵ (here ϵ is taken as 10^{-6}). Consider the Markov chain $\{(\boldsymbol{\mathcal{N}}'_1(t), \boldsymbol{\mathcal{N}}_2(t), \boldsymbol{\mathcal{I}}(t), \boldsymbol{\mathcal{N}}_3(t), \boldsymbol{\mathcal{S}}(t)), t \ge 0\}$ where $\boldsymbol{\mathcal{N}}'_1(t)$ = number of customers in the finite orbit at time t. Its state space is $\{(n'_1,0,S,0,0); 0 \le n'_1 \le K_1\} \bigcup \{(n'_1,n_2,S,S,1); 0 \le n'_1 \le K_1; 1 \le n_2 \le K\} \bigcup \{(n'_1,0,S,n_3,1); 0 \le n'_1 \le K_1; 1 \le n_3 \le S\} \bigcup \{\tilde{\Delta}\}$ where $\{\tilde{\Delta}\}$ is an absorbing state which denotes the realization of CLT. The infinitesimal generator of the Markov chain is

$$\tilde{\boldsymbol{\mathcal{W}}}_{K_1} = \begin{bmatrix} \tilde{\boldsymbol{\mathcal{T}}}_{K_1} & \tilde{\boldsymbol{\mathcal{T}}}^0_{K_1} \\ \mathbf{0} & 0 \end{bmatrix}$$

where

$$\tilde{\mathcal{T}}_{K_1} = \begin{bmatrix} \tilde{G}_{00} & \tilde{G}_0 & & & \\ \tilde{G}_2 & \tilde{G}_1 & \tilde{G}_0 & & \\ & \ddots & \ddots & \ddots & \\ & & \tilde{G}_2 & \tilde{G}_1 & \tilde{G}_0 \\ & & & \tilde{G}_2 & \tilde{G}'_1 \end{bmatrix}, \tilde{\mathcal{T}}^0_{K_1} = \begin{bmatrix} \tilde{G} \\ \vdots \\ \tilde{G} \end{bmatrix}$$

with

$$(\tilde{G})_{ij} = \begin{cases} \alpha & j = i, i = 1, \\ \alpha + \mu & j = 1, 2 \leq i \leq K + S + 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$(\tilde{G}_{00})_{ij} = \begin{cases} -(\lambda + \alpha) & j = i, i = 1, \\ -(\lambda + \alpha + \mu) & j = i, 2 \leq i \leq K + S, \\ -(p\lambda + \alpha + \mu) & j = i, i = K + S + 1, \\ \lambda & j = i + 1, 1 \leq i \leq K + S, \\ 0 & \text{otherwise,} \end{cases}$$

$$(\tilde{G}'_1)_{ij} = \begin{cases} -(\lambda + \alpha) & j = i, i = 1, \\ -(\lambda + \alpha + \mu) & j = i, 2 \leq i \leq K + S, \\ -(\alpha + \mu) & j = i, i = K + S + 1, \\ \lambda & j = i + 1, 1 \leq i \leq K + S, \\ 0 & \text{otherwise,} \end{cases}$$

$$(\tilde{G}_1)_{ij} = \begin{cases} -(\eta + \lambda + \alpha) & j = i, i = 1, \\ -(\lambda + \alpha + \mu) & j = i, 2 \leq i \leq K + S, \\ -(p\lambda + \alpha + \mu) & j = i, i = K + S + 1, \\ \lambda & j = i + 1, 1 \leq i \leq K + S, \\ 0 & \text{otherwise,} \end{cases}$$

$$(\tilde{G}_0)_{ij} = \begin{cases} p\lambda & i = j = K + S + 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$(\tilde{G}_2)_{ij} = \begin{cases} \eta & j = i+1, i = 1, \\ 0 & \text{otherwise}, \end{cases}$$

$\tilde{G}_{00}, \tilde{G}_0, \tilde{G}_1, \tilde{G}_2, \tilde{G}'_1$ are square matrices of order $K+S+1$.

The expected sojourn time in the maximum inventory levels in a cycle is given by $E^{(S)}(\mathbf{T}) = -\boldsymbol{\gamma}_{K_1}\tilde{\boldsymbol{\mathcal{T}}}_{K_1}^{-1}\mathbf{e}$ where $\boldsymbol{\gamma}_{K_1} = (\mathbf{x}_{n'_1}(0, S, 0, 0), \mathbf{x}_{n'_1}(0, S, n_3, 1);$ $0 \leq n'_1 \leq K_1, 1 \leq n_3 \leq S, \mathbf{x}_{n'_1}(n_2, S, S, 1); 0 \leq n'_1 \leq K_1, 1 \leq n_2 \leq K)$ is a row vector of order $(K_1+1)(K+S+1)$. It may be noted that the set of states with inventory level at the maximum could be revisited several times due to cancellations in a cycle. This maximum does not arise in classical queueing-inventory models.

5.2.4 Expected sojourn time in zero inventory level in a cycle before realization of common life time

Now we compute the expected time the system stays with no items in inventory in finite orbit case. Consider the Markov Chain $\{(\boldsymbol{\mathcal{N}}'_1(t), \boldsymbol{\mathcal{N}}_2(t), \boldsymbol{\mathcal{I}}(t)), t \geq 0\}$ where $\boldsymbol{\mathcal{N}}'_1(t)$ is the number of customers in the finite orbit at time t. The state space of the system is given by $\{(n'_1, n_2, 0); 0 \leq n'_1 \leq K_1; 0 \leq n_2 \leq K\} \bigcup \{\tilde{\Delta}'\}$. The absorbing state of the Markov Chain is $\{\tilde{\Delta}'\}$ which denotes the realization of CLT. Thus the infinitesimal generator $\tilde{\boldsymbol{\mathcal{W}}}'_{K_1}$ of the Markov Chain is of the form

$$\tilde{\boldsymbol{\mathcal{W}}}'_{K_1} = \begin{bmatrix} \tilde{\boldsymbol{\mathcal{T}}}'_{K_1} & \tilde{\boldsymbol{\mathcal{T}}}'^{0}_{K_1} \\ \mathbf{0} & 0 \end{bmatrix}$$

where

$$\tilde{\boldsymbol{\mathcal{T}}}'_{K_1} = \begin{bmatrix} \hat{G}_1 & \hat{G}_0 & & & \\ & \hat{G}_1 & \hat{G}_0 & & \\ & & \ddots & \ddots & \\ & & & \hat{G}_1 & \hat{G}_0 \\ & & & & \hat{G}'_1 \end{bmatrix}, \tilde{\boldsymbol{\mathcal{T}}}'^{0}_{K_1} = \begin{bmatrix} \hat{G} \\ \vdots \\ \hat{G} \end{bmatrix}$$

with $\hat{G} = (S\beta + \alpha)\mathbf{e}$,

$$(\hat{G}_1)_{ij} = \begin{cases} -(\lambda + S\beta + \alpha) & j = i, 1 \leq i \leq K, \\ -(p\lambda + S\beta + \alpha) & j = i, i = K+1, \\ \lambda & j = i+1, 1 \leq i \leq K, \\ 0 & \text{otherwise}, \end{cases}$$

$$(\hat{G}_0)_{ij} = \begin{cases} p\lambda & j = i, i = K+1, \\ 0 & \text{otherwise}, \end{cases}$$

$$(\hat{G}'_1)_{ij} = \begin{cases} -(\lambda + S\beta + \alpha) & j = i, 1 \leq i \leq K, \\ -(S\beta + \alpha) & j = i, i = K+1, \\ \lambda & j = i+1, 1 \leq i \leq K, \\ 0 & \text{otherwise}. \end{cases}$$

$\hat{G}_1, \hat{G}_0, \hat{G}'_1$ are square matrices of order $K+1$.

Thus the expected sojourn time in zero inventory level during a cycle is given by $E^{(0)}(\mathbf{T}) = -\boldsymbol{\gamma}'_{K_1} \tilde{\boldsymbol{\mathcal{T}}}'^{-1}_{K_1} \mathbf{e}$ where $\boldsymbol{\gamma}'_{K_1} = (\mathbf{x}_{n'_1}(n_2, 0, 0, 0); 0 \leq n'_1 \leq K_1, 0 \leq n_2 \leq K)$ is a row vector of order $(K_1+1)(K+1)$.

5.2.5 Expected number of revisits to S in a cycle before the realization of common life time

We compute the expected number of revisits of inventory level to S in a cycle. Consider a Markov chain $\{(\boldsymbol{\mathcal{N}}_4(t), \boldsymbol{\mathcal{N}}'_1(t), \boldsymbol{\mathcal{N}}_2(t), \boldsymbol{\mathcal{I}}(t), \boldsymbol{\mathcal{N}}_3(t), \boldsymbol{\mathcal{S}}(t)), t \geq 0\}$ on the states $\{(n_4, n'_1, 0, i, 0, 0); n_4 \geq 0; 0 \leq n'_1 \leq K_1; 0 \leq i \leq S\} \bigcup \{(n_4, n'_1, n_2, 0, 0, 0); n_4 \geq 0; 0 \leq n'_1 \leq K_1; 1 \leq n_2 \leq K\} \bigcup \{(n_4, n'_1, 0, i, n_3, 1); n_4 \geq 0; 0 \leq n'_1 \leq K_1; 1 \leq i \leq S; 1 \leq n_3 \leq i\} \bigcup \{(n_4, n'_1, n_2, i, i, 1); n_4 \geq 0; 0 \leq n'_1 \leq K_1; 1 \leq i \leq S; 1 \leq n_2 \leq K\} \bigcup \{*'\}$ where $\{*'\}$ is an absorbing state which denotes the realization of the CLT. Here we consider $\boldsymbol{\mathcal{N}}_4(t)$ (see Section 5.1 for its definition) as the level and $\boldsymbol{\mathcal{N}}'_1(t), \boldsymbol{\mathcal{N}}_2(t), \boldsymbol{\mathcal{I}}(t), \boldsymbol{\mathcal{S}}(t)$ are referred to as phases. Thus the infinitesimal generator is

$$\check{\boldsymbol{\mathcal{W}}} = \begin{bmatrix} 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots \\ \check{G} & \check{G}_1 & \check{G}_0 & & & \dots \\ \check{G} & & \check{G}_1 & \check{G}_0 & & \dots \\ \check{G} & & & \check{G}_1 & \check{G}_0 & \dots \\ \vdots & & & & \ddots & \ddots \end{bmatrix}$$

where $\check{G}_1, \check{G}_0$ are square matrices of order $(K_1+1)[U_1+(S+1)K+1]$ with $\check{G} = \alpha\mathbf{e}$,

$$\check{G}_1 = \begin{bmatrix} \mathcal{G}_{00} & \mathcal{G}_0 & & & \\ \mathcal{G}_2 & \mathcal{G} & \mathcal{G}_0 & & \\ & \ddots & \ddots & \ddots & \\ & & \mathcal{G}_2 & \mathcal{G} & \mathcal{G}_0 \\ & & & \mathcal{G}_2 & \mathcal{G}_1 \end{bmatrix},$$

$\check{G}_0 = diag(\mathcal{G}^0, ..., \mathcal{G}^0), \mathcal{G}_0 = diag(0, ..., 0, L_1),\ \mathcal{G}_2 = diag(N, 0, ..., 0),$

$$\mathcal{G}_{00} = \begin{bmatrix} \check{H}_{00} & L_0 & & & & \\ \check{M}_0 & H & L & & & \\ & \check{M} & H & L & & \\ & & \ddots & \ddots & \ddots & \\ & & & \check{M} & H & L \\ & & & & \check{M} & H_1 \end{bmatrix}, \mathcal{G} = \begin{bmatrix} \check{H}_0 & L_0 & & & & \\ \check{M}_0 & H & L & & & \\ & \check{M} & H & L & & \\ & & \ddots & \ddots & \ddots & \\ & & & \check{M} & H & L \\ & & & & \check{M} & H_1 \end{bmatrix},$$

$$\mathcal{G}_1 = \begin{bmatrix} \check{H}_0 & L_0 & & & & \\ \check{M}_0 & H & L & & & \\ & \check{M} & H & L & & \\ & & \ddots & \ddots & \ddots & \\ & & & \check{M} & H & L \\ & & & & \check{M} & H_1' \end{bmatrix}, \mathcal{G}^0 = \begin{bmatrix} \check{B}_0 & & & & \\ \check{B}_1 & 0 & & & \\ & \check{B} & 0 & & \\ & & \ddots & \ddots & \\ & & & \check{B} & 0 \end{bmatrix}$$

with

$$\left(\check{M}_0\right)_{jk} = \begin{cases} (S-j+1)\beta & 1 \le j \le S-1, k = \sum_{i=1}^{j+1} i + j \\ 0 & \text{otherwise,} \end{cases}$$

$$\left(\check{M}\right)_{jk} = \begin{cases} (S-j+1)\beta & 1 \le j \le S-1, k = j+1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\left(\check{B}_0\right)_{jk} = \begin{cases} \beta & k = j + S + 1, \frac{S(S+1)}{2} - 1 \leq j \leq \frac{S(S+1)}{2} + S - 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\left(\check{B}_1\right)_{jk} = \begin{cases} \beta & j = S, k = \frac{(S+2)(S+1)}{2} + S \\ 0 & \text{otherwise,} \end{cases}$$

$$\left(\check{B}\right)_{jk} = \begin{cases} \beta & k = S + 1, j = S \\ 0 & \text{otherwise.} \end{cases}$$

The following matrices give transition rates from the state $(i, n_3, k_1) \rightarrow (j, m_3, k_2)$ where $i(j)$ represents the number of items in the inventory; $n_3(m_3)$, the number of customers in the buffer and k_l, for $l = 1, 2$, are status of the server.

$$\check{H}_{00(i,j)}^{(k_1,k_2)}(n_3, m_3) = \begin{cases} S\beta & j = i + 1, m_3 = n_3, k_2 = k_1; \\ & i = 0, n_3 = 0, k_1 = 0 \\ (S - i)\beta & j = i + 1, m_3 = n_3 = 0, k_2 = k_1; \\ & 1 \leq i \leq S - 2, n_3 = 0, k_1 = 0 \\ (S - i)\beta & j = i + 1, m_3 = n_3, k_2 = k_1; \\ & 1 \leq i \leq S - 2, 1 \leq n_3 \leq i, k_1 = 1 \\ \mu & j = i - 1, m_3 = n_3 - 1, k_2 = k_1 - 1; \\ & 1 \leq i \leq S, n_3 = 1, k_1 = 1 \\ \mu & j = i - 1, m_3 = n_3 - 1, k_2 = k_1; \\ & 2 \leq i \leq S, 2 \leq n_3 \leq i, k_1 = 1 \\ \lambda & j = i, m_3 = n_3 + 1, k_2 = k_1 + 1; \\ & 1 \leq i \leq S, n_3 = 0, k_1 = 0 \\ \lambda & j = i, m_3 = n_3 + 1, k_2 = k_1; \\ & 1 \leq i \leq S, 1 \leq n_3 \leq i - 1, k_1 = 1 \\ -(\lambda + \alpha + (S - i)\beta) & j = i, m_3 = n_3, k_2 = k_1; \\ & 0 \leq i \leq S, n_3 = 0, k_1 = 0 \\ -(\lambda + \mu + \alpha + (S - i)\beta) & j = i, m_3 = n_3, k_2 = k_1; \\ & 1 \leq i \leq S, 1 \leq n_3 \leq i, k_1 = 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\check{H}^{(k_1,k_2)}_{0(i,j)}(n_3,m_3) = \begin{cases} S\beta & j=i+1, m_3=n_3, k_2=k_1; \\ & i=0, n_3=0, k_1=0 \\ (S-i)\beta & j=i+1, m_3=n_3, k_2=k_1; \\ & 1\le i\le S-2, n_3=0, k_1=0 \\ (S-i)\beta & j=i+1, m_3=n_3, k_2=k_1; \\ & 1\le i\le S-2, 1\le n_3\le i, k_1=1 \\ \mu & j=i-1, m_3=n_3-1, k_2=k_1-1; \\ & 1\le i\le S, k_1=1, n_3=1 \\ \mu & j=i-1, m_3=n_3-1, k_2=k_1; \\ & 2\le i\le S, 2\le n_3\le i, k_1=1 \\ \lambda & j=i, m_3=n_3+1, k_2=k_1+1; \\ & 1\le i\le S, n_3=0, k_1=0 \\ \lambda & j=i, m_3=n_3+1, k_2=k_1; \\ & 1\le i\le S, 1\le n_3\le i-1, k_1=1 \\ -(\lambda+\alpha+S\beta) & j=i, m_3=n_3, k_2=k_1; \\ & i=0, n_3=0, k_1=0 \\ -(\lambda+\alpha+\eta+(S-i)\beta) & j=i, m_3=n_3, k_2=k_1; \\ & 1\le i\le S, n_3=0, k_1=0 \\ -(\lambda+\mu+\alpha+(S-i)\beta) & j=i, m_3=n_3, k_2=k_1=1; \\ & 1\le i\le S, 1\le n_3\le i, k_1=1 \\ 0 & \text{otherwise,} \end{cases}$$

$\check{H}_{00}$, $\check{H}_0$, $\check{B}_0$ are square matrices of order U_1 and dimension of the matrices $\check{B}_1$, $\check{M}_0$ are $(S+1)\times U_1$. $\check{B}$ is a square matrix of order $S+1$ and all other sub-matrices are given in Section 5.1.

If p_k is the probability that absorption occurs with exactly k revisits, then

$$p_k = \check{\delta}_{K_1}(-\check{G}_1^{-1}\check{G}_0)^k(-\check{G}_1^{-1}\check{G}), k\ge 0$$

with $\check{\delta}_{K_1} = (\mathbf{x}_0, \mathbf{x}_1, ..., \mathbf{x}_{K_1})$ is a row vector of order $(K_1+1)[U_1+K(S+1)+1]$. Therefore the expected number of revisits to S before realization of CLT is $\check{E}^{(S)}(N) = \sum_{k=0}^{\infty} kp_k$ (see Krishnamoorthy et al. [22]).

5.2.6 Additional Performance Measures

1. Expected number of customers in the orbit

$$E_O = \sum_{n_1=1}^{\infty} n_1 \left[\sum_{i=0}^{S} x_{n_1}(0,i,0,0) + \sum_{n_2=1}^{K} x_{n_1}(n_2,0,0,0) \right.$$

$$\left. + \sum_{n_2=1}^{K} \sum_{i=1}^{S} x_{n_1}(n_2,i,i,1) + \sum_{i=1}^{S} \sum_{n_3=1}^{i} x_{n_1}(0,i,n_3,1) \right].$$

2. Expected number of customers in the waiting room

$$E_W = \sum_{n_1=0}^{\infty} \sum_{n_2=1}^{K} n_2 \left[x_{n_1}(n_2,0,0,0) + \sum_{i=1}^{S} x_{n_1}(n_2,i,i,1) \right].$$

3. Expected number of customers in the buffer

$$E_B = \sum_{n_1=0}^{\infty} \left[\sum_{n_2=1}^{K} \sum_{i=1}^{S} i x_{n_1}(n_2,i,i,1) + \sum_{i=1}^{S} \sum_{n_3=1}^{i} n_3 x_{n_1}(0,i,n_3,1) \right].$$

4. Expected number of items in the inventory before realization of CLT

$$E_I = \sum_{n_1=0}^{\infty} \sum_{i=1}^{S} i \left[x_{n_1}(0,i,0,0) + \sum_{n_3=1}^{i} x_{n_1}(0,i,n_3,1) + \sum_{n_2=1}^{K} x_{n_1}(n_2,i,i,1) \right].$$

5. Expected cancellation rate before realization of CLT

$$E_{CR} = \sum_{n_1=0}^{\infty} \left[\sum_{i=0}^{S-1} (S-i)\beta x_{n_1}(0,i,0,0) + \sum_{i=1}^{S-1} \sum_{n_3=1}^{i} (S-i)\beta x_{n_1}(0,i,n_3,1) \right.$$

$$\left. + \sum_{n_2=1}^{K} \left(S\beta x_{n_1}(n_2,0,0,0) + \sum_{i=1}^{S-1} (S-i)\beta x_{n_1}(n_2,i,i,1) \right) \right].$$

6. Expected number of items in the system immediately on realization of CLT (this is the inventory, which cannot be carried forward to the next cycle since their life time is expired – a typical case is: flights taking off with one or more vacant seats)

$$E_I' = \sum_{i=1}^{S} i \left[\mathcal{Z}_1^{(i)} x_0(0,i,0,0) + \mathcal{Z}_3^{(i)} \left(\sum_{n_2=1}^{K} x_0(n_2,i,i,1) + \sum_{n_3=1}^{i} x_0(0,i,n_3,1) \right) \right] + \sum_{n_1=1}^{\infty} \sum_{i=1}^{S} i \left[\mathcal{Z}_3^{(i)} \left(\sum_{n_2=1}^{K} x_{n_1}(n_2,i,i,1) + \sum_{n_3=1}^{i} x_{n_1}(0,i,n_3,1) \right) + \mathcal{Z}_2^{(i)} x_{n_1}(0,i,0,0) \right]$$

where $\mathcal{Z}_1^{(i)} = \frac{\alpha}{\lambda+\alpha+(S-i)\beta}$, $\mathcal{Z}_2^{(i)} = \frac{\alpha}{\lambda+\alpha+\eta+(S-i)\beta}$, $\mathcal{Z}_3^{(i)} = \frac{\alpha}{\lambda+\alpha+\mu+(S-i)\beta}$.

A Random Walk

In order to compute the measures given in a, b, c and d below we consider a random walk (a birth and death process) on the set $\{0, 1, 2, ..., S\}$ which is the set of possible values of the inventory. Here a left transition means purchase of an item and a right transition represents cancellation. This is similar to a situation of a transport system with S seats. Then

a. Probability that the transport system is fully vacant at the time of realization of CLT (all seats remain vacant at the time of realization of CLT which is the departure time):

$$P_{vacant} = \sum_{n_1=0}^{\infty} \left[x_{n_1}(0,S,0,0) + \sum_{n_2=1}^{K} x_{n_1}(n_2,S,S,1) + \sum_{n_3=1}^{S} x_{n_1}(0,S,n_3,1) \right].$$

b. Probability that the transport system goes with full capacity at the time of realization of CLT (all seats are filled at departure time):

$$P_{full} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{K} x_{n_1}(n_2, 0, 0, 0).$$

c. Expected number of purchases before realization of CLT

$$E_{NP} = \frac{\mu}{\alpha} \sum_{n_1=0}^{\infty} \left[\sum_{n_2=1}^{K} \sum_{i=1}^{S} x_{n_1}(n_2, i, i, 1) + \sum_{i=1}^{S} \sum_{n_3=1}^{i} x_{n_1}(0, i, n_3, 1) \right].$$

d. Expected number of cancellations before realization of CLT

$$E_{NC} = \frac{1}{\alpha} \sum_{n_1=0}^{\infty} \left[\sum_{i=1}^{S-1} (S-i)\beta \left(x_{n_1}(0, i, 0, 0) + \sum_{n_3=1}^{i} x_{n_1}(0, i, n_3, 1) + \sum_{n_2=1}^{K} x_{n_1}(n_2, i, i, 1) \right) + S\beta \sum_{n_2=0}^{K} x_{n_1}(n_2, 0, 0, 0) \right].$$

The significance of measures a, b, c and d above occur in the determination of the capacity of transport system.

5.3 Special Case

In this section we consider the case of negligible service time. This means that the service rate is infinite. Here we can combine the buffer and waiting room: if inventory is available then none will be in the waiting room; however, none in the waiting room need not necessarily mean that inventory is available. Thus the Markov chain is $\psi = \{(\boldsymbol{\mathcal{N}}_1(t), \boldsymbol{\mathcal{I}}(t)); t \geq 0\}$, where $\boldsymbol{\mathcal{N}}_1(t)$ and $\boldsymbol{\mathcal{I}}(t)$ have the same definition as in Section 5.1. The state space of the process ψ is given by $\{(n_1, i);\ n_1 \geq 0,\ -K \leq i \leq S\} \bigcup \{(n_1, S^*);\ n_1 \geq 0\}$. Here i can be positive, zero or negative. Negative value of i indicates that no item is available in the inventory and $|i|$ customers are in the waiting room. Once the CLT is realized the inventory level reaches its maximum, which is denoted by S^*. This is so because the next cycle starts at that moment with instantaneous replenishment of inventory. The infinitesimal generator $\boldsymbol{\mathcal{Q}}^*$ of this $CTMC$ is of the form:

$$\boldsymbol{\mathcal{Q}}^* = \begin{bmatrix} B'_{00} & B'_0 & & & \\ B'_2 & B'_1 & B'_0 & & \\ & B'_2 & B'_1 & B'_0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}.$$

B'_{00}, B'_0, B'_1 and B'_2 are square matrices of order $K+S+2$.

$$B'_{00} = \begin{bmatrix} b'_S & S\beta & & & & & & & \alpha \\ \lambda & b_S & S\beta & & & & & & \alpha \\ & \ddots & \ddots & \ddots & & & & & \vdots \\ & & \lambda & b_S & S\beta & & & & \alpha \\ & & & \lambda & b_{S-1} & (S-1)\beta & & & \alpha \\ & & & & \ddots & \ddots & \ddots & & \vdots \\ & & & & & \lambda & b_1 & \beta & \alpha \\ & & & & & & \lambda & b_0 & \alpha \\ & & & & & & \lambda & & -\lambda \end{bmatrix},$$

$$B'_2 = \begin{bmatrix} 0 & & & & & & & 0 \\ 0 & 0 & & & & & & 0 \\ \vdots & \ddots & \ddots & & & & & \vdots \\ 0 & & 0 & 0 & & & & 0 \\ 0 & & & \eta & 0 & & & 0 \\ \vdots & & & & \ddots & \ddots & & \vdots \\ 0 & & & & & \eta & 0 & 0 \\ 0 & & & & & \eta & 0 & 0 \end{bmatrix},$$

$$B_1' = \begin{bmatrix} b_S' & S\beta & & & & & & & \alpha \\ \lambda & b_S & S\beta & & & & & & \alpha \\ & \ddots & \ddots & \ddots & & & & & \vdots \\ & & \lambda & b_S & S\beta & & & & \alpha \\ & & & \lambda & b_{S-1}' & (S-1)\beta & & & \alpha \\ & & & & \ddots & \ddots & \ddots & & \vdots \\ & & & & & \lambda & b_1' & \beta & \alpha \\ & & & & & & \lambda & b_0' & \alpha \\ & & & & & & \lambda & & -(\eta+\lambda) \end{bmatrix},$$

$B_0' = diag(p\lambda, 0, ..., 0)$ where $b_S' = -(p\lambda + \alpha + S\beta)$; $b_i = -(\lambda + \alpha + i\beta)$ for $0 \le i \le S$ and $b_i' = -(\lambda + \alpha + \eta + i\beta)$ for $0 \le i \le S-1$.

5.3.1 Stability of the system

In this section we perform the steady state analysis of the queueing-inventory model under study by first establishing the stability condition of the system. Define $B' = B_0' + B_1' + B_2'$. This is the infinitesimal generator of the finite state $CTMC$ corresponding to the inventory level $\{-K, -K+1, ..., 0, ..., S, S^*\}$. Let $\boldsymbol{\phi}'$ denote the steady-state probability vector of B'. That is,

$$\boldsymbol{\phi}' B' = 0, \quad \boldsymbol{\phi}' \mathbf{e} = 1. \tag{5.11}$$

Write $\boldsymbol{\phi}' = (\phi'(-K), \phi'(-K+1), ..., \phi'(0), \phi'(1), ..., \phi'(S), \phi'(S^*))$. Then using relations in (5.11) we get the components of the vector $\boldsymbol{\phi}'$ explicitly as

$$\phi'(i) = \begin{cases} \frac{\alpha}{\alpha+\lambda+\eta}, & i = S^* \\ v_{S-1}^* v_{S^*} \phi'(S^*), & i = S \\ v_{S^*} \phi'(S^*), & i = S-1 \\ \prod\limits_{j=i+1}^{S-1} v_j v_{S^*} \phi'(S^*), & -K \le i \le S-2 \end{cases}$$

where $v_{S-1}^* = \frac{\beta}{(\lambda+\alpha+\eta)}$ and

$$v_i = \begin{cases} \frac{\lambda}{\alpha+S\beta}, & i = -K+1 \\ \frac{\lambda}{(\lambda+\alpha+S\beta)-S\beta v_{i+1}}, & -K+2 \leq i \leq 0 \\ \frac{\lambda+\eta}{(\lambda+\alpha+S\beta)-S\beta v_0}, & i = 1 \\ \frac{\lambda+\eta}{(\lambda+\alpha+\eta+(S-i+1)\beta)-(S-i+2)\beta v_{i+1}}, & 2 \leq i \leq S-1 \\ \frac{\lambda+\eta}{(\lambda+\alpha+\eta+\beta)-2\beta v_{S-1}-(\lambda+\eta)v_{S-1}^*}, & i = S^* \end{cases}$$

Since the Markov chain is an $LIQBD$, it is stable if and only if the left drift rate exceeds the right drift rate. That is,

$$\boldsymbol{\phi}' B_0' \mathbf{e} < \boldsymbol{\phi}' B_2' \mathbf{e}. \tag{5.12}$$

We have the following lemma:

Lemma 5.3.1. The stability condition of the queueing-inventory model is given by

$$p\lambda \prod_{i=-K+1}^{S-1} v_i < \eta \left[\sum_{j=1}^{S-2} \prod_{i=j+1}^{S-1} v_i + v_{S-1}^* + 1 \right]. \tag{5.13}$$

Proof: From the well known result in Neuts [33] on the positive recurrence of B', we have $\boldsymbol{\phi}' B_0' \mathbf{e} < \boldsymbol{\phi}' B_2' \mathbf{e}$. With a bit of computation, this simplifies to the result

$$p\lambda \prod_{i=-K+1}^{S-1} v_i v_{S^*} \phi'(S^*) < \eta \left[\sum_{j=1}^{S-2} \prod_{i=j+1}^{S-1} v_i + v_{S-1}^* + 1 \right] v_{S^*} \phi'(S^*).$$

5.3.2 Steady state analysis of the system

For computing the steady-state probability vector of the system, we first consider the rate matrix R which is the minimal nonnegative solution to the matrix quadratic equation:

$$B_0' + R\ B_1' + R^2\ B_2' = \mathbf{O}. \tag{5.14}$$

From the structure of the coefficient matrices B_0', B_1' and B_2', we observe that the rate matrix R has the form

$$R = \begin{bmatrix} r_{-K} & \cdots & r_0 & r_1 & \cdots & r_S & r_{S^*} \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \\ \vdots & & \vdots & \vdots & & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & 0 & 0 \end{bmatrix}.$$

Substituting R^2 and R into equation (5.14) we obtain

$$r_i = \begin{cases} q_{i+1}^* + q_{i+1}r_{i+1}, & -K \leq i \leq -1 \\ ((\eta r_{-K} + \lambda)r_1 + S\beta r_{-1})(\lambda + \alpha + S\beta)^{-1}, & i = 0 \\ ((\eta r_{-K} + \lambda)r_{i+1} + S\beta r_{i-1})(\lambda + \alpha + \eta + (S - i)\beta)^{-1}, & 1 \leq i \leq S - 2 \\ ((\eta r_{-K} + \lambda)(r_S + r_{S^*}) + 2\beta r_{S-2})(\lambda + \alpha + \eta + \beta)^{-1}, & i = S - 1 \\ \beta r_{S-1}(\lambda + \alpha + \eta)^{-1}, & i = S \\ \alpha(r_{-K} + ... + r_S)(\eta + \lambda)^{-1}, & i = S^* \end{cases}$$

where

$$q_i = \begin{cases} \lambda(p\lambda + \alpha + S\beta)^{-1}, & i = -K + 1 \\ \lambda((\lambda + \alpha + S\beta) - S\beta q_{i-1})^{-1}, & -K + 2 \leq i \leq 0 \end{cases}$$

and

$$q_i^* = \begin{cases} p\lambda(p\lambda + \alpha + S\beta)^{-1}, & i = -K + 1 \\ S\beta q_{i-1}^*((\lambda + \alpha + S\beta) - S\beta q_{i-1})^{-1}, & -K + 2 \leq i \leq 0. \end{cases}$$

Let $\mathbf{x}^*$ denote the steady-state probability vector of $\boldsymbol{\mathcal{Q}}^*$. Then $\mathbf{x}^*$ satisfies the relations $\mathbf{x}^* \ \boldsymbol{\mathcal{Q}}^* = 0$ and $\mathbf{x}^* \ \mathbf{e} = 1$.

Partitioning $\mathbf{x}^*$ as $\mathbf{x}^* = (\mathbf{x}_0^*, \mathbf{x}_1^*, \mathbf{x}_2^*, ...)$, each
$\mathbf{x}_{n_1}^* = (x_{n_1}^*(-K), x_{n_1}^*(-K + 1), ..., x_{n_1}^*(-1), x_{n_1}^*(0), x_{n_1}^*(1), ..., x_{n_1}^*(S), x_{n_1}^*(S^*))$
and

$$\mathbf{x}_{n_1}^* = \mathbf{x}_0^* \ R^{n_1}, \quad n_1 \geq 1, \tag{5.15}$$

which implies $\mathbf{x}_{n_1}^* = \mathbf{x}_0^* r_{-K}^{n_1 - 1} R, \quad n_1 \geq 1$ and $\mathbf{x}_0^*(I + (1 - r_{-K})^{-1}R)\mathbf{e} = 1$.

The normalizing condition expressed by $\mathbf{x}_0^*(I + (1 - r_{-K})^{-1}R)\mathbf{e} = 1$ indicates that the total probability adds to 1.

5.4 Numerical illustration

In this section we provide numerical illustration of the system performance with variation in values of underlying parameters.

In these the "expected inventory level immediately on realization of common life time", plays a significant role due to the fact that it gives an idea as to how much of inventory is 'wasted'.

Effect of the arrival rate λ

Table 5.1 indicates that the increase in λ makes expected number of customers in the orbit, that in the waiting room as well as in the buffer, expected cancellation rate and expected sojourn time in zero inventory all increase. As λ increases there is a comparatively high decrease in the expected number of items in the inventory immediately on realization of CLT and expected sojourn time in S. These are on expected lines.

λ	E_O	E_W	E_B	E_I	E_{CR}	E'_I	$E^{(S)}(\mathbf{T})$	$E^{(0)}(\mathbf{T})$
12	0.0290	0.0617	0.6827	4.2981	5.9389	0.3325	0.0234	1.4963×10^{-5}
13	0.0596	0.0944	0.8144	**4.2998**	6.3448	0.3094	0.0218	2.0955×10^{-5}
14	0.1151	0.1368	0.9536	4.2975	6.7365	0.2890	0.0205	2.8029×10^{-5}
15	0.2118	0.1890	1.0978	4.2927	7.1128	0.2709	0.0193	3.6044×10^{-5}
16	0.3766	0.2509	1.2446	4.2866	7.4730	0.2595	0.0184	4.4801×10^{-5}
17	0.6568	0.3218	1.3919	4.2803	7.8172	0.2395	0.0175	5.4032×10^{-5}

Table 5.1: Effect of the arrival rate λ: Fix $S = 6, K = 4, \mu = 15, \eta = 5, \alpha = 2, \beta = 7, p = 0.5, K_2 = 50$

Effect of the service rate μ

From Table 5.2 we observe that the increase in μ makes expected number of customers in the orbit, expected number of customers in the waiting room, expected number of items immediately on realization of CLT, expected sojourn time in S, expected number of customers in the buffer and expected number of items in the inventory to decrease. This is a consequence of decrease in traffic intensity. But as μ increases, expected cancellation rate, expected sojourn time in 0 inventory increase: more the number of customers served out higher the rate of cancellation. The expected number of unsold items (in the sense of items remaining in inventory immediately after the expiry of CLT) decrease because of a large number of customers purchasing the inventory.

μ	E_O	E_W	E_B	E_I	E_{CR}	E'_I	$E^{(S)}(\mathbf{T})$	$E^{(0)}(\mathbf{T})$
10	1.0984	0.8694	1.3674	3.2331	5.1532	0.1822	0.0218	0.0008
11	0.7211	0.7591	1.2192	3.1583	5.4146	0.1773	0.0191	0.0010
12	0.4920	0.6630	1.0870	3.0951	5.6395	0.1732	0.0169	0.0012
13	0.3465	0.5800	0.9699	3.0418	5.8323	0.1698	0.0152	0.0013
14	0.2510	0.5090	0.8665	2.9967	5.9975	0.1670	0.0139	0.0015
15	0.1865	0.4486	0.7758	2.9584	6.1395	0.1648	0.0128	0.0016

Table 5.2: Effect of the service rate μ: Fix $S = 5, K = 6, \lambda = 13, \eta = 5, \alpha = 1.5, \beta = 4, p = 0.5, K_2 = 50$

Effect of the retrial rate η

η	E_O	E_W	E_B	E_I	E_{CR}	E'_I	$E^{(S)}(\mathbf{T})$	$E^{(0)}(\mathbf{T})$
3	0.2895	0.1575	1.4938	6.1700	7.9673	0.1362	0.0537	3.8825×10^{-8}
4	0.2417	0.1574	1.4938	6.1703	7.9672	0.1361	0.0538	3.7693×10^{-8}
5	0.2137	0.1574	1.4938	6.1706	7.9671	0.1361	0.0538	3.7051×10^{-8}
6	0.1952	0.1574	1.4938	6.1709	7.9671	0.1361	0.0539	3.6638×10^{-8}
7	0.1822	0.1574	1.4937	6.1711	7.9671	0.1360	0.0539	3.6352×10^{-8}
8	0.1725	0.1574	1.4937	6.1713	7.9671	0.1360	0.0539	3.6142×10^{-8}

Table 5.3: Effect of the retrial rate η: Fix $S = 7, K = 5, \lambda = 11, \mu = 13, \alpha = 0.5, \beta = 15, p = 0.75, K_2 = 50$

From Table 5.3 we observe that as η increases there is a comparatively high decrease in expected number of customers in the orbit as is expected, slight decrease in expected number of items immediately on realization of CLT, expected cancellation rate, expected number of customers in the waiting room and buffer and slight increase in the expected number of items before realization of CLT. The latter is attributed to larger number of cancellations and the former due to more items taken away by waiting customers (with η increasing larger number of orbital customers get into the waiting room and buffer). The expected sojourn time in S increases with increase in η values. However, expected sojourn time in 0 shows a decreasing trend.

Effect of the common life time parameter α

Table 5.4 shows that an increase in α results in a decrease in expected number of customers in the orbit, in the waiting room and that in the buffer - the shorter the life time, lesser the number of cancellations. The expected length of a cycle turns out to be smaller and so the expected number of arrivals decrease. The expected sojourn time in 0 and expected cancellation rate also decrease with increase in α values. However, expected number of items in the inventory before realization, expected sojourn time in S, expected number of items in the inventory immediately on realization of CLT all show an increasing trend. This could be attributed to decrease in the number of demands during a shorter duration of time (with α increasing the CLT realizes faster).

α	E_O	E_W	E_B	E_I	E_{CR}	E'_I	$E^{(S)}(\mathbf{T})$	$E^{(0)}(\mathbf{T})$
0.25	1.6091	0.5753	1.4113	4.1614	8.0883	0.0415	0.0066	0.0010
0.5	0.7578	0.4074	1.2181	4.3553	7.0672	0.0910	0.0090	0.0007
0.75	0.4195	0.3013	1.0796	4.4709	6.2904	0.1436	0.0108	0.0005
1	0.2553	0.2294	0.9735	4.5390	5.6710	0.1989	0.0121	0.0004
1.25	0.1654	0.1784	0.8884	4.5760	5.1615	0.2496	0.0132	0.0003
1.5	0.1120	0.1411	0.8181	4.5913	4.7331	0.3010	0.0140	0.0002

Table 5.4: Effect of α: Fix $S = 7, K = 5, \lambda = 11, \mu = 13, \eta = 4, \beta = 3, p = 0.75, K_2 = 50$

Effect of β

From Table 5.5, we observe that the expected number of customers in the orbit and that in the waiting room, expected inventory level on realization of CLT and expected sojourn time in 0 decrease with increase in β value which is a consequence of positive inventory in the system over a longer duration of time in a cycle. Here expected number of customers in the buffer, expected cancellation rate, expected number of items in the inventory and expected sojourn time in S show a sharper upward trend on realization of CLT. This tendency is a natural consequence of higher cancellation rate for the same CLT parameter value.

β	E_O	E_W	E_B	E_I	E_{CR}	E_I'	$E^{(S)}(\mathbf{T})$	$E^{(0)}(\mathbf{T})$
8	0.0999	0.1155	1.1177	5.7946	8.4143	0.0626	0.0316	3.0820×10^{-6}
9	0.0945	0.1085	1.1285	5.9082	8.4433	0.0643	0.0357	1.4032×10^{-6}
10	0.0905	0.1032	1.1370	5.9996	8.4667	0.0658	0.0394	6.8522×10^{-7}
11	0.0873	0.0990	1.1437	6.0747	8.4859	0.0672	0.0427	3.5504×10^{-7}
12	0.0848	0.0956	1.1491	6.1376	8.5020	0.0684	0.0457	1.9361×10^{-7}
13	0.0827	0.0929	1.1537	6.1910	8.5157	0.0695	0.0484	1.1045×10^{-7}

Table 5.5: Effect of β: Fix $S = 7, K = 5, \lambda = 11, \mu = 15, \alpha = 0.25, \eta = 6, p = 0.75, K_2 = 50$

Effect of variation in α, β on P_{vacant} and P_{full}

We assign the following values to the parameters: $S = 6, K = 4, \lambda = 11, \mu = 15, \eta = 5, p = 0.75$. The effect of the parameters α and β are given in Table 5.6. Probability for the inventory being full (at S), increases with increasing value of α (for fixed β). Similarly probability for inventory being zero at the time of realization of CLT is a linearly decreasing function of α (for fixed β). P_{full} decreases with β for fixed α (a consequence of higher cancellation rate). On the other hand, probability for the other extreme shows negligible increase.

α	P_{vacant}	P_{full}
1.2	0.0690	0.4530
1.4	0.0784	0.3968
1.6	0.0871	0.3459
1.8	0.0952	0.2998
2	0.1027	0.2586
2.2	0.1097	0.2223

β	P_{vacant}	P_{full}
0	0.0828	0.3708
0.1	0.0839	0.3474
0.2	0.0850	0.3245
0.3	0.0862	0.3019
0.4	0.0874	0.2797
0.5	0.0887	0.2581

Table 5.6: Effect of α for $\beta = 0$ and that of β for $\alpha = 1.5$ on P_{vacant} and P_{full}

Effect of variation in α, β on E_{NP} and E_{NC}

We assign the same values as in the previous illustration, to the parameters $S = 6, K = 4, \lambda = 11, \mu = 15, \eta = 5, p = 0.75$. With increase in the values of α, keeping β fixed we notice that E_{NP} and E_{NC} decrease. However, keeping α fixed and increasing value of β result in E_{NP} and E_{NC} increasing (see Table 5.7).

α	E_{NP}	E_{NC}
1.3	4.6852	1.3007
1.35	4.5411	1.2270
1.4	4.4038	1.1590
1.45	4.2729	1.0960
1.5	4.1479	1.0376
1.55	4.0268	0.9834

β	E_{NP}	E_{NC}
1	4.5720	1.8305
1.1	4.6266	1.9591
1.2	4.6734	2.0787
1.3	4.7135	2.1899
1.4	4.7478	2.2934
1.5	4.7770	2.3897

Table 5.7: Effect of α for $\beta = 0.5$ and that of β for $\alpha = 1.5$ on E_{NP} and E_{NC}

5.4.1 Optimization Problem

Based on the above performance measures we construct a cost function for checking the optimality of the waiting room capacity K. It may be noted that cancellation to some extent prior to CLT realization results in higher profit to the system since there is a cancellation penalty imposed on the customer. Hence we define profit/revenue function $F(K, S)$ as

$$\boldsymbol{\mathcal{F}}(K, S) = \boldsymbol{\mathcal{C}}_1 E_{CR} - \boldsymbol{\mathcal{C}}_2 E_I + \boldsymbol{\mathcal{C}}_3 E_B - \boldsymbol{\mathcal{C}}_4 E_W$$

where
$\boldsymbol{\mathcal{C}}_1$ = Revenue to the system due to per unit cancellation of inventory purchased,
$\boldsymbol{\mathcal{C}}_2$ = Holding cost per unit time per inventory,
$\boldsymbol{\mathcal{C}}_3$ = Revenue to the system per customer per unit time in the buffer (this is an income on account of inventory being sold for sure),

$\mathcal{C}_4$ = Holding cost of customer per unit per unit time in the waiting room.

In order to study the variation in different parameters on expected total cost we first fix the costs $\mathcal{C}_1 = \$50, \mathcal{C}_2 = \$10, \mathcal{C}_3 = \$15, \mathcal{C}_4 = \$20$.

Effect of variation in S and K

We assign the following values to the parameters: $\lambda = 11, \mu = 15, \beta = 17, \eta = 5, \alpha = 0.25, p = 0.75$. For different values of S and K, the expected revenue are calculated and presented in Table 5.8. This table shows that the expected revenue decreases when S increases and for each value of S we get an optimum K.

K \ S	2	3	4	5	6
3	413.7963	404.6718	399.7146	394.1720	387.4320
4	**415.6992**	406.2415	400.7019	394.7882	387.8239
5	415.5614	406.7719	401.0563	394.9985	**387.9483**
6	414.4258	**406.7782**	**401.0896**	**395.0037**	387.9387
7	412.8614	406.5428	400.9743	394.9154	387.8690
8	411.1812	406.2183	400.8032	394.7937	387.7792

Table 5.8: Effect of S and K on expected revenue

Chapter 6

$GI/M/1$ type queueing-inventory systems with postponed work, reservation, cancellation and common life time

In the previous chapter the problem discussed was an $LIQBD$ process. This turned out to be so since we assumed that unsatisfied customers stay back in the system. In the present chapter we discuss two models in which items in the inventory have a random common life time. On realization of common

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Some results of this chapter are included in the following paper.
A. Krishnamoorthy, Dhanya Shajin and B. Lakshmy : $GI/M/1$ **type queueing-inventory systems with postponed work, reservation, cancellation and common life time**, Invited paper to the special issue "Stochastic Models" of Indian Journal of Pure and Applied Mathematics, Guest Editor: Professor M. K. Ghosh, 2015 (to appear).

life time, all customers in the system are flushed out. Subsequently the inventory reaches its maximum level S through a replenishment which follows an exponential distribution for the next cycle. Through cancellation of purchases, inventory gets added until their expiry time; inter cancellation times follow exponential distribution. Customers arrive according to a Poisson process and service time is exponentially distributed. On arrival, if a customer finds the server busy, he joins a buffer of varying size. If there is no inventory, the arriving customer first try to queue up in a finite waiting room of capacity K. Finding that full, he joins a pool of infinite capacity with probability γ $(0 < \gamma < 1)$; else it is lost to the system forever. We discuss two models based on 'transfer' of customers from the pool to the waiting room / buffer. In Model 1 when, at a service completion epoch the waiting room size drops to a preassigned number $L-1$ $(1 < L < K)$ or below, a customer is transferred from pool to waiting room with probability p $(0 < p < 1)$ and positioned as the last among the waiting customers. If at a departure epoch the waiting room turns out to be empty and there is at least one customer in the pool, then the one ahead of all waiting in the pool gets transferred to the waiting room with probability one. We introduce a totally different transfer mechanism in Model 2: when at a service completion epoch, the server turns idle with at least one item in the inventory, the pooled customer is immediately taken for service. At the time of a cancellation if the server is idle with none, one or more customers in the waiting room, then the head of the pooled customer goes to the buffer directly for service. Also we assume that no customer joins the system when there is no item in the inventory. Our computational experiments show that the second model is more cost effective.

With the assumption of flush out of all customers from the system on realization of CLT, what we get is $GI/M/1$ type Markov chain. Whereas for the model discussed in chapter 5 the stability condition was to be investigated, the present model is always stable. Here in reality we can have traffic intensity crossing 1, still the system is stable!

6.1 Mathematical formulation: Model 1

We have a single commodity inventory system with S items at the beginning of a cycle. Customers arrive according to a Poisson process of rate λ demanding exactly one unit of item (extension to demand for more than one item by a customer is straight forward). To deliver the item to the customer in service, it requires an exponentially distributed time with parameter μ. The inventoried items have a common life time which means that they all perish (unfit for use) together on realization of this time (example is drugs that are manufactured in a batch). We assume that this common life time is exponentially distributed with parameter α. On realization of common life time the process of ordering for inventory replenishment starts. The physical arrival of items takes an exponentially distributed amount of time having parameter η. The quantity of replenishment is S. A buffer of varying size, depending on the number of items in the inventory is available near the service counter. We call it varying size because at most as many customers as the number of items in the inventory are allowed to be in this buffer. In addition the possibility of cancellation of purchase (return of the item with a penalty) is introduced here. Inter cancellation time follows exponential distribution with parameter $i\beta$, when there are i items in the purchased list in the current cycle (that is, there are $(S - i)$ items in the inventory). Next in order is a finite waiting space of capacity K. When the buffer is full further arrivals wait in this room; as and when inventory level in the buffer goes above (due to cancellation), the head in the waiting room moves to the buffer and positions himself as the last there. When the waiting room is also full, further arrivals are directed to a pool (of customers) having infinite capacity. Whereas customers join with probability one in the buffer and waiting room whenever there is a vacancy, it is not the case with the pool. An arrival, finding waiting room also full, joins the pool with probability $\gamma(0 < \gamma < 1)$ or balks with complementary probability.

We introduce a transfer mechanism of customers from pool to waiting room as follows: when, at a departure epoch the number of customers in the waiting room drops to a preassigned number $L-1, (1 < L < K)$ or below, a customer is transferred from the pool to the waiting room with probability p $(0 < p < 1)$ and positioned as last among the waiting customers. If at a service completion epoch the waiting room turns out to be empty and there is at least one customer in the pool, the one ahead of all waiting in the pool gets transferred (with probability one) to the waiting room. Transfer of customers in a pure queueing theory perspective from a pool is introduced and analyzed in Deepak et al. [13].

It is in the transfer mechanism that the two models discussed in this chapter differ. This mechanism for Model 2 is discussed at the appropriate place in Section 6.2.

Further all customers are flushed out from the system (finite buffer, waiting room and pool) when the common life time is realized.

In the sequel we use the following notations:

$\boldsymbol{\mathcal{N}}_1(t)$	Number of customers in the pool at time t
$\boldsymbol{\mathcal{N}}_2(t)$	Number of customers in the waiting room at time t
$\boldsymbol{\mathcal{N}}_3(t)$	Number of customers in the buffer (including in service) at time t
$\boldsymbol{\mathcal{I}}(t)$	Number of items in the inventory at time t

$$u(t) = \begin{cases} 0, & \text{if server is idle at time } t \\ 1, & \text{pooled customer in service at time } t \\ 2, & \text{customer in service not from the pool at time } t \end{cases}$$

Cycle :	The time duration from the epoch at which we start with maximum inventory level S at a replenishment epoch, to the moment when the common life time is realized
Lead time :	On expiry of common life time, the inventory level reaches its maximum S through a replenishment for the next cycle. The time elapsed between realization of CLT of a batch to the epoch at which the replenishment takes place for the next cycle, is called lead time
U_1	$= (S+1)(S+2)/2 + K(S+1)$
U_2	$= K(S+1)$
U_3	$= (S+1)^2 + K(2S+1)$
U_4	$= S(S+1) + K(2S+1) + 1$

By the above assumptions $\Omega = \{(\boldsymbol{\mathcal{N}}_1(t), \boldsymbol{\mathcal{N}}_2(t), \boldsymbol{\mathcal{I}}(t), \boldsymbol{\mathcal{N}}_3(t)), t \geq 0\}$ is a $CTMC$. Its state space is given by

$$\{\Delta\} \bigcup \{(0, 0, i, n_3); 0 \leq i \leq S, 0 \leq n_3 \leq i\}$$

$$\bigcup \{(n_1, n_2, i, n_3); n_1 \geq 0, 1 \leq n_2 \leq K, 0 \leq i \leq S, n_3 = i\}$$

where $\{\Delta\}$ denotes the temporary absorbing state. Thus the infinitesimal generator $\boldsymbol{\mathcal{Q}}_1$ is of the form

$$\boldsymbol{\mathcal{Q}}_1 = \begin{bmatrix} A_\Delta & A_{\Delta 0} & & & & \\ A_{0\Delta} & A_{00} & A_{01} & & & \\ A_2' & A_{10} & A_1 & A_0 & & \\ A_2' & & A_2 & A_1 & A_0 & \\ \vdots & & & \ddots & \ddots & \ddots \end{bmatrix}.$$

The matrices A_0, A_1, A_2 are square matrices of the same order U_2 with A_0 representing transition from level n_1 to $n_1 + 1, n_1 \geq 1$; A_1 represents the transitions within the level $n_1, n_1 \geq 1$; and A_2 contains transition rates from level n_1 to $n_1 - 1, n_1 \geq 2$. Dimension of matrices A_{00}, A_{01}, A_{10} are $U_1 \times U_1, U_1 \times$

$U_2, U_2 \times U_1$ respectively. Matrices $A_{0\Delta}, A_2'$ are column vectors of orders U_1, U_2 respectively. $A_{\Delta 0}$ is a row vector of order U_1.

$$A_{\Delta} = -\eta, A_{\Delta 0} = \eta \mathbf{e}'_{(S(S+1)/2)+1}, A_{0\Delta} = \alpha \mathbf{e}, A_2' = \alpha \mathbf{e}.$$

Define $A_{k(n_2,i,n_3)}^{(m_2,j,m_3)}$, $k = 00, 01, 10, 0, 1, 2$ as the transition rates from $(n_2, i, n_3) \rightarrow (m_2, j, m_3)$ where n_2, m_2 represent the number of customers in the waiting room, i, j represent the number of items in the inventory and n_3, m_3 represent the number of customers in the buffer. These transition rates are

$$A_{1(n_2,i,n_3)}^{(m_2,j,m_3)} = \begin{cases} \lambda & 1 \leq n_2 \leq K-1, 0 \leq i \leq S, n_3 = i; \\ & m_2 = n_2 + 1, j = i, m_3 = n_3, \\ \mu & 1 \leq n_2 \leq K, 1 \leq i \leq S, n_3 = i; \\ & m_2 = n_2, j = i-1, m_3 = n_3 - 1, \\ (1-p)(S-i)\beta & 2 \leq n_2 \leq L, 0 \leq i \leq S-1, n_3 = i; \\ & m_2 = n_2 - 1, j = i+1, m_3 = n_3 + 1, \\ (S-i)\beta & L+1 \leq n_2 \leq K, 0 \leq i \leq S-1, n_3 = i; \\ & m_2 = n_2 - 1, j = i+1, m_3 = n_3 + 1, \\ -(\lambda + S\beta + \alpha) & 1 \leq n_2 \leq K-1, i = 0, n_3 = 0; \\ & m_2 = n_2, j = i, m_3 = n_3, \\ -(\lambda + \mu + (S-i)\beta + \alpha) & 1 \leq n_2 \leq K-1, 1 \leq i \leq S, n_3 = i; \\ & m_2 = n_2, j = i, m_3 = n_3, \\ -(\gamma\lambda + S\beta + \alpha) & n_2 = K, i = 0, n_3 = 0; \\ & m_2 = n_2, j = i, m_3 = n_3, \\ -(\gamma\lambda + \mu + (S-i)\beta + \alpha) & n_2 = K, 1 \leq i \leq S, n_3 = i; \\ & m_2 = n_2, j = i, m_3 = n_3, \\ 0 & \text{otherwise}, \end{cases}$$

$$A_{10(n_2,i,n_3)}^{(m_2,j,m_3)} = \begin{cases} (S-i)\beta & n_2 = 1, 0 \leq i \leq S-1, n_3 = i; \\ & m_2 = n_2, j = i+1, m_3 = n_3 + 1, \\ p(S-i)\beta & 2 \leq n_2 \leq L, 0 \leq i \leq S-1, n_3 = i; \\ & m_2 = n_2, j = i+1, m_3 = n_3 + 1, \\ 0 & \text{otherwise}, \end{cases}$$

$$A_{00(n_2,i,n_3)}^{(m_2,j,m_3)} = \begin{cases} \lambda & n_2 = 0, 0 \le i \le S, n_3 = i; \\ & m_2 = n_2 + 1, j = i, m_3 = n_3, \\ \lambda & n_2 = 0, 1 \le i \le S, 0 \le n_3 \le i - 1; \\ & m_2 = n_2, j = i, m_3 = n_3 + 1, \\ \lambda & 1 \le n_2 \le K - 1, 0 \le i \le S, n_3 = i; \\ & m_2 = n_2 + 1, j = i, m_3 = n_3, \\ \mu & n_2 = 0, 1 \le i \le S, 1 \le n_3 \le i; \\ & m_2 = n_2, j = i - 1, m_3 = n_3 - 1, \\ \mu & 1 \le n_2 \le K, 1 \le i \le S, n_3 = i; \\ & m_2 = n_2, j = i - 1, m_3 = n_3 - 1, \\ (S-i)\beta & n_2 = 0, 0 \le i \le S - 1, 0 \le n_3 \le i; \\ & m_2 = n_2, j = i + 1, m_3 = n_3, \\ (S-i)\beta & 1 \le n_2 \le K, 0 \le i \le S - 1, n_3 = i; \\ & m_2 = n_2 - 1, j = i + 1, m_3 = n_3 + 1, \\ -(\lambda + (S-i)\beta + \alpha) & n_2 = 0, 0 \le i \le S, n_3 = 0; \\ & m_2 = 0, j = i, m_3 = n_3, \\ -(\lambda + \mu + (S-i)\beta + \alpha) & n_2 = 0, 1 \le i \le S, 1 \le n_3 \le i; \\ & m_2 = 0, j = i, m_3 = n_3, \\ -(\lambda + S\beta + \alpha) & 1 \le n_2 \le K - 1, i = 0, n_3 = 0; \\ & m_2 = n_2, j = i, m_3 = n_3, \\ -(\lambda + \mu + (S-i)\beta + \alpha) & 1 \le n_2 \le K - 1, 1 \le i \le S, n_3 = i; \\ & m_2 = n_2, j = i, m_3 = n_3, \\ -(\gamma\lambda + S\beta + \alpha) & n_2 = K, i = 0, n_3 = 0; \\ & m_2 = n_2, j = i, m_3 = n_3, \\ -(\gamma\lambda + \mu + (S-i)\beta + \alpha) & n_2 = K, 1 \le i \le S, n_3 = i; \\ & m_2 = n_2, j = i, m_3 = n_3, \\ 0 & \text{otherwise,} \end{cases}$$

$$A_{2(n_2,i,n_3)}^{(m_2,j,m_3)} = \begin{cases} (S-i)\beta & n_2 = 1, 0 \le i \le S - 1, n_3 = i; \\ & m_2 = n_2, j = i + 1, m_3 = n_3 + 1, \\ p(S-i)\beta & 2 \le n_2 \le L, 0 \le i \le S - 1, n_3 = i; \\ & m_2 = n_2, j = i + 1, m_3 = n_3 + 1, \\ 0 & \text{otherwise,} \end{cases}$$

$$A_{01(n_2,i,n_3)}^{(m_2,j,m_3)} = \begin{cases} \gamma\lambda & n_2 = K, 0 \leq i \leq S, n_3 = i; \\ & m_2 = n_2, j = i, m_3 = n_3, \\ 0 & \text{otherwise,} \end{cases}$$

$$A_{0(n_2,i,n_3)}^{(m_2,j,m_3)} = \begin{cases} \gamma\lambda & n_2 = K, 0 \leq i \leq S, n_3 = i; \\ & m_2 = n_2, j = i, m_3 = n_3, \\ 0 & \text{otherwise.} \end{cases}$$

6.1.1 Analysis of the system

In this section, we perform the steady-state analysis of the queueing-inventory model described above.

Let $\mathbf{x}$ be the steady-state probability vector of generator $\boldsymbol{\mathcal{Q}}_1$. Then we have

$$\mathbf{x}\boldsymbol{\mathcal{Q}}_1 = 0, \quad \mathbf{xe} = 1. \tag{6.1}$$

First partitioning $\mathbf{x}$ as $\mathbf{x} = (x_\Delta, \mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, ...)$ and then each of the sub-vectors as

$$\mathbf{x}_0 = (x_0(0, i, n_3), x_0(n_2, i, i); 0 \leq i \leq S, 0 \leq n_3 \leq i, 1 \leq n_2 \leq K),$$
$$\mathbf{x}_{n_1} = (x_{n_1}(n_2, i, i); 0 \leq i \leq S, 1 \leq n_2 \leq K), \quad \text{for} \quad n_1 \geq 1,$$

we see that $\mathbf{x}$ is obtained as (see Neuts [33])

$$\mathbf{x}_{n_1} = \mathbf{x}_1 R^{n_1-1}, \quad n_1 \geq 2$$

where R is the minimal nonnegative solution to the matrix quadratic equation:

$$\sum_{k=0}^{2} R^k A_k = \mathbf{O}.$$

The boundary equations are given by

$$\begin{aligned} x_\Delta A_{\Delta 0} + \mathbf{x}_0 A_{00} + \mathbf{x}_1 A_{10} &= 0, \\ \mathbf{x}_0 A_{01} + \mathbf{x}_1 [A_1 + RA_2] &= 0, \\ x_\Delta &= \frac{\alpha}{\eta} \sum_{n_1=0}^{\infty} \mathbf{x}_{n_1} \mathbf{e}. \end{aligned}$$

The normalizing condition (6.1) gives

$$x_\Delta + \mathbf{x}_0\mathbf{e} + \mathbf{x}_1[I-R]^{-1}\mathbf{e} = 1.$$

The system state probabilities computed above provide the following useful information

1. Expected number of customers in the pool before realization of CLT
$$E_P(N) = \sum_{n_1=1}^{\infty} n_1 \mathbf{x}_{n_1}\mathbf{e}.$$

2. Expected number of customers in the waiting room before realization of CLT
$$E_W(N) = \sum_{n_1=0}^{\infty}\sum_{n_2=1}^{K}\sum_{i=0}^{S} n_2 x_{n_1}(n_2,i,i).$$

3. Expected number of customers in the buffer before realization of CLT
$$E_B(N) = \sum_{n_1=0}^{\infty}\sum_{n_2=1}^{K}\sum_{i=1}^{S} i x_{n_1}(n_2,i,i) + \sum_{i=1}^{S}\sum_{n_3=1}^{i} n_3 x_0(0,i,n_3).$$

4. Expected number of items in the inventory before realization of CLT
$$E_I(N) = \sum_{n_1=0}^{\infty}\sum_{n_2=1}^{K}\sum_{i=1}^{S} i x_{n_1}(n_2,i,i) + \sum_{i=1}^{S}\sum_{n_3=0}^{i} i x_0(0,i,n_3).$$

5. Expected number of items in the inventory immediately on realization of CLT
$$E_I'(N) = \sum_{n_1=0}^{\infty}\sum_{n_2=1}^{K}\sum_{i=1}^{S} i u_1(i) x_{n_1}(n_2,i,i) +$$
$$\sum_{i=1}^{S} i u_2((i) x_0(0,i,0) + \sum_{i=1}^{S}\sum_{n_3=1}^{i} i u_1(i) x_0(0,i,n_3)$$
where $u_1(i) = \frac{\alpha}{\alpha+\lambda+\mu+(S-i)\beta}$, $u_2(i) = \frac{\alpha}{\alpha+\lambda+(S-i)\beta}$.

6. Rate of addition to the pool is

$$\gamma\lambda \sum_{n_1=0}^{\infty} \sum_{i=0}^{S} x_{n_1}(K, i, i).$$

7. The probability that a customer enters service immediately on arrival

$$\sum_{i=1}^{S} x_0(0, i, 0).$$

8. The rate at which pooled customers are transferred to the waiting room

$$E_{PW}(R) = \sum_{n_1=1}^{\infty} \sum_{i=0}^{S-1} (S-i)\beta \left[\sum_{n_2=2}^{L} p x_{n_1}(n_2, i, i) + x_{n_1}(1, i, i) \right].$$

9. The rate at which customers abandon the system on arrival

$$E_{WL}(R) = (1-\gamma)\lambda \sum_{n_1=0}^{\infty} \sum_{i=0}^{S} x_{n_1}(K, i, i).$$

10. Expected cancellation rate

$$E_C(R) = \sum_{n_1=0}^{\infty} \sum_{n_2=1}^{K} \sum_{i=0}^{S-1} (S-i)\beta x_{n_1}(n_2, i, i) + \sum_{i=0}^{S-1} \sum_{n_3=0}^{i} (S-i)\beta x_0(0, i, n_3).$$

11. Expected inventory depletion rate

$$E_P(R) = \mu \left\{ \sum_{n_1=0}^{\infty} \sum_{n_2=1}^{K} \sum_{i=1}^{S} x_{n_1}(n_2, i, i) + \sum_{i=1}^{S} \sum_{n_3=1}^{i} x_0(0, i, n_3) \right\}.$$

12. Expected number of cancellations in a cycle

$$E_{NC} = \frac{1}{\alpha} E_C(R).$$

13. Expected number of purchases in a cycle

$$E_{NP} = \frac{1}{\alpha} E_P(R)$$

14. Expected number of transfers from the pool to the waiting room

$$E_{PW}(N) = \frac{1}{\alpha} \sum_{n_1=1}^{\infty} \sum_{i=0}^{S-1} (S-i)\beta \left[\sum_{n_2=2}^{L} p x_{n_1}(n_2, i, i) + x_{n_1}(1, i, i) \right].$$

15. The probability that the system has all the S items in the inventory at the time of realization of CLT

$$P_{vacant} = \sum_{n_3=1}^{S} x_0(0, S, n_3) + \sum_{n_1=0}^{\infty} \sum_{n_2=1}^{K} x_{n_1}(n_2, S, S) + x_0(0, S, 0).$$

This is equal to the probability, for example, that a bus with S seats depart without any passenger on board.

16. The probability that the system is left with no item in the inventory at the time of realization of CLT

$$P_{full} = x_0(0, 0, 0) + \sum_{n_1=0}^{\infty} \sum_{n_2=1}^{K} x_{n_1}(n_2, 0, 0).$$

This is equivalent to the probability that bus referred to in the previous item leaves with full capacity.

6.2 Mathematical formulation: Model 2

In this section we describe an inventory problem in which further restrictions are imposed on Model 1 described in Section 6.1. The following are the additional restrictions:

(a). No customer joins the system when inventory level is zero.

(b). At the instant when a cancellation of purchased inventory occurs with none, one or more customers waiting in the waiting room and server idle due to no item in the inventory the head of the pool is transferred to the buffer for immediate service.

(c). If at a service completion epoch, there is no customer in the buffer as well as the waiting room, then again the head of the pool is transferred to the buffer for immediate service.

(d). No transfer takes place from the pool / waiting room to the buffer when no item is in the inventory.

All other assumptions in Model 1 hold for the present model also. Assumptions (b), (c) are to reduce the waiting time of customers in the pool to the extent possible. Thus we get a $CTMC$ $\{(\boldsymbol{\mathcal{N}}_1(t), \boldsymbol{\mathcal{N}}_2(t), \boldsymbol{\mathcal{I}}(t), \boldsymbol{\mathcal{N}}_3(t), u(t)), t \geq 0\}$ with state space

$$\begin{aligned}\Omega' = \{\Delta\} & \bigcup \{(0,0,i,0,0)\,; 0 \leq i \leq S\} \bigcup \{(n_1,n_2,0,0,0)\,; n_1 \geq 0, 1 \leq n_2 \leq K\} \\ & \bigcup \{(n_1,0,i,n_3,k)\,; n_1 \geq 0, 1 \leq i \leq S, 1 \leq n_3 \leq i, k=1,2\} \\ & \bigcup \{(n_1,n_2,i,i,k)\,; n_1 \geq 0, 1 \leq n_2 \leq K, 1 \leq i \leq S, k=1,2\}\end{aligned}$$

where $\{\Delta\}$ denotes the temporary absorbing state. In this model $u(t)$ is brought in to identify whether the current service, if any, is of a pooled customer. This is introduced to explicitly compute certain system performance index that would help to control the number of customers in the pool. Thus the infinitesimal generator $\boldsymbol{\mathcal{Q}}_2$ is of form

$$\boldsymbol{\mathcal{Q}}_2 = \begin{bmatrix} B_\Delta & B_{\Delta 0} & & & & \\ B_{0\Delta} & B_{00} & B_{01} & & & \\ B_2' & B_{10} & B_1 & B_0 & & \\ B_2' & & B_2 & B_1 & B_0 & \\ \vdots & & & \ddots & \ddots & \ddots \end{bmatrix}.$$

The matrices B_0, B_1, B_2 are square matrices of the same order U_4 with B_0 representing transition from level n_1 to $n_1+1, n_1 \geq 1$, B_1 represents the transitions within the level $n_1, n_1 \geq 1$ and B_2 contains transition rates from level n_1 to $n_1 - 1, n_1 \geq 2$. Dimension of matrices B_{00}, B_{01}, B_{10} are $U_3 \times U_3, U_3 \times U_4, U_4 \times U_3$ respectively. Matrices $B_{0\Delta}, B_2'$ are column vectors of order U_3, U_4 respectively. $B_{\Delta 0}$ is a row vector of order U_3.

$$B_\Delta = -\eta, B_{\Delta 0} = \eta \mathbf{e}'_{S^2+1}, B_{0\Delta} = \alpha \mathbf{e}, B_2' = \alpha \mathbf{e}.$$

Define $B_l^{(n_2,i,n_3,k_1:m_2,j,m_3,k_2)}$, $l = 00, 01, 10, 0, 1, 2$ as the transition rates from $(n_2, i, n_3, k_1) \to (m_2, j, m_3, k_2)$ where n_2, m_2 represent the number of customers in the waiting room, i, j represent the number of items in the inventory, n_3, m_3 represent the number of customers in the buffer and k_1, k_2 represent the status of server. These transition rates are

$$B_1^{(n_2,i,n_3,k_1:m_2,j,m_3,k_2)} = \begin{cases} \lambda & n_2 = 0, 1 \leq i \leq S, n_3 = i, k_1 = 1,2; \\ & m_2 = n_2 + 1, j = i, m_3 = n_3, k_2 = k_1, \\ \lambda & n_2 = 0, 2 \leq i \leq S, 1 \leq n_3 \leq i-1, k_1 = 1,2; \\ & m_2 = n_2, j = i, m_3 = n_3 + 1, k_2 = k_1, \\ \lambda & 1 \leq n_2 \leq K-1, 1 \leq i \leq S, n_3 = i, k_1 = 1,2; \\ & m_2 = n_2 + 1, j = i, m_3 = n_3, k_2 = k_1, \\ \mu & n_2 = 0, i = 1, n_3 = 1, k_1 = 1,2; \\ & m_2 = n_2, j = i-1, m_3 = n_3 - 1, k_2 = 0, \\ \mu & n_2 = 0, 2 \leq i \leq S, 2 \leq n_3 \leq i, k_1 = 1,2; \\ & m_2 = n_2, j = i-1, m_3 = n_3 - 1, k_2 = 2, \\ \mu & 1 \leq n_2 \leq K, i = 1, n_3 = i, k_1 = 1,2; \\ & m_2 = n_2, j = i-1, m_3 = n_3 - 1, k_2 = 0, \\ \mu & 1 \leq n_2 \leq K, 2 \leq i \leq S, n_3 = i, k_1 = 1,2; \\ & m_2 = n_2, j = i-1, m_3 = n_3 - 1, k_2 = 2, \\ (S-i)\beta & n_2 = 0, 1 \leq i \leq S-1, 1 \leq n_3 \leq i, k_1 = 1,2; \\ & m_2 = n_2, j = i+1, m_3 = n_3, k_2 = k_1, \\ (S-i)\beta & 1 \leq n_2 \leq K, 1 \leq i \leq S-1, n_3 = i, k_1 = 1,2; \\ & m_2 = n_2 - 1, j = i+1, m_3 = n_3 + 1, k_2 = k_1, \\ -(S\beta + \alpha) & 0 \leq n_2 \leq K, i = 0, n_3 = i, k_1 = 0; \\ & m_2 = n_2, j = i, m_3 = n_3, k_2 = k_1, \\ -(\lambda + \mu + (S-i)\beta + \alpha) & n_2 = 0, 1 \leq i \leq S, 1 \leq n_3 \leq i, k_= 1,2; \\ & m_2 = 0, j = i, m_3 = n_3, k_2 = k_1, \\ -(\lambda + \mu + (S-i)\beta + \alpha) & 1 \leq n_2 \leq K-1, 1 \leq i \leq S, n_3 = i, k_1 = 1,2; \\ & m_2 = n_2, j = i, m_3 = n_3, k_2 = k_1, \\ -(\gamma\lambda + \mu + (S-i)\beta + \alpha) & n_2 = K, 1 \leq i \leq S, n_3 = i, k_1 = 1,2; \\ & m_2 = n_2, j = i, m_3 = n_3, k_2 = k_1, \\ 0 & \text{otherwise}, \end{cases}$$

$$B_{00}^{(n_2,i,n_3,k_1:m_2,j,m_3,k_2)} = \begin{cases} \lambda & n_2 = 0, 1 \le i \le S, n_3 = i, k_1 = 1,2; \\ & m_2 = n_2 + 1, j = i, m_3 = n_3, k_2 = k_1, \\ \lambda & n_2 = 0, 2 \le i \le S, 1 \le n_3 \le i-1, k_1 = 1,2; \\ & m_2 = n_2, j = i, m_3 = n_3 + 1, k_2 = k_1, \\ \lambda & n_2 = 0, 1 \le i \le S, n_3 = 0, k_1 = 0; \\ & m_2 = n_2, j = i, m_3 = n_3 + 1, k_2 = 2, \\ \lambda & 1 \le n_2 \le K-1, 1 \le i \le S, n_3 = i, k_1 = 1,2; \\ & m_2 = n_2 + 1, j = i, m_3 = n_3, k_2 = k_1, \\ \mu & n_2 = 0, 1 \le i \le S, n_3 = 1, k_1 = 1,2; \\ & m_2 = n_2, j = i-1, m_3 = n_3 - 1, k_2 = 0, \\ \mu & n_2 = 0, 2 \le i \le S, 2 \le n_3 \le i, k_1 = 1,2; \\ & m_2 = n_2, j = i-1, m_3 = n_3 - 1, k_2 = 2, \\ \mu & 1 \le n_2 \le K, i = 1, n_3 = i, k_1 = 1,2; \\ & m_2 = n_2, j = i-1, m_3 = n_3 - 1, k_2 = 0, \\ \mu & 1 \le n_2 \le K, 2 \le i \le S, n_3 = i, k_1 = 1,2; \\ & m_2 = n_2, j = i-1, m_3 = n_3 - 1, k_2 = 2, \\ (S-i)\beta & n_2 = 0, 0 \le i \le S-1, n_3 = 0, k_1 = 0; \\ & m_2 = n_2, j = i+1, m_3 = n_3, k_2 = 0, \\ (S-i)\beta & n_2 = 0, 1 \le i \le S-1, 1 \le n_3 \le i, k_1 = 1,2; \\ & m_2 = n_2, j = i+1, m_3 = n_3, k_2 = k_1, \\ S\beta & 1 \le n_2 \le K, i = 0, n_3 = i, k_1 = 0; \\ & m_2 = n_2 - 1, j = i+1, m_3 = n_3 + 1, k_2 = 2, \\ (S-i)\beta & 1 \le n_2 \le K, 1 \le i \le S-1, n_3 = i, k_1 = 1,2; \\ & m_2 = n_2 - 1, j = i+1, m_3 = n_3 + 1, k_2 = k_1, \\ -(S\beta + \alpha) & 0 \le n_2 \le K, i = 0, n_3 = i, k_1 = 0; \\ & m_2 = n_2, j = i, m_3 = n_3, k_2 = k_1, \\ -(\lambda + (S-i)\beta + \alpha) & n_2 = 0, 1 \le i \le S, n_3 = 0, k_1 = 0; \\ & m_2 = 0, j = i, m_3 = n_3, k-2 = k_1, \\ -(\lambda + \mu + (S-i)\beta + \alpha) & n_2 = 0, 1 \le i \le S, 1 \le n_3 \le i, k_{=}1,2; \\ & m_2 = 0, j = i, m_3 = n_3, k_2 = k_1, \\ -(\lambda + \mu + (S-i)\beta + \alpha) & 1 \le n_2 \le K-1, 1 \le i \le S, n_3 = i, k_1 = 1,2; \\ & m_2 = n_2, j = i, m_3 = n_3, k_2 = k_1, \\ -(\gamma\lambda + \mu + (S-i)\beta + \alpha) & n_2 = K, 1 \le i \le S, n_3 = i, k_1 = 1,2; \\ & m_2 = n_2, j = i, m_3 = n_3, k_2 = k_1, \\ 0 & \text{otherwise}, \end{cases}$$

$$B_{2}^{(n_2,i,n_3,k_1:m_2,j,m_3,k_2)} = \begin{cases} S\beta & 0 \le n_2 \le K, i = 0, n_3 = i, k_1 = 0; \\ & m_2 = n_2, j = i+1, m_3 = n_3 + 1, k_2 = 1, \\ \mu & n_2 = 0, 2 \le i \le S, n_3 = 1, k_1 = 1,2; \\ & m_2 = n_2, j = i-1, m_3 = n_3, k_2 = 1, \\ 0 & \text{otherwise}, \end{cases}$$

$$B_{10}^{(n_2,i,n_3,k_1:m_2,j,m_3,k_2)} = \begin{cases} S\beta & 0 \le n_2 \le K, i = 0, n_3 = i, k_1 = 0; \\ & m_2 = n_2, j = i+1, m_3 = n_3 + 1, k_2 = 1, \\ \mu & n_2 = 0, 2 \le i \le S, n_3 = 1, k_1 = 1,2; \\ & m_2 = n_2, j = i-1, m_3 = n_3, k_2 = 1, \\ 0 & \text{otherwise}, \end{cases}$$

$$B_{01}^{(n_2,i,n_3,k_1:m_2,j,m_3,k_2)} = \begin{cases} \gamma\lambda & n_2 = K, 1 \le i \le S, n_3 = i, k_1 = 1,2; \\ & m_2 = n_2, j = i, m_3 = n_3, k_2 = k_1, \\ 0 & \text{otherwise,} \end{cases}$$

$$B_{0}^{(n_2,i,n_3,k_1:m_2,j,m_3,k_2)} = \begin{cases} \gamma\lambda & n_2 = K, 1 \le i \le S, n_3 = i, k_1 = 1,2; \\ & m_2 = n_2, j = i, m_3 = n_3, k_2 = k_1, \\ 0 & \text{otherwise.} \end{cases}$$

6.2.1 Steady-state analysis

Note that the system described is always stable since realization of common life time results in all customers in the system being flushed out. In this section, we perform the steady-state analysis of the queueing-inventory model.

Let $\mathbf{y}$ be the steady-state probability vector of generator $\boldsymbol{\mathcal{Q}}_2$. Then we have

$$\mathbf{y}\boldsymbol{\mathcal{Q}}_2 = 0, \quad \mathbf{ye} = 1. \tag{6.2}$$

Partitioning $\mathbf{y}$ as $\mathbf{y} = (y_\Delta, \mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2, ...)$ and then each of the sub-vectors as

$$\mathbf{y}_0 = ((y_0(0,i,0,0), 0 \le i \le S); (y_0(0,i,n_3,k), 1 \le i \le S, 1 \le n_3 \le i, k = 1,2);$$

$$(y_0(n_2,0,0,0), 1 \le n_2 \le K); (y_0(n_2,i,i,k), 1 \le n_2 \le K, 1 \le i \le S, k = 1,2))$$

$$\mathbf{y}_{n_1} = ((y_{n_1}(n_2,i,i,k), 1 \le i \le S, 1 \le n_2 \le K, k = 1,2);$$

$$(y_{n_1}(0,i,n_3,k), 1 \le i \le S, 1 \le n_3 \le i, k = 1,2); (y_{n_1}(n_2,0,0,0), 1 \le n_2 \le K))$$

$$\text{for} \quad n_1 \ge 1,$$

we see that $\mathbf{y}$ is obtained as (see Neuts [33])

$$\mathbf{y}_{n_1} = \mathbf{y}_1 R^{n_1-1}, \quad n_1 \ge 2$$

where R is the minimal nonnegative solution to the matrix quadratic equation:

$$\sum_{k=0}^{2} R^k B_k = \mathbf{O}.$$

The boundary equations are given by

$$y_{\Delta}B_{\Delta 0} + \mathbf{y}_0 B_{00} + \mathbf{y}_1 B_{10} = 0,$$

$$\mathbf{y}_0 B_{01} + \mathbf{y}_1[B_1 + RB_2] = 0,$$

$$y_{\Delta} = \frac{\alpha}{\eta}\sum_{n_1=0}^{\infty} \mathbf{y}_{n_1}\mathbf{e}.$$

The normalizing condition (6.2) gives

$$y_{\Delta} + \mathbf{y}_0\mathbf{e} + \mathbf{y}_1[I-R]^{-1}\mathbf{e} = 1.$$

6.2.2 Random walk

We consider the model with negligible service time; reservation, cancellation and realization of common life time on the set $\{0,1,2,...,S\}$, the set of possible states of the inventory level process. No customer joins when the inventory level is zero and so there will be none in the waiting room and pool. The arrival process, cancellation and CLT are as described in Section 6.1. Let $\boldsymbol{\mathcal{I}}(t)$ be the inventory level at time t. Then $\{\boldsymbol{\mathcal{I}}(t), t \geq 0\}$ is a Markov chain on state space $\{0,1,2,...,S\} \bigcup \left\{\tilde{\Delta}\right\}$ where $\left\{\tilde{\Delta}\right\}$ is an absorbing state which denotes the realization of common life time. Thus the infinitesimal generator is

$$\tilde{\boldsymbol{\mathcal{W}}} = \begin{bmatrix} \tilde{\boldsymbol{\mathcal{T}}} & \tilde{\boldsymbol{\mathcal{T}}}^0 \\ \mathbf{0} & 0 \end{bmatrix}$$

where

$$\tilde{\boldsymbol{\mathcal{T}}} = \begin{array}{c} \\ 0 \\ 1 \\ \vdots \\ S-2 \\ S-1 \\ S \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & \cdots & S-2 & S-1 & S \end{array} \\ \begin{pmatrix} h_S & S\beta & & & & \\ \lambda & h_{S-1} & (S-1)\beta & & & \\ & \ddots & \ddots & \ddots & & \\ & & \lambda & h_2 & 2\beta & \\ & & & \lambda & h_1 & \beta \\ & & & & \lambda & h_0 \end{pmatrix} \end{array}, \quad \tilde{\boldsymbol{\mathcal{T}}}^0 = \alpha\mathbf{e}$$

with $h_i = -(\lambda + i\beta + \alpha),\ 0 \le i \le (S-1)$ and $h_S = -(S\beta + \alpha)$. The expected time $E_{\mathbf{T}}$ until absorption follows a Phase type distribution with representation $(\xi, \tilde{\boldsymbol{\mathcal{T}}})$ where $\xi = (0, ..., 0, 1)$ is the initial probability vector of order $(S+1)$. Hence $E_{\mathbf{T}} = -\xi \tilde{\boldsymbol{\mathcal{T}}}^{-1}\mathbf{e}$.

6.2.3 Expected number of pooled customers getting service in a cycle

In order to compute the number of pooled customers getting service in a cycle, we consider the case of a finite pool. For numerical procedure the truncation level $\boldsymbol{\mathcal{P}}_L$ (size of the pool) is taken such that the probability of the number of customers in the pool going above the truncation size is of the order less than ϵ (here ϵ is taken as 10^{-6}). Consider the Markov chain $\{(\boldsymbol{\mathcal{N}}(t), \boldsymbol{\mathcal{N}}'_1(t), \boldsymbol{\mathcal{N}}_2(t), \boldsymbol{\mathcal{I}}(t), \boldsymbol{\mathcal{N}}_3(t), u(t)), t \ge 0\}$ where $\boldsymbol{\mathcal{N}}(t)$ = number of pooled customers who received service upto realization of CLT in the present cycle and $\boldsymbol{\mathcal{N}}'_1(t)$ = number of customers in the finite pool at time t. Its state space is

$$\begin{gathered}
\{\Delta'\} \bigcup \{(n, n_1, 0, i, n_3, k)\,; n \ge 0, 0 \le n_1 \le \boldsymbol{\mathcal{P}}_L, 1 \le i \le S, 1 \le n_3 \le i, k = 1, 2\} \\
\bigcup \{(n, 0, 0, i, 0, 0)\,; n \ge 0, 0 \le i \le S\} \bigcup \\
\{(n, n_1, n_2, 0, 0, 0)\,; n \ge 0, 0 \le n_1 \le \boldsymbol{\mathcal{P}}_L, 1 \le n_2 \le K\} \\
\bigcup \{(n, n_1, n_2, i, i, k)\,; n \ge 0, 0 \le n_1 \le \boldsymbol{\mathcal{P}}_L, 1 \le n_2 \le K, 1 \le i \le S, k = 1, 2\}
\end{gathered}$$

where $\{\Delta'\}$ is an absorbing state which denotes the realization of common life time. The infinitesimal generator of the Markov chain is

$$\tilde{\boldsymbol{\mathcal{N}}}_{\boldsymbol{\mathcal{P}}_L} = \begin{bmatrix} 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots \\ \boldsymbol{\mathcal{H}} & \boldsymbol{\mathcal{H}}_1 & \boldsymbol{\mathcal{H}}_0 & & & \\ \boldsymbol{\mathcal{H}} & & \boldsymbol{\mathcal{H}}_1 & \boldsymbol{\mathcal{H}}_0 & & \\ \boldsymbol{\mathcal{H}} & & & \boldsymbol{\mathcal{H}}_1 & \boldsymbol{\mathcal{H}}_0 & \\ \vdots & & & & \ddots & \ddots \end{bmatrix}$$

where $\boldsymbol{\mathcal{H}}_0, \boldsymbol{\mathcal{H}}_1$ are square matrices of order $U_3 + \boldsymbol{\mathcal{P}}_L U_4$ with $\boldsymbol{\mathcal{H}} = \alpha\mathbf{e}$. The entries in $\boldsymbol{\mathcal{H}}_0$ and $\boldsymbol{\mathcal{H}}_1$ are as under:

$$\boldsymbol{\mathcal{H}}_1 = \begin{bmatrix} B'_{00} & B_{01} & & & & \\ B'_{10} & B'_1 & B_0 & & & \\ & B'_2 & B'_1 & B_0 & & \\ & & \ddots & \ddots & \ddots & \\ & & & B'_2 & B'_1 & B_0 \\ & & & & B'_2 & B_1^0 \end{bmatrix}, \boldsymbol{\mathcal{H}}_0 = \begin{bmatrix} M'_1 & & & & \\ M'_2 & M_1 & & & \\ & M_2 & M_1 & & \\ & & \ddots & \ddots & \\ & & & M_2 & M_1 \end{bmatrix}$$

where

$$B_{00}^{\prime(n_2,i,n_3,k_1:m_2,j,m_3,k_2)} = \begin{cases} \lambda & n_2 = 0, 1 \le i \le S, n_3 = i, k_1 = 1,2; \\ & m_2 = n_2+1, j = i, m_3 = n_3, k_2 = k_1, \\ \lambda & n_2 = 0, 2 \le i \le S, 1 \le n_3 \le i-1, k_1 = 1,2; \\ & m_2 = n_2, j = i, m_3 = n_3+1, k_2 = k_1, \\ \lambda & n_2 = 0, 1 \le i \le S, n_3 = 0, k_1 = 0; \\ & m_2 = n_2, j = i, m_3 = n_3+1, k_2 = 2, \\ \lambda & 1 \le n_2 \le K-1, 1 \le i \le S, n_3 = i, k_1 = 1,2; \\ & m_2 = n_2+1, j = i, m_3 = n_3, k_2 = k_1, \\ \mu & n_2 = 0, 1 \le i \le S, n_3 = 1, k_1 = 2; \\ & m_2 = n_2, j = i-1, m_3 = n_3-1, k_2 = 0, \\ \mu & n_2 = 0, 2 \le i \le S, 2 \le n_3 \le i, k_1 = 2; \\ & m_2 = n_2, j = i-1, m_3 = n_3-1, k_2 = 2, \\ \mu & 1 \le n_2 \le K, i = 1, n_3 = i, k_1 = 2; \\ & m_2 = n_2, j = i-1, m_3 = n_3-1, k_2 = 0, \\ \mu & 1 \le n_2 \le K, 2 \le i \le S, n_3 = i, k_1 = 2; \\ & m_2 = n_2, j = i-1, m_3 = n_3-1, k_2 = 2, \\ (S-i)\beta & n_2 = 0, 0 \le i \le S-1, n_3 = 0, k_1 = 0; \\ & m_2 = n_2, j = i+1, m_3 = n_3, k_2 = 0, \\ (S-i)\beta & n_2 = 0, 1 \le i \le S-1, 1 \le n_3 \le i, k_1 = 1,2; \\ & m_2 = n_2, j = i+1, m_3 = n_3, k_2 = k_1, \\ S\beta & 1 \le n_2 \le K, i = 0, n_3 = i, k_1 = 0; \\ & m_2 = n_2-1, j = i+1, m_3 = n_3+1, k_2 = 2, \\ (S-i)\beta & 1 \le n_2 \le K, 1 \le i \le S-1, n_3 = i, k_1 = 1,2; \\ & m_2 = n_2-1, j = i+1, m_3 = n_3+1, k_2 = k_1, \\ -(S\beta+\alpha) & 0 \le n_2 \le K, i = 0, n_3 = i, k_1 = 0; \\ & m_2 = n_2, j = i, m_3 = n_3, k_2 = k_1, \\ -(\lambda+(S-i)\beta+\alpha) & n_2 = 0, 1 \le i \le S, n_3 = 0, k_1 = 0; \\ & m_2 = 0, j = i, m_3 = n_3, k-2 = k_1, \\ -(\lambda+\mu+(S-i)\beta+\alpha) & n_2 = 0, 1 \le i \le S, 1 \le n_3 \le i, k_= 1,2; \\ & m_2 = 0, j = i, m_3 = n_3, k_2 = k_1, \\ -(\lambda+\mu+(S-i)\beta+\alpha) & 1 \le n_2 \le K-1, 1 \le i \le S, n_3 = i, k_1 = 1,2; \\ & m_2 = n_2, j = i, m_3 = n_3, k_2 = k_1, \\ -(\gamma\lambda+\mu+(S-i)\beta+\alpha) & n_2 = K, 1 \le i \le S, n_3 = i, k_1 = 1,2; \\ & m_2 = n_2, j = i, m_3 = n_3, k_2 = k_1, \\ 0 & \text{otherwise.} \end{cases}$$

$$M_1^{(n_2,i,n_3,k_1:m_2,j,m_3,k_2)} = \begin{cases} \mu & n_2=0, i=1, n_3=1, k_1=1; \\ & m_2=n_2, j=i-1, m_3=n_3-1, k_2=0, \\ \mu & n_2=0, 2\le i\le S, 2\le n_3\le i, k_1=1; \\ & m_2=n_2, j=i-1, m_3=n_3-1, k_2=2, \\ \mu & 1\le n_2\le K, i=1, n_3=i, k_1=1; \\ & m_2=n_2, j=i-1, m_3=n_3-1, k_2=0, \\ \mu & 1\le n_2\le K, 2\le i\le S, n_3=i, k_1=1; \\ & m_2=n_2, j=i-1, m_3=n_3-1, k_2=2, \\ 0 & \text{otherwise.} \end{cases}$$

$$B_{10}^{\prime(n_2,i,n_3,k_1:m_2,j,m_3,k_2)} = \begin{cases} S\beta & 0\le n_2\le K, i=0, n_3=i, k_1=0; \\ & m_2=n_2, j=i+1, m_3=n_3+1, k_2=1, \\ \mu & n_2=0, 2\le i\le S, n_3=1, k_1=2; \\ & m_2=n_2, j=i-1, m_3=n_3, k_2=1, \\ 0 & \text{otherwise.} \end{cases}$$

$$B_2^{\prime(n_2,i,n_3,k_1:m_2,j,m_3,k_2)} = \begin{cases} S\beta & 0\le n_2\le K, i=0, n_3=i, k_1=0; \\ & m_2=n_2, j=i+1, m_3=n_3+1, k_2=1, \\ \mu & n_2=0, 2\le i\le S, n_3=1, k_1=2; \\ & m_2=n_2, j=i-1, m_3=n_3, k_2=1, \\ 0 & \text{otherwise.} \end{cases}$$

$$M_2^{(n_2,i,n_3,k_1:m_2,j,m_3,k_2)} = \begin{cases} \mu & n_2=0, 2\le i\le S, n_3=1, k_1=1; \\ & m_2=n_2, j=i-1, m_3=n_3, k_2=1, \\ 0 & \text{otherwise.} \end{cases}$$

$$M_2^{\prime(n_2,i,n_3,k_1:m_2,j,m_3,k_2)} = \begin{cases} \mu & n_2=0, 2\le i\le S, n_3=1, k_1=1; \\ & m_2=n_2, j=i-1, m_3=n_3, k_2=1, \\ 0 & \text{otherwise.} \end{cases}$$

$$
B_1^{\prime(n_2,i,n_3,k_1:m_2,j,m_3,k_2)} = \begin{cases}
\lambda & n_2 = 0, 1 \leq i \leq S, n_3 = i, k_1 = 1, 2; \\
 & m_2 = n_2 + 1, j = i, m_3 = n_3, k_2 = k_1, \\
\lambda & n_2 = 0, 2 \leq i \leq S, 1 \leq n_3 \leq i - 1, k_1 = 1, 2; \\
 & m_2 = n_2, j = i, m_3 = n_3 + 1, k_2 = k_1, \\
\lambda & 1 \leq n_2 \leq K - 1, 1 \leq i \leq S, n_3 = i, k_1 = 1, 2; \\
 & m_2 = n_2 + 1, j = i, m_3 = n_3, k_2 = k_1, \\
\mu & n_2 = 0, i = 1, n_3 = 1, k_1 = 2; \\
 & m_2 = n_2, j = i - 1, m_3 = n_3 - 1, k_2 = 0, \\
\mu & n_2 = 0, 2 \leq i \leq S, 2 \leq n_3 \leq i, k_1 = 2; \\
 & m_2 = n_2, j = i - 1, m_3 = n_3 - 1, k_2 = 2, \\
\mu & 1 \leq n_2 \leq K, i = 1, n_3 = i, k_1 = 2; \\
 & m_2 = n_2, j = i - 1, m_3 = n_3 - 1, k_2 = 0, \\
\mu & 1 \leq n_2 \leq K, 2 \leq i \leq S, n_3 = i, k_1 = 2; \\
 & m_2 = n_2, j = i - 1, m_3 = n_3 - 1, k_2 = 2, \\
(S - i)\beta & n_2 = 0, 1 \leq i \leq S - 1, 1 \leq n_3 \leq i, k_1 = 1, 2; \\
 & m_2 = n_2, j = i + 1, m_3 = n_3, k_2 = k_1, \\
(S - i)\beta & 1 \leq n_2 \leq K, 1 \leq i \leq S - 1, n_3 = i, k_1 = 1, 2; \\
 & m_2 = n_2 - 1, j = i + 1, m_3 = n_3 + 1, k_2 = k_1, \\
-(S\beta + \alpha) & 0 \leq n_2 \leq K, i = 0, n_3 = i, k_1 = 0; \\
 & m_2 = n_2, j = i, m_3 = n_3, k_2 = k_1, \\
-(\lambda + \mu + (S - i)\beta + \alpha) & n_2 = 0, 1 \leq i \leq S, 1 \leq n_3 \leq i, k_{=}1, 2; \\
 & m_2 = 0, j = i, m_3 = n_3, k_2 = k_1, \\
-(\lambda + \mu + (S - i)\beta + \alpha) & 1 \leq n_2 \leq K - 1, 1 \leq i \leq S, n_3 = i, k_1 = 1, 2; \\
 & m_2 = n_2, j = i, m_3 = n_3, k_2 = k_1, \\
-(\gamma\lambda + \mu + (S - i)\beta + \alpha) & n_2 = K, 1 \leq i \leq S, n_3 = i, k_1 = 1, 2; \\
 & m_2 = n_2, j = i, m_3 = n_3, k_2 = k_1, \\
0 & \text{otherwise.}
\end{cases}
$$

$$
B_1^{0(n_2,i,n_3,k_1:m_2,j,m_3,k_2)} = \begin{cases}
\lambda & n_2 = 0, 1 \leq i \leq S, n_3 = i, k_1 = 1, 2; \\
 & m_2 = n_2 + 1, j = i, m_3 = n_3, k_2 = k_1, \\
\lambda & n_2 = 0, 2 \leq i \leq S, 1 \leq n_3 \leq i - 1, k_1 = 1, 2; \\
 & m_2 = n_2, j = i, m_3 = n_3 + 1, k_2 = k_1, \\
\lambda & 1 \leq n_2 \leq K - 1, 1 \leq i \leq S, n_3 = i, k_1 = 1, 2; \\
 & m_2 = n_2 + 1, j = i, m_3 = n_3, k_2 = k_1, \\
\mu & n_2 = 0, i = 1, n_3 = 1, k_1 = 2; \\
 & m_2 = n_2, j = i - 1, m_3 = n_3 - 1, k_2 = 0, \\
\mu & n_2 = 0, 2 \leq i \leq S, 2 \leq n_3 \leq i, k_1 = 2; \\
 & m_2 = n_2, j = i - 1, m_3 = n_3 - 1, k_2 = 2, \\
\mu & 1 \leq n_2 \leq K, i = 1, n_3 = i, k_1 = 2; \\
 & m_2 = n_2, j = i - 1, m_3 = n_3 - 1, k_2 = 0, \\
\mu & 1 \leq n_2 \leq K, 2 \leq i \leq S, n_3 = i, k_1 = 2; \\
 & m_2 = n_2, j = i - 1, m_3 = n_3 - 1, k_2 = 2, \\
(S - i)\beta & n_2 = 0, 1 \leq i \leq S - 1, 1 \leq n_3 \leq i, k_1 = 1, 2; \\
 & m_2 = n_2, j = i + 1, m_3 = n_3, k_2 = k_1, \\
(S - i)\beta & 1 \leq n_2 \leq K, 1 \leq i \leq S - 1, n_3 = i, k_1 = 1, 2; \\
 & m_2 = n_2 - 1, j = i + 1, m_3 = n_3 + 1, k_2 = k_1, \\
-(S\beta + \alpha) & 0 \leq n_2 \leq K, i = 0, n_3 = i, k_1 = 0; \\
 & m_2 = n_2, j = i, m_3 = n_3, k_2 = k_1, \\
-(\lambda + \mu + (S - i)\beta + \alpha) & n_2 = 0, 1 \leq i \leq S, 1 \leq n_3 \leq i, k_{=}1, 2; \\
 & m_2 = 0, j = i, m_3 = n_3, k_2 = k_1, \\
-(\lambda + \mu + (S - i)\beta + \alpha) & 1 \leq n_2 \leq K - 1, 1 \leq i \leq S, n_3 = i, k_1 = 1, 2; \\
 & m_2 = n_2, j = i, m_3 = n_3, k_2 = k_1, \\
-(\mu + (S - i)\beta + \alpha) & n_2 = K, 1 \leq i \leq S, n_3 = i, k_1 = 1, 2; \\
 & m_2 = n_2, j = i, m_3 = n_3, k_2 = k_1, \\
0 & \text{otherwise.}
\end{cases}
$$

$$M_1^{'(n_2,i,n_3,k_1:m_2,j,m_3,k_2)} = \begin{cases} \mu & n_2 = 0, 1 \leq i \leq S, n_3 = 1, k_1 = 1; \\ & m_2 = n_2, j = i-1, m_3 = n_3 - 1, k_2 = 0, \\ \mu & n_2 = 0, 2 \leq i \leq S, 2 \leq n_3 \leq i, k_1 = 1; \\ & m_2 = n_2, j = i-1, m_3 = n_3 - 1, k_2 = 2, \\ \mu & 1 \leq n_2 \leq K, i = 1, n_3 = i, k_1 = 1; \\ & m_2 = n_2, j = i-1, m_3 = n_3 - 1, k_2 = 0, \\ \mu & 1 \leq n_2 \leq K, 2 \leq i \leq S, n_3 = i, k_1 = 1; \\ & m_2 = n_2, j = i-1, m_3 = n_3 - 1, k_2 = 2, \\ 0 & \text{otherwise.} \end{cases}$$

If p_k is the probability that absorption occurs with exactly k pooled customers getting service, then

$$p_k = \delta_{\boldsymbol{\mathcal{P}}_L}(-\boldsymbol{\mathcal{H}}^{-1}\boldsymbol{\mathcal{H}}_0)^k(-\boldsymbol{\mathcal{H}}_1^{-1}\boldsymbol{\mathcal{H}}), \quad k \geq 0$$

with $\delta_{\boldsymbol{\mathcal{P}}_L} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_{\boldsymbol{\mathcal{P}}_L})$ is a row vector of order $U_3 + \boldsymbol{\mathcal{P}}_L U_4$. Therefore the expected number of pooled customers getting service before realization of common life time is

$$E_{\boldsymbol{\mathcal{P}}_L}(N) = \sum_{k=0}^{\infty} k p_k$$

(see Krishnamoorthy et al. [22]).

6.2.4 Additional performance measures

1. Expected number of customers in the pool before realization of CLT

$$E_P(N) = \sum_{n_1=1}^{\infty} n_1 \mathbf{y}_{n_1} \mathbf{e}.$$

2. Expected number of customers in the waiting room before realization of CLT

$$E_W(N) = \sum_{n_1=0}^{\infty} \sum_{n_2=1}^{K} n_2 \Bigg(y_{n_1}(n_2, 0, 0, 0) + \sum_{i=1}^{S} \Big[y_{n_1}(n_2, i, i, 1)$$

$$+y_{n_1}(n_2,i,i,2)\Big]\Bigg).$$

3. Expected number of customers in the buffer before realization of CLT

$$E_B(N)=\sum_{n_1=0}^{\infty}\Bigg(\sum_{n_2=1}^{K}\sum_{i=1}^{S}i\Big[y_{n_1}(n_2,i,i,1)+y_{n_1}(n_2,i,i,2)\Big]$$

$$+\sum_{i=1}^{S}\sum_{n_3=1}^{i}n_3\Big[y_{n_1}(0,i,n_3,1)+y_{n_1}(0,i,n_3,2)\Big]\Bigg).$$

4. Expected number of items in the inventory before realization of CLT

$$E_I(N)=\sum_{n_1=0}^{\infty}\sum_{i=1}^{S}i\Bigg(\sum_{n_2=1}^{K}\Big[y_{n_1}(n_2,i,i,1)+y_{n_1}(n_2,i,i,2)\Big]$$

$$+\sum_{n_3=0}^{i}\Big[y_{n_1}(0,i,n_3,1)+y_{n_1}(0,i,n_3,2)\Big]\Bigg)+\sum_{i=1}^{S}iy_0(0,i,0,0).$$

5. Expected number of items in the inventory immediately on realization of CLT

$$E_I'(N)=\sum_{n_1=0}^{\infty}\sum_{i=1}^{S}i\frac{\alpha}{\alpha+\lambda+\mu+(S-i)\beta}\Bigg(\sum_{n_2=1}^{K}\Big[y_{n_1}(n_2,i,i,1)$$

$$+y_{n_1}(n_2,i,i,2)\Big]+\sum_{n_3=0}^{i}\Big[y_{n_1}(0,i,n_3,1)+y_{n_1}(0,i,n_3,2)\Big]\Bigg)$$

$$+\sum_{i=1}^{S}i\frac{\alpha}{\alpha+\lambda+(S-i)\beta}y_0(0,i,0,0).$$

6. Rate of addition to the pool is

$$\gamma\lambda\sum_{n_1=0}^{\infty}\sum_{i=1}^{S}\Big[y_{n_1}(K,i,i,1)+y_{n_1}(K,i,i,2)\Big].$$

7. The probability that a customer enters service immediately on arrival

$$\sum_{i=1}^{S} y_0(0,i,0,0).$$

8. The rate at which pooled customers are transferred to the buffer

$$E_{PB}(R) = \sum_{n_1=1}^{\infty}\left(\sum_{i=2}^{S}\mu\Big[y_{n_1}(0,i,1,1)+y_{n_1}(0,i,1,2)\Big]\right.$$
$$\left.+\sum_{n_2=0}^{K} S\beta y_{n_1}(n_2,0,0,0)\right)$$

9. The rate at which customers abandon the system on arrival

$$E_{WL}(R) = (1-\gamma)\lambda\sum_{n_1=0}^{\infty}\sum_{i=1}^{S}\Big[y_{n_1}(K,i,i,1)+y_{n_1}(K,i,i,2)\Big].$$

10. Expected cancellation rate

$$E_C(R) = \sum_{n_1=0}^{\infty}\sum_{i=1}^{S-1}(S-i)\beta\left(\sum_{n_2=1}^{K}\Big[y_{n_1}(n_2,i,i,1)+y_{n_1}(n_2,i,i,2)\Big]\right.$$
$$\left.+\sum_{n_3=1}^{i}\Big[y_{n_1}(0,i,n_3,1)+y_{n_1}(0,i,n_3,2)\Big]\right)+\sum_{n_1=0}^{\infty}\sum_{n_2=0}^{K}S\beta y_{n_1}(n_2,0,0,0)$$
$$+\sum_{i=1}^{S-1}(S-i)\beta y_0(0,i,0,0)$$

11. Expected inventory depletion rate

$$E_P(R) = \mu\sum_{n_1=0}^{\infty}\sum_{i=1}^{S}\left(\sum_{n_2=1}^{K}\Big[y_{n_1}(n_2,i,i,1)+y_{n_1}(n_2,i,i,2)\Big]\right.$$
$$\left.+\sum_{n_3=1}^{i}\Big[y_{n_1}(0,i,n_3,1)+y_{n_1}(0,i,n_3,2)\Big]\right).$$

12. Expected number of cancellations in a cycle

$$E_{NC} = \frac{1}{\alpha} E_C(R)$$

13. Expected number of purchases in a cycle

$$E_{NP} = \frac{1}{\alpha} E_P(R)$$

14. Expected number of transfers from the pool to the buffer

$$E_{PB}(N) = \frac{1}{\alpha} E_{PB}(R)$$

15. The probability that the system has S items in the inventory at the time of realization of CLT

$$P_{vacant} = \sum_{n_1=0}^{\infty} \left(\sum_{n_3=1}^{S} \left[y_{n_1}(0, S, n_3, 1) + y_{n_1}(0, S, n_3, 2) \right] \right.$$
$$\left. + \sum_{n_2=1}^{K} \left[y_{n_1}(n_2, S, S, 1) + y_{n_1}(n_2, S, S, 2) \right] \right) + y_0(0, S, 0, 0)$$

16. The probability that the system is left with no item in the inventory at the time of realization of CLT

$$P_{full} = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{K} y_{n_1}(n_2, 0, 0, 0)$$

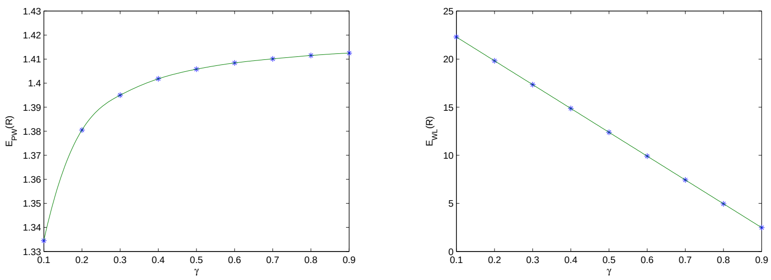
6.3 Numerical illustration

In this section we provide numerical illustration of the system performance with variation in values of underlying parameters.

Model 1

Effect of γ on $E_{PW}(R)$ and $E_{WL}(R)$

We consider the following values for the parameters $S = 12, K = 10, L = 6, \lambda = 20, \mu = 25, p = 0.75, \alpha = 0.25, \beta = 5, \eta = 5$. For this set of parameter values, Figure 6.1a shows the impact of the probability γ on measures $E_{PW}(R)$ and $E_{WL}(R)$. From Figure 6.1b, it is clear that $E_{PW}(R)$ is increasing and the loss rate $E_{WL}(R)$ is monotonically decreasing in γ. This is due to the fact that as γ increases inflow rate to the pool increases, thus the loss rate decreases. Also, as γ increases transfer rate from pool to waiting room increases. However, this increase is marginal because of the constrains in the transfer policy.



(a) Effect of γ on $E_{PW}(R)$ (b) Effect of γ on $E_{WL}(R)$

Figure 6.1: Effect of γ on $E_{PW}(R)$ and $E_{WL}(R)$

Effect of the arrival rate λ

From Table 6.1, we observe that an increase in the arrival rate makes a decrease in measures like expected number of items in the inventory before realization of common life time and expected number of items in the inventory immediately on realization of common life time. However, the expected number of

λ	$E_P(N)$	$E_W(N)$	$E_B(N)$	$E_I(N)$	$E'_I(N)$	$E_{PW}(R)$	E_{NP}	E_{NC}
15	60.1708	5.0662	0.6412	0.9744	0.0069	1.3788	7.3557	2.6579
20	74.3319	5.1698	0.6746	0.9534	0.0059	1.4108	7.5985	2.6662
25	88.5280	5.2447	0.7000	0.9397	0.0052	1.4321	7.8095	2.6717
30	102.7461	5.3015	0.7198	0.9301	0.0046	1.4470	7.9834	2.6756
35	116.9786	5.3460	0.7357	0.9230	0.0042	1.4578	8.1304	2.6784
40	131.2199	5.3819	0.7488	0.9176	0.0038	1.4660	8.2517	2.6806

Table 6.1: Effect of the arrival rate: $S = 8, K = 6, L = 4, \mu = 10, \eta = 5, \alpha = 0.25, \beta = 0.1, p = 0.75, \gamma = 0.75$

customers in the pool, waiting room and buffer, expected number of cancellations, expected number of purchases and rate of transfer from the pool to waiting room increase. These are on expected lines.

Effect of the service time parameter μ

Table 6.2 indicates that increase in μ makes expected number of customers in the pool, waiting room and buffer, expected number of items in the inventory before realization of common life time and expected number of items immediately on realization of common life time, all decrease. However, as μ increases, rate of transfer from pool to waiting room, expected number of purchases and expected number of cancellations increase: higher the common life realization time more the number of customers served out.

μ	$E_P(N)$	$E_W(N)$	$E_B(N)$	$E_I(N)$	$E'_I(N)$	$E_{PW}(N)$	E_{NP}	E_{NC}
15	74.2543	5.1679	0.4101	0.6893	0.0038	1.4461	7.4800	2.7719
20	74.2166	5.1666	0.2804	0.5599	0.0028	1.4576	7.5623	2.8237
25	74.1952	5.1658	0.2061	0.4858	0.0022	1.4619	7.8061	2.8533
30	74.1818	5.1651	0.1595	0.4398	0.0018	1.4636	8.1040	2.8719
35	74.1729	5.1647	0.1281	0.4081	0.0016	1.4644	8.3909	2.8844
40	74.1667	5.1644	0.1061	0.3862	0.0014	1.4649	8.6412	2.8931

Table 6.2: Effect of the service time parameter: $S = 8, K = 6, L = 4, \lambda = 30, \eta = 5, \alpha = 0.25, \beta = 0.1, p = 0.75, \gamma = 0.75$

Effect of the common life time parameter α

From Table 6.3, we observe that an increase in α results in a decrease in measures like expected number of customers in the pool and also in the waiting room, expected number of purchase, expected number of cancellation and rate of transfer from pool to waiting room. This is so since the mean value of common life time decreases with increase in value of α. However, the expected number of customers in the buffer, expected number of items in the inventory immediately on realization of common life time and expected number of items in the inventory before realization of common life time, all increase. These are also on expected lines.

α	$E_P(N)$	$E_W(N)$	$E_B(N)$	$E_I(N)$	$E'_I(N)$	$E_{PW}(N)$	E_{NP}	E_{NC}
0.1	204.8010	5.6348	0.3449	0.4618	0.0011	1.5978	12.8634	7.3807
0.2	95.9983	5.3198	0.5721	0.7985	0.0039	1.4703	8.5541	3.4469
0.3	59.9611	5.0246	0.7703	1.1000	0.0081	1.3541	6.9144	2.1491
0.4	42.1545	4.7481	0.9430	1.3697	0.0135	1.2481	5.9629	1.5094
0.5	31.6269	4.4891	1.0931	1.6112	0.0198	1.1513	5.3009	1.3231
0.6	24.7288	4.2463	1.2233	1.8273	0.0269	1.0629	4.7939	0.8859

Table 6.3: Effect of α: $S = 8, K = 6, L = 4, \lambda = 30, \mu = 10, \eta = 5, \beta = 0.1, p = 0.75, \gamma = 0.75$

Effect of the cancellation rate β

β	$E_P(N)$	$E_W(N)$	$E_B(N)$	$E_I(N)$	$E'_I(N)$	$E_{PW}(R)$	E_{NP}	E_{NC}
0.15	73.3620	5.1572	0.7203	0.9996	0.0062	0.5527	8.8455	3.9717
0.20	72.4069	5.1446	0.7682	1.0479	0.0064	0.7368	10.0703	5.2596
0.25	71.4672	5.1320	0.8182	1.0984	0.0067	0.9208	11.2715	6.5207
0.30	70.5440	5.1195	0.8705	1.1510	0.0070	1.1048	12.4478	7.7616
0.35	69.6382	5.1070	0.9250	1.2060	0.0073	1.2888	13.5977	8.9783
0.40	68.7507	5.0946	0.9817	1.2632	0.0076	1.4727	14.7199	10.1694

Table 6.4: Effect of β: $S = 8, K = 6, L = 4, \lambda = 30, \eta = 5, \alpha = 0.25, \mu = 10, p = 0.75, \gamma = 0.75$

Table 6.4 shows that the expected number of customers in the pool and that in the waiting room and rate of transfer from pool to waiting room decrease with increase in β value. Here expected number of customers in the buffer, expected number of purchase, expected number of cancellation, expected number of items in the inventory before realization of common life time and immediately on realization of common life time show a sharp upward trend. This is expected for large cancellation rates.

Effect of α, β on P_{full} and P_{vacant}

For $\beta = 0$, varying over α, we notice from Table 6.5 that, P_{full} decreases with increasing value of α – shorter the life time, lesser the chance for inventory being completely sold. Thus P_{vacant} increases with increasing value of α.

α	0.1	0.12	0.14	0.16	0.18	0.2	0.22	0.24	0.26
P_{full}	0.9398	0.9283	0.9169	0.9057	0.8947	0.8838	0.8730	0.8624	0.8520
P_{vacant}	0.0081	0.0097	0.0112	0.0128	0.0143	0.0158	0.0173	0.0188	0.0203

Table 6.5: Effect of α for $\beta = 0, S = 7, K = 5, L = 3, \lambda = 30, \eta = 5, \mu = 20, p = 0.75, \gamma = 0.75$

Table 6.6 shows the effect of β for fixed α value. It tells that higher cancellation rate results in reduction in probability of system being full (in the context of the bus / train / air plane, leaving with all seats occupied). However, the extreme case of P_{vacant} does not increase with increase in value of β. Rather P_{vacant} stays constant. This could be attributed to high arrival rate ($\lambda = 30$) and moderately high service rate ($\mu = 20$); cancelled items are resold before common life time realization.

β	0.1	0.12	0.14	0.16	0.18	0.2	0.22	0.24	0.26
P_{full}	0.9066	0.9000	0.8935	0.8869	0.8804	0.8739	0.8674	0.8609	0.8545
P_{vacant}	0.0082	0.0082	0.0082	0.0082	0.0082	0.0082	0.0082	0.0082	0.0082

Table 6.6: Effect of β for $\alpha = 0.1, S = 7, K = 5, L = 3, \lambda = 30, \eta = 5, \mu = 20, p = 0.75, \gamma = 0.75$

Model 2

Effect of the arrival rate λ

Table 6.7 indicates that the increase in λ makes a decrease in measures like expected number of purchases, expected number of items in the inventory before realization and also immediately on realization of common life time. As λ increases there is a moderate increase in the expected number of cancellations, expected number of customers in the pool and waiting room. The column on $E_{PB}(R)$ shows increase in value with λ increasing which could be attributed to increase in number of customers in the pool. There are some surprises in the column corresponding to E_{NP}. It shows an increasing trend with increase in value of λ upto a certain level and then it starts decreasing with further increase in value of λ. Still surprising is that the expected number of cancellations (E_{NC}) monotonically increase with λ. We do not have an explanation for these strange behaviour of E_{NP} and E_{NC}. However, in Model 1 this trend is not seen.

λ	$E_P(N)$	$E_W(N)$	$E_B(N)$	$E_I(N)$	$E'_I(N)$	$E_{PB}(R)$	E_{NP}	E_{NC}
5	0.0001	0.1987	0.3052	1.3901	0.0522	0.0001	7.3867	2.1106
10	0.0109	0.7014	0.3044	0.8075	0.0140	0.0045	8.1730	2.3437
15	0.1044	1.4440	0.2974	0.6139	0.0068	0.0333	8.2475	2.4211
20	0.4091	2.2433	0.2928	0.5250	0.0043	0.1002	7.8961	2.4567
25	1.0033	2.9272	0.2910	0.4772	0.0032	0.1908	7.2815	2.4758
30	1.8779	3.4396	0.2911	0.4486	0.0025	0.2819	6.5808	2.4872
35	2.9776	3.7991	0.2924	0.4301	0.0021	0.3597	5.9085	2.4946
40	4.2413	4.0449	0.2943	0.4174	0.0019	0.4205	5.3123	2.4997

Table 6.7: Effect of the arrival rate λ: $S = 7, K = 5, \mu = 20, \eta = 5, \alpha = 0.25, \beta = 0.1, \gamma = 0.75$

Effect of the service time parameter μ

From Table 6.8 we observe that as μ increases there is a moderate decrease in expected number of customers in the pool, waiting room and buffer, expected number of items in the inventory before realization of common life time and immediately on realization of common life time. But as μ increases there is a sharp increase in expected number of purchases and expected number of cancellations. $E_{PB}(R)$ decreases with increase in value of μ (see column $E_{PB}(R)$ of Table 6.8). The reason for this is the increase in probability of the server becoming idle with positive inventory in the system.

μ	$E_P(N)$	$E_W(N)$	$E_B(N)$	$E_I(N)$	$E'_I(N)$	$E_{PB}(R)$	E_{NP}	E_{NC}
22	1.3822	3.2058	0.2655	0.4219	0.0024	0.2367	7.0040	2.4979
24	1.0221	2.9720	0.2443	0.4002	0.0022	0.1962	7.3616	2.5066
26	0.7597	2.7450	0.2266	0.3823	0.0021	0.1610	7.6591	2.5137
28	0.5678	2.5292	0.2115	0.3674	0.0020	0.1312	7.9034	2.5197
30	0.4268	2.3277	0.1985	0.3549	0.0020	0.1065	8.1022	2.5247
32	0.3229	2.1417	0.1871	0.3443	0.0019	0.0862	8.2690	2.5290

Table 6.8: Effect of μ: $S = 7, K = 5, \lambda = 30, \eta = 5, \alpha = 0.25, \beta = 0.1, \gamma = 0.75$

Effect of common life time parameter α

From Table 6.9 we observe that as α increases there is high decrease in expected number of customers in the pool and that in the waiting room, rate of transfer from pool to buffer, expected number of cancellations and expected number of purchases. However, expected number of customers in the buffer, expected number of items in the inventory before and also immediately on realization of common life time, show a sharper upward trend. This is a consequence of higher rate of realization of CLT.

α	$E_P(N)$	$E_W(N)$	$E_B(N)$	$E_I(N)$	$E'_I(N)$	$E_{PB}(R)$	E_{NP}	E_{NC}
0.1	2.8942	3.9126	0.1443	0.2108	0.0005	0.3610	8.4422	6.6519
0.2	2.0721	3.5680	0.2441	0.3723	0.0017	0.3009	6.9775	3.1792
0.3	1.7348	3.3261	0.3361	0.5221	0.0035	0.2662	6.2731	2.0272
0.4	1.5289	3.1281	0.4208	0.6610	0.0060	0.2409	5.8045	1.4551
0.5	1.3794	2.9555	0.4989	0.7900	0.0090	0.2204	5.4446	1.1147
0.6	1.2607	2.8005	0.5707	0.9097	0.0124	0.2029	5.1467	0.8901

Table 6.9: Effect of α: $S = 7, K = 5, \lambda = 30, \mu = 20, \eta = 5, \beta = 0.1, \gamma = 0.75$

Effect of the cancellation rate β

Table 6.10 indicates that an increase in β makes expected number of customers in the pool, waiting room and buffer, expected number of items in the inventory before realization of common life time, rate of transfer from pool to buffer, expected number of cancellations and expected number of purchases, all increase. The high rate of arrival of customers results in the waiting room always occupied. Consequently pooled customers get very little access to the buffer as per the transfer policy.

β	$E_P(N)$	$E_W(N)$	$E_B(N)$	$E_I(N)$	$E_I'(N)$	$E_{PB}(R)$	E_{NP}	E_{NC}
0.05	1.6070	3.3412	0.2738	0.4302	0.0025	0.1326	6.0277	1.2473
0.06	1.6617	3.3636	0.2772	0.4339	0.0025	0.1616	6.1444	1.4959
0.07	1.7158	3.3845	0.2807	0.4375	0.0025	0.1909	6.2579	1.7442
0.08	1.7700	3.4040	0.2841	0.4412	0.0025	0.2208	6.3684	1.9921
0.09	1.8240	3.4223	0.2876	0.4449	0.0025	0.2511	6.4759	2.2398
0.10	1.8779	3.4396	0.2911	0.4486	0.0025	0.2819	6.5808	2.4872

Table 6.10: Effect of β: $S = 7, K = 5, \lambda = 30, \eta = 5, \alpha = 0.25, \mu = 20, \gamma = 0.75$

Effect of α, β on P_{full} and P_{vacant}

α	0.1	0.12	0.14	0.16	0.18	0.2	0.22	0.24	0.26
P_{full}	0.9398	0.9283	0.9169	0.9057	0.8947	0.8838	0.8730	0.8624	0.8520
P_{vacant}	0.0114	0.0136	0.0158	0.0179	0.0201	0.0222	0.0243	0.0264	0.0284

Table 6.11: Effect of α for $\beta = 0, S = 7, K = 5, \lambda = 30, \eta = 5, \mu = 20, \gamma = 0.75$

β	0.1	0.12	0.14	0.16	0.18	0.2	0.22	0.24	0.26
P_{full}	0.9057	0.8990	0.8923	0.8857	0.8790	0.8724	0.8658	0.8592	0.8526
P_{vacant}	0.00114	0.0114	0.0114	0.0115	0.0115	0.0115	0.0115	0.0115	0.0116

Table 6.12: Effect of β for $\alpha = 0.1, S = 7, K = 5, \lambda = 30, \eta = 5, \mu = 20, \gamma = 0.75$

The interpretation of results in Tables 6.11 and 6.12 are on the same lines as in Model 1 (see Tables 6.5, 6.6).

6.3.1 Cost analysis

Based on the above performance measures we construct a cost function for checking the optimality of the waiting room capacity K. It may be noted that we cannot arrive at an analytical form for the cost function since system state probabilities are not available in compact form.

We define a profit/revenue function as $\mathcal{F}(K,S)$ as

$$\mathcal{F}(K,S) = \mathcal{C}_1 E_C(R) + \mathcal{C}_2 E_P(R) - \mathcal{C}_3 E_B(N) - \mathcal{C}_4 E_W(N) - \mathcal{C}_5 E_P(N) - \mathcal{C}_6 E_I(N)$$

where

$\mathcal{C}_1$ =Revenue to the system due to cancellation of an inventory purchased

$\mathcal{C}_2$ =Revenue to the system due to unit purchase of item in the inventory

$\mathcal{C}_3$ =Holding cost of customer per unit per unit time in the buffer

$\mathcal{C}_4$ =Holding cost of customer per unit per unit time in the waiting room

$\mathcal{C}_5$ =Holding cost of customer per unit per unit time in the pool

$\mathcal{C}_6$ =Holding cost per unit time per item in the inventory

In order to study the variation in different parameters on profit function we first fix the costs $\mathcal{C}_1 = \$50, \mathcal{C}_2 = \$200, \mathcal{C}_3 = \$4, \mathcal{C}_4 = \$7, \mathcal{C}_5 = \$2, \mathcal{C}_6 = \$10$.

Effect of variation in S and K in Model 1

K \ S	6	7	8	9	10
5	121.2477	179.2218	237.3408	295.4081	353.2731
6	106.9024	163.3576	220.0913	276.9073	333.6499
7	94.3398	149.3635	204.7516	260.3190	315.9145
8	83.2438	136.9571	191.0771	245.4376	299.8984
9	73.3350	125.8753	178.8292	232.0525	285.4205
10	64.3745	115.8827	167.7849	219.9586	272.2969

Table 6.13: Effect of S and K on expected revenue

We assign the following values to the parameters: $\lambda = 30, \mu = 20, \beta = 0.1, \eta = 5, \alpha = 0.25, p = 0.75, \gamma = 0.75, L = 3$. For different values of S and

K, the expected profit is calculated and presented in Table 6.13. This table shows that the profit function decreases when K increases and increases with increasing value of S.

Effect of variation in p, γ on expected revenue in Model 1

We assign the following values to the parameters: $S = 8, K = 6, L = 3, \lambda = 30, \mu = 20, \beta = 0.1, \eta = 5, \alpha = 0.25, p = 0.75, \gamma = 0.75$. In Fig. 6.2, each curve is drawn keeping the other parameters fixed; these graphs show that there is decrease, though marginal, in revenue with increase in value of p. With γ increasing $\mathcal{F}(K, S)$ shows an increasing trend.

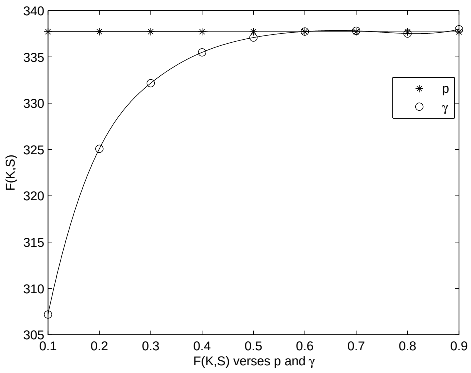


Figure 6.2: Effect of p and γ on expected revenue

Effect of variation in S and K in Model 2

We assign the following values to the parameters: $\lambda = 30, \mu = 20, \beta = 0.1, \eta = 5, \alpha = 0.25, \gamma = 0.75$. For different values of S and K, the expected revenue is calculated (see Table 6.14). This table shows that the profit function increases when S and K increase.

K \ S	3	4	5	6	7
5	157.9565	204.3360	247.6024	288.2351	326.6477
6	163.5330	212.5247	258.3252	301.3083	341.8402
7	167.6997	218.9368	267.0368	312.2501	354.8707
8	170.7896	223.9125	274.0451	321.3158	365.9338
9	173.0665	227.7434	279.6345	328.7579	375.2396
10	174.7352	230.6729	284.0582	334.8184	383.0017

Table 6.14: Effect of S and K on expected revenue

A comparison between Model 1 and 2 is in order. A look at the values in Tables 6.13 and 6.14 indicate that the expected revenue is lower for Model 1. This is due to the transfer policy. In Model 1 the expected number of customers in the pool is relatively larger than that in Model 2. This results in higher holding cost of customers in the former and hence a reduced revenue from that results (see columns 2 and 3 of Table 6.13 and last two columns of Table 6.14). It is interesting to note that in Model 1, $\mathcal{F}(K, S)$ decreases with increase in value of K; however, this trend is reversed in Model 2. These are consequences of the transfer policies adopted: Model 1: based on number of customers in the waiting room and Model 2: transfer from the pool only when server is idle with positive level of inventory on hand. With $\beta = 0$, P_{full} have the same values for different α values for both models; however, P_{vacant} does not show any similarity in behaviour.

Chapter 7

On a queueing-inventory system with postponed work, reservation, cancellation and common life time

In the present chapter we consider a single server queueing-inventory system in which items in the inventory have a random common life time. On realization of common life time, customers are flushed out from the finite buffer and waiting room, but not from the pool of postponed work. Subsequently the inventory reaches its maximum S through an instantaneous (zero lead time) replenishment for the next cycle. Through cancellation of purchases, inventory gets added until their expiry time, where inter cancellation time

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Some results of this chapter are included in the following paper.
A. Krishnamoorthy, Dhanya Shajin and B. Lakshmy : **On a queueing-inventory system with postponed work, reservation, cancellation and common life time** (communicated).

follows exponential distribution. Customers arrive according to a Poisson process and service time is exponentially distributed. On arrival if a customer finds the server busy, then he joins a buffer of varying size. If there is no inventory, the arriving customer first try to access a finite waiting room of capacity K. If that is found full, he joins a pool of infinite capacity with probability γ $(0 < \gamma < 1)$; else it is lost to the system forever. When, at a service completion epoch the waiting room size drops to a preassigned level $L-1$ $(1 < L < K)$ or below, a customer is transferred from pool to waiting room with probability p $(0 < p < 1)$ and positioned as the last among the waiting customers. If at a departure epoch the waiting room turns out to be empty and there is at least one customer in the pool, then the one ahead of all waiting in the pool gets transferred to the waiting room with probability one.

7.1 Mathematical formulation

We have a single commodity inventory system with S items at the beginning of a cycle. Customers arrive according to a Poisson process of rate λ, each demanding exactly one unit of the item. To deliver one unit of the item to the customer in service, it requires an exponentially distributed time with parameter μ. The inventoried items have a CLT which means that they all perish together on realization of this time. We assume that this CLT is exponentially distributed with parameter α. On realization of CLT, the inventory reaches its maximum level S (for identification purpose denoted by S^*) for the next cycle to commence, through an instantaneous replenishment. A buffer of varying size, depending on the number of items in the inventory is available at the service counter. We call it varying size because at most as many customers as the number of items in the inventory are allowed to be in this buffer. In addition, as in chapter 5 and 6, the possibility of cancellation of purchased item (return / cancellation of the item with a penalty) is permitted here also. Inter cancellation time follows exponential distribution with parameter $i\beta$, when there are $(S-i)$ items present in the inventory. Next in order is a finite waiting space of capacity K. When the buffer is full, further arrivals wait in this room; as and when inventory level in the buffer goes one step above (due to cancellation), the head in the waiting room moves to the buffer and positions himself as the last there. When the waiting room is also full, further arrivals are directed to a pool (of customers) having infinite capacity. Whereas customers join with probability one in the buffer and waiting room whenever there is a vacancy, it is not the case with the pool. An arrival, finding waiting room also full, joins the pool with probability $\gamma(0 < \gamma < 1)$ or balks with complementary probability. When, at a departure epoch the number of customers in the waiting room drops to a preassigned level $L-1, (1 < L < K)$ or below, a customer is transferred from the pool to the waiting room with probability p $(0 < p < 1)$ and positioned as last among the waiting customers. If at a service completion epoch the waiting room turns out to be empty and there is at least one customer in the pool, the one ahead of all waiting in the pool gets transferred (with probability one) to the waiting room. Transfer of customers from a pool of postponed work is introduced and analyzed in Deepak et al. [13]. Customers are flushed out from the finite buffer and waiting room, but not from the pool when the CLT is realized. Subsequently, $S+K$ pooled customers, if that many are available or all in the pool, whichever is less, are immediately transferred to the buffer and waiting room.

In the sequel we use the following notations:

$\boldsymbol{\mathcal{L}}(t)$	Level of the system at time t
$\boldsymbol{\mathcal{N}}_1(t)$	Number of customers within each level at time t
$\boldsymbol{\mathcal{N}}_2(t)$	Number of customers in the waiting room at time t
$\boldsymbol{\mathcal{N}}_3(t)$	Number of customers in the buffer at time t
$\boldsymbol{\mathcal{I}}(t)$	Number of items in the inventory at time t

Cycle : The time duration from the epoch at which we start with maximum inventory level S at a replenishment epoch, to the moment when the common life time is realized

U_1 $= (S+1)(S+2)/2 + K(S+1) + K + S + 1$

U_2 $= (S+K)(K(S+1)+1)$

By the above assumptions $\Omega = \{(\boldsymbol{\mathcal{L}}(t), \boldsymbol{\mathcal{N}}_1(t), \boldsymbol{\mathcal{N}}_2(t), \boldsymbol{\mathcal{I}}(t), \boldsymbol{\mathcal{N}}_3(t)), t \geq 0\}$ is a $CTMC$ which is **not** a $LIQBD$ when $\boldsymbol{\mathcal{L}}(t)$ is the number of customers in the pool. We redefine the state space to make the Markov chain a QBD. The level $\boldsymbol{\mathcal{L}}(t) = \ell, \ell \geq 1$, of the system at time t, represent the pool having $n = (\ell - 1)(S+K) + a; a = 1, 2, ..., S+K$ customers. That is, the level $\ell = 1$ means the pool contains $n = 1, 2, ..., S+K$ customers, level $\ell = 2$ represents the pool as having $n = S+K+1$ to $n = 2(S+K)$ customers and so on. Level $\ell = 0$ means no customer in the pool. Thus the state space

$$\{\Delta\} \bigcup \{(0,0,0,i,n_3)\,; 0 \leq i \leq S, 0 \leq n_3 \leq i\}$$

$$\bigcup \{(0,0,n_2,i,n_3)\,; 1 \leq n_2 \leq K, 0 \leq i \leq S, n_3 = i\}$$

$$\bigcup \{(\ell,n_1,n_2,i,n_3)\,; \ell \geq 1, 1 \leq n_1 \leq S+K, 1 \leq n_2 \leq K, 0 \leq i \leq S, n_3 = i\}$$

where

$$\{\Delta\} = \{(0,0,0,S^*,n_3); 0 \leq n_3 \leq S\} \bigcup \{(0,0,n_2,S^*,S^*); 1 \leq n_2 \leq K\}$$

$$\bigcup \{(\ell,n_1,K,S^*,S^*); \ell \geq 1, 1 \leq n_1 \leq (S+K)\}$$

denotes the temporary absorbing state. Thus the infinitesimal generator $\boldsymbol{\mathcal{Q}}$ is of the form

$$\boldsymbol{\mathcal{Q}} = \begin{bmatrix} A_{00} & A_{01} & & & \\ A_{10} & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}.$$

The matrices A_0, A_1, A_2 are square matrices of the same order U_2 with entries in A_0 representing transition rates from level ℓ to $\ell+1, \ell \geq 1$, A_1 represents the transition rates within the level $\ell, \ell \geq 1$ and A_2 contains transition rates from level ℓ to $\ell-1, \ell \geq 2$. Dimension of matrices A_{00}, A_{01}, A_{10} are $U_1 \times U_1, U_1 \times U_2, U_2 \times U_1$ respectively.

Define non-diagonal entries of $A_{k(n_1,n_2,i,n_3)}^{(m_1,m_2,j,m_3)}$, $k = 00, 01, 10, 0, 1, 2$ as the transition rates from $(n_1, n_2, i, n_3) \to (m_1, m_2, j, m_3)$ where n_1, m_1 represent the number of customers within the level, n_2, m_2 represent the number of customers in the waiting room, i, j represent the number of items in the inventory and n_3, m_3 represent the number of customers in the buffer. These transition rates are

$$A_{0(n_1,n_2,i,n_3)}^{(m_1,m_2,j,m_3)} = \begin{cases} \gamma\lambda & n_1 = S+K, n_2 = K, 0 \leq i \leq S, n_3 = i; \\ & m_1 = 1, m_2 = K, j = i, m_3 = j \\ \gamma\lambda & n_1 = S+K, n_2 = K, i = S^*, n_3 = S; \\ & m_1 = 1, m_2 = K, j = S, m_3 = S \\ 0 & \text{otherwise.} \end{cases}$$

$$A_{01(n_1,n_2,i,n_3)}^{(m_1,m_2,j,m_3)} = \begin{cases} \gamma\lambda & n_1 = 0, n_2 = K, 0 \leq i \leq S, n_3 = i; \\ & m_1 = n_1+1, m_2 = n_2, j = i, m_3 = n_3 \\ \gamma\lambda & n_1 = 0, n_2 = K, i = S^*, n_3 = S; \\ & m_1 = n_1+1, m_2 = n_2, j = S, m_3 = S \\ 0 & \text{otherwise.} \end{cases}$$

$$A_{10(n_1,n_2,i,n_3)}^{(m_1,m_2,j,m_3)} = \begin{cases} \alpha & 1 \leq n_1 \leq S, 0 \leq n_2 \leq K, 0 \leq i \leq S, n_3 = i; \\ & m_1 = 0, m_2 = 0, j = S^*, m_3 = n_1 \\ \alpha & S+1 \leq n_1 \leq S+K, 0 \leq n_2 \leq K, 0 \leq i \leq S, n_3 = i; \\ & m_1 = 0, m_2 = n_1 - S, j = S^*, m_3 = S \\ (S-i)\beta & n_1 = 1, n_2 = 1, 0 \leq i \leq S-1, n_3 = i; \\ & m_1 = 0, m_2 = n_2, j = i+1, m_3 = j \\ p(S-i)\beta & n_1 = 1, 2 \leq n_2 \leq L, 0 \leq i \leq S-1, n_3 = i; \\ & m_1 = 0, m_2 = n_2, j = i+1, m_3 = j \\ 0 & \text{otherwise.} \end{cases}$$

$$A_{2(n_1,n_2,i,n_3)}^{(m_1,m_2,j,m_3)} = \begin{cases} \alpha & 1 \le n_1 \le S+K, 0 \le n_2 \le K, 0 \le i \le S, n_3 = i; \\ & m_1 = n_1, m_2 = K, j = S^*, m_3 = S \\ (S-i)\beta & n_1 = 1, n_2 = 1, 0 \le i \le S-1, n_3 = i; \\ & m_1 = S+K, m_2 = n_2, j = i+1, m_3 = j \\ p(S-i)\beta & n_1 = 1, 2 \le n_2 \le L, 0 \le i \le S-1, n_3 = i; \\ & m_1 = S+K, m_2 = n_2, j = i+1, m_3 = j \\ 0 & \text{otherwise.} \end{cases}$$

$$A_{00(n_1,n_2,i,n_3)}^{(m_1,m_2,j,m_3)} = \begin{cases} \alpha & n_1 = 0, 1 \le n_2 \le K, 0 \le i \le S, n_3 = i; \\ & m_1 = 0, m_2 = 0, j = S^*, m_3 = 0 \\ \alpha & n_1 = 0, n_2 = 0, 0 \le i \le S, 0 \le n_3 \le i; \\ & m_1 = 0, m_2 = 0, j = S^*, m_3 = 0 \\ \lambda & n_1 = 0, 0 \le n_2 \le K, 0 \le i \le S, n_3 = i; \\ & m_1 = 0, m_2 = n_2+1, j = i, m_3 = n_3 \\ \lambda & n_1 = 0, n_2 = 0, 1 \le i \le S, 0 \le n_3 \le i-1; \\ & m_1 = 0, m_2 = n_2, j = i, m_3 = n_3+1 \\ \mu & n_1 = 0, 1 \le n_2 \le K, 1 \le i \le S, n_3 = i; \\ & m_1 = 0, m_2 = n_2, j = i-1, m_3 = n_3-1 \\ \mu & n_1 = 0, n_2 = 0, 1 \le i \le S, 1 \le n_3 \le i; \\ & m_1 = 0, m_2 = n_2, j = i-1, m_3 = n_3-1 \\ (S-i)\beta & n_1 = 0, 1 \le n_2 \le K, 0 \le i \le S-1, n_3 = i; \\ & m_1 = 0, m_2 = n_2-1, j = i+1, m_3 = n_3+1 \\ (S-i)\beta & n_1 = 0, n_2 = 0, 0 \le i \le S-1, 0 \le n_3 \le i; \\ & m_1 = 0, m_2 = 0, j = i+1, m_3 = n_3 \\ \lambda & n_1 = 0, n_2 = 0, i = S^*, 0 \le n_3 \le S-1; \\ & m_1 = 0, m_2 = 0, j = S, m_3 = n_3+1 \\ \lambda & n_1 = 0, 0 \le n_2 \le K-1, i = S^*, n_3 = S; \\ & m_1 = 0, m_2 = n_2+1, j = S, m_3 = S \\ \mu & n_1 = 0, n_2 = 0, i = S^*, 1 \le n_3 \le S; \\ & m_1 = 0, m_2 = 0, j = S-1, m_3 = n_3-1 \\ \mu & n_1 = 0, 1 \le n_2 \le K, i = S^*, n_3 = S; \\ & m_1 = 0, m_2 = n_2, j = S-1, m_3 = S-1 \\ 0 & \text{otherwise.} \end{cases}$$

$$A_{1(n_1,n_2,i,n_3)}^{(m_1,m_2,j,m_3)} = \begin{cases} \gamma\lambda & 1 \leq n_1 \leq S+K-1, n_2 = K, 0 \leq i \leq S, n_3 = i; \\ & m_1 = n_1 + 1, m_2 = K, j = i, m_3 = j \\ \gamma\lambda & 1 \leq n_1 \leq S+K-1, n_2 = K, i = S^*, n_3 = S; \\ & m_1 = n_1 + 1, m_2 = K, j = S, m_3 = j \\ (S-i)\beta & 2 \leq n_1 \leq S+K, n_2 = 1, 0 \leq i \leq S, n_3 = i; \\ & m_1 = n_1 - 1, m_2 = n_2, j = i+1, m_3 = j \\ p(S-i)\beta & 2 \leq n_1 \leq S+K, 2 \leq n_2 \leq L, 0 \leq i \leq S, n_3 = i; \\ & m_1 = n_1 - 1, m_2 = n_2, j = i+1, m_3 = j \\ \lambda & 1 \leq n_1 \leq S+K, 1 \leq n_2 \leq K-1, 0 \leq i \leq S, n_3 = i; \\ & m_1 = n_1, m_2 = n_2 + 1, j = i, m_3 = n_3 \\ \mu & 1 \leq n_1 \leq S+K, 1 \leq n_2 \leq K, 1 \leq i \leq S, n_3 = i; \\ & m_1 = n_1, m_2 = n_2, j = i-1, m_3 = n_3 - 1 \\ \mu & 1 \leq n_1 \leq S+K, n_2 = K, i = S^*, n_3 = S; \\ & m_1 = n_1, m_2 = n_2, j = S-1, m_3 = S-1 \\ (1-p)(S-i)\beta & 1 \leq n_1 \leq S+K, 2 \leq n_2 \leq L, 0 \leq i \leq S-1, n_3 = i; \\ & m_1 = n_1, m_2 = n_2 - 1, j = i+1, m_3 = n_3 + 1 \\ (S-i)\beta & 1 \leq n_1 \leq S+K, L+1 \leq n_2 \leq K, 0 \leq i \leq S-1, n_3 = i; \\ & m_1 = n_1, m_2 = n_2 - 1, j = i+1, m_3 = n_3 + 1 \\ 0 & \text{otherwise.} \end{cases}$$

The diagonal entries are determined by the fact that each row sum is zero.

7.2 Steady-state analysis

In this section, we perform the steady-state analysis of the queueing-inventory model under study by first establishing the stability condition of the queueing system.

7.2.1 Stability condition

To establish the stability condition, we consider the Markov chain $\{(\boldsymbol{\mathcal{N}}_1(t), \boldsymbol{\mathcal{N}}_2(t), \boldsymbol{\mathcal{I}}(t), \boldsymbol{\mathcal{N}}_3(t)), t \geq 0\}$ on the finite state space $\{(n_1, n_2, i, n_3), 1 \leq n_1 \leq S+K, 1 \leq n_2 \leq K, 0 \leq i \leq S, n_3 = i\} \bigcup \{(n_1, K, S^*, S), 1 \leq n_1 \leq S+K\}$.

Let $\boldsymbol{\pi} = (\boldsymbol{\pi}_1, \boldsymbol{\pi}_2, ..., \boldsymbol{\pi}_{S+K})$ be the steady-state probability vector of this Markov chain. Its infinitesimal generator is

$$A (= A_0 + A_1 + A_2) = \begin{bmatrix} B_1 & B_0 & & & & B_2 \\ B_2 & B_1 & B_0 & & & \\ & \ddots & \ddots & \ddots & & \\ & & & B_2 & B_1 & B_0 \\ B_0 & & & & B_2 & B_1 \end{bmatrix}$$

where B_0, B_1, B_2 are square matrices of order $K(S+1)+1$.

Define $B_k^{(n_2,i,n_3:m_2,j,m_3)}$, $k = 0,1,2$ as the transition rates from $(n_2,i,n_3) \to (m_2,j,m_3)$ where n_2, m_2 represent the number of customers in the waiting room, i, j represent the number of items in the inventory and n_3, m_3 represent the number of customers in the buffer. These transition rates are

$$B_1^{(n_2,i,n_3:m_2,j,m_3)} = \begin{cases} \lambda & 1 \le n_2 \le K-1, 0 \le i \le S, n_3 = i; \\ & m_2 = n_2+1, j = i, m_3 = n_3 \\ \mu & 1 \le n_2 \le K, 1 \le i \le S, n_3 = i; \\ & m_2 = n_2, j = i-1, m_3 = n_3-1 \\ \mu & n_2 = K, i = S^*, n_3 = S; \\ & m_2 = n_2, j = S-1, m_3 = S-1 \\ (1-p)(S-i)\beta & 2 \le n_2 \le L, 0 \le i \le S-1, n_3 = i; \\ & m_2 = n_2-1, j = i+1, m_3 = n_3+1 \\ (S-i)\beta & L+1 \le n_2 \le K, 0 \le i \le S-1, n_3 = i; \\ & m_2 = n_2-1, j = i+1, m_3 = n_3+1 \\ \alpha & 1 \le n_2 \le K, 0 \le i \le S, n_3 = i; \\ & m_2 = K, j = S^*, m_3 = S \\ -(\lambda+\alpha+S\beta) & 1 \le n_2 \le K-1, i = 0, n_3 = 0; \\ & m_2 = n_2, j = i, m_3 = n_3 \\ -(\gamma\lambda+\alpha+S\beta) & n_2 = K, i = 0, n_3 = 0; \\ & m_2 = n_2, j = i, m_3 = n_3 \\ -(\lambda+\alpha+\mu+(S-i)\beta) & 1 \le n_2 \le K-1, 1 \le i \le S, n_3 = i; \\ & m_2 = n_2, j = i, m_3 = n_3 \\ -(\gamma\lambda+\alpha+\mu+(S-i)\beta) & n_2 = K, 1 \le i \le S, n_3 = i; \\ & m_2 = n_2, j = i, m_3 = n_3 \\ -(\gamma\lambda+\mu) & n_2 = K, i = S^*, n_3 = S; \\ & m_2 = n_2, j = i, m_3 = n_3 \\ 0 & \text{otherwise.} \end{cases}$$

$$B_0^{(n_2,i,n_3:m_2,j,m_3)} = \begin{cases} \gamma\lambda & n_2 = K, 0 \le i \le S, n_3 = i; \\ & m_2 = K, j = i, m_3 = j \\ \gamma\lambda & n_2 = K, i = S^*, n_3 = S; \\ & m_2 = K, j = S, m_3 = S \\ 0 & \text{otherwise.} \end{cases}$$

$$B_2^{(n_2,i,n_3:m_2,j,m_3)} = \begin{cases} (S-i)\beta & n_2 = 1, 0 \le i \le S-1, n_3 = i; \\ & m_2 = n_2, j = i+1, m_3 = n_3+1 \\ p(S-i)\beta & 2 \le n_2 \le L, 0 \le i \le S-1, n_3 = i; \\ & m_2 = n_2, j = i+1, m_3 = n_3+1 \\ 0 & \text{otherwise.} \end{cases}$$

This finite state space Markov chain with infinitesimal generator A, has steady state probability given by

$$\boldsymbol{\pi} A = 0, \qquad \boldsymbol{\pi}\mathbf{e} = 1. \tag{7.1}$$

Now the original Markov chain of the whole system is stable if and only if the left drift rate is higher than that to the right.

That is,

$$\boldsymbol{\pi} A_0 \mathbf{e} < \boldsymbol{\pi} A_2 \mathbf{e} \tag{7.2}$$

where

$$A_0 = \begin{bmatrix} 0 & 0 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & 0 \\ B_0 & 0 & \dots & 0 \end{bmatrix}, A_2 = \begin{bmatrix} B_2' & 0 & \cdots & B_2 \\ 0 & B_2' & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & B_2' \end{bmatrix}$$

with

$$B_2^{\prime(n_2,i,n_3:m_2,j,m_3)} = \begin{cases} \alpha & 1 \leq n_2 \leq K, 0 \leq i \leq S, n_3 = i; \\ & m_2 = K, j = S^*, m_3 = S \\ 0 & \text{otherwise.} \end{cases}$$

Partition $\boldsymbol{\pi}_{n_1}$ for $1 \leq n_1 \leq S+K$ as

$$\boldsymbol{\pi}_{n_1} = (\pi_{n_1}(n_2, i, n_3); 1 \leq n_2 \leq K, 0 \leq i \leq S, n_3 = i) \bigcup (\pi_{n_1}(K, S^*, S)).$$

From equation (7.1) we get

$$\begin{aligned} \boldsymbol{\pi}_1 B_1 + \boldsymbol{\pi}_2 B_2 + \boldsymbol{\pi}_{S+K} B_0 &= 0, \\ \boldsymbol{\pi}_{i-1} B_0 + \boldsymbol{\pi}_i B_1 + \boldsymbol{\pi}_{i+1} B_2 &= 0, \qquad 2 \leq i \leq S+K-1, \\ \boldsymbol{\pi}_1 B_2 + \boldsymbol{\pi}_{S+K-1} B_0 + \boldsymbol{\pi}_{S+K} B_1 &= 0 \end{aligned}$$

Solving these equations we get

$$\boldsymbol{\pi}_i = \boldsymbol{\pi}_{S+K} \boldsymbol{\mathcal{W}}_i \boldsymbol{\mathcal{V}}_i, 1 \leq i \leq S+K-1$$

where

$$\boldsymbol{\mathcal{V}}_i = \begin{cases} B_1^{-1}, & i = 1 \\ (B_1 - B_2 \boldsymbol{\mathcal{V}}_{i-1} B_0)^{-1}, & 2 \leq i \leq S+K-1 \end{cases}$$

$$\boldsymbol{\mathcal{W}}_i = \begin{cases} -[\boldsymbol{\mathcal{W}}_2 \boldsymbol{\mathcal{V}}_2 B_2 + (-1)^2 B_0], & i = 1 \\ -[\boldsymbol{\mathcal{W}}_{i+1} \boldsymbol{\mathcal{V}}_{i+1} B_2 + (-1)^{i+1} B_0 \boldsymbol{\mathcal{U}}_{i-1}], & 2 \leq i \leq S+K-2 \\ -[B_2 + (-1)^{S+K} B_0 \boldsymbol{\mathcal{U}}_{S+K-2}], & i = S+K-1 \end{cases}$$

with

$$\boldsymbol{\mathcal{U}}_i = \begin{cases} \boldsymbol{\mathcal{V}}_1 B_0, & i = 1 \\ \boldsymbol{\mathcal{U}}_{i-1}\boldsymbol{\mathcal{V}}_i B_0, & 2 \le i \le S+K-2 \end{cases}$$

From the normalizing condition $\boldsymbol{\pi}\mathbf{e} = 1$ we have

$$\boldsymbol{\pi}_{S+K}\left[I + \sum_{i=1}^{S+K-1} \boldsymbol{\mathcal{W}}_i\boldsymbol{\mathcal{V}}_i\right]\mathbf{e} = 1.$$

Inequality (7.2) gives the stability condition as

$$\boldsymbol{\pi}_{S+K} B_0 \mathbf{e} < \boldsymbol{\pi}_{S+K}\left[\left(I + \sum_{i=1}^{S+K-1} \boldsymbol{\mathcal{W}}_i\boldsymbol{\mathcal{V}}_i\right) B_2' + \boldsymbol{\mathcal{W}}_1\boldsymbol{\mathcal{V}}_1 B_2\right]\mathbf{e}. \tag{7.3}$$

7.2.2 Steady-state probability vector

Assuming that (7.3) is satisfied, we briefly outline the computation of the steady state probability of the system state. Let $\mathbf{x}$ denote the steady-state probability vector of the generator $\boldsymbol{\mathcal{Q}}$. Then

$$\mathbf{x}\boldsymbol{\mathcal{Q}} = 1, \qquad \mathbf{x}\mathbf{e} = 1. \tag{7.4}$$

Partitioning $\mathbf{x}$ as $\mathbf{x} = (\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, ...)$ we see that $\mathbf{x}$, under the assumption that the stability condition (7.3) holds, is obtained as (see Neuts [33])

$$\mathbf{x}_i = \mathbf{x}_1 R^{i-1}, \quad i \ge 2 \tag{7.5}$$

where R is the minimal nonnegative solution to the matrix quadratic equation:

$$A_0 + RA_1 + R^2 A_2 = \mathbf{O} \tag{7.6}$$

and the boundary equations are given by

$$\begin{aligned} \mathbf{x}_0 A_{01} + \mathbf{x}_1[A_1 + RA_2] &= 0, \\ \mathbf{x}_0 A_{00} + \mathbf{x}_1 A_{10} &= 0. \end{aligned} \tag{7.7}$$

The normalizing condition (7.4) gives

$$\mathbf{x}_0\left[I - A_{01}\left(A_1 + RA_2\right)^{-1}\left(I - R\right)^{-1}\right]\mathbf{e} = 1. \tag{7.8}$$

Now we look at a few of the system performance measures.

7.2.3 Probability of a tagged customer in the pool being served in a cycle

Assume that at the time of realization of CLT there are at least r customers in the pool. We compute the probability that r^{th} customer in the pool get served before realization of CLT in the following cycle.

Case 1: $1 \leq r \leq S$

When the number of customers in the pool $r,\ 1 \leq r \leq S$, we need not consider new arrivals nor cancellations in the cycle that commences at this epoch. At the time of realization of CLT, customers are flushed out from the buffer and waiting room. Subsequently pooled customers are transferred immediately to the buffer and to waiting room, subject to a maximum of $S+K$. If $1 \leq r \leq S$, then at the time of realization of CLT the tagged customer gets transferred to the buffer. Thus we need look at the Markov chain with the state space $\Omega_1 = \{(\boldsymbol{\mathcal{N}}_3(t), \boldsymbol{\mathcal{I}}(t)), t \geq 0\}$ is $\{(r, S), (r-1, S-1), ..., (1, S-r+1)\} \bigcup \{\Delta\}$ where $\{\Delta\}$ is the absorbing state which denotes the realization of CLT. Thus the infinitesimal generator $\boldsymbol{\mathcal{H}}_1$ of the Markov chain Ω_1 is of the form

$$\boldsymbol{\mathcal{H}}_1 = \begin{bmatrix} \boldsymbol{\mathcal{T}}_1 & \boldsymbol{\mathcal{T}}_1^0 \\ \mathbf{0} & 0 \end{bmatrix}$$

where

$$\boldsymbol{\mathcal{T}}_1 = \begin{array}{c} \\ r \\ r-1 \\ \vdots \\ 2 \\ 1 \end{array} \begin{pmatrix} r & r-1 & r-2 & \cdots & 2 & 1 \\ -(\mu+\alpha) & \mu & & & & \\ & -(\mu+\alpha) & \mu & & & \\ & & \ddots & \ddots & & \\ & & & & -(\mu+\alpha) & \mu \\ & & & & & -(\mu+\alpha) \end{pmatrix}, \ \boldsymbol{\mathcal{T}}_1^0 = \alpha\mathbf{e}$$

with initial probability vector $\zeta_1 = (1, 0, ..., 0)$. $\boldsymbol{\mathcal{T}}_1$ is a square matrix of order r and ζ_1 has r elements. Therefore, when $1 \leq r \leq S$, the time till absorption to $\{\Delta\}$, denoted by τ_1, follows Phase-type distribution having representation $(\zeta_1, \boldsymbol{\mathcal{T}}_1)$ and expected value $E(\tau_1) = -\zeta_1 \boldsymbol{\mathcal{T}}_1^{-1}\mathbf{e}$.

Case 2: $\quad S+1 \leq r \leq S+K$

If the number of customers in the pool is such that $r = S+k,\ 1 \leq k \leq K$ then again external arrivals need not be considered. Here at the time of realization of CLT, S pooled customers are transferred to the buffer and the tagged customer gets transferred to the waiting room. The Markov chain that we have to now analyze has state space $\Omega_2 = \{(\boldsymbol{\mathcal{N}}_2(t), \boldsymbol{\mathcal{I}}(t), \boldsymbol{\mathcal{N}}_3(t)), t \geq 0\}$ is $\{(n_2, i, n_3); 1 \leq n_2 \leq k, 0 \leq i \leq S, n_3 = i\} \bigcup \{(0, i, n_3); 1 \leq i \leq S, 1 \leq n_3 \leq i\} \bigcup \{\Delta\}$ where $\{\Delta\}$ is the absorbing state which denotes the realization of CLT. Thus the infinitesimal generator $\boldsymbol{\mathcal{H}}_2$ of the Markov chain Ω_2 is of the form

$$\boldsymbol{\mathcal{H}}_2 = \begin{bmatrix} \boldsymbol{\mathcal{T}}_2 & \boldsymbol{\mathcal{T}}_2^0 \\ \mathbf{0} & 0 \end{bmatrix}$$

where

$$\boldsymbol{\mathcal{T}}_2 = \begin{array}{c} \\ k \\ k-1 \\ k-2 \\ \vdots \\ 2 \\ 1 \\ 0 \end{array} \!\!\! \begin{array}{c} \begin{array}{ccccccc} k & k-1 & k-2 & \cdots & 2 & 1 & 0 \end{array} \\ \begin{pmatrix} D_1 & D_2 & & & & & \\ & D_1 & D_2 & & & & \\ & & D_1 & \ddots & & & \\ & & & \ddots & \ddots & & \\ & & & & D_1 & D_2 & \\ & & & & & D_1 & D_{10} \\ & & & & & & D_{00} \end{pmatrix} \end{array}, \quad \boldsymbol{\mathcal{T}}_2^0 = \begin{bmatrix} D_0 \\ D_0 \\ D_0 \\ \vdots \\ D_0 \\ D_0 \\ D_0' \end{bmatrix}$$

with initial probability vector $\zeta_2 = (0, ..., 0, 1, 0, ..., 0)$ where 1 is in the $(S+1)^{th}$ place. $\boldsymbol{\mathcal{T}}_2$ is a square matrix of order $k(S+1) + S(S+1)/2$. Unlike in case 1, the present situation requires consideration of cancellation.

Define $D_{m(i,n_3)}^{(j,m_3)}$, for $m = 1, 2, 10, 00$ as the transition rates from $(i, n_3) \rightarrow (j, m_3)$ where i, j represent the number of items in the inventory and n_3, m_3

represent the number of customers in the buffer. These transition rates are

$$D_{1(i,n_3)}^{(j,m_3)} = \begin{cases} \mu & 1 \le i \le S, n_3 = i; j = i-1, m_3 = j \\ -(\alpha + S\beta) & i = 0, n_3 = i; j = i, m_3 = n_3 \\ -(\alpha + (S-i)\beta + \mu) & 1 \le i \le S, n_3 = i; j = i, m_3 = n_3 \\ 0 & \text{otherwise} \end{cases}$$

$$D_{2(i,n_3)}^{(j,m_3)} = \begin{cases} (S-i)\beta & 0 \le i \le S-1, n_3 = i; j = i+1, m_3 = j \\ 0 & \text{otherwise} \end{cases}$$

$$D_{00(i,n_3)}^{(j,m_3)} = \begin{cases} \mu & 2 \le i \le S, 2 \le n_3 \le i; j = i-1, m_3 = n_3 - 1 \\ (S-i)\beta & 1 \le i \le S-1, 1 \le n_3 \le i; j = i+1, m_3 = n_3 \\ -(\alpha + \mu + (S-i)\beta) & 1 \le i \le S, 1 \le n_3 \le i; j = i, m_3 = n_3 \\ 0 & \text{otherwise;} \end{cases}$$

$$D_{10(i,n_3)}^{(j,m_3)} = \begin{cases} (S-i)\beta & 0 \le i \le S-1, n_3 = i; j = i+1, m_3 = j \\ 0 & \text{otherwise;} \end{cases}$$

$$D_{0(i,n_3)}^{\Delta} = \begin{cases} \alpha & 0 \le i \le S, n_3 = i \\ 0 & \text{otherwise;} \end{cases}$$

$$D_{0(i,n_3)}^{'\Delta} = \begin{cases} \alpha & 2 \le i \le S, 2 \le n_3 \le i \\ (\alpha + \mu) & 1 \le i \le S, n_3 = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, when $S + 1 \le r \le S + K$, the time till absorption to $\{\Delta\}$, denoted by τ_2, follows Phase-type distribution having representation $(\zeta_2, \boldsymbol{\mathcal{T}}_2)$ and expected value $E(\tau_2) = -\zeta_2 \boldsymbol{\mathcal{T}}_2^{-1}\mathbf{e}$.

Case 3: $r > S + K$

When the number of customers in the pool is such that the tagged customer is in the r^{th} position with $r = S + K + k,\ k \ge 1$, we have to consider the future arrivals also. At the time of realization of CLT, customers are flushed out from the buffer and waiting room. At the beginning of the new cycle, $S + K$ customers from the pool are transferred to the buffer and to the waiting room. The state space $\Omega_3 = \{(\mathcal{N}_1(t), \mathcal{N}_2(t), \mathcal{N}_3(t), \mathcal{I}(t)), t \geq 0\}$ is $\{(n_1, n_2, i, n_3); 0 \leq n_1 \leq k, 1 \leq n_2 \leq K, 0 \leq i \leq S, n_3 = i\} \bigcup \{(0, 0, i, n_3); 1 \leq i \leq S, 1 \leq n_3 \leq i\} \bigcup \{\Delta\}$ where $\{\Delta\}$ is the absorbing state which denotes the realization of CLT. Thus the infinitesimal generator $\mathcal{H}_3$ of the Markov chain Ω_3 is of the form

$$\mathcal{H}_3 = \begin{bmatrix} \mathcal{T}_3 & \mathcal{T}_3^0 \\ \mathbf{0} & 0 \end{bmatrix}$$

where

$$\mathcal{T}_3 = \begin{array}{c} \\ k \\ k-1 \\ k-2 \\ \vdots \\ 2 \\ 1 \\ 0 \end{array} \begin{pmatrix} k & k-1 & k-2 & \cdots & 2 & 1 & 0 \\ E_1 & E_2 & & & & & \\ & E_1 & E_2 & & & & \\ & & E_1 & \ddots & & & \\ & & & \ddots & \ddots & & \\ & & & & E_1 & E_2 & \\ & & & & & E_1 & E_{10} \\ & & & & & & E_{00} \end{pmatrix}, \quad \mathcal{T}_3^0 = \begin{bmatrix} E_0 \\ E_0 \\ E_0 \\ \vdots \\ E_0 \\ E_0 \\ E_0' \end{bmatrix}$$

with initial probability vector $\zeta_3 = (0, ..., 0, 1, 0, ..., 0)$ where 1 is in the $(K(S+1))^{th}$ place. $\mathcal{T}_3$ is a square matrix of order $(k+1)K(S+1) + S(S+1)/2$ and ζ_3 has $(k+1)K(S+1) + S(S+1)/2$ elements.

Define non-diagonal elements of $E_{m(n_2,i,n_3)}^{(m_2,j,m_3)}$, for $m = 1, 2, 10, 00$ as the transition rates from $(n_2, i, n_3) \to (m_2, j, m_3)$ where n_2, m_2 represent the number of customers in the waiting room, i, j represent the number of items in the inventory and n_3, m_3 represent the number of customers in the buffer. These transition rates are

$$E_{2(n_2,i,n_3)}^{(m_2,j,m_3)} = \begin{cases} p(S-i)\beta & 2 \leq n_2 \leq L, 0 \leq i \leq S-1, n_3 = i; \\ & m_2 = n_2, j = i+1, m_3 = n_3 + 1 \\ (S-i)\beta & n_2 = 1, 0 \leq i \leq S-1, n_3 = i; \\ & m_2 = n_2, j = i+1, m_3 = n_3 + 1 \\ 0 & \text{otherwise;} \end{cases}$$

$$E^{(m_2,j,m_3)}_{10(n_2,i,n_3)} = \begin{cases} p(S-i)\beta & 2 \le n_2 \le L, 0 \le i \le S-1, n_3 = i; \\ & m_2 = n_2, j = i+1, m_3 = n_3+1 \\ (S-i)\beta & n_2 = 1, 0 \le i \le S-1, n_3 = i; \\ & m_2 = n_2, j = i+1, m_3 = n_3+1 \\ 0 & \text{otherwise;} \end{cases}$$

$$E^{(m_2,j,m_3)}_{1(n_2,i,n_3)} = \begin{cases} \lambda & 1 \le n_2 \le K-1, 0 \le i \le S, n_3 = i; \\ & m_2 = n_2+1, j = i, m_3 = n_3 \\ \mu & 1 \le n_2 \le K, 1 \le i \le S, n_3 = i; \\ & m_2 = n_2, j = i-1, m_3 = n_3-1 \\ (1-p)(S-i)\beta & 2 \le n_2 \le L, 0 \le i \le S-1, n_3 = i; \\ & m_2 = n_2-1, j = i+1, m_3 = n_3+1 \\ (S-i)\beta & L+1 \le n_2 \le K, 0 \le i \le S-1, n_3 = i; \\ & m_2 = n_2-1, j = i+1, m_3 = n_3+1 \\ 0 & \text{otherwise;} \end{cases}$$

$$E^{(m_2,j,m_3)}_{00(n_2,i,n_3)} = \begin{cases} \lambda & 1 \le n_2 \le K-1, 0 \le i \le S, n_3 = i; \\ & m_2 = n_2+1, j = i, m_3 = n_3 \\ \lambda & n_2 = 0, 1 \le i \le S, n_3 = i; \\ & m_2 = n_2+1, j = i, m_3 = n_3 \\ \lambda & n_2 = 0, 2 \le i \le S, 1 \le n_3 \le i-1; \\ & m_2 = n_2, j = i, m_3 = n_3+1 \\ \mu & 1 \le n_2 \le K, 1 \le i \le S, n_3 = i; \\ & m_2 = n_2, j = i-1, m_3 = n_3-1 \\ \mu & n_2 = 0, 2 \le i \le S, 2 \le n_3 \le i; \\ & m_2 = n_2, j = i-1, m_3 = n_3-1 \\ (S-i)\beta & 2 \le n_2 \le K, 0 \le i \le S-1, n_3 = i; \\ & m_2 = n_2-1, j = i+1, m_3 = n_3+1 \\ (S-i)\beta & n_2 = 1, 0 \le i \le S-1, n_3 = i; \\ & m_2 = n_2-1, j = i+1, m_3 = n_3+1 \\ (S-i)\beta & n_2 = 0, 1 \le i \le S-1, 1 \le n_3 \le i; \\ & m_2 = n_2, j = i+1, m_3 = n_3 \\ 0 & \text{otherwise;} \end{cases}$$

$$E^{\Delta}_{0(n_2,i,n_3)} = \begin{cases} \alpha & 1 \le n_2 \le K, 0 \le i \le S, n_3 = i \\ 0 & \text{otherwise;} \end{cases}$$

$$E'^{\Delta}_{0(n_2,i,n_3)} = \begin{cases} \alpha & 1 \le n_2 \le K, 0 \le i \le S, n_3 = i \\ \alpha & n_2 = 0, 2 \le i \le S, 2 \le n_3 \le i \\ (\mu+\alpha) & n_2 = 0, 1 \le i \le S, n_3 = 1 \\ 0 & \text{otherwise.} \end{cases}$$

The diagonal entries are such that each row sum is zero.

Thus, when $r \ge S+K+1$, the time till absorption to $\{\Delta\}$, denoted by τ_3, follows Phase-type distribution having representation $(\zeta_3, \boldsymbol{\mathcal{T}}_3)$ and expected value $E(\tau_3) = -\zeta_3 \boldsymbol{\mathcal{T}}_3^{-1}\mathbf{e}$.

Thus the probability that the customer who was occupying the r^{th} position in the pool at the time of realization of CLT in a cycle, gets served before the completion of the realization of immediately following cycle (realization of CLT)

$$= \sum_{r=1}^{S} P(n_3 = r) P(\text{at least } r \text{ services completed before CLT realization}$$

$$+ \underbrace{\sum_{k=1}^{K} P(n_2 = k)}_{r=S+k,\ 1\le k\le K} P(\text{at least } r \text{ services completed before CLT realization}$$

$$+ \underbrace{\sum_{k=1}^{\infty} P(n_1 = k)}_{r=S+K+k,\ k\ge 1} P(\text{at least } r \text{ services completed before CLT realization}$$

$$= \sum_{r=1}^{S} P(n_3 = r) P(\tau_1 \le CLT) + \sum_{k=1}^{K} P(n_2 = k) P(\tau_2 \le CLT)$$

$$+ \sum_{k=1}^{\infty} P(n_1 = k) P(\tau_3 \le CLT)$$

$$= \sum_{r=1}^{S} P(n_3 = r) PH(\zeta_1, \boldsymbol{\mathcal{T}}_1) + \underbrace{\sum_{k=1}^{K} P(n_2 = k)}_{r=S+k,\ 1\le k\le K} PH(\zeta_2, \boldsymbol{\mathcal{T}}_2)$$

$$+ \underbrace{\sum_{k=1}^{\infty} P(n_1 = k)}_{r=S+K+k,\ k\ge 1} PH(\zeta_3, \boldsymbol{\mathcal{T}}_3)$$

$$= \sum_{r=1}^{S} x_0(0,0,S,r) PH(\zeta_1, \boldsymbol{\mathcal{T}}_1) + \sum_{k=1}^{K} x_0(0,k,S,S) PH(\zeta_2, \boldsymbol{\mathcal{T}}_2)$$

$$+ \underbrace{\sum_{\ell=1}^{\infty} \sum_{h=1}^{S+K} x_\ell(h,K,S,S)}_{k=h+(\ell-1)(S+K),\ k\ge 1} PH(\zeta_3, \boldsymbol{\mathcal{T}}_3).$$

7.2.4 Expected sojourn time in zero inventory level in a cycle

In order to compute the sojourn time of the system in a cycle, with no inventory, we consider the case of a finite pool. For numerical procedure the truncation level K' (size of the pool) is taken such that the probability of the number of customers in the pool going above the truncated size is of order less than ϵ (here ϵ is taken as 10^{-6}). Consider the Markov chain $\{(\boldsymbol{\mathcal{N}}'_1(t), \boldsymbol{\mathcal{N}}_2(t), \boldsymbol{\mathcal{I}}(t)), t \ge 0\}$ where $\boldsymbol{\mathcal{N}}'_1(t)$ = number of customers in the finite pool at time t. Its state space is $\{(0, n_2, 0); 0 \le n_2 \le K\} \bigcup \{(n_1, n_2, 0); 1 \le n_1 \le K', 1 \le n_2 \le K\} \bigcup \{\Delta\}$ where $\{\Delta\}$ is an absorbing state which denotes either the realization of CLT or cancellation of purchased item. The infinitesimal generator of the Markov chain is

$$\boldsymbol{\mathcal{W}}_{K'} = \begin{bmatrix} \boldsymbol{\mathcal{T}}_{K'} & \boldsymbol{\mathcal{T}}^0_{K'} \\ \mathbf{0} & 0 \end{bmatrix}$$

where

$$\boldsymbol{\mathcal{T}}_{K'} = \begin{bmatrix} G_1^0 & G_0^0 & & & \\ & G_1 & G_0 & & \\ & & \ddots & \ddots & \\ & & & G_1 & G_0 \\ & & & & G_1' \end{bmatrix}, \ \boldsymbol{\mathcal{T}}^0_{K'} = \begin{bmatrix} G^0 \\ G \\ \vdots \\ G \\ G \end{bmatrix}$$

with

$$G^0 = (S\beta + \alpha)\mathbf{e}, \ G = (S\beta + \alpha)\mathbf{e},$$

$$G_1^0 = \begin{bmatrix} b_S & \lambda & & & \\ & b_S & \lambda & & \\ & & \ddots & \ddots & \\ & & & b_S & \lambda \\ & & & & b''_S \end{bmatrix}, G_1' = \begin{bmatrix} b_S & \lambda & & & \\ & b_S & \lambda & & \\ & & \ddots & \ddots & \\ & & & b_S & \lambda \\ & & & & b'_S \end{bmatrix},$$

$$G_0^0 = \begin{bmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & \gamma\lambda \end{bmatrix}, G_1 = \begin{bmatrix} b_S & \lambda & & & \\ & b_S & \lambda & & \\ & & \ddots & \ddots & \\ & & & b_S & \lambda \\ & & & & b''_S \end{bmatrix}, G_0 = \begin{bmatrix} 0 & & & \\ & \ddots & & \\ & & 0 & \\ & & & \gamma\lambda \end{bmatrix}$$

where $b_S = -(S\beta + \alpha + \lambda), b'_S = -(S\beta + \alpha), b''_S = -(S\beta + \alpha + \gamma\lambda)$. Matrices G_1, G_0, G_1' are square matrices of order K and G_1^0 is of order $K+1$. G_0^0 is of dimension $(K+1) \times K$. G^0, G are column matrices of order $K+1$ and K, respectively.

Expected sojourn time in zero inventory level during a cycle is given by $\mathcal{E}^0_{\mathbf{T}} = -\eta_{K'}\boldsymbol{\mathcal{T}}^{-1}_{K'}\mathbf{e}$ where $\eta_{K'} = \{x_0(0, n_2, 0, 0); 0 \leq n_2 \leq K\} \bigcup \{x_\ell(n_1, n_2, 0, 0); 1 \leq \ell \leq m+1, 1 \leq n_1 \leq h, 1 \leq n_2 \leq K\}$ is a row vector of order $(K'+1)K+1$.

7.2.5 Expected number of revisits to S in a cycle

Next we compute the expected number of revisits of inventory level to S in a cycle. For this we have to consider a Markov chain $\{((\boldsymbol{\mathcal{N}}), \boldsymbol{\mathcal{N}}'_1(t), \boldsymbol{\mathcal{N}}_2(t), \boldsymbol{\mathcal{I}}(t), \boldsymbol{\mathcal{N}}_3(t)), t \geq 0\}$ on the states $\{\Delta\} \bigcup \{(n, 0, 0, i, n_3); n \geq 0, 0 \leq i \leq S, 0 \leq n_3 \leq$

$i\} \bigcup \{(n, n_1, n_2, i, n_3); n \geq 0, 0 \leq n_1 \leq K', 1 \leq n_2 \leq K, 0 \leq i \leq S, n_3 = i\}$ where $\{\Delta\}$ is an absorbing state which denotes the realization of the CLT. Here $\boldsymbol{\mathcal{N}}(t)$ = number of revisits to S up to time t (within the same cycle). Thus the infinitesimal generator is

$$\boldsymbol{\mathcal{F}}_{K'} = \begin{bmatrix} 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots \\ H & H_1 & H_0 & & & \\ H & & H_1 & H_0 & & \\ H & & & H_1 & H_0 & \\ \vdots & & & & \ddots & \ddots \end{bmatrix}$$

where $H = \alpha\mathbf{e}$.

Define $H_{m(n_1,n_2,i,n_3)}^{(m_1,m_2,j,m_3)}$, for $m = 0, 1$ as the transition rates from $(n_1, n_2, i, n_3) \rightarrow (m_1, m_2, j, m_3)$ where n_1, m_1 represent the number of customers in the pool, n_2, m_2 represent the number of customers in the waiting room, i, j are the number of items in the inventory and n_3, m_3 the number of customers in the buffer.

The non-diagonal elements of H_1 are

$$H_{0(n_1,n_2,i,n_3)}^{(m_1,m_2,j,m_3)} = \begin{cases} \beta & n_1 = 0, n_2 = 0, i = S-1, 0 \leq n_3 \leq S-1; \\ & m_1 = n_1, m_2 = n_2, j = i+1, m_3 = n_3 \\ \beta & n_1 = 0, 1 \leq n_2 \leq K, i = S-1, n_3 = i; \\ & m_1 = n_1 - 1, m_2 = n_2 - 1, j = i+1, m_3 = n_3 + 1 \\ \beta & 1 \leq n_1 \leq K', n_2 = 1, i = S-1, n_3 = i; \\ & m_1 = n_1 - 1, m_2 = n_2, j = i+1, m_3 = n_3 + 1 \\ p\beta & 1 \leq n_1 \leq K', 2 \leq n_2 \leq L, i = S-1, n_3 = i; \\ & m_1 = n_1 - 1, m_2 = n_2, j = i+1, m_3 = n_3 + 1 \\ (1-p)\beta & 1 \leq n_1 \leq K', 2 \leq n_2 \leq L, i = S-1, n_3 = i; \\ & m_1 = n_1, m_2 = n_2 - 1, j = i+1, m_3 = n_3 + 1 \\ \beta & 1 \leq n_1 \leq K', L+1 \leq n_2 \leq K, i = S-1, n_3 = i; \\ & m_1 = n_1, m_2 = n_2 - 1, j = i+1, m_3 = n_3 + 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$H_{1(n_1,n_2,i,n_3)}^{(m_1,m_2,j,m_3)} = \begin{cases} \lambda & 0 \le n_1 \le K', 1 \le n_2 \le K-1, 0 \le i \le S, n_3 = i; \\ & m_1 = n_1, m_2 = n_2+1, j = i, m_3 = n_3 \\ \lambda & n_1 = 0, n_2 = 0, 0 \le i \le S, n_3 = i; \\ & m_1 = 0, m_2 = n_2+1, j = i, m_3 = n_3 \\ \lambda & n_1 = 0, n_2 = 0, 1 \le i \le S, 0 \le n_3 \le i-1; \\ & m_1 = n_1, m_2 = n_2, j = i, m_3 = n_3+1 \\ \gamma\lambda & 0 \le n_1 \le K'-1, n_2 = K, 0 \le i \le S, n_3 = i; \\ & m_1 = n_1+1, m_2 = n_2, j = i, m_3 = n_3 \\ \mu & 0 \le n_1 \le K', 1 \le n_2 \le K, 1 \le i \le S, n_3 = i-1; \\ & m_1 = n_1, m_2 = m_2, j = i-1, m_3 = j \\ \mu & n_1 = 0, n_2 = 0, 1 \le i \le S, 1 \le n_3 \le i; \\ & m_1 = n_1, m_2 = n_2, j = i-1, m_3 = n_3-1 \\ (S-i)\beta & n_1 = 0, 1 \le n_2 \le K, 0 \le i \le S-2, n_3 = i; \\ & m_1 = 0, m_2 = n_2-1, j = i+1, m_3 = n_3+1 \\ (S-i)\beta & 1 \le n_1 \le K', n_2 = 1, 0 \le i \le S-2, n_3 = i; \\ & m_1 = n_1-1, m_2 = n_2, j = i+1, m_3 = n_3+1 \\ p(S-i)\beta & 1 \le n_1 \le K', 2 \le n_2 \le L, 0 \le i \le S-2, n_3 = i; \\ & m_1 = n_1-1, m_2 = n_2, j = i+1, m_3 = n_3+1 \\ (1-p)(S-i)\beta & 1 \le n_1 \le K', 2 \le n_2 \le L, 0 \le i \le S-2, n_3 = i; \\ & m_1 = n_1, m_2 = n_2-1, j = i+1, m_3 = n_3+1 \\ (S-i)\beta & 1 \le n_1 \le K', L+1 \le n_2 \le K, 0 \le i \le S-2, n_3 = i; \\ & m_1 = n_1, m_2 = n_2-1, j = i+1, m_3 = n_3+1 \\ 0 & \text{otherwise}; \end{cases}$$

The diagonal entries of H_1 are such that each row sum of $\boldsymbol{\mathcal{F}}_{K'}$ is zero. Matrices H_0, H_1 are square matrices of order $(K'+1)(S+1)K + S(S+1)/2$.

If q_v is the probability that absorption occurs with exactly v revisits, then

$$q_v = \varsigma_{K'}(-H_1^{-1}H_0)^v(-H_1^{-1}H), \quad v \ge 0$$

with $\varsigma_{K'} = \{x_0(0,0,i,n_3); 0 \le i \le S, 0 \le n_3 \le i\} \bigcup \{x_0(0,n_2,i,n_3); 1 \le n_2 \le K, 0 \le i \le S, n_3 = i\} \bigcup \{x_\ell(n_1,n_2,i,n_3); 1 \le \ell \le m+1, 1 \le n_1 \le h, 1 \le n_2 \le K, 0 \le i \le S, n_3 = i\}$ is a row vector of order $(K'+1)(S+1)K + S(S+1)/2$. Therefore the expected number of revisits to S before realization of CLT is $E_{revisits}(S) = \sum_{v=0}^{\infty} vq_v$.

7.2.6 Additional performance measures

1. Expected number customers in the pool before realization of CLT

$$E_P(N) = \sum_{\ell=1}^{\infty} \sum_{n_1=1}^{S+K} \sum_{n_2=1}^{K} \sum_{i=0}^{S} [(\ell-1)(S+K) + n_1] x_\ell(n_1, n_2, i, i).$$

2. Expected number of customers in the waiting room before realization of CLT

$$E_W(N) = \sum_{\ell=1}^{\infty}\sum_{n_1=1}^{S+K}\sum_{n_2=1}^{K}\sum_{i=0}^{S} n_2 x_\ell(n_1,n_2,i,i) + \sum_{n_2=1}^{K}\sum_{i=0}^{S} n_2 x_0(0,n_2,i,i).$$

3. Expected number of customers in the buffer before realization of CLT

$$E_B(N) = \sum_{\ell=1}^{\infty}\sum_{n_1=1}^{S+K}\sum_{n_2=1}^{K}\sum_{i=1}^{S} i x_\ell(n_1,n_2,i,i) + \sum_{i=1}^{S}\left(\sum_{n_2=1}^{K} i x_0(0,n_2,i,i)\right.$$
$$\left. + \sum_{n_3=1}^{i} n_3 x_0(0,0,i,n_3)\right).$$

4. Expected number of items in the inventory before realization of CLT

$$E_I(N) = \sum_{\ell=1}^{\infty}\sum_{n_1=1}^{S+K}\sum_{n_2=1}^{K}\sum_{i=1}^{S} i x_\ell(n_1,n_2,i,i) + \sum_{i=1}^{S} i\left(\sum_{n_2=1}^{K} x_0(0,n_2,i,i)\right.$$
$$\left. + \sum_{n_3=0}^{i} x_0(0,0,i,n_3)\right).$$

5. The rate at which pooled customers are transferred to the waiting room

$$E_{PW}(R) = \sum_{\ell=1}^{\infty}\sum_{n_1=1}^{S+K}\sum_{i=0}^{S-1}(S-i)\beta\left(x_\ell(n_1,1,i,i) + p\sum_{n_2=2}^{L} x_\ell(n_1,n_2,i,i)\right).$$

6. Expected cancellation rate

$$E_C(R) = \sum_{\ell=1}^{\infty}\sum_{n_1=1}^{S+K}\sum_{n_2=1}^{K}\sum_{i=0}^{S-1}(S-i)\beta x_\ell(n_1,n_2,i,i) + \sum_{i=0}^{S-1}(S-i)\beta$$
$$\left(\sum_{n_2=1}^{K} x_0(0,n_2,i,i) + \sum_{n_3=0}^{i} x_0(0,0,i,n_3)\right).$$

7. Expected inventory depletion rate

$$E_P(R) = \mu \left[\sum_{\ell=1}^{\infty} \sum_{n_1=1}^{S+K} \sum_{n_2=1}^{K} \sum_{i=1}^{S} x_\ell(n_1, n_2, i, i) + \sum_{i=1}^{S} \left(\sum_{n_2=1}^{K} x_0(0, n_2, i, i) + \sum_{n_3=1}^{i} x_0(0, 0, i, n_3) \right) \right].$$

8. The probability that the system is left with no item in the inventory at the time of realization of CLT

$$P_{full} = \sum_{n_2=0}^{K} x_0(0, n_2, 0, 0) + \sum_{\ell=1}^{\infty} \sum_{n_1=1}^{S+K} \sum_{n_2=1}^{K} x_\ell(n_1, n_2, 0, 0).$$

9. The probability that the system has S items in the inventory at the time of realization of CLT

$$P_{vacant} = \sum_{\ell=1}^{\infty} \sum_{n_1=1}^{S+K} \sum_{n_2=1}^{K} x_\ell(n_1, n_2, S, S) + \sum_{n_2=1}^{K} x_0(0, n_2, S, S) + \sum_{n_3=0}^{S} x_0(0, 0, S, n_3).$$

7.3 Numerical illustration

In this section we provide numerical illustration of the system performance with variation in values of underlying parameters.

Effect of γ on $E_{PW}(R)$

We consider the following values for the parameters $S = 8, K = 6, L = 4, \lambda = 7, \mu = 10, p = 0.75, \alpha = 0.25, \beta = 1.5$. For this set of parameter values, Table 7.1 shows the impact of the probability γ on measure $E_{PW}(R)$. From Table 7.1, it is clear that $E_{PW}(R)$ is monotonically increasing in γ.

γ	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$E_{PW}(R)$	0.0150	0.0286	0.0413	0.0534	0.0650	0.0763	0.0875	0.0985	0.1097

Table 7.1: Effect of γ on $E_{PW}(R)$

Effect of the arrival rate λ

Table 7.2 indicates that increase in λ value makes expected number of customers in the buffer, expected number of items in the inventory before realization of common life time and expected purchase rate, all decrease. However, as λ increases, expected number of customers in the pool and waiting room increase.

λ	$E_P(N)$	$E_W(N)$	$E_B(N)$	$E_I(N)$	$E_P(R)$	$E_C(R)$
1	0	0.0002	2.5689	5.1026	6.6592	0.6487
2	0.0014	0.0239	2.3807	4.5328	8.1841	1.2877
3	0.0647	0.2594	1.8511	3.6279	7.5471	1.8192
4	0.5942	0.9429	1.5312	2.8534	6.6355	**1.9446**
5	2.5597	1.9519	1.4145	2.0790	5.9835	1.5755
6	8.4346	3.0149	1.3611	1.8237	5.4780	0.9497

Table 7.2: Effect of λ: $S = 8, K = 5, L = 3, \mu = 10, \alpha = 0.25, \beta = 0.5, p = 0.75, \gamma = 0.75$

Effect of the service time parameter μ

μ	$E_P(N)$	$E_W(N)$	$E_B(N)$	$E_I(N)$	$E_P(R)$	$E_C(R)$
11	52.4543	4.0173	1.1372	1.2172	5.6964	0.2905
12	43.6491	3.9545	1.0346	1.1300	5.8441	0.3600
13	37.7957	3.8986	0.9496	1.0589	5.9782	0.4248
14	33.6765	3.8491	0.8784	0.9998	6.1017	0.4840
15	30.6459	3.8056	0.8181	0.9501	6.3251	0.53376
16	28.3361	3.7671	0.7665	0.9079	6.3251	0.5859

Table 7.3: Effect of μ : $S = 7, K = 5, L = 3, \lambda = 8, \alpha = 0.25, \beta = 0.75, p = 0.75, \gamma = 0.75$

From Table 7.3, we observe that an increase in μ makes a decrease in measures like expected number of items in the inventory before realization of common life time, expected number of customers in the pool, waiting room and buffer: higher the realization time more the number of customers served out. However, the expected rate of purchase and cancellation increase. These are on expected lines.

Effect of common life time parameter α

Table 7.4 shows that an increase in α results in a decrease in expected number of customers in the pool and waiting room. However, expected number of items in the inventory before realization, expected number of customers in the buffer, expected cancellation rate, all show an increasing trend: the shorter the life time, lesser the number of cancellations.

α	$E_P(N)$	$E_W(N)$	$E_B(N)$	$E_I(N)$	$E_P(R)$	$E_C(R)$
0.3	22.1746	3.7151	1.3311	1.5223	5.7504	0.5789
0.4	7.5199	3.0006	1.4159	1.8945	6.0592	1.1647
0.5	3.9765	2.4532	1.4602	2.2005	6.2141	1.5101
0.6	2.4562	2.0322	1.4816	2.4523	**6.2717**	1.7054
0.7	1.6455	1.7034	1.4892	2.6616	6.2676	1.8075
0.8	1.1604	1.4424	1.4884	2.8372	6.2237	1.8506

Table 7.4: Effect of α: $S = 7, K = 5, L = 3, \lambda = 8, \mu = 10, \beta = 0.75, p = 0.75, \gamma = 0.75$

Effect of cancellation rate β

β	$E_P(N)$	$E_W(N)$	$E_B(N)$	$E_I(N)$	$E_P(R)$	$E_C(R)$
0.6	40.8548	3.9731	1.0824	1.1931	4.8864	0.3044
0.8	7.8415	3.0149	1.2846	1.6709	5.9967	1.2864
1	2.7985	2.1952	1.4792	2.1503	6.8642	2.4426
1.2	1.2410	1.5751	1.6510	2.5940	7.4822	3.4656
1.4	0.6360	1.1391	1.7908	2.9905	7.8652	4.2263
1.6	0.3681	0.8460	1.8972	3.3376	8.0454	4.7440

Table 7.5: Effect of β: $S = 7, K = 5, L = 3, \lambda = 7, \mu = 10, \alpha = 0.25, p = 0.75, \gamma = 0.75$

From Table 7.5, we observe that the expected number of customers in the pool and that in the waiting room decrease with increase in β value which is a consequence of presence of positive inventory in the system over a longer duration of time in a cycle. Here expected number of customers in the buffer, expected number of items in the inventory, expected cancellation rate, and expected purchase rate show a sharper upward trend on realization of common life time. This tendency is a natural consequence of higher cancellation rate for the same common life time parameter value.

Effect of α, β on P_{full} and P_{vacant}

α	P_{full}	P_{vacant}
0.4	0.6533	0.0200
0.45	0.5868	0.0260
0.5	0.5288	0.0318
0.55	0.4742	0.0373
0.6	0.4542	0.0426
0.65	0.3953	0.0476

Table 7.6: Effect of α for $\beta = 0, S = 7, K = 5, L = 3, \lambda = 5, \mu = 9, p = 0.75, \gamma = 0.75$

β	P_{full}	P_{vacant}
0.75	0.4539	0.0106
1	0.2878	0.0184
1.25	0.1728	0.0260
1.5	0.1016	0.0345
1.75	0.0600	0.0448
2	0.0361	0.0571

Table 7.7: Effect of β for $\alpha = 0.2, S = 7, K = 5, L = 3, \lambda = 7, \mu = 11, p = 0.75, \gamma = 0.75$

For $\beta = 0$, varying over α, we notice from Table 7.6 that, P_{full} decreases with increasing value of α – the shorter life time, lesser chance for inventory being completely sold. P_{vacant} increases with increasing value of α.

Table 7.7 shows the effect of β for fixed α value. It tells that higher cancellation rate results in reduction in probability of system being full. However, P_{vacant} increases with increase in value of β.

7.3.1 Optimization problem

Based on the above performance measures we construct a cost function for checking the optimality of the waiting room capacity K and the maximum inventory level S. It may be noted that cancellation to some extent prior to common life realization results in higher profit to the system since there is a cancellation penalty imposed on the customer. We define a revenue function

$$\mathcal{F}(K,S) = \mathcal{C}_C E_C(R) + \mathcal{C}_P E_P(R) - \mathcal{C}_B E_B(N) - \mathcal{C}_W E_W(N) - \mathcal{C}_P E_P(N) - \mathcal{C}_I E_I(N)$$

where

$\mathcal{C}_C$ = revenue to the system due to per unit cancellation of inventory purchased

$\mathcal{C}_P$ = revenue to the system due to per unit purchase of item in the inventory

$\mathcal{C}_B$ = holding cost of customer per unit per unit time in the buffer

$\mathcal{C}_W$ = holding cost of customer per unit per unit time in the waiting room

$\mathcal{C}_P$ = holding cost of customer per unit per unit time in the pool

$\mathcal{C}_I$ = holding cost per unit time per item in the inventory

In order to study the variation in different parameters on profit function we first fix the costs $\mathcal{C}_C = \$25, \mathcal{C}_P = \$125, \mathcal{C}_B = \$4, \mathcal{C}_W = \$6, \mathcal{C}_P = \$2, \mathcal{C}_I = \$8$.

Effect of variation in S and K

K \ S	4	5	6	7	8
4	676.2926	846.0144	997.8604	1091.2001	1236.7100
5	676.6111	**846.3291**	998.0183	1091.3711	1236.7987
6	675.3351	845.7577	**998.0245**	1091.4225	1237.0284
7	674.3501	845.7024	997.9805	**1091.4302**	1237.0879
8	673.5696	845.6392	997.9242	1091.4287	**1237.1224**
9	672.9314	845.5666	997.4116	1091.4116	1237.1021

Table 7.8: Effect of S and K on expected revenue for $\lambda = 5, \mu = 11, \alpha = 0.2, \beta = 1.5, p = 0.75, \gamma = 0.75$

For different values of S and K, the expected revenue is calculated and presented in Table 7.8. This table shows that the revenue function increases when S increases whereas the expected revenue function increases first with K and then keep going down.

Concluding remarks:

In this thesis we discussed several queueing-inventory models with blocking sets (partial or complete). In certain cases explicit product form solution of the system state could be arrived at, despite high correlation between the component random variables. Below we give a sketch of the findings in this thesis:

In chapter 2 we studied three queueing-inventory systems with positive service time, partial blocking and another model with complete blocking. In the case of partial blocking sets, stochastic decomposition turned out to be impossible. Product form solution was obtained only when completely blocking set was introduced, which was done in Section 2.6. We obtained steady state probability vector. Cost functions were constructed in each case to numerically investigate their optimal values. The effect of various parameters on the system performance measures were also investigated. Even when the lead time follows general distribution the product form solution is possible when a complete blocking is introduced. In a future work we propose to extend the present work to the case when the service process is arbitrarily distributed.

Chapter 3 considered a single server supply chain model with a (rQ, KQ)-production inventory system and a distribution centre which adopts (s, Q)-policy. All underlying distributions were assumed to be exponential that are independent of each other. In this model, the steady state distribution was obtained in product form. The effects of various performance measures were investigated. A cost function in s, Q, r and K was constructed to numerically investigate their optimal values. We also obtained the waiting time distribution of the distribution centre for realizing the replenishment order. Measures such as the expected number of up and down crossings of rQ at the production centre while production was on was of interest.

The model discussed in this chapter could be extended to consider supply of raw materials at the production centre. Further exponential distributions could be replaced by Erlang or even phase type distribution. This would be at a cost, namely, the product form nature of the solution will be extremely hard, if not impossible. So one has to resort to algorithmic approach for the analysis of the problem. These are a few of the proposed line of future work.

In the next chapter we analyzed two queueing-inventory models. In model II we assumed that the LP-customers joint he system only when the on-hand inventory was positive and no HP-customer present in the system. We obtained a product form solution for the steady-state probability vector for this system. But in model I, we assume LP-customers did not join only when the on-hand inventory was zero. The result was loss of stochastic decomposition of the system state. A relaxation of assumptions in model II resulted in major analytical loss, namely, the product form solution (see model I). Thus the blocking set was of highly specialized characteristic. For both models we derived several performance characteristics. We compared the two models by constructing a cost function. We can see form Table 4.4 that the cost involved in model I is much higher than that in model II. We can attribute the lost sales assumption (for LP-customers) in model II for this trend. In a follow up paper we study priority queueing-inventory with retrial of LP-customers and infinite capacity for the waiting space of HP-customers.

In chapters 5, 6 and 7 we considered an inventory problem where inventoried items had a common life time. The demands formed a Poisson process and service time exponentially distributed as well as negligible were considered. A finite buffer of varying size (with inventory depletion/ increase due to demand/ cancellation) was provided. The buffer size varied since at any time, number of customers sure of receiving inventory depends on number of items in the inventory at that moment. Further, a finite capacity waiting room and an orbit of infinite capacity for unsatisfied customers were provided. We obtained the system state distribution. A profit function was constructed and numerically analyzed.

It is interesting to note that in a recent investigation of a queueing-inventory system with reservation, cancellation and common life time having only one infinite capacity waiting line we could arrive at product form solution. In a way it looks surprising since a 2-dimensional object yields analytical solution whereas the more than 2-dimensional processes discussed in chapters 5 to 7 were not amenable to product form solution.

In yet another recent investigation where we introduced two distinct service rates of service - a higher rate when inventory level is above s and a lower rate when inventory level is below the reorder level s. With positive lead time for replenishment, we obtain the stability condition which involves the lead time parameter also; this is not the case for models discussed in chapter 2 to 4 of this thesis.

Further extensions of the problems discussed in this thesis to the case of Markovian arrival process, phase type service and / lead time are on the anvil.

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