A Study of Some Centrality Measures in Graphs

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Certificate

Certified that the work presented in this thesis entitled “A Study of Some Centrality Measures in Graphs ”is based on the authentic record of research carried out by Shri. Ram Kumar R under my guidance in the Department of Computer Applications, Cochin University of Science and Technology, Kochi-682 022 and has not been included in any other thesis submitted for the award of any degree.

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Declaration

I hereby declare that the work presented in this thesis entitled “A Study of Some Centrality Measures in Graphs ”is based on the original research work carried out by me under the supervision and guidance of Dr. B. Kannan, Associate Professor, Department of Computer Applications, Cochin University of Science and Technology, Kochi - 682 022, Kerala, India and has not been included in any other thesis submitted previously for the award of any degree.

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Certificate

Certified that the work presented in this thesis entitled "A Study of Some Centrality Measures in Graphs "submitted to Cochin University of Science and Technology by Sri. Ram Kumar R for the award of degree of Doctor of Philosophy under the faculty of technology, contains all the relevant corrections and modifications suggested by the audience during the pre-synopsis seminar and recommended by the Doctoral Committee.

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Conferences and Publications

Conferences

- **On the median number of a graph**-UGC Sponsored National seminar on Recent trends in Distances in Graphs held at Ayya Nadar Janaki Ammal college Sivakasi,12-13 march 2012
- **Connected Fair sets in Graphs**-Indo-Slovenia Conference on Graph Theory and Applications Organised by Department of Futures Studies, University of Kerala, Graph Theory Research Group, University of Maribor and Institute of Mathematics, Physics and Mechanics, Ljubljana, February 22-24,2013.
- **Pacifying Edges of a Graph**- UGC Sponsored National seminar on Emerging trends in Applied Mathematics held at Mar Ivanios College, Thiruvananthapuram, 5-7 September 2013.

Publications

- **Median Sets and Median Number of a Graph**- ISRN Discrete Mathematics, Volume 2012- Hindawi Publishing Corporation.

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Chapter 1

Introduction

1.1 Background of the problem

There has been a steady increase in the research relating to the study of graphs as they are the mathematical models of various real-world complex networks like the world-wide web, social networks, email networks, biological networks etc. One of the most important aspects of such networks that researchers have been trying to study is centrality, which measures the degree of influence or importance of an individual with in the network under consideration

Centrality is in fact one of the fundamental notions in graph theory which has established its close connection with various other areas like Social networks, Flow networks, Facility location problems etc. Even though a plethora of centrality measures have been introduced from time to time, according to the changing demands, the term is not well defined and we can only give some common qualities that a centrality measure is expected to have. Nodes with high centrality scores are often more likely to be very powerful, indispensable, influential, easy propagators of information, significant in maintaining the cohesion of the group and are easily susceptible to anything that disseminate in the network.

Nodes with low centrality are considered to be peripheral. They have very little significance in any kind of group activity and thus contributes very less in maintaining the cohesion of the group. While the above said are their disadvantages they are not without advantages. They are comparatively insulated from the spread of anything undesirable say, contagious diseases in the case of human networks, viruses in the case of computer networks etc and are usually subjected to lesser traffic flow.

Sabidussi [108] gave a set of conditions that a measure should possess in order to qualify to become a centrality measure. One of these was that adding an edge to the node should increase its centrality and another was that adding an edge anywhere in the network should not decrease the centrality of any node. These are not generally acceptable as many of the centrality measures do not possess these qualities. That is, Sabidussi's condition are insufficient to define centrality. Freeman in [43] categorised the class of all centrality measures in to three-degree, betweenness and closeness. Degree centrality of a node is the number of nodes to which a particular node is directly attached and it gives the extend of exposure of a node to attract anything that is spreading in the network. The closeness centrality gives an account of how close a node is to all the other nodes in the network and it measures the cost involved in spreading an information from a node to other nodes of the network. Betweenness centrality gives the frequency with which a particular node appears in the shortest path between other pairs of nodes. It reflects the capability of a node in controlling the flow of information between other pair of nodes. For more on the various centrality measures, see [20].

Facility location problems, where the purpose is to identify the locations for setting up a facility like hospital, fire station, library, ware house, depot etc for a given a set of customers, from the time of its inception, has been heavily relying on the concept of centrality. The locations chosen should be optimal and the criteria for optimality depends on the nature of the problem, but it is accepted that it depends on the distances between the various locations. When we are looking to place an emergency facility like fire station or hospital, the location is chosen in such a way that the maximum response time between the site of facility and the emergency is kept to a minimum. This is called the effectiveness oriented model.

When the facility is something like a shopping mall, where the objective is to minimise the total transportation cost from the facility point to all its customers, the location is chosen in such a way that sum of the distances to be covered is a minimum. This is usually referred to as efficiency oriented model. There is a third approach known as equity oriented model where the location for a facility is to be chosen such that it is more or less equally fair to all the customers. Issue of equity is relevant in setting public sector facilities where the distribution of travel distances among the recipients of the service is also of importance. That is, the inverse of measures of dispersion like range, mean deviation etc are used as the centrality measure in such models. In practice, we calculate the inequity measures and the location having the least inequity measures are considered to be the central points.

1.2 Preliminaries

This section introduces various graph theoretic terms that are being used in the coming chapters. The description of certain terms that are frequently used through out this thesis are given as definitions. A graph G consists of a finite nonempty set $V = V(G)$ of *vertices* to together with a set, $E = E(G)$, of unordered pairs of distinct vertices. A pair $e = \{u, v\}$ of vertices u and v of G is called an *edge* of G having *end vertices* u and v. We write $e = uv$ and say that u and v are adjacent vertices; vertex u and edge e are *incident* with each other, as are e and v. If two edges e_1 and e_2 are incident with a common vertex then they are *adjacent edges*. A graph H is a *subgraph* of G, if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. If G_1 is a subgraph of G then G is a *supergraph* of G. For any set S of vertices of G, the *induced subgraph* $\langle S \rangle$ is the maximal subgraph of G with vertex set S. If v is a vertex of a graph G then $G - v$ is the subgraph of G consisting of all vertices of G except v and all edges not incident with v. The removal of a set of vertices S, which is the removal of single vertices in succession, results in $G-S$. If u and v are nonadjacent vertices of G then $G+uv$ is the graph obtained by addition of the edge uv to G. A *walk* of a graph is an alternating sequence of vertices and edges $v_0e_1v_1e_2\ldots v_{n-1}e_nv_n$ beginning and ending with vertices, in which each edge is incident with two vertices immediately preceding and succeeding it. The integer n is the length of the walk. This walk is referred to as a v_0-v_n walk. Here v_0 and v_n are called the *origin* and *terminus* respectively and $v_1,\ldots,v_{n-1}$ its internal vertices. If the origin and terminus are identical the walk is called a *closed walk*. When all the edges of a walk are distinct then it is called a *trail* and further if all vertices are also distinct then it is called a *path*. A path on n vertices shall be denoted by P_n. A closed trail whose origin and internal vertices are all distinct is called a *cycle*. A cycle of length n, denoted by C_n, is called an *n-cycle*; an *n-cycle* is odd or even according as n is odd or even. A graph G is *connected* if there exists a path between any pair of vertices of G. An *acyclic* graph is one that contains no cycle. A *tree* is a connected acyclic graph. A graph in which each pair of distinct vertices are adjacent is called a *complete graph* and is denoted by K_n if it contain n vertices. A subset S of V is called a *clique* if every pair of vertices of S are adjacent. A graph is *bipartite* if its vertex set can be partitioned into two subsets V_1 and V_2 such that each edge has one end in V_1 and the other end in V_2. (V_1,V_2) is called a *bipartition* of G. A *complete bipartite graph*, $K_{m,n}$, has a bipartition (V_1,V_2) where $|V_1|=m$, $|V_2|=n$ and each vertex of V_1 is adjacent to every vertex of V_2. The *complement* G^c of a graph G is the graph with vertex set V, two vertices being adjacent in G^c if and only if they are not adjacent in G.

Definition 1. For two vertices u and v of G, distance between u and v denoted by $d_G(u,v)$, is the number of edges in a shortest u-v path.

Definition 2. The *eccentricity* $e_G(u)$ of a vertex u is $\max_{v\in V(G)} d_G(u,v)$.

When G is obvious, we write $d(u,v)$ and $e(u)$ for $d_G(u,v)$ and $e_G(u)$ respectively.

Definition 3. A vertex v is an *eccentric vertex* of u if $e(u) = d(u,v)$. A vertex v is an eccentric vertex of G if there exists a vertex u such that $e(u) = d(u,v)$.

The set of all vertices which are at a distance i from the vertex u is denoted by $N_i(u)$. The set of all vertices adjacent to x in a graph G, denoted by $N(x)$, is the *neighbourhood* of the vertex x. For an $S \subseteq V$, neighborhood of S denoted by $N(S) = \bigcup_{u \in S} N(u)$.

Definition 4. The diameter of the graph G, $diam(G)$, is $\max_{u \in V(G)} e(u)$. The *radius* of G, denoted by $rad(G)$, is $\min_{u \in V(G)} e(u)$. Two vertices u and v are said to be diametrical if $d(u,v) = diam(G)$.

Definition 5. The interval $I(u,v)$ between vertices u and v of G consists of all vertices which lie in some shortest path between u and v. The number of intervals of a graph is denoted by $in(G)$.

Definition 6. A vertex u of a graph G is called a *universal vertex* if u is adjacent to all other vertices of G.

Definition 7. A vertex v of a graph G is called a *cut-vertex* if $G - v$ is no longer connected. Any maximal induced subgraph of G which does not contain a cut-vertex is called a *block* of G.

Definition 8. [15] A finite sequence of vertices $\pi = (v_1, \ldots, v_k) \in V^k$ is called a $profile$. For the profile $\pi = (v_1, \ldots v_k)$ and $x \in V(G)$, the $remoteness$ $D_G(x,\pi)$ is $\sum_{1 \leqslant i \leqslant n} d(x, v_i)$. When the underlying graph is obvious we use $D(x,\pi)$ instead of $D_G(x,\pi)$ and further if the vertex is also obvious we use $D(\pi)$ instead of $D(x,\pi)$.

The *Hypercube* Q_n is the graph with vertex set $\{0,1\}^n$, two vertices

being adjacent if they differ exactly in one co-ordinate. A *subcube* of the hypercube Q_n is an induced subgraph of Q_n, isomorphic to Q_m for some $m \leqslant n$.
The graph on n vertices formed by joining all the vertices of a $(n-1)$-*cycle* to a vertex is a *wheel graph* and is denoted by W_n.
A graph G is a *block graph* if every block of G is complete. A graph G is *chordal* if every cycle of length greater than three has a chord; namely an edge connecting two non consecutive vertices of the cycle. Trees, k-trees, interval graphs, block graphs are all examples of chordal graphs.

Definition 9. [71] Let G be a graph with vertex set $\{v_1, \ldots, v_n\}$ and let $\{B_1, \ldots, B_r\}$ be the blocks of G. Then the *Skeleton* S_G of G is a graph with $V(S_G) = \{v_1, \ldots, v_n, B_1, \ldots, B_r\}$ and $E(S_G) = \{(v_i, B_j) | v_i \in V(B_j)\}$.

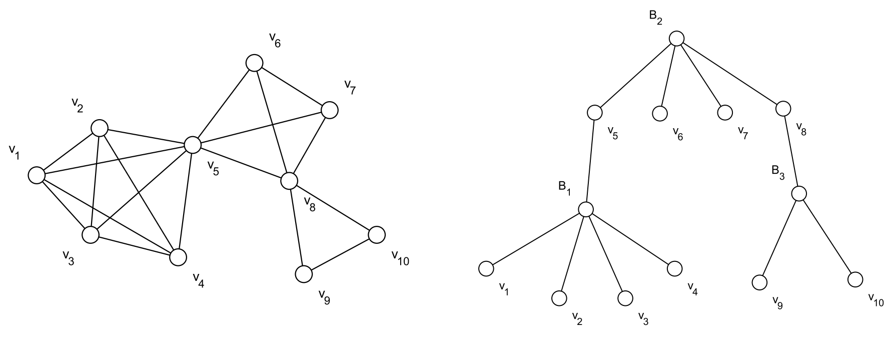


Figure 1.1: A block graph and its skeleton graph

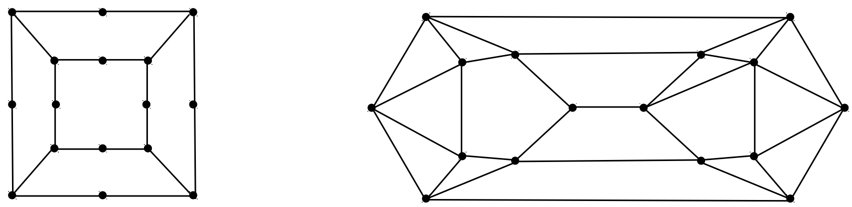
Definition 10. A graph is a *unique eccentric vertex* graph(written UEV graph) if every vertex has a unique eccentric vertex. The unique eccentric vertex of the vertex u is denoted by $\bar{u}$.

Definition 11. A graph G is *self centered* if all the vertices of G have the same eccentricity.

Definition 12. [54] A graph G is called *even* if for each vertex u of G there is a unique eccentric vertex $\bar{u}$, such that $d(u,\bar{u}) = diam(G)$. In other words even graphs are self centered, UEV graphs.

Definition 13. [54] An even graph G is called *balanced* if $deg(u) = deg(\bar{u})$ for each $u \in V$, *harmonic* if $\bar{u}\bar{v} \in E$ whenever $uv \in E$ and *symmetric* if $d(u,v) + d(u,\bar{v}) = diam(G)$ for all $u, v \in V$.

Gobel and Veldman in [54] proved that every harmonic even graph is balanced and every symmetric even graph is harmonic. They also gave examples of harmonic graphs that are not symmetric and balanced graphs that are not harmonic.



(a) Harmonic but not symmetric even graph

(b) Balanced but not harmonic even graph

Figure 1.2

Definition 14. The *Cartesian product* $G \square H$ of two graphs G and H has vertex set, $V(G) \times V(H)$, two vertices (u,v) and (x,y) being adjacent if either $u = x$ and $vy \in E(H)$ or $ux \in E(G)$ and $v = y$. For more on graph products see [57].

Given integers i and j, we introduce the following notations

$$\begin{aligned} i \oplus_n j &= i + j \text{ if } i + j \leqslant n. \\ &= i + j - n \text{ if } i + j > n \end{aligned}$$

$$\begin{aligned} i \ominus_n j &= i - j \text{ if } i - j \geqslant 1 \\ &= i - j + n \text{ if } i - j \leqslant 0 \end{aligned}$$

1.3 Synopsis

In this thesis graph theoretic studies on various centrality measures are being conducted. The rest of the thesis is organised as follows.

Chapter 2 is devoted to the literature survey on various centrality measures.

In **Chapter 3** we identify the S-center of different classes of graphs such as trees, complete graphs, block graphs, wheel graphs, complete bipartite graphs, odd cycles and symmetric even graphs. We give some results regarding centers of dominating boundary sets of symmetric even graphs. Center Number of a graph is introduced as the number of distinct center sets of a graph. Center number of the above classes of graphs are found out. We introduce a new class of graphs called Center Critical Graphs and characterise them.

Eccentricity measures how far is a vertex from the furthest in the graph. In some cases it is desirable to reduce the eccentricity of a vertex by introducing additional edges to the graph. One special case of this problem is when addition of only a single edge is permissible. In **chapter 4** we introduce the concepts Pacifying Edges and Shrinking Edges in a graph and the same are identified for paths, odd cycles an symmetric even graphs.

Chapter 5 discusses the median sets of various classes of graphs and enumerate them.

Chapter 6 focuses on equity based centrality, introduces the concept of Partiality, Fair Center and Fair Sets of graphs and fair sets of some specific classes of graphs are identified.

Chapter 7 is devoted to the study of Antimedian graphs and a generalisation of it called weakly antimedian graphs. Antimedian block graphs and

weakly antimedian trees are characterised and new classes of antimedian and weakly antimedian graphs are introduced.
Finally, **chapter 8** concludes the thesis by summarizing the results of the previous chapters and gives some problems for further study.

Chapter 2

Review of literature

In this chapter we make a detailed survey on the various graph theoretic centrality measures like center, median, antimedian etc. The survey is conducted on a structural rather than algorithmic point of view.

2.1 Center

The *center* of a graph consists of those vertices with minimum eccentricity, where eccentricity of a vertex is the maximum distance of the vertex among the set of all vertices. The problem of finding the center of a graph has been studied by many authors since the nineteenth century beginning with the classical result due to Jordan [70] that the center of a tree consists of a single vertex or a pair of adjacent vertices. The graph center problem is interesting from both a structural and an algorithmic point of view. Harary and Norman in [59] proved that the center of a connected graph lies with in a block of the graph. Kopylov and Timofeev in [80] stated without proof that given a graph G there exists a graph H such that center of H, $C(H) \cong G$.Buckley et al. in [24] demonstrated that for $n \geqslant 2$ and a graph G there exists a graph H such that vertex and edge connectivity of H equal to n, chromatic number of G, $\chi(G) = \chi(H) + n$ and $C(G) \cong C(H)$. A planar graph which can be drawn such that all vertices are on the outer face is called an *outerplanar graph*. A graph is *maximal outerplanar* if it is outerplanar and adding an edge makes it non-outerplanar. A.Proskurowski [103, 104] showed that only a finite number of graphs can be centers of maximal outerplanar graphs and generalized this result for the class of 2-trees which contains maximal outerplanar graphs. A graph is *chordal* if every cycle of length greater than 3 contains a chord. Laskar and Shier in

[83] proved that for a connected chordal graph the center always induces a connected subgraph. Soltan and Chepoi in [112] proved that the center of a connected chordal graph has diameter at most 3. Truszczyski [116] proved that the center $C(G)$ of a unicycle graph containing the cycle C is either K_1 or K_2 or $C(G) \subseteq C$.Chepoi in [29] characterised the centers of chordal graphs. It was shown by Nieminen in [90] that the center vertices of a chordal graph constitutes a convex vertex set. Chang [28] showed that the center of a connected chordal graph is distance invariant, biconnected and of diameter no more than 5. He proved that for any connected chordal graph with $diam(G) = 2\ rad(G)$, center of G, $C(G)$, is a clique and for any connected chordal graph with $diam(G) = 2\ rad(G) - 1$, $diam(C(G)) \leqslant 3$. He also gave a a necessary and sufficient condition for a biconnected chordal graph of diameter 2 and radius 1 to be the center of some chordal graph and further conjectured that $diam(C(G)) \leqslant 2$ for any connected chordal graph with $diam(G) = 2\ rad(G) - 2$. Vijayakumar et al. in [98] disproved this conjecture. Chepoi in [30] gave a linear time algorithm for finding the center of a chordal graph. If G is a nontrivial graph then its line graph $L(G)$ is the graph whose nodes are the edges of G and two nodes in $L(G)$ are adjacent if and only if the corresponding edges are adjacent in G. It was proved by Knor et al. [79] that given a graph G there exists a graph H such that G is the center of H and the Line graph of G is the center of Line graph of H. The *i-iterated line graph* of G, $L^i(G)$, is given by $L^0(G) = G$ and $L^i(G) = L(L^{i-1}(G)$ for $i \geqslant 1$. For a graph G such that $L^2(G)$ is not empty, Knor et al. [78] constructed a supergraph H such that $C(L^i(H)) = L^i(G)$ for all i, $0 \leqslant i \leqslant 2$. Buckley et al. [23] defined a graph G as an L-graph if all its diametrical paths contain a central vertex. They proved that $C(G \square H) = C(G) \times C(H)$, where $G \square H$ is the Cartesian product of the graphs G and H. They further proved that if either $C(G)$ is a bridge or $C(G) = \{x\}$ where x does not lie in a cycle then G is an L-graph. An L-graph is an L_1-graph if all its diametrical paths contain

all its central vertices, it is called an L_3-graph if G is an L-graph and no diametrical path of G contains all central vertices of G and it is called an L_2 graph if it is neither L_1 nor L_3. Gliviak and Kys [52] gave upper and lower bounds for the number of elements in the center of all L-graphs, that is, L_1-graphs, L_2-graphs and L_3-graphs. Gliviak et al. [51] showed that the central subgraph of any two-radially maximal graph contains an edge and that those of them that have a star as the central subgraph are sequential joins of complete graphs. If G is a simply connected set of lattice points with graph structure defined by 4-neighbour adjacency, Khuller et al. in [73] showed that the center of G is either a 2×2 square block, a diagonal staircase, or a (dotted) diagonal line with no gaps. Pramanik [102] proved that for every non-trivial connected graph H there exists a graph G such that H is the center of G and the inserted graph of H is the center of the inserted graph of G.

2.1.1 Self-centered graphs

Buckley in [21] determined the extremal sizes of a connected self-centered graph having p vertices and radius r. Akiyama and Ando [1] characterized graphs G for which both G and G^c are self-centered with diameter 2. Akiyama et al. in [2] characterised self-centered graphs with p vertices, radius 2 and minimum size. Laskar and Shier [83] showed that a connected self-centered chordal graph has radius $\leqslant 3$. Nandakumar and Parthasarathy [97] proved that a unique eccentric vertex graph is self-centered if and only if each vertex is eccentric. Das and Rao in [39] sowed that there are no self-centered chordal graphs with radius =3 and characterised all self-centered chordal graphs. Buckley in [22] showed that a self complementary graph with diameter d is self-centered if and only if $d = 2$. Klavzar et al.[75] introduced *Almost Self-Centered graphs* as the graphs in which all but two are central vertices. The block structure of these graphs is described and constructions for generating such graphs are proposed. They also showed that any graph can be embedded into an Almost self-centered graph graph of prescribed radius. Balakrishnan et al. in [7] characterised almost self centered median and chordal graphs.

2.1.2 Some generalizations of center

Slater in [109] generalized the concept of center of a graph to center of an arbitrary subset, say S, of the vertex set of the graph and called it *S-center*. He proved that the *S-center* of a tree consists of a single vertex or a pair of adjacent vertices. Chang in [120] studied the S-center of distance hereditary graphs and proved that the S-center of a distance hereditary graph is either a connected graph of diameter 3 or a cograph. He also proved that for a bipartite distance hereditary graph the S-center is either a connected graph of diameter $\leqslant 3$ or an independent set. The *Steiner distance* of a set S of vertices in a connected graph G is the minimum size among all connected subgraphs of G containing S. For $n \geqslant 2$, the *n-eccentricity* $e_n(v)$ of a vertex v of a graph G is the maximum Steiner distance among all sets of n vertices of G that contains v. The *n-center* of $G, C_n(G)$, is the subgraph induced by those vertices of G having minimum n-eccentricity. Oellerman [95] showed that every graph is the n-center of some graph. It was also shown that the n-center of a tree is a tree and characterized those trees that are n-centers of trees. In [94] he described an algorithm for finding $C_n(T)$ of a tree. Another generalisation of the center problem, called the *p-center* problem, was studied algorithmically by many authors [31, 42, 55, 58, 72, 86, 101, 121].

2.2 Median

The *Median* $M(G)$ of a graph G consists of those vertices that minimises the sum of the distances to all vertices of the graph. The first known result is by Jordan [70] who proved that the median of a tree consists of a single vertex or a pair of adjacent vertices. If v is a vertex of the tree T, the maximal number of vertices of a branch of T from v is called the weight at v. The vertex of T with the minimal weight is called the *Centroid* of T. Zelinka [122] proved that the median of a tree coincides with its centroid. Slater [110] showed that for every graph H there exists a graph G whose median is H, and that the median of a $2-tree$ is isomorphic to K_1, K_2 or K_3. Hendry [65] proved that for every two graphs G and H, there exists a connected graph F such that $C(F) \cong G$ and $M(F) \cong H$, where $C(F)$ and $M(F)$ are disjoint. Holbert [66] went a step further proving that for every two graphs G and H and positive integer k, there exists a connected graph F such that $C(F) \cong G$ and $M(F) \cong H$ and $d(C(H), M(H)) = k$. That is, even though both center and median are "centers" of a graph they can be arbitrarily far. On the other hand, they can also be arbitrarily close. Novotny and Tian [93] proved that for any three graphs G, H and K, where K is isomorphic to an induced subgraph of both G and H, there exists a connected graph F such that $C(F) \cong G$, $M(F) \cong H$ and $C(H) \cap M(H) \cong K$. The *periphery* $P(G)$ is the subgraph induced by those vertices of G having maximum eccentricity. Winters in [118] proved that for any graph G, there exists a connected graph H such that $M(H) \cong F$ and $d_H(u, v) \leqslant 2$ for all $u, v \in V(H)$. Given graphs G and H and an integer m, he gave a necessary and sufficient condition for G and H to be the median and periphery, respectively, of some connected graph such that the distance between the median and periphery is m. Necessary and sufficient conditions were also given for two graphs to be the median and periphery and to intersect in any common induced subgraph. Dankelmann and Sabidussi in [38] showed that given any connected graph H , there exists a connected graph G whose median is an isometric subgraph which is isomorphic to H. Soltan[111] showed that the median of a ptolemaic graph is connected and Niemenen in [90] established that the median of a ptolemaic graph is complete.

2.2.1 p-median

The concept of median has been generalised to p-median, where p is any positive integer. This is a set of p vertices that minimises the sum of the distances of each vertex to its nearest vertex in the p-vertex set. The p-median problem has been mostly approached algorithmically and Hakimi in [56] gave a method for solving the p-median problem and since then the problem has been approached algorithmically by many authors [3, 4, 25, 45, 55, 61, 67, 68, 72, 81, 85, 99, 106, 114].

2.2.2 Median of a set

A generalization of the median problem is to find the median of a subset of the vertex set. In this case a median is a vertex that minimizes the sum of the distances to all the elements of the subset. If S is any subset of V then the median of S was called as S-centroid by Slater [109]. He proved that S-centroid of a tree is a path and that if S contains odd number of elements then S-centroid contains a unique vertex.

2.2.3 Median of a profile

Another generalization was to find the median set of a *profile*, a sequence of vertices. In this case a median is vertex that minimizes the sum of the distances to all the elements of the profile, taking into consideration repetition of vertices in the profile, see [55]. The set of all medians of a profile is called the median set of the profile. If u and v are vertices of a graph G, then $I(u, v)$ consists of vertices of the shortest paths between u and v. A graph G is called a *Median graph* if for every triple of vertices $\{u, v, w\}$ of G, $I(u, v) \cap I(v, w) \cap I(u, w)$ contains a unique vertex. Bandelt and Barthelemy [13] proved that the median set of any profile of odd length in a median graph consists of a unique vertex and that the median set of any profile of even length is an interval. Mulder in [89] designed the

Majority Strategy for finding the median of any profile in a tree. Bandelt and Chepoi [14] conducted further studies on the median sets of profiles of a graph and they characterised the class of graphs with connected median sets. Medians of profiles with bounded diameter has been studied in [9] and it has been proved that medians of such a profile can be obtained locally, either in a properly bounded isometric subgraph or in an induced subgraph that contains the profile. A subgraph H of a (connected) graph G is an isometric subgraph if $d_H(u,v) = d_G(u,v)$ holds for any vertices $u, v \in H$. Let G be an isometric subgraph of some hypercube (such graphs are also called partial cubes). The smallest integer d such that G is an isometric subgraph of Q_d is called the *isometric dimension* of G and denoted $idim(G)$. Balakrishnan et al. in [6] designed an algorithm that computes the median of a profile in a median graph in $O(n \text{ idim}(G))$ time. Balakrishnan et al. in [11] considered another method called *plurality strategy* for finding the median set of a profile of a graph. They have showed that plurality, Hill climbing and steepest Ascent Hill Climbing [107] produces the median set of a profile if and only if the induced subgraph of the median set is connected. The concept of profiles has been generalised to signed profiles [12] where each vertex is assigned a positive or negative sign. This has significance in location theory where a particular facility may be preferred by some of the clients and may be rejected by some others. It is proved that hypercubes are the only graphs in which majority Strategy, starting from any initial vertex, produces the median set for any signed profile on the graph.

2.3 Antimedian and Anticenters

The main objectives in a facility location theory are usually the minimisation of the sum of the distances, minimisation of the maximum distance etc and they have been discussed earlier. But when we have to place a facility that is obnoxious or undesirable such as nuclear reactors or garbage dump sites we go for maximisation instead of minimisation. This has recently gained much importance due to rapid industrialisation and urbanisation. The graph induced by the set of vertices that maximises the sum of the distance to all other vertices of the graph is called the antimedian of a graph and the graph induced by the set of vertices with maximum eccentricity is called its anticenter.

Church and Garfinkel [32] studied the one-facility maximum median (maxian) problem, providing an $O(mnlogn)$ algorithm where m is the number of edges and n is the number of vertices. Minieka[88] proposed methods for finding the anticenter and antimedian of a graph. Antimedian and anticenter problems were later studied algorithmically by many authors [16, 33, 34, 35, 36, 37, 113, 115].

Bielak and Syslo [19] proved that every graph is the antimedian of some graph. Vijayakumar and S.B.Rao [105] showed that if G_1 and G_2 are any two cographs, then there is a cograph that is both Eulerian and Hamiltonian having G_1 as its median and G_2 as its antimedian. Balakrishnan et al. [5] proved that for an arbitrary graph G and $S \subseteq V(G)$ it can be decided in polynomial time whether S is the antimedian set of some profile. They further proved that if G and H are connected graphs with connected antimedian sets then $G \Box H$ has connected antimedian sets. Balakrishnan et al. in [8] showed that given graphs G and J and an integer $r \geqslant 2$, there exists a graph H such that G and J are the median and the antimedian of H and $d_H(G, J) = r$.

2.4 Distance related extremal graphs

Extremal graph theory focuses on the study of graphs that are extremal with respect to any particular property under consideration. Graphs having extremal properties with respect to distance based graph parameters like radius and diameter have been extensively studied.

Graphs having extremal properties with respect to distance parameters like radius and diameter have been studied extensively. Ore in [96] defined a graph to be *diameter maximal* if the addition of any edge to the graph decreases the diameter of the graph and gave a characterisation of such graphs. Caccetta and Smyth [27] gave a general form of diameter maximal graphs with edge connectivity k, diameter d, number of vertices n and having the maximum number of edges.
A graph G is *diameter minimal* if the deletion of any edge increases the diameter of G. This class of graphs were studied by many authors[26, 41, 47, 53, 62, 63, 64, 100, 119] .
A graph G is called *radius minimal* if radius of $G - e$ is greater than radius of G for every edge of G. Gliviak[46] proved that a graph is radius minimal if and only if it is a tree.
Any graph G such that radius of $G + e \leqslant$ radius of G for every $e \in G^c$ is called a *radially maximal* graph . Vizing in [117] found an upper bound on the number of edges in radially maximal graphs and a lower bound was found by Nishanov [92]. Nishanov in [91] studied some properties of radially maximal graphs with radius $r \geqslant 3$ and diameter $2r - 2$. Harary and Thomassen [60] characterized radially maximal graphs with radius two and showed that there exists infinitely many radially maximal graphs with radius three. Gliviak [50] proved that any graph can be an induced subgraph of a regular radially maximal graph with a prescribed radius $r \geqslant 3$.
A graph G is two-radially maximal if G is noncomplete and for each pair (u, v) of its nodes such that $d(u, v) = 2$ we have $r(G + uv) < r(G)$. Gliviak et al. in [51] proved that the central subgraph of any two-radially maximal graph contains an edge and showed that those of them that have a star as the central subgraph are sequential joins of complete graphs. Gliviak [48] gave an overview of results for radially maximal, minimal, critical and stable graphs. Knor [76] characterized unicyclic, non-selfcentric, radially-maximal graphs on the minimum number of vertices. He further proved

that the number of such graphs is $\frac{1}{48}r^3 + O(r^2)$. In [49] it was conjectured that if G is a non-selfcentric radially-maximal graph with radius $r \geqslant 3$ on the minimum number of vertices then G is planar, has exactly $3r - 1$ vertices, the maximum degree of G is 3 and the minimum degree of G is 1. Knor [77] with the help of exhaustive computer search proved this result for $r = 4$ and 5. Directed radially maximal graphs were studied in [44] and [49].

Chapter 3

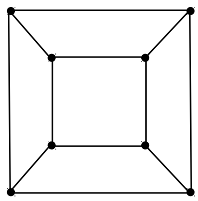
Center Sets and Center Number

3.1 Introduction

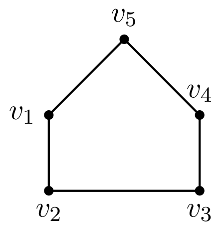
Slater in [109] generalized the concept of center of a graph to center of an arbitrary subset of the vertex set of the graph. For any subset S of V in the graph $G = (V, E)$, the S-*eccentricity*, $e_{G,S}(v)$ (in short $e_S(v)$) of a vertex v in G is $\max_{x \in S}(d(v, x))$. The S-center of G is $C_S(G) = \{v \in V | e_S(v) \leqslant e_S(x) \, \forall x \in V\}$. For a graph G, an $A \subseteq V$ is defined to be a *Center set* if there exists an $S \subseteq V$ such that $C_S(G) = A$. In this chapter we identify the center sets of some familiar classes of graphs such as block graphs, complete bipartite graphs, wheel graphs, odd cycles, symmetric even graphs etc and enumerate the number of distinct center sets of these classes of graphs. But before that we introduce a class of graphs called center critical graphs and characterise them.
It shall be interesting to find the a vertex set of minimum cardinality whose center is the same as the center of the whole graph. Searching on this line we stumbled up on a class of graphs where the center of none of the proper subset of the vertex set is the same as the center of the graph and they are defined as center critical graphs.

3.2 Center Critical graphs

Definition 3.2.1. A graph G is said to be *center critical* if for all proper subsets S of V, we have $C_S(G) \neq C(G)$.



(a) A center critical graph



(b) C_5, not center critical
$C_{\{v_1,v_2,v_3,v_4\}}(G) = \{v_1, v_2, v_3, v_4, v_5\}$
$= C(G)$

Now, we shall give characterisation of center critical graphs. For that we require the following theorem from [97]

Theorem 3.2.2. *A UEV graph G is self-centered if and only if each vertex of G is an eccentric vertex.*

Theorem 3.2.3. *A graph G is center critical if and only if G is both self-centered and a UEV graph.*

Proof. Let G be a center critical graph having vertex set $\{v_1, \dots, v_n\}$. First we shall prove that for every $v_i \in V$ there exists a $v_j \in V$ such that v_i is the unique eccentric vertex of v_j. Assume the contrary. Let there exist a vertex, say v_k, such that v_k is not an eccentric vertex of any vertex. Let $S = V \setminus \{v_k\}$. Then for every vertex v_i of G, $e_S(v_i) = e(v_i)$ since the eccentric vertices of v_i are in S. Since the eccentricities of none of the vertices change, $C_S(G) = C(G)$ contradicting our assumption that G is center critical. Hence every vertex of G is an eccentric vertex.
Let v_k be such that when ever v_k is an eccentric vertex of v_ℓ then there exists a vertex v'_k such that v'_k is also an eccentric vertex of v_ℓ. Again take $S = V \setminus \{v_k\}$. Since every vertex v_ℓ that has v_k as an eccentric vertex

has another eccentric vertex, we have $e_S(v_k) = e(v_k)$. As above we get that $C_S(G) = C(G)$, a contradiction. That is, we have proved that each vertex v_i, $1 \leqslant i \leqslant n$ is a unique eccentric vertex of a vertex, say v'_i, where $v'_i = v_j$ for some j, $1 \leqslant j \leqslant n$. Since $\{v'_1, \dots, v'_n\} = V$ and each v'_i has a unique eccentric vertex each vertex of G has a unique eccentric vertex. Now, it is also obvious that every vertex is an eccentric vertex. Therefore by Theorem 3.2.2, G is self-centered. Conversely assume that G is both self-centered and unique eccentric vertex graph, and let $rad(G) = r$. Then, again by Theorem 3.2.2, every vertex of G is an eccentric vertex. Therefore for every $x \in V$ there exists a $y \in V$ such that $x = \bar{y}$. Let $S \subseteq V$ and $x \in V \setminus S$. Then $e(y) = r$ and since $\bar{y} = x \in V \setminus S$, $e_S(y) < r$. Let $z \in S$. Then $e_S(\bar{z}) = r$. Hence $C_S(G) \neq V$ which shows that G is center critical. □

Remark 3.2.1. C_5 is a graph that is self centered but not center critical, as it is not a UEV graph. In fact all odd cycles are self centered but not UEV and hence are not center critical.

3.3 Center Sets of Some Graph Classes

Prior to identifying the center sets of various classes of graphs we recall the following lemma by Harary et al. in [59].

Lemma 3.3.1 (Lemma 1 of [59])**.** The center of a connected graph G is contained in a block of G.

We generalize this lemma to any S-center of a graph and the proof is almost similar to the proof given there.

Theorem 3.3.1. *Any S-center of a connected graph G is contained in a block of G.*

Proof. For an $S \subseteq V$, assume that $C_S(G)$ lies in more than one block of G. Then G contains a vertex v such that $G - v$ contains at least two components, say, G_1 and G_2, each of which contains a vertex belonging to $C_S(G)$. Let u be the vertex of S such that $d(u, v) = e_S(v)$ and P be the shortest $u - v$ path. Then P does not intersect at least one of G_1 and G_2, say G_1. Let w be the vertex of G_1 such that $w \in C_S(G)$. Then v belong to the shortest $w - u$ path and hence

$$e_S(w) \geqslant d(w, u) = d(w, v) + d(u, v) \geqslant 1 + e_S(v)$$

contradicting the fact that $w \in C_S(G)$. Thus for any $S \subseteq V$, $C_S(G)$ lies in a single block of G. □

3.3.1 Center sets of Block graphs

Proposition 3.3.2. Let G be a block graph with vertex set V and blocks $B_1, \ldots, B_r$. For $1 \leqslant i \leqslant r$, let $V(B_i) = V_i$. The center sets of G are singleton sets $\{v\}, v \in V(G)$ and V_i for $1 \leqslant i \leqslant r$.

Proof. If $S = \{v\}$, then $e_S(v) = 0 \leqslant e_S(x)$ for all $x \in V$. Therefore $C_{\{v\}}(G) = \{v\}$. Hence $\{v\}$, where $v \in V$ are all center sets.

Let S be a proper subset of V_i, $1 \leqslant i \leqslant r$ containing at least two elements. Hence $e_S(x) = 1$ for every $x \in V_i$ and $e_S(x) > 1$ for all $x \in V - V_i$. So $C_S(G) = V_i$. Therefore each V_i, $1 \leqslant i \leqslant r$ is a center set.

Consider $S \subseteq V(G)$ containing at least 2 elements from 2 different blocks, and let x be a cut vertex of G with $e_S(x) = k$. Also assume that $d(x, v) = k$ where $v \in S$. Let $P : x = x_0 x_1 \ldots x_r x_{r+1} \ldots x_k = v$ be the shortest $x - v$ path. See that $e_S(x_1) = k - 1$. Since the eccentricities will never decrease to zero, we can find two vertices in P (may be identical) say x_r, and x_{r+1} so that $e_S(x_r) = e_S(x_{r+1}) = k - r$. Then for every vertex y in the block containing x_r and x_{r+1}, $e_S(y) = k - r$ and as we move away from this block the S-eccentricity increases. Hence the S-center of G is the block containing x_r and x_{r+1}.

Now let $e_S(x_r) = k-r$ and $e_S(x_{r+1}) = k-r+1$. Then for every y other than x_r in the block containing x_r and x_{r+1}, $e_S(y) = k-r+1$ and as we move away from this block the S-eccentricity increases. Therefore S-center of G is x_r. Hence the center sets of block graphs are $\{v\}$, $v \in V(G)$ and $V_i, 1 \leqslant i \leqslant r$. □

As a consequence of Proposition 3.3.2, we have the following corollaries. Corollary 3.3.4, is a theorem of Slater in [109].

Corollary 3.3.3. *The center sets of the complete graph K_n with vertex set V are $\{u\}, u \in V$ and the whole set V.*

Corollary 3.3.4 (Theorem 4 of [109])**.** *The center sets of a tree $T = (V, E)$ are $\{u\}, u \in V$, and $\{u, v\}, uv \in E$.*

Corollary 3.3.5. *The induced subgraphs of all center sets of a block graph are connected.*

Now we shall find the center sets of some simple classes of graphs such as complete bipartite graphs, $K_n - e$, Wheel graphs, etc. First we identify the center sets of bipartite graphs $K_{m,n}$, $m, n > 1$. When m or n is 1, $K_{m,n}$ is a tree whose center sets have already been identified.

3.3.2 Center Sets of Complete bipartite graphs

Proposition 3.3.6. Let $K_{m,n}$ be a complete bipartite graph with bipartition (X, Y) where $|X| = m > 1$ and $|Y| = n > 1$. Then the center sets of $K_{m,n}$ are

1. $V = X \cup Y$
2. X

3. Y
4. $\{v\}, v \in V$
5. $\{x, y\}, x \in X, y \in Y$

Proof. First we shall show that each of the sets described in the theorem are center sets. Let $A \subseteq V(K_{m,n})$ and let $A_1 = A \cap X$, and let $A_2 = A \cap Y$

1. If $|A_1| > 1$ and $|A_2| > 1$, $C_A(K_{m,n}) = V$.
2. If $A_1 = \emptyset$ with $|A_2| > 1$ then $C_A(K_{m,n}) = X$.
3. If $A_2 = \emptyset$ with $|A_1| > 1$ then $C_A(K_{m,n}) = Y$.
4. If $|A_1| = 1$ and $|A_2| > 1$ then $C_A(K_{m,n}) = \{x\}$ where $A_1 = \{x\}$
5. If $|A_2| = 1$ and $|A_1| > 1$ then $C_A(K_{m,n}) = \{x\}$ where $A_2 = \{x\}$
6. If $|A_1| = |A_2| = 1$ then $C_A(K_{m,n}) = \{x, y\}$ where $A_1 = \{x\}$ and $A_2 = \{y\}$

Thus $C_A(K_{m,n})$ is one of the sets given in the theorem and the result follows. □

Illustration 3.3.1. Here we give the center sets of $K_{5,4}$ with vertex set $\{v_1, v_2, v_3, v_4, u_1, u_2, u_3, u_4, u_5\}$. The center sets are

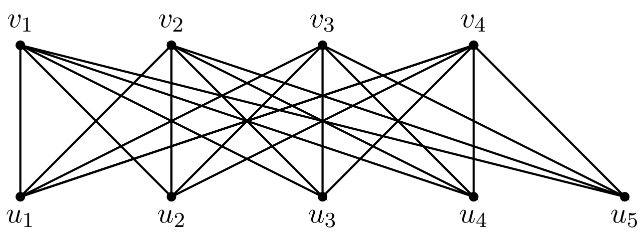


Figure 3.1: $K_{5,4}$

1. $\{v_1, v_2, v_3, v_4, u_1, u_2, u_3, u_4, u_5\}$
2. $\{v_1, v_2, v_3, v_4\}$

3. $\{u_1, u_2, u_3, u_4, u_5\}$
4. $\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{u_1\}, \{u_2\}, \{u_3\}, \{u_4\}, \{u_5\}$
5. $\{v_1, u_1\}, \{v_2, u_1\}, \{v_3, u_1\}, \{v_4, u_1\}, \{u_2, v_1\}, \{u_2, v_2\}, \{u_2, v_3\}, \{u_2, v_4\}, \{u_3, v_1\}, \{u_3, v_2\}, \{u_3, v_3\}, \{u_3, v_4\}, \{u_4, v_1\}, \{u_4, v_2\}, \{u_4, v_3\}, \{u_4, v_4\}, \{u_5, v_1\}, \{u_5, v_2\}, \{u_5, v_3\}, \{u_5, v_4\}$

3.3.3 Center sets of $K_n - e$

Next we shall find the center sets of another class of graphs, $K_n - e$. When $n = 2$, $K_n - e$ is a pair of isolated vertices and when $n = 3$, $K_n - e$ is path and center sets of this has been identified in Corollary 3.3.4. The following theorem identifies the center sets of $K_n - e$ for $n \geqslant 4$

Proposition 3.3.7. For the graph $K_n - e(= xy)$, $n \geqslant 4$, the center sets are

1. $\{v\}$, $v \in V$
2. $V \setminus \{x\}$
3. $V \setminus \{y\}$
4. $V \setminus \{x, y\}$
5. V

Proof. As in Proposition 3.3.6, initially we prove that all the sets described in the theorem are center sets.

1. For each $v \in V$, $C_{\{v\}}(K_n - e) = \{v\}$.
2. Let $A \subseteq V$ be such that $|A| > 1$, $y \in A$ and $x \notin A$, then $C_A(K_n - e) = V \setminus \{x\}$.
3. For $A \subseteq V$ such that $|A| > 1$, $x \in A$ and $y \notin A$, $C_A(K_n - e) = V \setminus \{y\}$.
4. Let $A \subseteq V$ be such that $x, y \in A$. Then $C_A(K_n - e) = V \setminus \{x, y\}$.
5. For $A \subseteq V$ be such that $|A| > 1$, $x, y \notin A$ $C_A(K_n - e) = V$.

Now we have found the centers of all types of subsets of V and therefore above mentioned sets are precisely the center sets of $K_n - e$. □

Illustration 3.3.2. Consider $K_6 - e$ with vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6\}$ and $e = v_1v_2$. Then the center sets are

1. $\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}, \{v_6\}$
2. $\{v_1, v_3, v_4, v_5, v_6\}$
3. $\{v_2, v_3, v_4, v_5, v_6\}$
4. $\{v_3, v_4, v_5, v_6\}$
5. $\{v_1, v_2, v_3, v_4, v_5, v_6\}$

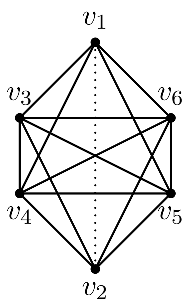


Figure 3.2: $K_6 - e$, $e = uv$

3.3.4 Center sets of Wheel graph

Now we shall identify the center sets of wheel graphs. The wheel graph W_4 is K_4 and their center sets have already been identified. First we prove the case for $n \geqslant 6$. The center sets of W_5, the only remaining case, will be given in the remark after the Proposition 3.3.8.

Proposition 3.3.8. Let W_n, $n \geqslant 6$, be wheel graph on the vertex set $\{v_1, \ldots, v_n\}$ where v_n is the universal vertex. Then the center sets of W_n are

1. $\{v_i\}$, $1 \leqslant i \leqslant n$
2. $\{v_i, v_n\}$, $1 \leqslant i \leqslant n-1$
3. $\{v_i, v_j, v_n\}$, where $v_iv_j \in E(C_{n-1})$

4. $\{v_i, v_j, v_k, v_n\}$ where $v_iv_j, v_jv_k \in E(C_{n-1})$

Proof. First we shall prove that each of the sets described above are center sets.

1. For $1 \leqslant i \leqslant n$, $C_{\{v_i\}}(G) = \{v_i\}$.
2. Let $S = \{v_{i\ominus_{n-1}1}, v_i, v_{i\oplus_{n-1}1}\}$. $e_S(v_i) = e_S(v_n) = 1$ and $e_S(v) = 2$ for all other $v \in V$ and therefore $C_S(G) = \{v_i, v_n\}$.
3. For $S = \{v_i, v_{i\oplus_{n-1}1}, v_n\}$, $C_S(G) = S = \{v_i, v_{i\oplus_{n-1}1}, v_n\}$.
4. For $S = \{v_i, v_n\}$, $C_S(G) = \{v_{i\ominus_{n-1}1}, v_i, v_{i\oplus_{n-1}1}, v_n\}$.

For all $S \subseteq V$ such that $S \neq \{v_n\}$, $e_S(v_n) = 1$ and hence for all $S \subseteq V$ such that $S \neq \{v_i\}$, $1 \leqslant i \leqslant n-1$, $v_n \in C_S(G)$. Now, let A be such that A contain v_i and v_j such that $d_{C_{n-1}}(v_i, v_j) > 2$. Let $S \subseteq V$ be such that $C_S(G) = A$ then obviously $S \neq \{v_i\}$, $1 \leqslant i \leqslant n$. We have $v_n \in C_S(G)$ with $e_S(v_n) = 1$ Therefore v_i and v_j belong to $C_S(G)$ implies there exist a vertex v_k in $V(C_{n-1})$ such that $d(v_i, v_k) = d(v_j, v_k) = 1$ which is impossible by the choice of v_i and v_j. Hence v_i and v_j of $V(C_{n-1})$ belong to a center set implies $d_{C_{n-1}}(v_i, v_j) \leqslant 2$. Also v_i, $v_{i\oplus_{n-1}2}$ belong to $C_S(G)$ implies $v_{i\oplus_{n-1}1}$ belong to $C_S(G)$. Hence the center sets are precisely those described in the theorem. □

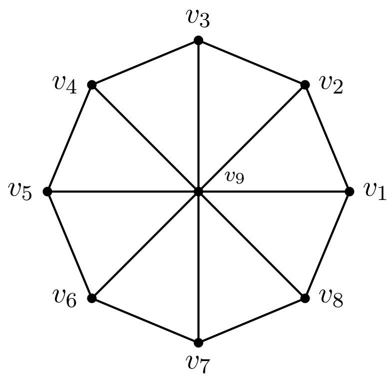


Figure 3.3: W_9

Illustration 3.3.3. Consider the wheel W_9 with vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_9\}$ and having v_9 as the universal vertex. The center sets are

1. $\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}, \{v_6\}, \{v_7\}, \{v_8\}, \{v_9\}$
2. $\{v_1, v_9\}, \{v_2, v_9\}, \{v_3, v_9\}, \{v_4, v_9\}, \{v_5, v_9\}$, $\{v_6, v_9\}, \{v_7, v_9\}$, $\{v_8, v_9\}$
3. $\{v_1, v_2, v_9\}, \{v_2, v_3, v_9\}, \{v_3, v_4, v_9\}, \{v_4, v_5, v_9\}$, $\{v_5, v_6, v_9\}$, $\{v_6, v_7, v_9\}$, $\{v_7, v_8, v_9\}, \{v_8, v_1, v_9\}$
4. $\{v_1, v_2, v_3, v_9\}, \{v_2, v_3, v_4, v_9\}, \{v_3, v_4, v_5, v_9\}, \{v_4, v_5, v_6, v_9\}$, $\{v_5, v_6, v_7, v_9\}$, $\{v_6, v_7, v_8, v_9\}, \{v_7, v_8, v_1, v_9\}, \{v_8, v_1, v_2, v_9\}$

Remark 3.3.1. Let $\{v_1, v_2, v_3, v_4, v_5\}$ be the vertex set of W_5 with v_5 as the universal vertex. All sets of the types given in the Proposition 3.3.8 are center sets in the same manner. Since the outer cycle is of length 4, $C_{\{v_1,v_3\}}(W_5) = \{v_2, v_4, v_5\}$ and $C_{\{v_2,v_4\}}(W_5) = \{v_1, v_3, v_5\}$. By the arguments similar to that given in the proof of Proposition 3.3.8, the center sets of W_5 are precisely,

1. $\{v_1\}, \{v_2\}, \{v_3\}, \{v_4\}, \{v_5\}$
2. $\{v_1, v_5\}, \{v_2, v_5\}, \{v_3, v_5\}, \{v_4, v_5\}$
3. $\{v_1, v_2, v_5\}, \{v_2, v_3, v_5\}, \{v_3, v_4, v_5\}, \{v_4, v_1, v_5\}$
4. $\{v_1, v_2, v_3, v_5\}, \{v_2, v_3, v_4, v_5\}, \{v_3, v_4, v_1, v_5\}, \{v_4, v_1, v_2, v_5\}$
5. $\{v_1, v_3, v_5\}$, $\{v_2, v_4, v_5\}$

Remark 3.3.2. The subgraph induced by any center set of a wheel graph is connected. In fact, the subgraphs induced by all center sets of any graph with a universal vertex are connected.

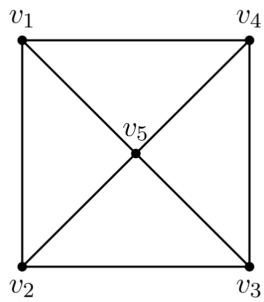


Figure 3.4: W_5

3.3.5 Center sets of Odd cycles

Theorem 3.3.9. *Let C_{2n+1}, $n \geqslant 2$ be an odd cycle with vertex set $V = \{v_1, \ldots, v_{2n+1}\}$. An $A \subseteq V$ is a center set of C_{2n+1} if and only if either $A = V$ or A does not contain a pair of alternate vertices.*

Proof. If $A = V$ then it is a center set namely, of itself. So assume $A \neq V$. Let $A \subset V$ be such that it contains three consecutive vertices say, v_1, v_2, v_3. Assume there exists an $S \subset V$ with $A = C_S(G)$. Let d be the S-eccentricity of a vertex of A. Then there exists a vertex v_i in S such that $d(v_1, v_i) = d$. $d(v_2, v_i) = d$ implies v_1 and v_2 are the eccentric vertices of v_i which means $d = n$ or $A = V$. Hence $d(v_2, v_i) \neq d$. $d(v_2, v_i) = d+1$ implies $e_S(v_2) \geqslant d+1$. Hence $d(v_2, v_i) = d-1$. Then there exists a vertex v_j such that $d(v_2, v_j) = d$ and $d(v_1, v_j) = d - 1$. Then as explained above $d(v_3, v_j)$ cannot be d and therefore $d(v_3, v_j) = d + 1$. This means that $e_S(v_2) \neq e_S(v_3)$. Hence any three consecutive vertices cannot be in a center set. Now, assume that $A \subset V$ is such that it contains a pair of alternate vertices and does not contain the middle vertex, say, contains v_1 and v_3 and does not contain v_2. Assume $A = C_S(G)$. Let $e_S(v_1) = e_S(v_3) = d$. Then $e_S(v_2) = d + 1$. Let v_i be a vertex in S such that $d(v_2, v_i) = d+1$. Obviously $d(v_1, v_i) = d(v_3, v_i) = d$ and this implies v_i is the eccentric vertex of v_2 or $d(v_2, v_i) = n$. But since

C_{2n+1} is an odd cycle either $d(v_1, v_i) = n$ or $d(v_3, v_i) = n$, a contradiction. Hence if A is a center set then it cannot contain a pair of alternate vertices. Conversely assume that A is such that it does not contain any pair of alternate vertices of the cycle. Now take S to be the set of all vertices of C_{2n+1} which are eccentric vertices of vertices of A^c and which are not eccentric vertices of any of the vertices of A. It is obvious by the choice of A that such vertices do exist. Since an eccentric vertex of at least one of the two neighbours of each vertex of A belong to S and none of the eccentric vertices of any vertex of A belong to S, for each vertex x of A, $e_S(x) = n - 1$. Since at least one of the eccentric vertices of each vertex of A^c belong to S, for each vertex y of A^c, $e_S(y) = n$. Thus $A = C_S(G)$. Hence the theorem. □

Corollary 3.3.10. *For the odd cycle C_{2n+1}, $n \geqslant 2$, if A is a center set then either $|A| \leqslant n$ or $A = V$.*

Proof. Let $C_{2n+1} = (v_1, v_2, \ldots, v_{2n+1}, v_1)$.
Case 1-n is odd.
Subcase 1.1: Only one among v_1,v_2 and v_3 is in A.
Let $A_1 = \{v_1, v_2, v_3\}$, $A_2 = \{v_4, v_6\}$, …, $A_{n-1} = \{v_{2n-2}, v_{2n}\}$, $A_n = \{v_{2n-1}, v_{2n+1}\}$. A contains at most one vertex from each A_i. Therefore $|A| \leqslant n$.
Subcase- 1.2: Exactly two vertices among v_1,v_2 and v_3 are in A.
With out loss of generality we can assume that they are v_1 and v_2. Then v_3, v_4, v_{2n} and v_{2n+1} are not in A. Let $A_1 = \{v_5, v_7\}$, $A_2 = \{v_6, v_8\}$, $A_3 = \{v_9, v_{11}\}$, …,$A_{n-3} = \{v_{2n-4}, v_{2n-2}\}$, $A_{n-2} = \{v_{2n-1}\}$. A contains at most one vertex from each A_i. Hence $|A| \leqslant n - 2 + 2 = n$.
Case 2: n is even.
Subcase 2.1: Only one of v_1, v_2 and v_4 is in A.
Let $A_1 = \{v_1, v_2, v_4\}$, $A_2 = \{v_3, v_5\}$, $A_3 = \{v_6, v_8\}$,$A_4 = \{v_7, v_9\}$ …,

$A_{n-1} = \{v_{2n-2}, v_{2n}\}$, $A_n = \{v_{2n-1}, v_{2n+1}\}$. A contains at most one vertex from each A_i. Therefore $|A| \leqslant n$.

Subcase 2.2: v_1 and v_2 are in A.

Then v_3, v_4, v_{2n} and v_{2n+1} are not in A. Let $A_1 = \{v_5, v_7\}$, $A_2 = \{v_6, v_8\}$, $A_3 = \{v_9, v_{11}\}$, ...,$A_{n-3} = \{v_{2n-3}, v_{2n-1}\}$, $A_{n-2} = \{v_{2n-2}\}$. A contains at most one vertex from each A_i. Hence $|A| \leqslant n - 2 + 2 = n$.

Subcase 2.3: v_1 and v_4 are in A. Then v_2, v_3, v_6 and v_{2n} are not in A. Let $A_1 = \{v_5, v_7\}$, $A_2 = \{v_8, v_{10}\}$, $A_3 = \{v_9, v_{11}\}$, ...,$A_{n-3} = \{v_{2n-3}, v_{2n-1}\}$, $A_{n-2} = \{v_{2n+1}\}$. A contains at most one vertex from each A_i. Hence $|A| \leqslant n - 2 + 2 = n$.

Thus in all the cases $|A| \leqslant n$. □

Corollary 3.3.11. *For any $m \leqslant n$, there exists an $S \subseteq V(C_{2n+1})$ such that $|C_S(C_{2n+1})| = m$.*

Proof. Let $C_{2n+1} = (v_1, v_2, \ldots, v_{2n+1}, v_1)$.

Given an $m \leqslant n$, we shall prove the existance of a subset of $V(C_{2n+1})$ of size m which does not contain any pair of alternate vertices. Take $2n+1-m$ circularly arranged 0's. Number these 0's $1, 2, \ldots, 2n+1-m$. If m is even put two 1's each between the first and the second 0's, third and the fourth 0's etc up to $(m-1)^{th}$ and the m^{th} 0's. If m is odd put two 1's each between the first and the second 0's, third and the fourth 0's etc., up to $(m-2)^{th}$ and the $(m-1)^{th}$ 0's and one 1 between m^{th} and $(m+1)^{th}$ 0's. In both these cases we get a circular arrangement of 0's and 1's that has m 1's and does not contain a pattern of the type 101 or 111. Starting at an arbitrary point represent these bits by $v_1, v_2, \ldots, v_{2n+1}$ and form the vertex set corresponding to the $1's$. This is a center set have m vertices. □

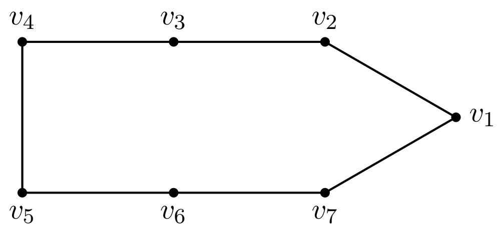


Figure 3.5: C_7

Illustration 3.3.4. Consider the odd cycle C_7.
Here $A = \{v_1, v_4, v_5\}$ is a center set, since it contains no pair of alternate vertices. $A^c = \{v_2, v_3, v_6, v_7\}$. The set of eccentric vertices of A^c which are not eccentric to any of the vertices of A is $\{v_3, v_6\}$.
$e_{\{v_3,v_6\}}(v_1) = 2$, $e_{\{v_3,v_6\}}(v_2) = 3$, $e_{\{v_3,v_6\}}(v_3) = 3$, $e_{\{v_3,v_6\}}(v_4) = 2$, $e_{\{v_3,v_6\}}(v_5) = 2$, $e_{\{v_3,v_6\}}(v_6) = 3$, $e_{\{v_3,v_6\}(v_7)} = 3$. Thus
$C_{\{v_3,v_6\}}(C_7) = \{v_1, v_4, v_5\}$. $\{v_1, v_4, v_5, v_6\}$ is not a center set since it contains (v_4, v_6) and (v_1, v_6), pairs of alternate vertices.

3.3.6 Center sets of Symmetric Even graphs

The following theorem gives the center sets of some familiar classes of graphs such as even cycles, hypercubes etc. Here we recall the following definition.

Definition 15. For an $S \subseteq V$, a vertex $x \in S$ is called an *interior vertex* if $N(x) \subseteq S$. An $S \subseteq V$ is called a *boundary set* of G if does not contain any interior vertices.

Theorem 3.3.12. *Let G be a symmetric even graph. An $A \subseteq V$ is a* *center set if and only if either $A = V$ or A is a boundary set of G.*

Proof. Since symmetric even graphs are self-centered $C_V(G) = V$. So assume $A \subset V$. Let A be such that $A = C_S(G)$ for an $S \subset V$ and let $x \in A$. Suppose $e_S(x) = k$ with $d(x, y) = k$ where $y \in S$. If $k = diam(G)$ then $A = V$. So assume $k < diam(G)$. Then since G is a symmetric even graph there exists a vertex z adjacent to x such that $d(y, z) = k + 1$. Therefore $e_S(z) \geqslant k + 1$ or $z \notin C_S(G)$. Hence if A is a center set such that $A \subset V$, then there exists an x in A such that $\{x\} \cup N(x) \cap S^c \neq \emptyset$.

Conversely, suppose that $A \subset V$ satisfies the condition given in the theorem. We need to find out an $S \subseteq V$ such that $A = C_S(G)$. Since G is symmetric even it is self-centered and unique eccentric vertex. Let $\overline{A^c}$ denote the set of eccentric vertices of A^c. Let $x \in A$. Then there exists a x' adjacent to x such that $x' \in A^c$. Then $\overline{x'} \in \overline{A^c}$. Since $d(x', \overline{x'}) = diam(G)$ and x and x' are adjacent $d(x, \overline{x'}) = diam(G) - 1$. Also since G is unique eccentric vertex there does not exist an z in $\overline{A^c}$ such that $d(x, z) = diam(G)$. Therefore, $e_{\overline{A^c}}(x) = diam(G) - 1$ and for every $y \in A^c$, $e_{\overline{A^c}}(y) = diam(G)$. Since G is self-centered for every $x \in A$, $e_{\overline{A^c}}(x) = diam(G) - 1$ and for every $y \in A^c$, $e_{\overline{A^c}}(x') = diam(G)$. Therefore $C_{\overline{A^c}}(G) = A$. Hence the theorem. □

Corollary 3.3.13. *For the even cycle C_{2n}, if A is a center set then either $|A| \leqslant \lfloor \frac{4n}{3} \rfloor$ or $A = V$.*

Proof. Suppose A is a center set such that $|A| < 2n$. To prove $|A| \leqslant \lfloor \frac{4n}{3} \rfloor$. Since A is a center set A cannot contain three consecutive vertices of the cycle. Let each vertex belonging to A be represented by 1 and each vertex not belonging to A be represented by 0. Thus we get a circular arrangement of 0's and 1's such that two successive 0's contains at most two 1's between them. From this we can conclude that m 0's can accommodate at most $2m$ 1's between them. If $A' \neq V$ is a center set of maximum cardinality then the binary representation of A' will have exactly $\lceil \frac{2n}{3} \rceil$ zeros and hence $2n - \lceil \frac{2n}{3} \rceil$ 1's. In other words $|A'| = 2n - \lceil \frac{2n}{3} \rceil = \lfloor \frac{4n}{3} \rfloor$. Since A' is a center set of maximum cardinality, we have $|A| \leqslant \lfloor \frac{4n}{3} \rfloor$. Hence the corollary. □

Next we have another corollary similar to the Corollary 3.3.11.

Corollary 3.3.14. *For any $m \leqslant \lfloor \frac{4n}{3} \rfloor$, there exists an $S \subseteq V(C_{2n})$ such that $|C_S(C_{2n})| = m$.*

Proof. Similar to the proof of Corollary 3.3.11 □

Now, we recall the following definitions.

Definition 16. An $S \subseteq V$ is a *dominating set* in G if every vertex in $V \setminus S$ is adjacent to a vertex in S.

Next, we shall prove a result regarding the centers of dominating sets of symmetric even graphs. But for that we require the following propositions from [54].

Proposition 3.3.15. Every harmonic even graph is balanced.

Proposition 3.3.16. Every Symmetric even graph is harmonic.

Combining the above two propositions we get the following proposition.

Proposition 3.3.17. Every Symmetric even graph is balanced.

Theorem 3.3.18. *Let G be a symmetric even graph and let $S \subseteq V$. Then $C_S(G) = \overline{S^c}$ if and only if S is a dominating set.*

Proof. Assume $C_S(G) = \overline{S^c}$. Suppose $S \cup N(S) \neq V$. Then there exists an $x \in V$ such that $x \notin S$ and $x \notin N(S)$. That is x and all its neighbours belong to S^c. Let $x_1, \ldots, x_k$ be the neighbours of x. By proposition 3.3.17, $deg(u) = deg(\bar{u})$. Let $y_1, y_2, \ldots, y_k$ be the neighbours of $\bar{x}$. We have $d(x_i, \bar{x}) = diam(G) - 1$ for $1 \leqslant i \leqslant k$. Since G is symmetric even there

exists a vertex adjacent to $\bar{x}$, say y_i, such that $d(x_i, y_i) = diam(G)$ for $1 \leqslant i \leqslant k$. Hence $\bar{x}$ and all its neighbours belong to $\overline{S^c}$. This contradicts the condition for $\overline{S^c}$ to be a center set.
Conversely suppose $S \cup N(S) = V$. Let $x \in \overline{S^c}$. Then $\bar{x} \in S^c$. Since $S \cup N(S) = V$, $\bar{x} \in N(S)$. Therefore there exists an $z \in S$ such that z is adjacent to $\bar{x}$. Then $d(x, z) = diam(G) - 1$. $d(x, z') = diam(G)$ for some $z' \in S$ implies both $y \in S^c$ and $z' \in S$ are the eccentric vertices of x a contradiction to the fact that the graph is unique eccentric vertex. Hence $e_S(x) = diam(G) - 1$. Now let $x \notin \overline{S^c}$. Then since every vertex is an eccentric vertex, $x \in \overline{S}$ and therefore there exists a w in S such that $d(x, w) = diam(G)$. Thus $C_S(G) = \overline{S^c}$. □

For a graph G, let $\mathcal{DB}(G)$ denote the class of dominating boundary sets, that is, dominating sets which are also boundary sets. We have the following theorem on the centers of sets which belong to such a class of sets in a symmetric even graph.

Theorem 3.3.19. *Let G be a symmetric even graph. Let $S \subseteq V$ be such that $S \in \mathcal{DB}(G)$. Then $C_S(G) = S'$ if and only if $C_{S'}(G) = S$.*

Proof. Suppose $C_S(G) = S'$. Since $S \cup N(S) = V$, $C_S(G) = \overline{S^c}$. That is $S' = \overline{S^c}$. For every $x \in S^c$, $e_{\overline{S^c}}(x) = diam(G)$. Since G is unique eccentric vertex graph and S is a boundary set, for every $x \in S$, $e_{\overline{S^c}}(x) = diam(G) - 1$. Hence $C_{S'}(G) = C_{\overline{S^c}}(G) = S$. Conversely assume $C_{S'}(G) = S$. To prove $C_S(G) = S'$. Since $C_S(G) = \overline{S^c}$ we need only prove that $S' = \overline{S^c}$. Let $x \in S'$. If $x \in \overline{S}$ then $x = \bar{y}$ where $y \in S$. Then we have $d(x, y) = diam(G)$. Since S is the S'-center of G this implies $C'_S(G) = V$. But this contradicts the fact that S is a boundary set. Hence $x \in \overline{S^c}$ or $S' \subseteq \overline{S^c}$. Now to prove that $\overline{S^c} \subseteq S'$. On the contrary assume that there exists an $x \in \overline{S^c}$ such that $x \notin S'$. Let $x = \overline{y}$ where $y \in S^c$. Since the eccentric vertex of y, x, does not belong to S', $e_{S'}(y) \leqslant diamG - 1$. If $z \in S'$ then $z \in \overline{S^c}$. Let

$z = \overline{w}$ where $w \in S^c$. Since $S \cup N(S) = V$ there exists a w' adjacent to w such that w' belong to S. We have $e_{S'}(w') = diamG - 1$. This implies $y \in S$, contradicting the choice of y. Therefore $S' = \overline{S^c}$. □

Theorem 3.3.20. *Let G be a symmetric even graph. Then*

i) $S \in \mathcal{DB}(G)$ if and only if $C_S(G) \in \mathcal{DB}(G)$.

ii) For $S_1, S_2 \in \mathcal{DB}(G)$, $C_{S_1}(G) = S_2$ if and only if $C_{S_2}(G) = S_1$.

Proof. i) Suppose $S \subseteq V$ is such that $S \in \mathcal{DB}(G)$ and let $S' = C_S(G)$. Since S' is a center set of a symmetric even graph if and only if it is a boundary set, to prove that $S' \in \mathcal{DB}(G)$ we need only prove that $S' \cup N(S') = V$. Since $S \cup N(S) = V$, $S' = \overline{S^c}$. Let $x \notin S'$. Therefore $x \in \overline{S}$ since the graph is symmetric even. Let $x = \bar{y}$ where $y \in S$. Since S is a boundary set there exists a vertex y' adjacent to y such that $y' \in S^c$. We have $d(x, y') = diam(G) - 1$. Since G is symmetric even there exists a vertex x' adjacent to x such that $d(x', y') = diam(G)$. That is $x' \in \overline{S^c}$ or $x' \in S'$. In other words $x \in N(S')$. Hence $S' \cup N(S') = V$. Conversely suppose $S' \subseteq V$ is such that $S' \in \mathcal{DB}(G)$ and $C_S(G) = S'$ for an $S' \subseteq V$. To prove $S \in \mathcal{DB}(G)$. By the previous theorem $C_S(G) = S'$ implies $C_{S'}(G) = S$. Now $S' \subseteq V$ is such that $S' \in \mathcal{DB}$ and $C_{S'}(G) = S$ and hence as proved earlier we can prove that $S \cup N(S) = V$ or $S \in \mathcal{DB}(G)$.

ii) This part is obvious from Theorem 3.3.19.

□

Illustration 3.3.5. Let G be the 12-cycle with vertex set $\{v_1, \ldots, v_{12}\}$.

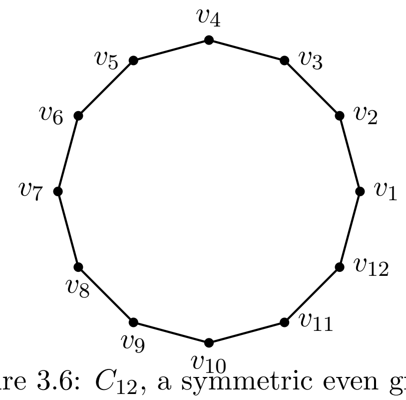


Figure 3.6: C_{12}, a symmetric even graph

Take the vertex set $S = \{v_1, v_3, v_5, v_8, v_{11}\}$. This is a dominating boundary set. We have that $C_S(G) = \{v_1, v_3, v_4, v_6, v_8, v_{10}, v_{12}\} = \overline{S^c}$ and this is again a dominating boundary set It can also be verified that $C_{\overline{S^c}}(G) = S$. Consider $A = \{v_1, v_3, v_4, v_7\}$.
$A^c = \{v_2, v_5, v_6, v_8, v_9, v_{10}, v_{11}, v_{12}\}$. $\overline{A^c} = \{v_8, v_{11}, v_{12}, v_2, v_3, v_4, v_5, v_6\}$.
$e_{\overline{A^c}}(v_1) = 5e_{\overline{A^c}}(v_2) = 6$, $e_{\overline{A^c}}(v_3) = 5$, $e_{\overline{A^c}}(v_4) = 5$, $e_{\overline{A^c}}(v_5) = 6$,
$e_{\overline{A^c}}(v_6) = 6$, $e_{\overline{A^c}}(v_7) = 5$, $e_{\overline{A^c}}(v_8) = 6$, $e_{\overline{A^c}}(v_9) = 6$, $e_{\overline{A^c}}(v_{10}) = 6$,
$e_{\overline{A^c}}(v_{11}) = 6$, $e_{\overline{A^c}}(v_{11}) = 6$. Hence $C_{\overline{A^c}}(C_{12}) = A$.

3.4 Enumerating Center Sets

In designing and modelling networks it is important to have more center sets to locate facilities. Therefore the number of center sets is a good indication to the structural well-behaviour of the graph. In this section we enumerate the center sets of various classes of graphs. We first give the following definition.

Definition 3.4.1. The number of distinct center sets of a graph G is defined as the *Center number* of G and is denoted by $cn(G)$.

The following lemma gives the center numbers of some familiar classes of graphs. The proof of the lemma follows from the Corollary 3.3.3, Theorem 3.3.6, Corollary 3.3.4, Theorem 3.3.7 and Proposition 3.3.8 respectively, so we omit the proofs.

Lemma 3.4.1. Let G be a graph on n vertices then,

1. $cn(G) = n + 1$ when $G = K_n$ the complete graph on n vertices.

2. $cn(G) = n + 3$ when $G = K_{p,q}$ the complete bipartite graph with $p, q > 1$.

3. $cn(G) = 2n - 1$ when G is a tree.

4. $cn(G) = n + 4$ when $G = K_n - e$, $e \in E$, $n \geqslant 4$.

5. If G is the wheel graph W_n then

$$\begin{aligned} cn(W_n) &= 4n - 3 \text{ if } n \geqslant 6 \\ &= 4n - 1 \text{ if } n = 5 \end{aligned}$$

3.4.1 Center number of Even and Odd cycles

We now find the center number of odd and even cycles. For that we introduce the following terms. Suppose we have n *linearly arranged* objects. Let $L(n, k)$ denote the number of ways of choosing k objects from these n objects so that no three consecutive objects are simultaneously chosen. Let $L_1(n, k)$ denote the number ways to choose k objects from these n objects so that no two objects from alternate positions are simultaneously chosen and let $L_2(n, k)$ denote the number of ways to choose k objects from these n objects so that no two objects from consecutive positions are simultaneously chosen.

Consider n *circularly arranged* objects where $n \geqslant 4$. Let $R(n, k)$ denote the number of ways to choose k objects from these n objects so that three objects from three consecutive positions are not chosen and $R_1(n,k)$ denote the number ways to choose k objects from these n objects so that no two objects from alternate positions are simultaneously chosen. Here we assume $n \geqslant 4$ since we are interested only in cycles of length greater than 3.

Lemma 3.4.2. $L(n,k) = \binom{n}{k}\binom{n-k+1}{0} - \binom{n-3}{k-3}\binom{n-k+1}{1} + \binom{n-6}{k-6}\binom{n-k+1}{2} - \binom{n-9}{k-9}\binom{n-k+1}{3} + \cdots$.

Proof. A particular choice of k objects from n objects can be represented by a binary string of size n where a 1 at the i^{th} position indicates that the i^{th} object is chosen and a 0 at the j^{th} position indicates that the j^{th} object is not chosen. So the number of choices of the required type is actually the number of binary strings of size n having k 1's and not containing three consecutive 1's. Let x_0 denote the number of 1's before the first 0, for $1 \leqslant i \leqslant n-k-1$, let x_i denote the number of 1's between the i^{th} 0 and the $(i+1)^{th}$ 0 and let x_{n-k} denote the number of 1's after the $(n-k)^{th}$ 0. Therefore the total number of 1's in a binary string is $x_0 + x_1 + \cdots + x_{n-k}$. For a binary string of our choice, $0 \leqslant x_i \leqslant 2$. Hence $L_1(n,k)$ is the number of different solutions of the equation

$$x_0 + x_1 + \cdots + x_{n-k} = k, 0 \leqslant x_i \leqslant 2 \tag{3.1}$$

Now consider the product

$$\underbrace{(1+t+t^2) \times \cdots \times (1+t+t^2)}_{(n-k+1) \text{ times}} \tag{3.2}$$

In the expansion of this product, taking t^{y_0} from the first term, t^{y_1} from the second term, ..., $t^{y_{n-k}}$ from the $n-k+1^{th}$ term we get $t^{y_0+y_1+\ldots+t_{n-k}}$.

Therefore any solution of the equation

$$y_0 + y_1 + \ldots + y_{n-k} = k, 0 \leqslant y_i \leqslant 2 \tag{3.3}$$

gives us the term y^k in the expansion. In other words the number of solutions of equation 3.3 is the coefficient of t^k in expression 3.2. Since the Equations 3.1 and 3.3 are same, we get that $L(n,k)$ is the coefficient of t^k in $(1+t+t^2)^{n-k+1}$.

$$\begin{aligned}(1+t+t^2)^{n-k+1} &= \left(\frac{1-t^3}{1-t}\right)^{n-k+1} \\ &= (1-t^3)^{n-k+1}(1-t)^{-(n-k+1)} \\ &= \left(1 - \tbinom{n-k+1}{1}t^3 + \tbinom{n-k+1}{2}t^6 + \cdots\right) \\ &\quad \times \left(1 + \tbinom{n-k+1}{1}t + \tbinom{n-k+1}{2}t^2 + \cdots + \tbinom{n}{k}t^k + \cdots\right)\end{aligned}$$

Therefore $L(n,k)= \binom{n}{k}\binom{n-k+1}{0} - \binom{n-3}{k-3}\binom{n-k+1}{1} + \binom{n-6}{k-6}\binom{n-k+1}{2} - \binom{n-9}{k-9}\binom{n-k+1}{3} + \cdots$.

The series on the right hand side is finite as all the terms after a finite number of terms shall be zero. □

Lemma 3.4.3. $R(n,k) = L(n-1,k) + 2L(n-4,k-2) + L(n-3,k-1)$, $n \geqslant 4,\ k \geqslant 2$.

Proof. Let the n circularly arranged objects be $v_1, \ldots, v_n$. The set of all objects such that no 3 objects from 3 consecutive positions are chosen can be divided in to the following types.

Type 1 The object v_n is chosen and the objects v_{n-1} and v_1 are not chosen. Then the total number of choices is $L(n-3,k-1)$. (See Figure 3.7)

Type 2 The objects v_n and v_{n-1} are chosen and v_1 is not chosen. v_n and v_{n-1} are chosen implies v_{n-2} is not chosen. In this case the number of

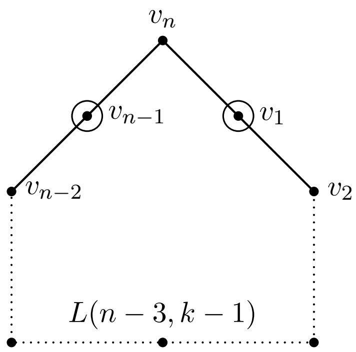


Figure 3.7

choices is $L(n-4, k-2)$.

The objects v_n and v_1 are chosen and v_{n-1} is not chosen. Again as in the previous case the total number of choices is $L(n-4, k-2)$.

The object v_n is not chosen. Here the total number of choices is $L(n-1, k)$. Therefore $R(n, k) = L(n-1, k) + 2L(n-4, k-2) + L(n-3, k-1)$. □

It is obvious that

$$\begin{aligned} R(n, k) &= 1 \text{ when } k = 0 \\ &= n \text{ when } k = 1 \end{aligned}$$

Now we have determined $R(n, k)$ for all $n \geqslant 4$ and $k \geqslant 0$.

Theorem 3.4.2. *The center number of the even cycle C_{2n} is* $1 + \sum_{k=1}^{\lfloor \frac{4n}{3} \rfloor} R(2n, k)$.

Proof. By the Corollary 3.3.13, the maximum cardinality among the center sets other than V is $\lfloor \frac{4n}{3} \rfloor$ and by the Theorem 3.3.12, $R(2n, k)$ gives the number of center sets of size k where $k \leqslant \lfloor \frac{4n}{3} \rfloor$. Also V is a center set.

Hence $cn(C_{2n}) = 1 + \sum_{k=1}^{\lfloor \frac{4n}{3} \rfloor} R(2n, k)$. □

Illustration 3.4.1. Consider the even cycle C_{12}. Here $n = 6$. Then

$$cn(C_{12}) = 1 + \sum_{k=1}^{8} R(12, k) \tag{3.4}$$

It is obvious that

$$R(12, 1) = 12. \tag{3.5}$$

$R(12, 2) = L(11, 2) + 2L(8, 0) + L(9, 1)$
$L(11, 2) = \binom{11}{2} = 55$, $L(8, 0) = 1$ and $L(9, 1) = 9$. Hence,

$$R(12, 2) = 55 + 2 + 9 = 66 \tag{3.6}$$

$R(12, 3) = L(11, 3) + 2L(8, 1) + L(9, 2)$
$L(11, 3) = \binom{11}{3}\binom{9}{0} - \binom{8}{0}\binom{9}{1} = 165 - 9 = 156.$
$L(8, 1) = 8$ and $L(9, 2) = \binom{9}{2} = 36$. Hence,

$$R(12, 3) = 156 + 16 + 36 = 208 \tag{3.7}$$

$R(12, 4) = L(11, 4) + 2L(8, 2) + L(9, 3)$
$L(11, 3) = \binom{11}{4}\binom{8}{0} - \binom{8}{1}\binom{8}{1} = 330 - 64 = 266$
$L(8, 2) = \binom{8}{2} = 28$
$L(9, 3) = \binom{9}{3}\binom{7}{0} - \binom{6}{0}\binom{7}{1} = 84 - 7 = 77.$ Hence,

$$R(12, 4) = 266 + 56 + 77 = 399 \tag{3.8}$$

$R(12, 5) = L(11, 5) + 2L(8, 3) + L(9, 4)$
$L(11, 5) = \binom{11}{5}\binom{7}{0} - \binom{8}{2}\binom{7}{1} = 462 - 196 = 266$
$L(8, 3) = \binom{8}{3}\binom{6}{0} - \binom{5}{0}\binom{6}{1} = 56 - 6 = 50$

$L(9,4) = \binom{9}{4}\binom{6}{0} - \binom{6}{1}\binom{6}{1} = 126 - 36 = 90$. Hence,

$$R(12,5) = 266 + 100 + 90 = 456 \tag{3.9}$$

$R(12,6) = L(11,6) + 2L(8,4) + L(9,5)$
$L(11,6) = \binom{11}{6}\binom{6}{0} - \binom{8}{3}\binom{6}{1} + \binom{5}{0}\binom{6}{2} = 462 - 336 + 15 = 141$
$L(8,4) = \binom{8}{4}\binom{5}{0} - \binom{5}{1}\binom{5}{1} = 70 - 25 = 45$
$L(9,5) = \binom{9}{5}\binom{5}{0} - \binom{6}{2}\binom{5}{1} = 126 - 75 = 51$. Hence,

$$R(12,6) = 141 + 90 + 51 = 282 \tag{3.10}$$

$R(12,7) = L(11,7) + 2L(8,5) + L(9,6)$
$L(11,7) = \binom{11}{7}\binom{5}{0} - \binom{8}{4}\binom{5}{1} + \binom{5}{1}\binom{5}{2} = 330 - 350 + 50 = 30$
$L(8,5) = \binom{8}{5}\binom{4}{0} - \binom{5}{2}\binom{4}{1} = 56 - 40 = 16$
$L(9,6) = \binom{9}{6}\binom{4}{0} - \binom{6}{3}\binom{4}{1} + \binom{3}{0}\binom{4}{2} = 84 - 80 + 6 = 10$. Hence,

$$R(12,7) = 30 + 32 + 10 = 72 \tag{3.11}$$

$R(12,8) = L(11,8) + 2L(8,6) + L(9,7)$
$L(11,8) = \binom{11}{8}\binom{4}{0} - \binom{8}{5}\binom{4}{1} + \binom{5}{2}\binom{4}{2} = 165 - 224 + 60 = 1$
$L(8,6) = \binom{8}{6}\binom{3}{0} - \binom{5}{3}\binom{3}{1} + \binom{2}{0}\binom{3}{2} = 28 - 30 + 3 = 1$
$L(9,7) = \binom{9}{7}\binom{3}{0} - \binom{6}{4}\binom{3}{1} + \binom{3}{1}\binom{3}{2} = 36 - 45 + 9 = 0$. Hence,

$$R(12,8) = 1 + 2 + 0 = 3 \tag{3.12}$$

Using equations 3.5 to 3.12 in 3.4 we get
$cn(C_{12}) = 1 + 12 + 66 + 208 + 399 + 456 + 282 + 72 + 3 = 1499$

Before proving the center number of odd cycles, we prove the following lemmata. We first find $L_2(n,k)$ for given values of n and k.

Lemma 3.4.4. $L_2(n,k) = \binom{n-k+1}{k}$.

Proof. As in Lemma 3.4.2, we give a binary representation for a particular choice of k objects that conforms to the conditions specified in the definition of $L_2(n,k)$. For each 1 in this binary representation we count the total number of 0's preceding this 1. So if we have k 1's then we get k numbers from $\{0,1,\ldots,n-k\}$ and all these are distinct since there should be at least one 0 between any two successive 1's. Thus corresponding to each choice of k objects of the desired type we get a unique set of k distinct numbers from $\{0,1,\ldots,n-k\}$. Conversely each choice of k distinct numbers from $\{0,1,\ldots,n-k\}$ gives us a unique choice of k objects from n linearly arranged objects satisfying the specified condition. Thus we get a one-to-one correspondence between the k-element subsets of $\{0,1,\ldots,n-k\}$ and the choices of k objects as specified in the definition of $L_2(n,k)$. Hence $L_2(n,k) = \binom{n-k+1}{k}$. □

Lemma 3.4.5. $L_1(n,k) = \sum_{\ell=0}^{k} L_2(\lfloor \frac{n}{2} \rfloor, \ell) L_2(\lceil \frac{n}{2} \rceil, k-\ell)$.

Proof. Consider n linearly arranged objects. Choosing k objects from these n objects such that no two objects are from alternate positions can be done as follows. First choose ℓ objects from $\lceil \frac{n}{2} \rceil$ objects in the odd positions such that no two objects are consecutive among these $\lceil \frac{n}{2} \rceil$ objects. This can be done in $L_2(\lceil \frac{n}{2} \rceil, \ell)$ ways. Now choose $k-\ell$ objects from the remaining $\lfloor \frac{n}{2} \rfloor$ objects in the even positions, such that no two objects are consecutive among these $\lfloor \frac{n}{2} \rfloor$ objects. This can be done $L_2(\lfloor \frac{n}{2} \rfloor, k-\ell)$ ways. Hence $L_1(n,k) = \sum_{\ell=0}^{k} L_2(\lceil \frac{n}{2} \rceil, \ell) L_2(\lfloor \frac{n}{2} \rfloor, k-\ell)$. □

Lemma 3.4.6. $L_1(n,k) = \sum_{\ell=0}^{k} \binom{\lfloor \frac{n}{2} \rfloor - \ell + 1}{\ell} \binom{\lceil \frac{n}{2} \rceil - (k-\ell) + 1}{k-\ell}$.

Proof. The proof follows from Lemma 3.4.5 and Lemma 3.4.4. □

Lemma 3.4.7. $R_1(n,k) = L_1(n-2,k)+2L_1(n-5,k-1)+3L_1(n-6,k-2)$, $n \geqslant 6, k \geqslant 2$.

Proof. Let the n circularly arranged objects be $v_1, \ldots, v_n$. The set of all choices of k objects such that no two objects occupy alternate positions can be divided in to various types.

Type I: Both v_n and v_{n-1} are not chosen. In this case the total number of choices is $L_1(n-2,k)$ (See Figure 3.8).

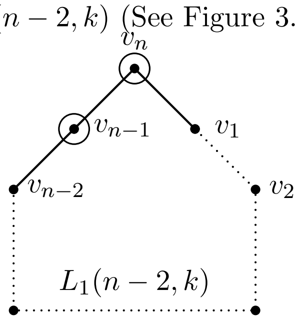


Figure 3.8

Type II: v_n is selected and v_{n-1} is not selected. v_n is selected implies v_{n-2} and v_2 are not selected. The number of choices where v_1 is selected is $L_1(n-6,k-2)$ and the number of choices where v_1 is not selected is $L_1(n-5,k-1)$. Hence the total number of such choices is $L_1(n-6,k-2)+L_1(n-5,k-1)$.

Type III: v_n is not selected and v_{n-1} is selected. As in the previous case the total number of such choices is
$L_1(n-6,k-2)+L_1(n-5,k-1)$.

Type IV: Both v_n and $v_{n-1}$are selected. v_n and v_{n-1} are selected implies v_1, v_2, v_{n-2} and v_{n-3} are not selected. Therefore the number of choices of this type is $L_1(n-6,k-2)$.

$$\text{Hence, } R_1(n,k) = L_1(n-2,k)+2L_1(n-5,k-1)+3L_1(n-6,k-2). \quad (3.13)$$

□

Now it is easy to see that

$$\begin{aligned} R_1(n,k) &= 1, \text{ when } k=0 \\ &= n, \text{ when } k=1 \text{ or } k=2 \text{ and } n=4 \text{ or } 5 \\ &= 0, \text{ when } k \geqslant 3,\ n=4 \text{ or } 5 \end{aligned}$$

Thus we have determined $R_1(n,k)$ for all $n \geqslant 4$ and $k \geqslant 0$.

Now with the help of Theorem 3.3.9 and Corollary 3.3.10, we have the center number of the odd cycle C_{2n+1}, $n \geqslant 2$.

Theorem 3.4.3. *The center number of the odd cycle C_{2n+1}, $n \geqslant 2$, is* $1 + \sum_{k=1}^{n} R_1(2n+1,k)$.

Illustration 3.4.2. We shall find out the center number of the odd cycle C_{11}. We have that

$$cn(C_{11}) = 1 + \sum_{k=1}^{5} R_1(11,k) \quad (3.14)$$

It is obvious that

$$R_1(11,1) = 11 \quad (3.15)$$

From equation 3.13 we have that
$R_1(11,2) = L_1(9,2) + 2L_1(6,1) + 3L_1(5,0)$

$$\begin{aligned} L_1(9,2) &= \sum_{\ell=0}^{2} \binom{4-\ell+1}{\ell}\binom{5-(2-\ell)+1}{2-\ell} \\ &= \binom{5}{0}\binom{4}{2} + \binom{4}{1}\binom{5}{1} + \binom{3}{2}\binom{5}{0} = 6+20+3 = 29 \end{aligned}$$

$L_1(6,1) = 6$ and $L_1(5,0) = 1$. Therefore,

$$R_1(11,2) = 29 + 12 + 3 = 44 \tag{3.16}$$

$R_1(11,3) = L_1(9,3) + 2L_1(6,2) + 3L_1(5,1)$
$L_1(9,3) = \binom{5}{0}\binom{3}{3} + \binom{4}{1}\binom{4}{2} + \binom{3}{2}\binom{5}{1} = 1 + 24 + 15 = 40$
$L_1(6,2) = \binom{4}{0}\binom{2}{2} + \binom{3}{1}\binom{3}{1} + \binom{2}{2}\binom{4}{0} = 1 + 9 + 1 = 11$
$L_1(5,1) = 5$. Therefore,

$$R_1(11,3) = 40 + 2 \times 11 + 3 \times 5 = 77 \tag{3.17}$$

$R_1(11,4) = L_1(9,4) + 2L_1(6,3) + 3L_1(5,2)$
$L_1(9,4) = \binom{4}{1}\binom{3}{3} + \binom{3}{2}\binom{4}{2} = 4 + 18 = 22$
$L_1(6,3) = \binom{3}{1}\binom{2}{2} + \binom{2}{2}\binom{3}{1} = 3 + 3 = 6$
$L_1(5,2) = \binom{3}{0}\binom{2}{2} + \binom{2}{1}\binom{3}{1} = 1 + 6 = 7$. Therefore,

$$R_1(11,4) = 22 + 2 \times 6 + 3 \times 7 = 55 \tag{3.18}$$

$R_1(11,5) = L_1(9,5) + 2L_1(6,4) + 3L_1(5,3)$
$L_1(9,5) = \binom{3}{2}\binom{3}{3} = 3$
$L_1(6,4) = \binom{2}{2}\binom{2}{2} = 1$
$L_1(5,3) = \binom{2}{1}\binom{2}{2} = 2$. Therefore,

$$R_1(11,5) = 3 + 2 \times 1 + 3 \times 2 = 11 \tag{3.19}$$

Using equations 3.15 to 3.19 in 3.14 we get
$cn(C_{11}) = 1 + 11 + 44 + 77 + 55 + 11 = 199$

3.5 Conclusion

In this chapter the generalisation of the center of a graph to the center of arbitrary vertex sets have been explored particularly with reference to some special classes of graphs like K_n, $K_{m,n}$, $K_n - e$, odd cycles and a more general class of graphs called symmetric even graphs. In the process of identification of center sets of odd cycles and symmetric even graphs we have devised methods for finding a set whose center is a prescribed set. The duality property of dominating boundary sets of symmetric even graphs with respect to the center function has been also brought to light. For any graph there may exist subsets of the vertex set whose center is the same as the center of the graph and therefore we can look for such sets with minimum cardinality. Searching on this line we came across a class of graphs where none of the proper subsets of the vertex sets has center equal to the center of the graph. We called them the center critical graphs and characterised them as self centred, unique eccentric vertex graphs. Finally we have enumerated the number of distinct center sets of some of the graphs mentioned above.

Chapter 4

Pacifying and Shrinking edges

4.1 Introduction

Extremal graph theory mostly deals with studying the classes of graphs that are minimal or maximal with respect to certain conditions. Most of the literature on distance related extremal graph theory is concerned with identifying the class of graphs that are radially maximal, radially minimal, diameter minimal, diameter maximal etc[see section 2.4]. The eccentricity of a vertex can be decreased by adding edges and it shall be interesting to identify such edges particularly in networking problems where, by adding a minimum number of edges we may be able to reduce the distances of an actor from other actors in the network remarkably and thus can increase its significance in the network. This is useful for the actor as well as the whole network as it increases the cohesion of the network at a minimal cost. In this chapter we take a particular case of this problem where we add a single edge. Given that we are allowed to add a single edge, we identify the edge(s) that when added to a graph reduces the eccentricity of a vertex the most. We also identify the edge(s) that reduces the radius of the graph the most. Such edges are being introduced as pacifying and shrinking edges respectively.

Definition 4.1.1. For a vertex $w \in G$, an edge $uv \notin E(G)$ is defined to be a *pacifying edge* of w if $e_{G+uv}(w) \leqslant e_{G+xy}(w)$ for all $xy \in E(G^c)$.

It is not necessary that every vertex of a graph has pacifying edges. One trivial example is the complete graph where every vertex has eccentricity one. There are other non trivial examples. Take the complete bipartite graph $K_{m,n}$ where $m, n > 2$. Each vertex of this graph has eccentricity

two. Since $m, n > 2$ by adding an edge between any single pair of non-adjacent vertices the eccentricity of none of the vertices reduces. In other words no vertex of $K_{m,n}$ has a pacifying edge. C_5 is another example of a graph in which no vertex has a pacifying edge.
The following is an example of a graph in which some vertices have pacifying edges while some others do not have any pacifying edge.

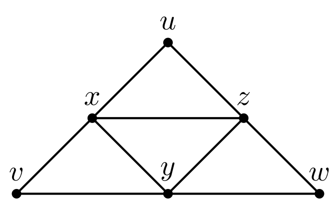


Figure 4.1: Graph having vertices with and without pacifying edges

Here, xw, uy and zv are the pacifying edges of x,y and z respectively as they reduce the eccentricity of these vertices from 2 to 1. But, the vertices u,v and w do not have any pacifying edge.

Observations
The following are some simple observations that can be made on the pacifying edges of the vertices of a graph with more than two vertices.

1. Every vertex having a unique eccentric vertex has a pacifying edge.

2. Every vertex whose eccentricity is greater than 2 and whose eccentric vertices are all mutually adjacent has a pacifying edge.

3. A vertex of a graph of diameter 2 has a pacifying edge if and only if its degree is $|V| - 2$.

4. C_5 is a graph in which no vertex has a pacifying edge. In fact, it is later shown later that in all other cycles every vertex has at least one pacifying edge.

4.2 Pacifying edges of some classes of graphs

4.2.1 Pacifying edges of a path

In the following theorem we identify the pacifying edges of the vertices of a path.

Theorem 4.2.1. *Consider the path P_n with end vertices a and b and let $w \in V(P_n)$. Assume that $d(w,a) \leqslant d(w,b)$.*

1. *If $d(w,a) = d(w,b)$ then w has no pacifying edges.*
2. *Let $d(w,b) < 2d(w,a)$ with $d(w,b) = d(w,a) + t, 0 < t < d(w,a)$. Then the pacifying edges of w are given by the following*

   i. *Edges w_1w_2's such that $w_1 \in w$-b path, $w_2 \in w_1$-b path, $d(w_1,w) = m$ where $0 \leqslant m \leqslant d(w,a) - \frac{t+1}{2}$ and*
   $t+1 \leqslant d(w_1,w_2) \leqslant d(w_1,b)$ when $0 \leqslant m < d(w,a) - t$,
   $t+1 \leqslant d(w_1,w_2) < 2(d(w,a) - m)$ when
   $d(w,a) - t \leqslant m \leqslant d(w,a) - \frac{t+1}{2}$

   ii. *Edges w_1w_2's such that $w_1 \in w$-a path, $w_2 \in w_1$-b path, $d(w_1,w) = m$ where*
   $0 \leqslant m \leqslant d(w,a) - \frac{t+1}{2}$ and $t+2m+1 \leqslant d(w_1,w_2) \leqslant d(w_1,b)$ when $0 \leqslant m < d(w,a) - t$,
   $t+2m+1 \leqslant d(w_1,w_2) \leqslant 2d(w,a)$ when
   $d(w,a) - t \leqslant m \leqslant d(w,a) - \frac{t+1}{2}$.

3. *When $2d(w,a) \leqslant d(w,b) < 3d(w,a)$ with $d(w,b) = d(w,a) + t$, $d(w,a) \leqslant t < 2d(w,a)$, the pacifying edges of w are given by the following*

i. *Edges* w_1w_2*'s such that* $w_1 \in w$-b *path,* $w_2 \in w_1$-b *path,* $d(w_1, w) = m$ *where* $0 \leqslant m \leqslant d(w,a) - \frac{t+1}{2}$ *and*
$t + 1 \leqslant d(w_1, w_2) \leqslant 2(d(w,a) - m)$

ii. *Edges* w_1w_2 *such that* $w_1 \in w$-a *path,* $w_2 \in w_1$-b *path,* $d(w_1, w) = m$ *where* $0 \leqslant m \leqslant d(w,a) - \frac{t+1}{2}$ *and*
$t + 2m + 1 \leqslant d(w_1, w_2) \leqslant 2d(w,a)$

4. *When* $d(w,b) \geqslant 3d(w,a)$,

i. *If* $d(w,b) = 3n$ *for some integer* n*, then the pacifying edges are*

a. w_1w_2 *where* $w_1 = w$ *and* $2n \leqslant d(w_1, w_2) \leqslant 2n + 2$.

b. w_1w_2 *where* w_1 *is the vertex adjacent to* w *on* w-b *path and* $d(w_1, w_2) = 2n$

c. w_1w_2 *where* w_1 *is the vertex adjacent to* w *on* w-a *path and* $d(w_1, w_2) = 2n + 2$

ii. *If* $d(w,b) = 3n + 1$ *for some integer* n *then the pacifying edges are* w_1w_2 *where* $w_1 = w$*,* $w_2 \in w$-b *path and*
$2n + 1 \leqslant d(w_1, w_2) \leqslant 2n + 2$.

iii. *If* $d(w,b) = 3n+2$ *for some integer* n *then the only pacifying edge is* w_1w_2 *where* $w_1 = w$*,* $w_2 \in w$-b *path and* $d(w_1, w_2) = 2n + 2$

Proof. In a path with end vertices a and b every vertex has either a or b as its eccentric vertex. By adding a single edge we can reduce the distance of a vertex to at most one of a and b. That is by adding a single edge eccentricity of a vertex can be at most reduced to the smaller of its distances to a and b. Let $d(w,a) = y$.

Case 1: $d(w,a) = d(w,b)$.

By the above statements w has no pacifying edges.

Case 2:$d(w,b) < 2d(w,a)$.

That is $d(w,b) = y + t$ where $0 < t < y$. Consider the graph $G + (w,b)$. Let w' be the eccentric vertex of w in the unique cycle of the graph

$G+(w,b)$. $d_{G+(w,b)}(w,a)=y$ and $d_{G+(w,b)}(w,w')<y$ and $d_{G+(w,b)}(w,b)=1$. Therefore $e_{G+(w,b)}(w)=y$. Hence by the observations that we made at the beginning of the proof (w,b) is a pacifying edge of the vertex w.

i. Now take a vertex w_1 in the w-b path at a distance $m(\geqslant 0)$ from w. Join it to vertex w_2 at a distance ℓ from w_1 in the w_1-b path. Let w_1' be the eccentric vertex of w_1 in the unique cycle of $G+w_1w_2$. Assume $\ell<t+1$. Then $t+1-\ell>0$ or $y+t+1-\ell-m+m-y>0$. That is $(y+t+\ell-m)+m+1>y$. But $(y+t+\ell-m)+m+1$ is $d_{G+w_1w_2}(w,b)$. Therefore the $e_{G+w_1w_2}(w)>y$. In other words, w_1w_2 is not a pacifying edge. That is if w_1w_2 is a pacifying edge of w, $d(w_1,w_2)=\ell\geqslant t+1$. If $\ell>2(y-m)$ then $d(w_1,w_1')>y-m$ and therefore $d(w,w_1')>y$. Hence $\ell\leqslant 2(y-m)$. That is $t+1\leqslant\ell\leqslant 2(y-m)$.
Also $m>y-\frac{t+1}{2}\Leftrightarrow 2(y-m)<t+1$. Therefore $0\leqslant m\leqslant y-\frac{t+1}{2}$.
Now $m\leqslant y-t$ if and only if $2(y-m)\geqslant y+t-m$. That is $m\leqslant y-t$ if and only if $2(y-m)\geqslant d(w_1,b)$. In this case $t+1\leqslant\ell\leqslant d(w_1,b)$.
When $m>y-t$, $t+1\leqslant\ell\leqslant 2(y-m)$.
$d(w_1,b)<t+1\Longleftrightarrow y+t-m<t+1\Longleftrightarrow m>y-1$.
Therefore when $m<y-t$, $d(w_1,b)\geqslant t+1$.
Conversely, let w_1 and w_2 be such that $d(w_1,w)=m<y-t$ and $t+1\leqslant d(w_1,w_2)\leqslant d(w_1,b)$. Since $d(w_1,w_2)\geqslant t+1$, $t-d(w_1,w_2)+1\leqslant 0$ or $y+t-d(w_1,w_2)+1\leqslant y$. That is $y+t-d(w_1,w_2)-m+m+1\leqslant y$. In other words $d_{G+w_1w_2}(w,b)\leqslant y$. Since $m\leqslant y-t$, $d(w_1,b)\leqslant 2(y-m)$ and therefore $d(w_1,w_2)\leqslant 2(y-m)$. Hence $d(w,w_1')\leqslant y$ where w_1' is the eccentric vertex of w_1 in the unique cycle of $G+w_1w_2$. That is $e_{G+w_1w_2}=y$. That is, the edge w_1w_2 is a pacifying edge of w.
Now assume that $y-\frac{t+1}{2}\geqslant d(w_1,w)=m>y-t$ and $t+1\leqslant d(w_1,w_2)\leqslant 2(y-m)$. Since $y-\frac{t+1}{2}\geqslant d(w_1,w)=m$ we have that $t+1\leqslant 2(y-m)$. We have already proved that when $d(w_1,w_2)\geqslant t+1$, $d(w,b)\leqslant y$. Since $d(w_1,w_2)\leqslant 2(y-m)$, $d(w,w_1')\leqslant y$. That is

$e_{G+w_1w_2}(w) = y$. Hence w_1w_2 is a pacifying edge of w.

ii. Take w_1 at a distance m from w in the w-a path and let w_2 be at a distance ℓ from w_1 in the w_1-b path. Let $\ell < t + 2m + 1$. Then $t - \ell + 2m + 1 > 0$ or $y + t - \ell + m + m + 1 > y$. This gives, $y + t - (\ell - m) + m + 1 > y$. That is $d_{G+w_1w_2}(w, b) > y$. Therefore w_1w_2 is not a pacifying edge of w. In other words for a pacifying edge w_1w_2 of w, $d(w_1, w_2) = \ell \geqslant t + 2m + 1$. $\ell > 2y \implies d(w, w') > y$ where w' is the eccentric vertex of w in the unique cycle of $G + w_1w_2$. Hence $\ell \leqslant 2y$. Therefore $t + 2m + 1 \leqslant \ell \leqslant 2y$.
$2y < t + 2m + 1 \Leftrightarrow m > y - \frac{t+1}{2}$. Hence $m \leqslant y - \frac{t+1}{2}$. When $m \leqslant y - t$, $m + y + t \leqslant 2y$. That is $d(w_1, b) \leqslant 2y$. Therefore when $0 < m \leqslant y - t$, $t + 2m + 1 \leqslant \ell \leqslant d(w_1, b)$ and when $y - \frac{t+1}{2} \geqslant m > y - t$, $t + 2m + 1 \leqslant \ell \leqslant 2y$. Here $d(w_1, b) < t + 2m + 1 \implies m + y + t < t + 2m + 1 \implies y < m + 1$ or $m > y - 1$. Therefore when $m \leqslant y - t$, $d(w_1, b) \geqslant t + 2m + 1$.
Now we shall prove that if w_1 and w_2 are such that $w_1 \in w$-a path, $d(w_1, w) = m$, $0 < m \leqslant y - t$ and $t + 2m + 1 \leqslant d(w_1, w_2) \leqslant d(w_1, b)$ then w_1w_2 is pacifying edge of w. $d(w_1, w_2) \geqslant t + 2m + 1$ then $y + t - d(w_1, w_2) + m + m + 1 \leqslant y$ or $d_{G+w_1w_2}(w, b) \leqslant y$. Also
$d(w_1, w_2) \leqslant d(w_1, b) \leqslant 2y \implies d_{G+w_1w_2}(w, w') \leqslant y$ where w' is the eccentric vertex of w in the unique cycle of $G + w_1w_2$. Hence $e_{G+w_1w_2}(w) = y$. or w_1w_2 is a pacifying edge of w. Let w_1 and w_2 be such that $w_1 \in w$-a path, $d(w_1, w) = m$, $y - \frac{t+1}{2} \geqslant m \geqslant y - t$ and $t + 2m + 1 \leqslant d(w_1, w_2) \leqslant 2y$. It can be easily seen that w_1w_2 is pacifying edge of w.

Thus for $w \in V(P_n)$ such that $d(w, b) < 2d(w, a)$ the pacifying edges are precisely those given above.

Case 3: $2d(w, a) \leqslant d(w, b) < 3d(w, a)$.

i. Let w_1 be a vertex in the w-b path at a distance $m(\geqslant 0)$ from w and

w_2 be at a distance ℓ from w_1 in the w_1-b path. It can be seen as above that if $d(w_1, w_2) < t+1$ or $> 2(y-m)$ then w_1w_2 cannot be a pacifying edge of w. That is for an edge w_1w_2 to be pacifying edge $t+1 \leqslant d(w_1, w_2) \leqslant 2y$. As in previous case, when $m > y - \frac{t+1}{2}$, $2(y-m) < t+1$. Hence m should be such that $0 \leqslant m \leqslant y - \frac{t+1}{2}$. Conversely if w_1 and w_2 are such that w_1 is in the w-b path, $d(w, w_1) = m$, $0 \leqslant m \leqslant y - \frac{t+1}{2}$ and $t+1 \leqslant d(w_1, w_2) \leqslant 2(y-m)$ then it can be shown as in the previous case that w_1w_2 is a pacifying edge.

ii. w_1 is in the a-w path w_2 is in the w_1-b path, $d(w_1, w_2) = \ell$ and $d(w_1, w) = m$. It can be easily proved that w_1w_2 is a pacifying edge if and only if $0 < m \leqslant y - \frac{t+1}{2}$ and $t + 2m + 1 \leqslant \ell \leqslant 2y$.

The above two cases precisely give the pacifying edges of w when
$2d(w, a) \leqslant d(w, b) < 3d(w, a)$.

Case 4: $d(w, b) \geqslant 3d(w, a)$.
We have the following subcases.
Subcase 4.1: $d(w, b) = 3n$.
By adding an edge between a pair of vertices at a distance less than $2n$ the eccentricity of w can be reduced at most to $n+2$. Now, if we join a pair of vertices at a distance greater than $2n+2$, since the resulting cycle has radius at least $n+2$, the eccentricity of w is at least $n+2$. Also if we join w to v where $d(w, v) = 2n$ then $d_{G+(w,v)}(w, w') = n$, $d_{G+(w,v)}(w, b) = n+1$ and $d_{G+(w,v)}(w, a) \leqslant n$. Hence $e_{G+(v,w)}(w) = n+1$. If we join a pair of vertices at a distance $2n+1$ or $2n+2$ the resulting cycle has radius $n+1$ and therefore eccentricity of w is at least $n+1$. Let $w_1(\neq w)$ and w_2 be pair of vertices at a distance $2n$ such that w_1 belong to w-b path and w_2 belong to w_1-b path. Then $d_{G+w_1w_2}(w_1, w_1') = n$ where w_1' is the eccentric vertex of w_1 in the unique cycle of $G + w_1w_2$ and therefore $d_{G+w_1w_2}(w, w_1') \geqslant n+1$. Hence $e_{G+w_1w_2}(w) \geqslant n+1$.

Let $w_1(\neq w)$ and w_2 be pair of vertices at a distance $2n$ such that w_1 belong to w-a path and w_2 belong to w_1-b path. Then $d_{G+w_1w_2}(w,b) \geqslant n+3$ and therefore $e_{G+w_1w_2}(w) \geqslant n+3$. From these observations we can conclude that by adding a single edge the eccentricity of w can be reduced at most to $n+1$.

Let w_1 and w_2 be pair of vertices such that w_1 belongs to w-b path, w_2 belongs to w_1-b path and $d(w_1,w)=m$. Join w_1 and w_2. $d_{G+w_1w_2}(w,b) = 3n-(d(w_1,w_2)-1) = 3n+1-d(w_1,w_2)$. If w_1w_2 has to be pacifying edge of w then $d_{G+w_1w_2} \leqslant n+1$. That is, $3n+1-d(w_1,w_2) \leqslant n+1$ or $d(w_1,w_2) \geqslant 2n$. $d(w_1,w_2) > 2((n+1)-m) \implies d(w,w_1') > n+1$. Therefore, if (w_1,w_1') is to be a pacifying edge of w then $d(w_1,w_2) \leqslant 2((n+1)-m)$. Hence we have, $2n \leqslant d(w_1,w_2) \leqslant 2((n+1)-m)$. This is possible only when $m=0$ or 1.

When $m=0$, $2n \leqslant d(w_1,w_2) \leqslant 2n+2$ and when $m=1$, $2n \leqslant d(w_1,w_2) \leqslant 2n$. In fact it is easy to verify that when $m=0$ and $2n \leqslant d(w_1,w_2) \leqslant 2n+2$ or $m=1$ and $d(w_1,w_2)=2n$, $e_{G+w_1w_2}(w)$ is $n+1$.

Let w_1 and w_2 be such that w_1 belongs to w-a path, w_2 belongs to w_1-b path and $d(w_1,w)=m(>0)$. $d(w_1,w_2) > 2(n+1)$ implies that the cycle formed by joining w_1 and w_2 has radius greater than $n+1$. That is, $d_{G+w_1w_2}(w,w') > n+1$ where w' is the eccentric vertex of w in the unique cycle of $G+w_1w_2$. In other words $e_{G+w_1w_2}(w) > n+1$. Hence if w_1w_2 has to be pacifying edge of w then $d(w_1,w_2) \leqslant 2(n+1)$.

$2n+2m > d(w_1,w_2) \implies 3n-(d(w_1,w_2)-m)+1+m > n+1$.

That is, $d_{G+w_1w_2}(w,b) > n+1$ or $e_{G+w_1w_2}(w) > n+1$. Hence $d(w_1,w_2) \geqslant 2n+2m$. Thus we get $2n+2m \leqslant d(w_1,w_2) \leqslant 2n+2$. This is possible only when $m=1$ and we get $d(w_1,w_2)=2n{+}2$. When $d(w_1,w_2) = 2n{+}2$ and $m=1$, we have $d_{G+w_1w_2}(w,b)=n{+}1$ and $d_{G+w_1w_2}(w,w')=n{+}1$ where w' is the eccentric vertex of w in the unique cycle of $G+w_1w_2$. That

is, $e_{G+w_1w_2}(w) = n+1$ or w_1w_2 is a pacifying edge of w.
Thus the pacifying edges of w are precisely the following.

a. w_1w_2 where $w_1 = w, w_2$ belong to w-b path and $d(w_2, w) = 2n$.
b. w_1w_2 where $w_1 = w, w_2$ belong to w-b path and $d(w_2, w) = 2n+1$.
c. w_1w_2 where $w_1 = w, w_2$ belong to w-b path and $d(w_2, w) = 2n+2$.
d. w_1w_2 where w_1 belong to w-b path, w_2 belong to w_1-b path, $d(w_1, w) = 1$ and $d(w_2, w) = 2n+1$.
e. w_1w_2 where w_1 belong to w-a path, w_2 belong to w-b path, $d(w_1, w) = 1$ and $d(w_2, w) = 2n+1$.

Subcase 4.2: $d(w, b) = 3n+1$.
Since $d(w, b) \geqslant 3d(w, a)$ we have $n \geqslant d(w, a)$. Joining w to a vertex v in the w-b path such that $d(w, v) = 2n+1$ reduces the eccentricity of w to $n+1$. If we join two vertices at a distance less than $2n+1$ then the eccentricity of w reduces at most to $n+2$. If we join two vertices at a distance greater than $2n+2$ then the resulting cycle has radius at least $n+2$ and therefore there exists at least one vertex whose distance from w is at least $n+2$. Therefore a pacifying edge should be between two vertices at distance $2n+1$ or $2n+2$. In both these cases we get cycles having radii $n+1$ and therefore eccentricity of w in the resulting graph is at least $n+1$. Thus, the pacifying edges of w are precisely those edges that reduces its eccentricity to $n+1$.
Let w_1 and w_2 be such that $d(w_1, w) = m(\geqslant 0)$, w_1 belongs to the w-b path and w_2 belongs to the w_1-b path. For w_1w_2 to be a pacifying edge of w, $d_{G+w_1w_2}(w, w_1',) \leqslant n+1$ where w_1' is the eccentric vertex of w_1 in the unique cycle of $G + w_1w_2$. That is, the radius of the cycle should be less than or equal to $n+1-m$. Hence $d(w_1w_2) \leqslant 2(n+1-m)$. Similarly, $d_{G+w_1w_2}(w, b)$ should be less than or equal to $n+1$. Hence $3n+1-d(w_1, w_2)-m+m+1 \leqslant n+1$ or $d(w_1, w_2) \geqslant 2n+1$. Thus, we get $2n+1 \leqslant d(w_1, w_2) \leqslant 2(n+1-m)$. But this is possible only when $m = 0$

and in this case $2n+1 \leqslant d(w_1, w_2) \leqslant 2n+2$. It can be easily verified that when $m = 0$ and $d(w_1, w_2) = 2n+1$ or $2n+2$, $e_{G+w_1w_2}(w) = n+1$.
Let w_1 and w_2 be such that $d(w_1, w) = m(> 0)$, w_1 belongs to the w-a path and w_2 belongs to the w_1-b path. For w_1w_2 to be pacifying edge of w, $d_{G+w_1w_2}(w, w')$ should be less than or equal to $n+1$ where w' is the eccentric vertex of w in the unique cycle of $G+w_1w_2$. That is, $d(w_1, w_2) \leqslant 2(n+1)$. Also, $d_{G+w_1w_2}(w, b) \leqslant n+1$ gives $3n+1-(d(w_1, w_2)-m)+1+m \leqslant n+1$ or $d(w_1, w_2) \geqslant 2n+2m+1$. Thus we get $2n+2m+1 \leqslant d(w_1, w_2) \leqslant 2n+2$. This is not possible for any positive values of m. Hence the pacifying edges of w are w_1w_2 where $w_1 = w$ and $d(w_1, w_2) = 2n+1$ or $2n+2$.
Subcase 4.3: $d(w, b) = 3n+2$.
Joining w to a vertex v in the w-b path such that $d(w, v) = 2n+2$ reduces the eccentricity of w to $n+1$. If we join two vertices at a distance less than $2n+2$ then the eccentricity of w reduces at most to $n+2$. If we join two vertices at a distance greater than $2n+2$ then the resulting cycle has radius at least $n+2$ and therefore there exists at least one vertex whose distance from w is at least $n+2$. Hence a pacifying edge should be between two vertices at distance $2n+2$. In this case we get a cycle having radius $n+1$ and therefore eccentricity of w in the resulting graph is at least $n+1$. Thus, the pacifying edges of w are precisely those edges that reduces its eccentricity to $n+1$.
Let w_1 and w_2 be such that $d(w_1, w) = m(\geqslant 0)$, w_1 belongs to the w-b path and w_2 belongs to the w_1-b path. For w_1w_2 to be a pacifying edge of w, $d_{G+w_1w_2}(w, w'_1,) \leqslant n+1$ where w'_1 is the eccentric vertex of w_1 in the unique cycle of $G+w_1w_2$. That is the radius of the cycle should be less than or equal to $n+1-m$. Hence $d(w_1, w_2) \leqslant 2(n+1-m)$. Similarly, $d_{G+w_1w_2}(w, b)$ should be less than or equal to $n+1$. That is $3n+2-d(w_1, w_2)-m+m+1 \leqslant n+1$ or $d(w_1, w_2) \geqslant 2n+2$. Thus we get $2n+2 \leqslant d(w_1, w_2) \leqslant 2(n+1-m)$. But this is possible only when $m = 0$

and in this case $d(w_1, w_2) = 2n + 2$. It can be easily verified that when $m = 0$ and $d(w_1, w_2) = 2n + 2$, $e_{G+w_1w_2}(w) = n + 1$.
Let w_1 and w_2 be such that $d(w_1, w) = m(> 0)$, w_1 belongs to the w-a path and w_2 belongs to the w_1-b path. For w_1w_2 to be pacifying edge of w, $d_{G+w_1w_2}(w, w')$ should be less than or equal to $n+1$ where w' is the eccentric vertex of w in the unique cycle of $G + w_1w_2$. That is, $d(w_1, w_2) \leqslant 2(n+1)$. Also, $d_{G+w_1w_2}(w, b) \leqslant n+1$ gives $3n+2-(d(w_1, w_2)-m)+1+m \leqslant n+1$ or $d(w_1, w_2) \geqslant 2n+2m+2$. Thus we get $2n+2m+2 \leqslant d(w_1, w_2) \leqslant 2n+2$. This is not possible for any positive values of m. Hence the pacifying edge of w is w_1w_2 where $w_1 = w$ and $d(w_1, w_2) = 2n + 2$. Thus we have listed the pacifying edges of all the different types of vertices of a path. □

As an illustration of the above theorem 4.2.1, consider the following example.

Example 4.2.1. Consider the path $P_{17} = v_1v_2 \ldots v_{17}$.

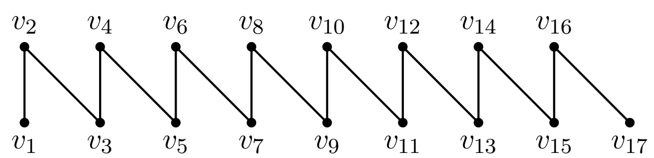


Figure 4.2: Path P_{17}

The following tables give the pacifying edges and the reduced eccentricities of certain vertices of this path.

Table 4.1: Pacifying edges of vertices where $d(w,b) < 3d(w,a)$

Vertex (w)	$d(w,a)$	$d(w,b)$	t	m	$d(w_1,w_2)$	pacifying edges	$e_{P_{17}}$	$e_{P_{17}+w_1w_2}$
v_7 $d(w,b) < 2d(w,a)$	6	10	4		$w_1 \in w-b$ path		10	6
				0	$5 \leqslant d(w_1,w_2) \leqslant 10$	$v_7v_{17}, v_7v_{16}, v_7v_{15}, v_7v_{14}, v_7v_{13}, v_7v_{12}$		
				1	$5 \leqslant d(w_1,w_2) \leqslant 9$	$v_8v_{13}, v_8v_{14}, v_8v_{15}, v_8v_{16}, v_8v_{17}$		
				2	$5 \leqslant d(w_1,w_2) \leqslant 8$	$v_9v_{14}, v_9v_{15}, v_9v_{16}, v_9v_{17}$		
				3	$5 \leqslant d(w_1,w_2) \leqslant 7$	$v_{10}v_{15}, v_{10}v_{16}, v_{10}v_{17}$		
					$w_1 \in w-a$ path			
				1	$7 \leqslant d(w_1,w_2) \leqslant 11$	$v_6v_{13}, v_6v_{14}, v_6v_{15}, v_6v_{16}, v_6v_{17}$		
				2	$9 \leqslant d(w_1,w_2) \leqslant 12$	$v_5v_{14}, v_5v_{15}, v_5v_{16}, v_5v_{17}$		
				3	$11 \leqslant d(w_1,w_2) \leqslant 13$	$v_4v_{15}, v_4v_{16}, v_4v_{17}$		
v_6, $2d(w,a) \leqslant d(w,b) < 3d(w,a)$	5	11	6		$w_1 \in w-b$ path		11	5
				0	$7 \leqslant d(w_1,w_2) \leqslant 10$	$v_6v_{13}, v_6v_{14}, v_6v_{15}, v_6v_{16}$		
				1	$7 \leqslant d(w_1,w_2) \leqslant 8$	v_7v_{14}, v_7v_{15}		
					$w_1 \in w-a$ path			
				1	$9 \leqslant d(w_1,w_2) \leqslant 10$	v_5v_{14}, v_5v_{15}		

Table 4.2: Pacifying edges of vertices where $d(w,b) \geqslant 3d(w,a)$

Vertex(w)	$d(w,a)$	$d(w,b)$	n	m	$d(w_1,w_2)$	pacifying edges	$e_{P_{17}}$	$e_{P_{17}+w_1w_2}$
v_5	4	12	4		$w_1 \in w-b$ path		12	5
				0	$8 \leqslant d(w_1,w_2) \leqslant 10$	$v_5v_{13}, v_5v_{14}, v_5v_{15}$		
				1	$d(w_1,w_2) = 8$	v_6v_{14}		
					$w_1 \in w-a$ path			
				1	$d(w_1,w_2) = 10$	v_4v_{14}		
v_4	3	13	4	0	$9 \leqslant d(w_1,w_2) \leqslant 10$	v_4v_{13}, v_4v_{14}	13	5
v_3	2	14	4	0	$d(w_1,w_2) = 10$	v_3v_{13}	14	5

4.2.2 Pacifying edges of Odd Cycles

Theorem 4.2.2. *Let G be the odd cycle $C_{2n+1}(n > 2)$ with vertex set $\{v_1, \dots, v_{2n+1}\}$.*

1. *If n is even the pacifying edges of a vertex v_i are*

   (a) $v_i v_{i\oplus_{2n+1}n}$

   (b) $v_i v_{i\oplus_{2n+1}(n+1)}$

   (c) $v_i v_{i\oplus_{2n+1}(n+2)}$

   (d) $v_i v_{i\oplus_{2n+1}(n-1)}$

   (e) $v_{i\oplus_{2n+1}1} v_{i\oplus_{2n+1}(n+1)}$

   (f) $v_{i\ominus_{2n+1}1} v_{i\oplus_{2n+1}n}$

   (g) $v_{i\oplus_{2n+1}1} v_{i\oplus_{2n+1}n}$

   (h) $v_{i\ominus_{2n+1}1} v_{i\oplus_{2n+1}(n+1)}$

2. *If n is odd the pacifying edges v_i are $v_i v_{i\oplus_{2n+1}n}$ and $v_i v_{i\oplus_{2n+1}(n+1)}$.*

Proof. 1. Suppose n is even. Add the edge $v_i v_{i\oplus_{2n+1}n}$. Then we get two cycles, say C_1' and C_2', both containing v_i and having $n+2$ and $n+1$ edges respectively. $n+2$ is even and v_i has eccentricity $\frac{n}{2}+1$ in C_1' and consequently $e_{G+v_i v_{i\oplus_{2n+1}n}} = \frac{n}{2}+1$. Similarly by adding the edge $v_i v_{i\oplus_{2n+1}(n+1)}$ the eccentricity of v_i reduces to $\frac{n}{2}+1$. Adding the edge $v_i v_{i\oplus_{2n+1}(n+2)}$ we get cycles C_1' and C_2' where C_1' has $n+3$ edges, C_2' has n edges and both contain the vertex v_i. C_1' has radius $\frac{n}{2}+1$ and C_2' has radius $\frac{n}{2}$. Therefore v_i has eccentricity $\frac{n}{2}+1$ in the new graph. Similarly adding the edge $v_i v_{i\oplus_{2n+1}(n-1)}$ reduces the eccentricity of v_i to $\frac{n}{2}+1$. Adding an edge between v_i and a vertex other than $v_{i\oplus_{2n+1}n}$, $v_{i\oplus_{2n+1}(n+1)}$, $v_{i\oplus_{2n+1}(n+2)}$, $v_{i\oplus_{2n+1}(n-1)}$ we get two cycles C_1' and C_2', both containing v_i, and one of them having radius greater than $\frac{n}{2}+1$. Therefore eccentricity of v_i in such a graph is greater than $\frac{n}{2}+1$. Now we add an edge between v_j and v_k such that $j,k \neq i$. Let C_1' and C_2' be the resulting two cycles. Take two cases.

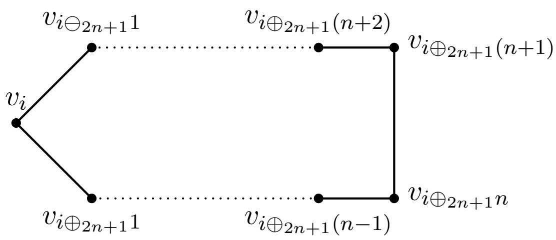


Figure 4.3: Odd Cycle C_{2n+1}

Case 1: $v_i \in C_1'$ where $|E(C_1')| < |E(C_2')|$. That is, C_2' has at least $n+2$ edges or radius of C_2' is at least $\frac{n}{2}+1$. Assume $d(v_i, v_j) \leqslant d(v_i, v_k)$. Let $\bar{v_j}$ be the eccentric vertex of v_j in C_2'. That is $d(v_j, \bar{v_j}) \geqslant \frac{n}{2}+1$. Therefore $d(v_k, \bar{v_j}) \geqslant \frac{n}{2}$. Since $n > 2$, $\frac{n}{2} > 1$.

$$\begin{aligned} d(v_i, \bar{v_j}) &= \min\{d(v_i, v_j) + d(v_j, \bar{v_j}), d(v_i, v_k) + d(v_k, \bar{v_j})\} \\ &\geqslant \min\{d(v_i, v_j) + \frac{n}{2} + 1, d(v_i, v_k) + \frac{n}{2}\} \end{aligned}$$

$d(v_i, \bar{v_j}) = \frac{n}{2}+1$ only when $d(v_i, v_k) = d(v_i, v_j) = 1$ and this implies $n = 2$. Since $n > 2$ we have $d(v_i, \bar{v_j}) > \frac{n}{2}+1$. Hence $v_j v_k$ is not a pacifying edge of v_i.

Case 2: $v_i \in V(C_2')$ where $E(C_2') > E(C_1')$. Here we shall consider two sub cases.

Subcase 2.1: $|E(C_2')| = n+2$ and $|E(C_1')| = n+1$.

Assume $d(v_i, v_j) \leqslant d(v_i, v_k)$. Let $\bar{v_j}$ be the vertex that is eccentric to both v_j and v_k in C_1'. Then $d(v_i, \bar{v_j}) = d(v_i, v_j) + \frac{n}{2}$. But $d(v_i, \bar{v_j}) = \frac{n}{2}+1$ when v_i is adjacent to v_j. In this case we have that the eccentricity of v_i is $\frac{n}{2}+1$. In other words, for the vertex v_i, the edge $v_j v_k$ such that v_j is adjacent to v_i and $d_{C_{2n+1}}(v_j, v_k) = d_{C_{2n+1}}(v_i, v_k) = n$ is a pacifying edge of v_i. Consequently, the edges $v_{i\oplus_{2n+1}1} v_{i\oplus_{2n+1}(n+1)}$ and $v_{i\ominus_{2n+1}1} v_{i\oplus_{2n+1}n}$ are pacifying edges of the vertex v_i.

Subcase 2.2: $|E(C_2')| = n+3$ and $|E(C_1')| = n$.
Let $\bar{v_j}$ be the vertex eccentric to v_j in C_1'. Then

$$d(v_i, \bar{v_j}) = \begin{cases} d(v_i, v_j) + \frac{n}{2} \text{ if } d(v_i, v_j) < d(v_i, v_k) \\ d(v_i, v_k) + d(v_j, \bar{v_j}) - 1 = d(v_i, v_j) + \frac{n}{2} - 1 \text{ if } d(v_i, v_j) = d(v_i, v_k) \end{cases}$$

$d(v_i, v_j) = d(v_i, v_k) = 2$ implies $n = 2$. But we have $n > 2$. Hence $d(v_i, v_j) = d(v_i, v_k)$ implies both are greater than 2 or $d(v_i, \bar{v_j}) \geqslant \frac{n}{2} + 2$. This gives, $v_j v_k$ is not a pacifying edge. Hence we assume that $d(v_i, v_j) < d(v_i, v_k)$. Then $d(v_i, \bar{v_j}) = d(v_i, v_j) + \frac{n}{2}$. Thus $d(v_i, \bar{v_j}) = \frac{n}{2} + 1$ if and if only if v_i is adjacent to v_j. In other words, for the vertex v_i, the edge $v_j v_k$ such that v_j is adjacent to v_i, $d_{C_{2n+1}}(v_j, v_k) = n - 1$ and $d(v_i, v_k) = n$ is a pacifying edge of v_i. Consequently, the edges $v_{i \oplus_{2n+1} 1} v_{i \oplus_{2n+1} n}$ and $v_{i \ominus_{2n+1} 1} v_{i \oplus_{2n+1} (n+1)}$ are pacifying edges of the vertex v_i.

Subcase 2.3: $|E(C_2')| > n+3$.
In this case $e_{C_2'}(v_i) \geqslant \frac{n}{2} + 2$ or $e_{C_{2n+1}}(v_i) \geqslant \frac{n}{2} + 2$.
Thus we get that the pacifying edges of v_i are precisely

(a) $v_i v_{i \oplus_{2n+1} n}$
(b) $v_i v_{i \oplus_{2n+1} (n+1)}$
(c) $v_i v_{i \oplus_{2n+1} (n+2)}$
(d) $v_i v_{i \oplus_{2n+1} (n-1)}$
(e) $v_{i \oplus_{2n+1} 1} v_{i \oplus_{2n+1} (n+1)}$
(f) $v_{i \ominus_{2n+1} 1} v_{i \oplus_{2n+1} n}$
(g) $v_{i \oplus_{2n+1} 1} v_{i \oplus_{2n+1} n}$
(h) $v_{i \ominus_{2n+1} 1} v_{i \oplus_{2n+1} (n+1)}$

Assume n is odd. Joining v_i to $v_{i \oplus_{2n+1} n}$ we get two cycles C_1' and C_2' having $n+1$ and $n+2$ edges respectively. Then C_1' and C_2' have radii $\frac{n+1}{2}$. Therefore the eccentricity of v_i in both C_1' and C_2' is $\frac{n+1}{2}$

or eccentricity of v_i in the $G + v_i v_{i \oplus_{2n+1} n}$ is $\frac{n+1}{2}$. Similarly by adding the edge $v_i v_{i \oplus_{2n+1}(n+1)}$ the eccentricity of v_i reduces to $\frac{n+1}{2}$. Now, let v_i be joined to any vertex other than $v_{i \oplus_{2n+1} n}$ and $v_{i \oplus_{2n+1}(n+1)}$. Then one of the cycles formed contains at least $n+3$ edges. That is, the radius of that cycle is $\frac{n+3}{2}$ or eccentricity of v_i in the new graph is at least $\frac{n+3}{2}$. Hence any such edge cannot be a pacifying edge of v_i. Suppose we join v_j and v_k where $j, k \neq i$. Let C'_1 and C'_2 be the two cycles formed where $|E(C'_1)| \leqslant |E(C'_2)|$. Here we shall consider two cases.

Case 1: Suppose $v_i \in V(C'_1)$. Take the following subcases.

Subcase 1.1: $|E(C'_1)| = n+1$ and $|E(C'_2)| = n+2$.
Then C'_1 is an even cycle. Let $d(v_i, v_j) < d(v_i, v_k)$. Let $\bar{v_j}$ be the eccentric vertex of v_j in C'_2.
$d(v_i, \bar{v_j}) = d(v_i, v_j) + d(v_j, \bar{v_j}) = d(v_i, v_j) + \frac{n+1}{2} > \frac{n+1}{2}$. Hence $e_{G+v_j v_k} < \frac{n+1}{2}$. That is, $v_j v_k$ is not a pacifying edge.

Subcase 1.2: If $|E(C'_2)| \geqslant n+3$ then the radius of $C'_2 \geqslant \frac{n+1}{2} + 1$. Then $d(v_i, \bar{v_j}) \geqslant \frac{n+1}{2} + 1$ where $\bar{v_j}$ is the eccentric vertex of v_j in C'_2. That is, the eccentricity of v_i in $G + v_j v_k$ is at least $\frac{n+1}{2} + 1$ or $v_j v_k$ is not a pacifying edge.

Case 2: Suppose $v_i \in V(C'_2)$. We have that $|E(C'_2)| \geqslant n+2$. Again we consider two sub cases.

Subcase 2.1: $|E(C'_2)| = n+2$. Let $\bar{v_j}$ be the eccentric vertex of v_j in C'_1.

$$d(v_i, \bar{v_j}) = \begin{cases} d(v_i, v_j) + d(v_j, \bar{v_j}) \text{ if } d(v_i, v_k) > d(v_i, v_j) \\ d(v_i, v_j) + d(v_j, \bar{v_j}) - 1 \text{ if } d(v_i, v_k) = d(v_i, v_j) \end{cases}$$

$d(v_i, v_k) = d(v_i, v_j) \implies d(v_i, \bar{v_j}) = d(v_i, v_j) + \frac{n+1}{2} - 1$.
$d(v_i, v_k) = d(v_i, v_j) = 1 \implies$ our cycle is C_3 which is not the case.

Hence $d(v_i, v_j) > 1$ or $d(v_i, \bar{v_j}) < \frac{n+1}{2}$. So $v_j v_k$ cannot be a pacifying edge of v_i.
If $d(v_i, v_k) > d(v_i, v_j)$ then

$$\begin{aligned} d(v_i, \bar{v_j}) &= d(v_i, v_j) + d(v_j, \bar{v_j}) \\ &= d(v_i, v_j) + \frac{n+1}{2} \\ &> \frac{n+1}{2} \end{aligned}$$

Hence $e_{G+v_j v_k}(v_i) > \frac{n+1}{2}$. That is, $v_j v_k$ is not a pacifying edge.
Subcase 2.2: $|E(C_2')| \geqslant n+3$. This implies $d(v_i, \bar{v_i}) \geqslant \frac{n+3}{2}$ where $\bar{v_i}$ is the eccentric vertex of v_i in C_2'.
In other words $v_i v_{i \oplus_{2n+1} n}$ and $v_i v_{i \oplus_{2n+1} (n+1)}$ are the only pacifying edges of v_i.

□

4.2.3 Pacifying edges of Symmetric Even graphs

Next we shall identify the pacifying edges of vertices of a symmetric even graph. Let G be a symmetric even graph. From the definition it is clear that if diameter of G is d, then for every $u, w \in G$, $d(u, \bar{u}) = d(u, w) + d(w, \bar{u}) = d$. That is $d(u, w) = m$ implies $d(\bar{u}, w) = d - m$. Since $d(w, \bar{w}) = d$ we have $d(\bar{u}, \bar{w}) = d - (d - m) = m$.
Now we shall find the pacifying edges of vertices of a symmetric even graph.

Theorem 4.2.3. *Let G be a Symmetric Even graph having diameter d. Then*

1. *If d is even then the only pacifying edge of a vertex v is $v\bar{v}$.*

2. *If d is odd the pacifying edges of v are*

(a) *All edges vy such that y is either $\bar{v}$ or a vertex adjacent to $\bar{v}$.*
(b) *All edges $x\bar{v}$ such that x is either v or a vertex adjacent to v.*

Proof. Let v_1 and v_2 be vertices such that $d(v_1, v) = r_1$, $d(v_2, \bar{v}) = r_2$ and $d(v_1, v) \leqslant d(v_2, v)$. Now consider the graph $G + v_1v_2$. If $d(v_1, v) = d(v_2, v)$, then $d_{G+v_1v_2}(v, \bar{v}) = d$ and hence the eccentricity of v does not decrease. So we can assume that $d(v_1, v) < d(v_2, v)$. Let u be a vertex belonging to a shortest v-v_2 path. If $d(u, v) = m$, then, since G is symmetric even, $d(\bar{u}, \bar{v}) = m$. Therefore $d(v_2, \bar{u}) \leqslant r_2 + m$. $d(v_2, \bar{u}) = r_2 + m - \ell$ implies $d(u, \bar{u}) = d - r_2 - m + r_2 + m - \ell = d - \ell$, a contradiction to fact that G is self-centered. Hence $d(v_2, \bar{u}) = r_2 + m$. That is, the length of the shortest path from v to $\bar{u}$ in $G+v_1v_2$ passing through the edge v_1v_2 is r_1+1+r_2+m. Thus $d_{G+v_1v_2}(v, \bar{u}) = \min\{d - m,\ r_1 + 1 + r_2 + m\}$.
Let w be a vertex in the shortest v-v_2 path such that $d(w, v) = k$(ie $d(\bar{w}, \bar{v}) = k$) and $r_1 + 1 + r_2 + k = d - k$ or $d - k - 1$ according to the parity of $r_1 + r_2 + 1$ and d. For any vertex x such that $d(\bar{v}, x) < k$, we have that $d_{G+v_1v_2}(v, x) < r_1 + r_2 + 1 + k$ and for any vertex x such that $d(\bar{v}, x) > k$ we have $d_{G+v_1v_2}(v, x) < d - k$. That is $\bar{w}$ is an eccentric vertex of v in $G + v_1v_2$. Hence the eccentricity of v is $d_{G+v_1v_2}(\bar{w}, v)$. Now we shall consider two cases.

1. Assume d is even. When $r_1 + r_2$ is odd $r_1 + r_2 + 1$ is even and hence $r_1 + r_2 + 1 + k = d - k$ or $k = \frac{d}{2} - \frac{r_1+r_2+1}{2}$ and therefore $e_{G+v_1v_2}(v) = r_1 + r_2 + 1 + k = \frac{d}{2} + \frac{r_1+r_2+1}{2}$.
   When r_1+r_2 is even r_1+r_2+1 is odd and hence $r_1+r_2+1+k = d-k-1$ or $k = \frac{d}{2} - \frac{r_1+r_2+2}{2}$ and therefore $e_{G+v_1v_2}(v) = r_1 + r_2 + 1 + d = r_1 + r_2 + 1 + \frac{d}{2} - \frac{r_1+r_2+2}{2} = \frac{d}{2} + \frac{r_1+r_2}{2}$.
   Thus $e_{G+v_jv_k}(v) = \frac{d}{2} + \lceil \frac{r_1+r_2}{2} \rceil$. This is a minimum when $r_1 = r_2 = 0$. That is, the only pacifying edge is $v\bar{v}$.

2. Assume d is odd. When $r_1 + r_2$ is odd, $r_1 + r_2 + 1$ is even and hence

$r_1 + r_2 + 1 + k = n - k - 1$ or $x = \frac{d-1}{2} - \frac{r_1+r_2+1}{2}$ and therefore $e_{G+v_1v_2}(v_i) = r_1 + r_2 + 1 + d = \frac{d-1}{2} + \frac{r_1+r_2+1}{2}$. When $r_1 + r_2$ is even r_1+r_2+1 is odd and hence $r_1+r_2+1+k = n-k$ or $x = \frac{d-1}{2} - \frac{r_1+r_2}{2}$ and therefore $e_{G+v_1v_2}(v) = r_1 + r_2 + 1 + k = \frac{d-1}{2} + \frac{r_1+r_2+2}{2}$.
Thus $e_{G+v_1v_2}(v) = \frac{d-1}{2} + \lfloor\frac{r_1+r_2+2}{2}\rfloor$. This is a minimum when $r_1 = r_2 = 0$ or $r_1 = 1, r_2 = 0$ or $r_1 = 0, r_2 = 1$. Consequently, the pacifying edges are

(a) All edges vy such that y is either $\bar{v}$ or a vertex adjacent to $\bar{v}$.

(b) All edges $x\bar{v}$ such that x is either v or a vertex adjacent to v. □

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Remark 4.2.1. The theorems 4.2.2 and 4.2.3 prove that every vertex of a cycle $C_n(n > 5)$ has at least one pacifying edge.

4.3 Shrinking Edges

In this section we consider the problem of identifying the edge(s) when added to a graph decreases its radius the most. This helps in having centers which are more effective than the previous centers. We call such edges the shrinking edges and shrinking edges of paths, odd cycles and symmetric even graphs are identified.

Definition 4.3.1. For a graph G, an edge $uv \in E(G^c)$ is called a *Shrinking Edge* if $rad(G + uv) \leqslant rad(G + xy)$ for every $xy \in E(G^c)$.

We shall identify the shrinking edges of certain classes of graphs.

Corollary 4.3.2. *(to Theorem 4.2.1) Let P_m be a path vertex set $\{v_1, \ldots, v_m\}$. Then*

1. *if $m = 4n+1$ for some integer n then the shrinking edges of P_m are the pacifying edges of $v_{n-1}, v_n, v_{n+1}, v_{n+2}, v_{3n}, v_{3n+1}, v_{3n+2}$ and v_{3n+3}.*

2. *if $m = 4n + 2$, the shrinking edges of P_m are the pacifying edges of* $v_n, v_{n+1}, v_{n+2}, v_{3n+1}, v_{3n+2}$ and v_{3n+3}.
3. *if $m = 4n + 3$, the shrinking edges of P_m are the pacifying edges of* $v_{n+1}, v_{n+2}, v_{3n+2}$ and v_{3n+3}.
4. *if $m = 4n + 4$, the shrinking edges of P_m are the pacifying edges of* v_{n+2} and v_{3n+3}.

Proof. Let $m = 4n+1$. Consider an edge uv in P_m^c. If uv is a pacifying edge of any of the vertices mentioned in the theorem, then by the theorem 4.2.1 the eccentricity of this vertex in $P_m + uv$ is $n+1$ and the eccentricity of all other vertices is $> n+1$. Therefore $rad(P_m + uv) = n+1$. Also if uv is not a pacifying edge of any vertices of P_m then the eccentricity of all vertices of $P_m + uv > n+1$. Therefore $rad(P_m + uv) > n+1$. Therefore, shrinking edges are precisely the pacifying edges of $v_{n-1}, v_n, v_{n+1}, v_{n+2}, v_{3n}, v_{3n+1}, v_{3n+2}$ and v_{3n+3}. All other cases can be proved in exactly the same way. □

The table 4.3 gives the shrinking edges of P_m when $m = 4n+1, 4n+2, 4n+3$ and $4n+4$. In each of theses cases the radius is reduced to $n+1$.

The following corollary identify the shrinking edges of an odd cycle.

Corollary 4.3.3. *(to Theorem 4.2.2) Consider the cycle C_{2n+1} having vertex set $\{v_1, \ldots, v_{2n+1}\}$. An edge v_iv_j in C_{2n+1}^c is a shrinking edge if and only if it is the pacifying edge of some vertex v_i.*

Proof. Let n be even. If v_iv_j, an edge of C_{2n+1}^c, is a pacifying edge of a vertex v_k then $e_{G+v_iv_j}(v_k) = \frac{n}{2} + 1$ and also for all $v_\ell \neq v_k$, we have $e_{G+v_iv_j}(v_\ell) \geqslant \frac{n}{2} + 1$. Therefore $rad(G + v_iv_j) = \frac{n}{2} + 1$. By adding a single edge(any of the pacifying edges) the eccentricity of every vertex can be reduced exactly to $\frac{n}{2} + 1$. Therefore an edge is a shrinking edge if and only if it is a pacifying edge of some vertex. Similarly the case when n is odd. Here instead of $\frac{n}{2} + 1$ we have $\frac{n+1}{2}$. □

Table 4.3: Shrinking edges of path P_m

m	shrinking Edges	$C(P_m + uv)$
$4n+1$	$v_{n-2}v_{3n}$	v_{3n}
	$v_{n-1}v_{3n-1}$	v_{3n}
	$v_{n-1}v_{3n}$	v_{3n}
	$v_{n-1}v_{3n+1}$	$v_{n-1}, v_{3n}, v_{3n+1}$
	v_nv_{3n-1}	v_{3n}
	v_nv_{3n}	v_{3n}, v_{3n+1}
	v_nv_{3n+1}	v_n, v_{3n}, v_{3n+1}
	v_nv_{3n+2}	$v_n, v_{n+1}, v_{3n+1}, v_{3n+2}$
	$v_{n+1}v_{3n}$	v_{3n}
	$v_{n+1}v_{3n+1}$	v_{n+1}, v_{3n+1}
	$v_{n+1}v_{3n+2}$	$v_{n+1}, v_{n+2}, v_{3n+2}$
	$v_{n+1}v_{3n+3}$	$v_{n+1}, v_{n+2}, v_{3n+3}$
	$v_{n+2}v_{3n+1}$	v_{n+2}
	$v_{n+2}v_{3n+2}$	v_{n+1}, v_{n+2}
	$v_{n+2}v_{3n+3}$	v_{n+2}
	$v_{n+2}v_{3n+4}$	v_{n+2}
	$v_{n+3}v_{3n+2}$	v_{n+2}
	$v_{n+3}v_{3n+3}$	v_{n+2}
$4n+2$	$v_{n-1}v_{3n+1}$	v_{3n+1}
	v_nv_{3n}	v_{3n+1}
	v_nv_{3n+1}	v_{3n+1}
	v_nv_{3n+2}	v_n, v_{3n+1}, v_{3n+2}
	$v_{n+1}v_{3n+1}$	v_{3n+1}
	$v_{n+1}v_{3n+2}$	v_{n+1}, v_{3n+2}
	$v_{n+1}v_{3n+3}$	$v_{n+1}, v_{n+2}, v_{3n+3}$
	$v_{n+2}v_{3n+2}$	v_{n+2}
	$v_{n+2}v_{3n+3}$	v_{n+2}
	$v_{n+2}v_{3n+4}$	v_{n+2}
	$v_{n+3}v_{3n+3}$	v_{n+2}
$4n+3$	v_nv_{3n+2}	v_{3n+2}
	$v_{n+1}v_{3n+2}$	v_{3n+2}
	$v_{n+1}v_{3n+3}$	v_{n+1}, v_{3n+3}
	$v_{n+2}v_{3n+3}$	v_{n+2}
	$v_{n+2}v_{3n+4}$	v_{n+2}
$4n+4$	$v_{n+1}v_{3n+3}$	v_{3n+3}
	$v_{n+2}v_{3n+4}$	v_{n+2}

Finally, we give the shrinking edges of symmetric even graphs

Corollary 4.3.4. *(to Theorem 4.2.3) Consider the symmetric even graph G. An edge uv in G^c is a shrinking edge if and only if it is the pacifying edge of some vertex v.*

4.4 Conclusion

In this chapter we introduced the concept of pacifying edges and shrinking edges of the vertices of a graph and the same has been identified for paths, odd cycles and symmetric even graphs. It is established that the pacifying edges of the vertices of a path depends on the ratio of the distance of the vertex to the end vertices. A far as the odd cycles and symmetric even graphs are considered, the pacifying edges of any vertex depends on the parity of the radius of the graph. Shrinking edges of the path depends on the remainder that we get on dividing the length of the path by four. Any edge that is a pacifying edge of some vertex of the odd cycle or symmetric even graph is shown to be a shrinking edge of the graph.

Chapter 5

Median Sets and Median Number

5.1 Introduction

In this chapter we study another centrality measure called median. In fact, the generalisation of the median of a graph to median of arbitrary profiles of a graph is being considered. Given a graph it is possible to have infinitely many profiles, but the number of distinct medians of these profiles is finite and in many cases it much less than the maximum possible number of $2^n - 1$. We make an enumeration of the number of distinct medians of all profiles of a graph.

For the profile $\pi = (v_1, \ldots, v_k)$ and $x \in V$, the set of all vertices x for which $D(x, \pi)$ is minimum is the $Median$ of π in G and is denoted by $M_G(\pi)$. When the underlying graph is obvious we write $M(\pi)$ instead of $M_G(\pi)$. A set S such that $S = M(\pi)$ for some profile π is called a $Median\ set$ of G.The number of distinct Median sets in G is called $Median\ number$ of graph G and is denoted by $mn(G)$. Here we identify and enumerate the median sets of various classes of graphs. But before that we have a small result connecting the median number and the interval number of a graph.

Proposition 5.1.1. For any graph $G = (V, E)$ on n vertices, $in(G) \leqslant mn(G) \leqslant 2^n - 1$.

Proof. The upper bound is obvious as it is the number of nonempty subsets of the vertex set. For every $v \in V$, v is a median set of the profile (v).

For every $u, v \in V$ the set $I(u, v)$ is the median set of the profile (u, v). Therefore $in(G) \leqslant mn(G) \leqslant 2^n - 1$. □

5.2 Median number of some classes of graphs

5.2.1 Median number of Complete graphs

Proposition 5.2.1. $mn(K_n) = 2^n - 1$, where K_n is the complete graph on n vertices.

Proof. In K_n, each nonempty subset of the vertex set is a median set, namely, of the profile formed by taking all the elements of the set exactly once. Therefore the number of distinct median sets is the number of non-empty subsets of V which is $2^n - 1$. □

5.2.2 Median number of $K_n - e$

Proposition 5.2.2. If $e = uv$ is an edge of K_n, $n \geqslant 3$, then the class of median sets of $K_n - e$ consists of V together with all subsets of V which do not simultaneously contain u and v.

Proof. Let $e = (u, v) \in E$. For every vertex set S such that $\{u, v\} \nsubseteq S$, there exist a profile which has S as its Median set, namely the profile formed by taking the vertices of S exactly once. Let π be a profile which does not simultaneously contain u and v. Then $M(\pi)$ is a subset of the set of vertices corresponding to the profile π and hence does not contain u and v. Now, let π be a profile which contain both u and v. Then if u or v is repeated more than the other in the profile then $D(u, \pi) \neq D(v, \pi)$ and so they cannot appear together in the $M(\pi)$. Assume that π contain both u and v where both are repeated the same number of times. Let the profile be $(x_1, \ldots, x_k, \underbrace{u, \ldots, u}_{m \text{ times}}, \underbrace{v, \ldots, v}_{m \text{ times}})$, $m \geqslant 1$. For $x_i, 1 \leqslant i \leqslant k$,

$D(x_i, \pi) \leqslant k - 1 + 2m$. Also, $D(u, \pi) = k + 2m$ and $D(v, \pi) = k + 2m$. Therefore $M(\pi)$ does not contain both u and v. Now the profile (u, v) has V as its Median set. Hence V is the only Median set which contain both u and v. Therefore the class of all Median sets of the graph consists of V and all subsets of V which do not simultaneously contain u and v. □

Corollary 5.2.3. $mn(K_n - e) = 3 \times 2^{n-2}$.

Proof. If $e = uv$, by the above proposition, the median number of $K_n - e$ is one more than the number subsets of V which do not simultaneously contain u and v.

$$\begin{aligned} mn(K_n - e) &= \binom{n}{1} + \binom{n}{2} - \binom{n-2}{0} + \binom{n}{3} - \binom{n-2}{1} + \ldots + \binom{n}{n-1} - \binom{n-2}{n-3} + \binom{n}{n} \\ &= 2^n - 1 - (2^{n-2} - 1) \\ &= 2^n - 2^{n-2} \\ &= 3 \times 2^{n-2} \end{aligned}$$

□

Illustration 5.2.1. Consider the graph $K_6 - e$ given in figure 3.2. Here $e = v_1 v_2$. All the subsets of V except the following are center sets.

1. $\{v_1, v_2\}$
2. $\{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_2, v_5\}, \{v_1, v_2, v_6\}$
3. $\{v_1, v_2, v_3, v_4\}$, $\{v_1, v_2, v_3, v_5\}, \{v_1, v_2, v_3, v_6\}$, $\{v_1, v_2, v_4, v_5\}$, $\{v_1, v_2, v_4, v_6\}$, $\{v_1, v_2, v_5, v_6\}$
4. $\{v_1, v_2, v_3, v_4, v_5\}$, $\{v_1, v_2, v_3, v_4, v_6\}, \{v_1, v_2, v_3, v_5, v_6\}$, $\{v_1, v_2, v_4, v_5, v_6\}$

Proposition 5.2.4. [13] Let $G = (V, E)$ be a Median graph. For any profile π in G the Median Set is an interval $I(u, v)$ in G.

5.2.3 Median number of Block graphs

First, we shall evaluate the median number of trees.

Proposition 5.2.5. The Median number of a tree T on n vertices is $n + \binom{n}{2}$.

Proof. Since T is a median graph, by the above proposition all Median sets are intervals. As observed in the proof of proposition 5.1.1 all intervals are Median sets. Therefore, class of Median sets of T is precisely the class of intervals of T which is the class of all paths in$T$. Hence the Median number is the number of distinct paths in T which is $n + \binom{n}{2}$. □

Now we shall identify the median sets of block graphs which are in fact generalisations of both complete graphs and trees.

Lemma 5.2.1. The median sets of a block graph are either intervals or cliques.

Proof. Let $G = (V, E)$ be a block graph and let S_G denote its skeleton graph which is a tree. Let $\pi = (v_1, \ldots v_k)$ be a profile in G. Consider the same profile π in S_G and let $M_{S_G}(\pi)$ be the median of π in S_G.
First assume that there exists a vertex v of G in S_G such that $v \in M_{S_G}(\pi)$. Then $D_{S_G}(v, \pi) \leqslant D_{S_G}(x, \pi)$ for every $x \in V(S_G)$. For each $u \in V$, $d_{S_G}(u, v_i) = 2d_G(u, v_i)$ and therefore $D_{S_G}(u, \pi) = 2D_G(u, \pi)$. Hence $D_G(v, \pi) \leqslant D_G(u, \pi)$ for every $u \in V(G)$. Hence,

$$v \in M_{S_G}(\pi) \quad \Longrightarrow \quad v \in M_G(\pi) \tag{5.1}$$

Conversely if $v \in M_G(\pi)$ then

$$D_G(v, \pi) \leqslant D_G(x, \pi) \tag{5.2}$$

for every $x \in V(G)$.

Consider π as a profile in the tree S_G. Since S_G is a tree, $M_{S_G}(\pi)$ is a path. If $u_1, \dots u_k$ are the vertices of G in this path in the order of occurrence, then by 5.1 and 5.2, $u_1, \dots, u_k$ form the median of π in G. There exists a block containing u_1 and u_2, say S_1, a block containing u_2 and u_3, S_2, ..., a block containing u_{k-1} and u_k, S_{k-1}. Hence $u_1 \dots u_k$ is an interval in G.

Now assume that π is a profile of G such that, $M_{S_G}(\pi)$ does not contain any vertex of G. Then $M_{S_G}(\pi) = \{S\}$ where S corresponds to a block of G. Let $u_1, u_2 \dots u_r$ be the vertices adjacent to S in S_G. That is, $u_1, u_2 \dots u_r$ are the vertices belonging to a block(corresponding to S) in G. Since S is the only median of π, $D_{S_G}(u_i, \pi) \geqslant D_{S_G}(x, \pi)$ for $1 \leqslant i \leqslant r$. This implies that as we move from S to any of its adjacent vertices in S_G, $D_{S_G}(\pi)$ increases and hence as we move further $D_{S_G}(\pi)$ further increases. In other words minimum of $D_G(\pi)$ is a subset of $\{u_1, \dots, u_r\}$. or the median of π in G is a clique. Hence the median sets of a block graph are either intervals or cliques. □

Theorem 5.2.6. *The median number of a block graph is the number of intervals + number of cliques of size greater than 2.*

Proof. Let G be a block graph. If G is complete then, since singleton sets and pairs of adjacent vertices are the intervals the theorem is obvious. So assume G is not complete. Consider the interval $I(u, v)$ where u and v are non adjacent. Then if $\pi = \{u, v\}$, $M(\pi) = I(u, v)$. Also any clique is a median set, namely, of itself. Hence the sets of intervals(this includes cliques of size 1 and 2) together with cliques of size greater than 2 forms the set of median sets of a block graph. In other words

$mn(G)$=number of intervals + number of cliques of size greater than 2. □

5.2.4 Median number of Hypercubes

Initially, we quote the following theorem.

Proposition 5.2.7. (Imrich et al.,[69]) Let Q_r be a hypercube. Then, for any pair of vertices $u, v \in Q_r$ the subgraph induced by the interval $I(u,v)$ is a hypercube of dimension $d(u,v)$.

Theorem 5.2.8. *For the Hypercube Q_r, $mn(Q_r) = 3^r$*

Proof. Since Q_r is a Median graph, by Propositions 5.2.4 and 5.2.7 every Median set of Q_r is a subcube. Also in any graph G, $I(u,v)$ is the median set of the profile (u,v), where $u,\ v \in V(G)$. Thus in a hypercube every subcube is a Median set. Therefore, the Median sets of Q_r are precisely the induced subcubes. So the Median number of Q_r is the number of subcubes of Q_r. Every vertex of Q_r contain r co-ordinates where each co-ordinate is either 0 or 1. Keeping k co-ordinates fixed and varying 0 and 1 over the other $r-k$ positions we get a subcube of dimension $r-k$. By varying 0's and 1's over these k positions we get 2^k such subcubes. The k positions to be fixed can be chosen in $\binom{r}{k}$ ways. So, the total number of subcubes of dimension $r-k$ is $2^k \times \binom{r}{k}$. Therefore the total number of subcubes of Q_r is $\sum\limits_{0 \leqslant k \leqslant r} \binom{r}{k} \times 2^k = 3^r$. □

5.2.5 Median number of Wheel graphs

Theorem 5.2.9. *Let W_n, $n \geqslant 7$ be the wheel graph with vertex set $\{v_1, v_2, \ldots, v_{n-1}, v_n\}$ and having v_n as the universal vertex. The median sets of W_n are*

(1) $\{v_i\}, 1 \leqslant i \leqslant n$
(2) $\{v_i, v_{i \oplus_{(n-1)} 1}\}, 1 \leqslant i \leqslant n-1$
(3) $\{v_i, v_n\}, 1 \leqslant i \leqslant n-1$

(4) $\{v_i, v_n, v_{i\oplus_{(n-1)}1}\}, 1 \leqslant i \leqslant n-1$
(5) $\{v_i, v_j, v_n\} 1 \leqslant i, j \leqslant n-1$, $d_{C_{n-1}}(v_i, v_j) \geqslant 3$
(6)$\{v_i, v_{i\oplus_{(n-1)}1}, v_{i\oplus_{(n-1)}2}, v_n\}, 1 \leqslant i \leqslant n-1$.

Proof. Let C_{n-1} be the cycle $v_1, v_2, \ldots, v_{n-1}, v_1$. Each singleton set $\{v_i\}$, $1 \leqslant i \leqslant n-1$, is a Median set. The sets $\{v_i, v_j\}$, where v_i and v_j are adjacent are also Median sets. The profile $(v_i, v_{i\oplus_{(n-1)}1}, v_n)$, $1 \leqslant i \leqslant n-1$,has $\{v_i, v_{i\oplus_{(n-1)}1}, v_n\}$ as Median set. The set $\{v_i, v_{i\oplus_{(n-1)}1}, v_{i\oplus_{n-1}2}, v_n\}$ is the Median set of the profile $(v_i, v_{i\oplus_{(n-1)}2})$. Let $\pi = (x_1, x_2, \ldots, x_k)$ be a profile of W_n which contain the universal vertex v_n. Then since π contain the vertex v_n, $D(v_n, \pi) \leqslant k-1$. If some v_i, $1 \leqslant i \leqslant n-1$, belong to $M(\pi)$ then $D(v_i, \pi) \leqslant k-1$ and this implies $x_j = v_i$ at least for some j. Also, the number of x_j's with $d(v_i, x_j) = 2$ is less than the number of repetitions of v_i in π. Let v_k be such that $d(v_k, v_i) = 2$. Then v_k belong to $M(\pi)$ implies number of repetitions of v_k is greater than the number of repetitions of v_i in the profile π. But these two statements are contradictory. Thus for a profile which contain the universal vertex the Median set cannot contain two vertices which are at distance 2. Hence the only possible Median sets for such a profile are

i) sets of type $\{v_i, v_{i\oplus_{(n-1)}1}\}$

ii) sets of type $\{v_i, v_n\}$

iii) sets of type $\{v_i, v_{i\oplus_{(n-1)}1}, v_n\}$.

Now, let $\pi = (x_1, \ldots, x_k)$ be a profile which does not contain v_n. Then $D(v_n, \pi) = k$. If some $v_i, 1 \leqslant i \leqslant n-1$ belong to $M(\pi)$, then $D(v_i, \pi) \leqslant k$. Let v_j be such that $v_j \in M(\pi)$ and $d(v_i, v_j) = 2$. Then $D(v_j, \pi) \leqslant k$. since $D(v_i, \pi) \leqslant k$,
number of zeroes in $\{d(v_i, x_1), \ldots, d(v_i, x_k)\} \geqslant$ number of twos in $\{d(v_i, x_1), \ldots, d(v_i, x_k\}$. Similarly, number of zeroes in $\{d(v_j, x_1), \ldots,\ d(v_j, x_k\} \geqslant$

number of twos in $\{d(v_j, x_1), \ldots, d(v_j, x_k\}$.
Thus, number of repetitions of v_i in π = number of repetitions of v_j in π.
Now, let $d_{C_{n-1}}(v_i, v_j) = 2$. Without loss of generality, we may assume that $j = i \oplus_{(n-1)} 2$. If some vertex other than $v_i, v_{i\oplus_{(n-1)}1}, v_{i\oplus_{(n-1)}2}$ belong to π then $D(v_i, \pi) = D(v_j, \pi) > D(v_n, \pi)$. If $v_{i\oplus_{(n-1)}1} \in \pi$ then $D(v_{i\oplus_{(n-1)}1}, \pi) < D(v_i, \pi)$. Therefore π can only be $(v_i, \ldots, v_i, v_j, \ldots, v_j)$ where v_i and v_j are repeated the same number of times. Since $j = i \oplus_{(n-1)} 2$, we have $D(v_i, \pi) = D(v_j, \pi) = D(v_n, \pi) = D(v_{i\oplus_{(n-1)}1}, \pi)$ and for all other $x \in V$, $D(x, \pi) > k$. Hence
$M(\pi) = \{v_i, v_{i\oplus_{(n-1)}1}, v_{i\oplus_{(n-1)}2}, v_n\}$. If $d_{C_{n-1}}(v_i, v_j) \neq 2$ then some vertex other than v_i and v_j belong to π will contradict the fact that v_i and v_j belong to $M(\pi)$. Therefore, in this case also $\pi = (v_i, \ldots, v_i, v_j, \ldots, v_j)$ where v_i and v_j are repeated the same number of times. Here $D(v_i, \pi) = k$, $D(v_j, \pi) = k$, $D(v_n, \pi) = k$ and for all other $x \in V$, $D(x, \pi) > k$. In other words $M(\pi) = \{v_i, v_j, v_n\}$.
Hence the only Median sets are

(1) $\{v_i\}, 1 \leqslant i \leqslant n$
(2) $\{v_i, v_{i\oplus_{(n-1)}1}\}, 1 \leqslant i \leqslant n-1$
(3) $\{v_i, v_n\}, 1 \leqslant i \leqslant n-1$
(4) $\{v_i, v_n, v_{i\oplus_{(n-1)}1}\}, 1 \leqslant i \leqslant n-1$
(5) $\{v_i, v_j, v_n\} 1 \leqslant i, j \leqslant n-1$, $d_{C_{n-1}}(v_i, v_j) \geqslant 3$
(6)$\{v_i, v_{i\oplus_{(n-1)}1}, v_{i\oplus_{(n-1)}2}, v_n\}, 1 \leqslant i \leqslant n-1$.

□

Remark 5.2.1. When $n = 6$ all the above mentioned sets except in item 5 are median sets. The sets mentioned in item 5 are not present in W_6

Corollary 5.2.10. *For the wheel graph W_n, $n \geqslant 6$, $mn(W_n) = \frac{n^2+3n-2}{2}$*

Proof. By Theorem 5.2.9, for $n \geqslant 6$,

$mn(W_n) = n + n - 1 + n - 1 + n - 1 + \frac{(n-1)(n-6)}{2} + n - 1 = \frac{n^2+3n-2}{2}$.
When $n = 6$, $\frac{(n-1)(n-6)}{2} = 0$. Thus by Remark 5.2.1 we have $mn(W_n) = \frac{n^2+3n-2}{2}$ for $n \geqslant 6$. □

Illustration 5.2.2. We shall list the median sets and thus find the median number of W_9(See figure 3.3).

1. $\{v_1\}$, $\{v_2\}$, $\{v_3\}$, $\{v_4\}$, $\{v_5\}, \{v_6\}, \{v_1\}, \{v_2\}$, $\{v_3\}$
2. $\{v_1, v_2\}$, $\{v_2, v_3\}$, $\{v_3, v_4\}$, $\{v_4, v_5\}$, $\{v_5, v_6\}\{v_6, v_7\}$, $\{v_7, v_8\}$, $\{v_8, v_1\}$
3. $\{v_1, v_9\}$, $\{v_2, v_9\}$, $\{v_3, v_9\}$, $\{v_4, v_9\}$, $\{v_5, v_9\}\{v_6, v_9\}$, $\{v_7, v_9\}$, $\{v_8, v_9\}$
4. $\{v_1, v_2, v_9\}$, $\{v_2, v_3, v_9\}$, $\{v_3, v_4, v_9\}$, $\{v_4, v_5, v_9\}$, $\{v_5, v_6, v_9\}\{v_6, v_7, v_9\}, \{v_7, v_8, v_9\}, \{v_8, v_1, v_9\}$
5. $\{v_1, v_4, v_9\}$, $\{v_1, v_5, v_9\}$, $\{v_1, v_6, v_9\}$, $\{v_2, v_5, v_9\}$, $\{v_2, v_6, v_9\}$, $\{v_2, v_7, v_9\}$, $\{v_3, v_6, v_9\}, \{v_3, v_7, v_9\}$, $\{v_3, v_8, v_9\}, \{v_4, v_7, v_9\}$, $\{v_4, v_8, v_9\}$,$\{v_5, v_8, v_9\}$
6. $\{v_1, v_2, v_3, v_9\}$, $\{v_2, v_3, v_4, v_9\}$, $\{v_3, v_4, v_5, v_9\}$, $\{v_4, v_5, v_6, v_9\}$, $\{v_5, v_6, v_7, v_9\}\{v_6, v_7, v_8, v_9\}, \{v_7, v_8, v_1, v_9\}, \{v_8, v_1, v_2, v_9\}$

Thus the median number of W_9 is 9+8+8+8+12+8 = 53. From the formulae for median number of wheel graphs we have $mn(W_n) = \frac{9\times9+3\times9-2}{2} = 53$

Now we shall identify the median sets and hence compute the median number of W_5 having vertex set $\{v_1, v_2, v_3, v_4, v_5\}$ where v_5 is the universal vertex. Each singleton set, pair of adjacent vertices and triple of vertices that induces a clique are median sets, namely of itself. As we proved in the theorem it can be shown that a profile containing the vertex v_5 cannot contain two vertices at distance 2.
Now assume that π is profile that does not contain v_5 and $M(\pi)$ contains v_1 and v_3. Then assume that v_i is repeated n_i times in the profile for $1 \leqslant i \leqslant 4$.

Then $D(v_1,\pi) = n_2 + 2n_3 + n_4, D(v_2,\pi) = n_1 + n_3 + 2n_4, D(v_3,\pi) = 2n_1 + n_2 + n_4$ and $D(v_4,\pi) = n_1 + 2n_2 + n_3$. Since v_1 and v_3 belong to $M(\pi)$ we have that $n_1 = n_3$ and $n_2 = n_4$. This gives $D(v_i,\pi) = 2n_1 + 2n_2$ for $i \leqslant i \leqslant 5$. Hence $M(\pi) = \{v_1, v_2, v_3, v_4, v_5\}$. That is, this is the only median set that contain two vertices at distance 2. Thus the only median sets of W_5 are

1. $\{v_i\}$, $1 \leqslant i \leqslant 5$
2. $\{v_i, v_j\}$ where v_i and v_j are adjacent.
3. $\{v_i, v_j, v_k\}$ where v_i, v_j and v_k induces a clique.
4. $\{v_1, v_2, v_3, v_4, v_5\}$

Hence the median number of W_5 is 18.(See figure 3.4)

5.2.6 Median number of Complete Bipartite graphs

Theorem 5.2.11. *For the complete bipartite graph $K_{m,n}$, $m \leqslant n$, $m > 2$, all nonempty subsets of $V(K_{m,n})$ are median sets.*

Proof. Let (X, Y) be a bipartition of $K_{m,n}$ with $|X| = m$ and $|Y| = n$. Let $X = \{x_1, \ldots, x_m\}$ and $Y = \{y_1, \ldots, y_n\}$. Let A be a k-element subset of X with $k \leqslant m$. Without loss of generality we may assume that $A = \{x_1, \ldots, x_k\}$.
If $k < n$, take $\pi = (x_1, \ldots, x_k, y_1, \ldots, y_n)$. For each x_i, $1 \leqslant i \leqslant k$, $D(x_i,\pi) = 2(k-1)+n$. For each x_i, $k+1 \leqslant i \leqslant m$, $D(x_i,\pi) = 2k+n$. For each y_i, $1 \leqslant i \leqslant n$, $D(y_i,\pi) = 2(n-1)+k$. Therefore, $A = \{x_1, \ldots, x_k\} = M(\pi)$.
If $k = n$, then $\pi = (y_1, \ldots, y_n)$ has Median set $A = \{x_1, \ldots, x_k\}$. Therefore, every subset of X is a Median set.
Now, let $B \subseteq Y$ with $B = \{y_1, \ldots, y_k\}$.

If $k < n$ then as in the previous case $\pi = (x_1, \ldots, x_m, y_1, \ldots, y_k)$ has Median set B.
Now, let $k \geqslant m$ and let π be the profile $(x_1, \ldots x_1, \ldots, x_m, \ldots, x_m, y_1, \ldots, y_k)$, where each x_i is repeated the same number of times,(say) r.
For each y_i, $1 \leqslant i \leqslant k$, $D(y_i, \pi) = 2(k-1) + mr$, for each y_i, $k+1 \leqslant i \leqslant q$, $D(y_i, \pi) = 2k + mr$, and for each x_i, $1 \leqslant i \leqslant n$, $D(x_i, \pi) = 2r(m-1) + k$.
Moreover, $2(k-1) + mr < 2r(m-1) + k \Leftrightarrow k - 2 < (m-2)r \Leftrightarrow r > \frac{k-2}{m-2}$ $(m > 2)$. That is, if each x_i is repeated r times where $r > \frac{k-2}{m-2}$ then $M(\pi)$=B.
Now, let $C = \{x_1, \ldots, x_k, y_1, \ldots, y_r\}$, $1 \leqslant k \leqslant m$, $1 \leqslant r \leqslant n$.
Take $\pi = (x_1, \ldots, x_1, \ldots, x_k, \ldots, x_k, y_1, \ldots, y_1, \ldots, y_r, \ldots, y_r)$ where each x_i is repeated s_x times and y_i is repeated s_y times.
For each x_i, $1 \leqslant i \leqslant k$, $D(x_i, \pi) = 2(k-1)s_x + rs_y$, for each y_i, $1 \leqslant i \leqslant r$, $D(y_i, \pi) = 2(r-1)s_y + ks_x$, for each x_i, $k+1 \leqslant i \leqslant p$, $D(x_i, \pi) = 2ks_x + rs_y$ and for each y_i, $r+1 \leqslant i \leqslant q$, $D(y_i, \pi) = 2rs_y + ks_x$. Any x_i, $k+1 \leqslant i \leqslant p$ or y_i, $r+1 \leqslant i \leqslant q$ cannot be in $M(\pi)$.
Now, $2(k-1)s_x + rs_y = 2(r-1)s_y + ks_x \Leftrightarrow (k-2)s_x = (r-2)s_y$. Hence for any s_x and s_y such that $(k-2)s_x = (r-2)s_y$, the profile $(x_1, \ldots x_1, \ldots, x_k, \ldots, x_k, y_1, \ldots, y_1, \ldots, y_r, \ldots, y_r)$, where each x_i is repeated s_x times and y_i is repeated s_y times, has C as its Median. Therefore, every nonempty subset of $X \cup Y$ is a Median set. □

The following corollary is an immediate conclusion of the theorem.

Corollary 5.2.12.

$$mn(K_{m,n}) = \begin{cases} 2^{m+n} - 1 \ \textit{when}\ m \leqslant n, m > 2 \\ 9 \ \textit{when}\ m = n = 2 \\ \frac{n^2+7n+8}{2} \ \textit{when}\ m = 2,\ n > 2 \end{cases}$$

Proof. When $m < 2$ and $m \leqslant n$ from the theorem we have that all nonempty subsets are median sets. That is, $mn(K_{m,n}) = 2^{m+n} - 1$ If $m = n = 2$ then we get C_4, a median graph and the median sets of such graphs have been identified as intervals and therefore its median number is 9..

So we assume that $m = 2$ and $m < n$. Let $(\{x_1, x_2\}, \{y_1, y_2, \ldots, y_n\})$ be the bipartition. It is clear that $\{x_1\}$ and $\{x_2\}$ are median sets. The profile $(y_1, y_2, \ldots, y_n)$ has $\{x_1, x_2\}$ as the median set. Thus all subsets of $\{x_1, x_2\}$ are median sets. Let π be a profile and let k_1 be the number of repetitions of y_1 in π, k_2 be the number of repetitions of y_2 in π, ..., k_n be the number of repetitions of y_n in π, ℓ_1 be the number of repetitions of x_1 in π and ℓ_2 be the number of repetitions of x_2 in π

$D(y_1, \pi) = \ell_1 + \ell_2 + 2(k_2 + k_3 + \cdots + k_n)$

$D(x_1, \pi) = 2\ell_2 + (k_1 + \cdots + k_n)$ and $D(x_2, \pi) = 2\ell_1 + (k_1 + \cdots + k_n)$ Let $y_1 \in M(\pi)$. Then,

$\ell_1 + \ell_2 + 2(k_2 + k_3 + \cdots + k_n) \leqslant \min\{2\ell_2 + (k_1 + \cdots + k_n), 2\ell_1 + (k_1 + \cdots + k_n)\}$.

Here shall take some cases

Case-1: $\ell_1 < \ell_2$. Then we have $\ell_1 + \ell_2 + 2(k_2 + k_3 + \cdots + k_n) \leqslant 2\ell_1 + (k_1 + \cdots + k_n)$ and $\ell_1 + \ell_2 > 2\ell_1$. Therefore, $2(k_2 + k_3 + \cdots + k_n) < k_1 + \cdots + k_n$ or $k_1 > k_2 + k_3 + \cdots + k_n$. Hence $k_1 + k_3 + \cdots + k_n > k_2 + k_3 \cdots + k_n$. Hence $y_2 \notin M(\pi)$. Thus, in this case no other y_i is in $M(\pi)$. If $k_2 + \cdots + k_n = \ell_1$ and $k_1 = \ell_2$ then we get that $M(\pi) = \{x_2, y_1\}$. This in facts gives that $\{x_i, y_j\} i = 1, 2, 1 \leqslant j \leqslant n$are all median sets.

Case-2: Assume $\ell_1 = \ell_2 = \ell$. $D(x_1, \pi) = 2\ell + (k_1 + \cdots + k_n)$ and $D(x_2, \pi) = 2\ell + (k_1 + \cdots + k_n)$ and $D(y_1, \pi) = 2\ell + 2(k_2 + k_3 + \cdots + k_n)$ and therefore

$2\ell + 2(k_2 + k_3 + \cdots + k_n) \leqslant 2\ell + (k_1 + \cdots + k_n)$. That is,

$k_2 + k_3 + \cdots + k_n \leqslant k_1$.

Subcase 2.1: $k_2 + k_3 + \cdots + k_n = k_1$. Further let $k_i = k_1$ for some i say

2. Then $M(\pi) = \{x_1, x_2, y_1, y_2\}$. That is, $\{x_1, x_2, y_i, y_j\} 1 \leqslant i, j, \leqslant n$ are all median sets. If $k_i \neq k_1$ for any i then $M(\pi) = \{x_1, x_2, y_1\}$. That is, $\{x_1, x_2, y_2\}$ is a median set. Hence $\{x_1, x_2, y_i\},\ 1 \leqslant i, j, \leqslant n$ are all median sets.

Subcase 2.2: $k_2 + k_3 + \cdots + k_n < k_1$. In this case $M(\pi) = \{y_1\}$. Therefore $\{y_i\},\ 1 \leqslant i \leqslant n$ are all median sets.

Thus the median sets of $K_{m,n},\ m = 2, n > 2$ are

1. $\{x_i\},\ i = 1, 2$
2. $\{x_1, x_2\}$
3. $\{x_i, y_j\}\ i = 1, 2,\ 1 \leqslant j \leqslant n$
4. $\{y_i\},\ 1 \leqslant i \leqslant n$
5. $\{x_1, x_2, y_i\},\ 1 \leqslant i, j, \leqslant n$
6. $\{x_1, x_2, y_i, y_j\},\ 1 \leqslant i, j, \leqslant n$
7. $\{x_1, x_2, y_1, y_2, \ldots, y_n\}$

Hence the median number of $K_{2,n}$ where $n \geqslant 3$ is given by $2 + 1 + 2n + n + n + \frac{n(n-1)}{2} + 1 = \frac{n^2+7n+8}{2}$ □

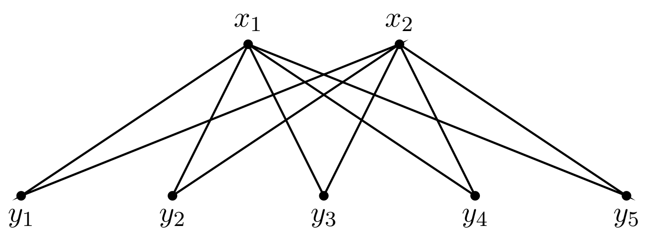


Figure 5.1: $K_{2,5}$

Illustration 5.2.3. Here we shall list the median sets of different types in $K_{2,5}$.

1. $M(x_1) = \{x_1\}$
2. $M((y_1, y_2, y_3, y_4, y_5)) = \{x_1, x_2\}$
3. Let $\pi = (x_1, x_1, x_2, y_1, y_2, y_2)$. Then $D(x_1, \pi) = 5$,$D(x_2, \pi) = 7$, $D(y_1, \pi) = 7$, $D(y_2, \pi) = 5$, $D(y_3, \pi) = 9$, $D(y_4, \pi) = 9$, $D(y_5, \pi) = 9$. Hence $M(\pi) = \{x_1, y_2\}$.
4. $M((y_1)) = \{y_1\}$
5. Let $\pi = (x_1, x_2, y_1, y_1, y_2, y_3)$, $D(x_1, \pi) = 6$, $D(x_2, \pi) = 6$ $D(y_1, \pi) = 6D(y_2, \pi) = 8$, $D(y_3, \pi) = 8$, $D(y_4, \pi) = 10$, $D(y_5, \pi) = 10$. Hence $M((x_1, x_2, y_1, y_1, y_2, y_3)) = \{x_1, x_2, y_1\}$.
6. $\pi = \{x_1, x_2, y_1, y_1, y_2, y_2\}$ $D(x_1, \pi) = 6$, $D(x_2, \pi) = 6$, $D(y_1, \pi) = 6$, $D(y_2, \pi) = 6$, $D(y_3, \pi) = 10$, $D(y_4, \pi) = 10$, $D(y_5, \pi) = 10$. Hence $M((x_1, x_2, y_1, y_1, y_2, y_3)) = \{x_1, x_2, y_1, y_2\}$

5.2.7 Median number of Cartesian Products

Definition 5.2.13. Let π_1 and π_2 be profiles in graphs G_1 and G_2 respectively with $\pi_1 = (u_1, \ldots, u_m)$ and $\pi_2 = (v_1, \ldots, v_n)$ then we define $\pi_1 \times \pi_2$ by $\pi_1 \times \pi_2 = ((u_i, v_j) | 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n)$.

$\pi_1 \times \pi_2$ is in fact a profile of $G_1 \Box G_2$.
If $V(G_1) = \{u_1, \ldots, u_m\}$, $V(G_2) = \{v_1, \ldots, v_n\}$, $\pi_1 = (u_1, u_1, u_2)$ and $\pi_2 = (v_1, v_2, v_2)$ then $\pi_1 \times \pi_2$ is
$((u_1, v_1), (u_1, v_2), (u_1, v_2), (u_1, v_1), (u_1, v_2), (u_1, v_2), (u_2, v_1), (u_2, v_2), (u_2, v_2))$

Lemma 5.2.2. Let π_1 and π_2 be profiles in the graphs G_1 and G_2 respectively. If $M_{G_1}(\pi_1) = M_1$ and $M_{G_2}(\pi_2) = M_2$, then $M_{G_1 \Box G_2}(\pi_1 \times \pi_2) = M_1 \times M_2$.

Proof. Let $\pi_1 = (u_1, u_2, \ldots, u_m)$, $\pi_2 = (v_1, v_2, \ldots, v_n)$ and $M = M(\pi_1 \times \pi_2)$. $\pi_1 \times \pi_2 = ((u_1, v_1), \ldots, (u_1, v_n), \ldots, (u_m, v_1), \ldots, (u_m, v_n))$. If $(x_1, y_1), (x_2, y_2) \in V(G_1 \Box G_2)$, then
$d_{G_1 \Box G_2}((x_1, y_1), (x_2, y_2)) = d_{G_1}(x_1, y_1) + d_{G_2}(x_2, y_2)$, see[9].

For an $(x, y) \in V(G_1 \Box G_2)$,
$D((x, y), \pi_1 \times \pi_2)= n \sum_{1\leqslant i\leqslant m} d(x, u_i) + m \sum_{1\leqslant i\leqslant n} d(y, v_i)$.
Let $(a, b) \in M_1 \times M_2$ ie $a \in M_1$ and $b \in M_2$. Then,

$$\sum_{1\leqslant i\leqslant m} d(a, u_i) \leqslant \sum_{1\leqslant i\leqslant m} d(x, u_i), \forall x \in V(G_1)$$
$$\sum_{1\leqslant i\leqslant n} d(b, v_i) \leqslant \sum_{1\leqslant i\leqslant n} d(y, v_i), \forall y \in V(G_2)$$

Therefore,

$$n \sum_{1\leqslant i\leqslant m} d(a, u_i) + m \sum_{21\leqslant i\leqslant n} d(b, v_i) \leqslant n \sum_{1\leqslant i\leqslant m} d(x, u_i) + m \sum_{1\leqslant i\leqslant n} d(y, v_i), \quad \forall (x, y) \in V(G_1 \Box G_2)$$

Hence, $D((a, b), \pi_1 \times \pi_2) \leqslant D((x, y), \pi_1 \times \pi_2)$, $\forall (x, y) \in V(G_1 \Box G_2)$
Thus, $(a, b) \in M_1 \times M_2 \Rightarrow (a, b) \in M$ or $M_1 \times M_2 \subseteq M$
Now, let $(a, b) \in M$

$$\begin{aligned} D((a, b), \pi_1 \times \pi_2) &= n \sum_{1\leqslant i\leqslant m} d(a, u_i) + m \sum_{1\leqslant i\leqslant n} d(b, v_i) \\ &\leqslant n \sum_{1\leqslant i\leqslant m} d(x, u_i) + m \sum_{1\leqslant i\leqslant n} d(y, v_i), \\ &\qquad \forall (x, y) \in V(G_1 \times G_2) \end{aligned}$$

If for some $x' \in V(G_1)$, $\sum_{1\leqslant i\leqslant m} d(x', u_i) < \sum_{1\leqslant i\leqslant m} d(a, u_i)$, then

$$n \sum_{1\leqslant i\leqslant m} d(x', u_i) + m \sum_{1\leqslant i\leqslant n} d(b, v_i) \quad < \quad n \sum_{1\leqslant i\leqslant m} d(a, u_i) + m \sum_{1\leqslant i\leqslant n} d(b, v_i)$$

This contradicts $(a, b) \in M = M(\pi_1 \times \pi_2)$.

Therefore $\sum_{1\leqslant i\leqslant m} d(a, u_i) \leqslant \sum_{1\leqslant i\leqslant m} d(x, u_i), \forall x \in V(G_1)$ and

$$\sum_{1\leqslant i\leqslant m} d(b, v_i) \leqslant \sum_{1\leqslant i\leqslant m} d(y, u_i), \forall y \in V(G_2)$$

Hence, $a \in M_1$ and $b \in M_2$ or $(a, b) \in M_1 \times M_2$. That is,
$M = M_1 \times M_2$. □

Theorem 5.2.14. *Consider the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. An $M \subseteq V(G_1 \Box G_2)$ is a median set if and only if $M = M_1 \times M_2$ where M_1 and M_2 are median sets of G_1 and G_2 respectively.*

Proof. By the above lemma the product of Median sets of G_1 and G_2 is again a Median set of $G_1 \Box G_2$. Now, let M be a median set of $G_1 \Box G_2$, with $M = M(\pi)$ where $\pi = ((u_1, v_1), \ldots, (u_k, v_k))$. Let $\pi_1 = (u_1, \ldots, u_k)$, $\pi_2 = (v_1, \ldots, v_k)$, $M_1 = M(\pi_1)$ and $M_2 = M(\pi_2)$. Let $(a, b) \in M$.
We have

$$\sum_{1\leqslant i\leqslant k} d(a, u_i) + \sum_{1\leqslant i\leqslant k} d(b, v_i) \leqslant \sum_{1\leqslant i\leqslant k} d(x, u_i) + \sum_{1\leqslant i\leqslant k} d(y, v_i), \forall x \in V(G_1), \forall y \in V(G_2).$$

$$\therefore k \sum_{1\leqslant i\leqslant k} d(a,u_i) + k \sum_{1\leqslant i\leqslant m} d(b,v_i) \leqslant k \sum_{1\leqslant i\leqslant k} d(x,u_i) + k \sum_{1\leqslant i\leqslant m} d(y,v_i), \quad \forall (x,y) \in V(G_1 \square G_2)$$

In other words, $D((a,b), \pi_1 \times \pi_2) \leqslant D((x,y), \pi_1 \times \pi_2)$, $\forall (x,y) \in V(G_1 \square G_2)$.
$\therefore (a,b) \in M(\pi_1 \times \pi_2)$ or $M \subseteq M(\pi_1 \times \pi_2)$.
Let $(a,b) \in M(\pi_1 \times \pi_2)$. Then $D((a,b), \pi_1 \times \pi_2) \leqslant D((x,y), \pi_1 \times \pi_2)$, $\forall x \in V(G_1), \forall y \in V(G_2)$. That is

$$k \sum_{1\leqslant i\leqslant k} d(a,u_i) + k \sum_{1\leqslant i\leqslant k} d(b,v_i) \leqslant k \sum_{1\leqslant i\leqslant k} d(x,u_i) + k \sum_{1\leqslant i\leqslant k} d(y,v_i), \quad \forall x \in V(G_1), \forall y \in V(G_2).$$

$$\sum_{1\leqslant i\leqslant k} d(a,u_i) + \sum_{1\leqslant i\leqslant k} d(b,v_i) \leqslant \sum_{1\leqslant i\leqslant k} d(x,u_i) + \sum_{1\leqslant i\leqslant k} d(y,v_i), \quad \forall x \in V(G_1), \forall y \in V(G_2).$$

$$\therefore \sum_{1\leqslant i\leqslant k} d((a,b),(u_i,v_i)) \leqslant \sum_{1\leqslant i\leqslant k} d((x,y),(u_i,v_i)), \forall (x,y) \in V(G_1 \square G_2).$$

Therefore, $(a,b) \in M$ which implies $M(\pi_1 \times \pi_2) \subseteq M$ or $M = M(\pi_1 \times \pi_2) = M(\pi_1) \times M(\pi_2)$. Thus, the class of all median sets of $G_1 \square G_2$ is the same as the class of all Cartesian products of median sets of G_1 and G_2. □

Corollary 5.2.15. $mn(G_1 \square G_2) = mn(G_1) \times mn(G_2)$

We can generalise the above result to the product of any (finite)number of graphs.

Corollary 5.2.16. *If $G_1, \ldots, G_K$ are k graphs, then*
$mn(G_1\Box\ldots\Box G_k) = mn(G_1) \times \ldots \times mn(G_k)$

The above corollary can be used to find the Median number of various classes of graphs.

Corollary 5.2.17. *For the hypercube Q_r, $mn(Q_r) = 3^r$.*

Proof. Since $Q_r = \underbrace{K_2\Box\ldots\Box K_2}_{r \text{ times}}$,

$$\begin{aligned} mn(Q_r) &= \underbrace{mn(K_2) \times \ldots \times mn(K_2)}_{r \text{ times}} \\ &= \underbrace{3 \times \ldots \times 3}_{r \text{ times}} \\ &= 3^r \end{aligned}$$

□

Corollary 5.2.18. *If G is the Grid graph $P_r\Box P_s$, $mn(G) = (\binom{r}{2} + r) \times (\binom{s}{2} + s)$.*

Corollary 5.2.19. *If G is the Hamming graph $K_{p_1}\Box K_{p_2}\Box\ldots\Box K_{p_r}$, $mn(G) = (2^{p_1} - 1) \times (2^{p_2} - 1) \times \ldots \times (2^{p_r} - 1)$.*

5.2.8 Median sets of Symmetric Even Graphs

Lemma 5.2.3. The only median sets of a symmetric even graph G which contains a vertex and its eccentric vertex is $V(G)$.

Proof. Let a and b be two eccentric vertices of the cycle G which belong to $M(\pi)$ where $\pi = (x_1, \ldots, x_k)$ is profile in G. Let $D(a, \pi) = D(b, \pi) = s$.

Then

$$\begin{aligned} D(a,\pi)+D(b,\pi) &= d(a,x_1)+\ldots+d(a,x_k)+d(b,x_1)+\ldots+d(b,x_k) \\ &= d(a,x_1)+d(b,x_1)+\ldots+d(a,x_k)+d(b,x_k) \\ &= \underbrace{d(a,b)+\ldots+d(a,b)}_{k \text{ times}} \\ &= kr \end{aligned}$$

Hence $2s = kr$ Now, suppose $M(\pi) \neq V$. Then there exists an $x \in V$ such that $D(x,\pi) > s$. That is, $d(x,v_1)+\ldots+d(x,v_k) > s$. Let y be the eccentric vertex of x. $d(y,v_1)+\ldots+d(y,v_k) \geqslant s$. Therefore, $d(x,v_1)+\ldots+d(x,v_k)+d(y,v_1)+\ldots+d(y,v_k) > 2s$. That is, $d(x,y)+\ldots+d(x,y)(k\ times)> 2s$ or $kr > 2s$, a contradiction. Therefore,any set distinct from V which is a Median set cannot contain two eccentric vertices. Also, $M((a,b)) = V$, since a and b are diametrical and $I(a,b) = V$. Hence the only median set of G which contain a vertex and and its eccentric vertex is, V. ☐

Corollary 5.2.20. *For a symmetric even graph G with $|V(G)| = 2r$, $mn(G) \leqslant 3^r$.*

Proof. Let V be the vertex set of G with $V = \{v_1,\ldots,v_{2r}\}$. Let $A=\{S : S \subseteq V$ and S does not contain any pair of eccentric vertices$\}$. By the above lemma the set of all Median sets is a subset of $A \cup \{V\}$. Hence $mn(C_{2r}) \leqslant |A|+1$. Let $B_i = \{v_i, v_{i+r}\}$, $1 \leqslant i \leqslant r$. Now A consists of all subsets of V which does not simultaneously contain both the elements from the same B_i, $1 \leqslant i \leqslant r$. The number ways of choosing a k-element subset of V so that it belongs to A is the product of the number of ways of choosing k B_i's from the r B_i's and the number of ways of choosing one element from each of these chosen B_i's. That is, $\binom{r}{k} \times 2^k$. Therefore $|A| = \sum_{k=1}^{r} \binom{r}{k} \times 2^k$. Hence $mn(G) \leqslant (\sum\limits_{1\leqslant k\leqslant r} \binom{r}{k} \times 2^k) + 1 = \sum\limits_{0\leqslant k\leqslant r} \binom{r}{k} \times 2^k = 3^r$. ☐

5.3 Conclusion

We have identified and enumerated the median sets of different classes of graphs. In the course of proving theorem 5.2.14 it was shown that median set of any profile in a Cartesian product graph is the product of the median sets of its projections. For symmetric even graphs, we proved that any $S \subseteq V$ such that S does not contain a pair of eccentric vertices is a median set. As far as the hypercubes are concerned all such sets are not median sets. In fact the median sets are precisely the subcubes. For the hypercube Q_r the median number is 3^r and this is much less than the bound, $3^{2^{r-1}}$, provided by the above corollary. In the case of even cycles, another class of symmetric even graphs, all the sets mentioned above were seen to be median sets with the help of computer programs. That is the median number of even cycles are seen to achieve this bound. We could not find a mathematical proof of this and hence we propose the following conjecture.

Conjecture 1. Given the cycle C_{2r} having $2r$ vertices, an $S \subseteq V(C_{2r})$ is a median set if and only if either $S = V(C_{2r})$ or S does not contain a pair of eccentric vertices and therefore $mn(C_{2r}) = 3^r$.

Chapter 6

Fair Sets

6.1 Introduction

The measures of centrality that we have discussed, center and median correspond to the effectiveness oriented model and the efficiency oriented model of the facility location problems. A third approach is the equity oriented model where equitable locations are chosen, that is locations which are more or less equally fair to all the customers. Issue of equity is relevant in locating public sector facilities where distribution of travel distances among the recipients of the service is also of importance. Most of the equity based study of location theory concentrate either on comparisons of different measures of equity [84] or on giving algorithms for finding the equitable locations [17, 18, 74, 82, 87]. Also in many optimization problems, we have a set of optimal vertices. If we want to choose among these, one of the important criteria can be equity or fairness. In this chapter we define a measure called *partiality* and various classes of graphs are studied in respect of this measure of centrality.

Definition 6.1.1. For an $x \in V$ and $S \subseteq V$, $min(x, S)$ denotes the minimum of the distances between x and vertices of S and $max(x, S)$ denotes the maximum of the distances between x and vertices of S. The *partiality* of x with respect to set S in G, denoted by $f(x, S) = max(x, S) - min(x, S)$. For a given vertex set S, the set $\{v \in V : f(v, s) \leqslant f(x, S)\ \forall x \in V\}$ is defined as the $fair\ center$ of S and is denoted by $FC(S)$. Any $S' \subseteq V$ such that $S' = FC(S)$ for some $S \subseteq V$, $|S| > 1$, is called a $fair\ set$ of G.

Example 6.1.1. Consider the following graph and the vertex set $S = \{v_1, v_3, v_5, v_6\}$.

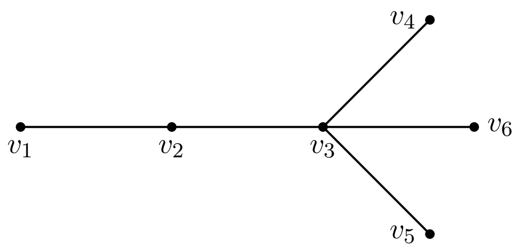


Figure 6.1

The table below shows the distances of all the vertices of the graph to vertices v_1, v_3, v_5 and v_6 and the last row shows the difference between the maximum and the minimum of each column.

Table 6.1

	v_1	v_2	v_3	v_4	v_5	v_6
v_1	0	1	2	3	3	3
v_3	2	1	0	1	1	1
v_5	3	2	1	2	1	2
v_6	3	2	1	2	2	0
$f(v,S)$	3	1	2	2	2	3

Here $f(v, S)$ is minimum for v_2 and hence $FC(S) = \{v_2\}$

6.2 Graphs with connected fair sets

In this section we characterize those chordal graphs for which the subgraph induced by fair sets are connected.

Theorem 6.2.1. *For any tree T, the subgraph induced by any fair set is connected.*

Proof. Let A be a fair set with $A = FC(S)$ where $S = \{v_1, \ldots, v_k\}$. Let $u, v \in A$. Assume that $v_1, \ldots, v_k$ are such that $d(u, v_1) \leqslant \cdots \leqslant d(u, v_k)$. Let P be the path $uu_1 \ldots u_m v$. At each stage as we move from u to v through the path P, let $d_1, \ldots, d_k$ denote the distance between the corresponding vertex of the path and $v_1, \ldots, v_k$ respectively. At u, $f(u, S) = d_k - d_1$. Since in any tree, the distances of two adjacent vertices from a given vertex differ by one, we have $f(u_1, S)$ is either $f(u, S)$ or $f(u, S)+1$ or $f(u, S)+2$. To prove $f(u_1, S) = f(u, S)$, we consider the following cases.

Case 1: $f(u_1, S) = f(u, S) + 2$.
We first consider $f(u_1, S) = f(u, S) + 2$. This is possible only when d_k increases by one and d_1 decreases by one as we move from u to u_1. Therefore, as we traverse from u to v through P, and the graph is a tree, d_k always increase by 1, so that the partiality cannot decrease. Hence $f(v, S) > f(u, S)$ which is a contradiction to the assumption that $u, v \in A$.

Case 2: $f(u_1, S) = f(u, S) + 1$.

Subcase 2.1: As we move from u to u_1, d_k increases by one and the role of v_1 is taken by some other vertex say v_2. Then similar to the Case 1, we can see that $f(v, S) > f(u, S)$, and a contradiction is obtained.

Subcase 2.2: The role of v_k is taken by another vertex, (say) v_{k-1}, so that the maximum distance remains the same(here d_{k-1}) and d_1 decreases by one. Now as we move from u_1 to u_2, since there was an increase in $d(u_1, v_{k-1})$ as compared to $d(u, v_{k-1})$ the maximum distance keeps on increasing so that the partiality becomes non decreasing. Hence the $f(v, S) > f(u, S)$, a contradiction to our assumption that $u, v \in A$.

From the contradictions of Cases 1 and 2, we obtain $f(u, S) = f(u_1, S)$. So that $u_1 \in A$ and in a similar fashion we can show that $V(P) \subseteq A$. Since u and v are arbitrary vertices of A, we can see that A is connected. Hence, we have the theorem. □

Next we prove that the above result can be extended to block graphs.

Corollary 6.2.2. *In a block graph, the subgraph induced by any fair set is connected.*

Proof. Let $G = (V, E)$ be a block graph. Let $v_1, v_2, \ldots, v_n$ be the vertices of G. Let $B_1, B_2, \ldots, B_r$ be the blocks of G. For any block graph G, its skeleton S_G is a tree [71]. (See figure 1.1). Also if $d_G(v_i, v_j) = d$ then $d_{S_G}(v_i, v_j) = 2d$. If $S = \{v_1, \ldots, v_k\}$ is a subset of $V(G)$, then for any vertex v_i, partiality $f_G(v_i, S) = \frac{1}{2} f_{S_G}(v_i, S)$. Hence if $v_l \in FC(S)$ with $f_G(v_l, S) = p$, then $f_{S_G}(v_l, S) = 2p$. Also for every $v_i \neq v_l$, $f_{S_G}(v_i, S) \geqslant 2p$. Now, let v_m be another vertex in G such that $f_G(v_m, S) = p$. Then $f_{S_G}(v_l, S) = 2p$, $f_G(v_m, S) = 2p$ and $f_{S_G}(v_i, S) \geqslant 2p$ for every $i = 1, \ldots, n$. Since S_G is connected there exists one path connecting v_l and v_m in S_G, say $v_l B_l v_{l+1} B_{l+1} \ldots B_{m-1} v_m$. Since we know that in a tree as we move along a path once partiality increases it cannot decrease $f_{S_G}(v_i, S) \leqslant 2p, i = l+1, \ldots, m-1$. But since partiality always greater than or equal to $2p$, $f_{S_G}(v_i, S) = 2p, i = l, l+1, \ldots, m-1, m$. Therefore $f_G(v_i, S) = p$, $i = l, l+1, \ldots, m-1, m$. Since v_l and v_{l+1} are adjacent to B_l they belong to same block in G. Therefore v_l and v_{l+1} are adjacent in G. Similarly v_{l+1} and v_{l+2} are adjacent in G. Hence we get a path $v_l, v_{l+1}, \ldots, v_{m-1} v_m$ in G connecting v_l and v_m all of whose partiality is p, the minimum. Therefore induced subgraph of any fair set is connected. □

The following theorem gives us an insight in to the structure of a chordal graph and this is being used to characterise chordal graphs with connected fair sets.

Theorem 6.2.3. *[40] A graph G is chordal if and only if it can be constructed recursively by pasting along complete subgraphs, starting from* *complete graphs.*

Theorem 6.2.4. *Let G be a chordal graph. Then G is a block graph if and only if the induced subgraph of any fair set of G is connected.*

Proof. Suppose G is a block graph. Then by Corollary 6.2.2, for any $S \subseteq V$ the induced subgraph of $FC(S)$ is connected. Conversely assume that the subgraphs induced by all fair sets of G are connected and assume that G is not a block graph. Since G is chordal, there exist two chordal graphs G_1 and G_2 such that G can be got by pasting G_1 and G_2 along a complete subgraph say, H, where $|V(H)| > 1$. Then there exists two vertices u and v such that $u \in V(G_1) \setminus V(H)$, $v \in V(G_2) \setminus V(H)$ and u and v are adjacent to all vertices of H. Consider the vertex set $V(H)$. Since u and v are adjacent to all vertices of H, $f(u, V(H)) = f(v, V(H)) = 1 - 1 = 0$. For all $x \in V(H)$, $f(x, V(H)) = 1$. Hence $FC(V(H))$ contains the vertices u and v and any path from u to v pass through the vertices of H which have partiality one. In other words the subgraph induced by the fair center of $V(H)$ is not connected, a contradiction. Therefore the subgraphs induced by all fair sets of G are connected implies G is a block graph. $\square$

As an illustration of Theorem 6.2.4, we have the following example.

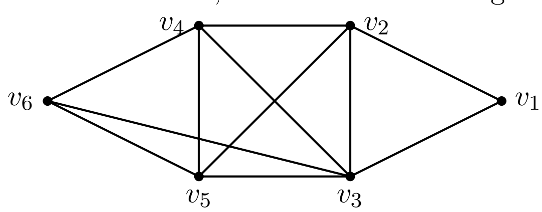


Figure 6.2: A Chordal graph with disconnected fair sets

For $V(H) = \{v_3, v_4, v_5\}$, we have $A = FC(V(H)) = \{v_2, v_6\}$, the induced subgraph of A is not connected.

6.3 Fair sets of some classes of graphs

In this section, we find the fair sets of some class of graphs, namely Complete graphs, $K_n - e$, $K_{m,n}$, the wheel graphs W_n, odd cycles and, symmetric even graphs. Before that we have the following lemma.

Lemma 6.3.1. For any graph $G = (V, E)$ all the fair sets A of G are of cardinality either $|V|$ or less than $|V| - 1$.

Proof. Let A be fair set of G and assume that $A \neq |V|$. To prove $|A| < |V| - 1$. If possible let $|A| = |V| - 1$. Let $A = FC(S)$ where $S \subseteq V$. Let y be the vertex which is not in A. For each $x \in A$ let $f(x, S) = k$. Also we have $f(y, S) > k$.
If $y \in S$ then we have $min(y, S) = 0$. So we must have $max(y, S) > k$. Therefore there exists an z in S such that $d(y, z) > k$ and this implies that $z \notin A$, a contradiction to the fact that $|A| = |V| - 1$. Hence for each x in S, $f(x, S) = k$.
Next let $y \notin S$. Let $min(y, S) = r$ and $max(y, S) = k + r + s$ where $r, s > 0$. Since $min(y, S) = r$ there exists a vertex w adjacent to y such that $min(w, S) = r - 1$. Since $f(w, S) = k$ we have $max(w, S) = k + r - 1 = k + r + s - (s + 1) = max(y, S) - (s + 1)$. Since $s \geqslant 1$, we have $|max(y, S) - max(w, S)| \geqslant 2$, a contradiction. □

6.3.1 Fair sets of Complete graphs

Now we identify the fair sets of complete graphs.

Proposition 6.3.1. For the complete graph on n vertices K_n, any $A \subseteq V$ such that $|A| \neq n - 1$, is a fair set.

Proof. Let $S \subseteq V$ with $|S| > 1$. Then for every $x \in S$, $f(x, S) = 1 - 0 = 1$ and for every $y \notin S$, $f(y, S) = 1 - 1 = 0$. Therefore $FC(S) = S^c$. Also if

$|S| = 1$ then $FC(S) = V$. Hence all $A \subseteq V$ such that $|A| \neq n-1$ is a fair set. □

6.3.2 Fair sets of $K_n - e$

The following proposition gives the fair sets of $K_n - e$

Proposition 6.3.2. Let G be the graph $K_n - e$ with $V(G) = \{v_1, \ldots, v_n\}$ and let e be the edge v_1v_2. Then $A \subseteq V$ is a fair set if and only if $|A| \neq n-1$ and either $\{v_1, v_2\} \subseteq A$ or $\{v_1, v_2\} \subseteq A^c$.

Proof. Let $\{v_1, v_2\} \subseteq A$ with $|A| < n-1$. Then $|A^c| \geqslant 2$. For each $x \in A$, $f(x, A^c) = 1 - 1 = 0$. For each $x \in A^c, f(x, A^c) = 1 - 0 = 1$. Therefore $FC(A^c) = A$. Now, let $\{v_1, v_2\} \subseteq A^c$. For each $x \in A$, $f(x, A^c) = 1-1 = 0$. $f(v_1, A^c) = 2 - 0 = 2$, $f(v_2, A^c) = 2 - 0 = 2$ and for every other x in A^c, $f(x, A^c) = 1 - 0 = 1$. Hence $FC(A^c) = A$.
Conversely, Let A be a fair set. We first prove that $|A| \neq n-1$. If $|A| = n-1$ then $|A^c| = 1$ so we have $FC(A^c) = V$. If B is any set such that $FC(B) = A$ then $|B| > 1$. If $\{v_1, v_2\} \subseteq B$, then $FC(B) = B^c \neq A$. If $v_1 \in B \cap A$ and $v_2 \notin B$ then $FC(B) = B^c \setminus \{v_2\} \neq A$. If $\{v_1, v_2\} \subseteq B^c$, then again $FC(B) = B^c \neq A$. Hence $|A| \neq n-1$.
Now let us assume that there is a set B with $FC(B) = A$. Suppose neither $\{v_1, v_2\} \subseteq A$ nor $\{v_1, v_2\} \subseteq A^c$. Without loss of generality, we assume $v_1 \in A$ and $v_2 \notin A$. If $\{v_1, v_2\} \subseteq B$, then $FC(B) = B^c \neq A$. If $v_1 \in B \cap A$ and $v_2 \notin B$ then $FC(B) = B^c \setminus \{v_2\} \neq A$. If $\{v_1, v_2\} \subseteq B^c$, then again $FC(B) = B^c \neq A$. From these we arrive at a contradiction to our assumption that $FC(B) = A$. Hence either $\{v_1, v_2\} \subseteq A$ or $\{v_1, v_2\} \subseteq A^c$. □

Illustration 6.3.1. Consider $K_6 - e$ given in figure 3.2. The fair sets of this graph are given bellow. First five are the fair sets that contain $\{v_1, v_2\}$ and the next four are the fair sets that exclude v_1 and v_2.

1. $\{v_1, v_2\}$
2. $\{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_2, v_5\}, \{v_1, v_2, v_6\}$
3. $\{v_1, v_2, v_3, v_4\}, \{v_1, v_2, v_3, v_5\}, \{v_1, v_2, v_3, v_6\}, \{v_1, v_2, v_4, v_5\}$, $\{v_1, v_2, v_4, v_6\}$, $\{v_1, v_2, v_5, v_6\}$
4. $\{v_1, v_2, v_3, v_4, v_5\}, \{v_1, v_2, v_3, v_4, v_6\}, \{v_1, v_2, v_3, v_5, v_6\}$, $\{v_1, v_2, v_4, v_5, v_6\}$
5. $\{v_1, v_2, v_3, v_4, v_5, v_6\}$
6. $\{v_3\}, \{v_4, \}, \{v_5\}, \{v_6\}$
7. $\{v_3, v_4\}, \{v_3, v_5\}, \{v_3, v_6\}, \{v_4, v_5\}, \{v_4, v_6\}, \{v_5, v_6\}$
8. $\{v_3, v_4, v_5\}, \{v_3, v_4, v_6\}, \{v_4, v_5, v_6\}, \{v_3, v_5, v_6\}$
9. $\{v_3, v_4, v_5, v_6\}$

6.3.3 Fair sets of Complete Bipartite graphs

The following proposition identifies the fair sets of complete bipartite graph $G = K_{m,n}$.

Proposition 6.3.3. Let G be a complete bipartite graph $K_{m,n}$ with bipartition (X, Y) where $|X| = m$ and $|Y| = n$. Let $A = A_1 \cup A_2$ where $A_1 \subseteq X$ and $A_2 \subseteq Y$. Then A is a fair set if and only if $|A_1| \neq m - 1$ and $|A_2| \neq n - 1$.

Proof. We prove the proposition case by case.
Case 1: $|A_1| < m - 1$ and $|A_2| < n - 1$.
Then $A^c = (X - A_1) \cup (Y - A_2)$. For, each $x \in A^c$, $f(x, A^c) = 2 - 0 = 2$ and for each $x \in A$ $f(x, A^c) = 2 - 1 = 1$. So, $FC(A^c) = A$.
Case 2: $A = X \cup Y$.
We can see that $FC(A) = A$.
Case 3: $|A_1| = m$ and $|A_2| < n - 1$.
Then as in the Case 1, we have $FC(A^c) = A$.
Case 4: $|A_1| = m - 1$ and $|A_2| = n - 1$.
Here $|A^c| = 2$ let it be $\{x_m, y_n\}$ where $x_m \in X$ and $y_n \in Y$. For each

$x \in A_1$, $f(x, A^c) = 2 - 1 = 1$, for each $x \in A_2$, $f(x, A^c) = 2 - 1 = 1$. $f(x_m, A^c) = f(y_n, A^c) = 1 - 0 = 1$. So $FC(A^c) = X \cup Y$.
Case 5: $|A_1| = m - 1$ and $|A_2| < n - 1$.
Let $A_1 = X \setminus \{x_1\}$. For each $x \in A_1$, $f(x, A^c) = 2-1 = 1$, for each $x \in A_2$, $f(x, A^c) = 2 - 1 = 1$. $f(x_1, A^c) = 1 - 0 = 1$ and for each $x \in Y \setminus A_2$, $f(x, A^c) = 2 - 0 = 2$. So $FC(A^c) = A_1 \cup A_2 \cup \{x_1\} = X \cup A_2$.
Case 6: $|A_1| = m - 1$ and $|A_2| = n$.
Then, we $FC(A^c) = X \cup Y$.
We can easily see that the cases 1 to 6, determine the fair centers of all types of subsets of V. Hence the proposition. □

As a simple illustration of Proposition 6.3.3, we have an example which discuss the Case 1 of the proposition.

Illustration 6.3.2. Consider the graph $K_{5,4}$ in figure 3.1 , with partitions $X = \{u_1, u_2, u_3, u_4, u_5\}$ and $Y = \{v_1, v_2, v_3, v_4\}$. By choosing $A_1 = \{u_1, u_2, u_3\}$, $A_2 = \{v_1, v_2\}$, $A^c = \{u_4, u_5, v_3, v_4\}$, we can see that $FC(A^c) = A$.

6.3.4 Fair sets of wheel graphs

Now we consider the case when the graph is a wheel W_n. We first prove the case when $n > 6$.

Theorem 6.3.4. *Let W_n, $(n \geqslant 6)$ be the wheel graph with vertex set $\{v_1, \ldots, v_{n-1}, v_n\}$, where v_n is the universal vertex. Let C_{n-1} be the cycle induced by $\{v_1, \ldots, v_{n-1}\}$. Then the fair sets of W_n are*

1. $\{v_i\}$, $1 \leqslant i \leqslant n$,

2. $\{v_i, v_j\}$ *such that* $v_i, v_j \in V(C_{n-1})$, $d_{C_{n-1}}(v_i, v_j) = 2$,

3. *$V(W_n)$,*

4. *All sets of the form $A_1 \cup \{v_n\}$ where $A_1 \subset V(C_{n-1})$ and $G[A_1]$ is not an induced path of length greater than $n-6$.*

Proof. We prove the theorem first for $n > 6$.We use the notation v_{i+k}(or v_{i-k}) for $v_{i+k-(n-1)}$(or $v_{i-k+(n-1)}$) when $i+k > n-1$(or $i-k < 1$). First, we prove that the four types of sets described in the theorem are indeed fair sets.

1. Let $S = \{v_{i\ominus_{n-1}1}, v_n, v_{i\oplus_{n-1}1}\}$, $1 \leqslant i \leqslant n-1$. $f(v_i, S) = 0$ and for all u other than v_i we have $f(u, S) > 0$ so that $FC(S) = \{v_i\}$. For $S = V$, $f(v_n, S) = 1$ and for all u other than v_n we have $f(u, S) = 2$, so in this case we can see that $FC(S) = \{v_n\}$. Hence $\{v_i\}$, $1 \leqslant i \leqslant n$ are all fair sets.

2. Let $S = \{v_n, v_i\}$, $1 \leqslant i \leqslant n-1$. $f(v_{i\ominus_{n-1}1}, S) = f(v_{i\oplus_{n-1}1}, S) = 0$ and for all other u, $f(u, S) > 0$. Hence $FC(S) = \{v_{i\ominus_{n-1}1}, v_{i\oplus_{n-1}1}\}$. In other words any $\{v_i, v_j\}$ such that $v_i, v_j \in V(C_{n-1})$, $d_{C_{n-1}}(v_i, v_j) = 2$ is a fair set.

3. Let $S = \{v_i, v_{i\oplus_{n-1}1}, v_n\}$. $f(v_i, S) = f(v_{i\oplus_{n-1}1}, S) = f(v_n, S) = 1 - 0 = 1$. For all other u, $f(u, S) = 2 - 1 = 1$. Hence $FC(S) = V$.

4. Now let $S \subseteq V$ be such that $v_n \in S$ and S contains at least one pair of vertices v_i and v_j such that $d_{C_{n-1}}(v_i, v_j) > 2$. Then for every $u \in S$ such that $u \neq v_n$, $f(u, S) = 2 - 0 = 2$, $f(v_n, S) = 1 - 0 = 1$ and for every $v \notin S$, $f(v, S) = 2 - 1 = 1$. Hence $FC(S) = S^c \cup \{v_n\}$. This gives us that for any $A \subseteq V$ such that $v_n \in A$ and $V(C_{n-1}) \setminus A$ is none of the following subsets of $V(C_{n-1})$ is a fair set.

   (a) $\{v_i\}$, $1 \leqslant i \leqslant n-1$.

(b) $\{v_i, v_{i\oplus_{n-1}1}\}$, $1 \leqslant i \leqslant n-1$.

(c) $\{v_i, v_{i\oplus_{n-1}2}\}$, $1 \leqslant i \leqslant n-1$.

(d) $\{v_i, v_{i\oplus_{n-1}1}, v_{i\oplus_{n-1}2}\}$, $1 \leqslant i \leqslant n-1$

The set $A_1 \cup \{v_n\}$ where A_1 is the complement in $V(C_{n-1})$ of a set mentioned in 4c above, is the fair center of $\{v_i, v_{i+1}, v_{i+2}, v_n\}$.

Therefore the only sets containing $\{v_n\}$ which have not been identified as fair sets are sets of the type $A_1 \cup \{v_n\}$ where A_1 is a path of length greater than $n-6$. Now let $A = A_1 \cup \{v_n\}$ where $A_1 \subseteq V(C_{n-1})$ forms a path of length greater than $n-6$. Assume there exists an $S \subseteq V$ such that $FC(S) = A$. If $S \subseteq V(C_{n-1})$ then $f(v_n, S) = 0$ and it is impossible to have $f(u, S) = 0$ for every $u \in A_1$. Hence S cannot be a subset of $V(C_{n-1})$ or $v_n \in S$. If S contains a v_i and v_j of C_{n-1} so that $d_{C_{n-1}}(v_i, v_j) > 2$ then $FC(S) = V(C_{n-1}) \setminus S \cup \{v_n\}$. So $FC(S) = A$ implies $V(C_{n-1}) \setminus S = A_1$. We have that v_i and v_j, two vertices such that $d_{C_{n-1}}(v_i, v_j) > 2$, does not belong to $V(C_{n-1}) \setminus S$. By the choice of A_1 such two vertices cannot be simultaneously absent from A_1. Hence $V(C_{n-1}) \setminus S \neq A_1$, a contradiction. So assume S does not contain two vertices v_i and v_j such that $d_{C_{n-1}}(v_i, v_j) > 2$. Hence S should be any one of the following:

a) $\{v_i, v_n\}$, $1 \leqslant i \leqslant n-1$.

b) $\{v_i, v_{i\oplus_{n-1}1}, v_n\}$, $1 \leqslant i \leqslant n-1$.

c) $\{v_i, v_{i\oplus_{n-1}2}, v_n\}$, $1 \leqslant i \leqslant n-1$.

d) $\{v_i, v_{i\oplus_{n-1}1}, v_{i\oplus_{n-1}2}, v_n\}$, $1 \leqslant i \leqslant n-1$

But we have already found out the fair centers of all these sets and none of them have A as its fair center. Hence an $A = A_1 \cup \{v_n\}$ where $A_1 \subseteq V(C_{n-1})$

forms a path of length greater than $n-6$ is not a fair set. Now we have the following observations

A. For an $S \subset V(C_{n-1})$, $v_n \in FC(S)$

B. If S contains v_i and v_j of C_{n-1} where $d_{C_{n-1}}(v_i, v_j) > 2$ then $v_n \in FC(S)$

C. Any $A = A_1 \cup \{v_n\}$ is fair set if and only if A_1 is not a path of length greater than $n-6$ in $V(C_{n-1})$.

D. For any $S \subseteq V$ such that $v_n \in S$ and $v_i, v_j \in S \implies d_{C_{n-1}}(v_i, v_j) \leqslant 2$, fair centers are sets of any of the following forms.

i) $\{v_i\}$

ii) $\{v_i, v_{i \oplus_{n-1} 2}\}$

iii) $V(W_n)$

iv) $\{v_1, \dots v_i, v_{i \oplus_{n-1} 2}, \dots, v_{n-1}, v_n\}$, a set of type described in C.

Hence the only fair sets of W_n are those described in C and D above.
When n=6 all sets described in item 4 of the theorem are same as those described in item 1. The rest of the proof is same as above. Hence the theorem. □

Illustration 6.3.3. Consider W_9 given in Figure 3.3. Here $\{v_1, v_2, v_3, v_4, v_5, v_9\}$ is not a fair set as the graph induced by $\{v_1, v_2, v_3, v_4, v_5\}$ is a path of length 4 and $n-6=3$. Similarly $\{v_1, v_2, v_3, v_4, v_5, v_6, v_9\}$, $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_9\}$ etc are also not fair sets. We are not listing the whole fair sets as the list is too long.

When $n=4$, we can see that $W_4 = K_4$, so we prove the case when the graph is a wheel W_n for $n=5$, where we get a proposition, which is entirely different from Theorem 6.3.4.

Proposition 6.3.5. If G is W_5 with $V = \{v_1, v_2, v_3, v_4, v_5\}$, where v_5 is adjacent to all other vertices and $v_1v_2v_3v_4v_1$ is the outer cycle, then the only fair sets of G are $\{v_5\}, \{v_1, v_3\}, \{v_2, v_4\}, \{v_1, v_3, v_5\}, \{v_2, v_4, v_5\}$ and V.

Proof. Given a non empty vertex set S, let $d_1, d_2, \dots d_k$ be the distances of the vertex v_1 from the vertices of S where $d_1 \leqslant d_2 \leqslant \dots \leqslant d_k$. Then the distances of v_3 from vertices of S are $2 - d_k, 2 - d_{k-1}, \dots, 2 - d_1$ where $2 - d_k \leqslant 2 - d_{k-1} \leqslant, \dots, \leqslant 2 - d_1$.
Hence $f(v_3, S) = d_k - d_1 = 2 - d_1 - (2 - d_k) = f(v_1, S)$. Hence if A is any fair set, $v_1 \in A$ implies $v_3 \in A$. Similarly $v_2 \in A$ implies $v_4 \in A$.
Now $f(v_i, V) = 2$ for $1 \leqslant i \leqslant 4$ and $f(v_5, V) = 1$. Hence $FC(V) = \{v_5\}$.
Similarly we can observe that $FC(\{v_5, v_4\}) = \{v_1, v_3\}$,
$FC(\{v_5, v_3\}) = \{v_2, v_4\}$, $FC(\{v_1, v_3\}) = \{v_2, v_4, v_5\}$,
$FC(\{v_2, v_4\}) = \{v_1, v_3, v_5\}$ and $FC(\{v_1, v_2, v_5\}) = V$. Hence the fair sets of W_5 are precisely those described in the theorem. □

6.3.5 Fair sets of Paths

Lemma 6.3.2. Let P_n be path $v_1v_2 \dots v_n$. Let $S = \{v_{i_1}, v_{i_2}, \dots, v_{i_k}\}$, where $1 \leqslant i_1 < i_2 < \dots < i_k \leqslant n$. Then $FC(S) \subset FC(S')$ where

$$S' = \begin{cases} \{v_{i_1}, v_{\frac{i_1+i_k}{2}}, v_{i_k}\} & \text{if } \frac{i_1+i_k}{2} \text{ is an integer.} \\ \{v_{i_1}, v_{\lfloor \frac{i_1+i_k}{2} \rfloor}, v_{\lceil \frac{i_1+i_k}{2} \rceil}, v_{i_k}\} & \text{if } \frac{i_1+i_k}{2} \text{ is not an integer.} \end{cases}$$

Proof. From theorem 6.2.1 the induced subgraph of any fair set in a path is connected and is therefore an interval. Suppose $\frac{i_1+i_k}{2}$ is not an integer. Let $d(v_{\lfloor \frac{i_1+i_k}{2} \rfloor}, v_{i_k}) = d(v_{\lceil \frac{i_1+i_k}{2} \rceil}, v_{i_1}) = d$. Then $f(v_{\lfloor \frac{i_1+i_k}{2} \rfloor}, S') = d$. As we move towards v_1 partiality remains to be d up to $v_{\lceil \frac{i_1 + \lfloor \frac{i_1+i_k}{2} \rfloor}{2} \rceil}$. After this partiality increases. Similarly $f(v_{\lceil \frac{i_1+i_k}{2} \rceil}, S') = d$. As we move towards

v_n partiality remains to be d up to $v_{\lfloor \frac{i_k+\lceil \frac{i_1+i_k}{2} \rceil}{2} \rfloor}$ and after wards partiality increases. Therefore

$$FC(S') = \{v_{\lceil \frac{i_1+\lfloor \frac{i_1+i_k}{2} \rfloor}{2} \rceil}, \ldots, v_{\lfloor \frac{i_k+\lceil \frac{i_1+i_k}{2} \rceil}{2} \rfloor}\} \tag{6.1}$$

$\max(x, S)$ is minimum for $x = v_{\lfloor \frac{i_1+i_k}{2} \rfloor}$ and $v_{\lceil \frac{i_1+i_k}{2} \rceil}$ and $max(x, S)$ increases by one as we move from these two vertices towards v_1 and v_n respectively. Also difference between $min(x, S)$ of two consecutive vertices can be at most one. Therefore at least one of $v_{\lfloor \frac{i_1+i_k}{2} \rfloor}$ and $v_{\lceil \frac{i_1+i_k}{2} \rceil}$ belong to $FC(S')$. Let $\lfloor \frac{i_k-i_1}{2} \rfloor = a$. Hence $\max(v_{\lfloor \frac{i_1+i_k}{2} \rfloor}, S) = \max(v_{\lceil \frac{i_1+i_k}{2} \rceil}, S) = a$. Now we shall take three cases.

Case 1:Both $v_{\lfloor \frac{i_1+i_k}{2} \rfloor}$ and $v_{\lceil \frac{i_1+i_k}{2} \rceil}$ belong to S. Let i_m be the largest integer such that $i_1 \leqslant i_m \leqslant \lceil \frac{i_1+i_k}{2} \rceil$, $v_{i_m} \in S$ and let i_l be the smallest integer such that $\lceil \frac{i_1+i_k}{2} \rceil \leqslant i_l \leqslant i_k$, $v_{i_l} \in S$. Then in this case
$FC(S) = \{v_{\lceil \frac{i_m+\lfloor \frac{i_l+i_k}{2} \rfloor}{2} \rceil}, \ldots, v_{\lfloor \frac{i_l+\lceil \frac{i_1+i_k}{2} \rceil}{2} \rfloor}\} \subset FC(S')$.

Case 2: One of $v_{\lfloor \frac{i_1+i_k}{2} \rfloor}$ and $v_{\lceil \frac{i_1+i_k}{2} \rceil}$ is present in S and the other is absent. Without loss of generality we may assume that $v_{\lfloor \frac{i_1+i_k}{2} \rfloor} \in S$ and $v_{\lceil \frac{i_1+i_k}{2} \rceil} \notin S$.
$f(v_{\lfloor \frac{i_1+i_k}{2} \rfloor}, S) = a$ and $f(v_{\lceil \frac{i_1+i_k}{2} \rceil}, S') = a - 1$. As we move away from $v_{\lceil \frac{i_1+i_k}{2} \rceil}$ towards v_n partiality remains the same up to $v_{\lfloor \frac{i_l+\lfloor \frac{i_1+i_k}{2} \rfloor}{2} \rfloor}$ and from there onwards partiality increases. Therefore
$FC(S) = \{v_{\lceil \frac{i_1+i_k}{2} \rceil}, \ldots, v_{\lfloor \frac{i_l+\lfloor \frac{i_1+i_k}{2} \rfloor}{2} \rfloor}\} \subset FC(S')$.

Case 3:Both $v_{\lfloor \frac{i_1+i_k}{2} \rfloor}$ and $v_{\lceil \frac{i_1+i_k}{2} \rceil}$ does not belong to S.
Assume $\lfloor \frac{i_1+i_k}{2} \rfloor - i_m \leqslant i_l - \lceil \frac{i_1+i_k}{2} \rceil$. Let $\lfloor \frac{i_1+i_k}{2} \rfloor - i_m = b$ and $i_l - \lceil \frac{i_1+i_k}{2} \rceil = c$. Then $f(v_{\lfloor \frac{i_1+i_k}{2} \rfloor}, S) = a - b$. $f(v_{\lceil \frac{i_1+i_k}{2} \rceil}, S) = a - (b+1)$ if $b < c$ and $f(v_{\lceil \frac{i_1+i_k}{2} \rceil}, S) = a - b$ if $b = c$. That is,
$f(v_{\lceil \frac{i_1+i_k}{2} \rceil}, S) \leqslant f(v_{\lfloor \frac{i_1+i_k}{2} \rfloor}, S)$.If $f(v_{\lceil \frac{i_1+i_k}{2} \rceil}, S) = a - (b+1)$ as we move

from $v_{\lceil \frac{i_1+i_k}{2} \rceil}$ towards v_n partiality remains to be $a-(b+1)$ up to $v_{\lfloor \frac{i_m+i_l}{2} \rfloor}$. If $b = c$ $f(v_{\lfloor \frac{i_1+i_k}{2} \rfloor}, S) = f(v_{\lceil \frac{i_1+i_k}{2} \rceil}, S) = a-b$. Hence in both the cases $FC(S) = \{v_{\lceil \frac{i_1+i_k}{2} \rceil}, \ldots, v_{\lfloor \frac{i_l+i_m}{2} \rfloor}\}$. In other words $FC(S) \subset FC(S')$. When $\frac{i_1+i_k}{2}$ is an integer $\lfloor \frac{i_1+i_k}{2} \rfloor = \lceil \frac{i_1+i_k}{2} \rceil$ and the proof is similar. □

Corollary 6.3.6. *Let S' be as in the Lemma 6.3.2 and t be an integer.*

$$\text{Length of } FC(S') = \begin{cases} 2t & \text{if } d(v_{i_1}, v_{i_k}) = 4t \text{ or } 4t+2 \\ 2t+1 & \text{if } d(v_{i_1}, v_{i_k}) = 4t+1 \text{ or } 4t+3 \end{cases}$$

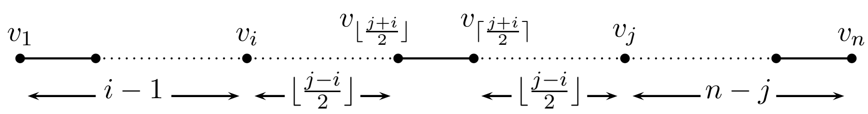
Proof. We shall assume that $v_{i_1} = v_1$.
Case 1: $d(v_{i_1}, v_{i_k}) = 4t$. So $i_k = 4t+1$ and
$FC(S') = \{v_{\lceil \frac{1+\lfloor \frac{1+4t+1}{2} \rfloor}{2} \rceil}, \ldots, v_{\lfloor \frac{4t+1+\lceil \frac{1+4t+1}{2} \rceil}{2} \rfloor}\} = \{v_{t+1}, \ldots, v_{3t+1}\}$, a path of length $2t$.
Case 2: $d(v_{i_1}, v_{i_k}) = 4t+1$. Then $i_k = 4t+2$ and hence,
$FC(S') = \{v_{\lceil \frac{1+\lfloor \frac{1+4t+2}{2} \rfloor}{2} \rceil}, \ldots, v_{\lfloor \frac{4t+2+\lceil \frac{1+4t+2}{2} \rceil}{2} \rfloor}\} = \{v_{t+1}, \ldots, v_{3t+2}\}$, a path of length $2t+1$.
Case 3: $d(v_{i_1}, v_{i_k}) = 4t+2 \Rightarrow i_k = 4t+3$ so that
$FC(S') = \{v_{\lceil \frac{1+\lfloor \frac{1+4t+3}{2} \rfloor}{2} \rceil}, \ldots, v_{\lfloor \frac{4t+3+\lceil \frac{1+4t+3}{2} \rceil}{2} \rfloor}\} = \{v_{t+2}, \ldots, v_{3t+2}\}$, a path of length $2t$.
Case 4: $d(v_{i_1}, v_{i_k}) = 4t+3$. Then $i_k = 4t+4$ and thus
$FC(S') = \{v_{\lceil \frac{1+\lfloor \frac{1+4t+4}{2} \rfloor}{2} \rceil}, \ldots, v_{\lfloor \frac{4t+4+\lceil \frac{1+4t+4}{2} \rceil}{2} \rfloor}\} = \{v_{t+2}, \ldots, v_{3t+3}\}$, a path of length $2t+1$. □

Corollary 6.3.7. *Let S' be as defined in the Lemma 6.3.2 with length of $FC(S')$ equal to d. If $S'' \subseteq V$ with $diam(S'') \leqslant diam(S')$ then the length of $FC(S'')$ is atmost $d+1$.*

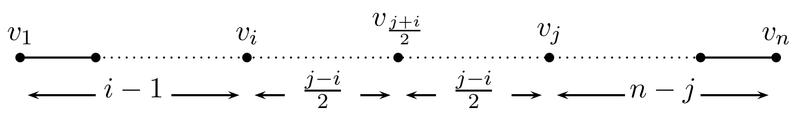
Theorem 6.3.8. *Let P be the path $v_1 v_2 \ldots v_n$. $A = \{v_i, v_{i+1}, \ldots, v_j\}$ is a fair set if and only if either*

(i) $i = 1$ *and* $j = n$ *or*
(ii) $\lfloor \frac{j-i}{2} \rfloor \leqslant \min\{i-1, n-j\}$

Proof. If $A = \{v_1, \dots, v_n\}$ then it is a fair set because $FC(\{v_1, v_2\}) = A$. Now assume that $A \neq \{v_1, \dots, v_n\}$. Suppose $j - i$ is odd and let $j - i = 2a + 1$.



$j - i$ is odd



$j - i$ is even

Figure 6.3

Without loss of generality we may assume that $i - 1 \leqslant n - j$. Suppose $i - 1 \geqslant a$. Let S be the set $\{v_{i-a}, v_{i+a}, v_{i+a+1}, v_{i+3a+1}\}$. Since $a \leqslant i - 1 \leqslant n - j$, $v_{i-a}, v_{i+3a+1} \in V(P)$. As in the proof of lemma 6.3.2 we get that $FC(S) = \{v_{\lceil \frac{i-a+\lfloor \frac{i-a+i+3a+1}{2} \rfloor}{2} \rceil}, \dots, v_{\lfloor \frac{i+3a+1+\lceil \frac{i-a+i+3a+1}{2} \rceil}{2} \rfloor}\} = \{v_i, \dots, v_{i+2a+1}\} = \{v_i, \dots, v_j\}$. Hence A is a fair set. Now Suppose

$i-1<a$. Let $S=\{v_1, v_{i+a}, v_{i+a+1}, v_{2i+2a}\}$. Then,
$FC(S)=\{v_{\lceil\frac{1+\lfloor\frac{2i+2a+1}{2}\rfloor}{2}\rceil},\ldots,v_{\lfloor\frac{2i+2a+\lceil\frac{2i+2a+1}{2}\rceil}{2}\rfloor}\}=\{v_{\lceil\frac{i+a+1}{2}\rceil},\ldots,v_{\lfloor\frac{3i+3a+1}{2}\rfloor}\}$.
Since $i-1<a$, $v_{\lceil\frac{i+a+1}{2}\rceil}$ lies towards the right of v_i and $v_{\lfloor\frac{3i+3a+1}{2}\rfloor}$ towards left of v_{i+2a+1}. Hence $FC(S)\subseteq A$ and length of $FC(S)\leqslant$ length of $A-2$. By lemma 6.3.2 any set with end vertices v_1 and v_{2i+2a} cannot have A as the fair center. From Corollary 6.3.7 it follows that any set with diameter less than that of S cannot have A as its fair center. If S is any set with end vertices v_p and v_q from equation 1 it is clear that if $p=1$ and $q>2i+2a$ or $p>1$ and $q>2i+2a$ then $FC(S)\neq A$. In other words A is not a fair set. Similarly we can prove the case when $j-i$ is even. Hence the theorem. □

Illustration 6.3.4. Consider the path P_8 with vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8\}$.

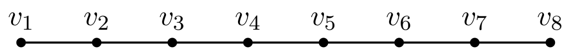


Figure 6.4: P_8

Here $\{v_2, v_3\}$, $\{v_2, v_3, v_4\}$, $\{v_2, v_3, v_4, v_5\}$ are all fair sets, but $\{v_2, v_3, v_4, v_5, v_6\}$ is not a fair set since $i-1=1$, and $\lfloor\frac{j-i}{2}\rfloor=\frac{6-2}{2}=2>1$. Similarly $\{v_3, v_4, v_5, v_6, v_7\}$ is also not a fair set.

6.3.6 Fair sets of Odd cycles

In the next theorem we find out the fair sets of odd cycles

Theorem 6.3.9. *Let the graph $G=C_{2n+1}$ be an odd cycle with vertex set $V=\{v_1,\ldots,v_{2n+1}\}$. $A\subseteq V$ is a fair set if and only if for every pair* *of consecutive vertices in A, the vertex which is eccentric to both theses vertices is also in A.*

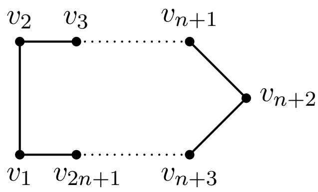


Figure 6.5: Odd cycle C_{2n+1}

Proof. Let v_1 and v_2 be a pair of adjacent vertices and let v_{n+2} be the vertex which is eccentric to both v_1 and v_2. Let $v_1, v_2 \in A \subseteq V$ where $A = FC(S)$ for some $S \subseteq V$. Let $min(v_1, S) = d_{min}$ and $max(v_1, S) = d_{max}$. Since $v_1, v_2 \in A$, $f(v_1, S) = f(v_2, S)$. We shall consider here two different cases.
Case 1: $min(v_2, S) = d_{min} + 1$ and $max(v_2, S) = d_{max} + 1$. Then there exists a vertex $v \in S$ such that $d(v_1, v) = d_{max}$ and $d(v_2, v) = d_{max} + 1$. Then $d(v_{m+2}, v) = m - d_{max}$. If there exists a vertex $v' \in S$ such that $d(v_{m+2}, v') < m - d_{max}$ then $d(v', v_1) > d_{max}$, a contradiction. Hence $min(v_{n+2}, S) = d(v_{n+2}, v)$. Similarly there exists a vertex u such that $d(u, v_1) = d_{min}$ and $d(u, v_2) = d_{min} + 1$. $max(v_{n+2}, S) = d(v_{n+2}, u) = m - d_{min}$. Therefore $f(v_{n+2}, S) = m - d_{min} - (m - d_{max}) = d_{max} - d_{min} = f(v_1, S) = f(v_2, S)$. That is $v_{n+2} \in A$.

Case 2: $min(v_2, S) = d_{min}$ and $max(v_2, S) = d_{max}$. Let u and u' be such that $min(v_1, S) = d(v_1, u)$ and $min(v_2, S) = d(v_2, u')$. $max(v_{n+2}, S) = m - d(v_1, u) = m - min(v_1, S) = m - d_{min}$. Let v and v' be such that $max(v_1, S) = d(v_1, v)$ and $max(v_2, S) = d(v_2, v')$. If $v = v'$ then $v = v_{n+2}$. In this case $min(v_{n+2}, S) = 0$. Therefore $f(v_{n+2}, S) = n - d_{min} =$

$max(v_1, S) - min(v_1, S) = f(v_1, S)$. If $v \neq v'$, $d(v_2, v) = d(v_1, v') = d_{max} - 1$. Hence $min(v_{n+2}, S) = d(v_{n+2}, v) = d(v_{n+2}, v') = m - (d_{max} - 1) = n - d_{max} + 1$. Therefore $f(v_{n+2}, S) = n - d_{min} - (n - d_{max} + 1) = d_{max} - d_{min} - 1 < f(v_1, S) = f(v_2, S)$. This contradicts the fact that $v_1, v_2 \in FC(S)$ and so we rule out this possibility. Hence in all the possible cases $v_1, v_2 \in A \Rightarrow v_{n+2} \in A$.

Conversely, assume that $A \subseteq V$ is such that for every pair of consecutive vertices v_i, v_{i+1} in A, v_{n+i+1} belong to A. Let $v_1, \ldots v_k$, $k > 1$, be consecutive vertices belonging to A. Then $v_{n+2}, v_{n+3}, \ldots, v_{n+k}$ belong to A. Without loss of generality we may assume that

i) A does not contain any consecutive set of vertices other than the above two.

ii) v_{n+1} does not belong to A.

Now, construct the set S as follows

step I) If $k = 3r$ or $3r + 1$ for some integer r then set $S = \{v_2, v_5, \ldots, v_{3r-1}\}$. If $k = 3r + 2$ then then set $S = \{v_3, v_6, \ldots, v_{3r}\}$.

step II) Add to S the vertices $v_i, n + 2 \leqslant i \leqslant n + k$ of A, which are not an eccentric vertex of any of the vertices in S.

step III) Add A^c to S.

Let $x \in V \setminus A$. Then $x \in S$ and therefore $min(x, S) = 0$. Let y and z be the eccentric vertices of x in G. Take note that yz is an edge. If $\{y, z\} \subseteq S^c$, then we have $\{y, z\} \subseteq A$. Since x is an eccentric vertex of y and z, we have $x \in A$, which is not true. Hence either y or z belongs to S, and we have $max(x, S) = n$, so that $f(x, S) = n$.

Let $x \in A$ be such that both the neighbours of x do not belong to A. Then $x \notin S$ and neighbours of x belong to S and $min(x, S) = 1$.

Let x_1 and x_2 be the eccentric vertices of x. Then $x_1, x_2 \notin S$ implies either $x_1, x_2 \in \{v_1, \ldots, v_k\}$ or $x_1, x_2 \in \{v_{n+2}, \ldots, v_{n+k}\}$. In the former case $x \in \{v_{n+1}, \ldots, v_{n+k}\}$ and in the latter case $x \in \{v_1, \ldots, v_k\}$ and this is not possible by the choice of x. Hence either x_1 or x_2 belong to S. Hence $max(x, S) = n$. Therefore $f(x, S) = n - 1$. By the way of choice of vertices $v_i, 1 \leqslant i \leqslant k$, in S either $min(v_i, S) = 1$ and $max(v_i, S) = n$ or $min(v_i, S) = 0$ and $max(v_i, S) = n-1$. Hence $f(v_i, S) = n-1$ for $1 \leqslant i \leqslant k$. For $n+2 \leqslant i \leqslant n+k$, $v_i \notin S$ implies eccentric of v_i belong to S. Therefore in this case $min(v_i, S) = 1$ and $max(v_i, S) = n$ or $f(v_i, S) = n - 1$. Now, for $n + 2 \leqslant i \leqslant n + k$, $v_i \in S$ implies $min(v_i, S) = 0$. Now an eccentric vertex of v_i, $m + 2 \leqslant i \leqslant n + k$, belong to S implies $v_i \notin S$. Hence for $v_i \in S$ eccentric vertices of $v_i \notin S$. Also there are no three consecutive vertices among v_i's, $1 \leqslant i \leqslant k$, absent from S. Hence $max(v_i, S) = n - 1$ for $n+2 \leqslant i \leqslant n+k$. Also the two eccentric vertices of v_{n+k+1}, v_k and v_{k+1} does not belong to S. Hence if $v_{n+k+1} \in S$, $f(v_{n+k+1}, S) = n-1$. Therefore for each $v_i \in A$, $f(v_i, S) = n - 1$ and for each $v_i \notin A$, $f(v_i, S) = n$. Hence $FC(S) = A$ or A is a fair set. ☐

We have an immediate corollary for Theorem 6.3.9 and the proof follows from the proof of Theorem 6.3.9.

Corollary 6.3.10. *If $A \subset V(C_{2n+1})$ contains no two adjacent vertices then A is a fair set of C_{2n+1}.*

Corollary 6.3.11. *The only connected fair sets of an odd cycle C_{2n+1} are singleton (vertex) sets and the whole vertex set V.*

Proof. By the Theorem 6.3.9, $\{v_i\}, 1 \leqslant i \leqslant 2n + 1$ and V are fair sets. Now let $A \subseteq V$ be a connected fair set of C_{2n+1} which contains more than one element. Let $v_i, v_j \in A$. Then there exists a path connecting v_i and v_j in A without loss of generality we may assume that it is $v_i, v_{i+1}, \ldots v_j$.

$v_i, v_{i+1} \in A$ implies $v_{n+i+1} \in A$. Therefore a path connecting v_i and v_{n+i+1} lies in A. Since this path contains $n+2$ consecutive vertices by the theorem we can conclude that A should also contain the other $n-1$ vertices or $A = V$. □

Illustration 6.3.5. Consider the odd cycle $C_{15} = v_1v_2, \ldots v_{15}v_1$. Let $A = \{v_1, v_2, v_3, v_5, v_6, v_9, v_{10}, v_{13}\}$. v_1 and v_2 are a pair of consecutive vertices and the vertex eccentric to both v_1 and v_2, v_9, also belong to A. Similarly, (v_2, v_3) and (v_5, v_6) are pairs of adjacent vertices and the vertices eccentric to these pairs namely, v_{10} and v_{13} also belong to A. Hence as per the theorem,A is a fair set. According to the construction given in the theorem we have
$S = \{v_2, v_4, v_7, v_8, v_{11}, v_{12}, v_{13}, v_{14}, v_{15}\}$. $e_S(v_1) = 6$, $e_S(v_2) = 6$, $e_S(v_3) = 6$, $e_S(v_4) = 7$, $e_S(v_5) = 6$, $e_S(v_6) = 6$, $e_S(v_7) = 7$, $e_S(v_8) = 7$, $e_S(v_9) = 6$, $e_S(v_{10}) = 6$, $e_S(v_{11}) = 7$, $e_S(v_{12}) = 7$, $e_S(v_{13}) = 6$, $e_S(v_{14}) = 7$, $e_S(v_{15}) = 7$. Therefore $C_S(C_{15}) = A$.

6.3.7 Fair sets of Symmetric Even graphs

Proposition 6.3.12 (Proposition 4 of [54])**.** Let u and v be vertices of a symmetric even graph G of diameter d. If $v \in N_i(u)$ and $\bar{v} \in N_j(u)$, then $i + j = d$.

The following theorem characterises the fair sets of symmetric even graphs.

Theorem 6.3.13. *Let G be a symmetric even graph. An $A \subseteq V$ is a fair set if and only if for every vertex $x \in A$, $\bar{x} \in A$.*

Proof. Let $diam(G) = d$. Assume $A \subseteq V$ is a fair set with $FC(S) = A$ where $S = \{v_1, \ldots, v_k\}$ and and let $x \in A$. Let $d(x, v_i) = d_i, 1 \leqslant i \leqslant k$. Without loss of generality we may assume that $d_1 \leqslant d_2 \leqslant \ldots \leqslant d_k$. Then

$f(x,S) = d_k - d_1$. By proposition 6.3.12, $d(x,v_i) = d_i \Rightarrow d(\bar{x},v_i) = d - d_i$. Therefore $f(\bar{x},S) = d - d_1 - (d - d_k) = d_k - d_1$. Hence $x \in A \Rightarrow \bar{x} \in A$.
Conversely, assume that $A \subseteq V$ is such that $x \in A \Rightarrow \bar{x} \in A$. To prove $A = FC(S)$ for some $S \subseteq V$. Let $A = \{x_1,\ldots,x_m,\bar{x_1},\ldots,\bar{x_m}\}$. Let $S = V \setminus \{x_1,\ldots,x_m\}$. Suppose for every x_i, $1 \leqslant i \leqslant m$ some neighbour of x_i is in S. Then $f(x_i,S) = d(x_i,\bar{x_i}) - 1 = d - 1$(Here the minimum distance is 1 since $x_i \notin S$ and some neighbour of x_i is in S). For each $\bar{x_i}, 1 \leqslant i \leqslant m$, $f(\bar{x_i},S) = d - 1 - 0 = d - 1$(Here the maximum distance is $d-1$ since $x_i \notin S$ and some neighbour of x_i is in S. The minimum distance is 0, since $\bar{x_i} \in S$). Now, for a y different from $x_i, \bar{x_i}$, $1 \leqslant i \leqslant m$, $min(y,S) = 0$ since $y \in S$ and $max(y,S) = d$ since $\bar{y} \in S$. Therefore $f(y,S) = d - 0 = d$. In other words $FC(S) = A$. Now, assume that there exists an x_j, say x_1, such that neither x_1 nor any of the vertices adjacent to x_1 are in S. That is the vertices adjacent to x_1 are among $x_2, x_3, \ldots, x_m$. Let x_i be a vertex adjacent to x_1. Then $min(x_1,S) > 1$ and $max(x_1,S) = d$. Therefore $f(x_1,S) = max(x_1,S) - min(x_1,S) \leqslant d - 2$. Let x_k be a vertex such that a neighbour of x_k is in S. Then $min(x_k,S) = 1$ and $max(x_k,S) = d$ and therefore $f(x_k,S) = d-1$. Therefore $FC(S) \neq A$. Let $S_1 = \{x_1\} \cup S \setminus \{\bar{x_1}\}$. Now, $min(x_1,S_1) = 0$ since $x_1 \in S_1$ and $max(x_1,S_1) = d - 1$ since $\bar{x_i}$ is adjacent to $\bar{x_1}$ and $barx_i \in S_1$. Hence $f(x_1,S_1) = d-1$. Also, $\min(\bar{x_1},S_1) = 1$ since $\bar{x_i} \in S_1$ and $\max(\bar{x_1},S_1) = d$ since $x_1 \in S_1$. Hence $f(\bar{x_1},S_1) = d - 1$. For a $y \in V$ such that $y \neq x_j, \bar{x_j}$, $1 \leqslant j \leqslant m$ we have that $\min(y,S_1) = 0$ and $\max(y,S_1) = d(y,\bar{y}) = d$ and therefore $f(y,S_1) = d$. If for every $y \in V$ either $y \in S_1$ or some neighbour of v is in S_1 then as above it can be shown that $FC(S_1) = A$. Otherwise, let x_2 be a vertex such that neither x_2 nor any of the vertices adjacent to x_2 are in S_1. It is clear that $x_2 \neq x_i, \bar{x_1}$. Let $S_2 = \{x_2\} \cup S_1 \setminus \{\bar{x_2}\}$. Then $\min(x_1,S_2) = 0$, $max(x_1,S_2) = d(x_1,\bar{x_i}) = d-1$ and therefore $f(x_1,S_1) = d - 1$. $\min(\bar{x_1},S_2) = d(\bar{x_1},\bar{x_i}) = 1$, $max(\bar{x_1},S_2) = d(x_1,\bar{x_1}) = d$ and

therefore $f(\bar{x_1}, S_2) = d-1$. we have $\min(x_2, S_2) = 0$. The vertices adjacent to x_2 are among $\bar{x_1}, x_3, \ldots, x_m$ and their eccentric vertices $x_1, \bar{x_3}, \ldots, \bar{x_m}$ are in S_2. Hence $\max(x_2, S_2) = d-1$. Thus $f(x_2, S_2) = d-1$. $\min(\bar{x_2}, S_2) = 1$ since $\bar{x_2}$ is adjacent to the vertices that are eccentric to the vertices adjacent to x_2 and $\max(\bar{x_2}, S_2) = d(\bar{x_2}, x_2) = d$. Hence $f(\bar{x_2}, S_2) = d-1$. If S_2 is such that for every $y \in V$ either $y \in S_1$ or some neighbour of v is in S_1 then $FC(S_2) = A$. Else, we continue the above process and it should be noted that the partiality of the vertices that are added and deleted at each stage is adjusted to $d-1$, the partiality of all the vertices that have been added and deleted in the previous stages are maintained to be $d-1$ and the partiality of all vertices different from x_j and $\bar{x_j}$ are equal to d . At most in m stages, we shall get an S' such that $f(x_j, S') = f(\bar{x_j}, S') = d-1$ for $1 \leqslant j \leqslant m$ and $f(y, S') = d$ for $y \neq x_j, \bar{x_j}$. That is, $FC(S') = A$ and that proves the theorem. $\square$

Corollary 6.3.14. *The only connected fair set of an even cycle C_{2n} is the whole vertex set V.*

Proof. Let A be a connected fair set of C_{2n}. Let $u \in A$. Then by the above theorem $\bar{u} \in A$. Since A is connected at least one of the paths connecting u and $\bar{u}$ should be in A. Again by the theorem the eccentric vertices of the vertices of this path should also be in A. Hence $A = V$. $\square$

Illustration 6.3.6. Consider the even cycle $C_{16} = v_1v_2 \ldots v_{15}v_1$ a symmetric even graph. We shall a find a vertex set whose center is
$A = \{v_1, v_2, v_3, v_4, v_5, v_9, v_{10}, v_{11}, v_{12}, v_{13}\}$.
Let $S = \{v_6, v_7, v_8, v_9, v_{10}, v_{11}\ , v_{12}, v_{13}, v_{14}, v_{15}, v_{16}\}$. Since neither v_2 nor any of the vertices adjacent to v_2 belong to S set $S_1 = \{v_2\} \cup S \setminus v_{10}$. Again, neither v_4 nor any of the vertices adjacent to v_4 are in S_1. Set, $S_2 = \{v_4\} \cup S_1 \setminus v_{12}$. Now for every $v \in V$ either v or a neighbour of v is in S_2. we have $FC(S_2) = A$.

6.4 Fair sets and Cartesian product of graphs

Next we have an expression for the fair center of product sets in the Cartesian product of two graphs

Theorem 6.4.1. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Let $S_1 \subseteq V_1$ and $S_2 \subseteq V_2$. Then $FC(S_1 \times S_2) = FC(S_1) \times FC(S_2)$ where $FC(S_1 \times S_2)$ is the fair center of $S_1 \times S_2$ in the graph $G_1 \square G_2$, $FC(S_1)$ is the fair center of S_1 in the graph G_1 and $FC(S_2)$ is the fair center of S_2 in the graph G_2.*

Proof. Let $(x, y) \in V_1 \times V_2$, $S_1 = \{u_{11}, u_{12}, \ldots, u_{1l}\}$ and
$S_2 = \{u_{21}, \ldots, u_{2m}\}$ where u_{11} is the vertex nearest to x and u_{1l} is the vertex farthest from x and u_{21} is the vertex nearest to y and u_{2m} is the vertex farthest from y. For $(u_{1i}, u_{2j}) \in S_1 \times S_2$

$$\begin{aligned} d((x,y),(u_{11},u_{21})) &= d(x,u_{11}) + d(y,u_{21}) \\ &\leqslant d(x,u_{1i}) + d(y,u_{2j}) (= d((x,y),(u_{1i},u_{2j}))) \\ &\leqslant d(x,u_{1l}) + d(y,u_{2m}) \\ &= d((x,y),(u_{1l},u_{2m})) \end{aligned}$$

That is, if u_{11} is the vertex nearest to x in $S_1 \subseteq V_1$, u_{21} is the vertex nearest to y in $S_2 \subseteq V_2$, u_{1l} is the vertex farthest from x in $S_1 \subseteq V_1$ and u_{2m} is the vertex farthest from y in $S_2 \subseteq V_2$ then (u_{11}, u_{21}) is the vertex nearest to (x, y) in $S_1 \times S_2$ and $(u_{1,l}, u_{2,m})$ is the vertex farthest from (x, y) in $S_1 \times S_2$

$$\begin{aligned} f_{G_1 \square G_2}((x,y), S_1 \times S_2) &= d((x,y),(u_{1l},u_{2m})) - d((x,y),(u_{11},u_{21})) \\ &= d(x,u_{1l}) + d(y,u_{2m}) - d(x,u_{11}) - d(y,u_{21}) \end{aligned}$$

$$\begin{aligned} &= d(x, u_{1l}) - d(x, u_{11} + d(y, u_{2m} - d(y, u_{21}) \\ &= f_{G_1}(x, S_1) + f_{G_2}(x, S_2) \end{aligned}$$

Now, let $u_1 \in FC(S_1)$ where $S_1 \subseteq V_1$ and let $u_2 \in FC(S_2)$ where $S_2 \subseteq V_2$. That is $f_{G_1}(u_1, S_1) \leqslant f_{G_1}(x, S_1)$, $\forall x \in V_1$ and $f_{G_2}(u_2, S_2) \leqslant f_{G_2}(y, S_2)$, $\forall y \in V_2$. Therefore $f_{G_1}(u_1, S_1) + f_{G_2}(u_2, S_2) \leqslant f_{G_2}(x, S_1) + f_{G_2}(y, S_2)$. So, $f_{G_1 \square G_2}((u_1, u_2), S_1 \times S_2) \leqslant f_{G_1 \square G_2}((x, y), S_1 \times S_2)$, $\forall (x, y) \in V_1 \times V_2$. Hence $(u_1, u_2) \in FC(S_1 \times S_2)$ in $G_1 \square G_2$.
Conversely, assume that $(u_1, u_2) \in FC(S_1 \times S_2)$ in $G_1 \square G_2$ where $S_1 \subseteq V_1$ and $S_2 \subseteq V_2$. That is, $f_{G_1 \square G_2}((u_1, u_2), S_1 \times S_2) \leqslant f_{G_1 \square G_2}((x, y), S_1 \times S_2)$, where $S_1 \subseteq V_1$ and $S_2 \subseteq V_2$, $\forall x \in V_1$, $y \in V_2$. Therefore, $f_{G_1}(u_1, S_1) + f_{G_2}(u_2, S_2) \leqslant f_{G_1}(x, S_1) + f_{G_2}(y, S_2)$, $\forall x \in V_1$ and $y \in V_2$ or $f_{G_1}(u_1, S_1) + fG_2(u_2, S_2) \leqslant f_{G_1}(x, S_1) + f_{G_2}(u_2, S_2)$, $\forall x \in V_1$. That is, $f_{G_1}(u_1, S_1) \leqslant f_{G_1}(x, S_1)$, $\forall x \in V_1$. Hence, $u_1 \in FC(S_1)$ in G_1. Similarly $u_2 \in FC(S_2)$ in G_2. Thus, $FC(S_1 \times S_2) = FC(S_1) \times FC(S_2)$. □

Corollary 6.4.2. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Then the subgraph induced by $FC(S_1 \times S_2)$, where $S_1 \subseteq V_1$ and $S_2 \subseteq V_2$ is connected in $G_1 \square G_2$ if and only if the subgraph induced by $FC(S_1)$ is connected in G_1 and the subgraph induced by $FC(S_2)$is connected in G_2.*

6.5 Conclusion

In this chapter we have initiated a structure based graph theoretical study on equity oriented centers which has been called fair centers. The difference between the maximum and minimum of distances from a given vertex to set of vertices has been chosen as the criteria for its fairness with respect to that set and hence repetition of vertices does not make any difference. Thus the concept of profile of vertices does not have any significance in this criteria of fairness. But we can consider a lot of other criteria for fairness like the sum of the deviations or mean deviation and in this case sets can be generalised to profiles. Fair centers of various classes of graphs have been determined. While identifying the fair sets of odd cycles and symmetric even graphs, methods for finding the set which has a given set as the fair set have been devised. It has been proved that all fair sets of a tree are connected and the result has been generalised for block graphs. Moreover block graphs have been characterised as the class of chordal graphs with connected fair sets. In the thousands of graphs that have been examined using computer programs, block graphs were the only graphs where the subgraphs induced by all fair sets were connected. So we put forward the following conjecture.

Conjecture 2. A graph G is a block graph if and only if the induced subgraph of all of its fair sets are connected.

Chapter 7

Antimedian and weakly Antimedian graphs

7.1 Introduction

The graphs in which every three vertex profile have a unique median is called a median graph. This has significance in minimisation problems. Maximisation problems have also gained importance owing to the growing need for locating undesirable facilities and the maximisation analogue of median graphs has been defined by Kannan et al. in [10]. Antimedian graphs were introduced by them as the graphs in which for every triple of vertices there exists a unique vertex x that maximizes the sum of the distances from x to the vertices of the triple.

For the profile $\pi = (v_1, \ldots v_k)$ and $x \in V$, the set of all vertices x for which $D(x, \pi)$ is maximum is the *Antimedian* of π in G and is denoted by $AM(\pi)$. A graph G is called an *Antimedian Graph* if every triple of G has a unique antimedian. Let $v_1, v_2, \cdots v_n$ be the vertex set of the path on n vertices $P = P_n$ and let $G_i, 2 \leqslant i \leqslant n-1$ be the rooted graphs with roots y_i respectively. Let G be the graph obtained from the disjoint union of P and the graphs G_i, such that for $i = 2, \cdots, n-1$, y_i is identified with v_i. Then G is a *belt*, with *support* P and *ears* G_is. A belt is even, if the support is an even path. If, in addition, the depth of G_i is at most $\lfloor \frac{i-2}{3} \rfloor$ for $i \leqslant \frac{n}{2}$ and at most $\lfloor \frac{n-i-1}{3} \rfloor$ for $i > \frac{n}{2}$, then it is called a *thin even belt.*

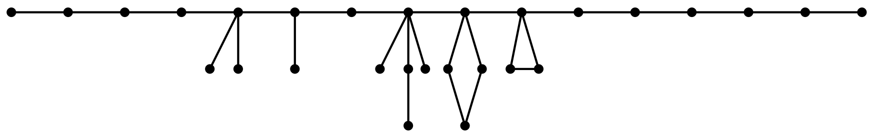


Figure 7.1: A Thin Even Belt

In a graph G, let (u_1, v_1) and (u_2, v_2) be two edges. Then $d((u_1, v_1), (u_2, v_2))$ is defined to be $min\{d(u_1, u_2), d(u_1, v_2), d(v_1, u_2), d(v_1, v_2)\}$.
Here we identify antimedian block graphs, define weakly antimedian graphs, identify weakly antimedian trees and construct a new class of antimedian and weakly antimedian graphs.

7.2 Some Antimedian graphs

Lemma 7.2.1. Let G be an antimedian block graph. Then G contains exactly two diametrical vertices a and b. If P is the shortest (a, b) path then G is a belt with P as support, $d(a, b)$ is odd and for any triple of vertices either a or b is its antimedian.

Proof. Let a and b be a pair of diametrical vertices. Suppose $d(a, b)$ is even. Let y be the middle vertex of P. Since no vertex can be farther away from y than a and b, a and b are the antimedians of the profile (y, y, y), contradicting the fact that G is antimedian. Therefore $d(a, b)$ is odd.
Now assume that G is not a belt with P as support. Then there exist a vertex x in G such that the shortest $(x\text{-}P)$ paths meet P at two adjacent vertices z and z'. Suppose it meet P at a pair of non adjacent vertices. Then we shall get a cycle involving vertices of more than one block a contradiction

to the fact that G is a block graph. Hence all the vertices at which the shortest x-P paths meet P are mutually adjacent or belong to a single block. Also, since P is a shortest a-b path it cannot contain more than two vertices from a single block. Hence the shortest x-P path meets P at at most 2 vertices and if it meets at two points they should be adjacent.

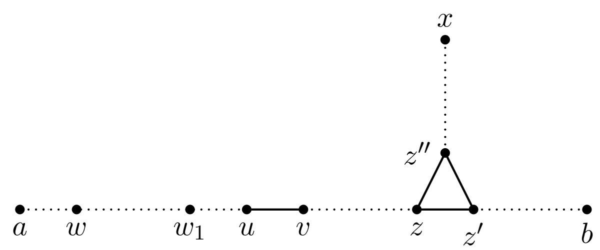


Figure 7.2: x-P path meeting P at a pair of adjacent vertices.

Let uv be the edge of P such that $d(a, u) = d(b, v)$. Assume that $d(zz', uv) \leqslant d(ww', uv)$ for every edge ww' such that $d(x', P) = d(x', w) = d(x', w')$ for some vertex x' in G. Also assume that $d(z, a) \geqslant d(z', b)$. Let w_1 be such that $d(w_1, u) = d(z, v)$.

Now, consider the three vertex profile (w_1, u, z). If $D(a, (w_1, u, z)) = k$ then $D(b, (w_1, u, z)) = k + 1, D(a, (w_1, u, z')) = k + 1$ and $D(b, (w_1, u, z')) = k$. If z'' lies on x-z and x-z' paths such that z'' is adjacent to both z and z' then $D(b, (w_1, u, z'')) = k + 1$ and $D(a, (w_1, u, z'')) = k + 1$.

Now we shall prove that for (w_1, u, z) and (w_1, u, z') either a or b is its antimedian. Assume that $c \in V(G)$ is the antimedian of (w_1, u, z). Let the shortest (c-P) path meet P at c'. Then c' cannot be on the a-w_1 part or b-z part of P since b is the only vertex diametrical to a and vice versa.

Let c' lie on (z, u) part of P. Vertex c is an antimedian of (w_1, u, z) implies c is an antimedian of (a, u, b). Let π be the profile (a, u, b). Assume that $d(c, a)$ is odd and $d(c, b)$ is even.

$$\begin{aligned}d(c,\pi)-d(b,\pi) &= d(c,a)+d(c,u)+d(c,b)-(d(b,a)+d(b,u)+d(b,b))\\ &= d(c,a)-d(b,a)+d(c,u)-d(b,u)+d(c,b)\end{aligned}$$

This being the sum of three even numbers is even. Let w_1' be the vertex adjacent to b on P and π' be the profile got by replacing b in π by w_1'. Then $D(c,\pi')-D(b,\pi')=D(c,\pi)-D(b,\pi)-2$. Repeat this process and if the profile at the i^{th} stage is $\pi^{(i)}$ then $D(c,\pi^{(i)})-D(b,\pi^{(i)})=D(c,\pi^{(i-1)})-D(b,\pi^{(i-1)})-2$. In this process no other pendant vertex y can be the antimedian of any $\pi^{(i)}$ since the sum of the distances to y will start to increase only when all vertices of the profile falls in the a-y path and in this case $D(b,\pi^{(i)})>D(y,\pi^{(i-1)})$ since b is the eccentric vertex of a. Finally we get a profile $\pi^{(k)}$in which $D(c,\pi^{(k)})=D(b,\pi^{(k)})$, that is $\pi^{(k)}$ has two antimedians c and b, a contradiction
Now, let c' lie on the (u,w_1) part of P. As in the previous case we can take the profile $\pi=(a,u,b)$ and continue the same process. Let $a,u_1,u_2,\ldots u_j,u$ be the a-u part of P. If the profile $\pi(k)=(a,u,u)$ is such that $D(c,\pi^{(k)}>D(b,\pi^{(k)})$ then take $\pi^{(k+1)}=(a,u_i,u)$, $\pi^{(k+2)}=(a,u_j,u_j)$, $\pi^{(k+3)}=(a,u_{j+1},u_j)$ etc. Here also $D(c,\pi^{(i)})-D(b,\pi^{(i)})=D(c,\pi^{(i-1)})-D(b,\pi^{(i-1)})-2$ and no other pendant vertex y can be the antimedian of any $\pi^{(i)}$. As in the previous case we get a profile $\pi^{(k)}$ such that $D(c,\pi^{(k)})=D(b,\pi^{(k)})$, that is $\pi^{(k)}$ has two antimedians c and b, a contradiction Therefore (w_1,u,z) has either a or b as its antimedian.
By a similar argument we can prove that (w_1,u,z') has antimedian a or b. Therefore $(w_1,u,z''))$ has also antimedian a or b. But since $D(a,(w_1,u,z))=k$ and $D(b,(w_1,u,z'))=k$ we have $D(a,(w_1,u,z''))=k+1$ and
$D(b,(w_1,u,z''))=k+1$. That is, $(w_1,u,z''))$ has two antimedians a and

b, a contradiction. Hence $z = z'$. In other words G is a belt with P as its support.

Let x be an arbitrary vertex different from a and b. Let x' be a vertex such that $x' \in P$ and $d(x, P) = d(x, x')$. Let $d(x'a) < d(x'b)$. Suppose $d(x, x') > d(a, x')$. Then $d(x, b) > d(a, b)$ which implies a and b are not diametrical. If $d(x, x') = d(a, x')$ then (b, b, b) has two antimedians. Hence $d(x, x') < d(a, x')$. Now it is clear that for every $x, y \in V$ such that at least one of x and y is different from a and b, $d(x, y) < d(a, b)$. That is a and b are the only diametrical vertices. Now, let $\pi = (u, v, w)$ be a triple of G. Suppose $z \neq a, b$ is the antimedian of π. Now clearly z should belong to an end block of G. If z belong to the block B and $B \neq K_2$ then there exists a non cut vertex z' in B. Let V_B denote the set of all non cut vertices of B. Hence $z, z' \in V_B$. Now we shall consider different cases .

Case 1: Let all of u, v and $w \notin V_B$. Then $D(z, \pi) = D(z', \pi)$ contradicting G is antimedian.

Case 2: $u, v \notin V_B$, $w \in V_B$. $D(z, \pi) = d(z, u) + d(z, v) + d(z, w) = d(z, u) + d(z, v) + 1$. Replace w by w' where w' is the cutvertex belonging to B. $D(z, \pi') = d(z, u) + d(z, v) + 1$. $D(w, \pi') = d(w, u) + d(w, v) + 1 = d(z, u) + d(z, v) + 1$. For every x in $V \setminus V_B$, $D(x, \pi') < d(z, u) + d(z, v) + 1$. Then, π' has two antimedians z and w, a contradiction.

Case 3: $u \notin V_B$, $v, w \in V_B$.

Subcase 3.1: $v = w = z$. $D(z, \pi) = 0 + 0 + d(z, u)$. $D(z', \pi) = d(z, u) + 2$. This contradicts the fact that z is antimedian.

Subcase 3.2: $v = z, w \neq z$. $D(z, \pi) = d(z, u) + 1, D(w, \pi) = 1 + d(z, u)$ contradicting G is antimedian.

Subcase 3.3: $v, w \neq z$

$D(z, \pi) = d(z, u) + 2, D(w, \pi) = d(z, u) + 1$. For any other vertex x, $d(x, \pi) < d(z, u) + 2$. Replace w by w' where w' is the cutvertex belonging to B. $D(z, \pi') = d(z, u) + 2$, $d(w, \pi') = d(z, u) + 2$ and for all x, $d(x, \pi) \leqslant d(z, u) + 2$. Therefore π' has two antimedians a contradiction.

Case 4: $u, v, w \in V_B$. Let w' be the cutvertex of B. Then $D(w', \pi) \geqslant D(z, \pi)$ contradicting that z is the unique antimedian of π. Hence $B = K_2$ and z should be a leaf.

Now in the way we proved that (w_1, u, z) has a as its antmedian we prove that for any three vertex profile π either a or b is its antimedian. □

Now we shall give the necessary and sufficient condition for a block graph to be antimedian. Before that we quote two theorems from [10].

Theorem 7.2.1. *Let G be a thin even belt. Then G is antimedian.*

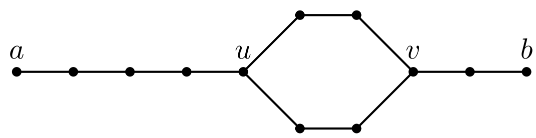
Theorem 7.2.2. *Let T be a tree. Then T is an antimedian graph if and only if it is an thin even belt.*

Theorem 7.2.3. *Let G be a block graph. Then G is an antimedian graph if and only if it is a thin even belt .*

Proof. By Theorem 7.2.1 we know that thin even belts are antimedian. It remains to prove that among block graphs thin even belts are the only antimedian graphs. Let G be an arbitrary antimedian block graph. By Lemma 7.2.1, G has exactly two diametrical vertices u and v and let $P : u = v_1 v_2 \ldots v_r = v$ be the u-v path in G. Let G_i , $1 \leqslant i \leqslant r$, be the maximal subgraph of G that contains v_i and no other vertex of P. We have that G is an even belt with P as support G_i's as the ears. Let d_i be the depth of G_i , $1 \leqslant i \leqslant r$. Suppose that for some $i \leqslant \frac{n}{2}$ the condition $d_i \leqslant \frac{(i-2)}{3}$ is not fulfilled. Hence $3d_i > i - 2$ and let w be a vertex from T_i with $3d(w, v_i) > i - 2$ or $3d(w, v_i) \geqslant i - 1$. Consider the triple $\pi = (u, v, v)$. Clearly $D(v, \pi) < D(u, \pi) = 2(r - 1)$. However $D(w, \pi) = 3d_i + i - 1 + 2(r - i) \geqslant 2r - 2$. We have a contradiction with Lemma 7.2.2 since w is also an antimedian vertex. By symmetry we have an analogue proof for $i > \frac{r}{2}$. □

Theorem 7.2.4. *Let H be a symmetric even graph of diameter ℓ and let u and v be a pair of diametric vertices of H. Let G be the graph obtained by adjoining to H paths of length m and n at u and v respectively. Then G is Antimedian if and only if*

1. *diameter of G is odd*
2. *$m + n > \ell$*
3. *$\frac{m}{3} + 3n > \ell$ and $\frac{n}{3} + 3m > \ell$*



Proof. First we shall assume that $diam(G)$ is odd, $m+n > \ell$, $\frac{m}{3}+3n > \ell$ and $\frac{n}{3} + 3m > \ell$. Let a and b be the diametrical vertices of G with $d(a, u) = m$ and $d(b, v) = n$. Let π be the profile (u_1, u_2, u_3). We shall prove that G is antimedian by showing that π has a unique antimedian. Here we shall take different cases.

Case 1: Each of u_1, u_2 and u_3 either belong to a-u path or b-v path. Then since $d(a, b)$ is odd and a path of odd length is antimedian π has a unique antimedian.

Case 2: u_1, u_2 belong to a-u path and u_3 belong to H. Let x be vertex in

H at distance k from v. Then $d(x, u_1) = d(v, u_1) - k,\ d(x, u_2) = d(v, u_2) - k$ and $d(x, u_3) \leqslant d(v, u_3) + k$. Hence $D(x, \pi) < D(v, \pi)$ for every $x \neq v$ in H. As we move from v to b, $D(\pi)$ increases by three at each step. As we move from u to u_2, $D(\pi)$ decreases by one at each step and as we move from from u_2 to u_1, $D(\pi)$ increases by one at each step and as we mover further $D(\pi)$ increases by three at each step. Therefore we can conclude that $D(\pi)$ attains the maximum at a or b. But since $d(a, b)$ is odd, $D(a, \pi) \neq D(b, \pi)$. Hence π has a unique antimedian.

Case 3: u_1 belong to a-u path, u_2 belong to H and u_3 belong to b-v path. We shall first prove that the antimedian of π is either a or b. If at all a vertex other than a and b is the antimedian of π, then it should be a vertex belonging to H. If that particular vertex is the antimedian of π then it should be the antimedian of (a, u_2, b). Hence without loss of generality we may assume that $\pi = (a, u_2, b)$. Let u_2' be the eccentric vertex of u_2 in H. Let $d(u, u_2') = d$ so that $d(v, u_2') = \ell - d$. Then $D(u_2', \pi) = \ell + d + m + \ell - d + n = 2\ell + m + n$. Let w be a vertex different from u_2' in H. Then $D(w, \pi) = d(w, u) + m + d(w, u_2') + d(w, v) + n < 2\ell + m + n,\ D(a, \pi) = 0 + m + \ell - d + m + \ell + n = 2m + 2\ell - d + n$, and $D(b, \pi) = 0 + n + d + m + \ell + n = m + 2n + \ell + d$. $D(a, \pi) \leqslant D(u_2', \pi)$ and $D(b, \pi) \leqslant D(u_2', \pi)$ implies

$$2m + 2\ell - d + n \leqslant 2\ell + m + n \tag{7.1}$$

$$m + 2n + \ell + d \leqslant 2\ell + m + n \tag{7.2}$$

Adding inequalities 7.1 and 7.2 we get $3m + 3n + 3\ell \leqslant 4\ell + 2m + 2n$ or $m + n \leqslant \ell$, a contradiction. Hence antimedian of π is either a or b. But since $d(a, b)$ is odd antimedian of π is unique.

Case 4: u_1 belong to a-u path, $u_2, u_3 \in H$. Assume that a vertex different from a and b is the antimedian of π. So it should be a vertex of H. Hence in π we may replace u_1 by a. Consider the profile $\pi' = (u, u_2, u_3)$. If w is the median of π' and if the w' is the eccentric vertex of w in H then,

antimedian of π' in H is w'. Therefore among the vertices of H, $D(\pi)$ is maximum for w' or Antimedian of π is w'. Since w' is the eccentric vertex of w, replacing u_2 and u_3 by w in π increases $D(w', \pi)$ by $d(u_2, w)+d(u_3, w)$ and therefore antimedian of (a, w, w) is also w'. Therefore without loss of generality we may assume that $\pi = (a, w, w)$. Let $d(u, w) = d$.
$D(u, \pi) = m + 2d$, $D(a, \pi) = m + d + m + d = 2m + 2d$, $D(w', \pi) = \ell + \ell + \ell - d + m = 3\ell - d + m = m + 2d + 3(\ell - d)$ and $D(b, \pi) = \ell - d + n + \ell - d + n + n + \ell + m = m + 2d + 3n + 3(\ell - d) - d$. $D(a, \pi) \leqslant D(w, \pi)$ and $D(b, \pi) \leqslant D(w, \pi)$ give

$$m \leqslant 3(\ell - d) \text{ or } \frac{m}{3} \leqslant \ell - d \tag{7.3}$$

$$3n + 3(\ell - d) - d \leqslant 3\ell - 3d \text{ or } 3n \leqslant d \tag{7.4}$$

Adding these two inequalities we get $\frac{m}{3} + 3n \leqslant \ell$, a contradiction. Hence $\pi = (u_1, u_2, u_3)$ has antimedian a or b.

$$\begin{aligned} D(a, \pi) &= d(a, u_1) + d(a, u_2) + d(a, u_3) \\ D(b, \pi) &= d(b, u_1) + d(b, u_2) + d(b, u_3) \\ &= 3(m + n + \ell) - (d(a, u_1) + d(a, u_2) + d(a, u_3)) \end{aligned}$$

$$D(a, \pi) = D(b, \pi) \implies 2(d(a, u_1) + d(a, u_2) + d(a, u_3)) = 3(m + n + \ell)$$

.

Therefore $3(m + n + \ell)$ is even or $m + n + \ell$ is even contradicting the fact that $d(a, b)$ is odd. In other words π has a unique antimedian.

Case 5: u_1, u_2 and u_3 belong to H. As in the previous cases initially we prove that a or b is the antimedian of π. Assume the contrary. Then the antimedian should be the antimedian vertex of π in H. Antimedian of π in H is the eccentric vertex of median of π in H. Let w be the median of π and let w' be the eccentric vertex of w in H. Therefore antimedian of π

is w'.
Let $d(u_1, w) = d_1,\ d(u_2, w) = d_2$ and $d(u_3, w) = d_3$.
Then $d(u_1, w') = \ell - d_1,\ d(u_2, w') = \ell - d_2$ and $d(u_3, w') = \ell - d_3$.
Therefore $D(w', \pi) = 3\ell - (d_1 + d_2 + d_3)$.
Let $d(u, u_1) = e_1,\ d(u, u_2) = e_2$ and $d(u, u_3) = e_3$
Then $d(v, u_1) = \ell - e_1,\ d(v, u_2) = \ell - e_2$ and $d(v, u_3) = \ell - e_3$.
Therefore $d(a, u_1) = m+e_1,\ d(a, u_2) = m+e_2,\ d(a, u_3) = m+e_3,\ d(b, u_1) = n + \ell - e_1,\ d(b, u_2) = \ell - e_2$ and $d(b, u_3) = \ell - e_3 + n$.
Hence $D(a, \pi) = 3m + (e_1 + e_2 + e_3)$ and $D(b, \pi) = 3n + 3\ell - (e_1 + e_2 + e_3)$
$D(w', \pi) \geqslant D(a, \pi)$ and $D(w', \pi) \geqslant D(b, \pi)$ gives

$$3\ell - (d_1 + d_2 + d_3) \geqslant 3m + (e_1 + e_2 + e_3) \tag{7.5}$$
$$3\ell - (d_1 + d_2 + d_3) \geqslant 3n + 3l - (e_1 + e_2 + e_3) \tag{7.6}$$

Adding these inequalities we get

$$6\ell - 2(d_1 + d_2 + d_3) \geqslant 3(m + n) + 3\ell$$

Therefore $3\ell \geqslant 3(m + n) + 2(d_1 + d_2 + d_3)$ or $\ell - \frac{2}{3}(d_1 + d_2 + d_3) \geqslant m + n$, a contradiction to the fact that $m + n > \ell$. Therefore antimedian of π is either a or b. Now, assume that $D(a, \pi) = D(b, \pi)$. Then

$$\begin{aligned} 3m + e_1 + e_2 + e_3 &= 3n + \ell - e_1 + \ell - e_2 + \ell - e_3. \\ \text{That is, } 2(e_1 + e_2 + e_3) &= 3n + 3\ell - 3m \end{aligned}$$

Therefore $3(n + \ell - m)$ is even or $n + \ell - m$ is even. This implies $n + \ell + m$ is even contradicting the fact that $d(a, b)$ is odd. Hence π has a unique antimedian.
Thus we have proved that for every three vertex profile π, antimedian of π is unique. In other words, G is an antimedian graph.

Conversely, assume that G is an antimedian graph. We shall prove the following

1. Diameter of G is odd
   Let diameter of G be even and let x be the vertex of G such that $d(x, a) = d(x, b)$. Then the profile (x, x, x) has two antimedians namely, a and b, a contradiction. Hence diameter of G is odd.

2. $m + n > \ell$
   On the contrary, assume that $m + n = \ell - p$ where $p \geqslant 0$. Let u' be a vertex of H such that $d(u', u) = n + p$. Since $m + n + p = \ell$, $n + p \leqslant \ell$ and hence such a u' exists in H. Let v' be the eccentric vertex of u' in H. For each $x \in V(H)$, $d(x, a) + d(x, b) = d(a, b)$ and $d(x, u') < d(v', u')$ for every $x \neq v'$ in H. Therefore $D(x, \pi) < D(v', \pi)$ for every $x \neq v'$ in $V(H)$. Also for every vertex y which is either in a-u path or b-v path, $D(y, \pi) < max(D(a, \pi), D(b, \pi))$. Now, $D(a, \pi) = d(a, b) + m + n + p$, $D(b, \pi) = d(a, b) + n + \ell - n - p = d(a, b) + \ell - p = d(a, b) + m + n$ and $D(v', \pi) = d(a, b) + \ell = d(a, b) + m + n + p$. Hence π has two antimedians a and v', a contradiction. Therefore $m + n > \ell$.

3. $\frac{m}{3} + 3n > \ell$ and $\frac{n}{3} + 3m > \ell$
   Assume that $\frac{m}{3} + 3n \leqslant \ell$. Let $u' \in V(H)$ be such that $d(u', u) = r$ and $d(u', v) = s$ where $r = 3n$. $\frac{m}{3} + 3n \leqslant \ell = r + s = 3n + s$. Therefore $\frac{m}{3} \leqslant s$. Now consider the profile $\pi = (a, u', u')$. Let v' be the eccentric vertex of u' in H. Then $D(v', \pi) = \ell + \ell + s + m$. Now, let $x \in V(H)$ be such that $d(x, v') = k$. Then $d(x, u') = \ell - k$ and $d(x, a) \leqslant d(v', a) + k$. Therefore $D(x, \pi) < D(v', \pi)$ for all $x \neq v'$ in H. Also it is obvious that for any vertex y in the a-u path or b-v path, $D(y, \pi) < max(D(a, \pi), D(b, \pi))$.
   $D(a, \pi) = 0 + m + r + m + r = 2m + 2r$, $D(b, \pi) = m + n + \ell + n +$

$s+n+s = m+3n+2s+\ell = m+r+s+s+\ell = m+2\ell+s$ and $D(v',\pi) = 2\ell+s+m$. Since $m \leqslant 3s$, $2m+2r \leqslant 2\ell+s+m$. Hence π has two antimedians, v' and b, a contradiction. Therefore $\frac{m}{3}+3n > \ell$ and similarly we can prove that $\frac{n}{3}+3m > \ell$.

□

None of the conditions in the above theorem is redundant.
Consider the following graphs.

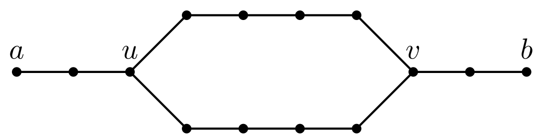


Figure 7.3: H_1

H_1 has $m = 2$, $n = 2$ and $\ell = 5$. Diameter is odd(9), $\frac{m}{3}+3n = \frac{n}{3}+3m = 6.66 > \ell = 5$ but $m+n = 4 < 5 = \ell$ and hence is not antimedian.

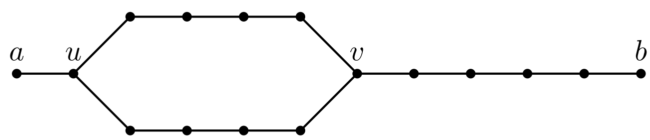


Figure 7.4: H_2

H_2 has $m = 1$, $n = 5$ and $\ell = 5$. Diameter is odd(11), $m+n = 6 > 5 = \ell$, $\frac{m}{3} + 3n = 15.33 > \ell$ but $\frac{n}{3} + 3m = 4.66 < \ell$ and hence is not antimedian.

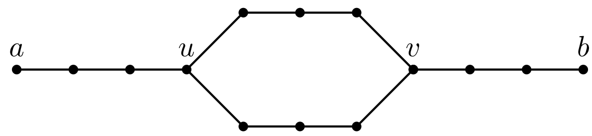
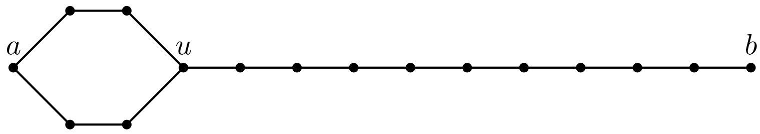


Figure 7.5: H_3

H_3 has $m = 3$, $n = 3$ and $\ell = 4$. $m+n > \ell$, $\frac{m}{3}+3n > \ell$ and $\frac{n}{3}+3m > \ell$, but diameter is even(10) and hence is not antimedian.

Theorem 7.2.5. *Let H be a symmetric even graph of diameter ℓ and let G be a graph obtained by joining a path P of length m to H. Then G is antimedian if and only if*

(1) diameter of G is odd
(2) $m > 3\ell$ or $m = 3\ell - 1$



Proof. Let the path P be joined to H at the vertex u and let b be eccentric vertex of u in H. Let a be the unique pendant vertex of G. That is, a

and b are the diametrical vertices of G. If d(a,b) is even then let u' be the vertex of G such that d(u',a)=d(u',b). Then $\pi = (a, u', b)$ has antimedians a and b. Therefore we assume that $d(a, b)$ is odd.

Case 1: $m < \ell$.

Let u_1 be a vertex at a distance m from b in H and let u'_1 be the eccentric vertex of u_1 in H. Consider the profile $\pi = (a, u_1, b)$.

$$\begin{aligned} D(a, \pi) &= 0 + d(a, u_1) + d(a, b) \\ &= m + \ell - m + \ell + m \\ &= 2\ell + m. \\ D(u'_1, \pi) &= d(u'_1, a) + d(u'_1, u_1) + d(u'_1, b) \\ &= d(a, b) + d(u'_1, u_1) \\ &= \ell + m + \ell \\ &= 2\ell + m \end{aligned}$$

Let $x \in V(H)$. Then

$$\begin{aligned} D(x, \pi) &= d(x, a) + d(x, b) + d(x, u_1) \\ &= d(a, b) + d(x, u_1) \\ &= \ell + m + d_x \text{ where } d_x \leqslant \ell \text{ and } d_x = \ell \text{ only when } x = u'_1. \end{aligned}$$

Therefore $D(x, \pi) \leqslant D(u'_1, \pi)$ for all $x \in V(H)$. Also, it is obvious that for any vertex x in the path P, $D(x, \pi) \leqslant max(D(a, \pi), D(b, \pi)$. Thus for any $x \in V(G)$, $D(x, \pi) \leqslant D(a, \pi) = D(u'_1, \pi)$ or π has two antimedians, namely, a and u'_1. Hence $m < \ell$ is not true.

Case 2: $\ell \leqslant m \leqslant 2\ell$.

Let $m = \ell + t$. Here we consider two subcases.

Subcase 2.1: t is even.

Let u_1 be a vertex such that $d(u_1, b) = \frac{t}{2}$ and let u'_1 be the eccentric vertex of u_1 in H. Let $\pi = (a, u_1, u)$.

$D(a,\pi) = 0 + m + m + \ell - \frac{t}{2} = 2m + \ell - \frac{t}{2}$
$D(b,\pi) = \ell + m + \ell + \frac{t}{2} = m - t + m + \ell + \frac{t}{2} = 2m + \ell - \frac{t}{2}$
For any $x \in H$ such that $d(x,b) = k$, $d(x,u) = d(b,u) - k$,
$d(x,a) = d(b,a) - k$ and $d(x,u_1) \leqslant d(b,u_1) - k$. Hence $D(x,\pi) \leqslant D(b,\pi)$.
Therefore π has two antimedians a and b.
Subcase 2.2: t is odd.
Let u_1 and u_2 be vertices of H such that $d(u_1,b) = \frac{t+1}{2}$ u_2 belong to u-u_1 path and is adjacent to u. Let $\pi = (a,u_1,u_2)$. $d(b,a) = d(u_2',a) + 1$, $d(b,u_2) = d(u_2',u_2) - 1$ and $d(b,u_1) = d(u_2',u_1) - 1$. Hence $D(b,\pi) = D(u_2',\pi) - 1$. Therefore $D(x,\pi) \leqslant D(u_2',\pi)$ for every $x \in V(H)$. In other words $D(x,\pi) \leqslant D(u_2',\pi)$ for every $x \in V(G)$.

$$\begin{aligned}
D(a,\pi) &= 0 + m + 1 + m + \ell - \frac{t+1}{2} \\
&= 2m + 1 + \ell - \frac{t+1}{2} \\
&= 2m + \ell + \frac{1-t}{2} \\
D(u_2',\pi) &= \ell - 1 + m + \frac{t+1}{2} + 1 + \ell \\
&= \ell + m + \frac{t+1}{2} + m - t \\
&= 2m + \ell + \frac{1-t}{2}
\end{aligned}$$

Therefore π has two antimedians, a and u_2', a contradiction. That is,
$\ell \leqslant m \leqslant 2\ell$ is not true.
Case 3: $2\ell \leqslant m \leqslant 3\ell,\ m \neq 3\ell - 1$
Let $m = 3\ell - t$, $t > 1$. Take two subcases
Subcase 3.1: t is even.
Let u_1 be such that $d(u_1,b) = \ell - \frac{t}{2}$. Let $\pi = (a,u_1,u)$. Let u_1 be such

that $d(u_1, b) = \ell - \frac{t}{2}$. Consider the profile $\pi = (a, u_1, u)$.

$$D(a, \pi) = 0 + m + \frac{t}{2} + m = 2m + \frac{t}{2}$$

$$\begin{aligned} D(b, \pi) &= m + \ell + \ell + \ell - \frac{t}{2} \\ &= m + 3\ell - \frac{t}{2} \end{aligned}$$

$$\begin{aligned} &= m + m + t - \frac{t}{2} \\ &= 2m + \frac{t}{2} \end{aligned}$$

Therefore $D(a, \pi) = D(b, \pi)$ As in case 2 $D(x, \pi) \leqslant D(a, \pi) = D(b, \pi)$ for every $x \in V(G)$. Hence π has two antimedians a and b, a contradiction.

Subcase 3.2: t is odd.

Let u_1 be such that $d(u_1, b) = \ell - \frac{t-1}{2}$ and let u_2 be such that u_2 is adjacent to u and u_2 lies on the shortest u-u_1 path. Let $\pi = (a, u_1, u_2)$. Then

$$\begin{aligned} D(a, \pi) &= 0 + m + \frac{t-1}{2} + m + 1 \\ &= 2m + 1 + \frac{t-1}{2} \\ &= 2m + \frac{t+1}{2} \\ D(u_2', \pi) &= \ell - 1 + m + \ell + \ell - \frac{t-1}{2} + 1 \\ &= 3\ell + m - \frac{t-1}{2} \\ &= m + t + m - \frac{t-1}{2} \\ &= 2m + \frac{t+1}{2} \end{aligned}$$

Also it can be seen that $D(x, \pi) \leqslant D(u_2', \pi)$ for every $x \in V(G)$. Hence π has two antimedians a and u_2', a contradiction.
Case 4: $m = 3\ell - 1$ or $m > 3\ell$
Let $\pi = (u_1, u_2, u_3)$. Here we shall consider some subcases.
Subcase 4.1: u_1, u_2 and u_3 belong to a-u path.
Since $d(a, b)$ is odd, π has a unique antimedian.
Subcase 4.2: $u_1, u_2 \in a$-u path and $u_3 \in H$.
Let $x \in H$ be such that $x \neq b$ and $d(x, b) = k$. Then $d(x, u_1) = d(b, u_1) - k$, $d(x, u_2) = d(b, u_2) - k$ and $d(x, u_3) < d(b, u_3) + k$. Therefore $D(x, \pi) < D(b, \pi)$. Similarly for every $y \in (a, u)$ path such that $y \neq a$, $D(y, \pi) < D(a, \pi)$. Hence antimedian of π is a or b. Since $d(a, b)$ is even $D(a, \pi) \neq D(b, \pi)$. Therefore π has a unique antimedian.
Subcase 4.3: $u_1 \in a$-u path and $u_2, u_3 \in H$.
First we shall prove that antimedian of π is a or b. If there exists a vertex different from a and b which is an antimedian of π then it should be a vertex of H. Hence we can assume that $\pi = (a, u_2, u_3)$. Consider the profile (u, u_2, u_3). Then its antimedian in H is the eccentric vertex of its median in H. Let x be the median and let y be the antimedian. Therefore Antimedian of π in G is also y. This implies the antimedian of (a, x, x) is also y. Hence without loss of generality we may assume that $\pi = (a, x, x)$. Let $d(u, x) = d$.

$$D(y, \pi) = \ell + \ell + \ell - d + m = 3\ell - d + m$$
$$D(a, \pi) = m + d + m + d + 0 = 2m + 2d$$

$D(a, \pi) \leqslant D(y, \pi) \implies m \leqslant 3\ell - 3d$. If $d \geqslant 1$ we get $m \leqslant 3\ell - 3$, a contradiction. So the only possibility is $d = 0$ and this implies $y = b$. We shall prove that $D(a, \pi) \neq D(b, \pi)$. On the contrary assume that $D(a, \pi) = D(b, \pi)$.
Let $d(u, u_2) = d_2$ and $d(u, u_3) = d_3$ so that $d(b, u_2) = \ell - d_2$ and $d(b, u_3) =$

$\ell - d_3$. Therefore,

$$\begin{aligned} D(a,\pi) &= d(a,a) + d(a,u_2) + d(a,u_3) \\ &= 0 + m + d_2 + m + d_3 \\ &= 2m + d_2 + d_3 \end{aligned}$$

$$\begin{aligned} D(b,\pi) &= d(b,a) + d(b,u_2) + d(b,u_3) \\ &= m + \ell + \ell - d_2 + \ell - d_3 \\ &= m + 3\ell - d_2 - d_3 \end{aligned}$$

Therefore $D(a,\pi) = D(b,\pi)$ implies $2m + d_2 + d_3 = m + 3\ell - d_2 - d_3$ or $2(d_2 + d_3) = 3\ell - m$. In other words $3\ell - m$ is even. This means that 3ℓ and m are of the same parity or ℓ and m are of the same parity. Hence we get that $\ell + m$ is even, a contradiction to the fact $d(a,b)$ is odd. That is π has either a or b as its antimedian and $D(a,\pi) \neq D(b,\pi)$.

Subcase 4.4: u_1, u_2 and u_3 belong to H.

Here also we first prove that antimedian of π is either a or b. If a vertex other than a or b is an antimedian of π, then it should be a vertex of H, infact, the eccentric vertex of a median of (u_1, u_2, u_3). Let x be the median of (u_1, u_2, u_3) and y be its antimedian. Let $d(u, u_1) = d_1$, $d(u, u_2) = d_2$, $d(u, u_3) = d_3$, $d(x, u_1) = e_1$, $d(x, u_2) = e_2$ and $d(x, u_3) = e_3$. Then $d(y, u_1) = \ell - e_1$, $d(y, u_2) = \ell - e_2$ and $d(y, u_3) = \ell - e_3$ and therefore $D(y,\pi) = 3\ell - (e_1 + e_2 + e_3)$. Also,we have $d(a, u_1) = m + d_1$, $d(a, u_2) = m + d_2$ and $d(a, u_3) = m + d_3$. Hence $D(a,\pi) = 3m + d_1 + d_2 + d_3$.
$D(a,\pi) \leqslant D(y,\pi)$
$\Longrightarrow 3m + d_1 + d_2 + d_3 \leqslant 3\ell - (e_1 + e_2 + e_3)$
$\Longrightarrow 3m \leqslant 3\ell - (e_1 + e_2 + e_3 + d_1 + d_2 + d_3)$
$\Longrightarrow m \leqslant \ell - \frac{1}{3}(e_1 + e_2 + e_3 + d_1 + d_2 + d_3)$, a contradiction to the fact that

$m = 3\ell - 1$ or $m > 3\ell$. Hence $D(a, \pi) > D(y, \pi)$ or a is the antimedian of π.

From these different cases we can conclude that G is antimedian if and only if $d(a, b)$ is even and either $m = 3\ell - 1$ or $m > 3\ell$. □

7.3 Weakly Antimedian Graphs

Balakrishnan et al. concluded [10] by suggesting a study on the class of graphs in which any triple of distinct vertices has a unique antimedian. It is this class of graphs that we consider in this section.

Definition 7.3.1. A Graph G is said to be *Weakly Antimedian* if any triple of distinct vertices has a unique antimedian.

An immediate conclusion is that every antimedian graph is weakly antimedian. The following is an example of a graph that is weakly antimedian, but not antimedian.

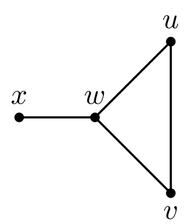


Figure 7.6: Weakly Antimedian graph that is not antimedian

Here each of the distinct triple has a unique antimedian and therefore is weakly antimedian. But consider the profile (w, w, w) .u, v, x are all its antimedian vertices. Hence it is not an antimedian graph.

Proposition 7.3.2. The path P_n is weakly antimedian if and only if n is even.

Proof. Since P_{2n} is antimedian, it is weakly antimedian. Also P_{2n+1} is not weakly antimedian. For, let $P_{2n} = \{x_1, x_2, \cdots x_{n-1}, x_n, x_{n+1}, \cdots x_{2n-1}\}$. Consider the triple $\pi = (x_{n-1}, x_n, x_{n+1})$. $d(x_1, x_{n-1}) = n - 2$, $d(x_1, x_n) = n - 1$, $d(x_1, x_{n+1}) = n$, $d(x_{2n-1}, x_{n-1}) = n$, $d(x_{2n-1}, x_n) = n - 1$ and $d(x_{2n-1}, x_{n+1}) = n - 2$. Therefore $D(x_1, \pi) = D(x_{2n-1}, \pi) = 3n - 3$
Also for all x_i, $i \neq 1, 2n+1$, $D(x_i, \pi) < 3n - 3$. Hence, the triple π has two antimedians or P_{2n+1} is not weakly antimedian. □

Proposition 7.3.3. C_n is weakly antimedian if and only if $n = 4$

Proof. C_4 is weakly antimedian since it is antimedian, see [10]. Let C_n be an odd cycle with vertex set $\{x_1, x_2, \cdots, x_{2r-1}\}$. Now consider the triple of vertices $\pi = (x_1, x_2, x_{2r-1})$. $d(x_1, x_r) = r - 1$, $d(x_2, x_r) = r - 2$ and $d(x_{2r-1}, x_r) = r - 1$. Therefore $D(x_r, \pi) = 3r - 4$. Similarly $D(x_r - 1, \pi) = 3r - 4$. It is obvious that for all other vertices the sum of the distances is less than $3r - 4$. Now let C_n be an even cycle with vertex set $\{x_1, x_2, \cdots, x_{2r}\}$, $r > 2$. Let $\pi = (x_1, x_3, x_{r+2})$. $d(x_1, x_{r+1}) = r$, $d(x_3, x_{r+1}) = r - 2$ and $d(x_{r+2}, x_{r+1}) = 1$. Therefore $D(x_{r+1}, \pi) = 2r - 1$. Similarly $D(x_{r+3}, \pi) = 2r - 1$. Also, for all other vertices sum of the distances is less than $2r - 1$. Hence $D(x, \pi)$ is maximum for x_{r+1} and x_{r+3}. Therefore C_n is not weakly antimedian when $n \neq 4$. □

Proposition 7.3.4. If the Cartesian product of two graphs G and H is weakly antimedian, then both G and H are weakly antimedian.

Proof. Suppose $G \Box H$ is weakly antimedian and G is not weakly antimedian. Then there exists a triple of distinct vertices, say (g_1, g_2, g_3) which has two antimedians. Let $a_1$and a_2 be the antimedians of the triple. Consider the vertex h of H. Let (h, h, h) have b as an antimedian. Then the triple $((g_1, h), (g_2, h), (g_3, h))$ of three distinct vertices of $G \Box H$ has two antimedians (a_1, b) and (a_2, b), a contradiction. □

The converse of the above theorem is not true. Figure 7.7 gives two graphs G and H and the corresponding $G\Box H$.

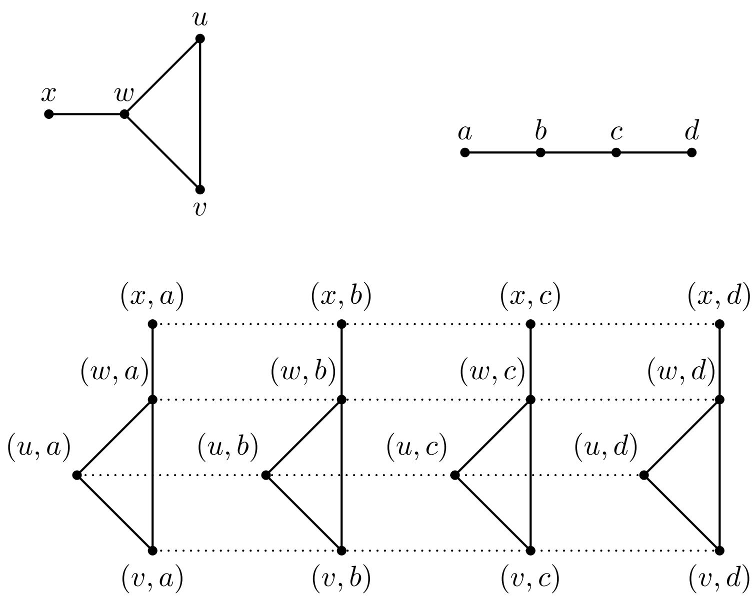


Figure 7.7: G,H and $G\Box H$

Consider the triple $\pi = ((w,a),(w,b),(w,c))$, $D((u,d),\pi) = 9$, $D((v,d),\pi) = 9$, and $D((x,d),\pi) = 9$. For all other vertices it is less than 9. Therefore, the above product graph is not Weakly antimedian even though both G and H are Weakly Antimedian.

Note: Let G be a graph and $u \in V(G)$. Then the multiplication of G with respect to u is the graph obtained from G by replacing u by two adjacent vertices u' and u" and joining them by an edge with all the neighbors of u. Contrary to the case of Antimedian graphs, multiplication of a Weakly antimedian graph with respect to a non antimedian vertex need not give a Weakly antimedian graph. The following serves as an example

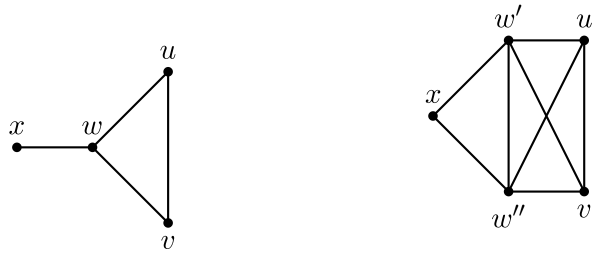


Figure 7.8: G and $G.w$

Here a is a non antimedian vertex. The multiplication of G with respect to a is
The triple (a', a'', d) does not have a unique antimedian.

Lemma 7.3.1. Let T be a weakly antimedian tree. Then T contains exactly 2 diametrical vertices a and b. Moreover $d(a, b)$ is odd and any triple of vertices have either a or b as its antimedian.

Proof. Let a and b be arbitrary diametrical vertices in T and let P be an a-b path in T. Suppose that $d(a, b)$ is even. Then, let y be the middle vertex of P. Let x be the vertex adjacent to y in the a-y path and let z be the vertex adjacent to y in the b-y path. Let $d(a, b) = 2k$, $d(a, y) = k$ and $d(b, y) = k$. Consider the profile $\pi = (x, y, z)$. Then $D(a, \pi) = k + 1 + k + k - 1 = 3k$ and $D(b, \pi) = 3k$. Let u be a vertex such that $u \neq b$ and $z \in (u, y)$ path. Let $d(u, y) = l$. Then $d(u, z) = l - 1$ and $d(u, x) = l + 1$ and therefore $D(u, \pi) = 3l$. $D(u, \pi) > D(b, \pi)$ implies $3l > 3k$ which implies $k > l$. We have $d(a, y) = k$ and $d(y, u) = l$. Therefore $d(a, u) = k + l > 2k$ which contradicts that a and b are diametrical vertices. Therefore if u is a vertex such that $z \neq b$ and $z \in (u, y)$ path then u cannot be the antimedian of $\pi = (x, y, z)$. Similarly if u is a vertex such that $u \neq a$ and $x \in (u, y)$ path, then u cannot be the antimedian of $\pi = (x, y, z)$. Now let u be a vertex such that neither x nor z belong to (u, y) path. Let $d(u, y) = r$ then $d(u, x) = r + 1$ and $d(u, z) = r + 1$ and hence $D(u, \pi) = 3r + 2$. Hence $3r + 2 > 3k$ or $3r > 3k - 2$ or $r > k - \frac{2}{3}$ or $r \geqslant k$. $r > k$ implies $d(a, u) > 2k$, a contradiction to the fact that b is a diametric vertex of a. Assume $r = k$. Then consider the profile (a, a_1, a_2) where a_1 and a_2 are the vertices immediately succeeding a in the path P. Now let a vertex v be such that neither a nor a_2 belong to (v, a_1) path. Then since a is a diametrical vertex of b, v should be adjacent to a_1. Then $D(v, (a, a_1, a_2)) = 5$ and if v has to be an antimedian of (a, a_1, a_2), T should be a star graph where a, a_2 and v are pendant vertices and a_1 is not a pendant vertex. If T does not contain any vertex different from these four then the profile (a, v, a_2) has three antimedians, a, v and a_2. This contradicts the fact that T is weakly antimedian. If T has a vertex different from v, say v', then the profile (a, a_1, a_2) has more than one antimedians say v and v'. Hence we can conclude that if v is the antimedian of (a, a_1, a_2) then v should be such that a_1 and a_2 lies in the a-v path. But for every such v, $D(v, (a, a_1, a_2)) \leqslant D(b, (a, a_1, a_2))$. Now, $D(b, (a, a_1, a_2)) = k + k - 1 + k - 2 = 3k - 3$ and since $r = k$, $D(u, (a, a_1, a_2)) = 3k - 3$. That is, the profile (a, a_1, a_2) has more than one antimedian, u and b, a contradiction. Therefore $r < k$ or $D(u, \pi) \leqslant k - 1 + k + k = 3k - 1$. In other words u cannot be the antimedian of $\pi = (x, y, z)$. Hence π has two antimedians a and b, a contradiction. Therefore $d(a, b)$ is odd.

Let x be an arbitrary vertex different from a and b. Let x' be a vertex such that $x' \in P$ and $d(x, P) = d(x, x')$. Let $d(x'a) < d(x'b)$. Suppose $d(x, x') > d(a, x')$. Then $d(x, b) > d(a, b)$, which implies a and b are not diametrical. If $d(x, x') = d(a, x')$ then (b, b_1, b_2), where b_1 and b_2 are the vertices immediately preceding b in the path, has two antimedians. Hence $d(x, x') < d(a, x')$. Now it is clear that for every $x, y \in V$ such that at least one of x and y is different from a and b, $d(x, y) < d(a, b)$. That is a and b

are the only diametrical vertices.
Now to prove that for any profile of distinct vertices either a or b is its antimedian. On the contrary assume that $z \notin \{a, b\}$ is the antimedian of $\pi = (u, v, w)$. Then z should be a leaf. If deg$z \geqslant 2$ then z has a neighbour x such that $d(x, \pi) > d(z, \pi)$ which is not possible. so z is a leaf of T. Let z' be the first vertex of the z-a path that is on P.
Suppose $u \notin P$ and let u' be the first vertex on the u-a path that is on P. Since $D(a, \pi) < D(z, \pi)$ and since $D(a, \pi)$ is reduced at least as much as $D(z, \pi)$ when we change u to u' we find that $D(a, (u', v, w)) < D(z, (u', v, w))$. So a is also not the antimedian of (u', v, w). Analogously b is also not the antimedian of (u', v, w). Therefore if u', v', w' are the vertices on the u-a, v-a, w-a paths then if neither a nor b is the antimedian of (u, v, w) then neither a nor b is the antimedian of (u', v', w'). Equivalently if for a given profile $\pi = (u, v, w)$, the profile $\pi' = (u', v', w')$ has either a or b as its antimedian then $\pi = (u, v, w)$ also has either a or b as its antimedian. So without loss of generality we may assume that $\pi = (u, v, w)$ where u, v, w belong to the path P.
Now, assume that the antimedian of π is z, a vertex different from a and b and let z meet the path P at the vertex at z'. If u, v and w belong to a-z'(b-z') path then $D(z', \pi) \leqslant D(b, \pi)$ ($D(z', \pi) \leqslant D(a, \pi)$) and therefore z' cannot be the antimedian of π. So assume that u belong to z'-a path and w belong to z'-b path. If z' is the antimedian of π then it is the antimedian of (a, u, b). Therefore we assume that $\pi = (a, u, b)$. Now we shall take two different cases.
Case 1: v belong to a-z' path. As in Lemma 7.2.1 we can see that $D(z', \pi) - D(b, \pi)$ is even and we follow the same procedure followed in Lemma 7.2.1. That is, Let w_1 be the vertex adjacent to b on P and π' be the profile got by replacing b in π by w_1. Then $D(z, \pi') - D(b, \pi') = D(z, \pi) - D(b, \pi) - 2$. Repeat this process and if the profile at the i^{th} stage is $\pi^{(i)}$ then $D(z, \pi^{(i)}) -$

$D(b, \pi^{(i)}) = D(z, \pi^{(i-1)}) - D(b, \pi^{(i-1)}) - 2$. As explained there, in this process no other pendant vertex y can be the antimedian of any $\pi^{(i)}$. Finally we get a profile of distinct vertices, $\pi^{(k)}$, in which $D(z, \pi^{(k)}) = D(b, \pi^{(k)})$. That is $\pi^{(k)}$ has two antimedians z and b, a contradiction.

Case 2: v belong to b-z' path. Here also we follow the same procedure to get $\pi^{(i)}$'s as described in the Case 2 of Lemma 7.2.1. If $\pi^{(k)} = (a, w_\ell, w_k)$ is such that $w_k \neq w_\ell$ and $D(b, \pi^{(k)}) = D(z, \pi^{(k)})$ then we have a profile of distinct vertices having more than one antimedian. Next, assume that $D(z, \pi^{(k)}) = D(b, \pi^{(k)})$ where $\pi^{(k)} = (a, w_k, w_k)$. Then consider the profile $\pi^{(k+1)} = (a, w_{k-1}, w_{k+1})$ where w_{k-1} and w_{k+1} are the vertices adjacent to w_k in the path P. Then $D(z, \pi^{(k+1)}) = D(z, \pi^{(k)})$ and $D(b, \pi^{(k)}) = D(b, \pi^{(k+1)})$. If at all a vertex different from z and b has to become the antimedian for $\pi^{(k+1)}$ it should be a vertex z_1 such that w_k lies both in z_1- w_{k-1} path and z_1- w_{k+1} path. Let $D(b, \pi^{(k)}) = D(z, \pi^{(k)}) = d$. Since $d(z_1, a) < d(b, a)$ and $d(z_1, w_k) < d(b, w_k)$ we get that $D(z_1, \pi^{(k)}) \leqslant d - 3$. Therefore $D(z_1, \pi^{(k+1)}) \leqslant d - 1$. But $D(b, \pi^{(k+1)}) = D(z, \pi^{(k+1)}) = d$. Therefore we get a profile of distinct vertices namely $\pi^{(k+1)}$ with two antimedians. Thus for any profile of vertices of P has either a or b as its antimedian. In other words, any profile of distinct vertices of T has either a or b as its antimedian. □

Theorem 7.3.5. *Let T be a tree.Then T is weakly antimedian if and only if it is a thin even belt.*

Proof. Thin even belts being antimedian are weakly antimedian Now to prove the converse. Let T be an arbitrary weakly antimedian tree. By the above lemma T has exactly two diametrical vertices, say a and b, and let $P : a = v_1 v_2, \ldots v_r = b$ be the u-v path in T. Let T_i, $1 \leqslant i \leqslant r$ be the maximal subtree of T that contains v_i and no other vertex of P. We can consider T_i as a rooted tree with root v_i. Moreover we can consider T as a

belt where P is its support and T_i are its ears. We know that T is an even belt. Let d_i be the depth of T_i, $1 \leqslant i \leqslant r$. Suppose that for some $i \leqslant n/2$ the condition $d_i \leqslant \lfloor (i-2)/3 \rfloor$ is not fulfilled. That is, $d_i > \lfloor (i-2)/3 \rfloor$. Therefore $3d_i > i-2$. Let w be a vertex from T_i with $3d(w, v_i) \geqslant i-1$. Now consider the triple $\pi = (v_1, v_{r-1}, v_r)$. Then $D(v_1, \pi) = 2r-3$, $D(v_r, \pi) = r$ and $D(w, \pi) \geqslant 3\lfloor (i-2)/3 \rfloor + i - 1 + r - i + r - 1 - i$.
Therefore
When $i = 3k$ for some integer k, $D(w, \pi) \geqslant 3k + 2r - i - 2 = 2r - 2$
When $i = 3k+1$, $D(w, \pi) \geqslant 3k + i - 1 + r - i + r - 1 - i = 2r - 3$
When $i = 3k+2$, $D(w, \pi) \geqslant 3(k+1) + i - 1 + r - i + r - 1 - i = 2r - 1$

That is,

$$D(w, \pi) \geqslant 2r - 2 \text{ when } i \equiv 0 (\text{mod} 3)$$
$$D(w, \pi) \geqslant 2r - 3 \text{ when } i \equiv 1 (\text{mod} 3)$$
$$D(w, \pi) \geqslant 2r - 1 \text{ when } i \equiv 2 (\text{mod} 3)$$

This is a contradiction to the above lemma that for any triple of vertices in a weakly antimedian tree v_1 or v_r is their antimedian. Hence the theorem. □

Theorem 7.3.6. *Let G be as given in Theorem 7.2.4. Then G is weakly antimedian if and only if*

1. *diameter of G is odd*
2. $m + n > \ell$
3. $\frac{m}{3} + 3n > \ell - \frac{2}{3}$ *and* $\frac{n}{3} + 3m > \ell - \frac{2}{3}$

Proof. Assume that $diam(G)$ is odd, $m + n > \ell$, $\frac{m}{3} + 3n > \ell - \frac{2}{3}$ and $\frac{n}{3} + 3m > \ell - \frac{2}{3}$. Let a and b be the diametrical vertices of G with $d(a, u) = m$ and $d(a, v) = n$. Let π be the profile (u_1, u_2, u_3) where u_1, u_2 and u_3 are all distinct. We shall prove that G is weakly antimedian by showing that π has a unique antimedian. Here we shall take different cases.

Case 1: Each of u_1, u_2 and u_3 either belong to a-u path or b-v path. Then since $d(a, b)$ is odd and a path of odd length is weakly antimedian π has a unique antimedian.

Case 2: u_1, u_2 belong to a-u path and u_3 belong to H.

As proved in Case 1 of Theorem 7.2.4 we can show that π has a unique antimedian which is either a or b.

Case 3: u_1 belong to a-u path, u_2 belong to H and u_3 belong to b-v path. The proof is the same as the proof of Case 3 of Theorem 7.2.4.

Case 4: $u_1 \in a$-u path and $u_2, u_3 \in H$.

Initially we prove that the antimedian of π is either a or b. So we assume that a vertex different from a and b is the antimedian of π. Hence we can replace u_1 by a in π. That is $\pi = (a, u_2, u_3)$ and the antimedian of π is the eccentric vertex of the median of π in H. Let x be the median of π in H and let y be the eccentric vertex of x in H. That is, antimedian of π is y. By arguments similar to what we used in case 4 of Theorem 7.2.4 we can assume that $\pi = (a, x, x')$ where x' is the vertex adjacent to x in the u_3-x path. Let $d(u, x) = d$ and $d(u, x') = d'$.

$$\begin{aligned} D(a, \pi) &= 0 + d(a, x) + d(a, x') \\ &= m + d + m + d' \\ &= (m + d + d') + m \end{aligned}$$

$$\begin{aligned}
D(b,\pi) &= d(b,a)+d(b,x)+d(b,x') \\
&= m+n+\ell+n+\ell-d+n+\ell-d' \\
&= m+3n+3\ell-(d+d') \\
&= (m+d+d')+3\ell-2(d+d')+3n \\
D(y,\pi) &= d(y,a)+d(y,x)+d(y,x') \\
&= d(y,u)+d(u,a)+d(y,x)+d(y,x') \\
&= d(y,x)-d(x,u)+d(a,u)+d(y,x)+d(y,x') \\
&= \ell-d+m+\ell+\ell-1 \\
&= (m+d+d')+3\ell-2d-d'-1
\end{aligned}$$

$D(a,\pi) \leqslant D(y,\pi) \implies m \leqslant 3\ell-2d-d'-1$ or $\frac{m}{3} \leqslant \ell-\frac{2d+d'+1}{3}$
$D(b,\pi) \leqslant D(y,\pi) \implies 3\ell-2(d+d')+3n \leqslant 3\ell-2d-d'-1$ or $3n \leqslant d'-1$
Adding these inequalities we get $\frac{m}{3}+3n \leqslant \ell+\frac{2d'-2d-4}{3}$. Since $d' \leqslant d+1$, we get , $\frac{m}{3}+3n \leqslant \ell-\frac{2}{3}$, contradiction. Hence π has antimedian a or b. As in case 4 of Theorem 7.2.4 we can show that $D(a,\pi)=D(b,\pi)$ implies $d(a,b)$ is even. Hence π has a unique antimedian.

Case 5: $u_1,\ u_2$ and u_3 belong to H

As proved in case 5 of Theorem 7.2.4 we can prove that π has a unique antimedian which is either a or b. Now we shall prove the converse. That is, assuming that G is weakly antimedian, we shall prove that diameter of G is odd, $m+n>\ell$, $\frac{m}{3}+3n>\ell-\frac{2}{3}$ and $\frac{n}{3}+3m>\ell-\frac{2}{3}$.

Let diameter of G be even. If a and b are the pendant vertices of G, let u' be the vertex such that $d(u'a)=d(u',b)$. Consider the profile $\pi=(a,u',b)$. π has two antimedians a and b, a contradiction.

it is proved that $m+n>\ell$ as in Theorem7.2.4.

Now to prove that $\frac{m}{3}+3n>\ell-\frac{2}{3}$. On the contrary assume that $\frac{m}{3}+3n \leqslant \ell-\frac{2}{3}$. Let u_1 and u_2 be vertices such that u_1 lies on the shortest u-u_2 path, $d(u_1,u)=r_1, d(u_2,u)=r_1+1, d(u_1,v)=k_1$ and $d(u_2,v)=k_1-1$. Assume

$r_1 = 3n$.

$$\begin{aligned}\frac{m}{3} + 3n \leqslant \ell - \frac{2}{3} &\implies \frac{m}{3} + 3n \leqslant r_1 + k_1 - \frac{2}{3} \\ . &\implies \frac{m}{3} \leqslant k_1 - \frac{2}{3} \\ &\implies m \leqslant 3k_1 - 2\end{aligned}$$

$$\begin{aligned}D(a,\pi) &= m + r_1 + m + r_1 + 1 = 2m + r_1 + r_1 + 1 = 2m + 2r_1 + 1 \\ &= m + 2r_1 + m + 1 \\ D(b,\pi) &= m + n + \ell + n + k_1 + n + k_1 - 1 = m + 3n + r_1 + k_1 + 2k_1 - 1 \\ &= m + 2r_1 + 3k_1 - 1 \\ D(u_1', \pi) &= 2\ell + k_1 - 1 + m = m + 2r_1 + 2k_1 + k_1 - 1 = m + 2r_1 + 3k_1 - 1\end{aligned}$$

Therefore $D(b,\pi) = D(u_1',\pi)$. Since u_1 is a median of (u, u_1, u_2), u_1' is an antimedian of (u, u_1, u_2) in H and hence $D(z,\pi) \leqslant D(u_1',\pi)$ for every $z \in V(H)$. Since $m \leqslant 3k_1 - 2$, we have $m + 1 \leqslant 3k_1 - 1$. Therefore $D(a,\pi) \leqslant D(b,\pi) = D(u_1',\pi)$. Thus π has two antimedians u_1' and π, a contradiction. Hence $\frac{m}{3} + 3n > \ell - \frac{2}{3}$. Similarly we can prove that $\frac{n}{3} + 3m > \ell - \frac{2}{3}$. □

As in Theorem 7.2.4 none of the three conditions are redundant.

H_1 is a graph where $\frac{m}{3} + 3n = \frac{n}{3} + 3m > \ell - \frac{2}{3}$ but $m + n < \ell$ and hence is not weakly antimedian.

H_3 has $m + n > \ell$ $\frac{m}{3} + 3n > \ell - \frac{2}{3}$, $\frac{n}{3} + 3m > \ell - \frac{2}{3}$, but diameter is even(10) and hence is not weakly antimedian.

Now consider the following graph

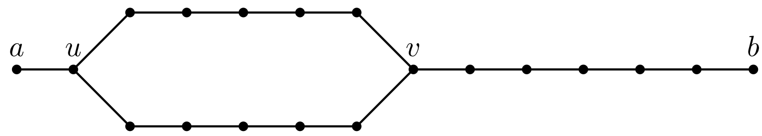


Figure 7.9: H_4

Here $m = 1$, $\ell = 6$ and $n = 6$. $m + n > \ell$, diameter is odd and $\frac{m}{3} + 3n > \ell - \frac{2}{3}$, but, $\frac{n}{3} + 3m = 2 + 3 = 5 < \ell - \frac{2}{3} = 5.33$. Hence H_4 is not weakly antimedian.

Theorem 7.3.7. *Let G be as defined in Theorem 7.2.5. Then G is weakly antimedian if and only if*

(1) diameter of G is odd.
(2) $m > 3\ell$ or $m = 3\ell - 1$ or $m = 3\ell - 3$.

Proof. Let the path P be joined to H at the vertex u and let b be eccentric vertex of u in H. Let a be the unique pendant vertex of G. That is, a and b are the diametrical vertices of G. If $d(a, b)$ is even, then the graph is obviously not weakly antimedian. Hence we assume that $d(a, b)$ is odd. We shall prove the theorem in various cases.

Case 1: $m < 3\ell - 3$

The profiles given in Case 1, Case 2 and Case 3 of Theorem 3 are profiles of distinct vertices which has more than one antimedians. Thus in this case G is not weakly antimedian.

Case 2: $m = 3\ell - 3$

Let $\pi = (u_1, u_2, u_3)$. When u_1, u_2 and u_3 are such that $u_1, u_2, u_3 \in (a, u)$

path or $u_1, u_2 \in a$-u path and $u_3 \in V(H)$ or $u_1, u_2, u_3 \in V(H)$, we can prove that π has a unique antimedian in exactly the same way as we proved Subcases 4.1, 4.2 and 4.4 of Theorem 7.2.5. So we shall assume that $u_1 \in (a, u)$ path and $u_2, u_3 \in V(H)$. First we shall prove that π has antimedian a or b. If a vertex different from a and b is the antimedian of π then it should be a vertex of H, in fact, the eccentric vertex of median of (u_1, u_2, u_3) in H. Let x be the antimedian of (u_1, u_2, u_3) and let y be the eccentric vertex of x. That is, y is an antimedian of π.
Let $d(x, u) = d, d(x, u_1) = d_1, d(x, u_2) = d_2, d(u, u_2) = e_2$ and $d(u, u_3) = e_3$. Then,

$$
\begin{aligned}
D(a, \pi) &= 0 + m + e_2 + m + e_3 = 2m + e_2 + e_3 \\
D(b, \pi) &= \ell - e_2 + \ell - e_3 + \ell + m = 3\ell + m - (e_2 + e_3) \\
D(y, \pi) &= \ell - d_2 + \ell - d_3 + +\ell - d + m = 3\ell + m - (d_2 + d_3 + d)
\end{aligned}
$$

$$
D(a, \pi) \leqslant D(y, \pi) \implies 2m + e_2 + e_3 \leqslant 3\ell + m - (d_2 + d_3 + d) \qquad (7.7)
$$

$$
D(b, \pi) \leqslant D(y, \pi) \implies 3\ell + m - (e_2 + e_3) \leqslant 3\ell + m - (d_2 + d_3 + d) \qquad (7.8)
$$

Adding inequalities 7.7 and 7.8 we get

$$
m \leqslant 3\ell - 2(d_2 + d_3 + d)
$$

$d_2 = d_3 = d = 0$ implies $u_2 = u_3 = u$ and this is not possible since we are considering profiles of distinct vertices. Hence at least one of d_2 and d_3 should be non zero. Let it be d_2. Now $d \neq 0$ implies $m \leqslant 3\ell - 4$, a contradiction. Hence $d = 0$ and this means $y = b$. Hence antimedian of π is either a or b. $D(a, \pi) = D(b, \pi)$ implies $d(a, b)$ is even. Hence π has a unique antimedian.

Case 3: $m = 3\ell - 2$.
In this case $d(a, b) = \ell + 3\ell - 2 = 4\ell - 2$, an even number. Therefore G is not weakly antimedian.
Case 4: $m = 3\ell$
In this case $d(a, b) = \ell + 3\ell = 4\ell$ again an even number. Hence G is not weakly antimedian.
Case 5: $m > 3\ell$ or $m = 3\ell - 1$.
When $m > 3\ell$ or $m = 3\ell - 1$, G is antimedian and hence weakly antimedian. Hence the theorem. □

Remark 7.3.1. Theorems 7.2.4, 7.2.5, 7.3.6 and 7.3.7 give us examples of graphs that are weakly antimedianbut not antimedian.

1. Let G be a graph described in theorem 7.2.4 and 7.3.6 with $n = 1, \ell \geqslant 5$ and $m = 3\ell - 10$. Then $\frac{m}{3} + 3n = \ell - \frac{1}{3}$. That is $\ell - \frac{2}{3} < \frac{m}{3} + 3n < \ell$. Hence G is weakly antimedian but not antimedian.
2. Let G be a graph described in theorem 7.2.5 and 7.3.7. If $m = 3\ell - 3$ then G is weakly antimedian but not antimedian.

7.4 Conclusion

Balakrishnan et.al in [10] characterised thin even belts as the antimedian trees. In this paper we have extended this result to block graphs. We have proved that a block graph is antimedian if and only if it is a thin even belt. We have given a generalisation of antimedian graphs called weakly antimedian graphs and proved that as far as cycles and trees are considered both are the same. We constructed a new class of graphs by attaching paths to a pair of eccentric vertices of a symmetric even graph and found necessary and sufficient conditions for such graphs to be antimedian and weakly antimedian. This also gave us examples of weakly antimedian graphs that are not antimedian.

Chapter 8

Conclusion and future works

This thesis has been devoted to the study of three different measures of centrality-center, median and fair center- and a class of graphs called anti-median graphs. We have found out these three centers of profiles of various classes of graphs like K_n, $K_{m,n}$, $K_n - e$, trees, cycles and a more general class of graphs called symmetric even graphs that includes hypercubes, even cycles, cocktail party graphs, crown graphs etc. While finding the center and fair center of a profile the repetition of vertices in the profile does not make any impact and so in these two cases we have taken sets of vertices instead of profiles. Two new graph parameters called the center number and median number, the number of distinct center sets and median sets of a graph, have been introduced and they have been evaluated for some of the above mentioned graphs. Two new concepts called pacifying edges and shrinking edges have been introduced and they have been identified for paths and symmetric even graphs. These concepts have very high significance in social networking where we can identify the persons to which a particular person should make a link so that his significance in the network increases to a maximum. We have put forward two conjectures, one in chapter 5 regarding the median number of even cycles and the other in chapter 6 pertaining to the characterisation of graphs with connected fair sets. In chapter 3 we proved that for a symmetric even graph the whole vertex set is the only median set which contains a vertex and its eccentric vertex while in chapter 7 it was proved that a vertex and its eccentric vertex appear together in a fair set. We have restricted our study to some particular graph classes and one can look for studying these centrality measures for more classes of graphs. It shall also be interesting to study the relationship among these centrality measures at least for some specific graph classes.

Another area of prospective study is related to multi criteria optimisation, that is, identifying the median which is most central, center of the graph which is most fair and so on.

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