A STUDY ON SEMIGROUP ACTION ON FUZZY SUBSETS, INVERSE FUZZY AUTOMATA AND RELATED TOPICS

Thesis submitted to the
Cochin University of Science and Technology
for the award of the degree of

DOCTOR OF PHILOSOPHY

under the Faculty of Science by

PAMY SEBASTIAN

Department of Mathematics
Cochin University of Science and Technology
Cochin - 682 022

JUNE 2014

Certificate

This is to certify that the thesis entitled '**A STUDY ON SEMIGROUP ACTION ON FUZZY SUBSETS, INVERSE FUZZY AUTOMATA AND RELATED TOPICS**' submitted to the Cochin University of Science and Technology by Ms. Pamy Sebastian for the award of the degree of Doctor of Philosophy under the Faculty of Science is a bonafide record of studies carried out by her under my supervision in the Department of Mathematics, Cochin University of Science and Technology. This report has not been submitted previously for considering the award of any degree, fellowship or similar titles elsewhere.

Dr. T. P. Johnson
(Research Guide)
Associate Professor
School of Engineering
Cochin University of Science and Technology
Kochi - 682 022, Kerala

Cochin-22
16-6-2014.

Declaration

I, Pamy Sebastian, hereby declare that this thesis entitled '**A STUDY ON SEMIGROUP ACTION ON FUZZY SUBSETS, INVERSE FUZZY AUTOMATA AND RELATED TOPICS**' contains no material which had been accepted for any other Degree, Diploma or similar titles in any University or institution and that to the best of my knowledge and belief, it contains no material previously published by any person except where due references are made in the text of the thesis.

Pamy Sebastian
Research Scholar
Registration No. 3652
Department of Mathematics
Cochin University of Science and Technology
Cochin-682 022, Kerala.

Cochin-22
16-06-2014.

To

My Daughters

Acknowledgement

First of all I thank God Almighty for guiding me throughout my life and showering me with His blessings.

I wish to express my sincere gratitude to Dr. T. P. Johnson, my guide and supervisor who encouraged me and gave me constant support, help and freedom through out my research work. With his timely help and inspiring words I could complete my research work in a peaceful atmosphere and in a fruitful manner. I thank him for his trust in me and allowing me to do research under him.

I would like to express my respect and thanks to Prof. T. Thrivikraman for his selfless nature and for the help he has given me in the initial stages of my research work.

My sincere thanks to Dr. P. Rameshan for the valuable discussions and new ideas for my work. Without him my research work would not have been complete.

Dr. Romeo P. G, Head, Dept of Mathematics, CUSAT was a constant support for me from the beginning to the end of my work. He helped me with discussions and in whatever way possible to carry on my research work. I thank him for his valuable suggestions.

Dr. A. Vijayakumar was the Head of the research center when I joined as a full time research scholar in the Dept of Mathematics, CUSAT. His approach and consideration helped me feel at home in this department and I thank him for all the facilities provided for me to start my work.

I sincere thanks to Prof. R. S. Chakravarthy, former Head of the Department and all other faculty members, Prof. M. N. N.Namboothiri, Prof. A. Krishnamoorthy, Dr. B. Lakshmi, Ms. Meena, Mr. Syam Sunder for all the support and help I received. I also thank the office staff and the librarian of the Department of Mathematics for their support and help provided to me.

I would like to thank Dr. Raju George, Principal, Mary Matha Arts & Science College, Mananthavady for the support and care he has always given me . I also thank Mr. Jose Cheriyan, Office Superindent and all my friends and colleagues in the college for the care, concern and help provided to me.

At this time I remember Ms. Deepthi C.P, a friend of everyone and an inspiration to me, who parted from us. I can not think of her without tears in my eyes.

I specially thank Dr.Varghese Jacob for helping me in every way. I thank Ms. Archana, Research Scholar, Dept of Mathematics, Kariyavattom, who helped me wholeheartedly by reading whatever I had written and giving suggestions. I specially thank Mr. Tijo James and Mr. Balesh for having patience to think and clear my doubts. I would like to thank my fellow research scholars Ms. Raji George, Mr. Jayaprasad, Dr. Sreenivasan, Dr. Kirankumar,Mr.Manjunath, Ms. Anu Varghese, Ms. Vineetha, Ms. Reshma, Ms. Chitra, Ms. Dhanya, Ms. Seethu, Mr. Pravas, Ms. Savitha, Ms. Jaya, Ms. Akhila, Ms. Reshmi and others for their love and support.

It will be unfair from my part if I don't express my heart felt gratitude to my daughters Tanya, Tiara and my husband Prasad for the support and help they have given me during my research period. I also thank all my relatives and friends especially my brother Lal, his wife Beena and their son Amith who made my stay in my hometown comfortable. I also thank my friends Sindu and Joseph for thier love and support.

Pamy Sebastian

A STUDY ON SEMIGROUP ACTION ON FUZZY SUBSETS, INVERSE FUZZY AUTOMATA AND RELATED TOPICS

Contents

1 Introduction **1**

1.1 Introduction 1

1.2 Preliminaries 8

1.3 Basics concepts and theorems in fuzzy automata and fuzzy languages 11

1.4 Fuzzy matrices and some basic operations 15

1.5 Summary of the Thesis 16

2 Category of S-Fuzzy subsets **19**

2.1 Introduction 19

2.2 Semigroup action on fuzzy subsets 20

2.3 General properties and special objects in $S - FSET_X$ . . . 25

2.4 Fuzzy semigroup action on fuzzy subsets 37

2.5 Application in fuzzy automata theory 39

2.6 Extending finite state automata to finite fuzzy state automata 40

3 Regular and Inverse Fuzzy Automata 45

3.1 Introduction 45

3.2 Preliminaries 46

3.3 Regular and inverse fuzzy automata 47

3.4 Construction of regular and inverse fuzzy automata 50

3.5 Inverse fuzzy languages 52

3.6 Homomorphic image of inverse fuzzy automata 55

3.7 Cartesian product of two inverse fuzzy automata 59

3.8 Properties of inverse fuzzy languages 63

4 Automorphism Group of an Inverse Fuzzy Automaton 73

4.1 Introduction 73

4.2 Preliminaries 74

4.3 Category $F - AUT$ 75

4.4 Automorphism group of a deterministic faithful inverse fuzzy automaton 78

5 Fuzzy Power Automata 85

5.1 Introduction 85

5.2 Preliminaries 86

5.3 Min-weighted power automata 87

5.4 Max-weighted power automaton 88

5.5 Some algebraic properties of fuzzy power automata 89

Concluding remarks and suggestions for further study 98

Bibliography 100

Chapter 1

Introduction

1.1 Introduction

"As far as laws of mathematics refer to reality they are not certain, and as far as they are certain they do not refer to reality." Albert Einestien

Fuzziness or Vagueness is a common phenomena in almost all situations in real life. Until Lotfi A. Zadeh [33] introduced the concept of fuzzy sets, Mathematics was purely based on set theory. In 1965, Zadeh through a seminal paper introduced a new theory called Fuzzy set theory claiming that many of the uncertainty problems could be solved through this new approach. He suggested a new concept called 'fuzzy sets' which are sets whose boundaries are not precise. While sets can be expressed using two valued logic with membership value 0 or 1, fuzzy set can take any value in the interval $[0, 1]$ as the degree of membership.

Fuzzy sets can be used to express gradual transition from membership to non membership and vice versa. It gives a meaningful representation of vague concepts expressed in natural languages. This is done by assigning to each element in the universal set a value representing its grade of membership in the fuzzy set. This grade corresponds to the degree to which that element is compatible with the concept represented by the fuzzy set. This element belong to the set to a greater or lesser degree as indicated by a larger or smaller membership grade. Then we can consider the concept of a crisp set as a particular case of the more general concept of a fuzzy set in which only two membership values 0 and 1 are allowed.

Fuzzy set theory has greater applications compared to set theory and so many researchers started to reconsider the various concepts and theorems in Mathematics and its applications in the broader frame work of fuzzy settings. Since the basics of Mathematics is set theory, all Mathematics can be rewritten based on fuzzy set theory.

Human thinking and reasoning frequently involve fuzzy information and we can give satisfactory answers. But our systems are unable to answer many questions. The reason is most systems are designed based upon classical set theory and two valued logic which is unable to cope with unreliable and incomplete informations and give expert opinions. Fuzzy sets have been able to provide solution to many real world problems. Zadeh formulated the fuzzy set theory in terms of standard operations such as complement, union, and intersection. Fuzzy set theory is applied in many scientific areas which includes linguistics, robotics, computer science, artificial intelligence, medical diagnosis and social sciences. George J. Klir and Bo Youn [9] and Zimmermann H.J [35] give basics for fuzzy set theory and its applications.

Since the introduction of fuzzy sets as a method for representing uncertainty, this idea has been applied to a wide range of scientific areas. One such area is Fuzzy automata and language theory. Fuzzy automata firstly introduced by W. G. Wee [32] and Lee and Zadeh introduced the concept of fuzzy languages [19]. There is an important reason to study fuzzy automata. Several states are fuzzy by nature as well as several languages. For example, wellness of a patient is a fuzzy state. The language on an alphabet $\{a, b\}$ which contains a large number of $a's$ is an example of a fuzzy language. The basic idea in the formation of fuzzy automata is that unlike classical case, a fuzzy automaton can switch from one state to another to a certain degree. In the case of fuzzy state automata we consider fuzzy subsets of the state set as the the fuzzy states. Analogous to different definition of classical automata there are several definitions of fuzzy automata. Fuzzy automata are the machines accepting fuzzy regular languages and used to define complex systems.

Semigroups are important in many areas of applied mathematics. The theory of finite semigroups has been of particular importance in theoretical computer science since 1950's because of the natural link between finite semigroups and finite automata. Correspond to every finite automata there exist a finite semigroup called transition semigroup of that automata. The word 'automata' comes from a greek word which means 'self acting'. In algebra, an action of a semigroup on a set is a rule which associate to each element of the semigroup a transformation of the set in such a way that the product of two elements of the semigroup is associated with the composite of the two corresponding transformations, which means that the elements of the semigroup are acting as transformations of the set. In computer science, semigroup actions are closely related to automata - the set models the state of automaton, the elements of semigroup are input symbols and the action models the transformations of that state in response to inputs. A set X on which S acts is known as an S-Set. A semigroup with identity is called a monoid. We always consider semigroups with identity. It is also common to work with right acts rather than left acts. Since every right S-act can be interpreted as a left act over the opposite monoid, which has the same element as S but the multiplication is defined by reversing the order, $s * r = r * s$. So the two notions are essentially equivalent.

An S-Set can be considered as a set with a structure. The additional structure on the set is the operation(action) of the semigroup on the set. An S-morphism from one S-Set X to another S-Set X' is a map $F : X \longrightarrow X'$ satisfying $F(\phi(s,x)) = \phi(s, F(x)) \forall s \in S, x \in X$. The set of all such S-homomorphisms is commonly written as $Hom(X, X')$. Thus from the categorical point of view, the set of all S-Sets together with the S-morphisms form a category S-SET and many results were proved by P. G. Romeo on the category of $S - SET$ and Functors [26]. Goguen J. A [10] [11] defined a category $SET(V)$ whose elements are functions $\mu : X \longrightarrow V$ where X is any set and V is any partially orderd set, a morphism from $\mu \longrightarrow \nu$ is a function $f : X \longrightarrow Y$ satisfying $\mu(x) \leqslant \nu(f(x)$ for all $x \in X$. With $V = [0, 1]$, this gives the category of fuzzy sets on a set X say $SET[0, 1]$ or $F - SET$. C. L Walker [31] studied further on $SET[0, 1]$ and Sergey A. Solovyov [27] [28] studied more on the properties of $SET(L)$. They proved that $SET[0, 1]$ and $SET(L)$ are both complete and cocomplete.

Any semigroup action $\delta : S \times X \longrightarrow X$ defines a transformation semigroup $S' = \{\delta_s : s \in S\}$ where $\delta_s(x) = (s, x)$ and every transformation semigroup can be turned into a semigroup action by defining $\delta : S \times X \longrightarrow X$ by $\delta(s, x) = sx$. Transformation semigroups are of essential importance for the structure theory of finite state machines in automata theory. The elements of X acts as states and the semigroup elements acts as input symbols. $\delta : X \times S \longrightarrow X$ is the next state function or transition function.

An automaton is supposed to run on some given sequence of inputs in discrete time steps. At each time step, an automaton gets one input that is picked from a set of symbols or letters which is called alphabet. The set of all finite sequences of letters is called the set of words. An automaton contains a finite set of states. At each time step when the automaton reads a symbol, it jumps or transits to a next state that is decided by the transition function. The automaton reads the symbols of the input word one after another and transit from one state to another according to the transition function until the word is read completely. The automaton starts from an initial state and once the input word has been read completely, it stops at a state called final state. There is a subset of the state set of the automaton called accepting state. If the final state is an accepting state then the automaton accepts the word. Otherwise the word is rejected. The set of all words accepted by an automaton is called the language recognized by that automaton. Thus an automaton is a mathematical object that takes a word as input and decides either to accept it or reject it.

An automaton has got two structures, one is the input structure and the other is the output structure. Output structure is more useful in practical purposes and it depends on the transition structure. But the input structure is independent of the output structure. So it is possible to study the input structure separately. Algebraic automata theory deals with the study of the transition structure or input structure of automata. Many results of semigroup theory is used in algebraic automata theory. One can refer [5], [13] for main results on semigroup theory.

Kleene's theorem [16] is considered to be the starting point of the automata theory. It says that the class of all languages recognized by a finite automata(recognizable languages) coincides with the rational languages, where the rational operations are union, product and star operation. Automata over infinite words are introduced by Buchi in early 1960s. Hopcroft J. E and Ullman J. D [12], Eilenberg [7] Lallement [17], Peter Linz [22], Dexter C. Kozen [6] are good references for Automata and Language theory. A. C. Fleck [8] studied homomorphisms and isomorphisms on automata and Chin-Hong Park [21] studied more on automata homomorphisms and power automata. The definition of syntactic monoid (a monoid canonically attached to each language) first appeared in a paper by Rabin and Scott [25] where the notion is credited to Myhill [20]. It was shown that a language is recognizable if and only if the syntactic monoid is finite.

The notion of variety is introduced by Birkoff [4] for infinite monoids. A Birkoff's variety of monoids is a class of monoids closed under taking submonoids, quotient monoids and direct products. He proved that varieties can be defined by a set of identities. For example, the identity $x * y = y * x$ characterizes the variety of commutative monoids. The collection of finite monoids does not form a variety since it is not closed under direct products. Elienberg defined a pseudovariety as a class of monoids closed under taking submonoids, quotient monoids and finite direct products. Elienberg's variety theorem states that the variety of finite monoids are in one to one correspondence with varieties of languages. This variety theorem has been extended in various directions by J. E Pin and many others [23], [24]. Inverse automata were first discussed by J. B. Stephen[29]. He proved that the transition monoid of an inverse automaton is an inverse monoid. Injective automata or reversible automata first appeared in Christopher Reutenaur's paper. He proved that a language is accepted by an injective automata if and only if the syntactic monoid of that language has commuting idempotents.

Like all other real life problems, impreciseness may occur in the case of machines also. Sometimes the state may not be clear-cut, or the transition from one state to the other may not be complete. There comes the importance of fuzzy automata theory.

Algebraic fuzzy automata theory deals with the study of the transition structure associated with a fuzzy automaton. As in the case of classical automata, corresponds to every fuzzy automaton there exists a finite monoid of fuzzy transition matrices and correspond to every finite monoid we can construct a fuzzy automaton. This one-one correspondence allow us to study the structure of a fuzzy automaton through the study of the structure of the associated transition monoid.

It is proved that every monoid is the syntactic monoid of some fuzzy language while this is not true in the case of crisp languages[17]. Eilenberg-type variety theorem is proved for fuzzy languages by Tatjana Petkovic [30] and it says that there is a one to one correspondence between the variety of finite monoids, variety of languages and the variety of fuzzy languages. Mordeson J. N, Malik D. S independently and together with

Nair P. S, Sen M. K proved many results on algebraic fuzzy automata theory and languages. These and many other references are available in [14].

In this thesis we concentrate on the following.

1. Extending semigroup action on sets to fuzzy framework.
2. Define regular and inverse fuzzy automata and corresponding fuzzy languages and study their algebraic properties.
3. Determine the automorphism group of an inverse fuzzy automaton.
4. Define min-weighted and max-weighted power automata and study the properties of their transition monoids.

1.2 Preliminaries

Definition 1.2.1. Let X be a nonempty set. A fuzzy subset of X is characterized by a function $\mu : X \longrightarrow [0, 1]$.
The set $\{x \in X : \mu(x) > 0\}$ is called the support of μ.

The set of all fuzzy subsets of X is denoted by $F(X)$ or I^X.

Definition 1.2.2. If μ and ν are two fuzzy subsets of X,then $\mu \vee \nu$ is a fuzzy subset of X defined by
$\mu \vee \nu(x) = \max\{\mu(x), \nu(x)\}$
$\mu \wedge \nu(x) = \min\{\mu(x), \nu(x)\}$

$\bar{\mu}(x) = 1 - \mu(x)$ for all $x \in X$.
For $\{\mu_\alpha : \alpha \in A\}$, $(\bigvee_{\alpha \in A} \mu)(x) = \sup_{\alpha \in A} \mu_\alpha(x)$
and
$(\bigwedge_{\alpha \in A} \mu_\alpha)(x) = \inf_{\alpha \in A} \mu_\alpha(x)$ for all $x \in X$

Definition 1.2.3 (Zadeh's Extension Principle)**.** Let $f : X \longrightarrow Y$ be a map where X and Y are two sets. Let μ be a fuzzy subset of X. Then f can be extended to a map $\tilde{f} : I^X \longrightarrow I^Y$ by the extension principle

$$\tilde{f}(\mu)(y) = \begin{cases} \bigvee \mu(x) & \text{where } x \in f^{-1}(y) \text{ if } f^{-1}(y) \neq \phi \\ 0 & \text{otherwise } \ \forall\, y \in Y \end{cases}$$

and if ν is a fuzzy subset of Y, then $\tilde{f}^{-1}(\nu)(x) = \nu(f(x))\ \forall\ x \in X$.
$\tilde{f}(\mu)$ is called image of μ under f and $\tilde{f}^{-1}(\nu)$ is called the pre-image of ν under f.

Definition 1.2.4. Let X and Y be two sets and let $f : X \longrightarrow Y$ be a map. Let μ be a fuzzy subset of X and ν a fuzzy subset of Y. A map $\tilde{f} : \mu \longrightarrow \nu$ is said to be a fuzzy morphism if $\mu(x) \leqslant \nu \circ f(x)\ \forall\ x \in X$.

Definition 1.2.5. A semigroup is a set equipped with an associative binary operation. A monoid is a semigroup with an identity element.

Definition 1.2.6. Let S be a semigroup with identity e. A set X is called an $S - Set$ if there exists a mapping $\phi : S \times X \longrightarrow X$ such that

1. For all $s_1, s_2 \in S$ and $x \in X$, $\phi(s_1 s_2, x) = \phi(s_1, \phi(s_2, x))$.
2. $\phi(e, x) = x$.

The mapping $\phi : S \times X \longrightarrow X$ is called the action of S on X and the

S-Set X is denoted by (X, ϕ).

Definition 1.2.7. Let X and Y be two S-sets. A mapping $f : X \longrightarrow Y$ is called an S-morphism from $X \longrightarrow Y$ if $f(\phi(s, x)) = \phi(s, f(x))$ $\forall s \in S, x \in X$. The collection of all $S - Sets$ together with S-morphisms is a category say $S - SET$.

Definition 1.2.8. A finite state automaton is a five tuple $M = (Q, X, \delta, F, s)$ where Q is finite set whose elements are called states, X is a finite set of input symbols called alphabet, δ is a function from $Q \times X \longrightarrow Q$ called transition function. The output function is $g : Q \times X \longrightarrow [0, 1]$ and $F = \{q \in Q, g(q, x) = 1\}$. The elements of F are called accepting states. $s \in Q$ is the initial state.

We can also represent a finite state automaton as $M = (Q, X, E, F, s)$ where E is a subset of $Q \times X \times Q$ and F and s are as above. Let X^* be the free monoid generated by X where the semigroup operation is concatenation and Λ denote the identity element. The elements of X^* are sequences of finite length of elements of X, called words or strings. Then $\delta : Q \times X \longrightarrow Q$ can be extended to a function $\delta : Q \times X^* \longrightarrow Q$ such that $\delta(q, \Lambda) = q$ and $\delta(q, xa) = \delta(\delta(q, x), a) \forall x \in X^*, a \in X$.

A language L over X is a subset of X^*. A string x is accepted by M if $\delta(s, x) \in F$ and rejected if $\delta(s, x) \notin F$. A language L said to be recognized by an automaton M if L is the set of all strings accepted by M. A language is recognizable if there exists a finite automata recognizing that language. A semigroup S recognizes a language L if there exist a subset P of S and a semigroup morphism $\phi : X^* \longrightarrow S$ such that $L = \phi^{-1}(P)$.

Definition 1.2.9. A deterministic automaton is said to be an inverse automaton if $\forall x \in \tilde{X}^*$, $(q, x, p) \in E \Longrightarrow (p, x^{-1}, q) \in E$ and $(p, x, q) \in E$ and $(p', x, q) \in E \Longrightarrow p = p'$ where $\tilde{X}^*$ is the free semigroup generated by $X \cup X^{-1}$[29].

1.3 Basics concepts and theorems in fuzzy automata and fuzzy languages

A fuzzy language over an alphabet X is a fuzzy subset of X^*. To each fuzzy language λ over X we associate a congruence P_λ called syntactic congruence as follows. For $u, v \in X^*$ $uP_\lambda v$ if and only if $\lambda(xuy) = \lambda(xvy)$ for all $x, y \in X^*$. The quotient monoid $Syn(\lambda) = X^*/P_\lambda$ is called the syntactic monoid of λ [14].

Theorem 1.3.1 (Myhill Nerode theorem)**.** *A fuzzy language λ is regular if and only if P_λ has finite index [30].*

For fuzzy languages $\lambda, \lambda_1, \lambda_2$ over an alphabet X, complement, union and intersection are defined respectively by

$$\overline{\lambda}(u) = 1 - \lambda(u), \lambda_1 \vee \lambda_2(u) = \lambda_1(u) \vee \lambda_2(u), \lambda_1 \wedge \lambda_2(u) = \lambda_1(u) \wedge \lambda_2(u).$$

Left and right quotients are defined respectively by

$$\lambda_1^{-1}\lambda_2(u) = \bigvee_{v \in X^*} \lambda_2(vu) \wedge \lambda_1(v) \ \text{ and } \ \lambda_2\lambda_1^{-1}(u) = \bigvee_{v \in X^*} \lambda_2(uv) \wedge \lambda_1(v).$$

Let $c \in [0,1]$ be arbitrary. Then the fuzzy language $c\lambda$ defined by $c\lambda(u) = c.\lambda(u)$ is called multiplication by constant c. Let X_1 and X_2 be two finite alphabets. $\phi : X_1^* \longrightarrow X_2^*$ be a homomorphism and ψ a fuzzy language in X_2^*. Then the inverse image of ψ under ϕ is a fuzzy language $\phi^{-1}\psi(u) = \psi(\phi(u))$. For a fuzzy language λ, a c-cut we mean $\lambda_c = \{u \in X^* | \lambda(u) \geqslant c\}$.

Theorem 1.3.2. *A fuzzy language λ is regular if and only if $Im(\lambda)$ is finite and λ_c is regular for every $c \in [0,1]$ [14].*

Definition 1.3.1. A family $\mathscr{F} = \mathscr{F}(X)$ of regular fuzzy languages is a variety of fuzzy languages in X^* if it is closed under unions, intersections, complements, multiplication by constants, quotients, inverse homomorphic images and cuts.

For a variety of fuzzy languages $\mathscr{F}$, let $\mathscr{F}^s$ be the family of finite monoids defined by

$$\mathscr{F}^s = \{Syn(\lambda) | \lambda \in \mathscr{F}(X), \text{ for some } X\}.$$

For a variety of finite monoids $\mathscr{S}$, let $\mathscr{S}^f(X)$ be the family of fuzzy languages defined by

$$\mathscr{S}^f(X) = \{\lambda, \text{ a fuzzy language over } X \text{ such that } Syn(\lambda) \in \mathscr{S}\}.$$

Theorem 1.3.3 (Elienberg's variety theorem)**.** *The mapping $\mathscr{F} \longrightarrow \mathscr{F}^s$ and $\mathscr{S} \longrightarrow \mathscr{S}^f$ are mutually inverse lattice isomorphism between the lattices of all varieties of fuzzy languages and all varieties of finite monoids.*

Definition 1.3.2. A fuzzy automaton can also be represented as a five tuple $(Q, X, \{T_u | u \in X\}, i, \tau)$ where $\{T_u | u \in X\}$ is the set of fuzzy transition matrices, $i = \{i_1, i_2, ...i_n\}, i_k \in [0,1], \tau = \{j_1, j_2, ...j_n\}^T, j_k \in [0,1]$ for $k = 1, 2, 3, ..., n$. μ can be extended to the set $Q \times X^* \times Q$ by

$$\mu(q,\Lambda,p)=\begin{cases} 1 & q = p \\ 0 & q \neq p \end{cases}$$

$$\mu(q,u,p)=\bigvee_{q_i \in Q}\{\mu(q,x_1,q_1) \wedge \mu(q_1,x_2,q_2) \wedge \ldots \wedge \mu(q_{k-1},x_k,p) | x_1 \ldots x_k = u\}.$$

The fuzzy language recognized by this fuzzy automaton is $f_M(u) = \bigvee_{q\in Q} \bigvee_{p \in Q} i(q) \wedge \mu(q,u,p) \wedge \tau(p)$ which can also written as $f_M(u) = i \circ T_u \circ \tau$, where the composition is the max-min composition of fuzzy matrices. The minimal fuzzy recognizer $M(\lambda)$ can be constructed in a way similar to the construction of minimal recognizer for a crisp language. The set of states will be $\{\lambda.u | u \in X^*\}$ where $\lambda.u$ is a fuzzy subset of X^* defined by $\lambda.u(w) = \lambda(uw)$ for $w \in X^*$ and $\delta(\lambda.u, x) = \lambda.(ux)$

Definition 1.3.3. A deterministic fuzzy automaton is fuzzy automaton $M = (Q, X, \mu, i, \tau)$ such that there exist a unique $s \in Q$ with $i(s) > 0$ and there exist a unique $q \in Q$ such that $\mu(s, x, q) > 0$ for all $x \in X^*$.

For each fuzzy automaton we can construct a deterministic fuzzy automaton such that the language recognized by them are same [14].

For a fuzzy automaton $A = (Q, X, \mu, i, \tau)$ define a congruence θ_A on X^* by $u\theta_A v \iff \mu(q, u, p) = \mu(q, v, p)\ \forall\ p, q \in Q$. Then the transition monoid $T(A)$ of A is isomorphic to X^*/θ_A. Let $M = (Q, X, \mu, i, \tau)$ be a fuzzy automaton. We say the triple (Q, X, μ) is the fuzzy finite state machine associated with M or a fuzzy automaton without outputs.

For $p, q \in Q$, p is called an immediate successor of q if there exists

$a \in X$ such that $\mu(q, a, p) > 0$. p is called a successor of q if there exist an $x \in X^*$ such that $\mu(q, x, p) > 0$. Let $S(q)$ denote the set of all successors of q. Let $T \subseteq Q$. The set of all successors of T denoted by $S(T) = \cup\{S(q) : q \in T\}$.

$N = (T, X, \nu)$ where $T \subseteq Q$, ν is a fuzzy subset of $T \times X \times T$ is called a submachine of M if $\mu|_{T\times X\times T} = \nu$ and $S(T) \subseteq T$. N is said to be separated if $S(Q - T) \cap T = \phi$.

Definition 1.3.4. A fuzzy automaton M is said to be connected if it has no proper sub machines. M is strongly connected if for every $p, q \in Q, p \in S(q)$. M is commutative if $\mu(p, ab, q) = \mu(p, ba, q)$ $\forall$ $a, b \in X^*, p, q \in Q$.

Definition 1.3.5. Let $M_1 = (Q_1, X_1, \mu_1, i_1, \tau_1)$ and $M_2 = (Q_2, X_2, \mu_2, i_2, \tau_2)$ be two fuzzy automata such that $Q_1 \cap Q_2 = \phi$, recognizing λ_1 and λ_2 respectively. The direct product of M_1 and M_2 is defined as
$M_1 \times M_2 = (Q_1 \times Q_2, X_1 \times X_2, \mu_1 \times \mu_2, i_1 \times i_2, \tau_1 \times \tau_2)$ where
$\mu_1 \times \mu_2((p_1, p_2), (x_1, x_2), (q_1, q_2)) = \mu_1(p_1, x_1, q_1) \wedge \mu_2(p_2, x_2, q_2)$ $\forall (x_1, x_2) \in X_1 \times X_2,$ $(p_1, p_2), (q_1, q_2) \in Q_1 \times Q_2$.
$i_1 \times i_2$ and $\tau_1 \times \tau_2$ are fuzzy subsets of $Q_1 \times Q_2$ defined by
$i_1 \times i_2(p_1, p_2) = i_1(p_1) \wedge i_2(p_2), \tau_1 \times \tau_2(q_1, q_2) = \tau_1(q_1) \wedge \tau_2(q_2)$.
For $M_1 = (Q_1, X, \mu_1, i_1, \tau_1)$ and $M_2 = (Q_2, X, \mu_2, i_2, \tau_2)$ the restricted direct product $M_1 \dot{\times} M_2 = (Q_1 \times Q_2, X, \mu_1 \dot{\times} \mu_2, i_1 \times i_2, \tau_1 \times \tau_2)$ where
$\mu_1 \times \mu_2((p_1, p_2), x, (q_1, q_2)) = \mu_1(p_1, x, q_1) \wedge \mu_2(p_2, x, q_2)$ $\forall x \in X,$ $(p_1, p_2),$ $(q_1, q_2) \in Q_1 \times Q_2$. and $i_1 \times i_2$ and $\tau_1 \times \tau_2$ are fuzzy subsets of $Q_1 \times Q_2$ defined by $i_1 \times i_2(p_1, p_2) = i_1(p_1) \wedge i_2(p_2),$ $\tau_1 \times \tau_2(q_1, q_2) = \tau_1(q_1) \wedge \tau_2(q_2)$. The language recognized by $M_1 \dot{\times} M_2$ is $\lambda_1 \wedge \lambda_2$ [14].

Definition 1.3.6. Let $M_1 = (Q_1, X, \mu_1), M_2 = (Q_2, X, \mu_2)$ be fuzzy finite state machines such that $Q_1 \cap Q_2 = \phi$. Their join is defined as $M_1 \vee M_2 = (Q_1 \cup Q_2, X, \mu_1 \vee \mu_2, i_1 \vee i_2, \tau_1 \vee \tau_2)$ where

$$\mu_1 \vee \mu_2(p, x, q) = \begin{cases} \mu_1(p, x, q) & \text{if } p, q \in Q_1 \\ \mu_2(p, x, q) & \text{if } p, q \in Q_2 \\ 0 & \text{otherwise} \end{cases}$$

$$i_1 \vee i_2(p) = \begin{cases} i_1(p) & \text{if } p \in Q_1 \\ i_2(p) & \text{if } p \in Q_2 \end{cases}$$

and

$$\tau_1 \vee \tau_2(q) = \begin{cases} \tau_1(q) & \text{if } q \in Q_1 \\ \tau_2(q) & \text{if } q \in Q_2 \end{cases}$$

1.4 Fuzzy matrices and some basic operations

Fuzzy matrices are matrices with entries from the unit interval $[0, 1]$. We can represent a fuzzy automaton with transition matrices which are fuzzy matrices, and their composition is $max - min$ operations on fuzzy matrices. There are some other elementary operations on the set of all fuzzy matrices.

Definition 1.4.1. If $A = [a_{ij}], B = [b_{ij}]$ are two fuzzy matrices with same number of rows and columns then $max(A, B) = [c_{ij}]$ where $c_{ij} = max\{a_{ij}, b_{ij}\}$. Similarly $min(A, B) = [c_{ij}]$ where $c_{ij} = min\{a_{ij}, b_{ij}\}$.

Definition 1.4.2. Let A and B be two matrices which are compatible as in the case of product, ie, number of columns of A is equal to the number of rows of B. Then $max - min(A, B) = [c_{ij}]$ where $c_{ij} = \bigvee_k a_{ik} \wedge b_{kj}$.

Similarly $min - max(A, B) = [c_{ij}], c_{ij} = \bigwedge_k a_{ik} \vee b_{kj}$.

The collection of all square fuzzy matrices of order n with $max - min$ operations is a monoid whose identity element is the unit matrix of order n.

1.5 Summary of the Thesis

This thesis comprises five chapters including the introductory chapter. This includes a brief introduction and basic definitions of fuzzy set theory and its applications, semigroup action on sets, finite semigroup theory, its application in automata theory along with references which are used in this thesis.

In the second chapter we defined an S-fuzzy subset of X with the extension of the notion of semigroup action of S on X to semigroup action of S on to a fuzzy subset of X using Zadeh's maximal extension principal and proved some results based on this. We also defined an S-fuzzy morphism between two S-fuzzy subsets of X and they together form a category $S - FSET_X$. Some general properties and special objects in this category are studied and finally proved that $S - SET$ and $S - FSET$ are categorically equivalent. Further we tried to generalize this concept to the action of a fuzzy semigroup on fuzzy subsets. As an application, using the above idea, we convert a finite state automaton to a finite fuzzy state automaton. A classical automata determine whether a word is accepted by the automaton where as a finite fuzzy state automaton determine the degree of acceptance of the word by the automaton.

In the third chapter we define regular and inverse fuzzy automata, its construction, and prove that the corresponding transition monoids are regular and inverse monoids respectively. The languages accepted by an inverse fuzzy automata is an inverse fuzzy language and we give a characterization of an inverse fuzzy language. We study some of its algebraic properties and prove that the collection IFL on an alphabet does not form a variety since it is not closed under inverse homomorphic images. We also prove some results based on the fact that a semigroup is inverse if and only if idempotents commute and every $\mathscr{L}$-class or $\mathscr{R}$-class contains a unique idempotent.

Fourth chapter includes a study of the structure of the automorphism group of a deterministic faithful inverse fuzzy automaton and prove that it is equal to a subgroup of the inverse monoid of all one-one partial fuzzy transformations on the state set.

In the fifth chapter we define min-weighted and max-weighted power automata, study some of its algebraic properties and prove that a fuzzy automaton and the fuzzy power automata associated with it have the same transition monoids.

The thesis ends with a conclusion of the work done and the scope of further study.

Chapter 2

Category of S-Fuzzy subsets

2.1 Introduction

Categories and functors were first introduced by Samuel Eilenberg and Saunders Mac Lane in 1945. Later by 1970s this concept is more developed and found application in many different areas of Mathematics. Many Mathematical results were proved in a much simpler way using categorical concepts. From 1980, Category theory occupies a central position in the field of theoretical computer science, theoretical physics and many other fields where Mathematics is applied. It is a powerful language which allows us to view various classes of objects with structures and their structure preserving relation in a more general frame work.

Some results of this chapter are included in the following paper.
Pamy Sebastian, T. P. Johnson. : Semigroup Action on Fuzzy subsets and the Category of S-Fuzzy subsets. International Review of Fuzzy Mathematics (2012), Vol.7, No.1, 27-34.

Many categorical properties of $G - Sets$ and $S - Sets$ were studied by various researchers [34] [26]. In this chapter we extend the notion of $S - Sets$ to fuzzy framework where objects are S-fuzzy subsets of an $S - Set$ X and morphisms are S-fuzzy morphisms between them. We study some categorical properties of the Category $S - FSET_X$ and prove that it is complete and cocomplete. Considering the Category $S - SET$ and $S - FSET$, we define a covariant functor between them and prove that these two categories are equivalent. Finally we give an application of this result in fuzzy automata theory.

2.2 Semigroup action on fuzzy subsets

Definition 2.2.1. Let X be an S-set where S is a semigroup with identity e and the action of S on X is defined by the function $\phi : S \times X \longrightarrow X$. A fuzzy subset μ of X is said to be an S-fuzzy subset of X if $\mu(\phi(s,x)) \geqslant \mu(x)\ \forall\ x \in X$. The semigroup action of S on X can be extended to I^X as

$$\text{for } x \in X,\ \phi(s,\mu)(x) = \begin{cases} \bigvee\limits_{y} \mu(y) \text{ where } y \in X : \phi(s,y) = x\} \\ 0 \text{ if no such y exists} \end{cases}$$

It is trivial that $\phi(e,\mu) = \mu$.

$$\begin{aligned} \phi(s_1, \phi(s_2,\mu))(x) &= \begin{cases} \bigvee\limits_{y} \{\phi(s_2,\mu)(y), y \in X, \phi(s_1,y) = x\} \\ 0 \text{ if no such y exists} \end{cases} \\ &= \begin{cases} \bigvee\limits_{y}\{\bigvee\limits_{z} \mu(z) : z \in X : \phi(s_2,z) = y\}, y \in X : \phi(s_1,y) = x\} \\ 0 \text{ if no such y exists} \end{cases} \end{aligned}$$

$$= \begin{cases} \bigvee_z \{\mu(z) : z \in X, \phi(s_1, \phi(s_2, z)) = x\} \\ 0 \text{ if no such y exists} \end{cases}$$

$$= \begin{cases} \bigvee_z \{\mu(z), z \in X : \phi(s_1 s_2, z) = x\} \\ 0 \text{ if no such y exists} \end{cases}$$

$$= \phi(s_1 s_2, \mu)(x)$$

This implies that $\phi(s_1, \phi(s_2, \mu)) = \phi(s_1 s_2, \mu)$.
μ is said to be an S-Fuzzy subset under this action.
Let $A \subseteq S$.
Define $\phi(A, \mu) = \bigcup\limits_{s \in A} \phi(s, \mu)$.
Then

$$\begin{aligned} \phi(A, \mu)(x) &= (\bigcup_{s \in A} \phi(s, \mu))(x) \\ &= \bigvee_{s \in A} \{\phi(s, \mu)(x)\} \end{aligned}$$

Property 1. If μ and ν are S-fuzzy subsets of an $S - Set\ X$, then

1. If $\mu \subseteq \nu$ then $\phi(s, \mu) \subseteq \phi(s, \nu)$.

2. $\phi(s, (\mu \cup \nu)) = \phi(s, \mu) \cup \phi(s, \nu)$.

3. $\phi(s, (\mu \cap \nu)) \subseteq \phi(s, \mu) \cap \phi(s, \nu)$.

4. $\phi(s, \phi(A, \mu)) = \phi(sA, \mu)$.

5. $\phi(A, \phi(B, \mu)) = \phi(AB, \mu)$.

Proof. Since $\mu \subseteq \nu$, $\mu(x) \leqslant \nu(x)\ \forall x \in X$.

$$\begin{aligned}
\phi(s,\mu)(x) &= \begin{cases} \bigvee_y \{\mu(y), y \in X : \phi(s,y) = x\} \\ 0 \text{ if no such y exists} \end{cases} \\
&\leqslant \begin{cases} \bigvee_y \{\nu(y), y \in X : \phi(s,y) = x\} \\ 0 \text{ if no such y exists} \end{cases} \\
&= \phi(s,\nu(x)) \;\; \forall x \in X
\end{aligned}$$

So $\phi(s,\mu) \subseteq \phi(s,\nu)$.

To prove (2),

$\phi(s,\mu \cup \nu)(x)$

$$\begin{aligned}
&= \begin{cases} \bigvee_y \{\mu \cup \nu(y), y \in X : \phi(s,y) = x\} \\ 0 \text{ if no such y exists} \end{cases} \\
&= \begin{cases} \bigvee_y \{\mu(y) \vee \nu(y), y \in X : \phi(s,y) = x\} \\ 0 \text{ if no such y exists} \end{cases} \\
&= \begin{cases} \bigvee_y \{\mu(y), y \in X : \phi(s,y) = x\} \vee \bigvee_y \{\nu(y), y \in X : \phi(s,y) = x\} \\ 0 \text{ if no such y exists} \end{cases} \\
&= \phi(s,\mu)(x) \vee \phi(s,\nu)(x) \\
&= (\phi(s,\mu) \cup \phi(s,\nu))(x)
\end{aligned}$$

So $\phi(s, \mu \cup \nu) = \phi(s, \mu) \cup \phi(s, \nu)$.

Now to prove (3),

$$\begin{aligned}
\phi(s, \mu \cap \nu)(x) &= \begin{cases} \bigvee\limits_{y}\{\mu \cap \nu(y), y \in X : \phi(s, y) = x\} \\ 0 \text{ if no such y exists} \end{cases} \\
&= \begin{cases} \bigvee\limits_{y}\{\mu(y) \wedge \nu(y), y \in X : \phi(s, y) = x\} \\ 0 \text{ if no such y exists} \end{cases} \\
&\leqslant \begin{cases} \bigvee\limits_{y}\{\mu(y), y \in X : \phi(s, y) = x\} \wedge \bigvee\limits_{y}\{\nu(y), y \in X : \phi(s, y) = x\} \\ 0 \text{ if no such y exists} \end{cases} \\
&= \phi(s, \mu)(x) \wedge \phi(s, \nu)(x) \\
&= \phi(s, \mu) \cap \phi(s, \mu)(x)
\end{aligned}$$

So $\phi(s, \mu \cap \nu) \subseteq \phi(s, \mu) \cap \phi(s, \nu)$.

To get (4),

$$\begin{aligned}
\phi(s, \phi(A, \mu))(x) &= \begin{cases} \bigvee\limits_{y}\{\phi(A, \mu)(y), y \in X : \phi(s, y) = x\} \\ 0 \text{ if no such y exists} \end{cases} \\
&= \begin{cases} \bigvee\limits_{y} \bigvee\limits_{u \in A}\{\phi(u, \mu)(y), y \in X : \phi(s, y) = x\} \\ 0 \text{ if no such y exists} \end{cases}
\end{aligned}$$

$$
\begin{aligned}
&= \bigvee_{u\in A} \phi(s,\phi(u,\mu))(x) \\
&= \bigvee_{u\in A} \phi(su,\mu)(x) \\
&= \bigcup_{u\in A} \phi(su,\mu)(x) \\
&= \phi(sA,\mu)(x).
\end{aligned}
$$

Now (5) follows from

$$
\begin{aligned}
\phi(A,\phi(B,\mu)) &= \bigcup_{x\in A} \phi(x,\phi(B,\mu)) \\
&= \bigcup_{x\in A} \phi(xB,\mu) \\
&= \phi(AB,\mu)
\end{aligned}
$$

□

Definition 2.2.2. Let X be an $S-Set$ and f be an S-morphism on X, let μ and ν be S-fuzzy subsets of X. A fuzzy morphism $\tilde{f}:\mu \longrightarrow \nu$ is said to be an S-fuzzy morphism if $\tilde{f}(\phi(s,\mu)) = \phi(s,\tilde{f}(\mu))\ \forall s \in S$.

The collection of all S-fuzzy subsets of X together with S-fuzzy morphisms between them is a category say $S-FSET_X$. Let $S-FSET_X$ and $S-FSET_Y$ be two categories of S-fuzzy subsets of X and Y respectively where $X,Y \in S-SET$

Let $f: X \to Y$ be an S-morphism, then f can be extended to S-fuzzy morphism $\tilde{f}: S-FSET_X \to S-FSET_Y$ by

$$\tilde{f}(\mu)(y) = \begin{cases} \bigvee_{x} \mu(x) & \text{where } x \in f^{-1}(y) \text{ if } f^{-1}(y) \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

The category $S-FSET$ can be defined as follows.
The objects of $S-FSET$ are the classes $S-FSET_X$ where $X \in S-SET$ and morphisms are $\tilde{f} : S-FSET_X \to S-FSET_Y$ defined as above.

2.3 General properties and special objects in $S-FSET_X$

In the category $S-FSET_X$, we consider $\mu : X \to [0,1]$ as an additional structure on the S-set X and the S-fuzzy morphisms are maps $\tilde{f} \in Hom(I^X, I^X)$ satisfying $f(\phi(s,\mu)) = \phi(s, f(\mu))$ and $\mu(x) \leqslant \nu \circ f(x)\ \forall\ x \in X, s \in S$. If $f \in Hom(X,X)$ is a bijection then $\tilde{f}$ is also a bijection. So $S-FSET_X$ can be considered as a set with structures whose objects are (X,μ) and morphisms are S-fuzzy morphisms. The composition is the ordinary composition of functions. The initial object in $S-FSET_X$ is the empty set with empty map to [0,1] and final objects are singleton S-fuzzy subsets which are fuzzy subsets on X such that for every $\{x\}$, $\mu(y) = 1$ at $y = x$ and 0 for all $y \neq x$.

Products in $S-FSET_X$

Let X and Y be two $S-Sets$. Then the product $X \times Y$ is defined as an $S-Set$ with the semigroup action on $X \times Y$ is defined as $\phi(s,(x,y)) = (\phi(s,x), \phi(s,y))$ together with the projection morphisms $p_1 : X \times Y \to X$

and $p_2 : X \times Y \to Y$. Here p_1 and p_2 are S-morphisms since $p_1(\phi(s,(x,y))) = p_1(\phi(s,x), \phi(s,y)) = \phi(s,x) = \phi(s, p_1(x,y))$ and $p_2(\phi(s,(x,y))) = p_2(\phi(s,x), \phi(s,y)) = \phi(s,y) = \phi(s, p_2(x,y))$.
In particular the product $X \times X$ is an $S - Set$.

Let μ and ν be S-fuzzy subset of X. The product $\mu \times \nu$ can be defined as a fuzzy subset θ of $X \times X$ defined as $\theta(x,y) = \mu(x) \wedge \nu(y)$ together with $\tilde{p_1} : I^{X\times X} \to I^X$ and $\tilde{p_2} : I^{X\times X} \to I^X$ defined by

$$\begin{aligned}
\tilde{p_1}(\theta)(a) &= \bigvee_{(x,y)} \{\theta(x,y) : p_1(x,y) = a\} \\
&= \bigvee_{(x,y)} \{\mu(x) \wedge \nu(y) : p_1(x,y) = a\} \\
&= \bigvee_{y} (\mu(a) \wedge \nu(y)) \\
&= \mu(a) \wedge \bigvee_{y} \nu(y) \text{ for every } a \in X
\end{aligned}$$

Similarly,

$$\tilde{p_2}(\theta)(b) = (\bigvee_{x\in X} \mu(x)) \wedge \ \nu(b).$$

Since $\theta(\phi(s,(x,y))) = \theta(\phi(s,x), \phi(s,y)) = \mu(\phi(s,x)) \wedge \nu(\phi(s,y)) \geqslant \mu(x) \wedge \nu(y) = \theta(x,y)$, θ is an S-fuzzy subset of $X \times X$ where the action of S on θ is defined as

$$\phi(s,\theta)(a,b) = \begin{cases} \bigvee_{(x,y)} \{\theta(x,y) : \phi(s,(x,y)) = (a,b)\} \\ 0 \text{ if no such } (x,y) \text{ exists} \end{cases}$$

$$= \begin{cases} \bigvee\limits_{(x,y)} \{\mu(x) \wedge \nu(y) : \phi(s,x) = a, \phi(s,y) = b\} \\ 0 \text{ if no such } (x,y) \text{ exists} \end{cases}$$

$$= \begin{cases} \bigvee\limits_{x} \{\mu(x), x \in X : \phi(s,x) = a\} \wedge \bigvee\limits_{y} \{\nu(y), y \in X : \phi(s,y) = b\} \\ 0 \text{ if no such } (x,y) \text{ exists} \end{cases}$$

$$= \phi(s,\mu)(a) \wedge \phi(s,\nu)(b)$$

$$\Longrightarrow \phi(s,(\mu \times \nu))(a,b) = \phi(s,\mu)(a) \wedge \phi(s,\nu)(b)$$

Also $\tilde{p}_1$ and $\tilde{p}_2$ are S-fuzzy morphisms, for,
$\theta(x,y) = \mu(x) \wedge \nu(y) \leqslant \mu(x) = \mu \circ p_1(x,y)\ \forall (x,y) \in X \times X$
$\Longrightarrow \tilde{p}_1$ is a fuzzy morphism.

$$\begin{aligned} \tilde{p}_1(\phi(s,\theta))(a) &= \bigvee_{(x,y)} \{\phi(s,\theta)(x,y) : p_1(x,y) = a\} \\ &= \bigvee_{y} \phi(s,\theta)(a,y) \\ &= \bigvee_{y} \bigvee_{(u,v)} \{\theta(u,v) : \phi(s,(u,v)) = (a,y), p_1(u,v) = a\} \\ &= \bigvee_{y} \bigvee_{(u,v)} (\mu(u) \wedge \nu(v) : \phi(s,u) = a, \phi(s,v) = y, p_1(u,v) = a\} \\ &= \bigvee_{y} \bigvee_{(u,v)} \{\mu(u) \wedge \nu(v) : p_1(u,v) = a, \phi(s,v) = y\} \\ &= \phi(s,\tilde{p}_1(\theta))(a) \end{aligned}$$

$\Longrightarrow \tilde{p_1}$ is an S-fuzzy morphism.
Similarly $\tilde{p}_2$ is also an S-fuzzy morphism.
The universality of the product will be followed by the universal mapping property(UMP) of the product in the category SET.
Thus $(\theta, \tilde{p}_1, \tilde{p}_2)$ is the product of μ and ν.

Example 2.3.1. Let $X = \{x_0, x_1, x_2\}$ and S be the subsemigroup $\{e, \delta\}$ of the full transformation semigroup on X where $\delta(x_0) = x_1, \delta(x_1) = x_1,\ \delta(x_2) = x_2$. Then S can be considered to be acting on X and the action ϕ of S on X is $\phi(\delta, x_0) = x_1,\ \phi(\delta, x_1) = x_1,\ \phi(\delta, x_2) = x_2$ and $\phi(e, x) = x\ \forall x \in X$.
Let μ and ν be fuzzy subsets on X defined by
$$\mu(x) = \begin{cases} 0.5 & \text{when } x = x_0 \\ 0.7 & \text{when } x = x_1, x_2 \end{cases}$$
and
$$\nu(x) = \begin{cases} 0.4 & \text{when } x = x_0 \\ 0.7 & \text{when } x = x_1 \\ 0.8 & \text{when } x = x_2 \end{cases}.$$
Then μ and ν are S-fuzzy subsets of X since $\mu(\phi(s, x)) \geqslant \mu(x)$ and $\nu(\phi(s, x) \geqslant \nu(x)$ for all $s \in S$.
Now θ is the fuzzy subset of $X \times X$defined by
$$\begin{aligned} \theta(x_i, x_j) &= 0.4 \text{ when } i = 0,\ j = 0 \\ &= 0.5 \text{ when } i = 0,\ j = 1 \\ &= 0.7 \text{ when } i = 0,\ j = 2 \\ &= 0.4 \text{ when } i = 1,\ j = 0 \\ &= 0.7 \text{ when } i = 1,\ j = 1 \\ &= 0.7 \text{ when } i = 1,\ j = 2 \\ &= 0.4 \text{ when } i = 2,\ j = 0 \\ &= 0.7 \text{ when } i = 2,\ j = 1 \end{aligned}$$

$= 0.7$ when $i = 2,\ j = 2$

θ is an S-fuzzy subset since it can be verified that $\theta(\phi(s,(x_i,x_j)) \geqslant \theta(x_i,x_j)\ \forall x_i,x_j \in X$ and $\tilde{p}_1$ and $\tilde{p}_2$ are S-fuzzy morphisms since $\theta(x_i,x_j) = \mu(x_i)\wedge\nu(x_j) \leqslant \mu(x_i) = \mu\circ p_1(x_i,x_j)$ and $\theta(x_i,x_j) = \mu(x_i)\wedge\nu(x_j) \leqslant \nu(x_j) = \nu\circ p_2(x_i,x_j)\ \forall x_i,x_j \in X$. Also $\tilde{p_k}\phi(s,\theta)(x_i) = \phi(s,\tilde{p_k}(\theta)(x_i)$ for all $k = 1,2,\ i = 0,1,2$.

Equalizers in $S-FSET_X$

Let X be an S-set. Let μ and ν be S-fuzzy subsets of X. Let $\tilde{f_1},\tilde{f_2} : I^X \to I^X$ be S-fuzzy morphisms from μ to ν. Then f_1 and f_2 are S-morphisms from $X \longrightarrow X$. Let $K = \{x \in X : f_1(x) = f_2(x)\}$. Let $i_K : K \to X$ be the inclusion.

Define $\theta : X \to [0,1]$ as $\theta(x) = \begin{cases} \mu(x) & \forall\ x \in K \\ 0 & \text{otherwise.} \end{cases}$

Then θ is an S-fuzzy subset of X for,

$$\begin{aligned} \theta(\phi(s,x) &= \mu(\phi(s,x) \\ &\geqslant \mu(x) \text{ since } \mu \text{ is an } S\text{-fuzzy subset} \\ &= \theta(x) \text{ if } x \in K \end{aligned}$$

For $x \notin K$, $\theta(x) = 0$ and it is trivial.

Let $\tilde{i} : I^X \longrightarrow I^X$ be the extension of i_K to I^X.

ie,

$$\tilde{i}(\mu)(x) = \begin{cases} \bigvee \mu(y) : y \in X, i_K(y) = x\} \\ 0 \text{ if no such y exists} \end{cases}$$

Then

$$\tilde{i}(\phi(s,\mu))(x) = \begin{cases} \bigvee\limits_{y} \{\phi(s,\mu)(y) : i_K(y) = x,\ \text{if } x \in K\} \\ 0 \text{ otherwise} \end{cases}$$

$$= \begin{cases} \bigvee_z \bigvee_y \{\mu(z) : \phi(s,z) = y, i_K(y) = x, \text{ if } x \in K\} \\ 0 \text{ otherwise} \end{cases}$$

$$= \begin{cases} \bigvee_z \{i(\mu)(z) : i_K\phi(s,z) = x, \text{ if } x \in K\} \\ 0 \text{ otherwise} \end{cases}$$

$$= \phi(s, \tilde{i}(\mu))(x) \text{ for all } \mu \in I^X, x \in X$$

$\Rightarrow \tilde{i}(\phi(s,\mu)) = \phi(s, \tilde{i}(\mu))$ for all $\mu \in I^X$.

Since θ is the restriction of μ to K, $\theta(x) = \mu \circ i_K(x)\ \forall\ x \in X$. Thus $\tilde{i}$ is an S-fuzzy morphism. The pair $(\tilde{i}, \theta)$ defined above satisfies universal mapping property by the universal mapping property in SET.

Example 2.3.2. Let μ and ν as in the example 2.3.1 and let $\tilde{f}_1, \tilde{f}_2 : \mu \longrightarrow \nu$ be two S-fuzzy morphisms,
ie, f_1, f_2 are two S-morphisms from $X \longrightarrow X$ such that $\mu(x) \leqslant \nu \circ f_1(x)$, $\tilde{f}_1(\phi(s,\mu)(x) = \phi(s, \tilde{f}_1(\mu)(x)$ and $\tilde{f}_2(\phi(s,\mu)(x) = \phi(s, \tilde{f}_2(\mu)(x)$ for all $s \in S, x \in X$.
Take $f_1, f_2 : X \longrightarrow X$ as $f_1(x_0) = x_2,\ f_1(x_1) = x_1,\ f_1(x_2) = x_2$
and $f_2(x_0) = x_1,\ f_2(x_1) = x_1,\ f_2(x_2) = x_1$. Then $\tilde{f}_1$ and $\tilde{f}_2$ are S-fuzzy morphisms since f_1 and f_2 are S-morphisms and
$\tilde{f}_1\phi(s,\mu)(x_i) = \phi(s, \tilde{f}_1(\mu))(x_i)\ \forall x_i \in X$ and $s \in S$ and $K = \{x_1\}$.

Define θ on X as $\theta(x) = \begin{cases} 0 & \text{when } x = x_0 \\ 0.7 & \text{when } x = x_1 \\ 0 & \text{when } x = x_2 \end{cases}$

$i_k(x_1) = x_1$ is the inclusion map.

$$\tilde{i}(\theta)(x_i) = \begin{cases} 0 & \text{when } x = x_0 \\ 0.7 & \text{when } x = x_1 \\ 0 & \text{when } x = x_2. \end{cases}$$

Thus $(\theta, \tilde{i})$ as defined above is the equalizer of $\tilde{f}_1, \tilde{f}_2$.

Theorem 2.3.1. *The Category $S-FSET_X$ is complete.*

Proof. A category is complete if and only if it has got products and equalizers. Since we have proved that $S-FSET_X$ has products and equalizers it is complete.

□

Coproducts in $S-FSET_X$

Let X_1 and X_2 be two S-Sets. Then $X_1 + X_2 = X_1 \times \{1\} \cup X_2 \times \{2\}$ is an S-Set under the action $\phi(s,(a,b)) = (\phi(s,a),b)$. The coproduct is the direct sum $X_1 + X_2$ together with the inclusions in_1 and in_2. Let μ_1 and μ_2 be two S-fuzzy subsets of X.

Consider the coproduct $X + X = X_1 \cup X_2$ where $X_1 = X \times \{1\}$ and $X_2 = X \times \{2\}$ and define μ on $X + X$ as $\mu(x_i, i) = \mu_i(x_i)\ \forall x_i, i \in \{1,2\}$. Let $in_1 : X_1 \longrightarrow X + X$ and $in_2 : X_2 \longrightarrow X + X$ be the inclusion maps. Consider the extension $\tilde{in}_1 : I^{X_1} \longrightarrow I^{X_1 \cup X_2}$ defined by
$\tilde{in}_1(\mu_1)(x,1) = \mu_1(x),\ \ \tilde{in}_1(\mu_1)(x,2) = 0$ and
$\tilde{in}_2(\mu_2)(x,2) = \mu_2(x),\ \ \tilde{in}_2(\mu_2)(x,1) = 0$.
We can easily prove that, as in the case of products, μ is an S-fuzzy subset of $X + X$ and $\tilde{in}_1, \tilde{in}_2$ are the injection S-fuzzy morphisms which satisfy UMP. Thus $(\mu, \tilde{in}_1, \tilde{in}_2)$ is the coproduct of μ_1 and μ_2.

Example 2.3.3. In example 2.3.1,
$X_1 \cup X_2 = \{(x_0, 1), (x_1, 1), (x_2, 1), (x_0, 2), (x_1, 2), (x_2, 2)\}$.
Define the fuzzy subset θ on $X_1 \cup X_2$ as

$$\begin{aligned}\theta(x_i, j) &= 0.5 \text{ if i=0, j=1}\\ &= 0.7 \text{ if i=1, j=1}\\ &= 0.7 \text{ if i=2, j=1}\\ &= 0.4 \text{ if i=0, j=2}\\ &= 0.7 \text{ if i=1, j=2}\\ &= 0.7 \text{ if i=2, j=2.}\end{aligned}$$

And $\theta(\phi(s, (x_i, j))) \geqslant \theta(x_i, j)$ for all $x_i \in X, s \in S, j = 1,\ 2$.
So θ is an S-fuzzy subset of $X_1 \cup X_2$.

$$\begin{aligned}\tilde{in}_1(\mu)(x, j) &= 0.5 \text{ if } x = x_0, \text{ j=1}\\ &= 0.7 \text{ if } x = x_1, \text{ j=1}\\ &= 0.7 \text{ if } x = x_2, \text{ j=1}\\ &= 0 \text{ for all } x \in X; \text{ j=2}\end{aligned}$$

and

$$\begin{aligned}\tilde{in}_2(\nu)(x, j) &= 0 \text{ for all } x \in X; \text{ j=1}\\ &= 0.4 \text{ if } x = x_0 \text{ ,j=2}\\ &= 0.7 \text{ if } x = x_1 \text{ ,j=2}\\ &= 0.8 \text{ if } x = x_2 \text{ ,j=2.}\end{aligned}$$

$\tilde{in}_1$ and $\tilde{in}_2$ are S-fuzzy morphisms since it can be verified that $\theta \circ in_1(x) \geqslant \mu(x)$ and $\tilde{in}_1(\phi(s, \mu))(x) = \phi(s, \tilde{in}_1(\mu))(x)$ for all $x \in X, s \in S$. Similarly, $\theta \circ in_2(x) \geqslant \nu(x)$ and $\tilde{in}_2(\phi(s, \nu))(x) = \phi(s, \tilde{in}_2(\nu))(x)$ for all $x \in X, s \in S$.

Thus $(\theta, \tilde{in}_1, \tilde{in}_2)$ is the coproduct of μ and ν.

Coequalizer in $S - FSET_X$

Let X be an $S - SET$. Let μ and ν be two S-fuzzy subset of X. Let $f, g : \mu \rightarrow \nu$ be S-fuzzy morphisms. For $x \in X, (f(x), g(x))$ is a relation R on X. Consider the smallest equivalence relation $\tilde{R}$ on X containing R. Let $Z = X/\tilde{R} = \{[y];\ y \in X\}$
Define θ on Z by $\theta[y] = \bigvee\{\nu(y) : y \in [y]\}$. The canonical onto map $h : X \longrightarrow X/\tilde{R}$ where $h(y) = [y]$ is an S-morphism since $h(\phi(s, y)) = [\phi(s, y)] = \phi(s, [y]) = \phi(s, h(y))$.

Consider the extension $\tilde{h} : I^X \longrightarrow I^{X/\tilde{R}}$ by
$\tilde{h}(\nu)[y] = \bigvee\{\nu(x) : h(x) = [y]\} = \theta(y)$.
Then θ is an S-fuzzy subset of Z.
Now $\tilde{h}$ is a fuzzy morphism since $\theta \circ h(y) = \theta[y] = \bigvee\{\nu(x) : x \in [y]\} \geqslant \nu(y)\ \forall\ y \in X$.
Also $\tilde{h}(\phi(s, \mu))[y] = \bigvee\limits_x\{\phi(s, \mu)(x) : h(x) = [y]\}$

$$= \bigvee_x \bigvee_z \{\mu(z) : \phi(s, z) = x, h(x) = [y]\}$$

$$= \bigvee_x \bigvee_z \{\mu(z) : h(\phi(s, z)) = [y]\}$$

$$= \bigvee_x \bigvee_z \{\mu(z) : \phi(s, h(z)) = [y]\}$$

$$= \bigvee_x \{\tilde{h}(\mu)(x) : \phi(s, x) = [y]\}$$

$$= \phi(s, \tilde{h}(\mu))[y] \qquad .$$

So $\tilde{h}$ is an S-fuzzy morphism.
Thus ($\tilde{h}$,θ) is the coequalizer of f and g satisfying UMP.

Example 2.3.4. In the example 2.3.2.
$R = \{(x_1, x_1), (x_2, x_1)\}$.
$\tilde{R} = \{(x_1, x_1), (x_2, x_1), (x_1, x_2), (x_2, x_2)\}$.
$Z = X/\tilde{R} = \{[x_0], [x_1]\}$ where $[x_0] = \{x_0\}, [x_1] = \{x_1, x_2\}$
Define θ on Z by $\theta[x] = \begin{cases} 0.4 & \text{if } x = x_0 \\ 0.8 & \text{if } x = x_1 \end{cases}$.
Then θ is an S-fuzzy subset of Z since $\theta(\phi(s, [x]) \geqslant \theta[x]$ for all $x \in X$ and $s \in S$.
Also $\tilde{h}$ is an S-fuzzy morphism since h is an S-morphism such that $\nu(x) \leqslant \theta \circ h(x)$ and $\tilde{h}(\phi(s, \nu))[x] = \phi(s, \tilde{h}(\nu))[x]$ for all $x \in X, s \in S$.
Thus $(\tilde{h}, \theta)$ is the coequalizer of f_1 and f_2.

Theorem 2.3.2. *The category $S - FSET_X$ is cocomplete.*

Proof. A category is cocomplete if and only if it has got coproducts and coequalizers. So the category $S - FSET_X$ is cocomplete.

□

Definition 2.3.1. We define a relation on $S - FSET_X$ by $\mu \sim_S \nu$ iff $\nu = \phi(s, \mu)$ for some $s \in S$.

Theorem 2.3.3. *The above defined relation is a quasi order relation.*

Proof. Since $e \in S$, and $\phi(e, \mu) = \mu, \mu \sim_S \mu$ Let $\mu \sim_S \nu$ and $\nu \sim_S \delta$ then $\nu = \phi(s_1, \mu)$ and $\delta = \phi(s_2, \nu)$ for some $s_1, s_2 \in S$.
Then $\phi(s_2 s_1, \mu) = \phi(s_2, \phi(s_1, \mu)) = \phi(s_2, \nu) = \delta$

$\Rightarrow \mu \sim_S \delta$ Thus $\sim_S$ is a quasi order relation. □

Theorem 2.3.4. *The fuzzy subset $[\mu]$ defined by $[\mu](x) = \bigvee\{\nu(x), \nu \sim_S \mu\}$ for $x \in X$ is an S-fuzzy subset of X under the action $\phi(s,[\mu]) = [\phi(s,\mu)]$ where $s \in S$ and $S - FSET_X/\sim_S$ is an S-Set.*

Proof.
$$
\begin{aligned}
[\mu](\phi(s,x)) &= \bigvee\{\nu(\phi(s,x)) : \nu \sim_S \mu\} \\
&\geqslant \bigvee\{\nu(x) : \nu \sim_S \mu\} \\
&= [\mu](x)
\end{aligned}
$$
ie, $[\mu]$ is an S-fuzzy subset of X.

Define $\phi : S \times S - FSET_X/\sim_S \to S - FSET_X/\sim_S$ by $\phi(s,[\mu]) = [\phi(s,\mu)]$ for every $\mu \in S - FSET_X$

Then (1). $\phi(e,[\mu]) = [\phi(e,\mu)] = [\mu]$.

$$
\begin{aligned}
(2).\ \phi(s_1s_2,[\mu]) &= [\phi(s_1s_2,\mu)] \\
&= [\phi(s_1,\phi(s_2,\mu)] \\
&= \phi(s_1,[\phi(s_2,\mu)])
\end{aligned}
$$

$\Rightarrow S - FSET_X/\sim_S$ is an S-Set.

□

Theorem 2.3.5. *The functor $F : S - SET \longrightarrow S - FSET$ defined by $F(X) = S - FSET_X$ $\forall X \in S - SET$ and for $f : X \longrightarrow Y, F(f) = \tilde{f}$ is a covariant functor.*

Proof. It is obvious that $F(1_X) = 1_{F(X)}$.

$$\tilde{f} \circ \tilde{g}(\mu)(y) = \begin{cases} \bigvee_x \tilde{g}(\mu)(x) & \text{where } x \in f^{-1}(y),\ \text{if } f^{-1}(y) \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \bigvee_x \bigvee_z (\mu)(z) & \text{where } z \in g^{-1}(x), x \in f^{-1}(y),\ \text{if } g^{-1}f^{-1}(y) \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \bigvee_z \mu(z) & \text{where } z \in g^{-1}f^{-1}(y),\ \text{if } g^{-1}f^{-1}(y) \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \bigvee_z \mu(z) & \text{where } z \in (f \circ g)^{-1}(y),\ \text{if } (f \circ g)^{-1}(y) \neq \phi \\ 0 & \text{otherwise} \end{cases}$$

$= (f \tilde{\circ} g)\mu(y)$

$\Longrightarrow \tilde{f} \circ \tilde{g} = (f \tilde{\circ} g)$

$\Longrightarrow F(f \circ g) = F(f) \circ F(g)$

$\Longrightarrow F$ is a covariant functor.

$\square$

Theorem 2.3.6. *The functor $F : S - SET \longrightarrow S - FSET$ defined above is full, faithful and surjective on objects. ie, $S - SET$ and $S - FSET$ are equivalent categories.*

Proof. Since every $\tilde{f} : S - FSET_X \longrightarrow S - FSET_Y$ is an extension of some $f : X \longrightarrow Y$, the mapping $\phi : Mor(X, Y) \longrightarrow Mor(F(X), F(Y)$ defined by $\phi(f) = F(f) = \tilde{f}$ is a surjection.
So the functor F is full.
Let f_1 and f_2 be two S-morphisms from X to Y such that $f_1 \neq f_2$. Then there exists an $x_0 \in X$ such that $f_1(x_0) \neq f_2(x_0)$.

Take the fuzzy subset μ of X as $\mu(x) = \begin{cases} 0 & \forall x \in X, x \neq x_0 \\ 1 & \text{for } x = x_0. \end{cases}$

$\tilde{f}_1(\mu)(f_1(x_0)) = \bigvee\{\mu(x) : f_1(x) = f_1(x_0)\} = \mu(x_0) = 1.$

But

$\tilde{f}_2(\mu)(f_1(x_0)) = \bigvee\{\mu(x) : f_2(x) = f_1(x_0)\} = 0$

$\Longrightarrow \tilde{f}_1 \neq \tilde{f}_2$

$\Longrightarrow$ the map $f \longrightarrow F(f)$ is injective.

So F is faithful.

The objects in $S-FSET$ are classes $S-FSET_X$. Corresponding to each object $S-FSET_X$, there exists an underlying set $X \in Obj\ S-SET$ such that $F(X) = S-FSET_X$. So F is surjective on objects. F is an equivalence functor between $S-SET$ and $S-FSET$. $\square$

2.4 Fuzzy semigroup action on fuzzy subsets

We know that a fuzzy subset λ of a semigroup S is said to be a fuzzy subsemigroup of S if $\lambda(xy) \geqslant \lambda(x) \wedge \lambda(y)$ for all $x, y \in S$[15].

Definition 2.4.1. Let $\phi : S \times X \longrightarrow X$ be the action of S on X. Let μ be a fuzzy subset of X such that $\mu(\phi(s,x)) \geqslant \mu(x)\ \forall\ x \in X$. ie, μ is an S-Fuzzy subset of X. Let λ be a fuzzy subsemigroup of S. We define the action of λ on μ as

$$\phi(\lambda,\mu)(x) = \begin{cases} \bigvee\limits_{y}\bigvee\limits_{s}\{\mu(y) \wedge \lambda(s) : \phi(s,y) = x, \text{ if } \phi^{-1}(x) \neq \phi\} \\ 0 \text{ otherwise} \end{cases}$$

We say μ is a λ-fuzzy subset of X.

Theorem 2.4.1. *Let X be an S-Set and f be an S-morphism on X. μ be an S-fuzzy subset of X and λ a fuzzy subsemigroup of S. Then $\tilde{f}(\mu)$ is a λ-fuzzy subset of X where $\tilde{f}$ is the maximal extension of f.*

Proof. $\tilde{f}(\mu)(\phi(s,x)) = \begin{cases} \bigvee\{\mu(y) : f(y) = \phi(s,x)\} \text{ if } f^{-1}(\phi(s,x)) \neq \phi \\ 0 \ \ \text{otherwise} \end{cases}$

$$= \begin{cases} \bigvee\{\mu(f^{-1}(\phi(s,x))) \text{ if } f^{-1}(\phi(s,x)) \neq \phi\} \\ 0 \ \ \text{otherwise} \end{cases}$$

$$= \begin{cases} \bigvee\{f(\mu)(\phi(s,x)) : f^{-1}(\phi(s,x)) \neq \phi\} \\ 0 \ \ \text{otherwise} \end{cases}$$

$$\geqslant \begin{cases} \bigvee\{f(\mu)(x) : f^{-1}(\phi(s,x)) \neq \phi\} \\ 0 \ \ \text{otherwise} \end{cases}$$

$$= \tilde{f}(\mu)(x)$$ □

Thus $\tilde{f}(\mu)$ is an S-Fuzzy subset of X.
And the action of λ on $\tilde{f}(\mu)$ is

$$\phi(\lambda, \tilde{f}(\mu))(x)) = \begin{cases} \bigvee_y \bigvee_s \{\tilde{f}(\mu)(y) \wedge \lambda(s) : \phi(s,y) = x\} \text{ if } \phi^{-1}(x) \neq \phi \\ 0 \ \ \text{otherwise} \end{cases}$$

$$= \begin{cases} \bigvee_y \bigvee_s \{(\bigvee_z \mu(z)) \wedge \lambda(s) : f(z) = y, \phi(s,y) = x\} \text{ if } \phi^{-1}(x) \neq \phi \\ 0 \ \ \text{otherwise} \end{cases}$$

If f is an S-morphism from an S-set X onto itself then $\tilde{f} : \mu \longrightarrow \nu$ is a λ-fuzzy morphism if $\mu(x) \leqslant \nu \circ f(x)$ and $\tilde{f}(\phi(\lambda, \mu)) = \phi(\lambda, \tilde{f}(\mu))$ $\forall$ S-fuzzy subset μ of X. The collection of all λ-fuzzy subsets of X together with the

λ-fuzzy morphisms is a category $\lambda - FSET_X$. As in the case of $S - FSET$, we can prove that the collection of all $\lambda - FSET_X, X \in S - SET$ for each λ is a category say $\lambda - FSET$ which is equivalent to $S - SET$.

2.5 Application in fuzzy automata theory

Since we have proved that $S - SET$ and $S - FSET$ are equivalent categories and the main application of semigroup action on sets is in automata theory, we can construct a fuzzy state automaton corresponding to a finite automaton. While an automaton accepts or reject a given word, a fuzzy state automaton accepts a word with a degree of acceptance.

Definition 2.5.1. Let $M = (Q, X, \delta, q_0, F)$ be a finite state automaton where Q is the set of states, X is an alphabet, δ is the action of X on Q, q_0 is the initial state, F the set of final states. A max-extended finite fuzzy state machine is a quintuple $M = (Q, X, \hat{\delta}, \mu_0, F)$. $\hat{\delta}$ is the action of X on I^Q defined by $\hat{\delta}(\mu_k, a)(q_i) = \begin{cases} \bigvee \mu_k(q) : \delta(q, a) = q_i, \\ 0 \text{ if no such } q \text{ exists} \end{cases} \quad \forall\, a \in X.$
μ_0 is a fuzzy subset of Q called the initial fuzzy state. As usual X^* be the free semigroup of all words of elements of X of finite length.
Extend $\hat{\delta}$ to X^* as $\hat{\delta}^* : I^Q \times X^* \longrightarrow I^Q$ as $\hat{\delta}^*(\mu, \Lambda)(q) = \mu(q)$
and $\hat{\delta}^*(\mu, xa)(q) = \hat{\delta}(\hat{\delta}^*(\mu, x), a)(q)$ for all $\mu \in I^Q, q \in Q, x \in X^*, a \in X$.

Lemma 2.5.1. Let $M = (Q, X, \hat{\delta}^*, \mu)$ be maximally extended fuzzy state machine. Then $\hat{\delta}^*(\mu, xy)(q) = \hat{\delta}^*(\hat{\delta}^*(\mu, x), y)(q)\ \forall\, q \in Q, x, y \in X^*$.

Proof. We prove this by induction on length of y.
Let y be the empty string, $q \in Q$

$\hat{\delta}^*(\mu, xy)(q) = \hat{\delta}^*(\mu, x\Lambda)(q) = \hat{\delta}^*(\mu, x)(q)$

Let the result is true for all words of length $n-1$

Let y be a word of length n. We can write $y = ua$ where $u \in X^*$ is of length $n-1$ and $a \in X$.

$$\begin{aligned} \hat{\delta}^*(\mu, xy)(q) &= \hat{\delta}^*(\mu, xua)(q) \\ &= \hat{\delta}(\hat{\delta}^*(\mu, xu), a)(q) \\ &= \hat{\delta}(\hat{\delta}^*(\hat{\delta}^*(\mu, x), u), a)(q) \\ &= \hat{\delta}^*(\hat{\delta}^*(\mu, x), ua)(q) \\ &= \hat{\delta}^*(\hat{\delta}^*(\mu, x), y)(q) \;\; \forall q \in Q, \; x, y \in X^*. \end{aligned}$$

□

2.6 Extending finite state automata to finite fuzzy state automata

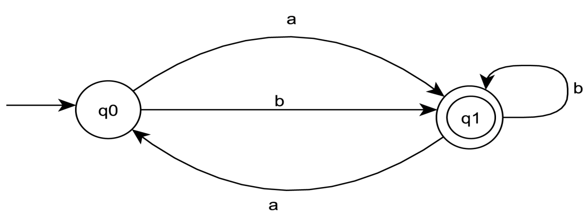
Consider a finite automaton $M = (Q, X, \delta, s, F)$. Extend δ to $\hat{\delta} : I^Q \times X \longrightarrow I^Q$ defined as $\hat{\delta}(\mu_i, a)(q_j) = \begin{cases} \bigvee\limits_k \{\mu_i(q_k) : \delta(q_k, a) = q_j\} \\ 0 \text{ if no such } a \text{ exists.} \end{cases}$

Let μ_0 be an initial fuzzy state. The degree of acceptance of a word by this fuzzy state automaton is given by

$$D(u) = \begin{cases} \bigvee\limits_{q \in F} \hat{\delta}(\mu_0, u)(q) & \forall u \in L \\ 0 & \text{for all } u \notin L. \end{cases}$$

where L is the language accepted by the finite automaton and $D(L) = \bigwedge\limits_{u \in L} D(u)$.

Example 2.6.1. Let $M = (Q, X, \delta, s, F)$ where $Q = \{q_0, q_1\}$, $X = \{a, b\}$, $s = q_0$, $F = q_1$ with the state transition function δ given as below $\delta(q_0, a, q_1)$, $\delta(q_0, b, q_1)$, $\delta(q_1, a, q_0)$, $\delta(q_1, b, q_1)$ Let μ_0 be a fuzzy subset of Q



defined by

$$\mu_0(q) = \begin{cases} 0.2 & \text{when } q = q_0 \\ 0.4 & \text{when } q = q_1 \end{cases}$$

Then

$$\hat{\delta}(\mu_0, a)(q) = \begin{cases} 0.4 & \text{when } q = q_0 \\ 0.2 & \text{when } q = q_1 \end{cases} \quad \text{say } = \mu_1$$

$$\hat{\delta}(\mu_0, b)(q) = \begin{cases} 0 & \text{when } q = q_0 \\ 0.4 & \text{when } q = q_1 \end{cases} \quad \text{say } = \mu_2$$

$$\hat{\delta}(\mu_1, a)(q) = \begin{cases} 0.2 & \text{when } q = q_0 \\ 0.4 & \text{when } q = q_1 \end{cases} \quad \text{say } = \mu_0$$

$$\hat{\delta}(\mu_1, b)(q) = \begin{cases} 0 & \text{when } q = q_0 \\ 0.4 & \text{when } q = q_1 \end{cases} \quad \text{say } = \mu_2$$

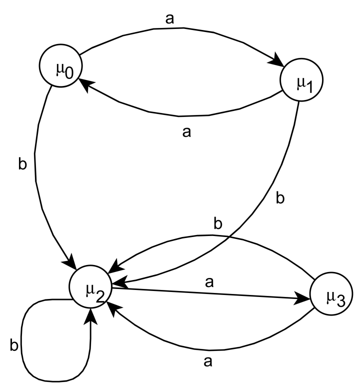
$$\hat{\delta}(\mu_2, a)(q) = \begin{cases} 0.4 & \text{when } q = q_0 \\ 0 & \text{when } q = q_1 \end{cases} \quad \text{say} = \mu_3$$

$$\hat{\delta}(\mu_2, b)(q) = \begin{cases} 0 & \text{when } q = q_0 \\ 0.4 & \text{when } q = q_1 \end{cases} \quad \text{say} = \mu_2$$

$$\hat{\delta}(\mu_3, a)(q) = \begin{cases} 0 & \text{when } q = q_0 \\ 0.4 & \text{when } q = q_1 \end{cases} \quad \text{say} = \mu_2$$

$$\hat{\delta}(\mu_3, b)(q) = \begin{cases} 0 & \text{when } q = q_0 \\ 0.4 & \text{when } q = q_1 \end{cases} \quad \text{say} = \mu_2$$

The fuzzy states are $\mu_0, \mu_1, \mu_2, \mu_3$ as defined above and the transition



function is $\hat{\delta}(\mu_0, a, \mu_1)$, $\hat{\delta}(\mu_0, b, \mu_2)$, $\hat{\delta}(\mu_1, a, \mu_0)$, $\hat{\delta}(\mu_1, b, \mu_2)$, $\hat{\delta}(\mu_2, a, \mu_3)$, $\hat{\delta}(\mu_2, b, \mu_2)$, $\hat{\delta}(\mu_3, a, \mu_2)$, $\hat{\delta}(\mu_3, b, \mu_2)$.

The language accepted by this fuzzy state automaton is $(aa)^*ab^*+b(aa)^*b^*+ab^*(aa)^*$ and the degree of acceptance $D(L) = 0.2$.

Example 2.6.2. Let μ_0 is a fuzzy subset of Q defined by

$$\mu_0(q) = \begin{cases} 1 & \text{when } q = q_0 \\ 0 & \text{otherwise} \end{cases}$$

ie, $\mu_0 = 1_{q_0}$.

Then

$$\hat{\delta}(\mu_0, a)(q) = \begin{cases} 0 & \text{when } q = q_0 \\ 1 & \text{when } q = q_1 \end{cases}$$

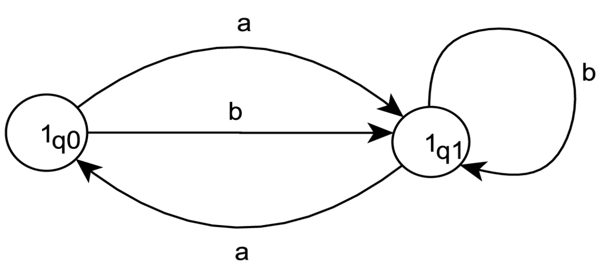
and

$$\hat{\delta}(\mu_0, b)(q) = \begin{cases} 0 & \text{when } q = q_0 \\ 1 & \text{when} q = q_1 \end{cases}$$

ie, $\hat{\delta}(\mu_0, a) = 1_{q_1}$ and $\hat{\delta}(\mu_0, b) = 1_{q_1}$

and

$\hat{\delta}(\mu_1, a) = 1_{q_0}$ and $\hat{\delta}(\mu_1, b) = 1_{q_1}$ which is the fuzzy state automaton associated with the crisp finite state automaton and the degree of acceptance of the language $(aa)^*ab^* + b(aa)^*b^* + ab^*(aa)^*$ is 1.



Chapter 3

Regular and Inverse Fuzzy Automata

3.1 Introduction

Algebraic approach to fuzzy automata theory mostly depends on the finite monoid theory because of the one-one correspondence between a fuzzy finite state automaton and its transition monoid. Eilenberg type variety theorem for fuzzy languages says that there is a one-one correspondence between variety of finite monoids and the variety of regular fuzzy languages. We know that the collection of finite inverse monoids does not form a variety since subalgebra of an inverse monoid need not be an in-

Some results of this chapter are included in the following paper.
Pamy Sebastian, T. P. Johnson. : Inverse Fuzzy Automata and Inverse Fuzzy Languages, Annals of Fuzzy Mathematics and Informatics(2013),Vol.6,No.2,447-453.

verse monoid. But they generate a variety called InV. InV consists of all finite monoids with commuting idempotents[2]. In this chapter we define an inverse fuzzy automaton such that its transition monoid is an inverse monoid and study some of its algebraic properties. We define an inverse fuzzy language, give a characterization for inverse fuzzy languages and prove some results. Also we prove some properties of inverse fuzzy languages based on the fact that the syntactic monoid is an inverse monoid.

3.2 Preliminaries

Definition 3.2.1. A semigroup S is called regular if for every element a in S there exist a b in S such that $a = aba$. A semigroup S is said to be an inverse semigroup if for every a in S there exists a unique b in S such that $aba = a$ and $bab = b$. We call b the inverse of a and denote by a^{-1}. If S has an identity then S is said to be an inverse monoid.

For any element a of an inverse monoid, aa^{-1} is an idempotent and idempotents of an inverse monoid commute. The collection of all finite inverse monoids generate a variety InV which is the collection of all semigroups with commuting idempotents. This is the smallest variety containing finite inverse monoids [2]. An analogues to Cayley's theorem for groups, Preston and Wagner proved that an inverse monoid S is isomorphic to a subinverse monoid of the monoid of all one-one partial transformations on S. A regular semigroup can be characterized by the property that for every x in S the $\mathscr{L}$-class ($\mathscr{R}$-class) containing x contains an idempotent and inverse semigroups can be characterized by the properties, (a) S

is regular and idempotents in S commute,(b) every $\mathscr{L}$-class and $\mathscr{R}$-class contains an idempotent.

3.3 Regular and inverse fuzzy automata

Let X be a nonempty finite set. Let X^* be the free monoid generated by X. Then X^* is regular if for every $x \in X^*$ there exist a $y \in X^*$ such that $x = xyx$ and X^* is inverse if $\forall\ x \in X^*$, there exist a unique $y \in X^*$ such that $x = xyx$, $y = yxy$.

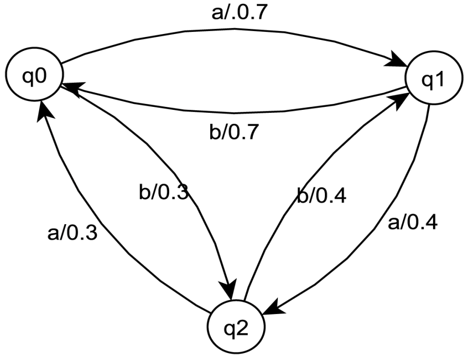
Definition 3.3.1. Let $M = (Q, X, \mu)$ be a fuzzy automaton. M is said to be regular if for every $x \in X^*$ there exist a $y \in X^*$ such that $\mu(p, x, q) = \mu(p, xyx, q)$ for all $p, q \in Q$. $M = (Q, X, \mu)$ is said to be an inverse fuzzy automaton if $\forall\ x \in X^*$, there exist a unique $y \in X^*$ such that $\mu(q, xyx, p) = \mu(q, x, p)$, $\mu(q, yxy, p) = \mu(q, y, p)$ $\forall p, q \in Q$.

In the case of a deterministic inverse fuzzy automaton this can be redefined as $\forall x \in X^*$, there exist a unique y such that $\mu(q, x, p) = \mu(p, y, q)$ and $\mu(p, x, q) = \mu(r, x, q) \implies p = r\ \forall p, q, r \in Q$. A deterministic fuzzy automaton can be represented by the transition matrices with each row contains atmost one nonzero entry (partial fuzzy transformations) and a deterministic inverse fuzzy automaton can be represented by transition matrices with each row and column contains atmost one nonzero entry (one-one partial fuzzy transformations). For an inverse fuzzy automaton we take $\tilde{X}^*$ to assure the existence of such a y. ie, $\forall\ x \in \tilde{X}^*$, $\mu(q, xx^{-1}x, p) = \mu(q, x, p)$ and $\mu(q, x^{-1}xx^{-1}, p) = \mu(q, x^{-1}, p)\ \forall\ p, q \in Q$.

Definition 3.3.2. A fuzzy language λ on an alphabet X is said to be an inverse fuzzy language if the minimal fuzzy automaton recognizing that language is an inverse fuzzy automaton.

Example 3.3.1. Let M=$(Q, \tilde{X}, \mu, i, \tau)$, where $Q = \{q_0, q_1, q_2\}$, $\tilde{X} = \{a, b\}, i = [1\ 0\ 0]$, $\tau = [0\ 0\ 1]^T$ and $\mu : Q \times X \times Q \longrightarrow [0, 1]$ as defined below
$\mu(q_0, a, q_1) = 0.7$, $\mu(q_1, a, q_2) = 0.4$, $mu(q_2, a, q_0) = 0.3$, $\mu(q_1, b, q_0) = 0.8$, $\mu(q_0, b, q_2) = 0.6$, $\mu(q_2, b, q_1) = 0.5$ and $= 0$ for all other elements of $Q \times \tilde{X} \times Q$. This is a deterministic regular fuzzy automaton for which $T_{aba} = T_a$. But this is not an inverse fuzzy automaton since b is not unique and $T_{bab} \neq T_b$

Example 3.3.2. Let M=$(Q, \tilde{X}, \mu, i, \tau)$, where $Q = \{q_0, q_1, q_2\}$, $\tilde{X} = \{a, b\}, i = [1\ 0\ 0]$, $\tau = [0\ 0\ 1]^{\mathrm{T}}$ and $\mu : Q \times \tilde{X} \times Q \longrightarrow [0, 1]$ as defined below
$\mu(q_0, a, q_1) = 0.7$, $\mu(q_1, a, q_2) = 0.4$, $\mu(q_2, a, q_0) = 0.3$, $\mu(q_1, b, q_0) = 0.7$, $\mu(q_0, b, q_2) = 0.3$, $\mu(q_2, b, q_1) = 0.4$ and $= 0$ for all other elements of $Q \times X \times Q$.



Then $T_a = \begin{bmatrix} 0 & 0.7 & 0 \\ 0 & 0 & 0.4 \\ 0.3 & 0 & 0 \end{bmatrix} \quad T_b = \begin{bmatrix} 0 & 0 & 0.3 \\ 0.7 & 0 & 0 \\ 0 & 0.4 & 0 \end{bmatrix}$

Then, the transition semigroup $T(M)$ is the semigroup generated by $\{T_a, T_b\}$ in which $T_{aba} = T_a, T_{bab} = T_b$.

Thus

$T(M) = \{T_a, T_{a^2}, T_{a^3}, T_{a^4}, T_{a^5}, T_b, T_{b^2},\ T_{ab}, T_{ba}, T_{ab^2}, T_{b^2a}, T_{a^2b},\ T_{ba^2}, T_{a^2b^2}, T_{b^2a^2}, T_{ab^2a}\}$. Here $T_{a^3}, T_{ab}, T_{ba}, T_{b^2a^2},\ T_{a^2b^2}, T_{ab^2a}$ are the idempotents.

$\mathscr{L}_{T_a} = \{T_a, T_{ba}\}$

$\mathscr{L}_{T_b} = \{T_b, T_{ab}\}$

$\mathscr{L}_{T_{a^2}} = \{T_{a^2}, T_{ba^2}, T_{b^2a^2}\}$

$\mathscr{L}_{T_{b^2}} = \{T_{b^2}, T_{ab^2}, T_{a^2b^2}\}$

$\mathscr{L}_{T_{a^2b}} = \{T_{a^2b}, T_{b^2a}, T_{ab^2a}\}$

$\mathscr{L}_{T_{a^3}} = \{T_{a^3}, T_{a^4}, T_{a^5}\}$.

Since every $\mathscr{L}$ class contains a unique idempotent, $T(M)$ is an inverse semigroup.

The fuzzy language accepted by this fuzzy automaton is

$$\lambda(x) = \begin{cases} 0.4 & \text{when } x = ((aabb)^* + (ab)^*)aa \\ 0.3 & \text{when } x = b((ab)^* + (ba)^*) + (bbb)^*(aa + b + bba) \\ 0 & \text{for all other } x \in X^* \end{cases}$$

which is an inverse fuzzy language.

Example 3.3.3. Let $Q = \{q_0, q_1, q_2\}, X = \{a, b\}$. Take $\tilde{X}^*$ as the free inverse monoid generated by X. Consider a deterministic fuzzy automaton $M = (Q, \tilde{X}, \mu)$ over $\tilde{X} = X \cup X^{-1}$ where μ is a fuzzy subset of $Q \times \tilde{X} \times Q$ with finite image C such that for every $p \in Q$, there exist atmost one $q \in Q$ such that $\mu(q, a, p) > 0$ and $\mu(q, a, p) = \mu(p, a^{-1}, q)\ \forall\ p, q \in Q, a \in X$. Then M is a deterministic fuzzy automaton which is inverse. Here $\tilde{X}^*$ acts on Q as one-one partial fuzzy transformations. The transition monoid is a subinverse monoid of the inverse monoid of all one-one partial fuzzy transformations on Q.

Theorem 3.3.1. *A fuzzy automaton $A = (Q, \tilde{X}, \mu, i, \tau)$ is inverse (regular) if and only if $\tilde{X}^*/\theta_A$ is an inverse (regular) monoid.*

Proof. Suppose A is an inverse fuzzy automaton
ie, for each $x \in \tilde{X}^*$ there exist a unique $x^{-1} \in \tilde{X}^*$ such that $\forall\ p, q \in Q$, $\mu(q, xx^{-1}x, p) = \mu(q, x, p)$ and $\mu(q, x^{-1}xx^{-1}, p) = \mu(q, x^{-1}, p)$
$\Longleftrightarrow xx^{-1}x\ \theta_A x$ and $x^{-1}xx^{-1}\ \theta_A x^{-1}$
$\Longleftrightarrow [xx^{-1}x] = [x]$ and $[x^{-1}xx^{-1}] = [x^{-1}]$.
Then, $[x][x^{-1}][x] = [x]$ and $[x^{-1}][x][x^{-1}] = [x^{-1}]$
$\Longleftrightarrow \tilde{X}^*/\theta_A$ is an inverse monoid.
As a particular case it is true that A is regular if and only if $\tilde{X}^*/\theta_A$ is regular.

□

3.4 Construction of regular and inverse fuzzy automata

Since every nondeterministic fuzzy automaton can be converted into a deterministic fuzzy automaton we give the construction of a deterministic inverse fuzzy automaton. There is a one to one correspondence between finite inverse (regular) monoids and inverse (regular) fuzzy automata on the set of generators. To construct a deterministic inverse fuzzy automaton on n states $Q = \{q_1, q_2, \ldots, q_n\}$, take $C = \{c_1, c_2, \ldots, c_k\}$, $k \leqslant n, c_i \in [0, 1]$. Consider the collection of all matrices with entries in C and such that there exists atmost one non-zero entry in each row and column. This collection represent the set of all one-one partial fuzzy transformations on Q with image in C, which is a monoid under the $max - min$ operation, denoted by FI_Q^C. FI_Q^C is finite since Q and C are finite. For every $A \in FI_Q^C$, there exists a unique inverse $B \in FI_Q^C$ such that $ABA = A$ and $BAB = B$. Here B will be the transpose of A. To construct an inverse fuzzy automaton on a finite alphabet m, take m matrices from FI_Q^C such that transpose of each matrices is included in the collection. Construct an automaton with these matrices as the transition matrices of the m alphabets. The automaton will be an inverse fuzzy automaton with the transition monoid as the monoid generated by the chosen fuzzy matrices. For a deterministic regular fuzzy automaton, we take a fuzzy matrix $A = [a_{ij}] \in FI_Q^C$ and $B = [b_{ij}]$ is another fuzzy matrix in FI_Q^C such that $b_{ij} \geqslant a_{ji}$ for all $a_{ji} \neq 0$ and $= 0$ for $a_{ji} = 0$. Then $ABA = A$ but B is not unique and BAB need not be equal to B.

Example 3.4.1. Let $c_0, c_1, c_2, c_4 \in [0, 1]$ $Q = \{q_0, q_1, q_2\}$, $X = \{a, b\}$

$$T_a = \begin{pmatrix} 0 & c_0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_1 \end{pmatrix} \quad T_{a^{-1}} = \begin{pmatrix} 0 & 0 & 0 \\ c_0 & 0 & 0 \\ 0 & 0 & c_1 \end{pmatrix}$$

$$T_b = \begin{pmatrix} c_2 & 0 & 0 \\ 0 & 0 & c_3 \\ 0 & c_4 & 0 \end{pmatrix} \quad T_{b^{-1}} = \begin{pmatrix} c_2 & 0 & 0 \\ 0 & 0 & c_4 \\ 0 & c_3 & 0 \end{pmatrix}$$

Then $M = (Q, \tilde{X}, \{T_a, a \in \tilde{X}\})$ is an inverse fuzzy automaton.

3.5 Inverse fuzzy languages

We have proved that the transition monoid of an inverse fuzzy automaton is an inverse monoid. If a fuzzy language is recognized by an inverse fuzzy automaton, the corresponding transition monoid should recognize that fuzzy language. So if λ is an inverse fuzzy language on $\tilde{X}$, there exist an inverse monoid I and a fuzzy subset δ of I and a homomorphism ϕ from $\tilde{X}^*$ to I such that $\phi^{-1}(\delta) = \lambda$. ie, $\phi^{-1}(\delta)(u) = \lambda(u)\ \forall\ u \in \tilde{X}^*$.

Theorem 3.5.1. *(Characterization of an inverse fuzzy language) A fuzzy language λ on $\tilde{X}$ is inverse if and only if for every $x \in \tilde{X}^*$ $\lambda(uxx^{-1}xv) = \lambda(uxv)$ and $\lambda(ux^{-1}xx^{-1}v) = \lambda(ux^{-1}v)$ for every $u, v \in \tilde{X}^*$.*

Proof. Let λ is a regular fuzzy language on $\tilde{X}$. The transition monoid of the minimal automaton $M(\lambda)$ is the syntactic monoid of λ. Let P_λ be the main congruence on $\tilde{X}^*$ defined by $xP_\lambda y$ if and only $\lambda(uxv) = \lambda(uyv)$ for all $u, v \in \tilde{X}^*$. The transition monoid of the quotient fuzzy automaton is isomorphic to the syntactic monoid of the fuzzy language λ. Thus $\tilde{X}^*/P_\lambda$ is isomorphic to $\tilde{X}^*/\theta_M$. Suppose λ is an inverse fuzzy language. Then the minimal fuzzy automaton $M(\lambda)$ recognizing λ is an inverse fuzzy automaton. ie, $\tilde{X}^*/\theta_M$ is an inverse monoid. Then for each $x \in \tilde{X}^*$ there exist a unique $x^{-1} \in \tilde{X}^*$ such that $\forall\ p, q \in Q$,

$\mu(q, xx^{-1}x, p) = \mu(q, x, p)$ and $\mu(q, x^{-1}xx^{-1}, p) = \mu(q, x^{-1}, p)$

$\Longrightarrow [xx^{-1}x]_{\theta_M} = [x]_{\theta_M}$ and $[x^{-1}xx^{-1}]_{\theta_M} = [x^{-1}]_{\theta_M}$

$\Longrightarrow [xx^{-1}x]_{P_\lambda} = [x]_{P_\lambda}$ and $[x^{-1}xx^{-1}]_{P_\lambda} = [x^{-1}]_{P_\lambda}$

$\Longrightarrow xx^{-1}xP_\lambda x$ and $x^{-1}xx^{-1}P_\lambda x^{-1}$

$\Longrightarrow \lambda(uxx^{-1}xv) = \lambda(uxv)$ and $\lambda(ux^{-1}xx^{-1}v) = \lambda(ux^{-1}v)\ \ \forall u, v \in X^*$.

Conversly, suppose for every $x \in \tilde{X}^*$, $\lambda(uxx^{-1}xv) = \lambda(uxv)$ and $\lambda(ux^{-1}xx^{-1}v) = \lambda(ux^{-1}v)$ for every $u, v \in \tilde{X}^*$.
$\Longrightarrow xx^{-1}xP_\lambda x$ and $x^{-1}xx^{-1}P_\lambda x^{-1}$
$\Longrightarrow [xx^{-1}x]_{P_\lambda} = [x]_{P_\lambda}$ and $[x^{-1}xx^{-1}]_{P_\lambda} = [x^{-1}]_{P_\lambda}$
$\Longrightarrow [xx^{-1}x]_{\theta_M} = [x]_{\theta_M}$ and $[x^{-1}xx^{-1}]_{\theta_M} = [x^{-1}]_{\theta_M}$
$\Longrightarrow \tilde{X}^*/\theta_M$ is an inverse monoid . ie, the minimal fuzzy automaton recognizing λ is an inverse automaton and thus λ is an inverse fuzzy language. $\square$

Lemma 3.5.1. The set of all inverse fuzzy languages on an alphabet $\tilde{X}$ is closed under intersection.

Proof. Let λ_1 and λ_2 be two inverse fuzzy languages in $\tilde{X}^*$.
Then there exist two inverse fuzzy automata $M_1 = (Q_1, \tilde{X}, \mu_1, i_1, \tau_1)$ and $M_2 = (Q_2, \tilde{X}, \mu_2, i_2, \tau_2)$ recognizing λ_1 and λ_2 respectively. Then the restricted direct product $M_1 \dot{\times} M_2$ is an inverse fuzzy automaton since
$\mu_1 \dot{\times} \mu_2((q_1, q_2), x, (p_1, p_2)) = \mu_1(q_1, x, p_1) \wedge \mu_2(q_2, x, p_2)$ $\forall x \in \tilde{X}^*$ and M_1 and M_2 are inverse fuzzy automata. The language recognized by $M_1 \dot{\times} M_2$ is $\lambda_1 \wedge \lambda_2$ [14].
So $\lambda_1 \wedge \lambda_2$ is an inverse fuzzy language.

$\square$

Lemma 3.5.2. Let $M_1 = (Q_1, \tilde{X}, \mu_1, i_1, \tau_1)$ and $M_2 = (Q_2, \tilde{X}, \mu_2, i_2, \tau_2)$ be two inverse fuzzy automata with $Q_1 \cap Q_2 = \phi$ and recognizing the inverse fuzzy languages λ_1 and λ_2 respectively. Then their join $M_1 \vee M_2$ is an inverse fuzzy automaton recognizing $\lambda_1 \vee \lambda_2$.

Proof. We have

$\lambda_1(x) = \bigvee_{p,q \in Q_1} i_1(p) \wedge \mu_1(p,x,q) \wedge \tau_1(q)$ and

$\lambda_2(x) = \bigvee_{p,q \in Q_2} i_2(p) \wedge \mu_2(p,x,q) \wedge \tau_2(q)$.

Now the fuzzy language recognized by $M_1 \vee M_2$ is

$$\begin{aligned}
\lambda(x) &= \bigvee_{p,q \in Q_1 \cup Q_2} (i_1 \vee i_2)(p) \wedge (\mu_1 \vee \mu_2)(p,x,q) \wedge (\tau_1 \vee \tau_2)(q) \\
&= \bigvee_{p,q \in Q_1} i_1(p) \wedge \mu_1(p,x,q) \wedge \tau_1(q) \vee \bigvee_{p,q \in Q_2} i_2(p) \wedge \mu_2(p,x,q) \wedge \tau_2(q) \vee \\
&\quad \bigvee_{p \in Q_1, q \in Q_2} i_1(p) \wedge 0 \wedge \tau_2(q) \\
&= \bigvee_{p,q \in Q_1} i_1(p) \wedge \mu_1(p,x,q) \wedge \tau_1(q) \vee \bigvee_{p,q \in Q_2} i_2(p) \wedge \mu_2(p,x,q) \wedge \tau_2(q) \\
&= \lambda_1(x) \vee \lambda_2(x) \\
&= (\lambda_1 \vee \lambda_2)(x) \text{ for all} x \in \tilde{X}^*.
\end{aligned}$$

Also we have $\forall\, x \in \tilde{X}^*,\ \mu_1 \vee \mu_2(p,x,q) = \begin{cases} \mu_1(p,x,q) & \text{if } p,q \in Q_1 \\ \mu_2(p,x,q) & \text{if } p,q \in Q_2 \\ 0 & \text{otherwise.} \end{cases}$

So if M_1 and M_2 are two inverse fuzzy automata then their join $M_1 \vee M_2$ is an inverse fuzzy automaton.

□

Theorem 3.5.2. *The class of all inverse fuzzy languages in $\tilde{X}^*$ is closed under finite boolean operations.*

Proof. Let λ is an inverse fuzzy language.

Then by the characterization of inverse fuzzy language for every $x \in \tilde{X}^*$ there exist $x^{-1} \in \tilde{X}^*$ such that $\lambda(uxx^{-1}xv) = \lambda(uxv)$ and

$\lambda(ux^{-1}xx^{-1}v) = \lambda(ux^{-1}v)\ \forall u, v \in \tilde{X}^*$.

Then $\lambda^c(uxx^{-1}xv) = 1 - \lambda(uxx^{-1}xv) = 1 - \lambda(uxv) = \lambda^c(uxv)$ and $\lambda^c(ux^{-1}xx^{-1}v) = 1 - \lambda(ux^{-1}xx^{-1}v) = 1 - \lambda(ux^{-1}v) = \lambda^c(ux^{-1}v)$.

Thus λ^c is an inverse fuzzy language.

Closure properties of union and intersection of inverse fuzzy languages follows by Lemma 3.5.1 and Lemma 3.5.2.

□

3.6 Homomorphic image of inverse fuzzy automata

Definition 3.6.1. Let $M_1 = (Q_1, X_1, \mu_1, i_1, \tau_1)$ and $M_2 = (Q_2, X_2, \mu_2, i_2, \tau_2)$ be two fuzzy automata. A pair (α, β) of mappings $\alpha : Q_1 \longrightarrow Q_2$ and $\beta : X_1 \longrightarrow X_2$ is called a homomorphism written as $(\alpha, \beta) : M_1 \longrightarrow M_2$, if $\mu_1(q, x, p) \leqslant \mu_2(\alpha(q), \beta(x), \alpha(p))$ $\forall p, q \in Q_1$ and $\forall x \in X_1$. The pair (α, β) is called a strong homomorphism if $\mu_2(\alpha(q), \beta(x), \alpha(p)) = \bigvee\{\mu_1(q, x, t : t \in Q_1, \alpha(t) = \alpha(p)\}$ $\forall q, p \in Q_1$ and $\forall x \in X_1$. β can be extended to $\beta^* : X_1^* \longrightarrow X_2^*$ by $\beta^*(\Lambda) = \Lambda$ and $\beta^*(ua) = \beta^*(u)\beta^*(a)$ $\forall u \in X_1^*, a \in X_1$ and $\beta^*(uv) = \beta^*(u)\beta^*(v)$ $\forall u, v \in X^*$ [14]. If α, β are one-one and onto then (α, β) is called an isomorphism.

Theorem 3.6.1. *Let M_1, M_2 be two fuzzy automata . Let $(\alpha, \beta) : M_1 \longrightarrow M_2$ be strong homomorphism. Then α is one-one if and only if $\mu_1(q, x, p) = \mu_2(\alpha(q), \beta^*(x), \alpha(p))$ $\forall q, p \in Q$ and $x \in X_1^*$ [14].*

Theorem 3.6.2. *If $M_1 = (Q_1, \tilde{X}_1, \mu_1)$ and $M_2 = (Q_2, \tilde{X}_2, \mu_2)$ be two fuzzy automata. Let $(\alpha, \beta) : M_1 \longrightarrow M_2$ be a strong homomorphism from*

$M_1 \longrightarrow M_2$ and if M_1 is inverse then $(\alpha, \beta)(M_1)$ is also inverse.

Proof. Since (α, β) is a strong homomorphism from $M_1 \longrightarrow M_2$
$\mu_2(\alpha(q), \beta(x), \alpha(p)) = \bigvee\{\mu_1(q, x, t) : t \in Q, \alpha(t) = \alpha(p)\}\ \forall p, q \in Q_1, x \in \tilde{X}_1$.
Since M_1 is inverse, for every $x \in \tilde{X_1}^*$ there exist a unique $y \in \tilde{X_1}^*$ such that $\mu_1(q, xyx, p) = \mu_1(q, x, p)$ and $\mu_1(q, yxy, p) = \mu_1(q, y, p)$ for every $p, q \in Q$.

$$\begin{aligned}
\mu_2(\alpha(q), \beta^*(x)\beta^*(y)\beta^*(x), \alpha(p)) &= \mu_2(\alpha(q), \beta^*(xyx), \alpha(p)) \\
&= \bigvee\{\mu_1(q, xyx, t) : t \in Q, \alpha(t) = \alpha(p)\} \\
&= \bigvee\{\mu_1(q, x, t) : t \in Q, \alpha(t) = \alpha(p)\} \\
&= \mu_2(\alpha(q), \beta^*(x), \alpha(p)
\end{aligned}$$

And,

$$\begin{aligned}
\mu_2(\alpha(q), \beta^*(y)\beta^*(x)\beta^*(y), \alpha(p)) &= \mu_2(\alpha(q), \beta^*(yxy), \alpha(p)) \\
&= \bigvee\{\mu_1(q, yxy, t) : t \in Q, \alpha(t) = \alpha(p)\} \\
&= \bigvee\{\mu_1(q, y, t) : t \in Q, \alpha(t) = \alpha(p)\} \\
&= \mu_2(\alpha(q), \beta^*(y), \alpha(p))
\end{aligned}$$

Thus the image of M_1 under (α, β) is an inverse fuzzy automata.
If λ is a fuzzy language recognized by M_1, then its image $\beta^*(\lambda)$ defined as

$$\beta^*(\lambda)(u) = \begin{cases} \bigvee\{\lambda(w) : \beta^*(w) = u \text{ if } {\beta^*}^{-1}(u) \neq \phi\} \\ 0 \text{ otherwise} \end{cases} \quad \text{for every } u \in \tilde{X_2}^*$$

Let $x \in \tilde{X_2}^*$.
Then $\beta^*(\lambda)(uxv) = \begin{cases} \bigvee\{\lambda(w) : \beta^*(w) = uxv \text{ if } {\beta^*}^{-1}(uxv) \neq \phi\} \\ 0 \ \text{ otherwise.} \end{cases}$

Then there exists $u', x', v' \in \tilde{X_1}^*$ such that $\beta^*(u'x'v') = uxv$.
Since $x' \in \tilde{X_1}^*$, there exists a unique inverse $y' \in \tilde{X_1}^*$ such that
$\lambda(u'x'v') = \lambda(u'x'y'x'v')$ and $\lambda(u'y'v') = \lambda(u'y'x'y'v')\ \ \forall u', v' \in \tilde{X_1}^*$.

So $\beta^*(\lambda)(uxv) = \begin{cases} \bigvee\{\lambda(u'x'v') : \beta^*(u'x'v') = uxv \text{ if } {\beta^*}^{-1}(uxv) \neq \phi\} \\ 0 \quad \text{otherwise} \end{cases}$

$=\begin{cases} \bigvee\{\lambda(u'x'y'x'v') : \beta^*(u'x'y'x'v') = uxyxv \text{ if } {\beta^*}^{-1}(uxyxv) \neq \phi\} \\ 0 \text{ otherwise} \end{cases}$

$=\beta^*(\lambda)(uxyxv)$

Similarly we can prove that $\beta^*(\lambda)(uyxyv) = \beta^*(\lambda)(uyv)$.

This says that $\beta^*(\lambda)$ is an inverse fuzzy language.

□

Theorem 3.6.3. *If $M_1 = (Q_1, \tilde{X}_1, \mu_1)$ and $M_2 = (Q_2, \tilde{X}_2, \mu_2)$ be two fuzzy automata. Let $(\alpha, \beta) : M_1 \longrightarrow M_2$ be a strong homomorphism from $M_1 \longrightarrow M_2$ with α, β are one-one and onto and if M_2 is inverse. Then $(\alpha, \beta)^{-1}(M_2)$ is also inverse.*

Proof. Suppose (α, β) be a strong homomorphism with α, being one one onto. Then $(\alpha, \beta) : M_1 \longrightarrow M_2$ has the property

$\mu_2(\alpha(q), \beta^*(x), \alpha(p)) = \mu_1(q, x, p)\ \forall x \in {\tilde{X}_1}^*$[14].

Let M_2 be an inverse fuzzy automata.

Then for every $x \in {\tilde{X}_2}^*$ there exists a unique $y \in {\tilde{X}_2}^*$ such that

$\mu_2(q, xyx, p) = \mu_2(q, x, p)$ and $\mu_2(q, yxy, p) = \mu_2(q, y, p)\ \forall q, p \in Q_2$.

ie, $\mu_1(\alpha^{-1}(q), {\beta^*}^{-1}(xyx), \alpha^{-1}(p)) = \mu_1(\alpha^{-1}(q), {\beta^*}^{-1}(x), \alpha^{-1}(p))$ and

$\mu_1(\alpha^{-1}(q), {\beta^*}^{-1}(yxy), \alpha^{-1}(p)) = \mu_1(\alpha^{-1}(q), {\beta^*}^{-1}(y), \alpha^{-1}(p))$

$\Rightarrow \mu_1(\alpha^{-1}(q), {\beta^*}^{-1}(x){\beta^*}^{-1}(y){\beta^*}^{-1}(x), \alpha^{-1}(p)) = \mu_1(\alpha^{-1}(q), {\beta^*}^{-1}(x), \alpha^{-1}(p))$

and

$\mu_1(\alpha^{-1}(q), {\beta^*}^{-1}(y){\beta^*}^{-1}(x)\beta^{*-1}(y), \alpha^{-1}(p)) = \mu_1(\alpha^{-1}(q), {\beta^*}^{-1}(y), \alpha^{-1}(p))$.

$\Rightarrow (\alpha, \beta)^{-1}(M_2)$ is inverse.

If λ is a fuzzy language recognized by M_2, then its inverse image $\beta^{*-1}(\lambda)$ defined as $\beta^{*-1}(\lambda)(u) = \lambda(\beta^*(u)) \ \ \forall u \in \tilde{X_1}^*$.

Let $u, x, v \in \tilde{X_1}^*$. Then since β^* is an isomorphism, $\beta^*(y)$ is the inverse of $\beta^*(x)$ where y is the inverse of x. Then,

$$\begin{aligned}
{\beta^*}^{-1}(\lambda)(uxv) &= \lambda(\beta^*(uxv)) \\
&= \lambda(\beta^*(u)\beta^*(x)\beta^*(v)) \\
&= \lambda(\beta^*(u)\beta^*(x)\beta^*(y)\beta^*(x)\beta^*(v) \\
&= \lambda(\beta^*(uxyxv) \\
&= {\beta^*}^{-1}\lambda(uxyxv)
\end{aligned}$$

And

$$\begin{aligned}
{\beta^*}^{-1}(\lambda)(uyv) &= \lambda(\beta^*(uyv)) \\
&= \lambda(\beta^*(u)\beta^*(y)\beta^*(v)) \\
&= \lambda(\beta^*(u)\beta^*(y)\beta^*(x)\beta^*(y)\beta^*(v) \ . \\
&= \lambda(\beta^*(uyxyv) \\
&= {\beta^*}^{-1}\lambda(uyxyv)
\end{aligned}$$

So ${\beta^*}^{-1}(\lambda)$ is an inverse fuzzy language.

$\square$

3.7 Cartesian product of two inverse fuzzy automata

Definition 3.7.1. Let $M_1 = (Q_1, X_1, \mu_1), M_2 = (Q_2, X_2, \mu_2)$ be fuzzy finite state machines such that $Q_1 \cap Q_2 = \phi$ and $X_1 \cap X_2 = \phi$. Then their direct sum is defined as $M_1 \oplus M_2 = (Q_1 \cup Q_2, X_1 \cup X_2, \mu_1 \oplus \mu_2)$ where

$$\mu_1 \oplus \mu_2(p, a, q) = \begin{cases} \mu_1(p, a, q) & \text{if } p, q \in Q_1, a \in X_1 \\ \mu_2(p, a, q) & \text{if } p, q \in Q_2, a \in X_2 \\ 1 & \begin{array}{l} \text{if either } (p, a) \in Q_1 \times X_1, q \in Q_2 \\ \text{or } (p, a) \in Q_2 \times X_2, q \in Q_1 \end{array} \\ 0 & \text{otherwise} \end{cases}$$

and

the cartesian product is defined as $M_1.M_2 = (Q_1 \times Q_2, X_1 \cup X_2, \mu_1.\mu_2)$ where

$$\mu_1.\mu_2((p_1, p_2), a, (q_1, q_2)) = \begin{cases} \mu_1(p_1, a, q_1) & \text{if } a \in X_1 \text{ and } p_2 = q_2 \\ \mu_2(p_2, a, q_2) & \text{if } a \in X_2 \text{ and } p_1 = q_1 \\ 0 & \text{otherwise} \end{cases} \quad [14]$$

Theorem 3.7.1. *Let $M_1 = (Q_1, \tilde{X}_1, \mu_1, i_1, \tau_1), M_2 = (Q_2, \tilde{X}_2, \mu_2, i_2, \tau_2)$ be two fuzzy automata. If M_1 and M_2 are inverse fuzzy automata then their Cartesian product $M_1.M_2$ is an inverse fuzzy automaton*

Proof. We have two theorems (see [14])

(1). If $M_1 = (Q_1, X_1, \mu_1), M_2 = (Q_2, X_2, \mu_2)$ be two fuzzy finite state machines such that $X_1 \cap X_2 = \phi$. Then for every $w \in (X_1 \cup X_2)^*$, there exist $u \in {X_1}^*, v \in {X_2}^*$ such that

$(\mu_1.\mu_2)((p_1, p_2), w, (q_1, q_2)) = (\mu_1.\mu_2)((p_1, p_2), uv, (q_1, q_2))$.

$w^* = uv$ is called the standard form of w.

(2). For every $u \in X_1^*, v \in X_2^*$,

$(\mu_1.\mu_2)((p_1,p_2),uv,(q_1,q_2)) = \mu(p_1,u,q_1) \wedge \mu_2(p_2,v,q_2)$

$\quad = (\mu_1.\mu_2)((p_1,p_2),vu,(q_1,q_2))$ for every $(p_1,p_2),(q_1,q_2) \in Q_1 \times Q_2$.

First suppose M_1 and M_2 are two inverse fuzzy automata.
Let $w \in (X_1 \tilde{\cup} X_2)^*$.
We know that $(X_1 \tilde{\cup} X_2)^*$ is the free semigroup on $(X_1 \cup X_2) \cup (X_1 \cup X_2)^{-1}$ in which $w = ww^{-1}w$ and $w^{-1} = w^{-1}ww^{-1}$ for all $w \in X_1 \cup X_2$.
If $w = \Lambda$ then the proof is trivial.
Suppose $w \neq \Lambda$.

Case 1. Let $w \in (X_1 \cup X_2)^*$.

By the above theorem there exist $u \in {X_1}^*, v \in {X_2}^*$ such that

$(\mu_1.\mu_2)((p_1,p_2),w,(q_1,q_2)) = (\mu_1.\mu_2)((p_1,p_2),uv,(q_1,q_2))$

$\quad = \mu_1(p_1,u,q_1) \wedge \mu_2(p_2,v,q_2)$ for every $(p_1,p_2),(q_1,q_2) \in Q_1 \times Q_2$. Since M_1 and M_2 are inverse fuzzy automata, there exist unique symmetric inverses $u^{-1} \in X_1^{-1*}$ and $v^{-1} \in X_2^{-1*}$ such that
$\mu_1(p_1,u,q_1) = \mu_1(p_1,uu^{-1}u,q_1)$ and $\mu_2(p_2,v,q_2) = \mu_2(p_2,vv^{-1}v,q_2)$ for every $p_1,q_1 \in Q_1, p_2,q_2 \in Q_2$.
Let $w^{-1} = v^{-1}u^{-1}$. Then clearly $w^{-1} \in (X_1 \cup X_2)^{-1*}$ and

$(\mu_1.\mu_2)((p_1,p_2),w,(q_1,q_2))$

$= \mu(p_1,u,q_1) \wedge \mu_2(p_2,v,q_2)$

$= \mu_1(p_1,uu^{-1}u,q_1) \wedge \mu_2(p_2,vv^{-1}v,q_2)$

$= (\mu_1.\mu_2)(p_1,p_2),uu^{-1}uvv^{-1}v,(q_1,q_2)$

$= \bigvee_{(r_1,r_2)\in Q_1\times Q_2} (\mu_1.\mu_2)(p_1,p_2),u,(r_1,r_2) \wedge (\mu_1.\mu_2)((r_1,r_2),u^{-1}uvv^{-1}v,(q_1,q_2))$

$$= \bigvee_{(r_1,r_2)\in Q_1\times Q_2} (\mu_1.\mu_2)(p_1,p_2),u,(r_1,r_2)\wedge(\mu_1.\mu_2)((r_1,r_2),vv^{-1}vu^{-1}u,(q_1,q_2))$$

$$= (\mu_1.\mu_2)((p_1,p_2),uvv^{-1}vu^{-1}u,(q_1,q_2))$$

$$= \bigvee_{(r_1,r_2)\in Q_1\times Q_2} (\mu_1.\mu_2)((p_1,p_2),uvv^{-1},(r_1,r_2)\wedge(\mu_1.\mu_2)((r_1,r_2),vu^{-1}u,(q_1,q_2))$$

$$= \bigvee_{(r_1,r_2)\in Q_1\times Q_2} (\mu_1.\mu_2)(p_1,p_2),uvv^{-1},(r_1,r_2)\wedge(\mu_1.\mu_2)((r_1,r_2),u^{-1}uv,(q_1,q_2))$$

$$= (\mu_1.\mu_2)((p_1,p_2),uvv^{-1}u^{-1}uv,(q_1,q_2))$$

$$= (\mu_1.\mu_2)((p_1,p_2),ww^{-1}w,(q_1,q_2)) \quad \text{for every } (p_1,p_2),(q_1,q_2)\in Q_1\times Q_2.$$

Similarly we can prove that

$$(\mu_1.\mu_2)((p_1,p_2),w^{-1},(q_1,q_2)) = (\mu_1.\mu_2)((p_1,p_2),w^{-1}ww^{-1},(q_1,q_2)).$$

Case 2. For $w \in (X_1\cup X_2)^{-1^*}$ the result follows as in the above case since $(X_1\cup X_2)^{-1^*} = (X_1^{-1}\cup X_2^{-1})^*$.

Case 3. Let $w \in ((X_1\cup X_2)\cup(X_1\cup X_2)^{-1})^*$. Then by the theorem there exist $u\in(X_1\cup X_2)^*, v\in(X_1\cup X_2)^{-1^*}$ such that $(\mu_1.\mu_2)((p_1,p_2),w,(q_1,q_2)) = (\mu_1.\mu_2)((p_1,p_2),uv,(q_1,q_2))$ and using case 1 and case 2 we get
$u_1\in X_1^*, u_2\in X_2^*, v_1\in X_1^{-1^*}, v_2\in X_2^{-1^*}$ such that

$$(\mu_1.\mu_2)((p_1,p_2),w,(q_1,q_2))$$

$$= (\mu_1.\mu_2)((p_1,p_2),u_1u_2v_1v_2,(q_1,q_2))$$

$$= \bigvee_{(r_1,r_2)\in Q_1\times Q_2} (\mu_1.\mu_2)((p_1,p_2),u_1u_2,(r_1r_2)\wedge(\mu_1.\mu_2)((r_1,r_2),v_1v_2,(q_1,q_2))$$

$$= \bigvee_{(r_1,r_2)\in Q_1\times Q_2} (\mu_1(p_1,u_1,r_1)\wedge\mu_2(p_2,u_2,r_2))\wedge(\mu_1(r_1,v_1,q_1)\wedge\mu_2(r_2,v_2,q_2))$$

$$= \bigvee_{(r_1,r_2)\in Q_1\times Q_2} (\mu_1(p_1,u_1u_1^{-1}u_1,r_1)\wedge\mu_2(p_2,u_2u_2^{-1}u_2,r_2)) \wedge(\mu_1(r_1,v_1v_1^{-1}v_1,q_1)\wedge\mu_2(r_2,v_2v_2^{-1}v_2,q_2)$$

$$= \bigvee_{(r_1,r_2\in Q_1\times Q_2} (\mu_1.\mu_2)((p_1,p_2),u_1u_1^{-1}u_1u_2u_2^{-1}u_2,(r_1,r_2))\wedge\mu_1.\mu_2((r_1,r_2),v_1v_1^{-1}$$

$$v_1 v_2 v_2^{-1} v_2, (q_1, q_2))$$

$$= \bigvee_{(r_1, r_2 \in Q_1 \times Q_2} (\mu_1.\mu_2)((p_1, p_2), u_1 u_2 u_2^{-1} u_1^{-1} u_1 u_2, (r_1 r_2) \wedge (r_1, r_2), v_1 v_2$$

$$v_2^{-1} v_1^{-1} v_1 v_2, (q_1, q_2))$$

$$= \bigvee_{(r_1, r_2 \in Q_1 \times Q_2} (\mu_1.\mu_2)((p_1, p_2), uu^{-1}u, (r_1, r_2) \wedge \mu_1.\mu_2((r_1, r_2), vv^{-1}v, (q_1, q_2))$$

$$= (\mu_1.\mu_2)((p_1, p_2), uu^{-1}uvv^{-1}v, (q_1, q_2)$$

$$= (\mu_1.\mu_2)((p_1, p_2), uvv^{-1}u^{-1}uv, (q_1, q_2))$$

$$= (\mu_1.\mu_2)((p_1, p_2), ww^{-1}w, (q_1, q_2)) \qquad \text{for all } (p_1, p_2), (q_1, q_2) \in Q_1 \times Q_2.$$

Similarly we can prove that

$$(\mu_1.\mu_2)((p_1, p_2), w^{-1}ww^{-1}, (q_1, q_2)) = (\mu_1.\mu_2)((p_1, p_2), w^{-1}, (q_1, q_2).$$

Thus the Cartesian product $M_1.M_2$ is an inverse fuzzy automaton. □

Theorem 3.7.2. *Let $M_1 = (Q_1, \tilde{X}_1, \mu_1, i_1, \tau_1)$, $M_2 = (Q_2, \tilde{X}_2, \mu_2, i_2, \tau_2)$ be two fuzzy automata. If their Cartesian product $M_1.M_2$ is an inverse fuzzy automaton then M_1 and M_2 are inverse fuzzy automata*

Proof. Suppose that $M_1.M_2$ is an inverse fuzzy automaton. Then for every $w \in (X_1 \tilde{\cup} X_2)^*$ there exist a unique $w^{-1} \in (X_1 \tilde{\cup} X_2)^*$ such that $\mu_1.\mu_2((p_1, p_2), ww^{-1}w, (q_1, q_2)) = \mu_1.\mu_2((p_1, p_2), w, (q_1, q_2))$ and $\mu_1.\mu_2((p_1, p_2), w^{-1}ww^{-1}, (q_1, q_2)) = \mu_1.\mu_2((p_1, p_2), w^{-1}, (q_1, q_2))$ for every $(p_1, p_2), (q_1, q_2) \in Q_1 \times Q_2$.
Let $p, q \in Q_1$ and $x \in X_1^*$.
Now, $\mu_1(p, x, q) = \mu_1.\mu_2((p, p'), x, (q, q'))$ for some $p' = q' \in Q_2$ and since there exist a unique $x^{-1} \in X_1 \tilde{\cup} X_2{}^*$ such that
$\mu_1.\mu_2((p, p'), x, (q, q') = \mu_1.\mu_2((p, p'), xx^{-1}x, (q, q')) = \mu_1(p, xx^{-1}x, q)$,
we get $\mu_1(p, x, q) = \mu_1(p, xx^{-1}x, q)$.

Clearly $x^{-1} \in \tilde{X_1}^*$. Also we can prove $\mu_1(p, x^{-1}, q) = \mu_1(p, x^{-1}xx^{-1}, q)$. So M_1 is an inverse fuzzy automaton.
Similarly we can prove that M_2 is an inverse fuzzy automaton.

□

3.8 Properties of inverse fuzzy languages

A fuzzy language on an alphabet $\tilde{X}$ is inverse if and only if for every $x \in \tilde{X}^*$, $\lambda(uxx^{-1}xv) = \lambda(uxv)$ and $\lambda(ux^{-1}xx^{-1}v) = \lambda(ux^{-1}v)$ for every $u, v \in \tilde{X}^*$(Theorem 3.5.1). Let us denote the family of all inverse fuzzy languages on an alphabet $\tilde{X}$ as IFL.

Lemma 3.8.1. IFL is closed under quotients.

Proof. Let $\lambda_1, \lambda_2 \in$ IFL. Let $x, u, v \in \tilde{X}^*$.
Then $\lambda_1(uxx^{-1}xv) = \lambda_1(uxv)$ and $\lambda_1(ux^{-1}xx^{-1}v) = \lambda_1(ux^{-1}v)$. Then,

$$\begin{aligned}
(\lambda_2^{-1}\lambda_1)(uxx^{-1}xv) &= \bigvee_{v_1 \in \tilde{X}^*} \{\lambda_1(v_1uxx^{-1}xv) \wedge \lambda_2(v_1)\} \\
&= \bigvee_{v_1 \in \tilde{X}^*} \{\lambda_1(v_1u)xx^{-1}xv \wedge \lambda_2(v_1)\} \\
&= \bigvee_{v_1 \in \tilde{X}^*} \{\lambda_1(v_1uxv) \wedge \lambda_2(v_1)\} \\
&= \lambda_2^{-1}\lambda_1(uxv)
\end{aligned}$$

and

$$
\begin{aligned}
(\lambda_2^{-1}\lambda_1)(ux^{-1}xx^{-1}v) &= \bigvee_{v_1\in\tilde{X}^*}\{\lambda_1(v_1ux^{-1}xx^{-1}v)\wedge\lambda_2(v_1)\}\\
&= \bigvee_{v_1\in\tilde{X}^*}\{\lambda_1(v_1u)x^{-1}xx^{-1}v\wedge\lambda_2(v_1)\}\\
&= \bigvee_{v_1\in\tilde{X}^*}\{\lambda_1(v_1ux^{-1}v)\wedge\lambda_2(v_1)\}\\
&= \lambda_1^{-1}\lambda_2(ux^{-1}v).
\end{aligned}
$$

Similarly we can prove that

$\lambda_1\lambda_2^{-1}(uxx^{-1}xv)=\lambda_1\lambda_2^{-1}(uxv)$ and $\lambda_1\lambda_2^{-1}(ux^{-1}xx^{-1}v)=\lambda_1\lambda_2^{-1}(ux^{-1}v)$.

Thus $\lambda_1^{-1}\lambda_2$ and $\lambda_2\lambda_1^{-1}\in$ IFL.

□

Lemma 3.8.2. IFL is closed under multiplication by constants.

Proof. Let λ be an inverse fuzzy language on $\tilde{X}$ and let $x\in\tilde{X}^*$.
Then $\lambda(uxx^{-1}xv)=\lambda(uxv)$ and $\lambda(ux^{-1}xx^{-1}v)=\lambda(ux^{-1}v)\ \ \forall u,v\in\tilde{X}^*$.
Let $c\in[0,1]$. Then
$(c\lambda)(uxx^{-1}xv)=c.\lambda(uxx^{-1}xv)=c.\lambda(uxv)=(c\lambda)(uxv)$ and
$(c\lambda)(ux^{-1}xx^{-1}v)=c.\lambda(ux^{-1}xx^{-1}v)=c.\lambda(ux^{-1}v)=(c\lambda)(ux^{-1}v)$.
Thus $c\lambda\in$ IFL.

□

Theorem 3.8.1. *If λ is an inverse fuzzy language on $\tilde{X}$, then $\forall c\in[0,1]$, λ_c is an inverse language on $\tilde{X}$.*

Proof. Let $M=(Q,\tilde{X},\mu,i,\tau)$ be an inverse fuzzy automaton recognizing λ. Then for every $x\in\tilde{X}^*$, $\mu(p,x,q)=\mu(p,xx^{-1}x,q)$ $\forall p,q\in Q$.
Let $c\in\lambda_c$ and let $uxv\in\lambda_c$. Then $\lambda(uxv)\geqslant c$.

$$\begin{aligned}
\lambda(uxx^{-1}xv) &= \bigvee_{p,q\in Q} i(p) \wedge \mu(p, uxx^{-1}xv, q) \wedge \tau(q) \\
&= \bigvee_{p,q\in Q} i(p) \wedge (\bigvee_{r,r'\in Q} \mu(p,u,r) \wedge \mu(r, xx^{-1}x, r') \wedge \mu(r', v, q)) \wedge \tau(q) \\
&= \bigvee_{p,q\in Q} i(p) \wedge (\bigvee_{r,r'\in Q} \mu(p,u,r) \wedge \mu(r, x, r') \wedge \mu(r', v, q)) \wedge \tau(q) \\
&= \bigvee_{p,q\in Q} i(p) \wedge \mu(p, uxv, q) \wedge \tau(q) \\
&= \lambda(uxv)
\end{aligned}$$

Thus $\lambda(uxv) \geqslant c$ iff $\lambda(uxx^{-1}xv) \geqslant c$.

ie, $uxv \in \lambda_c$ iff $uxx^{-1}xv \in \lambda_c$.

Similarly, we can prove that $ux^{-1}v \in \lambda_c$ iff $ux^{-1}xx^{-1}v \in \lambda_c$.

□

Theorem 3.8.2. *IFL is not closed under inverse homomorphic images.*

Proof. Let $\tilde{X_1} = \{a, b\}, \tilde{X_2} = \{c, d\}$.
Let $\beta : \tilde{X_1} \longrightarrow \tilde{X_2}$ defined as $\beta(a) = \beta(b) = c$.
Then β can be extended to a homomorphism $\beta^* : \tilde{X_1}^* \longrightarrow \tilde{X_2}^*$.
Let λ be an inverse fuzzy language on $\tilde{X_2}$.
Then $\lambda(u'cv') = \lambda(u'cdcv')\ \forall\ u', v' \in \tilde{X_2}^*$.
Suppose ${\beta^*}^{-1}\lambda$ is an inverse fuzzy language.
Then ${\beta^*}^{-1}\lambda(uav) = \beta^{*-1}(uabav)$ for all $u, v \in \tilde{X_1}^*$.
ie, $\lambda(\beta^*(uav) = \lambda(\beta^*(uabav)$.
which implies $\lambda(\beta^*(u)c\beta^*(v)) = \lambda(\beta^*(u)ccc\beta^*(v)$ and this says the inverse is not unique which is a contradiction. So $\beta^{*-1}\lambda$ is not an inverse fuzzy language.

□

Theorem 3.8.3. *IFL is not a variety of fuzzy languages.*

Proof. A collection of fuzzy languages is a variety if it is closed under finite boolean operations, homomorphic and inverse homomorphic images, quotients, multiplication by constants.

□

An inverse fuzzy language is defined as a regular fuzzy language with its syntactic monoid is an inverse monoid. A characterization for an inverse monoid by Wagner is that a monoid is an inverse monoid iff it is regular and any two idempotents commute each other. It is also proved that a monoid is regular iff every $\mathscr{L}$-class ($\mathscr{R}$-class) contains an idempotent. Thus a fuzzy language is an inverse fuzzy language then idempotents in the syntactic monoid commute each other and every $\mathscr{L}$-class ($\mathscr{R}$-class) contains an idempotent. This property is used to prove some results on inverse fuzzy languages.

Proposition 1. Let λ be an inverse fuzzy language on $\tilde{X}$. Let $[e]$ be an idempotent in $M(\lambda)$ then $\mu(p, xe, q) \leqslant \mu(p, x, q)$ and $\mu(p, ex, q) \leqslant \mu(p, x, q)$ for all $p, q \in Q, x \in \tilde{X}^*$.

Proof. Since λ is an inverse fuzzy language every element of $M(\lambda)$ acts as one-one partial fuzzy transformations on Q and idempotents in $M(\lambda)$ can be considered as fuzzy matrices with nonzero entries only in the diagonal and so T_e acts as a subidentity on Q.
Thus $\mu(p, e, q) \neq 0$ if $p = q$ and $= 0$ if $p \neq q$.

Now,

$$\begin{aligned}\mu(p, xe, q) &= \bigvee_{q' \in Q} \mu(p, x, q') \wedge \mu(q', e, q) \\ &= \mu(p, x, q) \wedge \mu(q, e, q) \\ &\leqslant \mu(p, x, q).\end{aligned}$$

Similarly, $\mu(p, ex, q) \leqslant \mu(p, x, q)$.

□

Proposition 2. If λ is an inverse fuzzy language then for every $x, u, \in \tilde{X}^*$ there exists an $n \in N$ such that $\lambda(xu^n) \leqslant \lambda(x)$.

Proof. Since λ is regular $M(\lambda)$ is finite and for $u \in \tilde{X}^*$, $[u] \in M(\lambda)$ and since $M(\lambda)$ is finite there exist an $n \in N$ such that $[u]^n$ is an idempotent. And $[u]^n = [u^n]$. By the above proposition $\mu(p, xu^n, q) \leqslant \mu(p, x, q)$.
Let i, τ be the initial and final fuzzy state in the minimal automaton recognizing λ

$$\begin{aligned}\lambda(xu^n) &= \bigvee_{p,q \in Q} i(p) \wedge \mu(p, xu^n, q) \wedge \tau(q) \\ &\leqslant \bigvee_{p,q \in Q} i(p) \wedge \mu(p, x, q) \wedge \tau(q) \\ &= \lambda(x)\end{aligned}$$

□

Theorem 3.8.4. *A regular fuzzy language λ is an inverse fuzzy language, then,*
(1) Idempotents of $M(\lambda)$ commute
(2) $\forall x, u, y \in \tilde{X}^*$, there exist an $n \in N$ such that $\lambda(xu^n y) \leqslant \lambda(xy)$.

Proof. Let λ be an inverse fuzzy language.

Then (1) is obvious since $M(\lambda)$ is an inverse monoid and idempotents in an inverse monoid commute.
To prove (2), let $x, u, y \in \tilde{X}^*$. Then $[x], [u], [y] \in M(\lambda)$.
Since $M(\lambda)$ is a finite inverse monoid, there exist an $n \in N$ such that $[u]^n$ is an idempotent in $M(\lambda)$.

$$\begin{aligned}
\lambda(xu^n y) &= \bigvee_{p,q\in Q} i(p) \wedge \mu(p, xu^n y, q) \wedge \tau(q) \\
&= \bigvee_{p,q\in Q} i(p) \wedge (\bigvee_{q'\in Q} \mu(p, xu^n, q') \wedge \mu(q', y, q)) \wedge \tau(q) \\
&\leqslant \bigvee_{p,q\in Q} i(p) \wedge (\bigvee_{q'\in Q} \mu(p, x, q') \wedge \mu(q', y, q)) \wedge \tau(q) \\
&= \bigvee_{p,q\in Q} i(p) \wedge \mu(p, xy, q) \wedge \tau(q) \\
&= \lambda(xy).
\end{aligned}$$

□

Let λ be a regular fuzzy language of $\tilde{X}^*$. Let $\pi : \tilde{X}^* \longrightarrow M(\lambda)$ be the syntactic morphism . Let $\lambda^+ = \{x \in \tilde{X}^* : \lambda(x) > 0\}$ be the support of λ. Let $\pi(\lambda^+)$ be the syntactic image of λ.

Theorem 3.8.5. *For every regular fuzzy language the following conditions are equivalent :*
(1) $\forall x, u, y \in \tilde{X}^*$, *there exist an* $n \in N$ *such that* $\lambda(xy) \geqslant \lambda(xu^n y)$.
(2) $\forall [x], [y] \in M(\lambda)$, *and for every idempotent* $[e] \in M(\lambda)$,
$[xey] \in \pi(\lambda^+)$ *implies* $[xy] \in \pi(\lambda^+)$.

Proof. Suppose (1) is satisfied. Let $[x], [e], [y] \in M(\lambda)$ such that $[xey] \in \pi(\lambda^+)$. Since π is onto, there exist $x, u, y \in \tilde{X}^*$ such that $\pi(x) = [x]$, $\pi(y) = [y]$, $\pi(u) = [e]$.
By (a), there exist an $n \in N$ such that $\lambda(xy) \geqslant \lambda(xu^n y)$.

$\pi(xu^n y) = \pi(x)\pi(u^n)\pi(y) = [x][e]^n[y] = [x][e][y] \in \pi(\lambda^+)$ by assumption.
So $xu^n y \in \lambda^+$ and since $\lambda(xy) \geqslant \lambda(xu^n y),\ xy \in \lambda^+$.
So $[x][y] = [xy] \in \pi(\lambda^+)$.

Conversely, suppose that for every $[x], [y] \in M(\lambda)$ and for every idempotent $[e] \in M(\lambda)$, $[xey] \in \pi(\lambda^+) \Longrightarrow [xy] \in \pi(\lambda^+)$
When $\lambda(xu^n y) = 0$ for all $n \in N$, the result is clear.
Suppose $x, u, y \in \tilde{X}^*$ such that $\lambda(xu^n y) > 0$ for all $n \in N$.
ie, $xu^n y \in \lambda^+$. Then $[x], [y], [u] \in M(\lambda)$ and since $M(\lambda)$ is finite there exist a $k \in N$ such that $[u]^k$ is an idempotent say $[e]$.
Now, $[xey] = [x][e][y] = \pi(x)\pi(u^k)\pi(y) = \pi(xu^k y) \in \pi(\lambda^+)$.
So $[xy] \in \pi(\lambda^+)$ by the assumption. ie, $\pi(xy) \in \pi(\lambda^+)$ and this implies $xy \in \lambda^+$. ie, $\lambda(xy) > 0$ which says that there exist a $k \in N$ such that $\lambda(xy) \geqslant \lambda(xu^k y)$.

□

Thus we have proved the theorem,

Theorem 3.8.6. *A regular fuzzy language λ is an inverse fuzzy language, then,*
(1) Idempotents of $M(\lambda)$ commute

(2) For every $[x], [y] \in M(\lambda)$, and for every idempotent $[e] \in M(\lambda)$, $[xey] \in \pi(\lambda^+)$ implies $[xy] \in \pi(\lambda^+)$.

Proof. From theorems3.8.4 and 3.8.5 we get 3.8.6. □

Theorem 3.8.7. *Let λ is an inverse fuzzy language. Then*
(1) idempotents of $M(\lambda)$ commute.

(2) For every $x, y \in \tilde{X}^*$, there exist an idempotent $[e] \in M(\lambda)$ such that $[xey] \in \pi(\lambda^+)$ if and only if $[xy] \in \pi(\lambda^+)$.

Proof. Since the syntactic monoid of an inverse fuzzy language is an inverse monoid, (1) is obvious.
For (2), let $x \in \tilde{X}^*$. Since $M(\lambda)$ is an inverse monoid, $\mathscr{L}_{[x]}$ contains a unique idempotent say $[e]$ which is a right identity for elements of $\mathscr{L}[x]$. Then $T_x \circ T_e = T_x$. Let $(Q, \tilde{X}, \mu, i, \tau)$ be the minimal fuzzy automata recognizing λ. Suppose $y \in \tilde{X}^*$ such that $[xey] \in \pi(\lambda^+)$.

$$\text{Now, } \lambda(xey) = i \circ T_x \circ T_e \circ T_y \circ \tau$$
$$= i \circ T_x \circ T_y \circ \tau$$
$$= \lambda(xy).$$

So $\lambda(xey) > 0 \iff \lambda(xy) > 0$

ie, $[xey] \in \pi(\lambda^+)$ if and only if $[xy] \in \pi(\lambda^+)$.

□

Theorem 3.8.8. *Let $M = (Q, \tilde{X}, \mu)$ be a fuzzy automaton. Then for every $x, y \in \tilde{X}^*$ and $m, n \in N$ with $[x]^m, [y]^n$ are idempotents in $\tilde{X}^*/\theta_M$, $\mu(p, x^m y^n, q) = \mu(p, y^n x^m, q)$ for every $p, q \in Q$ if and only $\tilde{X}^*/\theta_M$ has commuting idempotents.*

Proof. Since $\tilde{X}^*/\theta_M$ is a finite semigroup, for every $[x]$ in $\tilde{X}^*/\theta_M$ there exists an $n \in N$ such that $[x]^n$ is an idempotent. Let $[x]^m, [y]^n$ be two idempotents in $\tilde{X}^*/\theta_M$. Now $\mu(p, x^m y^n, q) = \mu(p, y^n x^m, q)$ for every $p, q \in Q$ iff $[x^m y^n] = [y^n x^m]$ iff $[x^m][y^n] = [y^n][x^m]$ iff $[x]^m[y]^n = [y]^n[x]^m$ iff $\tilde{X}^*/\theta_M$ has commuting idempotents.

□

Theorem 3.8.9. *If λ is a fuzzy language. Then for every $x, y \in \tilde{X}^*$ and $m, n \in N$ such that $[x]^m, [y]^n$ are idempotents, $\lambda(ux^m y^n v) = \lambda(uy^n x^m v)$ $\forall$ $u, v \in \tilde{X}^*$, if and only if the syntactic monoid of λ has commuting idempotents.*

Proof. Since for every $x, y \in \tilde{X}^*$ and $m, n \in N$ such that $[x]^m, [y]^n$ are idempotents, $\lambda(ux^m y^n v) = \lambda(uy^n x^m v)$,
$x^m y^n P_\lambda y^n x^m$ iff $[x]^m_{P_\lambda}[y]^n_{p_\lambda} = [y]^n_{p_\lambda}[x]^m_{P_\lambda}$ iff syntactic monoid of λ has commuting idempotents.

□

Thus we have proved the theorem.

Theorem 3.8.10. *If λ is an inverse fuzzy language, then,*

(1) for every $x, y \in \tilde{X}^*$ there exist $m, n \in N$ such that
$\lambda(ux^m y^n v) = \lambda(uy^n x^m v)$ $\forall$ $u, v \in \tilde{X}^*$.

(2) $\forall x, u, y \in \tilde{X}^*$, there exist an $n \in N$ such that $\lambda(xu^n y) \leqslant \lambda(xy)$.

By Eilegnberg-type variety theorem, the collection of all fuzzy languages such that for every $x, y \in \tilde{X}^*$ and $m, n \in N$ such that $[x]^m, [y]^n$ are idempotents, $\lambda(ux^m y^n v) = \lambda(uy^n x^m v)$ $\forall$ $u, v \in \tilde{X}^*$, form a variety of fuzzy languages and the associated psuedovariety is the variety generated by finite inverse monoids.

Chapter 4

Automorphism Group of an Inverse Fuzzy Automaton

4.1 Introduction

The transition semigroup of a fuzzy finite state automaton is a subsemigroup of the semigroup of all partial fuzzy transformations on Q. Park C. H studied automata homomorphisms in [21]. For a deterministic faithful inverse fuzzy automaton the transition monoid is a submonoid of the monoid FI_Q of all one one partial fuzzy transformations on Q and this fuzzy transformations can be represented as fuzzy matrices with atmost one non zero entry in each row and column. Since every group is isomor-

Some results of this chapter are included in the following paper.
Pamy Sebastian, T. P. Johnson. : Automorphism Group of an Inverse Fuzzy Automata. Annals of Pure and Applied Mathematics, Vol.2, No.1 (2012), 67-73

phic to a subgroup of the permutation group, we can find a subgroup of the permutation group which is isomophic to the automorphism group of an inverse fuzzy automaton. In this chapter we find the automorphism groups $AUT_X(M)$ and $Aut_X(M)$ of an inverse fuzzy automaton.

4.2 Preliminaries

Let $M_1 = (Q_1, X_1, \mu_1)$ and $M_2 = (Q_2, X_2, \mu_2)$ be fuzzy finite state machines. A pair (α, β) of mappings $\alpha : Q_1 \longrightarrow Q_2$ and $\beta : X_1 \longrightarrow X_2$ is called a homomorphism, written as $(\alpha, \beta) : M_1 \longrightarrow M_2$ if

$$\mu_1(q, x, p) \leqslant \mu_2(\alpha(q), \beta(x), \alpha(p)) \ \forall \ q, p \in Q \text{ and } \ \forall x \in X_1.$$

(α, β) is called a strong homomorphism if

$$\mu_2(\alpha(q), \beta(x), \alpha(p)) = \bigvee \{\mu_1(q, x, t) | t \in Q_1, \alpha(t) = \alpha(p)\}$$

$\forall p, q \in Q_1$ and $\forall x \in X_1$. A homomorphism is said to be an isomorphism if α and β are both one-one and onto. If $X_1 = X_2$ and β is the identity map, then we write $\alpha : M_1 \longrightarrow M_2$ is a homomorphism.
If (α, β) is a strong homomorphism with α one-one, then
$\mu_2(\alpha(q), \beta(x), \alpha(p)) = \mu_1(q, x, p) \ \forall q, p \in Q_1$ and $\forall x \in X_1$ [14].
Let $M = (Q, X, \mu)$ be fuzzy finite state machine. Consider the set of all strong homomorphisms $(\alpha, \beta) : M \longrightarrow M$ denoted by $END_X(M)$ and the set of all strong isomorphisms from $M \longrightarrow M$ by $AUT_X(M)$.
$END_X(M)$ form a monoid under the operation $(\alpha_1, \beta_1) \circ (\alpha_2, \beta_2) = (\alpha_1 \circ \alpha_2, \beta_1 \circ \beta_2)$ and $AUT_X(M)$ form a group where the inverse of (α, β) is

$(\alpha^{-1}, \beta^{-1})$. Composition is associative and identity is the pair of identity maps on Q and X. If β is the identity map on X, then we denote $END_X(M)$ as $End_X(M)$ and $AUT_X(M)$ as $Aut_X(M)$. Then $End_X(M)$ is a submonoid of $END_X(M)$ and $Aut_X(M)$ is a subgroup of $AUT_X(M)$.

Definition 4.2.1. A fuzzy automaton $M = (Q, X, \mu)$ is said to be faithful if for $a, b \in X,\ \mu(q, a, p) = \mu(q, b, p)\ \forall\ p, q \in Q \Longrightarrow a = b$ [14].

4.3 Category $F - AUT$

Considering the collection of all fuzzy finite state automata as a category $F - AUT$ with objects are fuzzy automata over finite set of states and morphisms are strong homomorphisms between them. Corresponds to every fuzzy automata $M = (Q, X, \mu)$ we get a finite monoid X^*/θ_M and every finite monoid is the syntactic monoid of some fuzzy language [30].

Lemma 4.3.1. If $(\alpha, \beta) \in AUT_X(M)$ then for any $u, v \in X^*$,
$u\theta_M v \Longleftrightarrow \beta(u)\theta_M\beta(v)$.

Proof. $u\theta_M v \Longleftrightarrow \mu(q, u, p) = \mu(q, v, p),\ \forall q, p \in Q$

$$\Longleftrightarrow \mu(\alpha(q), \beta(u), \alpha(p)) = \mu(\alpha(q), \beta(v),\ \alpha(p)),\ \forall \alpha(q),\ \alpha(p) \in Q$$

$$\Longleftrightarrow \mu(q, \beta(u), p) = \mu(q, \beta(v), p),\ \forall q, p \in Q, \text{ since } \alpha \text{ is one-one}$$

$$\Longleftrightarrow \beta(u)\theta_M\beta(v).$$

□

Let M_1 and M_2 be two fuzzy automata and let (α, β) be a morphism between them. Let X_1^*/θ_{M_1} and X_2^*/θ_{M_2} be the corresponding transition monoids. Let $f_\beta : X_1^*/\theta_{M_1}$ to X_2^*/θ_{M_2} defined by $f_\beta[u]_{M_1} = [\beta(u)]_{M_2}, \ \forall u \in \ X_1^*$.

Theorem 4.3.1. *Let $M_1 = (Q_1, X_1, \mu_1)$ and $M_2 = (Q_2, X_2, \mu_2)$ be two objects in the category $F - AUT$ and let $(\alpha_1, \beta_1), (\alpha_2, \beta_2)$ be strong morphisms from M_1 onto M_2. Then f_β is a semigroup morphisms and $f_{\beta_1\beta_2} = f_{\beta_1} f_{\beta_2}$. Thus the maps f_{β_1} and $f_{\beta_2} \in HOM(X_1^*/\theta_{M_1}, X_2^*/\theta_{M_2})$.*

Proof. f_{β_1} and f_{β_2} are well defined for, let $[u] = [v]$.
Then $\mu_1(q, u, p) = \mu_1(q, v, p) \ \forall \ q, p \in Q_1$

$$\Longrightarrow \bigvee\{\mu_1(q, u, t) | t \in Q_1, \alpha(t) = \alpha(p)\}$$
$$= \bigvee\{\mu_1(q, v, t) | t \in Q_1, \alpha(t) = \alpha(p)\} \ \forall \ q, p \in Q_1$$
$$\Longrightarrow \mu_2(\alpha(q), \beta(u), \alpha(p)) = \mu_2(\alpha(q), \beta(v), \alpha(p)) \ \forall \ \alpha(q), \alpha(p) \in Q_2$$
$$\Longrightarrow [\beta(u)]_{M_2} = [\beta(v)]_{M_2}$$
$$\Longrightarrow f_\beta[u]_{M_1} = f_\beta[v]_{M_1}.$$

So f_β is well defined.
Let $[u], [v] \in {X_1}^*/\theta_{M_1}$, where $u, v \in X^*$.

Then $f_\beta([u][v])_{M_1} = f_\beta[uv]_{M_1} = [\beta(uv)]_{M_2} = [\beta(u)\beta(v)]_{M_2}$
$$= [\beta(u)]_{M_2}[\beta(v)]_{M_2} = f_\beta[u]_{M_1} f_\beta[v]M_1.$$

So f_β is a semigroup morphism and
$f_{\beta_1\beta_2}[u]_{M_1} = [\beta_1\beta_2(u)]_{M_2} = f_{\beta_1}[\beta_2(u)]_{M_1} = f_{\beta_1} f_{\beta_2}[u]_{M_1}$.

□

We can define a covariant functor F between the category of fuzzy automata and the category of finite semigroups as $F(M) = X^*/\theta_M$ and $F(\alpha, \beta) = f_\beta$ for $(\alpha, \beta) \in HOM(M_1, M_2)$. The set of all inverse fuzzy automata form a full subcategory of $F - AUT$ and F as defined above is a covariant functor from this category to the category of finite inverse monoids which is a subcategory of finite monoids.

Theorem 4.3.2. *Let $M = (Q, X, \mu)$ be a faithful fuzzy automata. Let X^*/θ_M be the transition monoid. Consider $AUT(X^*/\theta_M)$ of all automorphisms on X^*/θ_M. Let $h : AUT_X(M) \longrightarrow AUT(X^*/\theta_M)$ be a map defined by $h(\alpha, \beta) = f_\beta$. Then h is a group homomorphism and $Ker\ h = Aut_X(M)$.*

Proof. $Ker\ h = \{(\alpha, \beta) \in AUT_X(M) : h(\alpha, \beta) = f_\beta\}$
where $f_\beta[u] = [u]$ for all $u \in X^*$.

$$
\begin{aligned}
[\beta(u)] = [u] &\Longleftrightarrow \beta(u)\theta_M u \\
&\Longleftrightarrow \mu(p, \beta(u), q) = \mu(p, u, q)\ \forall\ p, q \in Q, u \in X \\
&\Longleftrightarrow \beta(u) = u\ \forall u \in X \\
&\Longleftrightarrow \beta \text{ is the identity map on } X.
\end{aligned}
$$

Thus $Ker\ h = Aut_X(M)$.

$\square$

Corollary 1. By homomorphism theorem for groups $AUT_X(M)/Aut_X(M)$ is isomorphic to a subgroup of $AUT(X^*/\theta_M)$.

4.4 Automorphism group of a deterministic faithful inverse fuzzy automaton

Consider the set of all one-one partial fuzzy transformations on Q denoted as FI_Q. For each $\nu \in FI_Q$ there exist a unique inverse $\nu^{-1} \in FI_Q$ such that $\nu^{-1}(p,q) = \nu(q,p), \forall q \in Dom(\nu),\ p \in Q$. We can consider FI_Q as a collection of fuzzy matrices of cardinality $|Q|$ with atmost one nonzero entry in each row and column. This in an inverse monoid under the max-min operation of fuzzy matrices. For every matrix A in FI_Q, there exist another matrix B in FI_Q such that $ABA = A$. Here B is the transpose of the fuzzy matrix corresponding to ν. The transition monoid $\tilde{X}^*/\theta_M$ of an inverse fuzzy automaton M is a subinverse monoid of FI_Q. Consider $N(\tilde{X}^*/\theta_M) = \{\nu \in FI_Q : \nu \circ \tilde{X}^*/\theta_M \circ \nu^{-1} = \tilde{X}^*/\theta_M\}$ and $C(\tilde{X}^*/\theta_M) = \{\nu \in FI_Q : \nu \circ T_a \circ \nu^{-1} = T_a\ \forall\ T_a \in \tilde{X}^*/\theta_M\}$ where the composition is the $max - min$ composition of fuzzy matrices.

Lemma 4.4.1. Let $M = (Q, X, \mu)$ be a faithful inverse fuzzy automaton and $\nu \in N(\tilde{X}^*/\theta_M)$. Then for any $T_a \in \tilde{X}^*/\theta_M$ there exist a unique $T_b \in \tilde{X}^*/\theta_M$ such that $\nu \circ T_b \circ \nu^{-1} = T_a$.

Proof. Since $\nu \in N(\tilde{X}^*/\theta_M) \implies \nu \circ (\tilde{X}^*/\theta_M) \circ \nu^{-1} = \tilde{X}^*/\theta_M$, for $T_a \in \tilde{X}^*/\theta_M$ there exist a $T_b \in \tilde{X}^*/\theta_M$ such that $\nu \circ T_b \circ \nu^{-1} = T_a$.
To prove the uniqueness suppose there exist another $T_c \in \tilde{X}^*/\theta_M$ such that $\nu \circ T_c \circ \nu^{-1} = T_a$.

Then $\nu \circ T_b \circ \nu^{-1} = \nu \circ T_c \circ \nu^{-1}$

$$\implies \nu \circ T_b \circ \nu^{-1}(q,p) = \nu \circ T_c \circ \nu^{-1}(q,p)\ \forall\ q, p \in Q.$$

$$\Longrightarrow \bigvee_{q'\in Q}\bigvee_{q''\in Q}(\nu(q,q')\wedge\mu(q',b,q'')\wedge\nu^{-1}(q'',p))$$
$$=\bigvee_{q'\in Q}\bigvee_{q''\in Q}(\nu(q,q')\wedge(\mu(q',c,q'')\wedge\nu^{-1}(q'',p)).$$

Since ν and μ has got atmost one nonzero entry in each row and column, $\nu(q,q')\wedge\mu(q',b,q'')\wedge\nu^{-1}(q'',p) = \nu(q,q')\wedge\mu(q',c,q'')\wedge\nu^{-1}(q'',p) \neq 0,$ for some $q',q''\in Q$ and 0 for all other states in Q.
$\nu(q,q')\wedge\mu(q',b,q'')\wedge\nu^{-1}(q'',p) = \nu(q,q')\wedge\mu(q',c,q'')\wedge\nu^{-1}(q'',p)$ for all $q',q''\in Q$ and so $\mu(q,b,p)=\mu(q,c,p)$ for all $q,p\in Q$.
Thus $\mu(q,b,p)=\mu(q,c,p)\ \forall\ q,p\in Q$.
$\Longrightarrow b=c,$ since M is faithfull. □

Let $N^*(\tilde{X}^*/\theta_M)=\{\nu\in N(\tilde{X}^*/\theta_M):\nu$ has excatly one 1 in each row and column $\}$ and $C^*(\tilde{X}^*/\theta_M)=\{\gamma\in C(\tilde{X}^*/\theta_M):\gamma$ has exactly one 1 in each row and column $\}$.

Theorem 4.4.1. *Let $M=(Q,X,\mu)$ be a faithful inverse fuzzy automaton, then $Aut_X(M)=C^*(\tilde{X}^*/\theta_M)$.*

Proof. Let $(\alpha,\beta)\in Aut_X(M)$. Then β is the identity map on X and α is a one-one mapping from Q onto Q satisfying
$\mu(\alpha(p),a,\alpha(q))=\mu(p,a,q)\ \forall\ p,q\in Q,a\in X$.
ie, $T_a(\alpha(p),\alpha(q))=T_a(p,q)\ \forall\ p,q\in Q,a\in X$.

$$\text{Now } \alpha\circ T_a\circ\alpha^{-1}(p,q)=\bigvee_{q'\in Q}\bigvee_{q''\in Q}\alpha(p,q')\wedge T_a(q',q'')\wedge\alpha^{-1}(q'',q)$$
$$=\bigvee_{q'\in Q}\bigvee_{q''\in Q}\alpha(p,q')\wedge T_a(q',q'')\wedge\alpha^{-1}(q'',q)$$
$$=\alpha(p,q')\wedge T_a(q',q'')\wedge\alpha(q,q'')$$
$$=T_a(\alpha(p),\alpha(q))$$

$$= T_a(p,q) \text{ since } \alpha(p,q) = \alpha(q,q'') = 1$$

$\Longrightarrow \alpha \circ T_a \circ \alpha^{-1} = T_a$. Since α is a one-one mapping from Q onto Q, it is a permutation.

$\Longrightarrow \alpha \in C^*(\tilde{X}^*/\theta_M)$.

Conversely, let $\alpha \in C^*(\tilde{X}^*/\theta_M)$.

Then $\alpha \circ T_a \circ \alpha^{-1} = T_a \; \forall \; T_a \in \tilde{X}^*/\theta_M$.

$\Longrightarrow \alpha \circ T_a \circ \alpha^{-1}(p,q) = T_a(p,q) \; \forall \; T_a \in \tilde{X}^*/\theta_M, \;\; p,q \in Q$

$\Longrightarrow \bigvee_{q' \in Q} \bigvee_{q'' \in Q} \alpha(p,q') \wedge T_a(q',q'') \wedge \alpha^{-1}(q'',q) = T_a(p,q)$

$\Longrightarrow \alpha(p,q') \wedge T_a(q',q'') \wedge \alpha(q,q'') = T_a(p,q)$ for some $q', q'' \in Q$

$\Longrightarrow T_a(\alpha(p),\alpha(q)) = T_a(p,q)$ with $\alpha(p) = q'$ and $\alpha(q) = q''$

$\Longrightarrow \mu(\alpha(p),a,\alpha(q)) = \mu(p,a,q) \; \forall \; p,q \in Q$ and also α is one-one and onto.

Thus $\alpha \in Aut_X(M)$.

$\square$

Theorem 4.4.2. *Let $M = (Q, X, \mu)$ be a faithful inverse fuzzy automaton, then $AUT_X(M) = N^*(\tilde{X}^*/\theta_M)$.*

Proof. Let $(\alpha, \beta) \in AUT_X(M)$.
Then $\mu(\alpha(p),\beta(a),\alpha(q)) = \mu(p,a,q) \; \forall \; p,q \in Q$.
Equivalently, $T_{\beta(a)}(\alpha(p),\alpha(q)) = T_a(p,q) \; \forall \; p,q \in Q$.
Let $T_a \in \tilde{X}^*/\theta_M$.

$$\alpha \circ T_a \circ \alpha^{-1}(p,q) = \bigvee_{q' \in Q} \bigvee_{q'' \in Q} \alpha(p,q') \wedge T_a(q',q'') \wedge \alpha^{-1}(q'',q)$$
$$= \alpha(p,q') \wedge T_a(q',q'') \wedge \alpha^{-1}(q'',q) \text{ for some } q',q'' \in Q$$

$$= \alpha(p,q') \wedge T_a(q',q'') \wedge \alpha(q,q'') \text{ for some } q',q'' \in Q$$

$$= T_a(q',q'')$$

(since α is a one-one mapping from Q onto $Q, \alpha(p,q') = \alpha(q,q'') = 1$)

$$= T_{\beta(a)}(p,q) \;\forall\; p,q \in Q.$$

$\Longrightarrow \alpha \circ \tilde{X}^*/\theta_M \circ \alpha^{-1} = \tilde{X}^*/\theta_M$

$\Longrightarrow \alpha \in N^*(\tilde{X}^*/\theta_M)$.

Conversely, let $\alpha \in N^*(\tilde{X}^*/\theta_M)$. Then by lemma 4.4.1 $\forall\, T_a \in \tilde{X}^*/\theta_M$ there exist a unique $T_b \in \tilde{X}^*/\theta_M$ such that $\alpha \circ T_b \circ \alpha^{-1} = T_a$.
Define $\beta : X \longrightarrow X$ as $\beta(a) = b$. Then β is a well defined bijection.
For,let $\beta(t) = \beta(u) = c$. Then $\alpha \circ T_c \circ \alpha^{-1} = T_t$ and $\alpha \circ T_c \circ \alpha^{-1} = T_u$ and so $T_t = T_u \Longrightarrow T_t(p,q) = T_u(p,q)$ for every $p,q \in Q \Longrightarrow \mu(p,t,q) = \mu(p,u,q)$ for every $p,q \in Q \Longrightarrow t = u$, since M is faithful.
β is onto since for any $b \in X, T_b \in \tilde{X}^*/\theta_M$ and by lemma there exist a unique $T_a \in \tilde{X}^*/\theta_M$ such that $\alpha \circ T_b \circ \alpha^{-1} = T_a$.
ie, there exist an $a \in X$ such that $\beta(a) = b$.
Now, (α, β) is a homomorphism for,
let $p,q \in Q, a \in X$.
$\mu(\alpha(p), \beta(a), \alpha(q)) = \mu(\alpha(p), b, \alpha(q))$ with $\alpha \circ T_b \circ \alpha^{-1} = T_a$.

ie, $T_{\beta(a)}(\alpha(p), \alpha(q)) = T_b(\alpha(p), \alpha(q))$ with $\alpha \circ T_b \circ \alpha^{-1}(p,q) = T_a(p,q)$

$\Longrightarrow \bigvee_{q' \in Q} \bigvee_{q'' \in Q} \alpha(p,q') \wedge T_b(q',q'') \wedge \alpha^{-1}(q'',q) = T_a(p,q)$

$\Longrightarrow \alpha(p,q') \wedge T_b(q',q'') \wedge \alpha(q,q'') = T_a(p,q)$

$\Longrightarrow T_b(\alpha(p), \alpha(q) = T_a(p,q)$.

$\Longrightarrow T_{\beta(a)}(\alpha(p), \alpha(q)) = T_a(p,q)$

$\Longrightarrow \mu(\alpha(p), \beta(a), \alpha(q)) = \; mu(p,a,q) \;\forall\; p,q \in Q$.

$\Longrightarrow (\alpha, \beta)$ is an isomorphism.

Thus we have proved that $(\alpha, \beta) \in AUT_X(M)$.

$\square$

Theorem 4.4.3. *$C^*(\tilde{X}^*/\theta_M)$ is a normal subgroup of $N^*(\tilde{X}^*/\theta_M)$ or equivalently $Aut_X(M)$ is a normal subgroup of $AUT_X(M)$.*

Proof. Let $\alpha \in N^*(\tilde{X}^*/\theta_M)$ and $\nu \in C^*(\tilde{X}^*/\theta_M)$ and let $T_a \in \tilde{X}^*/\theta_M$. Since $\alpha \in N^*(\tilde{X}^*/\theta_M)$, by lemma 4.4.1 there exist a unique $T_b \in \tilde{X}^*/\theta_M$ such that $\alpha \circ T_b \circ \alpha^{-1} = T_a$ and since $\nu \in C^*(\tilde{X}^*/\theta_M), \nu \circ T_b \circ \nu^{-1} = T_b$.
Then $\alpha \circ \nu \circ \alpha^{-1} \circ T_a \circ (\alpha \circ \nu \circ \alpha^{-1})^{-1}$

$$
\begin{aligned}
&= \alpha \circ \nu \circ \alpha^{-1} \circ T_a \circ \alpha \circ \nu^{-1} \circ \alpha^{-1} \\
&= \alpha \circ \nu \circ T_b \circ \nu^{-1} \circ \alpha^{-1} \\
&= \alpha \circ T_b \circ \alpha^{-1} \\
&= T_a
\end{aligned}
$$

$\Longrightarrow \alpha \circ \nu \circ \alpha^{-1} \in C^*(\tilde{X}^*/\theta_M)$.

Thus $C^*(\tilde{X}^*/\theta_M)$ is a normal subgroup of $N^*(\tilde{X}^*/\theta_M)$.

$\square$

Corollary 2. By the theorem 4.4.3 and corollary (1),
$N^*(\tilde{X}^*/\theta_M)/C^*(\tilde{X}^*/\theta_M)$ is isomorphic to a subgroup of $AUT(\tilde{X}^*/\theta_M)$.

Example 4.4.1. In example 3.3.2, the $AUT_X(M)$ and $Aut_X(M)$ are trivial subgroups of $AUT(\tilde{X}^*/\theta_M)$.
Consider another example of an inverse fuzzy automata where $Q = \{q_0, q_1, q_2\}$, $X = \{a, b\}$ with transition matrices are

$$T_a = \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0 & 0.6 \\ 0 & 0.4 & 0 \end{bmatrix} \quad \text{and } T_b = \begin{bmatrix} 0.3 & 0 & 0 \\ 0 & 0 & 0.4 \\ 0 & 0.6 & 0 \end{bmatrix} \quad .$$

The transition monoid is $\{T_\Lambda, T_a, T_{a^2}, T_{a^3}, T_b, T_{ab}, T_{ba}\}$.

$$AUT_X(M) \cong \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\}.$$

and

$$Aut_X(M) \cong \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right\}.$$

Thus $AUT_X(M) = \{(\alpha_1, \beta_1), (\alpha_2, \beta_2)\}$ where (α_1, β_1) is the identity morphism. (α_2, β_2) is the morphism $\alpha_1 : Q \longrightarrow Q, \beta_1 : X \longrightarrow X$ defined by $\alpha(q_0) = q_0, \alpha(q_1) = q_2, \alpha(q_2) = q_1, \ \ \beta(a) = b, \beta(b) = a$.

This is an automorphism on M since

$\mu(\alpha(q_0), \beta(a), \alpha(q_0)) = 0.3 = \mu(q_0, a, q_0)$

$\mu(\alpha(q_0), \beta(b), \alpha(q_0)) = 0.3 = \mu(q_0, b, q_0)$

$\mu(\alpha(q_1), \beta(a), \alpha(q_2)) = 0.6 = \mu(q_1, a, q_2)$

$\mu(\alpha(q_1), \beta(b), \alpha(q_2)) = 0.4 = \mu(q_1, b, q_2)$

and

$\mu(\alpha(q_i), \beta(a), \alpha(q_j)) = 0 = \mu(q_i, a, q_j)$

$\mu(\alpha(q_i), \beta(b), \alpha(q_j)) = 0 = \mu(q_i, b, q_j)$ for all other i, j.

Chapter 5

Fuzzy Power Automata

5.1 Introduction

Corresponding to every fuzzy automaton M, we define min-weighted and max-weighted power automata $\mathcal{P}(M)^{\wedge}$ and $\mathcal{P}(M)^{\vee}$ such that the state set is the power set $\mathcal{P}(Q)$ of the state set Q of M and study some algebraic properties of it. The membership value of the transition function between two states of $\mathcal{P}(Q)$ is the minimum(maximum) of the membership values of the transition function between the elements of the state set if the image is the set of all elements with non zero membership value of the transition function from the domain and zero otherwise. If M is a deterministic connected inverse fuzzy automaton then the transition monoid is isomorphic to a subinverse monoid of the inverse monoid of all fuzzy matrices with each row and column containing excatly one non zero entry. If the fuzzy automata is commutative then the transition monoid will also be commutative. In this chapter we prove that the fuzzy automaton and its min-weighted and max-weighted power automata have the same transition monoids and so if the fuzzy automaton is inverse and commutative, then the corresponding fuzzy power automaton has a transition monoid which is inverse and commutative.

5.2 Preliminaries

Definition 5.2.1. Let $M_1 = (Q_1, X_1, \mu_1), M_2 = (Q_2, X_2, \mu_2)$ be fuzzy finite state machines such that $Q_1 \cap Q_2 = \phi$ and $X_1 \cap X_2 = \phi$. Then the direct sum is defined as $M_1 \oplus M_2 = (Q_1 \cup Q_2, X_1 \cup X_2, \mu_1 \oplus \mu_2)$ where

$$\mu_1 \oplus \mu_2(p,a,q) = \begin{cases} \mu(p,a,q) & \text{if } p,q \in Q_1, a \in X_1 \\ \mu_2(p,a,q) & \text{if } p,q \in Q_2, a \in X_2 \\ 1 & \begin{array}{l} \text{if either } (p,a) \in Q_1 \times X_1, q \in Q_2 \\ \text{or } (p,a) \in Q_2 \times X_2, q \in Q_1 \end{array} \\ 0 & \text{otherwise.} \end{cases}$$

and

the cartesian composition is defined as $M_1.M_2 = (Q_1 \times Q_2, X_1 \cup X_2, \mu_1.\mu_2)$ where

$$(\mu_1.\mu_2)((p_1,p_2),a,(q_1,q_2)) = \begin{cases} \mu_1(p_1,a,q_1) & \text{if } a \in X_1 \text{ and } p_2 = q_2 \\ \mu_2(p_2,a,q_2) & \text{if } a \in X_2 \text{ and } p_1 = q_1 \quad [14] \\ 0 & \text{otherwise.} \end{cases}$$

5.3 Min-weighted power automata

Definition 5.3.1. Let $M = (Q, X, \mu)$ be a fuzzy automaton. Let $\mathcal{P}(Q)$ be the power set of Q. Define $\mu^\wedge : \mathcal{P}(Q) \times X \times \mathcal{P}(Q) \longrightarrow [0,1]$ as

$$\mu^\wedge(A, a, B) = \begin{cases} \bigwedge\limits_{q\in A} \bigwedge\limits_{p\in B} \mu(q, a, p) & \text{if } B = \{p : \mu(q, a, p) > 0\} \\ 0 & \text{otherwise} \end{cases}$$

for all$A \neq \phi, B \neq \phi, A, B \in \mathcal{P}(Q)$,

$\mu^\wedge(\phi, a, \phi) = 1$ and $\mu^\wedge(A, a, B) = 0$ if $A = \phi$ or $B = \phi$, for all $a \in X$. Then $\mathcal{P}^\wedge(M) = (\mathcal{P}(Q), X, \mu^\wedge)$ is called the min-weighted power automaton. We can extend $\mu^\wedge$ to $\mathcal{P}(Q) \times X^* \times \mathcal{P}(Q)$ as

$$\mu^\wedge(A, xa, B) = \bigvee_{C\in\mathcal{P}(Q)} \mu^\wedge(A, x, C) \wedge \mu^\wedge(C, a, B) \text{ where } A, B, C \in \mathcal{P}(Q).$$

M can be embedded in $\mathcal{P}^\wedge(M)$ with the isomorphism $p \longrightarrow \{p\}$.

Theorem 5.3.1. *Every mapping (α, β) of a fuzzy automaton $A = (Q_1, X_1, \mu_1)$ into a fuzzy automaton $B = (Q_2, X_2, \mu_2)$ can be extended to a mapping from $\mathcal{P}^\wedge(A)$ into $\mathcal{P}^\wedge(B)$ such that (α, β) is an isomorphism if and only if the extended map is an isomorphism.*

Proof. Consider the extension $\hat{\alpha}$ of $\alpha : Q_1 \longrightarrow Q_2$ to $\mathcal{P}(Q_1) \longrightarrow \mathcal{P}(Q_2)$ such that for $A \in \mathcal{P}(Q)$ define $\hat{\alpha}(A) = \{\alpha(q), q \in A)\}$.
If (α, β) is a homomorphism, then $(\hat{\alpha}, \beta)$ is a homomorphism, since

for $a \in X_1$, $\mu_1{}^\wedge(A, a, B) = \begin{cases} \bigwedge\limits_{q\in A} \bigwedge\limits_{p\in B} \mu_1(q, a, p) & \text{if } B = \{p : \mu_1(q, a, p) > 0\} \\ 0 & \text{otherwise.} \end{cases}$

$$\leqslant \begin{cases} \bigwedge_{q\in A} \bigwedge_{p\in B} \mu_2(\alpha(q), \beta(a), \alpha(p)) & \text{if } B = \{p : \mu_1(q,a,p) > 0\} \\ 0 & \text{otherwise} \end{cases}$$

$$= {\mu_2}^{\wedge}(\hat{\alpha}(A), \beta(a), \hat{\alpha}(B)$$

$\Longrightarrow (\hat{\alpha}, \beta)$ is a homomorphism.

Suppose (α, β) is one-one. Then $(\hat{\alpha}, \beta)$ is one-one for, let $\hat{\alpha}(A) = \hat{\alpha}(B)$ for $A, B \in \mathcal{P}(Q)$.
Then $\{\alpha(q) : q \in A\} = \{\alpha(q) : q \in B\}$. For $q \in A$, $\alpha(q) \in \hat{\alpha}(A) = \hat{\alpha}(B) \Longrightarrow \alpha(q) = \alpha(q')$ for some $q' \in B$. Since α is one-one $q = q' \Longrightarrow q' \in B$. So $A \subseteq B$. Similarly we can prove that $B \subseteq A$. Hence $A = B$. Thus $\hat{\alpha}$ is one-one and so $(\hat{\alpha}, \beta)$ is one-one. Similarly $\hat{\alpha}$ is onto since α is onto. Converse is clear since α is the restriction of $\hat{\alpha}$ to Q.

□

5.4 Max-weighted power automaton

As in the case of min-weighted power automaton, we can define max-weighted power automaton for a fuzzy automaton $M = (Q, X, \mu)$.
For $A, B \in \mathcal{P}(Q)$ define $\mu^{\vee}(A, a, B)$ as follows.

$$\mu^{\vee}(A, a, B) = \begin{cases} \bigvee_{q\in A} \bigvee_{p\in B} \mu(q, a, p) & \text{if } B = \{p : \mu(q,a,p) > 0\} \\ 0 & \text{otherwise.} \end{cases}$$

for all $A \neq \phi, B \neq \phi$ in $\mathcal{P}(Q)$,

$\mu^{\vee}(\phi, a, \phi) = 1$ and $\mu^{\vee}(A, a, B) = 0$ if $A = \phi$ or $B = \phi$, for all $a \in X$. Then $\mathcal{P}^{\vee}(M) = (\mathcal{P}(Q), X, \mu^{\vee})$ is called the max-weighted power automaton. We can extend $\mu^{\vee}$ to $\mathcal{P}(Q) \times X^* \times \mathcal{P}(Q)$ as

$$\mu^{\vee}(A, xa, B) = \bigvee_{C\in\mathcal{P}(Q)} \mu^{\vee}(A, x, C) \wedge \mu^{\vee}(C, a, B)$$

where $A, B, C \in \mathcal{P}(Q)$. M can be embedded in $\mathcal{P}^{\vee}(M)$ with the isomorphism $p \longrightarrow \{p\}$.

5.5 Some algebraic properties of fuzzy power automata

Theorem 5.5.1. *If M_1 and M_2 are fuzzy automata, then* $\mathcal{P}^{\wedge}(M_1 \oplus M_2) \cong \mathcal{P}^{\wedge}(M_1).\mathcal{P}^{\wedge}(M_2)$.

Proof. $\mathcal{P}^{\wedge}(M_1 \bigoplus M_2) = (\mathcal{P}(Q_1 \cup Q_2), X_1 \cup X_2, (\mu_1 \oplus \mu_2)^{\wedge})$ and
$\mathcal{P}^{\wedge}(M_1).\mathcal{P}^{\wedge}(M_2) = (\mathcal{P}(Q_1) \times \mathcal{P}(Q_2), X_1 \cup X_2, \mu_1^{\wedge}.\mu_2^{\wedge})$.
Define a mapping $\alpha : \mathcal{P}(Q_1) \times \mathcal{P}(Q_2) \longrightarrow \mathcal{P}(Q_1 \cup Q_2)$ as $\alpha(A, B) = A \cup B$ where $A \in \mathcal{P}(Q_1)$ and $B \in \mathcal{P}(Q_2)$ and β is the identity map on $X_1 \cup X_2$.
We claim that (α, β) is an isomorphism from $\mathcal{P}^{\wedge}(M_1).\mathcal{P}^{\wedge}(M_2) \longrightarrow \mathcal{P}^{\wedge}(M_1 \oplus M_2)$.
We have (α, β) is a isomorphism iff α and β are one-one onto and
$\mu_1{}^{\wedge}.\mu_2{}^{\wedge}((A_1, B_1), a, (A_2, B_2)) \leqslant (\mu_1 \oplus \mu_2)^{\wedge}(\alpha(A_1, B_1), a, \alpha(A_2, B_2))$
$\forall\ (A_1, B_1), (A_2, B_2) \in \mathcal{P}(Q_1) \times \mathcal{P}(Q_2)$.
Clearly α and β are one-one onto.
Let $(A_1, B_1), (A_2, B_2) \in \mathcal{P}(Q_1) \times \mathcal{P}(Q_2)$ and $a \in X_1 \cup X_2$.
Then $\alpha(A_1, B_1) = A_1 \cup B_1$ and $\alpha(A_2, B_2) = A_2 \cup B_2$

$$\mu_1 \oplus \mu_2{}^{\wedge}(\alpha(A_1, B_1), a, \alpha(A_2, B_2)) = \mu_1 \oplus \mu_2{}^{\wedge}(A_1 \cup B_1, a, A_2 \cup B_2)$$

$$= \begin{cases} \displaystyle\bigwedge_{q\in A_1\cup B_1} \bigwedge_{p\in A_2\cup B_2} \mu_1 \oplus \mu_2(q, a, p) & \text{if } A_2 \cup B_2 = \{p : \mu_1 \oplus \mu_2(q, a, p) > 0\} \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \bigwedge_{q\in A_1} \bigwedge_{p\in A_2} \mu_1(q,a,p) & \text{if } a \in X_1, A_2 = \{p : \mu_1(q,a,p) > 0\} \\ \bigwedge_{q\in B_1} \bigwedge_{p\in B_2} \mu_2(q,a,p) & \text{if } a \in X_2, B_2 = \{p : \mu_2(q,a,p) > 0\} \\ 1 & \begin{array}{l}\text{if either } q \in A_1, a \in X_1, p \in B_2 \\ \text{or } q \in B_1, a \in X_2, p \in A_2\end{array} \\ 0 & \text{otherwise.} \end{cases} \tag{5.1}$$

Now,

$\mu_1{}^{\wedge}.\mu_2{}^{\wedge}(A_1, B_1), a, (A_2, B_2)$

$$= \begin{cases} \bigwedge_{(p_1,q_1)\in(A_1,B_1)} \bigwedge_{(p_2,q_2)\in(A_2,B_2)} \mu_1.\mu_2(p_1,q_1), a, (p_2,q_2) \\ \qquad \text{if}(A_2,B_2) = \{(p_2,q_2) : \mu_1\mu_2((p_1,q_1), a, (p_2,q_2)) > 0\} \\ 0 \qquad \text{otherwise} \end{cases}$$

$$= \begin{cases} \bigwedge_{p_1\in A_1} \bigwedge_{p_2\in A_2} \mu_1(p_1,a,p_2) & \text{if } a \in X_1, A_2 = \{p_2 : \mu_1(p_1,a,p_2) > 0\} \\ \bigwedge_{q_1\in B_1} \bigwedge_{q_2\in B_2} \mu_2(q_1,a,q_2) & \text{if } a \in X_2, B_2 = \{q_2 : \mu_2(q_1,a,q_2) > 0\} \\ 0 & \text{otherwise} \end{cases} \tag{5.2}$$

From equations (5.1) and (5.2), we get

$\mu_1.\mu_2{}^{\wedge}((A_1, B_1), a, (A_2, B_2)) \leqslant \mu_1 \oplus \mu_2{}^{\wedge}(\alpha(A_1, B_1), \beta(a), \alpha(A_2, B_2))$

$\forall (A_1, A_2), (A_2, B_2) \in \mathcal{P}(Q_1) \times \mathcal{P}(Q_2), a \in X_1.$

Thus (α, β) is an isomorphism from $\mathcal{P}^{\wedge}(M_1 \oplus M_2) \longrightarrow \mathcal{P}^{\wedge}(M_1).\mathcal{P}^{\wedge}(M_2)$.

□

Theorem 5.5.2. *A fuzzy automaton M and its min-weighed power automaton $\mathcal{P}^{\wedge}(M)$ have the same transition monoids.*

Proof. The transition monoid of the fuzzy automaton is X^*/μ_M where μ_M is the congruence defined on X^* by
$a\mu_M b$ if and only if $\mu(q,a,p) = \mu(q,b,p)\ \forall\ q,p \in Q$.
$\mu_{\mathcal{P}^\wedge(M)}$ is defined on X^* by
$a\mu_{\mathcal{P}^\wedge(M)}b$ if and only if $\mu^\wedge(A,a,B) = \mu^\wedge(A,b,B)\ \forall\ A,B \in \mathcal{P}(Q)$ and the transition monoid of $\mathcal{P}^\wedge(M)$ is $X^*/\mu_{\mathcal{P}^\wedge(M)}$.
Let $[a]_{\mu_M} \in X^*/\mu_M$ and $[a]_{\mu_{\mathcal{P}^\wedge(M)}} \in X^*/\mu_{\mathcal{P}^\wedge(M)}$.
First suppose $a \in X$ and let $b \in [a]_{\mu_M}$.
Then $\mu(q,a,p) = \mu(q,b,p)\ \forall\ q,p \in Q$.
Let $A,B \neq \phi \in \mathcal{P}(Q)$ (for A or B or both equal to ϕ then it is clear that $\mu^\wedge(A,a,B) = \mu^\wedge(A,b,B)$).

$$\begin{aligned}
\mu^\wedge(A,a,B) &= \begin{cases} \bigwedge\limits_{q\in A}\bigwedge\limits_{p\in B}\mu(q,a,p) & \text{if } B = \{p : \mu(q,a,p) > 0\} \\ \quad 0 & \text{otherwise} \end{cases} \\
&= \begin{cases} \bigwedge\limits_{q\in A}\bigwedge\limits_{p\in B}\mu(q,b,p) & \text{if } B = \{p : \mu(q,b,p) > 0\} \\ \quad 0 & \text{otherwise} \end{cases} \\
&= \mu^\wedge(A,b,B)
\end{aligned}$$

$\Longrightarrow b \in [a]_{\mu^\wedge_{\mathcal{P}(M)}}$.

Thus $[a]_{\mu_M} \subseteq [a]_{\mu^\wedge_{\mathcal{P}(M)}}$.

Now let $a = a_1a_2$ where $a_1, a_2 \in X$.

$\mu^\wedge(A,a,B) = \bigvee\limits_{C\in\mathcal{P}(Q)} \mu^\wedge(A,a_1,C) \wedge \mu^\wedge(C,a_2,B)$.

If $b \in [a]_{\mu_M}, b \in [a_1a_2]_{\mu_M} = [a_1]_{\mu_M}[a_2]_{\mu_M}$. Then there exist $b_1, b_2 \in X$ such that $b = b_1b_2$ and $[b_1] \in [a_1]_{\mu_M}, [b_2] \in [a_2]_{\mu_M}$.
This implies $\mu(p,a_1,q) = \mu(p,b_1,q)$ and $\mu(p,a_2,q) = \mu(p,b_2,q)$ for all $p,q \in Q$.

$\mu^\wedge(A,a,B) = \bigvee\limits_{C\in\mathcal{P}(Q)} \mu^\wedge(A,a_1,C) \wedge \mu^\wedge(C,a_2,B)$

$$= \bigvee_{C \in \mathcal{P}(Q)} \mu^{\wedge}(A, b_1, C) \wedge \mu^{\wedge}(C, b_2, B)$$

$$= \mu^{\wedge}(A, b_1 b_2, B)$$

$$= \mu^{\wedge}(A, b, B)$$

Thus $b \in [a]_{\mu_{\mathcal{P}^{\wedge}(M)}}$ and so $[a]_{\mu_M} \subseteq [a]_{\mu_{\mathcal{P}^{\wedge}(M)}}$

Conversely, suppose $b \in [a]_{\mu^{\wedge}(\mathcal{P}(M)}$ Then
$\mu^{\wedge}(A, a, B) = \mu^{\wedge}(A, b, B)\ \forall\ A, B \in \mathcal{P}(Q)$. Take $A = \{p\}$ and $B = \{q\}$.
Then $\mu(\{p\}, a, \{q\}) = \mu(\{p\}, b, \{q\})\ \forall\ p, q \in Q$.
$\Longrightarrow b \in [a]_{\mu_M}$.
Thus $[a]_{\mu_{\mathcal{P}^{\wedge}(M)}} \subseteq [a]_{\mu_M}$.
$\Longrightarrow\ [a]_{\mu_{\mathcal{P}^{\wedge}(M)}} = [a]_{\mu_M}$.

□

Illustration 5.5.1. For the inverse fuzzy automata in example 3.3.2, the transition semigroup is the semigroup generated by T_a and T_b where $T_a = T_{aba}$ and $T_b = T_{bab}$.

$\mathcal{P}(Q)$ has elements, say ϕ, $A_1 = \{q_0\}$, $A_2 = \{q_1\}$, $A_3 = \{q_2\}$, $A_4 = \{q_0, q_1\}$, $A_5 = \{q_1, q_2\}$, $A_6 = \{q_0, q_2\}$, $A_7 = \{q_0, q_1, q_2\}$.

Now, $\bar{\mu}(\phi, a, \phi) = \bar{\mu}(\phi, b, \phi) = 1$, $\bar{\mu}(\phi, a, A_i) = \bar{\mu}(A_i, a, \phi) = \bar{\mu}(\phi, b, A_i) = \bar{\mu}(A_i, b, \phi) = 0$ for $i = 1, 2, \ldots, 7$. The other values of $\bar{\mu}(A_i, a, A_j)$ and $\bar{\mu}(A_i, b, A_j)$ can be calculated by the formula

$$\begin{cases} \bigwedge_{q \in A_i} \bigwedge_{p \in A_j} \mu(q, a, p) & \text{if } A_j = \{p : \mu(q, a, p) > 0\} \\ 0 & \text{otherwise} \end{cases}$$

The transition monoid of the min-weighted power automaton
$\mathcal{P}^{\wedge}(M) = (\mathcal{P}(Q), X, \mu^{\wedge})$ is the semigroup generated by

$$T_a = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & 0.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.3 \end{bmatrix}$$

and

$$T_b = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.3 & 0 \\ 0 & 0 & 0 & 0 & 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.3 \end{bmatrix}$$

It is easy to verify that $T_{aba} = T_a$ and $T_{bab} = T_b$.
Also,
$T(\mathcal{P}^{\wedge}(M))$ is $\{T_a, T_{a^2}, T_{a^3}, T_{a^4}, T_{a^5}, T_b, T_{b^2}, T_{ab}, T_{ba}, T_{ab^2}, T_{b^2a}, T_{a^2b}, T_{ba^2}, T_{a^2b^2}, T_{b^2a^2}, T_{ab^2a}\}$ which is same as $T(M)$

Theorem 5.5.3. *A fuzzy automaton M and its max-weighed power automaton $\mathcal{P}^{\vee}(M)$ have the same transition monoids.*

Proof is similar to the min-weighted power automata. In the same example 3.3.2, the transition semigroup of the max-weighted power automaton is the semigroup generated by

$$T_a = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.7 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.4 & 0 & 0 & 0 & 0 \\ 0 & 0.3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.7 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.4 & 0 \\ 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.7 \end{bmatrix}$$

and

$$T_b = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.3 & 0 & 0 & 0 & 0 \\ 0 & 0.7 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0.7 & 0 \\ 0 & 0 & 0 & 0 & 0.7 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0.7 \end{bmatrix}$$

Here also the transition semigroup is
$\{T_a, T_{a^2}, T_{a^3}, T_{a^4}, T_{a^5}, T_b, T_{b^2}, T_{ab}, T_{ba}, T_{ab^2}, T_{b^2a}, T_{a^2b}, T_{ba^2}, T_{a^2b^2}, T_{b^2a^2}, T_{ab^2a}\}$.

Theorem 5.5.4. *If M is a commutative fuzzy automaton then the min-weighted (max-weighted) power automaton is commutative.*

Proof. Let $M = (Q, X, \mu, i, \tau)$ be a commutative fuzzy automaton.

Then $\mu(p, xy, q) = \mu(p, yx, q)$ for all $x, y \in X^*, p, q \in Q$.
Then $[xy]_{\mu_M} = [yx]_{\mu_M}$ for all $x, y \in X^*$.
By Theorem 5.5.2, we get $[xy]_{\mu^{\wedge}_{\mathcal{P}}(M)} = [yx]_{\mu^{\wedge}_{\mathcal{P}}(M)}$.
So $\mu^{\wedge}(A, xy, B) = \mu^{\wedge}(A, yx, B)$. Similarly, we get
$\mu^{\vee}(A, xy, B) = \mu^{\vee}(A, yx, B)$

□

Theorem 5.5.5. *If $M = (Q, X, \mu)$ is a commutative inverse fuzzy automaton then the min-weighted (max-weighted) power automaton is also a commutative inverse fuzzy automaton.*

Proof. From theorems 4.4.3, 5.5.3 and 5.5.4,
$\mathcal{P}^{\wedge}(M)(\mathcal{P}^{\vee}(M))$ is a commutative inverse fuzzy automaton.

□

Definition 5.5.1. For a fuzzy automaton M and for every $H \in \mathcal{P}(Q)$, define $H^{\vee}_{T(M)} = \{[x] \in T(M), \mu^{\vee}(H, x, H) > 0\}$ and $H^{\wedge}_{T(M)} = \{[x] \in T(M), \mu^{\wedge}(H, x, H) > 0\}$.

Theorem 5.5.6. *$H^{\vee}_{T(M)}(H^{\wedge}_{T(M)})$ is a subsemigroup of $T(M)$.*

Proof. Let $[x], [y] \in H^{\vee}_{T(M)}$.
Then $\mu^{\vee}(H, x, H) > 0, \mu^{\vee}(H, y, H) > 0$.
We have

$$\begin{aligned} \mu^{\vee}(H, xy, H) &= \bigvee_{C \in \mathcal{P}(Q)} \mu^{\vee}(H, x, C) \wedge \mu^{\vee}(C, y, H) \\ &\geqslant \mu^{\vee}(H, x, H) \wedge \mu^{\vee}(H, y, H) \\ &> 0 \end{aligned}.$$

Which implies $[xy] \in H^{\vee}_{T(M)}$.

$\square$

Theorem 5.5.7. *If M is an inverse fuzzy automaton, then $H^{\vee}_{T(M)}(H^{\wedge}_{T(M)})$ is a subinverse semigroup of $T(M)$.*

Proof. We have proved that $H^{\vee}_{T(M)}$ is a subsemigroup of $T(M)$.
Let $[x] \in H^{\vee}_{T(M)}$. ie, $\mu^{\vee}(H, x, H) > 0$.
Since the transition semigroup of a fuzzy automaton and its max-weighted power automaton are isomorphic, transition semigroup of $\mathcal{P}^{\vee}(M)$ is an inverse semigroup. $\mu^{\vee}(H, x, H)$ is a diogonal element in the transition matrix T_x of $\mathcal{P}^{\vee}(M)$ and $T_{x^{-1}}$ is the transpose of T_x. So $\mu^{\vee}(H, x^{-1}, H) = \mu^{\vee}(H, x, H) > 0$.

Thus $[x]^{-1} \in H^{\vee}_{T(M)}$ and so $H^{\vee}_{T(M)}$ is an inverse semigroup.
Similarly we can prove that $H^{\wedge}_{T(M)}$ is also an inverse semigroup. $\square$

Example 5.5.1. In example 3.3.2, $H^{\vee}_{T(M)} = \{T_{a^3}, T_{ab}, T_{ba}, T_{b^2a^2}, T_{a^2b^2}, T_{ab^2a}\}$ and this is a subinverse semigroup of $T(M)$ which is the set of all idempotents in $T(M)$.

Concluding remarks and suggestions for further study

In this work we extended the notion of semigroup action on sets to semigroup action on fuzzy subsets and studied some categorical properties of $S - FSET$. More studies can be carried out in this direction. We also introduced the concept of inverse fuzzy automata and studied the algebraic properties associated with its transition matrices. Further we studied the algebraic properties of the family of inverse fuzzy languages and proved that the collection of all inverse fuzzy languages does not form a variety. We defined a family of fuzzy languages which form a variety such that the associated pseudovariety is the set of all finite monoids with commuting idempotents. We can study more on this fuzzy language. Further studies can be carried out for the topological properties of inverse fuzzy languages and for many other fuzzy languages.

Bibliography

[1] *Almeida J. (1990)* : On pseudovarieties,varieties of languages,filters of congruences,pseudoidentities and related topics, *Algebra Universalities 27, 330-350.*

[2] *Ash C. J. (1987)* : Finite semigroups with commuting Idempotents, *J.Austral.Math.Soc (Series A) 43, 81-90.*

[3] *Benjamin Steinberg (2002)* : Inverse Automata and Profinite Topologies on a free group, *Journal of Pure and Applied Algebra, 167, 341-354.*

[4] *Birkoff G. (1935)* : On the structure of abstract algebra, *Proc.Cambridge Phil.Soc.31, 433-454.*

[5] *Clifford A. H and Preston G. B. (1961)* : The Algebraic Theory of Semigroups, Volume 1, *Amer. Math. Soc.7, Providence, RI.*

[6] *Dexter C. Kozen(1997)* : Automata and Computability, *Springer-Verlag, New York, Inc.* .

[7] *Eilenberg S. (1974)(1976)* : Automata,Languages and Machines, Vol A, B, *Academic Press, New York* .

[8] *Fleck A. C. (1962)* : Isomorphism groups of automata, *J. ACM 9, 469-476.*

[9] *George J. Klir and Bo Youn (1995)* : Fuzzy Sets and Fuzzy Logic, Theory and Applications, *Prentice-Hall, Inc.*

[10] *Goguen, J .A (1967)* : L-fuzzy sets *J. Math. Anal. appl. 18, 145-174.*

[11] *Goguen, J. A. (1969)* : Categories of L-sets *Bull Amer Math Soc 75:622*

[12] *Hopcroft J.E. and Ullman J.D. (1969)* : Formal Languages and Their Relation to Automata, *Addison-Wesley, Reading,M A* .

[13] *Howie J.M. (1976)* : An introduction to semigroup theory, *Academic Press.*

[14] *John N. Moderson and Devender S. Malik (2002)* : Fuzzy Automata and Languages, Theory and Applications, *Chapman and Hall/CRC.*

[15] *John N. Moderson Kiran R. Bhutani, Azriel Rosenfield (2005)* : Fuzzy Group Theory, *Springer-Verlag Berlin Heidelberg.*

[16] *Kleene S.C. (1956)* : Representation of events in nerve nets and finite automata, *Annals of Mathematics studies, 34, 3-41.*

[17] *Lallement G. (1976)* : Semigroup and Combinatorial Applications, *Wiely and Sons.*

[18] *Lawson M. V. (1998)* : Inverse semigroups, the theory of partial symmetries, *World Scientific.*

[19] *Lee E. T. and Zadeh L. A. (1969)* : Note on fuzzy languages, *Inform. Sci., 1, 421-434.*

[20] *Myhill J. (1957)* : Finite automata and the representation of events, *Technical Note, WADD 57-624, Wright Patterson AFB, Dayton, Ohio.*

[21] *Park C.H. (1993)* : Some Remarks on the Automata-Homomorphisms *Comm. Korean Math. Soc.8, 799-809.*

[22] *Peter Linz (2001)* : An introction to Formal Languages and Automata *Jones and Bartlett Publishers.Inc*

[23] *Pin J. E. (1986)* : Varietis of formal languages, *Plenum Publishing corp.New York.*

[24] *Pin J. E. (1995)* : Finite semigroups and recognizable languages-an introduction, *Kluwer Acad. Publ., Dordrecht. 1-32.*

[25] *Rabin M.O and Scott, D. (1959)* : Finite automata and their decision problems, *Rap.Tech., IBM J. Res. and Develop.*

[26] *Romeo P.G. (2011)* : On Catergory of S-Sets and Functors *J. Algebra and Discrete St. Vol.9.*

[27] *Sergey A Solovyov. (2003)* : Some remarks on the Category SET(L), Part II, *Mathematicki Vesnik, Vol 55, 65-82.*

[28] *Sergey A Solovyov.(2004)* : Some remarks on the Category SET(L), Part III, *Glasnik Mathematicki ,Vol 39 (59), 1-20.*

[29] *Stephen J.B. (2004)* : Applications of automata theory to presentations of inverse monoids,*J.Pure and Appl. Algebra, 63 (1990), 81-112.*

[30] *Tatjana Petkovic (2005)* : Varieties of Fuzzy languages, *Proc.* 1^{st} *International Conference on Algebraic Informatics, Aristotle University of Thessaloniki.*

[31] *Walker C. L.(2003)* : Categories of fuzzy sets, *Soft Computing 8, 299-304.*

[32] *Wee W. G. and Fu K. S.(1969)* : A formulation of fuzzy automata and its application as a model of learning systems, *IEEE Trans. on Systems Science and Cybernetics, SSC-5, 215-223.*

[33] *Zadeh L. A. (1965)* : Fuzzy Sets , *Inform. and Control, 338-353.*

[34] *Zaidi S. M. A, Irfan M and Gulam Muhiuddin (2009)* : On th Category of G-Sets-I, *International Mathematical Forum 4, No. 8, 383-393.*

[35] *Zimmerman H. J. (1991)* : Fuzzy Set Theory and its applications, *Kluwer Academic Publishers, Boston.*

List of Papers accepted

- *Pamy Sebastian, T. P. Johnson.* : Semigroup Action on Fuzzy subsets and the Category of S-Fuzzy subsets. *International Review of Fuzzy Mathematics (2012), Vol.7, No.1, 27-34.*

- *Pamy Sebastian, T. P. Johnson.* : Inverse Fuzzy Automata and Inverse Fuzzy Languages, *Annals of Fuzzy Mathematics and Informatics (2013), Vol.6, No.2, 447-453.*

- *Pamy Sebastian, T. P. Johnson.* : Automorphism Group of an Inverse Fuzzy Automata. *Annals of Pure and Applied Mathematics, Vol.2, No.1 (2012), 67-73*

Paper presented

- ***Pamy Sebastian***, *T. P. Johnson.* : Some Properties of Inverse Fuzzy Languages. *International Conference on Semigroups Algebras and Operator Theory - 2014*, February 26-28, 2014, CUSAT, India.

CURRICULUM VITAE

Name :	Pamy Sebastian
Present Address :	Department of Mathematics, Cochin University of Science and Technology, Cochin, Kerala, India – 682 022.
Official Address :	Assistant Professor, Department of Mathematics, Mary Matha Arts & Science College, Mananthavady, Kerala, India – 670 645.
Permanent Address :	Punnolil House, Vemom P.O. Mananthavady, Kerala, India – 670 645.
Email :	pamyseb@gmail.com
Qualifications :	**B.Sc.** (Mathematics), 1986, Calicut University, **M.Sc.** (Mathematics), 1988, M.G.University **M.Phil.**(Mathematics), 1991, Calicut University
Research Interest :	Fuzzy Automata and Language Theory.