A QUANTILE BASED ANALYSIS OF INCOME DATA

Thesis Submitted to the
Cochin University of Science and Technology
for the Award of the Degree of

Doctor of Philosophy

under the Faculty of Science

by

SREELAKSHMI N.

Department of Statistics

Cochin University of Science and Technology

Kochi-682022

FEBRUARY 2014

To My Loving Parents

CERTIFICATE

Certified that the thesis entitled **"A quantile based analysis of income data"** is a bonafide record of work done by Smt.Sreelakshmi N. under my guidance in the Department of Statistics, Cochin University of Science and Technology and that no part of it has been included anywhere previously for the award of any degree or title.

Kochi- 22

February 2014

Prof. K. R. Muraleedharan Nair

Professor (Retd.),

Department of Statistics,

Cochin University of

Science and Technology.

DECLARATION

This thesis contains no material which has been accepted for the award of any other Degree or Diploma in any University and to the best of my knowledge and belief, it contains no material previously published by any other person, except where due references are made in the text of the thesis.

Kochi- 22 **Sreelakshmi N.**

February 2014

CERTIFICATE

Certified that all the relevant corrections and modifications suggested by the audience during presynopsis seminar and recommended by the Doctoral committee of the candidate has been incorporated in the thesis and one research article entitled "Some properties of the new Zenga curve" has been published in the referred journal "Statistica and Applicazioni", in 2012 and one article entitled "The new Zenga curve in the context of reliability analysis" has been accepted for publication in "Communication in Statistics-Theory and methods".

Kochi- 22

February 2014

Prof. K. R. Muraleedharan Nair

Professor(Retd.),

Department of Statistics,

Cochin University of

Science and Technology.

Acknowledgments

This dissertation would not have come to the form and shape it is today without the help and support of several people whom I would like to thank and to whom I am happily in debt.

First of all, I would like to thank my supervising guide, Dr. K. R. Muraleedharan Nair, Former Head and Professor, Department of Statistics, Cochin University of Science and Technology for supporting, guiding, and working with me for the last four years. I am very grateful for his patience, persistence, encouragement, and optimism. He has been very generous and, under his supervision, I have learned not only about the topics this thesis deals with but also his personal experiences helped me so much in pursuing my research.

I wish to express my sincere gratitude to Dr. N. Unnikrishnan Nair, Former Vice chancellor, CUSAT, Dr. K.C. James, Professor and Head, Department of Statistics, Dr. N. Balakrishna, Professor and Former Head, Department of Statistics, CUSAT for the support extended to me during the research period. I am also thankful to Dr. P. G. Sankaran, Professor and Dr. Asha Gopalakrishnan, Professor, Department of Statistics, CUSAT for their valuable suggestions and help, to complete the work.

I also take this opportunity to thank all the faculty members of the Department of Statistics, CUSAT for their timely advice and suggestions during the entire period of my research.

I express my earnest gratitude to the non-teaching staff, Department of Statistics, CUSAT for the co-operation and help they had rendered.

Discussions with my friends and other research scholars of the department helped me in some situations during the course. I express my sincere thanks to all of them for their valuable suggestions and help.

I remember with deep gratefulness all my former teachers.
I would like to acknowledge the Kerala state council for science, technology and environment for the financial support rendered to me for the past three years of my research.

I am deeply indebted to my beloved father, mother and my sister for their encouragement, prayers, and blessings given to me. I am also indebted to my husband for his tolerance and constant encouragement for the timely completion of the thesis.

Above all, I bow before the grace of the Almighty.

Sreelakshmi N.

Contents

Chapter 1

Introduction

Curves that measure inequality in incomes have been a topic of immense interest for more than a century ever since the work of Lorenz in 1905. A measure of income inequality is designed to provide an index that can abridge the variations prevailing in income among the individuals in a group. Different forms of curves and summary indices of inequality measurement were discussed along with their justifications through applications to real data. Dalton (1920) has set up a set of desirable properties for a good measure of income inequality. The important requirements are

1. Principle of transfers: If a portion of income is transferred from rich to poor the inequality measure should decrease.

2. Scale independence: Proportionate addition or subtraction of incomes should leave the measure unaffected.

3. Principle of normalization: The range of the measure should be in the interval $[0, 1]$ with zero for perfect equality and one for perfect inequality.

4. Principle of symmetry: Invariance of the measure to any permutation of income among income receivers.

The celebrated Lorenz curve is the most widely used and extensively studied measure of inequality. The measure of inequality is provided through a graphical representation of incomes obtained by plotting a curve with co-ordinates $(p, L(p))$ where $L(p)$ is the share of total income received by a particular percentage of lower income households. Let X be a non-negative random variable admitting an absolutely continuous distribution function $F(x)$, with finite mean μ. The Lorenz curve is defined in terms of two parametric equations,

$$p = F(x)$$

and

$$L(p) = F_1(x) = \frac{1}{\mu}\int_0^x tdF(t).$$

Later Gini (1912) proposed an index which is defined as twice the area between the Lorenz curve and the line of equality. Although several inequality measures such as coefficient of variation, variance of logarithms, Atkinson measures [Atkinson (1970)], generalized entropy measures [Rohde (2008)], Theil coefficients [Theil (1967)] etc have been in use the Lorenz curve still occupies an important role when it comes to the measurrement of income inequality.

Lorenz curve and Gini index find application in a variety of fields. They have been extensively used in connection with studies on distribution of income as described by Kakwani & Podder (1973) and Gastwirth (1971), regional disparities in the house hold consumption in India by Bhattacharya & Mahalanobis (1967), and Chatterjee (1976) and concentration of domestic manufacturing establishment output by Einhorn (1962).

Another major problem encountered in connection with the study of income data is that of finding an appropriate model followed by the observations. Even though several models are suggested to describe income data, the one which is most widely used is the Pareto distribution named after Vilfredo Pareto (1848-1923). The distribution has originated from the famous Pareto's income law which states that in all places and at all times the distribution of incomes is governed by the empirical formula

$$P(X > x) = \left(\frac{x}{\sigma}\right)^{-\alpha}; x > \sigma (> 0)$$

where σ and α are constants. With regard to income data, it has been observed that while the Pareto curve is a rather good fit at the extremities of the income range, the fit over the whole range is poor. The lognormal distribution fits well over a large part of income range, but diverges markedly at the extremities. The classic books by Arnold (1983) and Kleiber & Kotz (2003) covers most of the works on modelling and analysis of income data.

The relationship between certain measures of inequality and notions in reliability theory have reviewed much attention among researchers, recently. Chandra & Singpurwalla (1981) has pointed out some relationships that are common to reliability theory and Economics. For a review on the application of reliability ideas to modelling issues in Economics and Political Science we refer to Bhattacharjee (1993). Many recent works on inequality measures are now interpreted in the reliability framework. The works of Giorgi & Crescenzi (2001a), Pundir et al. (2005) etc proceed in this direction.

Zenga (2007) proposed a curve and index based on the conditional expectation of concerned distribution. Specifically if X is a non-negative random variable defined over $0 \leq a < b \leq \infty$, with distribution function $F(x)$, density function $f(x)$ and finite positive mean μ, the Zenga measure of inequality is defined as

$$A(x) = 1 - \frac{\mu^{-}(x)}{\mu^{+}(x)}, \tag{1.1}$$

where $\mu^{-}(x) = E(X|X \leq x)$ and $\mu^{+}(x) = E(X|X > x)$. The superiority of this measure in comparison with existing inequality measures enables one to use the same as a potential alternative in the context of measurement of income inequality.

A probability distribution can be specified in two ways namely (i) in terms of the distribution function $F(x)$ and (ii) in terms of the quantile function defined by

$$Q(p) = \begin{cases} \inf\{x : F(x) \geq p, 0 \leq p \\ \inf\{x : F(x) > 0, p = 0 \end{cases}$$

There are many properties for quantile functions that are not shared by distribution functions even though both of them convey the same information about the distribution. In many cases, quantile function provides a straight forward analysis and in many situations it permits the use of distributions which have no closed form. Further analysis using quantile functions is more mathematically tractable. In terms of quantile functions, the inequality measure (1.1) can be written as

$$\begin{aligned} I(p) &= A(Q(p)) \\ &= 1 - \tfrac{(1-p)}{p} \tfrac{\int_0^p Q(u)du}{\int_p^1 Q(u)du} \\ &= 1 - \tfrac{(1-p)}{p} \tfrac{\int_0^p Q(u)du}{\mu - \int_0^p Q(u)du} \end{aligned}$$

The Zenga curve measures the inequality between i) the poorest $p \times 100\%$ of the population and ii) the richer remaining $(1-p) \times 100\%$ part of the population by comparing the mean incomes of the two disjoint and exhaustive sub-populations. The Zenga curve ($I(p)$ curve) is functionally related to Lorenz curve through the relationship

$$I(p) = \frac{p - L(p)}{p(1 - L(p))}.$$

Zenga (2007) while concluding his work, suggested that relating to the measure $A(x)$, it is necessary to analyze its behavior for the theoretical random variables usually employed to represent income distributions. Although several representation for the Zenga curve are feasible, the representation in terms of quantile functions is more mathematically tractable. Further very little work seems to have been done in quantile frame work. Motivated by this, in the present work, we study more aspects on the Zenga measure as well as other income inequality measures using quantile function approach. We also look into the application and interpretation of the inequality measure, $I(p)$, in the reliability context.

The thesis is organized into six chapters. After the present introductory chapter, in Chapter 2 we give a brief review of the background materials needed for the discussions in subsequent chapters. In addition to a discussion on the definition and properties of quantile functions, we also provide discussions on basic reliability concepts such as hazard rate, mean residual life, reversed hazard rate, reversed mean residual life and L moments in both the distribution function frame work and in the quantile function setup. We also provide a brief review of the widely used income inequality measures, their interrelationships and their properties.

In Chapter 3, we discuss properties of the Zenga measure denoted by $I(p)$. If $Q(p)$ represents the quantile function then $I(p)$ determines the distribution through the relationship

$$Q(p) = \mu \frac{d}{dp}\Big[\frac{p(1 - I(p))}{1 - pI(p)}\Big].$$

This enables one to arrive at the distribution followed by the data through the knowledge of $I(p)$. In a practical situation, by postulating a functional form for $I(p)$, the distribution of income shall be identified uniquely. Also it is interesting to see that if we multiply the income in one population by a constant amount, the inequality measure of the resulting population is same as that of first population. Unlike Lorenz curve, $I(p)$ curve can be increasing, decreasing and non-monotonic. To study the behavior of $I(p)$ curve we propose some equivalent conditions in terms of an average inequality measure at p. Porro (2008) has shown that the ordering based on $I(p)$ curve and Lorenz curve are equivalent. We give an alternate proof for this result which uses an approach that is more general. Further some sufficient conditions to check whether one distribution has lesser inequality than another are also studied.

The different measures of inequality are studied in relation to the notions and concepts in reliability theory in chapter 4. In this chapter, we examine the possible relationships of the

Zenga curve with other inequality measures as well as reliability concepts like mean residual life function and reversed mean residual life function. Then functional relationships enable us to establish characterization results for probability distributions. Giorgi & Crescenzi (2001a) has examined how the Bonferroni curve, another measure of income inequality, can be applied in reliability theory by proving analogous relationships between the Bonferroni curve and total time on test transform. Pundir et al. (2005) provide further methodological developments on the Bonferroni curve in the reliability framework. Zenga M. (2008) represented $A(x)$ as a function of mean residual life $m(x)$ and mean waiting time $r(x)$ in the form

$$A(x) = \frac{r(x) + m(x)}{m(x) + x}$$

In terms of $m(x)$, expression for $A(x)$ becomes

$$A(x) = \frac{1}{F(x)} \left[1 - \frac{\mu}{x + m(x)} \right].$$

The interpretation of the Zenga curve in the context of reliability analysis using the quantile based approach is more realistic and this paves way for further work in this area. We also discuss how the income inequality measure $I(p)$ can be used in studying the ageing concepts parallel to the corresponding results in reliability theory. An illustration for the behavior of Zenga curve in the context of survival analysis is also provided using the survival data given in Bryson & Siddiqui (1969) which gives the survival time of 43 patients suffering chronic granulocytic leukemia. We compute the average survival time of the least fortunate $p \times 100\%$ of the patients is $I(p) \times 100\%$ lower than that of remaining $(1 - p) \times 100\%$ of the patients suffering chronic granulocytic leukemia.

Chapter 5 deals with the interpretation of the truncated L moments in the context of wealth. We first examine the relationship of L moments with other income inequality measures. Bonferroni curve is used to measure the variability in income distribution. The second L moment of reversed residual life is also successfully used as a measure of variability in the reliability theory. The behavior of Bonferroni curve need not be similar to that of second L moment of reversed residual life. However the behavior of former can be inferred from later. As an example, Bonferroni curve is directly proportional to second L moment of reversed residual life if and only if the income of the population is distributed as power distribution. Bhattacharjee (1993) has observed that the distribution of land holdings obey anti-ageing properties like DFR, DFRA, IMRL, NWUE etc. Usually in reliability based works, the ageing properties are studied using the monotonic behaviour of certain reliability concepts. Here we look at the problem from another point of view by utilizing the truncated L moments which have their own economic interpretations.

Stochastic orders enable global comparison of two distributions in terms of their characteristics. More specifically for a given characteristic A, stochastic order says that the distribution of X has lesser (greater) A than the distribution of Y when certain inequalities in terms of the characteristic holds. Nair et al. (2013) considered such stochastic orders and results relevant to reliability analysis using quantile functions. Details of other orderings, proofs of results using the distribution function approach etc. are well documented in Shaked & Shanthikumar (2007). We define the ordering based on L moment and the implications between this and ordering based on certain income inequality measures.

One of the major problems encountered when we extend a univariate concept to higher dimensions is that it can not be done in a unique way. Accordingly several extensions are possible for a univariate notion. Multi-dimensional generalization of univariate quantiles have been done by Mosler (2002) and Fernández-Ponce & Suarez-Llorens (2003). A bivariate extension of the basic reliability concepts given in Nair & Sankaran (2009) is possible using copulas which is carried out in Chapter 6. We also derive the relationships connecting the bivariate concepts and those are advantageously used to obtain characterization theorems for probability distributions.

Finally, we mention certain problems that has originated during the present study. These works shall be undertaken in a future work.

Chapter 2

Basic concepts and Review of literature

The present chapter provides a brief review of some of the existing works on quantile functions, reliability theory and stochastic ordering which are of use in the subsequent chapters. The contents include definition and properties of quantile functions, some basic concepts in reliability theory, a discussion on certain criteria for ageing and stochastic ordering as well as a review on the summary measures of income inequality.

2.1 Quantile functions

As pointed out in the introduction, representation of a probability distribution in terms of quantile functions has the advantage that it can be used in situations where conventional distribution function approach fails. In several instances, further analysis using this approach is mathematically more tractable. A study based on quantile functions thus provides simpler and clearer perspective for solving problems in statistical modelling.

Historically the idea of quantiles seems to have been originated by Galton in 1875 in connection with his study on the " law of frequency of error" published in a Philosophy magazine. However, the term quantile was introduced by Kendall (1940). Subsequently quantile based family of distributions were studied by Hastings et al. (1947), Tukey (1962), Hogben (1963) and Gilchrist (2000). The role of quantile functions in modelling and analysis of statistical data was emphasized in the work of Parzen (1979). The recent book by Nair et al. (2013) provides an extensive discussion on properties and uses of quantile functions. The formal definition of quantile function and quantile density function are given below.

Definition 2.1

Let X be a nonnegative continuous random variable defined over $-\infty < x < \infty$ with distribution function $F(x)$ and density function $f(x)$. The quantile function, denoted by $Q(p)$ is defined as

$$Q(p) = \inf\{x : F(x) \geq p\}; 0 \leq p \leq 1. \tag{2.1}$$

It may be noted that $Q(p)$ is same as $F^{-1}(p)$. Also by the strict monotonicity of $F(x)$, we have $x = Q(p)$. To describe a probability distribution one can also use the derivative of quantile function defined in (2.1), which is termed as the quantile density function.

Definition 2.2

The quantile density function associated with a probability distribution is defined as

$$q(p) = Q'(p). \tag{2.2}$$

The quantile density function is non negative and can be interpreted as the slope of quantile function. Setting $x = Q(p)$ in the probability density function, the density quantile function turns out to be $f(Q(p))$. It may be noticed that the quantile density function and the density quantile function are connected through the relationship

$$f(Q(p))q(p) = 1. \tag{2.3}$$

Gilchrist (2000) has established the following properties for the quantile function.

1. If $Q_1(p)$ and $Q_2(p)$ are quantile functions,then $Q_1(p) + Q_2(p)$ and $Q_1(p) * Q_2(p)$ are also quantile functions.

2. $Q(p)$ is nondecreasing on $(0, 1)$, with $Q[F(x)] \leq x$ for all $-\infty < x < \infty$ for which $0 < F(x) < 1$.

3. $F[Q(p)] \geq p$ for any $0 < p < 1$.

4. $Q(p)$ is continuous from the left, i.e. $Q(p^-) = Q(p)$.

5. $Q(p^+) = \inf \{x : F(x) > p\}$ so that $Q(p)$ has limits from above.

6. Any jumps of $F(x)$ are flat points of $Q(p)$ and flat points of $F(x)$ are jumps of $Q(p)$.

7. The distribution $-Q(1-p)$ is the reflection of the distribution $Q(p)$ along the line $x = 0$.

8. If $T(x)$ is a non decreasing function of x, then $T[Q(p)]$ is a quantile function. Conversely if $T(x)$ is a non increasing function of x, then $T[Q(1-p)]$ is a quantile function.

9. The quantile function for the variable ${}^1\!/_X$ is ${}^1\!/_{Q(1-p)}$.

2.2 Basic concepts in Reliability theory

The theory of reliability focuses attention on the dependability, successful operation or performance as well as the study of failure pattern of components or devices put in operation. The reliability of a component or device is the probability that it will adequately perform its specified purpose for a specified period of time under specified operating conditions. The input in reliability analysis is the data pertaining to the lifetime of a device and the main problem in this scenario is that of finding an appropriate model to represent the lifetime data. In the sequel, we give a brief review of the basic reliability concepts. For details we refer to Galambos & Kotz (1978) and Lai & Xie (2006).

2.2.1 Hazard rate

Let X be a continuous nonnegative random variable with distribution function $F(x)$, survival function $\bar{F}(x)$ and p.d.f. $f(x)$. The hazard rate of X is defined as

$$h(x) = \lim_{\delta \to 0} \frac{P\{x \leq X \leq x + \delta | X > x\}}{\delta}.$$

For small δ, $\delta h(x)$ is approximately the conditional probability of failure in the interval $(x, x+\delta)$ given that the component has survived upto time x. In the continuous case, hazard rate takes the form

$$h(x) = \frac{f(x)}{\bar{F}(x)} = -\frac{d}{dx} log\bar{F}(x). \tag{2.4}$$

The hazard rate uniquely determines the distribution through the relationship

$$\bar{F}(x) = \frac{\mu}{h(x)} e^{-\int_0^x \frac{dt}{h(t)}}.$$

The hazard rate is also known as the failure rate. In actuarial science it is known as the force of mortality and in Economics the reciprocal of hazard rate is referred to as the Mill's ratio.

2.2.2 Mean residual function

If X represents the lifetime of a component or device, $X_t = X - t | X > t, t > 0$ represents the lifetime remaining for a component which has survived upto time t with distribution function

$$\bar{F}_t(x) = \frac{\bar{F}(x+t)}{\bar{F}(t)}.$$

The mean residual life function (MRLF) represents the average life remaining for the component conditional on the event the component has survived upto time t. If X is absolutely continuous with $E(X) < \infty$, the MRLF takes the form

$$m(t) = \frac{1}{\bar{F}(t)} \int_t^\infty \bar{F}(x)dx, \tag{2.5}$$

for all t for which $\bar{F}(t) > 0$. Clearly $m(0) = \mu = E(x)$. The function $m(t)$ uniquely determines the distribution through the relationship

$$\bar{F}(t) = \frac{\mu}{m(t)} e^{-\int_0^t \frac{dx}{m(x)}}.$$

Further the following relationship exists between the hazard rate and the MRLF

$$h(t) = \frac{1 + m^{'}(t)}{m(t)}$$

provided $m(t)$ is differentiable.

2.2.3 Reversed hazard rate

Earlier studies on reliability was centered around the truncated random variable $X_t = X - t|X > t$ which represent the lifetime of the components which has attained an age of t. Subsequently works associated with the random variable $X|X \leq t$, which represents the variable pertaining to lifetimes of components which has failed before attaining an age t were originated. The works of Keilson & Sumita (1982), Block et al. (1998), Nair & Asha (2004) proceed in this direction.

For a non negative random variable X admitting an absolutely continuous distribution with respect to Lebesgue measure, the reversed hazard rate is defined as

$$\lambda(x) = \frac{\lim\limits_{\Delta \to 0} P\left\{x - \Delta < X \leq x | X \leq x\right\}}{\Delta}.$$

In the continuous case, the reversed hazard rate takes the form

$$\lambda(x) = \frac{f(x)}{F(x)}.$$

The function $\lambda(x)$ uniquely determines the distribution through the relationship

$$F(x) = e^{-\int_x^\infty \lambda(t)dt}.$$

2.2.4 Reversed mean residual life function

The random variable $x - X|X \leq x$ represents the time elapsed since the failure of a unit given that its lifetime is atmost x with the distribution function

$${}_xF(t) = \frac{F(x) - F(x-t)}{F(x)}.$$

The reversed mean residual life of X is defined as

$$\begin{aligned} r(x) &= E(x - X|X \leq x) \\ &= \tfrac{1}{F(x)} \int_0^x F(t)dt \end{aligned}$$

It has been established that $r(x)$ uniquely determines the distribution through the relationship

$$F(x) = e^{-\int_x^\infty \frac{1-r'(t)}{r(t)}dt}.$$

2.3 Quantile based reliability concepts

The definition and properties of the quantile functions were discussed in section 2.1. Quantile measures are less influenced by the extreme observations and therefore there is no need to continue the life testing experiments until the failure of all items, but only upto the failure of a percentage of items. Also there are some models which have representation only in terms of quantile functions . Distributions such as Lambda distribution which are extensively used in analyzing income data do not have a closed form for the distribution function, but has a nice form for the quantile function. In view of the above, there is scope for detailed study on quantile based reliability analysis. Nair & Sankaran (2009) has formulated the important reliability concepts using the quantile function approach which are reproduced below.

2.3.1 Hazard quantile function

Setting $x = Q(p)$ in (2.4), we get

$$H(p) = h[Q(p)] = \frac{f[Q(p)]}{1-p}.$$

Using the relation (2.3), the above equation can be written as

$$H(p) = \frac{1}{(1-p)q(p)}. \tag{2.6}$$

$H(p)$ defined by (2.6) is the hazard quantile function. $H(p)$ uniquely determines the distribution through the relationship

$$Q(p) = \int_0^p \frac{du}{(1-u)H(u)}.$$

$H(p)$ shall be interpreted as the conditional probability of failure of a unit in the next small interval of time given the survival of unit upto $100(1-p)\%$ point of the distribution.

2.3.2 Mean residual quantile function

Setting $x = Q(p)$ in (2.5), we have

$$\begin{aligned} M\left(p\right) &= m\left(Q\left(p\right)\right) \\ &= \left(1-p\right)^{-1} \textstyle\int_p^1 \left[Q\left(u\right) - Q\left(p\right)\right] du. \end{aligned} \tag{2.7}$$

The representation given in (2.7) is the mean residual quantile function. Mean residual quantile function uniquely determines the distribution through the relationship,

$$M(p) + Q(p) - \mu = \int_0^p (1-u)^{-1} M(u) du \tag{2.8}$$

or

$$Q\left(p\right) = \mu - M\left(p\right) + \int_0^p \left(1-p\right)^{-1} M\left(u\right) du.$$

$M(p)$ can be expressed in terms of $H(p)$ through the relationship

$$M\left(p\right) = \left(1-p\right)^{-1} \int_p^1 \left(H\left(u\right)\right)^{-1} du$$

or

$$\left(H\left(p\right)\right)^{-1} = M\left(p\right) - \left(1-p\right) M'\left(p\right).$$

Mean residual quantile function is the mean remaining life beyond the 100(1-p)% of the distribution.

2.3.3 Reversed hazard quantile function

Analogous to reversed hazard rate, reversed hazard quantile function is defined as

$$A\left(p\right) = \left(p\, q\left(p\right)\right)^{-1}.$$

The quantile function is uniquely determined by $A(p)$ through the relationship

$$Q(p) = \int_0^p (uA(u))^{-1} du.$$

Further

$$H(p) = (1-p)^{-1} \, p \, A(p).$$

2.3.4 Reversed mean residual quantile function

The reversed mean residual quantile function $R(p)$ has the form

$$R(p) = p^{-1} \int_0^p (Q(p) - Q(u)) \, du.$$

$R(p)$ determines the distribution uniquely through the relationship

$$Q(p) = R(p) + \int_0^p u^{-1} R(u) \, du. \tag{2.9}$$

Further, there exists the following relationships between $R(p)$, $A(p)$ and $M(p)$

$$R(p) = p^{-1} \int_0^p (A(u))^{-1} \, du.$$

or

$$((1-p) \, M(p)) = \mu + p \, R(p) - Q(p).$$

2.4 L moments

The competing alternatives to moments are the L moments which are the expected values of linear functions of order statistics. A unified theory on L moments was presented by Hosking (1990) even though the work on linear combination of order statistics was introduced by Sillitto

(1969) and Greenwood et al. (1979). One of the advantages of L moments over the conventional moments is that the existence of first L moment ensures the existence of others. Also they have generally lower sampling variances and are robust against outliers. The r^{th} L moment is defined as

$$L_r = r^{-1} \sum_{k=0}^{r-1} (-1)^k \begin{pmatrix} r-1 \\ k \end{pmatrix} E\left(X_{r-k:r}\right) \;, \quad r = 1, 2, ... \tag{2.10}$$

We have

$$\begin{aligned} E\left(X_{r:n}\right) &= \int x \, f_r\left(x\right) dx \\ &= \tfrac{n!}{r!(n-r)!} \int_0^1 u^{r-1} \left(1-u\right)^{n-r} Q_r\left(u\right) \, du. \end{aligned} \tag{2.11}$$

Using (2.11) in (2.10), we have

$$L_r = r^{-1} \sum_{k=0}^{r-1} (-1)^k \begin{pmatrix} r-1 \\ k \end{pmatrix} \frac{r!}{k!\left(r-k-1\right)!} \int_0^1 u^{r-k-1} \left(1-u\right)^k Q\left(u\right) \, du.$$

Using the binomial expansion of $(1-p)^k$ in powers of p, the expression of L_r becomes

$$L_r = \int_0^1 \sum_{k=0}^{r-1} (-1)^{r-k} \begin{pmatrix} r \\ k \end{pmatrix} \begin{pmatrix} r+k \\ k \end{pmatrix} u^k Q\left(u\right) \, du.$$

The first four L moments are

$$L_1 = \int_0^1 Q\left(u\right) \, du \; = \; \mu$$

$$L_2 = \int_0^1 \left(2u-1\right) \, Q\left(u\right) \, du$$

$$L_3 = \int_0^1 \left(6u^2 - 6u + 1\right) Q\left(u\right) \, du$$

$$L_4 = \int_0^1 \left(20u^3 - 30u^2 + 12u - 1\right) Q\left(u\right) \, du.$$

Also L_1 and L_2 represent the measures of location and spread respectively. Nair & Vineshkumar (2010) has pointed out that the study of the measures of residual life based on L moments is worthy as the L moments are more advantageous than usual moments. The authors concentrate on studying the properties of first two L moments of residual life and their importance in reliability analysis as well as Economics. Also the second L moment of residual life has found to be a better measure of variability when compared to variance residual quantile function. Yitzhaki (2003) compared the merits of mean difference and variance in the context of measuring variability. Even though the second L moment of residual life and variance residual quantile function are measures of variability, the two functions may not exhibit same kind of monotonic behaviour. The truncated random variable $X_t = X|X > t$ has survival function $\overline{F_t}\,(x) = \frac{\overline{F}(x)}{\overline{F}(t)}$, and (2.10) simplifies to

$$L_r\,(t) = \sum_{k=0}^{r-1} (-1)^k \begin{pmatrix} r-1 \\ k \end{pmatrix}^2 \int_t^\infty x \left(\frac{\overline{F}\,(t) - \overline{F}\,(x)}{\overline{F}\,(t)} \right)^{r-k-1} \left(\frac{\overline{F}\,(x)}{\overline{F}\,(t)} \right)^k \frac{f\,(x)}{\overline{F}\,(t)} dx. \quad (2.12)$$

Setting $r = 1$ in (2.12), we get

$$L_1(t) = \frac{1}{\overline{F}(t)} \int_t^\infty x f(x) dx = E\,(X\,|X > t\,).$$

$L_1(t)$ is the vitality function discussed in Kupka & Loo (1989). When $r = 2$, we get

$$\begin{aligned} L_2(t) &= \sum_{k=0}^{1} (-1)^k \begin{pmatrix} 1 \\ k \end{pmatrix}^2 \int_t^\infty x \left(\frac{\overline{F}(t) - \overline{F}(x)}{\overline{F}(t)} \right)^{1-k} \left(\frac{\overline{F}(x)}{\overline{F}(t)} \right)^k \frac{f(x)}{\overline{F}(t)} dx \\ &= L_1(t) - t - \left(\overline{F}(t)\right)^{-2} \int_t^\infty \overline{F}^2(x) dx \end{aligned}$$

Setting $F(x) = p$ and $F(t) = p$ in (2.12), we have

$$\alpha_r(p) = L_r(Q(p)) = \sum_{k=0}^{r-1} (-1)^k \begin{pmatrix} r-1 \\ k \end{pmatrix}^2 \int_p^1 \left(\frac{u-p}{1-p}\right)^{r-k-1} \left(\frac{1-u}{1-p}\right)^k \frac{Q(u)}{1-p} du \quad (2.13)$$

which is the expression for the r^{th} L moment residual quantile function. Setting $r = 1$ and $r = 2$ in (2.13), we get

$$\alpha_1(p) = (1-p)^{-1} \int_p^1 Q(u)du$$

and

$$\alpha_2(p) = (1-p)^{-2} \int_p^1 (2u-p-1)Q(u)du.$$

$\alpha_1(p)$ uniquely determines $Q(p)$ through the relationship

$$Q(u) = \alpha_1(u) - (1-u)\alpha_1'(u).$$

$\alpha_1(p)$,$\alpha_2(p)$ and $M(p)$ are interrelated as

$$M(p) = (1-p)\alpha_1'(p)$$

$$M(p) = 2\alpha_2(p) - (1-p)\alpha_2'(p)$$

and

$$\alpha_2(p) = (1-p)^{-2} \int_p^1 (1-u)M(u)du. \quad (2.14)$$

Similarly one can define L moments for ${}_tX = X|X \le t$ with distribution function $\frac{F(x)}{F(t)}; 0 < x < t$. The r^{th} L moment of ${}_tX$ has the expression

$$B_r(t) = \sum_{k=0}^{r-1} (-1)^k \begin{pmatrix} r-1 \\ k \end{pmatrix}^2 \int_0^t x \left(\frac{F(x)}{F(t)}\right)^{r-k-1} \left(1 - \frac{F(x)}{F(t)}\right)^k \frac{f(x)}{F(t)} dx$$

In particular

$$B_1(t) = \int_0^t x \frac{f(x)}{F(t)} dx = E(X|X \leq x)$$

and

$$B_2(t) = \frac{1}{F^2(t)} \int_0^t (2F(x) - F(t))\, x f(x) dx.$$

In quantile setup, the corresponding definitions are given by

$$\beta_1(p) = \frac{1}{p} \int_0^p Q(u) du$$

and

$$\beta_2(p) = \frac{1}{p^2} \int_0^p (2u - p) Q(u) du.$$

Also $\beta_2(p)$ and $R(p)$ are connected through the relationship

$$\beta_2(p) = \frac{1}{p^2} \int_0^p u R(u) du \tag{2.15}$$

Q-Q Plot

The Q-Q plot is the graph of $(Q(p_r), x_{r:n})\,, r = 1, 2, ...n$ and $p_r = \frac{r-.5}{n}$ where $x_{r:n}$ is the r^{th} ordered observation, when the observations are arranged in ascending order of magnitude. While fitting a model for a data, $Q(p_r)$ is replaced by fitted quantile function. If the fitted values of $Q(p)$ lies along the straight line that bisects the axes of co-ordinates, the model can be taken as a satisfactory one. Q-Q plot is more applicable when the sample size is not small enough to construct various classes.

2.5 Ageing concepts

In studies pertaining to ageing concepts, the problem is to examine how a component or system improves or deteriorate with age. In the reliability context, life distributions are classified into different classes based on the monotonic behaviour of the failure rate and mean residual life function. The works of Abouammoh & El-Neweihi (1986),Deshpande et al. (1986), Gupta & Kirmani (1990), Ahmad & Mugdadi (2004) proceed in this direction. There can be no ageing, positive ageing or negative ageing. Positive ageing means the residual life time of a unit decreases with the increase in the age of the unit. Negative ageing is the dual concept of positive ageing which has a beneficial effect on life of the unit as the age increases and no ageing means that the age of a component has no effect on the distribution of residual lifetime of the unit. A detailed discussion on stochastic ageing and related notions are available in Barlow & Proschan (1975) and Lai & Xie (2006). Since lifetime distributions can be classified based on the ageing properties, the study paves way for the selection of the appropriate model.

Nair & Sankaran (2009) has identified some quantile functions as suitable models for lifetime data analysis. These models do not have closed form for the distribution function. So in order to study their ageing properties, the existing definitions based on distribution functions have to be modified in an appropriate manner. To facilitate a quantile based analysis, Nair & Vineshkumar (2011) expressed the basic ageing concepts listed in Lai & Xie (2006) in terms of quantile functions. Various ageing concepts like increasing (decreasing) hazard rate-IHR(DHR), increasing(decreasing) average hazard rate-IHRA(DHRA), new better than used in hazard rate(NBUHR), new better than used in hazard rate average(NBUHRA), increasing(decreasing) mean residual life-IMRL(DMRL),increasing (decreasing) variance residual life IVRL(DVRL), new better (worse) than used-NBU(NWU) etc are presented in the paper. Mainly the ageing concepts are studied under three broad heads, those based on hazard functions, residual quantile functions and survival functions. We list the ageing concepts based on these broad heads in the distribution function setup as well as quantile setup.

2.5.1 Ageing Concepts based on hazard quantile function

The concept of increasing and decreasing failure rates for univariate distributions have been used as a useful tool in the study of failure pattern of components / devices. A random variable X or its distribution function $F(x)$ is said to belong to increasing failure rate(IFR) if its hazard rate $h(x)$ is increasing. If $h(x)$ is decreasing in x, then $F(x)$ is said to belong to the decreasing failure rate (DFR) class. If the hazard rate is neither increasing nor decreasing, the lifetime model is exponential and vice versa. In terms of conditional survival function

$$\bar{F}(x|t) = \frac{\bar{F}(x+t)}{\bar{F}(x)},$$

F is said to be IFR(DFR) if $\bar{F}(x|t)$ is decreasing(increasing) in $0 \leq t < \infty$ for each $x \geq 0$. Also F is IFR(DFR) if and only if $-\log \bar{F}(t)$ is convex (concave).

In the quantile framework, a random variable X is said to have increasing hazard quantile function IHR(decreasing hazard quantile function DHR) if and only if

$$H(p_2) \geq (\leq) H(p_1)$$

for all $p_2 \geq p_1; 0 \leq p_1, p_2 < 1$ where $H(.)$ is defined as in (2.6).

2.5.2 Ageing concepts based on mean resdiual quantile function

A random variable X with mean residual life function $m(x)$ is said to be in the increasing mean residual life or IMRL (decreasing mean residual life or DMRL) class if $m(x)$ is increasing (decreasing) in $x > 0$. In other words, F is said to be DMRL if $m(s) \geq m(t)$ for $0 \leq s \leq t$.

The equivalent definition in quantile framework is given as follows. A random variable X

with $E(X) < \infty$ is said to be IMRL(DMRL) if and only if

$$M(p_1) \geq (\leq)\, M(p_2); p_1 \geq p_2$$

where $M(.)$ is defined as in (2.7). The definition can be given equivalently in terms of quantile functions $\int_0^1 [Q(p+(1-p)u) - Q(p)]du$ is a increasing (decreasing) in p. Also if $M(p)$ is differentiable, X is IMRL(DMRL) according as $M'(p) \geq (\leq) 0$. The properties of the classes generated by monotonic mean residual life function are extensively studied by Abouammoh & El-Neweihi (1986), Ahmad & Mugdadi (2004) and Gupta & Kirmani (1990). Similarly using hazard quantile function, the ageing concept based on mean residual quantile function is defined as X is IMRL(DMRL) if and only if

$$M(p) \geq (\leq) \frac{1}{H(p)}. \tag{2.16}$$

Other ageing property involving mean residual life is used better (worse) than aged (UBA(UWA)). A random variable X is said to the UBA(UWA) class if

$$\bar{F}(x|t) \geq (\leq) \exp\left[\frac{-x}{m(\infty)}\right]; m(\infty) < \infty$$

An extension of the above class is the used better than average in expectation (UBAE) and its dual namely used worse than average in expectation(UWAE) class. A random variable X is in the UBAE(UWAE) class if and only if

$$m(x) \geq (\leq) m(\infty).$$

For more details we refer to Alzaid (1988), Willmot & Cai (2000) and Ahmad & Mugdadi

(2004). Also it has been shown that

$$DMRL \Rightarrow UBA \Rightarrow UBAE.$$

In the quantile setup, a lifetime random variable X with $M(1) = \lim_{p \to 1^-} M(p) < \infty$ is

- UBA(UWA) if and only if

$$Q(p_1 + (1 - p_1)p_2) - Q(p_1) \geq (\leq) \frac{-1}{M(1)} \log(1 - p_1)$$

  for all $0 \leq p_1, p_2 < 1$.

- UBAE(UWAE) if and only if

$$M(p) \geq M(1) \tag{2.17}$$

  for all $0 < p < 1$.

2.5.3 Concepts based on survival function

The ageing properties in this class are obtained by comparing survival function at different points of time. New better (worse) than used (NBU(NWU)) is the most cited one in this category and new better(worse) than used in expectation(NBUE(NWUE)) and harmonic new better (worse) than used in expectation(HNBUE(HNWUE)) are the classes derived from NBU(NWU). We say that X is NBU(NWU) if and only if

$$\bar{F}(x + t) \leq (\geq) \bar{F}(x)\bar{F}(t)$$

for all $x, t > 0$.

Considering the expectations,instead of comparing the residual life distribution as such will lead to new better (worse) than used in expectation (NBUE(NWUE)) class. If $E(X) < \infty$, X is said to be NBUE(NWUE) if and only if

$$\mu \geq \int_0^\infty \frac{\bar{F}(x+t)}{\bar{F}(t)} dx = m(x)$$

for all $t \geq 0$. Using mean residual quantile function $M(p)$, X is NBUE(NWUE) if and only if

$$M(p) \leq (\geq) \mu. \tag{2.18}$$

Rolski (1975) introduced the class harmonically new better(worse) than used in expectation HNBUE(HNWUE) class which is defined by the relationship

$$\int_x^\infty \bar{F}(t) dt \leq (\geq) \mu e^{-x/\mu}; x > 0.$$

Klefsjö (1982) has extensively studied the properties of HNBUE and HNWUE classes. The random variable X is in HNBUE(HNWUE) class if and only if

$$\int_p^1 (1-u) q(u) du \leq (\geq) \mu e^{-\frac{Q(p)}{\mu}} \tag{2.19}$$

or alternatively

$$\frac{\int_0^p \frac{q(u)}{M(u)} du}{\int_0^p q(u) du} \geq (\leq) \frac{1}{\mu}.$$

2.6 Stochastic Orders

Stochastic orders are useful for a global comparison of two distributions in terms of certain characteristics. Suppose we have a characteristic A. Then by stochastic order it means that the distribution F_X of X has lesser (greater) A than the distribution F_Y of another random variable Y denoted by $X \leq (\geq)_A Y$. Mainly the concept of stochastic order is used to compare the characteristics of two distributions or to assess the relative behaviour of the properties of distributions. Also the characteristic of measure should have an appropriate measure $\delta(A)$ and the comparison using stochastic order is denoted by $\delta_X(A) \leq (\geq)\delta_Y(A)$. The stochastic orders reviewed in this section include the usual stochastic order, hazard rate order, mean residual life order, reversed mean residual life order etc. The details on orderings in the distribution function set up are described in Shaked & Shanthikumar (2007).

2.6.1 Stochastic ordering using distribution function

Let X and Y be two random variables. If $P(X > x) \leq P(Y > x)$ for all $x \in (-\infty, \infty)$,then X is said to be smaller than Y in stochastic order denoted by $X \leq_{st} Y$. This can equivalently be given as

$$P(X \leq x) \geq P(Y \leq x).$$

Let $F_X(x)$ and $F_Y(x)$ be the distribution functions and $\bar{F}_X(x)$ and $\bar{F}_Y(x)$ be the reliability functions of random variables X and Y respectively. Then the following conditions are equivalent.

1. $X \leq_{st} Y$
2. $\bar{F}_X(x) \leq \bar{F}_Y(x)$ or $F_X(x) \geq F_Y(x)$ for all x.

The definition of stochastic order has been translated in terms of quantile functions by Nair et al. (2013). Let X and Y be random variables with quantile functions $Q_X(p)$ and $Q_Y(p)$ respectively. We say that X is smaller than Y in stochastic order if and only if

$$Q_X(p) \leq Q_Y(p)$$

for all p in $(0, 1)$. The usual stochastic ordering can be used either to compare the distributions of two random variables X and Y or to compare the distribution of a random variable X at two different parameter values. For more properties of usual stochastic orders, we refer to Scarsini & Shaked (1990), Ma (1997) and Müller & Stoyan (2002).

2.6.2 Hazard rate order

Let X and Y be two random variables with absolutely continuous distributions and hazard rate functions $h_X(x)$ and $h_Y(x)$ respectively. Then X is said to be smaller than Y in the hazard rate order denoted by $X \leq_{hr} Y$ if and only if

$$h_X(x) \geq h_Y(x); x \in R. \tag{2.20}$$

Also (2.20) holds if and only if

$$\frac{\bar{F}_Y(x)}{\bar{F}_X(x)} \text{ increases in } x. \tag{2.21}$$

When X and Y have absolutely continuous distributions with densities f and g respectively, (2.20) is equivalent to

$$\frac{f(x)}{\bar{F}(y)} \geq \frac{g(x)}{\bar{G}(y)}$$

for all $x \leq y$. The idea behind the comparison based on hazard rate is that when the hazard rate becomes larger the variable becomes stochastically smaller.

We now consider the definition in quantile setup. We say that $X \leq_{hr} Y$ if $H_X(p) \leq H^*_Y(p)$ where $H_X(p) = h_X(Q_X(p))$ and $H^*_Y(p) = h_Y(Q_X(p))$. In terms of quantile function $Q_X(p)$ we get similar condition corresponding to (2.21) as $X \leq_{hr} Y$ if and only if $p^{-1}F_Y(Q_X(1-p))$ is decreasing in p.

Observe that

$$X \leq_{hr} Y \Rightarrow X \leq_{st} Y,$$

but not conversely.

2.6.3 Mean residual life order and reversed mean residual life order

For the random variables X and Y with mean residual life functions $m_X(x)$ and $m_Y(x)$ respectively, we say that X is smaller than Y in the mean residual life order, denoted by $X \leq_{mrl} Y$, if and only if $m_X(x) \leq m_Y(x)$ for all x.

Also X is smaller than Y in mean residual life order if and only if

$$M_X(p) \leq M_Y^*(p) \tag{2.22}$$

where $M_X(p) = m_X(Q_X(p))$ and $M_Y^*(p) = m_Y(Q_X(p))$. (2.22) is equivalent to

$$\frac{1}{\bar{F}_Y(Q_X(p))} \int_{Q_X(p)}^{\infty} \bar{F}_Y(x)dx \geq \frac{1}{1-p} \int_p^1 (1-u)q_X(u)du.$$

Nair et al. (2013) has established the implications between stochastic ordering, hazard rate ordering, mean residual life ordering which are reproduced below.

1. If $\frac{M_X(p)}{M_Y^*(p)}$ is increasing in p, then

$$X \leq_{mrl} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st} Y.$$

2. If $\frac{M_X(p)}{M_Y^*(p)} \geq \frac{E(X)}{E(Y)}$,then

$$X \leq_{mrl} Y \Rightarrow X \leq_{st} Y$$

2.6.4 Convex order

Another type of ordering which is extensively used is the convex ordering. For the two random variables X and Y if the condition

$$E\left[\phi(x)\right] \leq E\left[\phi(y)\right] \tag{2.23}$$

holds for all convex functions $\phi : R \to R$, provided the expectations exist, then X is said to be smaller than Y in the convex order. This is denoted by $X \leq_{cx} Y$. For example, let the functions ϕ_1 and ϕ_2 are defined as $\phi_1(x) = x$ and $\phi_2(x) = -x$, ϕ_1 and ϕ_2 are convex functions. Further from (2.23)

$$X \leq_{cx} Y \Rightarrow E(X) = E(Y).$$

In terms of distribution and survival functions, if $X \leq_{cx} Y$,then

$$\int_x^{\infty} \bar{F}_X(t)dt \leq \int_x^{\infty} \bar{F}_Y(t)dt; \text{for all } x \tag{2.24}$$

and

$$\int_{-\infty}^{x} F_X(t)dt \leq \int_{-\infty}^{x} F_Y(t)dt; \text{for all } x, \tag{2.25}$$

provided the integrals exist. When $E(X) = E(Y)$, (2.24) and (2.25) are equivalent to $X \leq_{cx} Y$.

In terms of quantile functions, $X \leq_{cx} Y$ if and only if

$$\int_0^p Q_X(u)du \geq \int_0^p Q_Y(u)du$$

or

$$\int_p^1 Q_X(u)du \leq \int_p^1 Q_Y(u)du.$$

Several modifications for convex ordering have came out. One among them is the dilation order. When X and Y have finite means, we say that X is less variable than Y in dilation order if

$$[X - E(X)] \leq_{cx} [Y - E(Y)]$$

and denoted by $X \leq_{dil} Y$.

Another category of ordering is based on comparing random variables according to their location and spread.Most important among them is the increasing(decreasing) convex order. Let X and Y be two random variables such that

$$E[\phi(x)] \leq E[\phi(y)] \tag{2.26}$$

for all increasing convex[concave] functions $\phi : R \to R$ provided the expectations exist.Then X is said to be smaller than Y in the increasing convex order denoted by $X \leq_{icx} Y$. One can also define a decreasing convex order by requiring (2.26) to hold for all decreasing convex functions. Similarly monotone concave orders can be also defined.

2.7 Quantiles in higher dimensions

The definition of quantile functions in the univariate set up has been discussed in section 2.1. Several attempts are available in literature to generalize the concept to higher dimensions. Using the characterizations based on minimizing distances, Tukey (1977) defined the concept of depth function and later Averous & Meste (1997) defined the median balls, Koshevoy & Mosler (1997) defined the zonoid quantiles. Using the accumulated probability characterization, Nolan (1992) and Massé & Theodorescu (1994) defined the multivariate quantiles as half planes and the central region as convex hull. For more details on the generalizations of univariate quantiles, we refer to Serfling (2002), Mosler (2002), Koltchinskii (1997), Chen & Welsh (2002).

In multivariate quantile approach, the authors have observed the following difficulties.

1. The nonexistence of a natural ordering in n-dimensions, $n > 1$.

2. The choice of the shape of the central region for non-symmetrical distributions.

3. The nonparametric estimation of the new concepts.

Fernández-Ponce & Suarez-Llorens (2003) clears these problems by proposing the concept of quantile curves which is the multivariate quantile defined as a set of points which accumulate the same probability for a fixed orthant.

Let $\mathbf{X} = (X_1, X_2)$ be a bivariate absolutely continuous random vector and $\underset{\sim}{x} = (x_1, x_2)$ be point in R^2. Denote the four directions in the two dimensional plane denoted by $\varepsilon = (\varepsilon_1, \varepsilon_2)$ with $\varepsilon_i \in \{-1, 1\}\,; i = 1, 2$. To simplify the notation , – and + can be respectively used for -1 and +1. For example,

$$F_{\varepsilon_{-+}}(x_1, x_2) = P\left[X_1 \le x_1, X_2 \le x_2\right].$$

The p^{th} bivariate quantile curve for the direction ε denoted by $Q_X(p, \varepsilon)$ defined as

$$Q_X(p, \varepsilon) = \left\{(x_1, x_2) \in R^2 : F_\varepsilon\left(x_1, x_2\right) = p\right\}.$$

Belzunce et al. (2007) showed how the accumulated probability of the central region given by the notion of quantile curves depends on the dependence structure of underlying bivariate distribution.

The concept of copula was first introduced by Sklar (1959). Copulas are simply multivariate distribution functions whose one dimensional marginals are uniform on the interval $(0, 1)$. A copula is a function $C(u, v)$ from I^2 to I where $I^2 = \{(x_1, x_2)\,|0 < x_i < 1, i = 1, 2\}$ with the following properties.

1. $C(u,v)$ is grounded function. That means for all u, v in I,

$$C(u,0) = 0 = C(0,v)$$

2. $C(u,1) = u, C(1,v) = v$

3. For every u_1, u_2, v_1, v_2 in I such that $u_1 \leq u_2$ and $v_1 \leq v_2$,

$$C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \geq 0$$

If F is a distribution function on R^n with one dimensional distribution $F_1, F_2, ..., F_n$ .Then there is a copula C so that

$$\begin{aligned} F(x_1, ..., x_n) = P\left[F_1(X_1) \leq F_1(x_1), ..., F_n(X_n) \leq F_n(x_n)\right] \\ = \; C\left[F_1(x_1), ..., F_n(x_n)\right]. \end{aligned}$$

If F is continuous ,then F is unique and is given by

$$C\left[u_1, u_2, ..., u_n\right] = F\left[Q_1(u_1), Q_2(u_2), ..., Q_n(u_n)\right].$$

Observe that C represents the distribution function of the random variable $[u_1, u_2, ..., u_n] = F_1(x_1), F_2(x_2), ..., F_n(x_n)$ and futher

$$X =_{st} (Q_1(u_1), ..., Q_n(u_n)).$$

where *st* means the same distribution. Also the copula of independence can be expressed as the one associated with n independent variables which is given by

$$C(u_1, ..., u_n) = \prod_{i=1}^{n} u_i.$$

For more details on copula, we refer to Nelsen (1999).

The probability $F_\varepsilon(\phi(u), \psi(v))$ depends only on the copula C for the direction ε and is given by the expression

$$F_\varepsilon(\phi(u), \psi(v)) = \begin{cases} C(u,v); & \varepsilon = \varepsilon_{--} \\ u - C(u,v); & \varepsilon = \varepsilon_{-+} \\ v - C(u,v) & \varepsilon = \varepsilon_{+-} \\ 1 - u - v - C(u,v) & \varepsilon = \varepsilon_{++} \end{cases}$$

$0 \le u \le 1; 0 \le v \le 1.$

where $\phi(u)$ and $\psi(v)$ are the inverses of the marginal distribution functions $F(x_1)$ and $G(x_2)$ respectively. Belzunce et al. (2007) proposed an alternate way to express the quantile curves by means of the quantiles for the conditional distributions $[Y|X \le x]$ and $[Y|X \ge x]$ as follows,

$$\begin{aligned} Q_X(p, \varepsilon_{--}) &\to \left\{ \left(Q_X(u), Q_{Y|X \le Q_X(u)}(\tfrac{p}{u})\right) : u > p \right\}, \\ Q_X(p, \varepsilon_{+-}) &\to \left\{ \left(Q_X(u), Q_{Y|X \ge Q_X(u)}(\tfrac{p}{1-u})\right) : u < 1 - p \right\}, \\ Q_X(p, \varepsilon_{-+}) &\to \left\{ \left(Q_X(u), Q_{Y|X \le Q_X(u)}(1 - \tfrac{p}{u})\right) : u > p \right\}, \\ Q_X(p, \varepsilon_{++}) &\to \left\{ \left(Q_X(u), Q_{Y|X \ge Q_X(u)}(1 - \tfrac{p}{1-u})\right) : u < 1 - p \right\}. \end{aligned}$$

In chapter 6, we use these ideas to extend the quantile based reliability concepts given in Nair & Sankaran (2009) to the bivariate setup.

2.8 Measures of income inequality

As pointed out in the introduction, the measurement and comparison of inequality of income in different populations had been a mjor problem of interest among researchers for more than a decade. There are so many questions regarding income inequality like (i) Is the distribution of income more equal than it was in the past? (ii) Are underdeveloped countries characterized by greater inequality than advanced countries? (iii) Do taxes lead to greater equality in the distribution of income or wealth? Income inequality measures are used to answer these wide range of questions. In other words, income inequality measures are focused on the economic inequality among the participants in an economy. These measures are defined over the entire population, eventhough inequality measures are derived to measure inequality among the poor as well as among the rich.

Eventhough several inequality measures have been used, the Lorenz curve still enjoys an important place in the context of measurement of income inequality. The basic idea behind the Lorenz curve is to utilize the share of total income received by a particular percentage lower income households to arrive at a measure of inequality. Suppose we have n incomes in the population. Let these be arranged in ascending order of magnitude as $x_{1:n} \leq x_{2:n} \leq ... \leq x_{n:n}$. Then for $i = 0, 1, 2, ...n$,

$$L\left(\frac{i}{n}\right) = \frac{\sum\limits_{j=1}^{i} x_{j:n}}{\sum\limits_{j=1}^{n} x_{j:n}}$$

Then the points $\left({}^{i}\!/_{n}, L\left({}^{i}\!/_{n}\right)\right)$ are linearly interpolated to complete the Lorenz curve. That is, ${}^{i}\!/_{n} \times 100\%$ of individuals in the population shares $L\left({}^{i}\!/_{n}\right) \times 100\%$ of total income. It is a bow shaped curve below the diagonal. As the bow is more bent, the inequality increases.

If n is very large, the distribution of incomes within the population can be approximated using a continuous distribution function $F(x)$ with a density function $f(x)$. $F(x)$ can be interpreted as the proportion of individuals having incomes less than or equal to x. The first moment distribution is specified by

$$F_{(1)}(x) = \frac{\int_0^x tf(t)dt}{\int_0^\infty tf(t)dt},$$

provided the denominator is finite.

$F_{(1)}(x)$ represents the proportional share of total income of individuals having an income less than or equal to x. The Lorenz curve corresponding to distribution F can be described as the set of points $\big(F(x), F_{(1)}(x)\big)$, which is defined in the unit square. The Lorenz curve can also be defined in terms of quantile functions. This will be useful especially when the distribution function do not have a closed form.

This approach was initiated by Gastwirth (1971). Let X be a non negative random variable with finite positive mean μ. Setting $F(x) = p$, we get

$$L(p) = \frac{1}{\mu}\int_0^p Q(u)du$$

where $\mu = \int_0^1 Q(u)du$. Differentiating the above expression, we get

$$Q(p) = \mu L'(p).$$

Since any distribution can be characterized by the quantile function, it is clear from the above expression that Lorenz curve uniquely determines the distribution.

Lorenz curve has the following properties.

1. $L(p)$ is continuous on $[0, 1]$, with $L(0) = 0$ and $L(1) = 1$.

2. $L(p)$ is increasing.

3. $L(p)$ is convex.

Also any function satisfying the above properties is the Lorenz curve associated with a statistical distribution. Lorenz curve itself can be considered as a distribution function and the moments of Lorenz curve distribution can be used as measures of income inequality.(Aaberge (2000)).

There exists two approaches for the construction of Lorenz curve models. First approach consists of, starting from an income distribution function and, obtaining corresponding Lorenz curve by using above mentioned representations. A second approach consists of selecting simple curves satisfying the required conditions for the Lorenz curve. This approach usually leads to complicated distribution functions, but may be flexible enough for empirical Lorenz curves.

The works of Kakwani & Podder (1973),Gupta (1984), Chotikapanich (1993), Rohde (2009) Sarabia et al. (2010b) proceed in this direction.

Sarabia (1997) obtained a hierarchy of Lorenz curves based on the generalized Tukey lambda distribution. Sarabia (1997) proposed a family of Lorenz curves which starts with a generating Lorenz curve and then creates a family by increasing the number of parameters. Later Sarabia et al. (2005) put forward a new class of Lorenz curves using a mixture of an initial Lorenz curve with a known probability density function. For more works on families of Lorenz curves, we refer to Sarabia et al. (2001) and Wang et al. (2007).

The concept of Lorenz curve have been extended to multivariate case. Taguchi (1972) defined the concentration surface of a two dimensional random vector (X_1, X_2) with density function $f(x_1, x_2)$ and mean values μ_1 and μ_2 for X_1 and X_2 respectively by the following implicit function

$$L(p_1, p_2, p_3) = 0$$

where

$$p_1 = \int_{-\infty}^{x_2} \int_{-\infty}^{x_1} f(u, v) du dv$$

$$p_2 = \frac{1}{\mu} \int_{-\infty}^{x_2} \int_{-\infty}^{x_1} u f(u, v) du dv$$

and

$$p_3 = \frac{1}{\mu} \int_{-\infty}^{x_2} \int_{-\infty}^{x_1} v f(u, v) du dv.$$

Koshevoy & Mosler (1996) has defined Lorenz zonoids and studied its properties.

One can also order distributions using the Lorenz curve. For nonnegative random variables X and Y with finite means, X is said to be less variable than Y in the Lorenz order if

$$\frac{X}{E(X)} \leq_{cx} \frac{Y}{E(Y)}.$$

The Lorenz ordering can also be given as

$$X \leq_L Y \Leftrightarrow L_X(p) \geq L_Y(p); 0 \leq p \leq 1.$$

When the Lorenz curves intersect, the Lorenz dominance has no interpretation. In such cases, the need of Gini index naturally arises. Using the fact that variability can be measured by considering the average difference between two independent observations from the distribution, the Gini index is defined.

Let X_1 and X_2 be i.i.d. random variables with corresponding order statistics be $X_{1:2}$ and $X_{2:2}$. Gini's mean difference is given by

$$T(x) = E|X_1 - X_2| = E\left(X_{2:2}\right) - E\left(X_{1:2}\right).$$

Since $T(x)$ does not satisfy the desirable properties of a good inequality measure, it shall be standardized by dividing by twice the mean. The resulting measure is Gini index. That is,

$$G = \frac{T(x)}{E(X)}$$

An alternative expression for G is

$$\begin{aligned} G &= 1 - \tfrac{E(X_{1:2})}{E(X_{2:2})} \\ &= 1 - \tfrac{2}{\mu} \int_0^\infty x\bar{F}(x) f(x) dx \end{aligned}$$

This one satisfies most of the properties except decomposability. G is defined geometrically as

$$G = \tfrac{Area\ between\, the\, line\, of\, equality\, and\ Lorenz\, curve}{Area\ below\, the\, line\, of\, equality}$$

That is the area between the diagonal and the Lorenz curve. Gini index is given by

$$G = 1 - 2\int_0^1 L(p)dp$$

G measures the extend to which the distribution of income among individuals within an economy deviates from perfectly equal distribution. Ord et al. (1983) considered the truncated form of Gini index defined by

$$G(t) = 2\int_t^\infty F(x,t)dF_1(x,t) - 1$$

where $F(x,t)$ is the distribution function of $X_1(t) = X|X > t$ and $F_1(x,t)$ is the first moment distribution given by

$$F_1(x,t) = \frac{\int_t^x \frac{yf(y)}{\bar{F}(t)}dy}{\int_t^\infty \frac{yf(y)}{\bar{F}(t)}dy}$$

Also $G(t)$ truncation invariant if and only if X follows the Pareto type I distribution. Gini index has also been extended to higher dimensions. Mosler (2002) defined the Gini zonoid index as the volume of Lorenz zonoid. Multivariate Gini indices are discussed in Koshevoy & Mosler (1996), Gajdos & Weymark (2005). Sathar et al. (2007) has extended the Gini index to the bivariate set up in the truncated setup.

For a bivariate random vector be (X_1, X_2) admitting an absolutely continuous distribution function, the bivariate Gini index for the truncated distribution is defined as the vector

$$G(t_1,t_2) = (G_1(t_1,t_2), G_2(t_1,t_2))$$

where

$$G_1(t_1,t_2) = 2\int_{t_1}^\infty F(x_1,t_1,t_2)dF_1(x_1,t_1,t_2) - 1$$

with

$$F_1(x_1,t_1,t_2) = \frac{\int_{t_1}^{x_1} y_1 \frac{f(y_1|X_2>t_2)}{\bar{F}(t_1|X_2>t_2)}dy_1}{\int_{t_1}^\infty y_1 \frac{f(y_1|X_2>t_2)}{\bar{F}(t_1|X_2>t_2)}dy_1}$$

and

$$F(x_1,t_1,t_2)=\frac{\int_{t_1}^{x_1} f(y_1|X_2>t_2)dy_1}{\int_{t_1}^{\infty} f(y_1|X_2>t_2)dy_1}$$

Similarly $G_2(t_1,t_2)$ is defined for the random variable $Y_2=X_2|X_1>t_1$.

Truncated Gini index defined in Takayama (1979), combined with other inequality measures can be used as poverty measures. Several other poverty measures are suggested by Sen (1976), Foster et al. (1984) and Sen (1986). However the most popular poverty index is the Sen index. Nair & Vineshkumar (2010) has formulated the truncated Gini index in the quantile framework as

$$\eta(p)=1-\frac{2}{\beta_1(p)}\int_0^p Q(u)\left(\frac{p-u}{p^2}\right)du.$$

or

$$\eta(p)=\frac{\beta_2(p)}{\beta_1(p)}.$$

Analogously the Sen index takes the form

$$S(p)=p\left[\frac{p\beta_1'(p)+\beta_2(p)}{p\beta_1'(p)+\beta_1(p)}\right].$$

It is also established that the poverty index $S(p)$ is constant if and only if the distribution of income is power.

Bonferroni (1930) proposed an income inequality measure, known as Bonferroni curve, based on the first moment distribution. Let X ba a non negative and absolutely continuous random variable with distribution function $F(x)$ and finite mean μ. The first incomplete moment and the partial mean of the probability distribution are given by

$${}_1F(x)=\frac{1}{\mu}\int_0^x tf(t)dt$$

and

$$\mu_x=\mu\left(\frac{{}_1F(x)}{F(x)}\right).$$

The Bonferroni measure is defined as

$$B\left[F(x)\right] = \frac{\mu_x}{\mu}$$

The Bonferroni curve is defined in the orthogonal plane $[F(x), B\left(F(x)\right)]$. Denoting by $p = F(x)$, the parametric expression of the curve is

$$B(p) = \frac{1}{p\mu}\int_0^p Q(u)du$$

$B(p)$ is related to $L(p)$ through the relationship

$$B(p) = \frac{L(p)}{p}$$

The values of $L(p)$ are fractions of total income while the values of $B(p)$ refer to relative income levels. The peculiarity of the $B(p)$ curve that it is sensitive to low levels of income and hence paves the way to use it in poverty measurement (Giorgi & Crescenzi (2001b)). The inferential properties of Bonferroni curve are discussed in Pundir et al. (2005).

Leimkuhler curve is an important tool in the field of Informetrics and information sciences. (Burrell (1991), Burrell (2005), Rousseau (1987). It plots the cumulative proportion of total productivity against the cumulative proportion of sources. The Leimkuhler curve is defined as

$$K(p) = \frac{1}{\mu}\int_{1-p}^1 Q(u)du$$

or

$$K(p) = 1 - L(1-p)$$

Important references about Leimkuhler curve are Egghe & Rousseau (1988), Egghe (2002), Egghe (2005b), Egghe (2005a). Sarabia et al. (2010a) proposed a general methodology for obtaining new classes of Lorenz and Leimkuhler curves starting from an ordered sequence of power Lorenz curves. The advantage of the new class is that it includes the curves discussed in Bradford (1985) and Kakwani & Podder (1973).

Zenga (1984) proposed a point concentration measure and an index based on the ratio between population and income fractiles. Let X be a nonnegative continuous random variable with probability density function $f(x)$ and finite mean μ and first moment distribution function $F_1(x)$. Assume that $F(x)$ and $F_1(x)$ are invertible.

The point concentration measure Z_p is defined as

$$Z_p = 1 - \frac{Q(p)}{Q_1(p)}$$

where $p \in [0, 1]$ and $Q(p) = F^{-1}(p)$ which is the inverse of distribution function (population fractile) and $Q_1(p) = F_1^{-1}(p)$ which is the inverse of first moment distribution(income fractile) and the index is given by

$$Z = \int_0^1 Z_p dp.$$

The behaviour of Z_p curve has been studied by Dancelli (1990), Berti & Rigo (2006), Zenga (1990). To calculate this Zenga curve, the evaluation of the inverse of cumulative distribution function and the inverse of incomplete first moment is necessary. More details about this Zenga curve and the corresponding index are provided in Kleiber & Kotz (2003).

Zenga (2007) proposed a new inequality measure, which is more realistic, based on the conditional expectations of the concerned distribution. The main feature of this inequality measure is the comparison with parts of the population and these compared parts are always two disjoint and adjascent groups. Here the two groups are differentiated in the value assumed by observed random variable, lower group is composed of the values of $X \leq x$ and upper group includes the values of $X > x$. Also the comparison is made on the ratio between the arithmetic mean of two groups.

Let X be a non negative continuous random variable with distribution function $F(x)$, density function $f(x)$ which is strictly positive on the support (a, b);$0 \leq a < b \leq \infty$ and finite positive mean μ.The lower mean ,$\mu^{-}(x)$ and upper mean $\mu^{+}(x)$ are respectively given by

$$\mu^{-}(x) = [F(x)]^{-1} \int_{a}^{x} tf(t)dt$$

and

$$\mu^{+}(x) = [1 - F(x)]^{-1} \int_{x}^{b} tf(t)dt.$$

The inequality measure is defined as

$$A(x) = 1 - \frac{\mu^{-}(x)}{\mu^{+}(x)}. \tag{2.27}$$

The inequality index I is given by

$$I = \int_{a}^{b} A(x)f(x)dx. \tag{2.28}$$

To represent the inequality measure given in (2.27) in terms of quantile function, setting $F(x) = p$, we have $x = F^{-1}(p) = Q(p)$. This gives,

$$\mu^{-}(x) = \mu^{-}[Q(p)] = M^{-}(p) = \tfrac{1}{p} \int_{0}^{p} Q(u)du$$

and

$$\mu^{+}(x) = \mu^{+}[Q(p)] = M^{+}(p) = \frac{1}{1-p} \int_{p}^{1} Q(u)du.$$

The inequality measure given in (2.27) is obtained as

$$\begin{aligned} I(p) &= 1 - \tfrac{M^-(p)}{M^+(p)} \\ &= 1 - \tfrac{(1-p)}{p} \tfrac{\int_0^p Q(u)du}{\int_p^1 Q(u)du} \end{aligned} . \tag{2.29}$$

The inequality index I is given by

$$I = \int_0^1 I(p)dp,$$

where the index I is given by the area under the curve $I(p)$. Further Zenga (2007) has established the relationship between Gini index G, Bonferroni index B and the Zenga index I as

$$G \leq B \leq I.$$

Polisicchio (2008) provided a distribution model with uniform inequality $I(p)$ curve. For a fixed k, $0 < k < 1$, the random variable X has the $I(p)$ curve given by

$$I(p) = k$$

for every p, $0 < p < 1$, if and only if the distribution function of X is

$$F(x) = \begin{cases} 0 & x \leqslant \mu k \\ (1-k)^{-1}\left[1 - (\mu k)^{\frac{1}{2}} x^{\frac{-1}{2}}\right] & \mu k < x \leqslant \mu/k \\ 1 & x \leqslant \mu/k \end{cases}$$

The above model is the truncated Pareto distribution with lower limit μk, upper limit $\frac{\mu}{k}$ and inequality parameter $\theta = \frac{1}{2}$.

Porro (2011) has extended the above work to investigate the model with linear form for the $I(p)$ curve.

Let X be nonnegative continuous random variable with finite and positive expectation μ, and with $I(p)$ curve given by

$$I(p) = ap + b \ \forall p \in (0,1);\ \ a, b \in R$$

then the distribution function $F(x)$ of X satisfies the relationship

$$x = \frac{\mu\left[aF^2(x) - 2aF(x) - b + 1\right]}{\left[-aF^2(x) - bF(x) + 1\right]^2}.$$

Setting $F(x) = p$, the above relationship can also be written as

$$x = \frac{\mu\left[ap^2 - 2ap - b + 1\right]}{\left[-ap^2 - bp + 1\right]^2}.$$

The author also presented real situations in social sciences where the empirical $I(p)$ curves are very similar to straight line. Zenga (2008) used reliability tools such as mean residual life MRL and mean waiting time MWT to represent the new Zenga curve $A(x)$. The MRL and MWT can be represented as $MRL = \mu^+(x) - x$ and $MWT = x - \mu^-(x)$. Substituting these expressions in the definition of $A(x)$, we get

$$A(x) = \frac{MRL + MWT}{MRL + x}.$$

Maffenini & Polisicchio (2010) made comparison between Lorenz curve and $I(p)$ curve by analyzing the effect of translation and transfer from rich to poor.

The main difference between the Lorenz curve and $I(p)$ curve is $I(p)$ curve compares adjacent and disjoint parts of distribution, but Lorenz curve makes the comparison of inequality based on cumulative, ordered and relative values.

The inferential aspects in connection with the $I(p)$ curve and I index are discussed by Greselin & Pasquazzi (2009), Greselin et al. (2010) and Greselin et al. (2009). Also the decomposition of the Zenga index by subgroups is presented in Radaelli (2010).

Chapter 3

Properties of the Zenga curve

3.1 Introduction

The Zenga curve, reviewed in section 2.8, differs from Lorenz curve in several aspects, even though there exists a functional relationship between them. The Zenga curve has no pre-established behaviour. This inequality measure stands out among others because of its ease in computation and simple interpretation. Also the dominating behaviour of Zenga curve as compared with the Lorenz curve in analyzing the effect of translation and transfer from rich to poor enables the same as a potential measure when it comes to measurement of income inequality. The main difference between the Lorenz curve and $I(p)$ curve is $I(p)$ curve compares adjacent and disjoint parts of distribution, but Lorenz curve makes the comparison of inequality based on cumulative, ordered and relative values.

Motivated by the above, in the present chapter we provide a detailed study on the salient features of the Zenga curve for different distributions, their equivalent forms in terms of quantile functions and some results based on a stochastic order defined using the Zenga curve.

For a non negative continuous random variable X defined over $0 \leq a < b \leq \infty$ with distribution function $F(x)$, density function $f(x)$ and with $E(X) < \infty$, denote the conditional expectations by $\mu^{-}(x) = E(X|X \leq x)$ and $\mu^{+}(x) = E(X|X > x)$. Zenga (2007) defines the measure of inequality as

$$A(x) = 1 - \frac{\mu^-(x)}{\mu^+(x)}. \tag{3.1}$$

If x is the level of income discriminating poor and rich, (3.1) shall be interpretted as the difference in average income among rich and that among poor. Zenga (2007) observes that relating to the measure $A(x)$, it is necessary to analyze its behaviour for theoretical distributions, usually employed to represent income data, and a method to obtain the density of X by knowing the inequality measure will be advantageous from the point of view of modelling the data.

Definition (3.1) can be reformulated in terms of the quantile functions, discussed in section 2.8. For the random variable X considered above, define the quantile function $Q(p)$ as

$$Q(p) = \begin{cases} \inf\{x : F(x) \geq p, 0 \leq p \\ \inf\{x : F(x) > 0, p = 0 \end{cases}$$

It may be noted that $Q(p)$ is same as $F^{-1}(p)$. Since $F(x)$ is continuous, $FoQ(p) = p$,where o deotes the composition of functions. Also by the strict monotonicity of $F(x)$, $x = Q(p)$. Setting $x = Q(p)$ in the expressions for $\mu^+(x)$ and $\mu^-(x)$ namely

$$\mu^+(x) = (1 - F(x))^{-1} \int_x^\infty tf(t)dt$$

and

$$\mu^-(x) = \frac{1}{F(x)} \int_0^x tf(t)dt,$$

we get

$$M^+(p) = \mu^+ \circ (Q(p)) = \frac{1}{1-p} \int_p^1 Q(u)du$$

for every $0 \leq p < 1$. and

$$M^{-}(p) = \mu^{-} \circ (Q(p)) = \frac{1}{p} \int_0^p Q(u)du \tag{3.2}$$

respectively. Thus in terms of quantiles for all p in $(0, 1)$,

$$\begin{aligned} I(p) &= A(Q(p)) \\ &= 1 - \tfrac{(1-p)}{p} \tfrac{\int_0^p Q(u)du}{\int_p^1 Q(u)du} \\ &= 1 - \tfrac{(1-p)}{p} \tfrac{\int_0^p Q(u)du}{\mu - \int_0^p Q(u)du} \\ &= \tfrac{\mu p - J(p)}{\mu p - pJ(p)} \end{aligned} \tag{3.3}$$

where $J(p) = \int_0^p Q(u)du$.

From (3.3), we have

$$\int_0^p Q(u)du = \mu \left[1 + \frac{1-p}{p(1-I(p))} \right]^{-1}.$$

That is,

$$J(p) = \frac{\mu p \left(1 - I(p)\right)}{1 - pI(p)}. \tag{3.4}$$

Relationships (3.3) and (3.4) express the Zenga inequality measure in terms of quantiles and vice versa. These relationships form the basis to establish characterization results in the sequel. Some of the results in this chapter are published in Nair et al. (2012).

3.2 Properties of the inequality measure

The Zenga curve $I(p)$ possess several interesting properties.

(i) $I(p)$ lies between 0 and 1. $I(p) = 0$ if and only if X is degenerate, in which situation there is complete equality.

(ii) The following functional relationship exists between the Lorenz curve $L(p)$ and the Zenga curve $I(p)$,

$$I(p) = \frac{p - L(p)}{p(1 - L(p))}. \tag{3.5}$$

(3.5) can be rewritten as

$$L(p) = \frac{p(1 - I(p))}{1 - p\,I(p)}. \tag{3.6}$$

In view of (3.5) and (3.6) one of the functions $I(p)$ or $L(p)$ determine the other uniquely.

(iii) The distribution of X is uniquely determined by $I(p)$ as

$$Q(p) = \mu \frac{d}{dp}\left[\frac{p(1 - I(p))}{1 - pI(p)}\right].$$

The above relationship follows by differentiating (3.4).

(iv) $I(p)$ is scale invariant.

To prove this, consider two populations with corresponding income variables X and Y, quantile functions $Q_X(p)$ and $Q_Y(p)$, mean incomes μ_x and μ_Y and $I(p)$ curves $I_X(p)$ and $I_Y(p)$ respectively.

When $Y = aX$, $a > 0$, we have

$$Q_Y(p) = aQ_X(p) \tag{3.7}$$

and

$$\mu_Y = a\mu. \tag{3.8}$$

From the definition (3.3), we get the expression for the $I(p)$ curve associated with the random variable Y as

$$I_Y(p) = 1 - \frac{(1-p)}{p} \frac{\int_0^p Q_Y(u)du}{\int_p^1 Q_Y(u)du}.$$

Substituting the transformations given in (3.7) and (3.8) in the above expression we get

$$I_Y(p) = 1 - \frac{(1-p)}{p}\frac{\int_0^p aQ_X(u)du}{\int_p^1 aQ_X(u)du}.$$

This gives

$$I_Y(p) = 1 - \frac{(1-p)}{p}\frac{\int_0^p Q_X(u)du}{\int_p^1 Q_X(u)du}$$

or

$$I_Y(p) = I_X(p).$$

The implication of the above result is that if we multiply the incomes of one population by a constant amount, the inequality measure of the resulting population is same as that of first population. In other words, a proportional increase or decrease in income does not have any effect on the inequality measure. This result has much utility as far as the Economists are concerned. For instance, the increase in price of petroleum products effects a proportional increase in the cost of transportation. This necessitates an increase in the price of consumables and other utility articles. The ultimate effect is a proportional increase in dearness allowance and hence salary. If the Zenga curve is used to measure inequality of income among the salaried group, it is ensured that the measure is unaltered by such changes in Economy.

For the two populations considered above, assume that the income of the first population is increased by a constant amount. It is of interest to look into the effect of this increase on the inequality measure of the second population. This aspect is examined below.

(v) If $Y = X + a$, $I_Y(p)$ can be expressed in terms of $I_X(p)$ through the relationship

$$I_Y(p) = \left[1 + \frac{a(1 - pI_X(p))}{\mu}\right]^{-1} I_X(p).$$

When $Y = X + a$, we have

$$Q_Y(p) = Q_X(p) + a \tag{3.9}$$

and

$$\mu_Y = a + \mu. \tag{3.10}$$

From (3.9) and (3.10), we get

$$J_Y(p) = ap + J_X(p).$$

Using the above relation in (3.4), we get

$$I_Y(p) = \frac{\mu p - J_X(p)}{p(\mu + a - ap - J_X(p))}.$$

Also for every p in $(0, 1)$

$$I_X(p) - I_Y(p) = \frac{a(1-p)(\mu p - J_X(p))}{p(\mu - J_X(p))(\mu - J_X(p) + a - ap)}.$$

In view of (3.4), the above equation can be written as

$$I_Y(p) = I_X(p) - \frac{aI_X(p)\,(1 - pI_X(p))}{\mu - apI_X(p) + a}$$

or equivalently,

$$I_Y(p) = \left[1 + \frac{a(1 - pI_X(p))}{\mu}\right]^{-1} I_X(p).$$

The above result shows that if the income is increased by a constant amount, the resulting inequality measure can be expressed in terms of the inequality measure of the original population.

(vi) Denote by $M^-(p)$ and $M^+(p)$ the corresponding lower and upper income means. The computation of $I(p)$ does not require the expressions of both $M^-(p)$ and $M^+(p)$. In fact, from (3.2) and (3.4), we have

$$M^-(p) = \frac{\mu(1 - I(p))}{1 - pI(p)} \tag{3.11}$$

(3.11) gives,

$$I(p) = \frac{\mu - M^-(p)}{\mu - pM^-(p)}.$$

Similarly using

$$(1 - p)M^+(p) + pM^-(p) = \mu,$$

we also have $I(p)$ in terms of $M^+(p)$ as

$$I(p) = \frac{M^+(p) - \mu}{pM^+(p)}$$

and

$$M^+(p) = \frac{\mu}{1 - p\,I(p)}. \tag{3.12}$$

(vii) The absolute Bonferroni curve $B^-(p) = M^-(p)$ and its dual [Greselin et al.(2010)] $B^+(p) = M^+(p)$, has expressions in terms of the Zenga curve given by (3.11) and (3.12).

(viii) Unlike the Lorenz curve which is increasing and convex on $[0, 1]$, the $I(p)$ curve can have different types of monotonicity properties and shapes, as is evident from the following examples.

Example 3.1

For the uniform distribution specified by

$$F(x) = \frac{x}{\alpha}, 0 \leq x \leq \alpha,$$

direct calculation gives,

$$I(p) = (1 + p)^{-1}.$$

Note that $I(p)$ is decreasing in p.

Example 3.2

For the Pareto I distribution with distribution function

$$F(x) = 1 - (\frac{k}{x})^2, \quad x > k > 0;$$

the quantile function is

$$Q(p) = k(1-p)^{-\frac{1}{2}}, \text{ with } \mu = 2k.$$

Direct calculations give

$$I(p) = 1 - p^{-1}[(1-p)^{\frac{1}{2}}(1-(1-p)^{\frac{1}{2}})].$$

Now

$$I'(p) = p^{-2}\{\frac{p}{2}(1-p)^{-\frac{1}{2}} + (1-p)^{\frac{1}{2}} - 1\}$$

$$= \frac{\left[1-(1-p)^{\frac{1}{2}}\right]^2}{2p^2(1-p)^{\frac{1}{2}}} > 0.$$

Hence $I(p)$ is increasing for the Pareto I model considered above.

Example 3.3

Consider the Pareto II distribution with distribution function

$$F(x) = 1 - \alpha^c (x+\alpha)^{-c},\ x;\ \alpha, c > 0.$$

The quantile function is

$$Q(p) = \alpha[(1-p)^{-1/c} - 1]$$

and the mean is $\mu = \alpha(c-1)^{-1}$. Using (3.3), we get

$$I(p) = \frac{c((1-p)^{-1/c} - 1)}{p[1 - c + c(1-p)^{-1/c}]}.$$

Specializing for $c = 2$ and differentiating the resulting expression, the condition $I'(p) = 0$ simplifies to

$$6 - 5p - 6(1-p)^{1/2} + 2p(1-p)^{1/2} = 0.$$

The solution is $p = 0.75$. Also $I'(p) < 0$ for $p < 0.75$ and $I'(p) > 0$ for $p > 0.75$. Thus for the Pareto II distribution with $c = 2$, $I(p)$ is first decreasing, reaches a minimum at $p = 0.75$ and then increases.

Example 3.4

Consider a random variable X with distribution specified by the quantile function

$$Q(p) = \theta + \sigma((\beta+1)p^\beta - \beta p^{\beta+1}),\ \beta, \theta, \sigma > 0.$$

It may be noted that this distribution does not possess a tractable distribution function. We have

$$J(p) = \theta p + \sigma(p^{\beta+1} - \frac{\beta}{\beta+2} p^{\beta+2})$$

and

$$\mu = \theta + 2(\beta + 2)^{-1}\sigma.$$

Using the above results in (3.3) we get

$$I(p) = \frac{\sigma[2 - p^{\beta}(\beta + 2 - \beta p)]}{\theta(\beta + 2)(1 - p) + \sigma[2 - p^{\beta+1}(\beta + 2 - \beta p)]}.$$

Taking $\theta = 3$, $\sigma = 3.3$, $\beta = 3$ and differentiating, we see that $I(p)$ is initially increasing, reaches a maximum at $p \doteq 0.53724$ and then decreases.

Remark 3.1

From (3.11) and (3.12)

$$M^{-}(p)M^{+}(p) = \frac{\mu^2(1 - I(p))}{(1 - pI(p))^2}. \tag{3.13}$$

Using the relationship between $I(p)$ and $L(p)$,(3.13) becomes

$$M^{-}(p)M^{+}(p) = \frac{\mu^2 L(p)(1 - L(p))}{p(1 - p)}.$$

The above equation expresses the product of mean incomes of upper and lower income groups in terms of the Lorenz curve.

To summarize, the properties of $I(p)$ discussed above make it a favorable choice among competing alternatives in terms of simplicity, logical soundness, flexibility and ease of interpretation.

A derived function that is of relevance in the measurement of inequality is the average inequality measure at p,

$$A_1(p) = \frac{1}{p}\int_0^p I(u)du, \quad 0 < p < 1. \tag{3.14}$$

We examine equivalent conditions to study the behavior of $I(p)$ curve using the average inequality measure at p. Obviously if $I(p)$ is increasing $A_1(p)$ is also an increasing function, but not conversely. The Zenga inequality index $I = \int_0^1 I(p)dp = A_1(1)$, and is obtained as the limit of $A_1(p)$ as $p \to 1$. In ascertaining the monotonicity of $I(p)$ the following equivalent conditions may be useful. We state the results for increasing $I(p)$ [II(p)] and the corresponding decreasing $I(p)$ [DI (p)] cases can be derived by reversing the inequalities. We say that X is II(p) [DI(p)] if $I(p)$ is nondecreasing (non increasing).

Theorem 3.1. *X is II(p) if and only if any one of the following conditions hold.*

(i) $pA_1(p)$ *is convex and twice derivable.*

(ii) $\begin{vmatrix} \mu^-(x) & \mu^+(x) \\ \mu^-(x+t) & \mu^+(x+t) \end{vmatrix} \geq 0 \qquad$ *for all* $\quad x, t > 0.$

(iii) $\dfrac{d}{dx}(\log \mu^+(x)) \geq \dfrac{d}{dx}(\log \mu^-(x)).$

Proof. (i) Assume that $pA_1(p)$ is convex and twice differentiable.

This implies

$$\frac{d^2}{dp^2}\int_0^p I(u)du \geq 0.$$

That is $I'(p) \geq 0$ or X is II(p).

(ii) When X is II(p), $A(x)$ is increasing.

This implies

$$\frac{\mu^-(x)}{\mu^+(x)} \text{ is decreasing.}$$

That is

$$\frac{\mu^-(x+t)}{\mu^+(x+t)} - \frac{\mu^-(x)}{\mu^+(x)} \leq 0.$$

This is same as (ii).

(iii) If X is II(p),

$$\frac{\mu^-(x)}{\mu^+(x)} \text{ is decreasing.}$$

This implies

$$\mu^+(x)\left[(\mu^-(x))'\right]-\mu^-(x)\left[(\mu^+(x))'\right]\leq 0.$$

This is same as (iii). The proof of the converse in each case can be obtained by retracing the steps.

□

3.3 Stochastic orders based on $I(p)$ curve

As mentioned in section 2.6, stochastic ordering of random variables provide a method for a global comparison of two distributions in terms of their characteristics. Let X and Y be two non-negative random variables with distribution functions $F_X(x)$ and $F_Y(x)$ and survival functions $\bar{F}_X(x)$ and $\bar{F}_Y(x)$ respectively. In this section we look into the problem of ordering random variables using the magnitude of Zenga measure $I(p)$. We now define the orderings using the $I(p)$ measure.

Definition 3.1

Let X and Y be two non-negative random variables with positive means μ_X and μ_Y and inequality measures $I_X(p)$ and $I_Y(p)$ respectively. Then X has lesser inequality than Y in terms of $I(p)$ ordering, denoted by $X\leq_I Y$ if $I_X(p)\leq I_Y(p)$ for all p in $(0,1)$.

It is natural to compare $I(p)$ ordering with other types of ordering. This aspect is examined below. Theorem 3.2 establishes that Lorenz ordering and $I(p)$ ordering are equivalent.

Theorem 3.2. *Let X and Y be two non negative random variables, then*

$$X\leq_L Y\Leftrightarrow X\leq_I Y.$$

Proof.

$$X\leq_I Y\Leftrightarrow I_X(p)\leq I_Y(p)$$

$$\begin{aligned}
&\Leftrightarrow \frac{M_X^+(p)-\mu_X}{pM_X^+(p)} \leq \frac{M_Y^+(p)-\mu_Y}{pM_Y^+(p)} \\
&\Leftrightarrow \mu_Y M_X^+(p) \leq \mu_X M_Y^+(p) \\
&\Leftrightarrow \mu_Y[\mu_X - \int_0^p Q_X(u)du] \leq \mu_X[\mu_Y - \int_0^p Q_Y(u)du] \\
&\Leftrightarrow \frac{1}{\mu_Y}\int_0^p Q_Y(u)du \leq \frac{1}{\mu_X}\int_0^p Q_X(u)du \\
&\Leftrightarrow L_Y(p) \leq L_X(p) \\
&\Leftrightarrow X \leq_L Y.
\end{aligned}$$

□

Remark 3.2 This result was proved in Porro (2008) using a different method. However the above proof uses an alternate approach. It now follows from the definition of convex ordering, discussed in section 2.6.4, that

$$\frac{X}{\mu_X} \leq_{cx} \frac{Y}{\mu_Y} \Leftrightarrow X \leq_L Y \Leftrightarrow X \leq_I Y. \tag{3.15}$$

Our next result provides a sufficient condition to check whether one distribution has lesser inequality than another.

Theorem 3.3. *If* $\mu_X \leq \mu_Y$ *then,*
(i) $M_X^+(p) \geq M_Y^+(p), \forall p \in (0,1) \Rightarrow X \geq_I Y$.
(ii)$M_X^-(p) \geq M_Y^-(p), \forall p \in (0,1) \Rightarrow X \leq_I Y$.

Proof. Since $M_X^+(p) = \frac{\mu_X}{1-pI_X(p)}$, we have by assumption,

$$\begin{aligned}
M_Y^+(p) \leq M_X^+(p) &\Rightarrow \frac{\mu_Y}{1-pI_Y(p)} \leq \frac{\mu_X}{1-pI_X(p)} \\
&\Rightarrow \frac{1-pI_X(p)}{1-pI_Y(p)} \leq \frac{\mu_X}{\mu_Y} \leq 1
\end{aligned}$$

$$\Rightarrow I_X(p) \geq I_Y(p).$$

The proof of (ii) is similar on using (3.11) and hence omitted. □

Sometimes it is easier to use the usual stochastic ordering instead of the convex ordering. In such cases an equivalent result in terms of the equilibrium distributions becomes more handy. If X is a non-negative random variable with finite positive expectation μ, the random variable X^* with distribution function

$$F_{X^*}(x) = \mu^{-1} \int_0^x \bar{F}(t)dt$$

is called the equilibrium distribution corresponding to X. For a detailed discussion on weighted distributions as well as equilibrium distributions, we refer to Hesselager et al. (1998), Gupta (2007), Sunoj & Maya (2008).

When X and Y have equal means, Shaked & Shanthikumar (2007) has established that

$$X \leq_{cx} Y \Leftrightarrow X^* \leq_{st} Y^*.$$

From (3.15), when the above expression holds, we have $I_X(p) \leq I_Y(p)$ ($A_X(x) \leq A_Y(x)$) and hence $Q_{X^*}(p) \leq Q_{Y^*}(p)$ ($\bar{F}_{X^*}(x) \leq \bar{F}_{Y^*}(x)$).

If $Q^*(p)$ is the quantile function of X^*,we have

$$Q_X(p) = Q^*(\mu_X^{-1} \int_0^p (1-u)q(u)du) = Q^*(\mu_X^{-1} T(p))$$

where

$$T(p) = \int_0^p (1-u)\, q(u)\, du,$$

and $q(p) = Q^{'}(p)$, is the total time on test transform extensively used in reliability analysis. For more details we refer to Bergman (1979), Nair et al. (2008). Thus

$$Q^{*}(p) = \mu Q_X(T_X^{-1}(p)).$$

For example, if X has generalized Pareto distribution with distribution function

$$F(x) = 1 - (1 + \frac{ax}{b})^{-\frac{a}{a+1}},$$

then

$$Q(p) = \frac{b}{a}[(1-p)^{-\frac{a}{a+1}} - 1].$$

Using the above formula

$$Q^{*}(p) = \frac{b}{a}[(1-p)^{-a} - 1].$$

Thus

$$X \leq_I Y \Leftrightarrow X \leq_{cx} Y \Leftrightarrow Q_X(T_X^{-1}(p)) \leq Q_Y(T_Y^{-1}(p)).$$

Note that this family of distributions contains the exponential, Pareto II and rescaled beta distributions as members.

Cox (1962) examined the role of length biased distributions in the context of renewal theory. The length biased distributions arises as a special case of weighted distributions. The random variable Y with p.d.f. specified by

$$g(x) = \frac{w(x)f(x)}{E[w(x)]} \; ; w(x) > 0 \tag{3.16}$$

is the weighted distribution corresponding to random variable X with weight function $w(x)$.

When $w(x) = x$, we get the length biased distribution. In this scenario (3.16) takes the form

$$f_B(x) = \frac{xf(x)}{\mu}. \tag{3.17}$$

The distribution function of the random variable X_B with p.d.f. (3.17) is

$$F_B(x) = \frac{1}{\mu}\int_0^x tf(t)dt.$$

The possibility of expressing the distribution function of the weighted distribution using the quantile function is yet to be studied in detail. Setting $x = Q(p^{'})$,$0 \leq p^{'} \leq 1$, we have

$$F_B\left(Q(p^{'})\right) = \frac{1}{\mu}\int_0^{p^{'}} Q(u)du.$$

From the definition of Lorenz curve, $F_B\left(Q(p^{'})\right) = L(p^{'})$ or $Q(p^{'}) = Q_B\left(L(p^{'})\right)$ where $Q_B\left(L(p^{'})\right) = F_B^{-1}\left(L(p^{'})\right)$. Again setting$L(p^{'}) = p$, we have

$$Q_B\left(p\right) = Q\left(L^{-1}(p)\right). \tag{3.18}$$

(3.18) provides the expression for the quantile function for the length biased model in terms of the Lorenz curve and this relationship helps us to identify the quantile function of the length biased model through a knowledge of the Lorenz curve of X.

As an example,for the power distribution specified by

$$Q(p) = \sigma p^{\frac{1}{\phi}}; \sigma, \phi > 0,$$

the Lorenz curve is given by $L(p^{'}) = \left(p^{'}\right)^{\frac{1}{\phi}+1}$ or $L^{-1}(p) = p^{\frac{\phi}{\phi+1}}$. Using (3.18), the quantile function of the corresponding length biased random variable is obtained as

$$Q_B\left(p\right) = \sigma p^{\frac{1}{\phi+1}}.$$

It may be observed that X_B also follows power distribution.

Chapter 4

The Zenga curve in the context of reliability analysis

4.1 Introduction

As mentioned in Section 2.3, the focal theme of interest in reliability analysis is the modelling and analysis of lifetime data. To enable this, certain concepts such as failure rate, mean residual life function etc. which are capable of describing the failure pattern are formulated and are used to obtain lifetime models. If X represents the lifetime of a component or device, a random variable which has received much interest in reliability analysis is the truncated random variable $X|X > x$ as well as $X|X \leq x$. The average values namely, $\mu^{+}(x) = E\left(X|X > x\right)$ and $\mu^{-}(x) = E\left(X|X \leq x\right)$ represents the average lifetime of components which has attained age x and the average lifetime of components which has failed before attaining age x. The former, namely $\mu^{+}(x)$, is the vitality function and is extensively studied by Kupka & Loo (1989) and Nair & Rajesh (2000).

The Zenga curve, defined in (3.1), is given in terms of $\mu^{+}(x)$ and $\mu^{-}(x)$. Observing that one can write (3.1) as

$$A(x) = \frac{\mu^{+}(x) - \mu^{-}(x)}{\mu^{+}(x)}.$$

$A(x)$ shall be interpreted as the difference in average age of components which has survived beyond age x from those which has failed before attaining age x, expressed in terms of average

age of components exceeding age x. $A(x)$ shall be viewed as a measure of proportional change in average age while switching over from survival before and after attaining age x. In this sense, the Zenga curve has a lot of significance in the study of reliability of components. If $m(x) = E\left(X - x|X > x\right)$ represents the mean residual life function and $r(x) = E\left(x - X|X \leq x\right)$ represents the mean waiting time, Zenga (2008) has represented (3.1) in the form

$$A(x) = \frac{r(x) + m(x)}{m(x) + x}.$$

Using the relationships

$$\mu^{-}(x) = \frac{\mu - \bar{F}(x)\left[x + m(x)\right]}{F(x)}$$

and

$$\mu^{+}(x) = x + m(x)$$

in the definition of $A(x)$, one can get alternate representation for $A(x)$ as

$$A(x) = \frac{1}{F(x)}\left[1 - \frac{\mu}{x + m(x)}\right],$$

where $\mu = E(X)$ represents the average lifetime.

Although several representations for the Zenga curve are feasible, the representation in terms of the quantile function given in (3.3) is more mathematically tractable. Further very little work seems to have been done on the Zenga curve in the quantile framework. Motivated by this, in the present chapter we look into the problem of (i) determining the possible relationships of the curve with other inequality measures as well as reliability concepts ii) characterization of probability distributions using these relationships iii) classification of lifetime distributions using the Zenga curve and iv) examining the behaviour of the curve using an empirical data on survival times. Some of the results of this chapter are included in Nair & Sreelakshmi (2012).

4.2 Zenga curve and other inequality measures

The relationship between the Zenga curve and the Lorenz curve was examined in section 3.2. It was observed there exists an explicit relationship between them, which is given in (3.5). In view of this relationship the knowledge of one of them enables to determine the other and hence the results especially characterization theorems for the Lorenz curve can be reformulated in terms of the Zenga curve. Bonferroni (1930) has proposed another measure of inequality which is referred to as the Bonferroni curve. For a nonnegative random variable X admitting an absolutely continuous distribution, the Bonferroni curve is defined as

$$B(p) = \frac{1}{\mu p} \int_0^p Q(u)du.$$

Observing that the Bonferroni curve is connected to the Lorenz curve through the relationship $L(p) = pB(p)$, from (3.5), we get

$$I(p) = \frac{1 - B(p)}{1 - pB(p)}$$

or

$$B(p) = \frac{1 - I(p)}{1 - pI(p)}.$$

In view of the above relationships it is inherent that $I(p)$ and $B(p)$ determine each other uniquely.

Another inequality measure extensively used in Informetrics is the Leimkuhler curve defined by

$$K(p) = \frac{1}{\mu} \int_p^1 Q(u)du.$$

The Lorenz curve $L(p)$ and $K(p)$ are connected through the relationship

$$\begin{aligned} K(p) &= 1 - \tfrac{1}{\mu}\int_0^{1-p} Q(u)du \\ &= 1 - L(1-p). \end{aligned}$$

Using (3.5) and the above expression, we get

$$I(p) = \frac{1}{p}\left(1 - \frac{1-p}{K(1-p)}\right)$$

or

$$K(p) = \frac{p}{1-(1-p)I(1-p)}.$$

The above relations enables one to evaluate $K(p)$ through the knowledge of $I(p)$ and vice versa.

Table 4.1 :The three curves for different distributions

Distribution	Lorenz curve	Bonferroni curve	I(p) curve
Power	$p^{\frac{1}{\beta}+1}$	$p^{\frac{1}{\beta}}$	$\frac{1-p^{\frac{1}{\beta}}}{1-p^{\frac{1}{\beta}+1}}$
Exponential	$p+(1-p)\log(1-p)$	$1+\frac{(1-p)}{p}\log(1-p)$	$\frac{1}{p[1-(\log(1-p))^{-1}]}$
Pareto II	$c\left(1-(1-p)^{1-\frac{1}{c}}\right)-p(c-1)$	$\frac{c\left(1-(1-p)^{1-\frac{1}{c}}\right)}{p}-(c-1)$	$\frac{c[(1-p)^{\frac{-1}{c}}-1]}{p\left[c\{(1-p)^{\frac{-1}{c}}-1\}+1\right]}$
Pareto I	$\alpha\left[1-(1-p)^{1-\frac{1}{\alpha}}\right]$	$\frac{\alpha}{p}\left[1-(1-p)^{1-\frac{1}{\alpha}}\right]$	$\frac{1-(1-p)^{\frac{1}{\alpha}}}{p}$
Rescaled beta	$c\left(1-(1-p)^{\frac{1}{c}}\right)+p(c+1)$	$\frac{c\left(1-(1-p)^{\frac{1}{c}}\right)}{p}+(c+1)$	$\frac{c[1-(1-p)^{\frac{1}{c}}]}{p\left[c\{1-(1-p)^{\frac{1}{c}}\}+1\right]}$
Govindarajulu	$p^{\beta+1}\left[\frac{\beta+2-\beta p}{\beta+2}\right]$	$p^{\beta}\left[\frac{\beta+2-\beta p}{\beta+2}\right]$	$\frac{2-p^{\beta}(\beta+2-\beta p)}{p\left[2-p^{\beta+1}(\beta+2-\beta p)\right]}$

The following graphs give $I(p)$ curve for different distributions

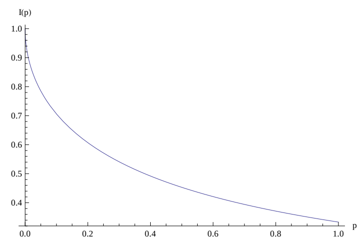


Figure 4.1: power distribution; $\beta = 2$

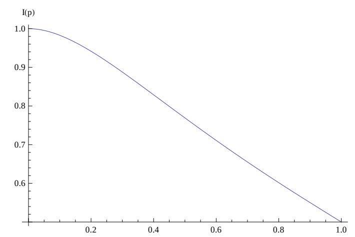


Figure 4.2: Govindarajulu distribution;$\beta = 6$

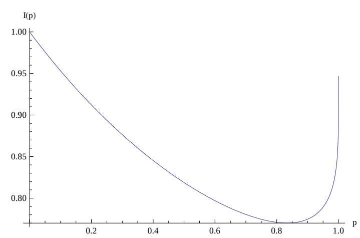


Figure 4.3: unit exponential distribution

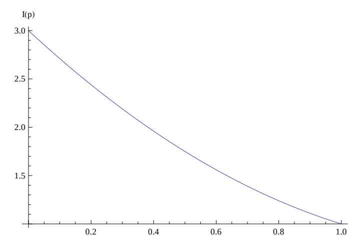


Figure 4.4: Pareto I distribution; $\alpha = 3$

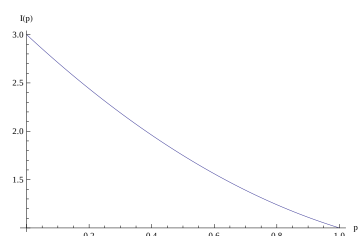


Figure 4.5: Pareto II distribution; $c = 3$

4.3 Relationship between the Zenga curve and certain reliability measures

The utility of reliability concepts in Economic analysis had been the focal theme of investigation in several works. Concepts such as moments of residual life, vitality function etc. has been advantageously used to identify the models to represent income data. In this section, we establish certain relationships between the Zenga curve and certain reliability concepts such as mean residual quantile function and reversed mean residual quantile function. Note that the approach used here is the representation using the quantile functions. These relationships are used subsequently to arrive at characterization results for certain distributions. If $M(p)$ represents the mean residual quantile function reviewed in section 2.3.2, there exists the relationship

$$\int_0^p Q(u)du = \mu\left(M(p) + Q(p)(1-p)\right). \tag{4.1}$$

Using the definition of $I(p)$ curve given in (3.3) and (4.1),it follows that

$$I(p) = \frac{M(p) + Q(p) - \mu}{p\left(M(p) + Q(p)\right)}. \tag{4.2}$$

From (2.8) and (4.2) , we have ,

$$I(p) = p^{-1}\left[1 + \frac{\mu}{\int_0^p \frac{M(u)}{1-u}du}\right]^{-1}. \tag{4.3}$$

Rearranging the terms in the above equation, we get

$$\int_0^p \frac{M(u)}{1-u}du = \frac{\mu p I(p)}{1 - pI(p)}.$$

Differentating the above expression with respect to p, we get

$$\frac{M(p)}{1-p} = \frac{\mu\left[pI'(p) + I(p)\right]}{\left[1 - pI(p)\right]^2}$$

or

$$M(p) = (1-p)\mu\frac{d}{dp}\left[\frac{1}{1-pI(p)}\right]. \tag{4.4}$$

Similarly from the definition of reversed mean residual quantile function $R(p)$ given in section 2.3.4, we get

$$\int_0^p Q(u)du = p\left(Q(p) - R(p)\right). \tag{4.5}$$

From (4.5) and (2.9), we get

$$\int_0^p Q(u)du = p\int_0^p \frac{R(u)}{u}du.$$

Substituting the above expression in (3.3) we have

$$I(p) = \frac{\mu - \int_0^p \frac{R(u)}{u}du}{\mu - p\int_0^p \frac{R(u)}{u}du}.$$

or

$$\int_0^p \frac{R(u)}{u}du = \frac{\mu\left[1 - I(p)\right]}{1 - pI(p)}.$$

Differentating the above expression with respect to p, we get

$$R(p) = p\mu\left[\frac{(p-1)I^{'}(p) + I(p)\left(1 - I(p)\right)}{\left(1 - pI(p)\right)^2}\right]. \tag{4.6}$$

(4.6) provides the relationship between $I(p)$curve and reversed mean residual quantile function.

Examples:

1. For the uniform distribution with quantile function

$$Q(p) = bp,$$

by direct calculations we get,
$I(p) = \frac{1}{1+p}$ and $M(p) = \frac{b}{2}(1-p)$.
The relationship (4.4) is immediate since $M(p) = \mu(1-p)$ where μ is the mean.

2. Rohde (2009) has examined the potential of the truncated Pareto distribution as a suitable model for income data and studied the properties of the corresponding distribution. Subsequently Sarabia et al. (2010b) showed that the model proposed by Rohde is a re parameterization of the model proposed by Aggarwal (1984) and they have discussed some important economic properties of the model.
The distribution is specified by

$$F(x) = \eta - \alpha^{\frac{1}{2}} x^{-\frac{1}{2}}; \frac{\alpha}{\eta^2} \leq x \leq \frac{\alpha}{(\eta-1)^2}$$

where $\alpha = \eta(\eta-1)\mu$. The quantile function associated with $F(x)$ is

$$Q(p) = \frac{\alpha}{(\eta-p)^2}.$$

The $I(p)$ curve and $R(p)$simplifies to

$$R(p) = \frac{\alpha p}{\eta(\eta-p)^2} \tag{4.7}$$

and

$$I(p) = \frac{1}{\eta}(= c, \ a \ constant).$$

From (4.5), we get

$$R(p) = \frac{p\mu c(1-c)}{(1-cp)^2}$$

which is same as the $R(p)$ given in (4.6) with $\alpha = \frac{\mu(1-c)}{c^2}$ and $\eta = \frac{1}{c}$.

In the sequel, we look into the problem of characterizing probability distributions using possible relationships between $I(p)$ and certain reliability concepts. Our first characterization result pertains to the class of distributions considered in Nair & Sankaran (2009) using a relationship between the $I(p)$ curve and hazard quantile function $H(p)$.

Theorem 4.1. *Let Xbe a nonnegative continuous random variable with $E(X) < \infty$. Then there exists a function g(.) satisfying*

$$I(p) = \frac{H(p)g\left[Q(p)\right]}{p\left(\mu + H(p)g\left[Q(p)\right]\right)} \tag{4.8}$$

if and only if

$$\frac{f'\left[Q(p)\right]}{f\left[Q(p)\right]} = \frac{\mu - Q(p) - g'\left[Q(p)\right]}{g\left[Q(p)\right]}.$$

Proof. When (4.8) holds, using (4.2) we have

$$\frac{M(p) + Q(p) - \mu}{p\left(M(p) + Q(p)\right)} = \frac{H(p)g\left[Q(p)\right]}{p\left(\mu + H(p)g\left[Q(p)\right]\right)}.$$

The above equation simplifies to,

$$M(p) + Q(p) = \mu + H(p)g\left[Q(p)\right]. \tag{4.9}$$

Nair & Sankaran (2009) showed that (4.9) holds if and only if the density quantile function has the form

$$\frac{f'\left[Q(p)\right]}{f\left[Q(p)\right]} = \frac{\mu - Q(p) - g'\left[Q(p)\right]}{g\left[Q(p)\right]},$$

as claimed.

□

Remark 4.1. The above general class of distributions include the Pearson, beta, gamma distributions etc. For instance, for the exponential distribution with quantile function,

$$Q(p) = -\frac{1}{\lambda}\ln(1-p),\ \lambda > 0,$$

the following relation exists between $I(p)$ and $H(p)$.

$$I(p) = \frac{1}{p\left[1 + (Q(p)H(p))^{-1}\right]}.$$

For the gamma distribution specified by the p.d.f.

$$f(x) = \frac{m^n}{)n} e^{-mx} x^{n-1}, x > 0,$$

the form of $g[Q(p)]$ can be identified as $g[Q(p)] = \frac{Q(p)}{m}$ and the $I(p)$ curve is related to $H(p)$ through the relationship

$$I(p) = \frac{1}{p[\frac{\mu m}{H(p)Q(p)} + 1]}.$$

Next two theorems provide characterization results using the mean residual and reversed mean residual quantile functions.

Theorem 4.2. *For a nonnegative continuous random variable X, the relationship*

$$pI(p) = \frac{A - M(p)}{B - M(p)} \tag{4.10}$$

holds if and only if Xfollows the distribution specified by the quantile function

$$Q(p) = \frac{\mu B}{B - A} + C(1-p)^{\frac{B-A}{\mu}} \tag{4.11}$$

provided $C(A-B)>0.$

Proof. When (4.10) holds, using (3.5), we get

$$\frac{p-L(p)}{1-L(p)}=\frac{A-M(p)}{B-M(p)}$$

This gives,

$$L(p)=\frac{A-pB-(1-p)M(p)}{A-B}.$$

Using the definition of $L(p)$ and $M(p)$, we have

$$\frac{1}{\mu}\int_0^p Q(u)du=\frac{A-pB}{A-B}+\frac{1-p}{A-B}\left(\frac{1}{1-p}\int_p^1 Q(u)du-Q(u)\right).$$

Differentiating the above expression with respect to p and rearranging the terms, we get

$$q(p)-\frac{A-B}{(1-p)\mu}Q(p)-\frac{B}{1-p}=0$$

The solution of the above differential equation is,

$$Q(p)=\frac{\mu B}{B-A}+C(1-p)^{\frac{B-A}{\mu}}.$$

For $Q(p)$ is an increasing function, $C(A-B)>0$. The proof of the converse is straight forward and hence omitted. □

Remark 4.2 Setting $B=0$ in (4.10) and (4.11), we get

$$pI(p)=1-\frac{A}{M(p)}$$

and

$$Q(p)=C(1-p)^{\frac{-A}{\mu}}$$

Put $C = k$ and $A = \frac{\mu}{\alpha}$ in the above expression, we get Pareto distribution of first kind with quantile function,

$$Q(p) = k(1-p)^{\frac{-1}{\alpha}}.$$

Further setting $Q(0) = 0$ in (4.11),we get

$$B\mu = C(A - B).$$

Using the above expression in (4.11) we get,

$$Q(p) = C\left[(1-p)^{\frac{-B}{C}} - 1\right].$$

If $C = \frac{b}{a}$ and $B = -\frac{b}{a+1}$, we get the quantile function of generalised Pareto distribution. For the ranges $0 < a < 1$ and $-1 < a < 0$, we get Pareto II and Rescaled Beta distributions respectively as special cases. But exponential distribution is not a special case.

Theorem 4.3. *For a non negative random variable X with reversed mean residual quantile function R(p), the relationship*

$$I(p) = \frac{1 - \beta R(p)}{1 - \beta p R(p)};\ \beta > 0 \tag{4.12}$$

holds if and only if X follows power distribution specified by the quantile function

$$Q(p) = \sigma p^{\frac{1}{\phi}}; \sigma, \phi > 0. \tag{4.13}$$

Proof. For the quantile function given in (4.13), direct calculations give

$$R(p) = \frac{\sigma}{\phi + 1} p^{\frac{1}{\phi}}$$

and

$$I(p) = \frac{1 - \beta R(p)}{1 - \beta p R(p)}$$

where $\beta = \frac{\phi+1}{\sigma}$.

Conversely, suppose that (4.12) holds, using (3.5) and (4.12), we have

$$\frac{p - L(p)}{p\,(1 - L(p))} = \frac{1 - \beta R(p)}{1 - \beta p R(p)}.$$

The above equation gives,

$$L(p) = \beta p R(p).$$

That is,

$$\frac{1}{\mu}\int_0^p Q(u)du = \beta p\left(Q(p) - \frac{1}{p}\int_0^p Q(u)du\right).$$

Differentiating the above equation with respect to p, we get

$$\frac{q(p)}{Q(p)} = \frac{1}{p\beta\mu}.$$

The solution to the above differential equation is

$$Q(p) = Cp^{\frac{1}{\beta\mu}}. \tag{4.14}$$

Put $C = \sigma$ and $\beta = \frac{\phi}{\mu}$, we get the quantile function given in (4.13) and the theorem follows. $\square$

4.4 Classification of Lifetime distributions

In the reliability context, concepts of ageing describe how a component or system improves or deteriorates with age. As an extension to the work of Chandra & Singpurwalla (1981), Klefsjö (1984) and Kochar & Xu (2009) have discussed the ageing properties such as IFR, IFRA, NBUE and HNBUE based on the Lorenz curve and their related partial orderings. In this section we obtain certain limits for $I(p)$ using certain criteria based on ageing. Our first result focuses attention on a necessary and sufficient condition for a distribution to be IFR.

Theorem 4.4. *Let X be continuous random variable with distribution function $F(x)$, finite mean μ and $I(p)$ curve denoted by $I_F(p)$. Assume that G(.) is the distribution function of a random variable following exponential distribution with same mean μ and $I(p)$ curve specified by*

$$I_G(p) = \frac{1}{p\left[1 - (\ln(1-p))^{-1}\right]} .$$

F is IFR if and only if

$$I_F(p) \leq I_G(p).$$

Proof. Barlow & Proschan (1975) has shown that F is IFR if and only if $F \leq_C G$. Hence

$$\begin{aligned} F \ is \ IFR &\Leftrightarrow F \leq_c G \\ &\Leftrightarrow \textstyle\int_p^1 Q_F(u)du \leq \int_p^1 Q_G(u)du \\ &\Leftrightarrow \textstyle\int_0^p Q_F(u)du \geq \int_0^p Q_G(u)du \\ &\Leftrightarrow I_F(p) \leq I_G(p). \end{aligned}$$

□

We have discussed some ageing concepts using the quantile based representation in chapter 2. Based on these definitions, lifetime distributins can be classified using the Zenga curve. The Next two theorems provide sufficient conditions for distributions belong to different ageing classes in terms of the $I(p)$curve.

Theorem 4.5. *A nonnegative continuous random variable X is IMRL (DMRL) if*

$$I(p) \geq (\leq) \frac{1}{p}\left[\frac{Q(p)}{Q(p)+\mu}\right].$$

Proof. Using (2.16) and (4.4) , we get

$$X\ is\ IMRL(DMRL) \Rightarrow (1-u)\mu\frac{d}{du}\frac{1}{1-uI(u)} \geq (\leq)\frac{1}{H(u)}$$

$$\Rightarrow \frac{d}{du}\frac{1}{1-uI(u)} \geq (\leq)\frac{1}{(1-u)\mu H(u)} = \frac{q(u)}{\mu} \quad . \tag{4.15}$$

On integration from 0 to p, (4.15) becomes

$$\frac{1}{1-pI(p)} \geq (\leq)\frac{Q(p)}{\mu} + 1.$$

This implies

$$I(p) \geq (\leq)\frac{1}{p}\left[\frac{Q(p)}{Q(p)+\mu}\right]$$

as claimed. □

Theorem 4.6. *Let X be a lifetime random variable with finite positive mean μ. A sufficient condition for X to be UBAE (UWAE) is that*

$$I(p) \geq (\leq)\frac{1}{p}\left\{1 - \frac{\mu}{M(1)\log(1-p)}\right\}^{-1}.$$

Proof. From (2.17) and (4.4), we have

$$\text{X is UBAE(UWAE)} \Rightarrow \frac{d}{du}\frac{1}{1-uI(u)} \geq (\leq)\frac{M(1)}{\mu(1-u)}. \tag{4.16}$$

Integrating (4.16) from 0 to p, we get

$$\frac{1}{1-pI(p)} \geq (\leq)1 - \frac{M(1)\log(1-p)}{\mu} \quad \Rightarrow I(p) \geq (\leq)\frac{1}{p}\left\{1 - \frac{\mu}{M(1)\log(1-p)}\right\}^{-1} .$$

The proof is complete. □

Analogous to the above theorems, one can find the sufficient conditions for ageing classes like increasing hazard rate average, IHRA (decreasing hazard rate average, DHRA), harmonic new better than used in expectation, HNBUE (harmonic new worse than used in expectation, HNWUE) in terms of $I(p)$curve using the basic definitions given in Nair & Vineshkumar (2011) and the relationships discussed in section 4.3. The results are mentioned below. The proof of the results are analogous to that of theorem 4.5 and theorem 4.6 and hence not included.

1) If X is IHRA (DHRA), then

$$I(p) \geq (\leq) \frac{z(p) - p}{p\,(z(p) - 1)},$$

where $z(p) = -\frac{1}{\mu} \int_0^p \frac{\log(1-u)}{H(u)} du$.

2) If X is HNBUE (HNWUE), then

$$I(p) \geq (\leq) \frac{1}{p} \left\{ 1 - \left[\frac{1}{1-p} - \int_0^p \frac{1}{(1-u)^2} e^{-\frac{Q(u)}{\mu} du} \right]^{-1} \right\}.$$

4.5 Illustration

In the context of reliability theory, the form of the hazard function enables to find the appropriate model for the lifetime data. In the sequel we illustrate the behaviour of the Zenga curve empirically using a survival data considering the quantile model having linear hazard quantile form with quantile function specified by

$$Q(p) = \frac{1}{a+b} \log \left[\frac{a+bp}{a(1-p)} \right] \tag{4.17}$$

We fit the model specified by (4.17) to the survival data given in Bryson & Siddiqui (1969) and examine the behaviour of $I(p)$ curve. The data contains survival time of 43 patients suffering from chronic granulocytic leukemia. We fit the model (4.17) for the data by the method of L moments.

The first two L moments are given by

$$l_1 = \frac{1}{a+b}\log\left(\frac{a+b}{a}\right)$$

$$l_2 = \frac{a\log\left(\frac{a+b}{a}\right)+b}{b^2}$$

Equating the population L moments to the sample L moments, the estimates of parameters are evaluated as

$$a = 0.000666573, b = 0.000972694.$$

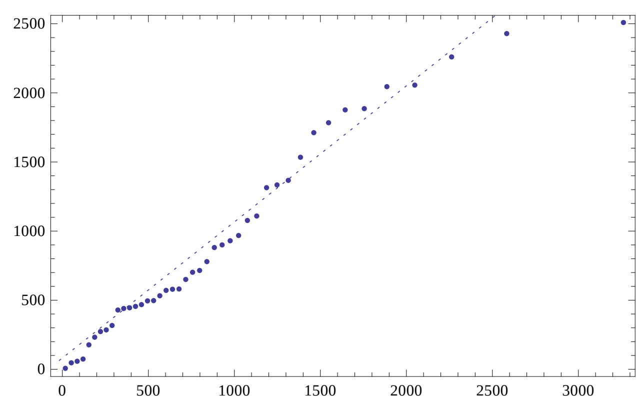


Figure 4.6: Q-Q Plot

Here the data is divided into 5 groups. Corresponding x values with respect to the values of p obtained by $p_i = \frac{i}{5}; i = 1, 2, ..., 5$ are used to get the observed and expected frequencies. Thus the chisquare value obtained here is 3.2 which is admissible so that the model (4.17) fits well to the data. This fact is also evidenced by the Q-Q plot given as figure 4.6

The plot of $\hat{I}(p)$ for different values of p is given below.

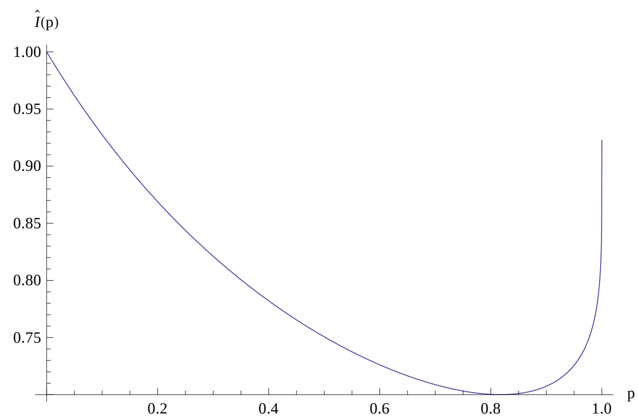


Figure 4.7:

From figure 4.7, one can observe that the curve is bathtub shaped. Further one can compute the average survival time of the least fortunate $p100\%$ of the patients is $I(p)\,100\%$ lower than that of remaining $(1-p)\,100\%$ of the patients suffering chronic granulocytic leukemia and for example $I(.8) = .70$ can be interpreted as the average survival time of the least fortunate 80% of patients is 70% lower than that of remaining 20% of the patients.

4.6 Quantile based income models

In this section, we consider three distributions expressed in terms of quantile functions, which are potential models to represent income data. It may be noticed that only very little work has been done in modelling income data using quantile functions and hence the properties of these models are examined in detail. Inequality measures such as Lorenz curve, Bonferroni curve etc. are calculated for these models. Further characterization results associated with these models are also discussed. Eventhough the results discussed below have no specific reference to the Zenga curve, the results are useful in view of the relationships between the inequality measures and reliability concepts.

4.6.1 Govindarajulu distribution

In the context of reliability analysis, Govindarajulu (1977) proposed a lifetime model with quantile based representation

$$Q(p) = \theta + \sigma \left\{ (\beta + 1)\, p^{\beta} - \beta p^{\beta+1} \right\};\ \ \theta, \sigma, \beta > 0. \tag{4.18}$$

When $\theta = 0$, (4.18) reduces to

$$Q(p) = \sigma \left\{ (\beta + 1)\, p^{\beta} - \beta p^{\beta+1} \right\};\ \ \sigma, \beta > 0. \tag{4.19}$$

We now look into some popular measures of income inequality for the Govindarajulu model. Using (4.18), $L(p)$ simplifies to

$$L(p) = \frac{\beta + 2}{(\beta + 2)\theta + \sigma} \left\{ \theta p + \sigma p^{\beta+1} \left(\frac{\beta + 2 - \beta p}{\beta + 2} \right) \right\}.$$

In the special case when $\theta = 0$,we get

$$L(p) = p^{\beta+1} \left(\frac{\beta + 2 - \beta p}{\beta + 2} \right). \tag{4.20}$$

For $\beta = 1$, the above expression becomes the Lorenz curve of the rescaled beta distribution. The Bonferroni curve is given by

$$B(p) = p^{\beta} \left(\frac{\beta + 2 - \beta p}{\beta + 2} \right). \tag{4.21}$$

It may be observed that as $\beta \to 0$, $B(p) \to 1$. Also as β gets large, $B(p) \to 0$.This means that as the value of β decreases ,there is a tendency to reach maximum equality. The Bonferroni index defined by

$$B_1 = \int_0^1 B(p)dp,$$

and for the model (4.19), B_1 simplifies to

$$B_I = \frac{1}{(\beta+1)(\beta+2)}.$$

The expression for $R(p)$ for the model (4.19) is given by

$$R(p) = p^{\beta}\left(\beta+1-\beta p-\frac{\beta+2-\beta p}{\beta+2}\right). \tag{4.22}$$

From (4.21) and (4.22), we can see that the ratio of Bonferroni curve to the reversed mean residual quantile function is in the bilinear form. The following theorem provides a characterization result for the Govindarajulu distribution based on a relationship between $B(p)$ and $R(p)$.

Theorem 4.7. *Let X be a nonnegative random variable with finite positive mean μ.The relationship*

$$B(p) = \left[\frac{A-Bp}{C-Dp}\right] R(p);\ \ A, B, C, D > 0 \tag{4.23}$$

holds if and only if X follow the Govindarajulu distribution with quantile function specified by

$$Q(p) = \sigma\left\{(\beta+1)\,p^{\beta} - \beta p^{\beta+1}\right\}.$$

Proof. By direct calculations using (4.21) and (4.22), we get $B(p)$ is of the form (4.23) with $A = (\beta+2)^2$,$B = C = \beta(\beta+2)$,$D = \beta(\beta+1)$.

Conversely, suppose that (4.23) holds.

Using the following relation between $L(p)$ and $R(P)$

$$R(p) = \mu\left[L'(p) - \frac{1}{p}L(p)\right]$$

in view of (4.23) and the fact that $B(p) = p^{-1}L(p)$, we get

$$\frac{L'(p)}{L(p)} = \frac{C - Dp + \mu(A - Bp)}{\mu p(A - Bp)}.$$

Integrating the above equation from 0 to p,we get

$$L(p) = Zp^{\frac{C}{\mu A}+1}(A - Bp)^{\frac{c}{\mu A}-\frac{D}{\mu B}}, \tag{4.24}$$

where Z is the constant of integration.Since the Lorenz curve determines the distribution uniquely, it is clear from (4.24) that the model is Govindarajulu distribution when $A = (\beta{+}2)$,$B = \beta$,$C = \beta\mu(\beta + 2)$ and $D = \beta\mu(\beta + 1)$. □

For the model (4.19), G simplifies to

$$G = \frac{\beta^2 + 4\beta + 1}{\beta^2 + 4\beta + 3}.$$

To describe the size distribution of income, Esteban (1986) introduced the concept of income share elasticity which provides the rate of change of total income at each income level. The income share elasticity is defined as

$$\pi(x) = 1 + \frac{xf'(x)}{f(x)}.$$

Setting $x = Q(p)$ and using (2.3) ,we get

$$E(p) = \pi\left[Q(p)\right] = 1 - \frac{Q(p)q'(p)}{q(p)}.$$

For Govindarajulu model,

$$E(p) = 1 - \frac{\sigma p^{\beta}(\beta + 1 - \beta p)\left[\beta(1 - p) - 1\right]}{1 - p}.$$

It may be observed from the above expression that the income share elasticity of Govindarajulu distribution is non monotonic in behaviour.

Another measure which has attracted a lot of interest is the poverty index described in Sen (1976). Nair & Vineshkumar (2010) has given the following definition for the above notions in terms of quantile function.The expression for the truncated Gini index and Sen index are respectively given as

$$\eta(p) = 1 - \frac{2p}{\int_0^p Q(u)du} \int_0^p \frac{Q(u)(p-u)du}{p^2}$$

and

$$S(p) = p[g(p) + (1 - g(p))\,\eta(p)]$$

where $g(p)$ is the income gap ratio given by

$$g(p) = 1 - \frac{1}{pQ(p)} \int_0^p Q(u)du.$$

For the model (4.19),the expression for $\eta(p)$ and $S(p)$ are given as

$$\eta(p) = 1 - \frac{\beta + 2 - \beta p}{(\beta + 2)(\beta + 1 - \beta p)}$$

and

$$S(p) = p\left[1 - \frac{p^2\beta^2 - p(2\beta^2 + 4\beta + 1) + (\beta^2 + 4\beta + 5)}{(\beta + 2(\beta + 1 - \beta p)^2}\right].$$

Remark 4.3 When the division of population is based on the mean income μ instead of x, we can represent the Zenga measure as a function of Frigyes' measure defined by

$$F = (F_1, F_2, F_3) = \left(\frac{\mu}{m_1}, \frac{m_2}{m_1}, \frac{m_2}{\mu}\right)$$

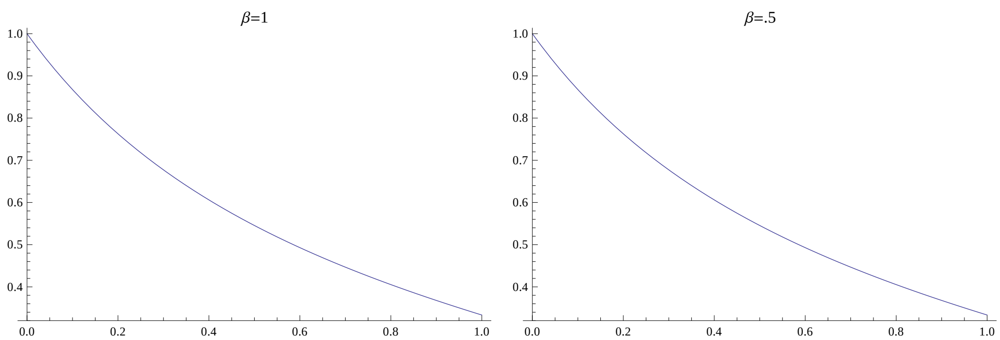
where $m_1 = E(X|X < \mu)$ and $m_2 = E(X|X > \mu)$. Using (3.1), the Zenga measure is

$$A(\mu) = 1 - \frac{1}{F_2}.$$

The $I(p)$ curve corresponding to (4.19) simplifies to

$$I(p) = \frac{2 - p^{\beta}(\beta + 2 - \beta p)}{2 - p^{\beta+1}(\beta + 2 - \beta p)}.$$

The following graphs show the behavior of the $I(p)$ curve for different values of β .



The behavior of the curve can be easily ascertained from the sign of $I'(p)$.

$$I'(p) \leqslant 0 \Rightarrow 2p(1-p)\beta p + (\beta + 2 - \beta p)(2p - 2(1-p)\beta) \leqslant p^{\beta+1}(\beta + 2 - \beta p)^2 \quad (4.25)$$

Since

$$p^{\beta+1} < \frac{1}{1 + \frac{1-p}{\beta-1}},$$

(4.25) becomes

$$2p(1-p)\beta p + (\beta + 2 - \beta p)(2p - 2(1-p)\beta) \leqslant \left[1 + \frac{1-p}{\beta+1}\right]^{-1} (\beta + 2 - \beta p)^2 \leqslant 0.$$

The above condition is true only when $\beta > 0$. Thus we can say that for all β, $I(p)$ curve is decreasing.

Remark 4.4

From theorem 4.7 and the relation between $B(p)$ and $I(p)$, it can be noted that the expression connecting $I(p)$ curve and $R(p)$ is not in simple form.

That is, we have

$$I(p) = \frac{1 - B(p)}{1 - pB(p)}$$

and for model (4.19)

$$I(p) = \frac{1 - k(p)R(p)}{1 - pk(p)R(p)}$$

where $k(p) = \frac{A - Bp}{C - Dp}$. But when $k(p) = 1$, the above expression connecting $I(p)$ and $R(p)$ provides a characterization for the power distribution.

Estimation

Gilchrist (2000) provides a detailed discussion on various estimation procedures of parameters in the quantile function. However L moments can be used for estimating the parameters in an easier manner. We fit the three parameter Govindarajulu model given in (4.18) to a real data set using the method of L moments. For the Govindarajulu model (4.18), the first three L moments are given as

$$l_1 = \theta + \frac{2\sigma}{\beta + 2}$$

$$l_2 = \frac{2\beta\sigma}{(\beta + 2)(\beta + 3)}$$

and

$$l_3 = \frac{2\beta\sigma(\beta - 2)}{\beta^3 + 9\beta^2 + 26\beta + 24}.$$

In this method ,we equate the sample L- moments to population L-moments to obtain the estimates of θ,β, and σ.To illustrate the application of the model in practical situation, we consider a data collected from the site of Beuro of Economic Analysis.The data includes 255 values denoting quarterly state personal incomes of Michigan state from the year 1948 up to 2011,third quarter.

The estimates for the parameters are obtained as

$$\hat{\theta} = 14257.3; \hat{\sigma} = 3.77; \hat{\beta} = 336762.$$

Here the data is divided into 10 groups.Corresponding x values with respect to the values of u

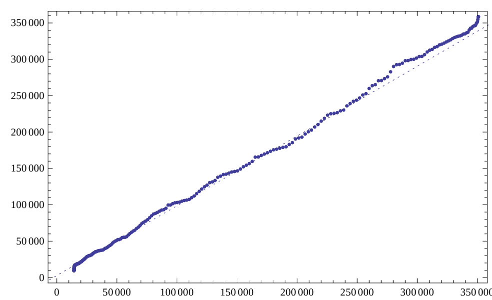


Figure 4.8: Q-Q Plot

obtained by $u_i = \frac{i}{10}; i = 1, 2, ..., 10$ are used to get the observed and expected frequencies.Thus the chisquare value obtained here is 8.4 which is admissible so that the model follows Govindarajulu distribution. The Q-Q plot for the model is given as figure 4.3 below. The graph also reveals the appropriateness of the model.

4.6.2 Quantile model with linear hazard quantile form

Development of new models assigning different functional forms for various concepts in reliability theory is a potential area of research.Many well known distributions that exist in the literature have been arisen in this way. Recently, Nair & Vineshkumar (2011) have proposed a new quantile function using linear form of the hazard quantile function. In this section, we investigate its application in modelling income data. The distribution is specified by the quantile function

$$Q(p) = \frac{1}{a+b} \log\left[\frac{a+bp}{a(1-p)}\right]. \tag{4.26}$$

When $a = \frac{1}{\lambda}$ and $b = 0$, (4.26) becomes,

$$Q(p) = -\frac{1}{\lambda} \log(1-p).$$

The above quantile function corresponds to the exponential distribution. When $a = b = \frac{1}{2\sigma}$, the quantile function takes the form

$$Q(p) = \sigma \log\left[\frac{1+p}{1-p}\right]$$

and the distribution is the half logistic distribution. Also for $a = \frac{\lambda}{1-u}$ and $b = -\frac{u\lambda}{1-u}$, the distribution is exponential-geometric with quantile function

$$Q(p) = \frac{1}{\lambda} \log\left[\frac{1-pu}{1-p}\right].$$

Moving onto the inequality measures as defined earlier, Lorenz curve for the class of models given in (4.26) takes the form

$$L(p) = \frac{(a+bp)\log(a+bp) - \log a(1+a) + (b-1)\log(1-p)}{(a+b)\log\left(\frac{a+b}{a}\right)}.$$

Also we get the income share elasticity as

$$E(p) = 1 - \frac{a+b(2p-1)}{(a+b)^2(1-p)} \log\left[\frac{a+bp}{a(1-p)}\right].$$

It may be observed that the income share elasticity of the distributions in the class (4.26) is decreasing. Unlike Govindarajulu distribution, the Gini index of this model does have a simple expression. The Gini index for the model (4.26) simplifies to,

$$G = \frac{1}{4b}\left[2(a+b)^2\log(a+b) - b\left(2a+5b+4\log a(1-a)-4\right) - 2a^2\log a\right].$$

The Zenga curve is obtained as

$$I(p) = \frac{p(a+b)\log\left(\frac{a+b}{a}\right) - (a+bp)\log(a+bp) + \log a(1+a) - (b-1)\log(1-p)}{p\left[1-(a+bp)\log(a+bp) + \log a(1+a) - (b-1)\log(1-p)\right]}.$$

The $I(p)$ curve of the above class is always decreasing for any values of the parameters. Figure 4.9 plots the $I(p)$ curve for different values of the parameters.

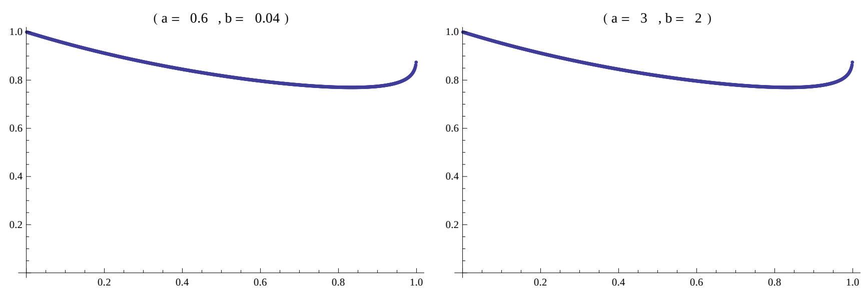


Figure 4.9: $I(p)$ curve at different values of parameters

Estimation

The first two L-moments of this model is obtained as

$$l_1 = \frac{1}{a+b}\log\left(\frac{a+b}{a}\right)$$

and

$$l_2 = \frac{a\log\left(\frac{a+b}{a}\right)+b}{b^2}.$$

To illustrate the procedure, we consider the 42 revised annual personal income estimates of united states from the year 1969 to 2010. Revised estimates for 2007-2010 were released June 22, 2011. (data source: U.S. Department of Commerce, Bureau of Economic Analysis, http://www.bea.doc.gov/). The parameter estimates of a and b are obtained as

$$a = 0.000092, b = 0.000236.$$

A reasonable model for the distribution of the personal income shall be taken as (4.26) with values of a and b be given above. The Q-Q plot for the model is given as figure 4.10 below.

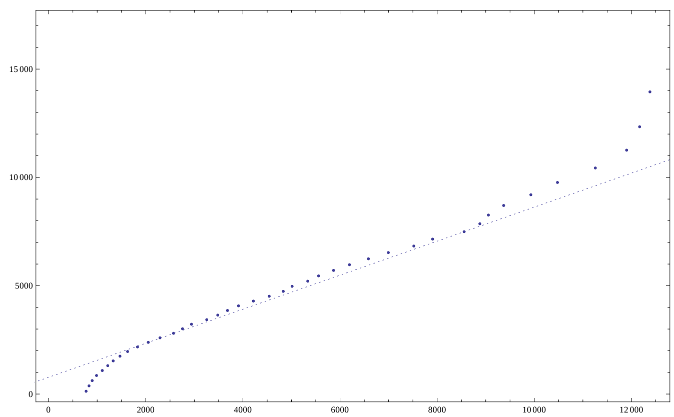


Figure 4.10: Q-Q Plot

4.6.3 Power x Pareto distribution

Unlike distribution function, quantile functions have an interesting property that they can be added or multiplied to generate new ones. Gilchrist (2000) considered the multiplied form of Power and Pareto distributions under the name Power x Pareto distribution. The distribution is specified by the quantile function $Q(p; c, \lambda_1, \lambda_2)$ and specified by

$$Q(p; c, \lambda_1, \lambda_2) = cp^{\lambda_1}(1-p)^{-\lambda_2}; c, \lambda_1, \lambda_2 \geqslant 0. \tag{4.27}$$

It may be noticed that (4.27) includes Power distribution as $\lambda_2 \to 0$ with distribution function

$$F(x) = \left(\frac{x}{c}\right)^{\lambda_1}; 0 < x < c$$

and the Pareto model specified by

$$F(x) = 1 - \left(\frac{c}{x}\right)^{\frac{1}{\lambda_1}}; x > c > 0$$

when $\lambda_2 \to 0$. Further Log-logistic distribution becomes a special case when $\lambda_1 = \lambda_2 = \lambda$ with thedistribution function

$$F(x) = \frac{1}{1 + \left({}^{c}/_{x}\right)^{\frac{1}{\lambda}}}; x > 0$$

Since uniform distribution is a special case of power distribution, Uniform x Pareto distribution can be considered as a special case of the quantile model (4.27). The uniform x Pareto distribution is given as

$$Q(p; c, 1, \lambda_2) = cp(1-p)^{-\lambda_2}. \tag{4.28}$$

Hankin & Lee (2006) has studied the properties of the model (4.27). We look into the economic measures of the model (4.27). The income share elasticity can be obtained as

$$E(p) = \frac{c(1-p)^{\lambda_2 - 1}}{\lambda_1 + p(\lambda_2 - \lambda_1)} \left[(\lambda_1 - \lambda_2 - 1)\left(p^2(\lambda_1 - \lambda_2) - 2\lambda_1 p\right) + \lambda_1(1 - \lambda_1)\right].$$

The Lorenz curve takes the form

$$L(p) = \frac{\beta_p(\lambda_1+1, 1-\lambda_2)}{\beta(\lambda_1+1, 1-\lambda_2)}; \quad \lambda_2 < 1$$

where

$$\beta_p(\lambda_1 + 1, 1 - \lambda_2) = \int_0^p u^{\lambda_1}(1-u)^{-\lambda_2} du.$$

Also the Zenga curve of Power x Pareto distribution can be obtained from the relation between $I(p)$ and $L(p)$ namely

$$I(p) = \frac{p - L(p)}{p\,(1 - L(p))}.$$

The $I(p)$ curve for different values of parameters are given in figure 4.6 below. It may be noted that the $I(p)$ curve and $E(p)$ take different shapes for different values of c, λ_1 and λ_2.

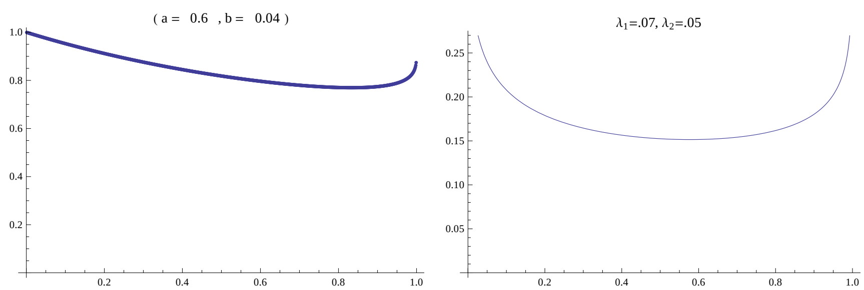


Figure 4.11: $I(p)$ curve at different values of parameters

Next theorem provides a characterization result for the Pareto distribution which is a member of the family defined by (4.27).

Theorem 4.8. *A non negative continuous random variable X follows the distribution with quantile function $Q(p; c, 0, \lambda_2)$ if and only if the relationship holds*

$$1 - E(p) = \frac{A}{1-p} Q(p); \quad A > 1 \tag{4.29}$$

holds for all $p \in (0, 1)$.

Proof. Suppose that (4.29) holds. From the definition of income share elasticity, we get

$$\frac{q'(p)}{q(p)} = \frac{A}{1-p}.$$

The solution of the above differential equation is

$$Q(p) = \frac{\delta}{A-1}(1-p)^{-(A-1)}$$

where δ is the constant of integration. Setting left end point to c, we get

$$Q(p) = c(1-p)^{-(A-1)};\ A > 1$$

as claimed. Proof of the converse is straight forward and hence not pursued here.

□

Estimation

Power x Pareto distribution is fitted to a personal income data set by using the method of L-moments. The data includes 179 observations describing the dollar estimates of 179 areas(includes all local areas and states) of United States of the year 2009.
For Power x Pareto model , the first three L-moments are given by

$$l_1 = C\beta_p(\lambda_1 + 1, 1 - \lambda_2)$$

$$l_2 = C\beta_p(\lambda_1 + 2, 2 - \lambda_2)$$

and

$$l_3 = C\left[3\beta_p(\lambda_1 + 3, 1 - \lambda_2) - \beta_p(\lambda_1 + 2, 1 - \lambda_2) - 2\beta_p(\lambda_1 + 4, 1 - \lambda_2)\right].$$

The parameter estimates are obtained as

$$c = 33.79; \lambda_1 = .8138; \lambda_2 = .6232.$$

The Q-Q plot is given as figure 4.12 below and the graph ensures that the model is a good fit.

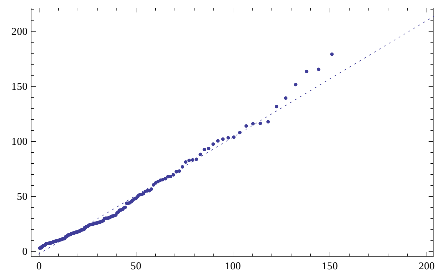


Figure 4.12: Q-Q Plot

Chapter 5

L moments and measures of income inequality

5.1 Introduction

As pointed out in section 2.4, the concept of L moments, introduced by Sillitto (1969), has been extensively used in reliability analysis. One of the interesting aspects of L moments is that it generally dominates the conventional moments in the sense that it provides smaller variance. Further the robustness of L moments against outliers enables the same as a potential tool when it comes to the modelling of lifetime data. Recently Nair & Vineshkumar (2010) studied L moments of residual life and has obtained characterization results for certain life distributions.They have also expressed the truncated Gini index and the celebrated Sen index for the poor in terms of the first two L moments.

Motivated by this in the current chapter we study the interrelationships of the L moments with other inequality measures such as Lorenz curve, Bonferroni curve and Zenga curve. We also look into the problem of characterizing distributions using possible functional relationships among these measures, in the truncated set up. We also address the problem of ordering of distributions based on L moments and compare the same with other types of orderings.

5.2 Relationship with other inequality measures

In this section, we look into possible functional relationships between the income inequality measures and truncated L moments. It follows easily from the definition of Lorenz curve that of the first residual quantile, $\alpha_1(p)$ is related to $L(p)$ through the equation

$$L(p) = 1 - \frac{(1-p)\alpha_1(p)}{\mu}. \tag{5.1}$$

For the line of equal distribution, the income variable X is degenerate and $L(p) = p$. In this set up, (5.1) takes the form

$$\alpha_1(p) = \mu.$$

From the above relationship, it may be observed that when there is equality of income the first residual quantile is equal to the mean income. It may also be noted that $\alpha_1(p)$ can be interpreted as the mean income among households with upper income level.

From the following definition of Pietra index

$$T = F(\mu) - \frac{1}{\mu}\int_0^{F(\mu)} Q(p)dp,$$

we get the mean income of lower income households, $\beta_1(p)$, is related to the Pietra index through the relationship

$$T = F(\mu)\left[1 - \frac{\beta_1\left(F(\mu)\right)}{\mu}\right].$$

Also the Bonferroni curve is related to $\beta_1(p)$ as

$$B(p) = \frac{\beta_1(p)}{\mu}.$$

$\beta_1(p)$ is the mean income of the truncated random variable ${}_xX$ and so greater the first L moment $\beta_1(p)$ indicates more equality in the population. Thus $\beta_1(p)$ can be used to infer the inequality of the population. Evidently as $\beta_1(p) \to \mu$, $B(p) \to 1$, the curve equality. Bonferroni curve uniquely determines the distribution through the relationship

$$Q(p) = \mu \left[pB'(p) + B(p) \right]$$

Substituting the above expression in the definition of reversed mean residual quantile function, we get

$$R(p) = \mu p B'(p).$$

Using the above equation in (2.15), we get

$$\begin{aligned} \beta_2(p) &= \tfrac{1}{p^2} \int_0^p \mu u^2 B'(u) du \\ &= \tfrac{\mu}{p^2} \left[p^2 B(p) - 2 \int_0^p u B(u) du \right] \\ &= \mu \left[B(p) - \tfrac{2}{p^2} \int_0^p u B(u) du \right]. \end{aligned}$$

Differentiating the above expression we get

$$p^2 \beta_2(p) = \mu \left[p^2 B(p) - 2 \int_0^p u B(u) du \right]$$

$$p^2 \beta_2^{'}(p) + 2p\beta_2(p) = \mu \left[p^2 B'(p) + 2pB(p) - 2pB(p) \right]$$

or

$$p\beta_2^{'}(p) + 2\beta_2(p) = \mu p B'(p)$$

That is,

$$B(p) = \frac{1}{\mu} \int_0^p \left(\beta_2^{'}(u) + \frac{2}{u} \beta_2(u) \right) du. \tag{5.2}$$

Evidently the case of equality reflects no variability so that $\beta_2(p) = 0$ if and only if $B(p) = 1$.

In light of the fact that L moments find application in reliability theory, we look into the scope of studying the relationship between L moments of the truncated variables $X_x = X|X >$

x and ${}_xX = X|X \leq x$ with the Zenga curve. From the definitions of Zenga curve and $\beta_1(p)$, we get

$$I(p) = \frac{\mu - \beta_1(p)}{p\left[\mu - p\beta_1(p)\right]}.$$

Now making use of (3.5) and (5.1), we get

$$I(p) = \frac{\alpha_1(p) - \mu}{p\alpha_1(p)} \tag{5.3}$$

or

$$\alpha_1(p) = \frac{\mu}{1 - pI(p)}. \tag{5.4}$$

This implies,

$$\alpha_1(p) \propto \frac{1}{1 - pI(p)}.$$

From the above equation it is evident that $I(p)$ uniquely determines the distribution upto a constant. Under the assumption that μ is known $I(p)$ uniquely determines the underlying distribution. This result provides a justification for using the $I(p)$ curve in the context of modelling income data. Recall that the L moments $\alpha_2(p)$ and $\beta_2(p)$ are measures of variability of X_x and ${}_xX$. The relationship between the $I(p)$ curve and $\alpha_2(p)$ and $\beta_2(p)$ are given below.
From (2.14) and (4.4), we get

$$\alpha_2(p) = (1-p)^{-2}\,\mu \int_p^1 \frac{(1-u)^2}{1 - uI(u)} du.$$

From section 4.2, the relationship between $I(p)$ and $B(p)$ curve is

$$I(p) = \frac{1 - B(p)}{1 - pB(p)}. \tag{5.5}$$

From (5.5) and (5.2) , we have

$$I(p) = \frac{\mu - \int_0^p \left(\beta_2'(u) + \frac{2}{u}\beta_2(u)\right) du}{\mu - p\int_0^p \left(\beta_2'(u) + \frac{2}{u}\beta_2(u)\right) du}.$$

The interrelationships discussed in this section will be used in sequel to investigate characterization results and stochastic orders in the forthcoming sections.

5.3 Characterization Results

In this section, we present characterization results for probability distributions based on functional relationships between the L moments and the different measures of inequality. Among the income inequality measures, Bonferroni curve is widely used as the measure of poverty since it is very sensitive to low level incomes [Giorgi & Crescenzi (2001b)]. Bonferroni curve is used to measure the variability in income distribution. Also the second L moment of reversed residual life is considered as the measure of variation. It is now of interest to investigate if the behaviour of the former can be inferred from the latter. However the behaviour of $B(p)$ need not be necessarily similar to that of $\beta_2(p)$. But the power distribution exhibits such a behaviour. Theorem 5.1 characterizes the power distribution using the functional relationship between $B(p)$ and $\beta_2(p)$.

Theorem 5.1. *Let X be nonnegative continuous random variable with Bonferroni curve $B(p)$ and second truncated L moment $\beta_2(p)$. The relationship*

$$B(p) = K\beta_2(p); K > 0$$

holds if and only if X follows the quantile function specified by

$$Q(p) = ap^{\frac{1}{b}}; a, b > 0.$$

Proof. Using the relationship between $B(p)$ and $\beta_2(p)$ in the previous section, we get

$$B(p) = K\mu \left[B(p) - \frac{2}{p^2} \int_0^p uB(u)du \right].$$

Differentiaiting the above equation with respect to p, we get

$$\frac{B'(p)}{B(p)} = \frac{2}{p(K\mu - 1)}.$$

The solution to the above differential equation is

$$B(p) = Cp^{\frac{2}{K\mu - 1}},$$

where C is a constant.Since $B(1) = 1$, we get $C = 1$. Thus $B(p) = p^{\frac{2}{K\mu-1}}$. The proof of the converse is straight forward and hence omitted. □

Illustration

In the sequel we fit the power model to the US department, Proprietors income-Quarterly data in the state Newhamphsire of the period 1948-1950 and examine the behaviour of $B(p)$ and $\beta_2(p)$. (Data source: U.S. Department of Commerce, Bureau of Economic Analysis). We fit the power model for the above mentioned data by the method of L moments.

The first two L moments are given by

$$l_1 = \frac{1}{a+b} \log \left(\frac{a+b}{a} \right)$$

and

$$l_2 = \frac{a \log \left(\frac{a+b}{a} \right) + b}{b^2}.$$

The population L moments are equated to sample L moments to obtain the estimates of parameters as

$$\hat{a} = 17.96, \hat{b} = 3.265.$$

The KS statistic is .156 against the table value as .361 at 5% level of significance which shows that power distribution gives suitable fit to data. For the power model,

$$B(p) = p^{\frac{1}{b}}$$

and

$$\beta_2(p) = \frac{abp^{\frac{1}{b}}}{1 + 3b + 2b^2}.$$

Plotting $(\beta_2(p), B(p))$ for the data we can estimate from Figure 5.1 that k=1.8259.

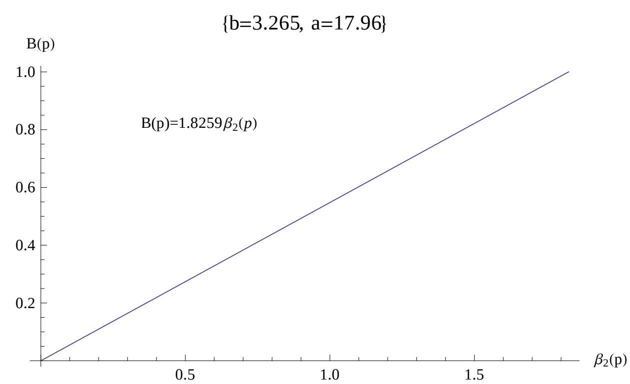


Figure 5.1:

Another distribution characterized by the functional relationship between $\beta_2(p)$ and $B(p)$ is the distribution with Lorenz curve

$$L(p) = pa^{p-1}; a > 1$$

The application of the distribution in modelling income data is studied in detail by Kakwani & Podder (1973) and Gupta (1984).

Theorem 5.2. *For the random variable X considered in theorem 5.1, the relationship*

$$B(p)\left[p^2(\log a)^2 - 2p\log a + 2\right] = \frac{p^2(\log a)^2}{\mu}\beta_2(p) - \frac{2}{a} \tag{5.6}$$

holds if and only if the quantile function is given by

$$Q(p) = \mu a^{p-2}\left[p(p-1) + a\right]. \tag{5.7}$$

Proof. From (5.6) and the relation between $B(p)$ and $\beta_2(p)$, we get

$$B(p)\left[p^2(\log a)^2 - 2p\log a + 2\right] = p^2(\log a)^2\left[B(p) - \frac{2}{p^2}\int_0^p uB(u)du\right] - \frac{2}{a}.$$

Differentiating and rearranging the terms, we get

$$\frac{B'(p)}{B(p)} = \log a.$$

This gives,

$$B(p) = Ca^p \tag{5.8}$$

where C is the constant of integration. The relationship $B(1) = 1$ gives $C = \frac{1}{a}$. From (5.8),we have

$$B(p) = a^{p-1}.$$

Bonferroni curve uniquely determines the distribution through the relationship,

$$Q(p) = \mu\left[pB'(p) + B(p)\right].$$

For the above expression of $B(p)$, we get the corresponding quantile function as given in (5.7).

□

The above discussions show that $\beta_2(p)$ can be used to study the variability in the lower income group and $\alpha_2(p)$ can be used to study the variability in the upper income group. Recall that $I(p)$ deals with the comparison of mean incomes of subgroups of population to measure the income inequality. When the income distribution is uniform it is intutive to infer that a measure of inequality comparing the lower income group and higher income group is inversely proportional to the mean incomes of higher income group. This is mathematically confirmed by the following theorem.

Theorem 5.3. *For the random variable considered in theorem 5.1, the relationship*

$$I(p) = \frac{k}{\alpha_1(p)};\ k > 0$$

holds if and only if X follows the uniform distribution specified by the quantile function

$$Q(p) = \theta p;\ 0 \leqslant p \leqslant 1.$$

Proof. For the uniform distribution specified by $Q(p) = \theta p$, direct calculations gives

$$I(p) = \frac{1}{1+p}$$

and

$$\alpha_1(p) = \frac{\theta}{2}(1+p).$$

Directly we get $I(p) = \frac{k}{l_1(p)}$ with $k = \frac{\theta}{2}$. Conversely using the relation (5.4) ,we get

$$\alpha_1(p) = \mu + kp.$$

Corresponding quantile function is

$$Q(p) = \mu - k + 2kp.$$

$Q(0) = 0$ yields $\mu = k$. So the resulting quantile function is $Q(p) = 2kp$ as claimed. $\square$

5.4 Ageing concepts

In this section we study how the L moments can be applied to study the ageing behaviour of a random variable.Concept of ageing is an important notion not only in the field of Reliability theory but also in Economics. Bhattacharjee (1993) has observed that the distribution of land holdings obey anti ageing properties like DFR,DFRA,IMRL,NWUE etc. Usually in reliability based works, the ageing properties are studied using concepts such as failure rate, mean residual life function etc. In this section we explain the problem from another point of view. The ageing properties are examined using truncated L moments which have their own economic interpretations.

Theorem 5.4. *Let X be a random variable representing income with finite positive mean μ. Denote by the first residual quantile $\alpha_1(p)$, then X is IFR(DFR) if and only if*

$$\alpha_1(p) \leq (\geq) \mu \left[1 - ln(1-p)\right].$$

Proof. From theorem 4.4, we get the necessary and sufficient condition for a distribution to be IFR(DFR) in terms of $I(p)$ curve as

$$I_F(p) \leq (\geq) \frac{1}{p\left[1 - (\ln(1-p))^{-1}\right]} \tag{5.9}$$

From (5.9) and (5.4), we get

$$\frac{\alpha_1(p)-\mu}{p\alpha_1(p)} \leq (\geq) \frac{1}{p\left[1-(\ln(1-p))^{-1}\right]}.$$

Rearranging the terms in the above expression, we get the result. The converse can be obtained by retracing the steps. □

Theorem 5.5. *For an IMRL(DMRL) distribution, if $\alpha_1(p)$ and $Q(p)$ respectiely denote the first residual quantile and quantile function, then*

$$\alpha_1(p)-\mu \geq (\leq) Q(p).$$

Proof. From theorem 4.5 we get the sufficient condition for a distribution to be IMRL(DMRL) as

$$I(p) \geq (\leq) \frac{1}{p}\left[\frac{Q(p)}{Q(p)+\mu}\right] \tag{5.10}$$

From (5.10) and (5.4), we get

$$\frac{\alpha_1(p)-\mu}{p\alpha_1(p)} \leq (\geq) \frac{1}{p}\left[\frac{Q(p)}{Q(p)+\mu}\right].$$

That is,

$$1-\frac{\mu}{\alpha_1(p)} \geq (\leq) \frac{Q(p)}{Q(p)+\mu}.$$

This in turn implies

$$\alpha_1(p)-\mu \geq (\leq) Q(p).$$

□

Remark 5.1

The above theorem illustrates that, for a population if the average excess holding over a particular lower threshold increases, then the difference between upper mean incomes and total mean incomes corresponding to each p is greater than the income corresponding to that p^{th} percentile for that population.

Theorem 5.6. *A distribution is HNBUE(HNWUE) if and only if*

$$\alpha_1(p) \leq (\geq) Q(p) + \frac{\mu}{1-p} e^{\frac{-Q(p)}{\mu}}.$$

Proof. From (2.19),the random variable X is in HNBUE(HNWUE) class if and only if

$$\int_p^1 (1-u)q(u)du \leq (\geq) \mu e^{\frac{-Q(p)}{\mu}}.$$

From the above equation, integrating by parts yields

$$\int_p^1 Q(u)du \leq (\geq) \mu e^{\frac{-Q(p)}{\mu}} + (1-p)Q(p).$$

or

$$\alpha_1(p) \leq (\geq) Q(p) + \frac{\mu}{1-p} e^{\frac{-Q(p)}{\mu}}$$

□

The above results focuses attention on ageing concepts using $\alpha_1(p)$. Our next result seeks a sufficient condition for a distribution to be NBUE which is given in terms of Gini's mean difference of residual random variable X_t.

Theorem 5.7. *For a distribution to be NBUE,*

$$\eta(p) \leq \mu$$

where $\eta(p)$ is the Gini's mean difference of residual random variable X_x.

Proof. From (2.18), a distribution is NBUE if and only if

$$M(p) \leq \mu \tag{5.11}$$

where $M(p)$ represents the mean residual quantile function. Multiplying (5.11) by $(1-p)$ and integrating from p to 1, we get

$$\int_p^1 (1-u)M(u)du \leq \frac{\mu}{2}(1-p)^2. \tag{5.12}$$

Using (2.14) and (5.12), we have

$$2\alpha_2(p) \leq \mu.$$

Nair & Vineshkumar (2010) has shown that

$$2\alpha_2(p) = \eta(p).$$

Therefore the above inequality becomes,

$$\eta(p) \leq \mu.$$

□

Remark 5.2

1. Equality holds when X has distribution with quantile function $Q(p) = -\frac{1}{\lambda}\log[1-p]; \lambda > 0$. Note that for this distribution $M(p) = \mu$.

2. From the above theorem, it is clear that as the sum of Gini's mean difference of residual random variable X_t and the total mean income becomes very negligible, the corresponding income distribution is exponential. So the above theorem can form a basis of forming

a statistic using the empirical version of $\eta(p)$ as well as the sample mean income to check the exponentiality of a population. This work will be taken up elsewhere.

5.5 Stochastic orders based on L moments

Nair & Vineshkumar (2010) has shown the second L moment of residual as well as reversed residual lives, $\alpha_2(p)$ or $\beta_2(p)$ can be used in distinguishing lifetime distributions based on its monotonic behaviour.The condition for $\alpha_2(p)$ to be increasing (decreasing) is

$$\alpha_2(p) \geq (\leq) \frac{M(p)}{2}$$

and the change point of $\beta_2(p)$ will be the solution of $R(p) = 2\beta_2(p)$. L moments can be used to give alternative definitions of ageing concepts. We consider two random variables X and Y. All the notations discussed in sequel are as defined in earlier sections corresponding to the suffixed random variable.To compare the orderings based on L moments ,inequality measures and reliability concepts ,we require the definitions of different kinds of orderings which are discussed in Chapter 2. We now define the following orderings based on L moments.

Definition 5.1

Let X and Y be two random variables with r^{th} L moment residual quantile functions $\alpha_{rX}(p) = \lambda_{rX}\left(Q_X(p)\right)$ and $\alpha^*_{rY}(p) = \lambda_{rY}\left(Q_X(p)\right)$. Then X is said to be smaller than Y in r^{th} L moment residual quantile function if and only if

$$\alpha_{rX}(p) \leq \alpha^*_{rY}(p)$$

for all p in $(0, 1)$.

For $r = 1$, we get $\alpha_{1X}(p) \leq \alpha^*_{1Y}(p)$ denoted by $X \leq_{FL} Y$. Similarly we can obtain $X \leq_{SL} Y$ for $r = 2$.

Definition 5.2

For the random variables X and Y with r^{th} L moment reversed residual quantile function $\beta_{rX}(p)$ and $\beta^*_{rY}(p)$ respectively, we can say that X is smaller than Y in r^{th} L moment quantile function if and only if

$$\beta_{rX}(p) \leq \beta^*_{rY}(p).$$

Analogous to the Definition 5.1, here also we can define $X \leq_{FLR} Y$ and $X \leq_{SLR} Y$ for $r = 1$ and $r = 2$ respectively.

We say that X dominates Y by second order stochastic dominance denoted by $X \geq_{SSD} Y$ if

$$\int_0^p Q_X(u)du \leq \int_0^p Q_Y(u)du.$$

Then for the first truncated L moments of X and Y denoted by $\lambda_{1X}(t)$ and $\lambda_{1Y}(t)$ respectively, we have

$$X \geq_{SSD} Y \Leftrightarrow \lambda_{1X}(t) \leq \lambda_{1Y}(t).$$

Theorem 5.8. *The following stochastic orders are equivalent.*

1. $X \leq_{MRL} Y \Leftrightarrow X \leq_{FL} Y.$

2. $X \leq_{RMRL} Y \Leftrightarrow X \leq_{FLR} Y.$

Proof. 1. We have,

$$M_X(p) \leq (\geq) M^*_Y(p) \Leftrightarrow \tfrac{1}{1-p}\int_p^1 Q_X(u)du - Q_X(p) \leq (\geq) \tfrac{1}{\bar{F}_Y(Q_X(p))}\int_{Q_X(p)}^1 \bar{F}_Y(x)dx$$
$$\Leftrightarrow X \leq (\geq)_{FL} Y$$

2. can be proved on similar lines.

□

The $\leq_{FL}$ order may not imply $\leq_{st}$ order. This can be shown using the following example.

Example 5.1

Let the random variable X be distributed as exponential with p.d.f.

$$f_X(x) = \frac{1}{2}e^{-\frac{x}{2}}$$

and Y follows gamma distribution with density function

$$f_Y(x) = xe^{-x}.$$

We have the mean residual life functions $M_X(x) = (x+2)e^{-x}$ and $M_Y(x) = 2$. Here $X \leq_{MRL} Y$ or $X \leq_{FL} Y$

However X and Y are not in usual stochastic order as is evidenced from the figure 5.2 below.

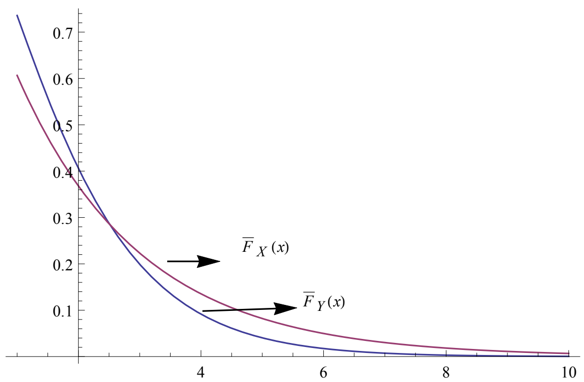


Figure 5.2:

For the exmple given above, we have the hazard functions, $h_X(x) = \frac{1}{2}$ and $h_Y(x) = \frac{x}{1+x}$. It may be observed from Figure 5.3 that $h_X(x) \geq h_Y(x)$.

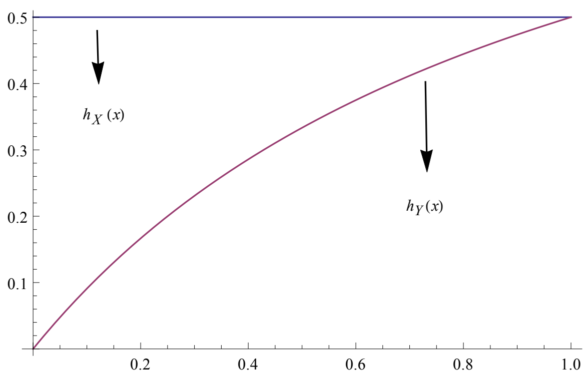


Figure 5.3:

The following theorem provides a sufficient condition for two random variables X and Y to have $\leq_{hr}$ order.

Theorem 5.9. *If* $\frac{\int_p^1 Q_X(u)du}{\int_p^1 Q_Y(u)du}$ *is increasing in* p *and* $X \leq_{FL} Y$*, then* $X \leq_{hr} Y$*.*

Proof. We have,

$$X \leq_{FL} Y \Rightarrow \frac{\int_p^1 Q_X(u)du}{\int_p^1 Q_Y(u)du} \leq \frac{\bar{F}_X(x)}{\bar{F}_Y(x)}. \tag{5.13}$$

If $\frac{\int_p^1 Q_X(u)du}{\int_p^1 Q_Y(u)du}$ is increasing in p we have

$$\frac{\int_p^1 Q_X(p)dp}{\int_p^1 Q_Y(p)dp} \geq \frac{1}{q_X(p) f_Y\left(Q_X(p)\right)}. \tag{5.14}$$

(5.13) and (5.14) give,

$$\frac{1}{q_X(p) f_Y\left(Q_X(p)\right)} \leq \frac{\bar{F}_X(x)}{\bar{F}_Y(x)}$$

or

$$\frac{f_X(x)}{\bar{F}_X(x)} \leq \frac{f_Y(x)}{\bar{F}_Y(x)}$$

or

$$X \leq_{hr} Y.$$

□

Theorem 5.10. *If $X \leq_{NBRURH} Y$ and $X \leq_{FLR} Y$, then $X \geq_L Y$.*

Proof. Assume that

$$\frac{\int_0^p Q_X(u)du}{\int_0^p Q_Y(u)du} \leq \frac{F_X(x)}{F_Y(x)}. \tag{5.15}$$

Since $X \leq_{NBRURH} Y$,we have $\frac{F_X(x)}{F_Y(x)} \leq \frac{\mu_X}{\mu_Y}$. Therefore (5.15) becomes

$$\frac{1}{\mu_X}\int_0^p Q_X(u)du \leq \frac{1}{\mu_Y}\int_0^p Q_Y(u)du \Rightarrow X \geq_L Y.$$

□

Chapter 6

Copula based reliability concepts

6.1 Introduction

A brief review of the works pertaining to the extension of quantiles to higher dimensions was discussed in section 2.7. Let $F_1(x_1)$ and $F_2(x_2)$ be marginal distribution functions of X_1 and X_2 respectively. Suppose $F_1(x_1) = u$ so that $x_1 = F_1^{-1}(u) = \phi(u)$. Also $x_2 = F_2^{-1}(v) = \psi(v)$. We have $\partial x_1 = \phi^{'}(u)\partial u$ and $\partial x_2 = \psi'(v)\partial v$ where $\phi^{'}(u) = \frac{d\phi(u)}{du}$ and $\psi'(v) = \frac{d\psi(v)}{dv}$.

Proceeding on the lines initiated by Belzunce et al. (2007), the probability $F_\varepsilon\left(\phi(u), \psi(v)\right)$ depends only on the copula C for the direction ε as detailed below,

$$F_\varepsilon\left(\phi(u), \psi(v)\right) = \begin{cases} C(u,v); & \varepsilon = \varepsilon_{--} \\ u - C(u,v); & \varepsilon = \varepsilon_{-+} \\ v - C(u,v) & \varepsilon = \varepsilon_{+-} \\ 1 - u - v - C(u,v) & \varepsilon = \varepsilon_{++} \end{cases}$$

$0 \leq u \leq 1; 0 \leq v \leq 1.$

For the bivariate random vector **X**, the joint distribution function is

$$F_{\varepsilon_{--}}\left(\phi(u), \psi(v)\right) = C(u,v). \tag{6.1}$$

Denote the joint survival function by $\bar{F}_{\varepsilon_{--}}(x_1, x_2)$ and the univariate survival functions by $\bar{F}_1(x_1)$ and $\bar{F}_2(x_2)$ respectively. Then consider

$$\bar{F}_{\varepsilon_{--}}(x_1, x_2) = \hat{C}(\bar{F}_1(x_1), \bar{F}_2(x_2))$$

where $\hat{C}$ is the survival copula. That is,

$$\bar{F}_{\varepsilon_{--}}(x_1, x_2) = \hat{C}(1-u, 1-v). \tag{6.2}$$

Nelsen (1999) describes the relation between C and $\hat{C}$ as

$$\hat{C}(1-u, 1-v) = 1 - u - v - C(u, v). \tag{6.3}$$

Also C satisfies the following properties, $C(u, 1) = u$ and $C(1, v) = v$.

Nair & Sankaran (2009) defines the hazard quantile function of the random variables X_1 and X_2 as

$$h_1(u) = \frac{1}{\phi'(u)du}$$

and

$$h_2(v) = \frac{1}{\psi'(v)dv}.$$

Also the mean residual quantile functions of X_1 and X_2 are defined as

$$m_1(u) = \frac{1}{1-u}\int_u^1 \phi(p)dp - \phi(u)$$

and

$$m_2(v) = \frac{1}{1-v}\int_v^1 \psi(p)dp - \psi(v).$$

In the reversed setup, the reversed hazard quantile function and reversed mean residual quantile functions are defined for the random variables X_1 and X_2 as

$$a_1(u) = \frac{1}{u\phi'(u)},$$

$$a_2(v) = \frac{1}{v\psi'(v)},$$

$$r_1(u) = \phi(u) - \frac{1}{u}\int_0^u \phi(p)dp,$$

and

$$r_2(v) = \psi(v) - \frac{1}{v}\int_0^v \psi(p)dp.$$

In the following we provide a bivariate extension of the basic quantile based reliability concepts given in Nair & Sankaran (2009) using the copula. Also we look for possible relationships connecting the bivariate concepts and use the same to derive various characterization theorems for bivariate distributions.

The extension of univariate quantile based reliability concepts to the bivariate setup can be done only on the basis of the four directions. Here we consider the bivariate copula based definitions of reliability concepts in the direction ε_{--}.

We define the bivariate hazard rate(bivariate reversed hazard rate) and bivariate mean residual life(bivariate reversed mean residual life) in a copula setup for the direction ε_{--} and study their relationships. We also check for the independence property of the concepts using the product copula.

6.2 Bivariate copula based reliability concepts

Different versions of bivariate hazard rate are discussed in Basu (1971), Cox (1992),Marshall (1975), Shaked & Shanthikumar (2007), Sun & Basu (1995) and Finkelstein (2003). However the most commonly used bivariate hazard rate is the vactor valued hazard rate defined in Johnson & Kotz (1975). For a bivariate random vector absolutely continuous distribution function with survival function $\bar{F}(\underset{\sim}{x})$, the hazard rate is defined as the vector

$$\mathbf{H}\left(\underset{\sim}{x}\right) = \left[H_1(\underset{\sim}{x}), H_2(\underset{\sim}{x})\right]$$

where

$$H_i(\underset{\sim}{x}) = -\frac{\partial}{\partial x_i} \log \bar{F}(\underset{\sim}{x});\ \ i = 1, 2. \tag{6.4}$$

An analogous expression of $\mathbf{H}\left(\underset{\sim}{x}\right)$ in the direction ε_{--} in terms of copula is given by

$$\underset{\sim}{h}_{\varepsilon_{--}}(u, v) = (h_1(u, v), h_2(u, v))$$

where

$$h_1(u, v) = H_1\left[\phi(u), \psi(v)\right] = -\frac{1}{\phi'(u)}\frac{\partial}{\partial u} \log \hat{C}(1-u, 1-v). \tag{6.5}$$

Similarly

$$h_2(u, v) = -\frac{1}{\psi'(v)}\frac{\partial}{\partial v} \log \hat{C}(1-u, 1-v). \tag{6.6}$$

This is obtained as follows. Since $x_1 = \phi(u)$ and $x_2 = \psi(v)$, we have $\partial x_1 = \phi'(u)\partial u$. Thus $\frac{\partial x_1}{\partial u} = \phi'(u)$. Also we have $\partial x_2 = \psi'(v)\partial v$. This gives $\frac{\partial x_2}{\partial v} = \psi'(v)$.
Substituting (6.2) in the expression (6.4), we get (6.5) and (6.6)

Remark 6.1

When X_1 and X_2 are independent, the bivariate hazard rate reduces to a vector with components equal to the hazard rates of X_1 and X_2. As an example, when $\hat{C}(1-u, 1-v) = (1-u)(1-v)$, we have

$$\frac{\partial}{\partial u}\log\left(\hat{C}(1-u,1-v)\right)=\frac{-1}{1-u},$$

and from (6.5), we get

$$h_1(u,v)=\frac{1}{(1-u)\phi'(u)}=h_1(u).$$

Also we have

$$\frac{\partial}{\partial v}\log\left(\hat{C}(1-u,1-v)\right)=\frac{-1}{1-v}.$$

Therefore (6.6) yields

$$h_2(u,v)=\frac{1}{(1-v)\psi'(v)}=h_2(v).$$

Thus we get,

$$\underset{\varepsilon_{--}}{\underline{h}}(u,v)=(h_1(u),h_2(v))=(h_1(u,0),h_2(0,v))$$

Example 6.1

Consider the Gumbel bivariate exponential distribution with survival copula specified by

$$\hat{C}(u,v)=uve^{-\theta\ln u\ln v}$$

with univariate marginals $\phi(u)=-\ln(1-u)$ and $\psi(v)=-\ln(1-v)$ respectively. Direct calculations yield the bivariate hazard rate as

$$\underset{\varepsilon_{--}}{\underline{h}}(u,v)=\left(\theta\ln(1-v)-1,\theta\ln(1-u)-1\right).$$

Next theorem discusses the uniqueness property of the bivariate hazard function.

Theorem 6.1. *For the bivariate random vector **X**, copula $C(u,v)$ can be expressed uniquely in* *terms of* $\underset{\varepsilon_{--}}{\underline{h}}(u, v)$ *through*

$$C(u, v) = (1 - v)(1 - exp[-\int_0^u \frac{h_1(p, v)}{(1 - p)h_1(p, 0)} dp]) - u \tag{6.7}$$

and

$$C(u, v) = (1 - u)(1 - exp[-\int_0^v \frac{h_2(u, p)}{(1 - p)h_2(0, p)} dp] - v. \tag{6.8}$$

Proof. From (6.5), we have

$$h_1(u, v)\phi'(u) = -\frac{\partial}{\partial u} \log \hat{C}(1 - u, 1 - v).$$

Integrating the above expression from 0 to u, we get

$$\int_0^u h_1(p, v)\phi'(p)dp = \log \frac{1 - v}{\hat{C}(1 - u, 1 - v)}$$

or

$$\hat{C}(1 - u, 1 - v) = (1 - v)e^{-\int_0^u h_1(p,v)\phi'(p)dp}. \tag{6.9}$$

Since $\phi'(u) = \frac{1}{(1-u)h_1(u,0)}$, (6.9) gives

$$\hat{C}(1 - u, 1 - v) = (1 - v)e^{-\int_0^u \frac{h_1(p,v)}{(1-p)h_1(p,0)} dp}. \tag{6.10}$$

Using (6.3), (6.10) can be written as

$$C(u, v) = (1 - v)(1 - exp[-\int_0^u \frac{h_1(p, v)}{(1 - p)h_1(p, 0)} dp]) - u.$$

Also $h_2(u, v)$ characterizes the distribution specified by the copula,

$$\hat{C}(1 - u, 1 - v) = (1 - u)e^{-\int_0^v \frac{h_2(u,p)}{(1-p)h_2(0,p)} dp}.$$

(6.8) follows from the above equation. □

Alternate definitions for the bivariate mean residual life function is provided independently by Shanbhag & Kotz (1987) and Arnold & Zahedi (1988). Consider the variable

$$X_{x_i} = X_i - x_i | X_i > x_i, X_j > x_j; i, j = 1, 2; i \neq j.$$

The bivariate mean residual life function of the variable X_{x_i} is defined as the vector

$$\mathbf{M}\left(\underset{-}{x}\right) = \left[M_1(\underset{-}{x}), M_2(\underset{-}{x})\right]$$

where

$$M_1(\underset{-}{x}) = \frac{1}{\bar{F}(\underset{-}{x})} \int_{x_1}^{\infty} \bar{F}(t, x_2) dt$$

and

$$M_2(\underset{-}{x}) = \frac{1}{\bar{F}(\underset{-}{x})} \int_{x_2}^{\infty} \bar{F}(x_1, t) dt.$$

On simplification, $M_1(\underset{-}{x})$ and $M_2(\underset{-}{x})$ becomes

$$M_1(\underset{-}{x}) = -x_1 - \frac{1}{\bar{F}(\underset{-}{x})} \int_{x_1}^{\infty} t \frac{\partial \bar{F}(t, x_2)}{\partial t} dt$$

and

$$M_2(\underset{-}{x}) = -x_2 - \frac{1}{\bar{F}(\underset{-}{x})} \int_{x_2}^{\infty} t \frac{\partial \bar{F}(x_1, t)}{\partial t} dt.$$

Substituiting $x_1 = \phi(u)$, $x_2 = \psi(v)$ and $\bar{F}(\underset{-}{x}) = \hat{C}(1-u, 1-v)$ in the above expressions, we get the copula analogue of bivariate mean residual functions as

$$\underset{\varepsilon_{--}}{\underset{-}{m}}(u, v) = (m_1(u, v), m_2(u, v)) \tag{6.11}$$

where

$$m_1(u,v) = M_1\left[\phi(u),\psi(v)\right] = -\phi(u) - \frac{1}{\hat{C}(1-u,1-v)}\int_u^1 \phi(p)\frac{\partial}{\partial p}\hat{C}(1-p,1-v)dp \quad (6.12)$$

and

$$m_2(u,v) = M_2\left[\phi(u),\psi(v)\right] = -\psi(v) - \frac{1}{\hat{C}(1-u,1-v)}\int_v^1 \psi(p)\frac{\partial}{\partial p}\hat{C}(1-u,1-p)dp. \quad (6.13)$$

Theorem 6.2. *Let **X** be a bivariate random vector admitting an absolutely continuous distribution function and bivariate copula mean residual life function defined by* (6.11). *Then the bivariate copula mean residual life function determines the underlying copula uniquely.*

Proof. Differentiating (6.12) with respect to u, we get

$$\frac{\frac{\partial}{\partial u}\hat{C}(1-u,1-v)}{\hat{C}(1-u,1-v)} = \frac{-\frac{\partial}{\partial u}m_1(u,v) - \phi'(u)}{m_1(u,v)} \quad (6.14)$$

or

$$\frac{\partial}{\partial u}\log\hat{C}(1-u,1-v) = \frac{-\frac{\partial}{\partial u}m_1(u,v) - \phi'(u)}{m_1(u,v)}.$$

That is,

$$\frac{\partial}{\partial u}\log\left(\hat{C}(1-u,1-v)\right) = -\frac{\partial}{\partial u}\log\left(m_1(u,v)\right) - \frac{\phi'(u)}{.}m_1(u,v) \quad (6.15)$$

Nair & Sankaran (2009) has shown that the univariate mean residual quantile function uniquely determines the quantile function through the expression

$$\phi(u) = \mu_1 - m_1(u) + \int_0^u \frac{m_1(p)}{1-p}dp. \quad (6.16)$$

On integration from 0 to u and using (6.16), equation (6.15) becomes

$$\log\left(\frac{\hat{C}(1-u,1-v)}{1-v}\right)=\log\left(\frac{m_1(0,v)}{m_1(u,v)}\right)-\int_0^u\frac{-\frac{\partial m_1(p,0)}{\partial p}+\frac{m_1(p,0)}{1-p}-m_1(0,0)}{m_1(p,v)}dp. \quad (6.17)$$

That is,

$$\hat{C}(1-u,1-v)=\frac{(1-v)m_1(0,v)}{m_1(u,v)}exp\left[-\int_0^u\frac{-\frac{\partial}{\partial p}m_1(p,0)+\frac{m_1(p,0)}{1-p}-m_1(0,0)}{m_1(p,v)}dp\right]$$

or

$$C(u,v)=1-u-v-\frac{(1-v)m_1(0,v)}{m_1(u,v)}exp\left[-\int_0^u\frac{-\frac{\partial}{\partial p}m_1(p,0)+\frac{m_1(p,0)}{1-p}-m_1(0,0)}{m_1(p,v)}dp\right].$$

Also $m_2(u,v)$ uniquely determines the underlying copula $C(u,v)$ as

$$C(u,v)=1-u-v-\frac{(1-u)m_2(u,0)}{m_2(u,v)}exp\left[-\int_0^v\frac{-\frac{\partial}{\partial p}m_2(0,p)+\frac{m_1(0,p)}{1-p}-m_2(0,0)}{m_2(u,p)}dp\right].$$

The theorem is immediate from the above expression for $C(u,v)$. □

Remark 6.2

For the product copula given by

$$C(u,v)=uv,$$

$\hat{C}$ takes the form

$$\hat{C}(1-u,1-v)=(1-u)(1-v).$$

Using the above expression for $\hat{C}$, (6.12) and (6.13) become

$$m_1(u,v) = \frac{1}{1-u}\int_u^1 \phi(p)dp - \phi(u) = m_1(u)$$

and

$$m_2(u,v) = \frac{1}{1-v}\int_v^1 \psi(p)dp - \psi(v) = m_2(v).$$

Therefore

$$\underset{\varepsilon_{--}}{\underline{m}}(u,v) = (m_1(u), m_2(v))$$

The implication of the above is that when we consider the product copula, the corresponding bivariate copula based mean residual function in the direction ε_{--} becomes the vector with components, the marginal mean residual quantile functions of X_1 and X_2 respectively.

Next theorem provides a relationship connecting the bivariate copula based hazard rate and mean residual life functions.

Theorem 6.3. *For the random vector* ***X*** *defined in theorem 6.2, the bivariate copula hazard rate is related to bivariate copula mean residual life function through the relationship*

$$h_1(u,v)m_1(u,v) = (1-u)h_1(u,0)\frac{\partial}{\partial u}m_1(u,v) + 1$$

and

$$h_2(u,v)m_2(u,v) = (1-v)h_2(0,v)\frac{\partial}{\partial v}m_2(u,v) + 1.$$

Proof. From (6.14) and (6.5), we get

$$-h_1(u,v)\phi'(u) = \frac{-\frac{\partial}{\partial u}m_1(u,v) - \phi'(u)}{m_1(u,v)}.$$

The above equation can be written as

$$h_1(u,v) = \frac{\frac{\partial}{\partial u}m_1(u,v) + \phi'(u)}{m_1(u,v)\phi'(u)}. \tag{6.18}$$

Since $\frac{1}{\phi'(u)} = (1-u)h_1(u,0)$, the above equation becomes,

$$h_1(u,v)m_1(u,v) = (1-u)h_1(u,0)\frac{\partial}{\partial u}m_1(u,v) + 1.$$

Proceeding on similar lines, we get

$$h_2(u,v)m_2(u,v) = (1-v)h_2(0,v)\frac{\partial}{\partial v}m_2(u,v) + 1.$$

□

The expression for the joint density function of **X** is given by

$$f(\underline{x}) = \frac{\partial^2}{\partial x_1 x_2}F(\underline{x}).$$

That is,

$$f(\underline{x}) = \frac{\partial}{\partial x_2}\left[\frac{\partial}{\partial x_1}F(\underline{x})\right]. \tag{6.19}$$

The copula analogue of (6.19) is given by

$$f\left[\phi(u),\psi(v)\right] = \frac{1}{\phi'(u)\psi'(v)}\frac{\partial}{\partial u\partial v}C(u,v).$$

Define

$$\mu = (\mu_1(v), \mu_2(u))$$

where

$$\mu_1(v) = \int_0^1 \phi(u)\frac{\partial}{\partial u}\hat{C}(1-u, 1-v)du$$

and

$$\mu_2(u) = \int_0^1 \psi(v)\frac{\partial}{\partial v}\hat{C}(1-u, 1-v)dv.$$

The bivariate copula vitality function can be defined as

$$d_{\varepsilon--}(u, v) = [d_1(u, v), d_2(u, v)]$$

where

$$d_1(u, v) = m_1(u, v) + \phi(u)$$

and

$$d_2(u, v) = m_2(u, v) + \psi(v).$$

That is,

$$d_1(u, v) = \frac{-1}{\hat{C}(1-u, 1-v)} \int_u^1 \phi(p)\frac{\partial}{\partial p}\hat{C}(1-p, 1-v)dp \tag{6.20}$$

and

$$d_2(u, v) = \frac{-1}{\hat{C}(1-u, 1-v)} \int_v^1 \psi(p)\frac{\partial}{\partial p}\hat{C}(1-u, 1-p)dp. \tag{6.21}$$

Following result focuses attention on characterization of copulas using the relation between bivariate hazard function and bivariate vitality function.

Theorem 6.4. *Let $g(.,.)$ be any positive real valued function and $d_{\varepsilon--}(u, v)$ be the bivariate vitality function. Then the relationship*

$$h_1(u, v)g\left[\phi(u), \psi(v)\right] = d_1(u, v) - \mu_1(v) \tag{6.22}$$

holds if

$$\frac{\partial}{\partial u}\hat{C}(1-u,1-v) = \frac{B^*(v)\phi'(u)}{g\left[\phi(u),\psi(v)\right]\phi'(0)}e^{-\int_0^u \frac{\phi'(p)[\phi(p)+\mu_1(u,v)]}{g[\phi(p),\psi(v)]}dp} \tag{6.23}$$

where $B^*(v) = g\left[0,\psi(v)\right]k(v)$ *in which* $k(v) = \frac{\partial}{\partial u}\hat{C}(1-u,1-v)$ *given* $u = 0$ *provided* $\phi'(0) \neq 0$,$g\left[0,\psi(v)\right] \neq 0$ *and* $k(v) \neq 0$.

Proof. From (6.22), (6.5) and (6.20), we get

$$\frac{-\frac{\partial}{\partial u}\hat{C}(1-u,1-v)}{\hat{C}(1-u,1-v)}\frac{g\left[\phi(u),\psi(v)\right]}{\phi'(u)} = \frac{-1}{\hat{C}(1-u,1-v)}\int_u^1 \phi(p)\frac{\partial}{\partial p}\hat{C}(1-p,1-v)dp - \mu_1(v)$$

Differentiating the above expression w.r.t. u, we get

$$\frac{\partial^2}{\partial u^2}\hat{C}(1-u,1-v)\left[\frac{g\left[\phi(u),\psi(v)\right]}{\phi'(u)}\right]$$

$$= \frac{\partial}{\partial u}\hat{C}(1-u,1-v)\left[-g'\left[\phi(u),\psi(v)\right] + \frac{g\left[\phi(u),\psi(v)\right]\phi''(u)}{\left(\phi'(u)\right)^2} - \phi(u) - \mu_1(v)\right].$$

That is,

$$\frac{\partial}{\partial u}\log\left[\frac{\partial}{\partial u}\left(\hat{C}(1-u,1-v\right)\right] = -\frac{\partial}{\partial u}\log\left[g\left(\phi(u),\psi(v)\right)\right] + \frac{\partial}{\partial u}\log\phi'(u) - \frac{\phi'(u)\left[\phi(u)+\mu_1(u,v)\right]}{g\left(\phi(u),\psi(v)\right)}. \tag{6.24}$$

Denote by $G(u,v) = \frac{\partial}{\partial u}\left(\hat{C}(1-u,1-v\right)$, (6.24) can be written as

$$\frac{\partial}{\partial u}\log G(u,v) = -\frac{\partial}{\partial u}\log\left[g\left(\phi(u),\psi(v)\right)\right] + \frac{\partial}{\partial u}\log\phi'(u) - \frac{\phi'(u)\left[\phi(u)+\mu_1(u,v)\right]}{g\left(\phi(u),\psi(v)\right)}. \tag{6.25}$$

Integrating (6.25) from 0 to u and rearranging the terms, we get

$$G(u,v) = \frac{k(v)g\left[0,\psi(v)\right]\phi'(u)}{\phi'(0)g\left[\phi(u),\psi(v)\right]}e^{-\int_0^u \frac{\phi'(p)[\phi(p)+\mu_1(v)]}{g[\phi(p),\psi(v)]}dp}.$$

The above equation can be written as

$$\frac{\partial}{\partial u}\hat{C}(1-u,1-v)=\frac{B^{*}(v)\phi'(u)}{g\left[\phi(u),\psi(v)\right]\phi'(0)}e^{-\int_0^u \frac{\phi'(p)[\phi(p)+\mu_1(v)]}{g[\phi(p),\psi(v)]}dp}$$

where $B^{*}(v)=g\left[0,\psi(v)\right]k(v)$, which is same as (6.23) □

Example 6.2

Consider the Gumbel type dependence with tukey lambda marginal. That is,
$\hat{C}(1-u,1-v)=(1-u)(1-v)e^{-\theta\ln(1-u)\ln(1-v)}$ and $\phi(u)=\frac{u^{\lambda}-(1-u)^{\lambda}}{\lambda}$
Then $h_1(u,v)$ and $d_1(u,v)$ are related as in (6.22) where $g\left[\phi(u),\psi(v)\right]$ can be obtained as the solution of the following differential equation,

$$\left[(1-u)\left(\frac{u^{\lambda-1}+(1-u)^{\lambda-1}}{\theta\ln(1-v)}\right)\right]\frac{\partial}{\partial u}g\left(\phi(u),\psi(v)\right)-\left[\frac{1+(1-u)(\lambda-1)\left(u^{\lambda-2}+(1-u)^{\lambda-2}\right)}{\theta\ln(1-v)\left(u^{\lambda-1}+(1-u)^{\lambda-1}\right)}\right]g\left(\phi(u),\psi(v)\right)$$

$$=\frac{(1-u)\left(u^{\lambda-1}+(1-u)^{\lambda-1}\right)\left((1-u)^{\lambda}-u^{\lambda}-\lambda\mu_1(v)\right)}{\lambda\theta\ln(1-v)}.$$

Example 6.3

When $\hat{C}(1-u,1-v)=(1-u)(1-v)$ and the tukey lambda marginal as $\phi(u)$, we get the univariate expressions for all concepts and $g\left[\phi(u),\psi(v)\right]$ has a closed form given by

$$g\left[\phi(u),\psi(v)\right]=\frac{\left[u^{\lambda-1}+(1-u)^{\lambda-1}\right]\left[u^{\lambda}-(1-u)^{\lambda}\right]\left[u^{\lambda-1}+(1-u)^{\lambda-1}+\mu\right]}{\lambda(\lambda-1)\left[u^{\lambda-2}+(1-u)^{\lambda-2}\right]}.$$

6.3 Concepts in reversed time

Roy (2002) has defined the bivariate reversed hazard rate as a vector analogous to the definition of vector valued hazard rate extensively discussed in Johnson & Kotz (1975) and examined its properties. Also the author has proposed a class of bivariate distributions using this vector.

Later Sankaran & Gleeja (2006) developed a more general class of bivariate distributions which extends the result given in Roy (2002). Roy (2002) defines the bivariate reversed hazard rate as the vector

$$\mathbf{A}(\underset{\sim}{x}) = \left(A_1(\underset{\sim}{x}), A_2(\underset{\sim}{x})\right),$$

where

$$A_i\left(\underset{\sim}{x}\right) = \frac{\partial}{\partial x_i}\log F\left(\underset{\sim}{x}\right), i = 1, 2.$$

Setting $F(\underset{\sim}{x}) = C(u, v)$, $x_1 = \phi(u)$ and $x_2 = \psi(v)$, we get the bivariate reversed hazard rate, in the copula setup as

$$\underset{\sim}{a}_{\varepsilon_{--}}(u, v) = (a_1(u, v), a_2(u, v))$$

where

$$a_1(u, v) = \frac{\frac{\partial}{\partial u}C(u, v)}{\phi'(u)C(u, v)} \tag{6.26}$$

and

$$a_2(u, v) = \frac{\frac{\partial}{\partial u}C(u, v)}{\psi'(v)C(u, v)}. \tag{6.27}$$

Example 6.4 For the Gumbel bivariate logistic distribution with copula specified by

$$C(u, v) = \frac{uv}{1 - (1 - u)(1 - v)}$$

and marginal quantile functions of X_1 and X_2 with $\phi(u) = -\log\left[1 - \frac{1}{u}\right]$ and $\psi(v) = -\log\left[1 - \frac{1}{v}\right]$ respectively, we calculate the bivariate copula reversed hazard rate as follows.

$$\frac{\partial}{\partial u}\log C(u, v) = \frac{1}{u} - \frac{1 - v}{1 - (1 - u)(1 - v)}.$$

From (6.26) and (6.27), we get

$$\underset{\sim}{a}_{\varepsilon_{--}}(u, v) = \left[\frac{(1 - u)v}{1 - (1 - u)(1 - v)}, \frac{v^2(1 - u)}{u\left[1 - (1 - u)(1 - v)\right]}\right].$$

The definition and properties of the reversed mean residual life are given in Finkelstein (2003) and Nanda et al. (2003). Nair & Asha (2008) has discussed the definition and properties of the bivariate reversed mean residual life and studied the relationship between bivariate reversed hazard rate and bivariate reversed mean residual life. Nair & Asha (2008) defines the bivariate reversed mean residual life as

$$\mathbf{R}(\underline{x}) = \Big(R_1(\underline{x}), R_2(\underline{x})\Big)$$

where

$$R_1(\underline{x}) = \frac{1}{\bar{F}(\underline{x})} \int_0^{x_1} F(t, x_2)dt$$

and

$$R_2(\underline{x}) = \frac{1}{\bar{F}(\underline{x})} \int_0^{x_2} F(x_1, t)dt.$$

We define the bivariate copula reversed mean residual life denoted by $\underline{r}_{\underline{\varepsilon}}(u, v)$ as

$$\underline{r}_{\underline{\varepsilon}}(u, v) = (r_1(u, v), r_2(u, v))$$

where

$$r_1(u, v) = \phi(u) - \frac{1}{C(u, v)} \int_0^u \phi(p) \frac{\partial}{\partial p} C(p, v)dp \tag{6.28}$$

and

$$r_2(u, v) = \psi(v) - \frac{1}{C(u, v)} \int_0^v \psi(p) \frac{\partial}{\partial p} C(p, v)dp. \tag{6.29}$$

Next we examine whether the bivariate copula reversed hazard rate and bivariate copula reversed mean residual life determine the copula uniquely.

Theorem 6.5. *Bivariate copula reversed hazard rate and the bivariate copula reversed mean residual function determine the underlying copula uniquely.*

Proof. From (6.26), it is observed that

$$\frac{\partial}{\partial u}\log C(u,v) = \phi'(u)a_1(u,v).$$

Integrating the above equation from u to 1 and rearranging the terms, we get

$$C(u,v) = ve^{-\int_u^1 a_1(p,v)\phi'(p)dp}. \tag{6.30}$$

From the definition of reversed hazard quantile function, we have

$$a_1(u,0) = \frac{1}{u\phi'(u)}.$$

Therefore (6.30) becomes,

$$C(u,v) = ve^{-\int_u^1 \frac{a_1(p,v)}{pa_1(p,0)}dp}.$$

Proceeding on the similar lines, we can also get

$$C(u,v) = ue^{-\int_v^1 \frac{a_2(u,p)}{pa_2(p,0)}dp}.$$

Further differentiating (6.28) with respect to u, we get

$$\frac{\partial}{\partial u}\log C(u,v) = \frac{\phi'(u) - \frac{\partial}{\partial u}r_1(u,v)}{r_1(u,v)}. \tag{6.31}$$

Integrating from u to 1 and simplifying we get

$$C(u,v) = ve^{-\int_u^1 \frac{\frac{\partial}{\partial p}r_1(p,v)-\phi'(p)}{r_1(p,v)}dp}. \tag{6.32}$$

Also $r_2(u,v)$ uniquely determines the copula by the relation

$$C(u,v) = ue^{-\int_v^1 \frac{\frac{\partial}{\partial p} r_2(u,p) - \psi'(p)}{r_2(u,p)} dp}. \tag{6.33}$$

In (6.32) and(6.33), $\phi(u)$ and $\psi(v)$ can be replaced by the following expressions given in Nair & Sankaran (2009),

$$\phi(u) = r_1(u) + \int_0^u p^{-1} r_1(p) dp$$

and

$$\psi(v) = r_2(v) + \int_0^v p^{-1} r_2(p) dp.$$

□

The following theorem discusses the relation between reversed hazard rate and reversed mean residual life in copula setup.

Theorem 6.6. *The bivariate copula reversed hazard rate is related to bivariate copula reversed mean residual life by the expression*

$$r_1(u,v)a_1(u,v) = 1 - ua_1(u,0)\frac{\partial}{\partial u} r_1(u,v)$$

and

$$r_2(u,v)a_2(u,v) = 1 - va_2(0,v)\frac{\partial}{\partial v} r_2(u,v).$$

Proof. Using (6.26) and (6.31), we get

$$a_1(u,v) = \frac{\phi'(u) - \frac{\partial}{\partial u} r_1(u,v)}{r_1(u,v)\phi'(u)}$$

That is,

$$a_1(u,v) = \frac{1 - ua_1(u,0)\frac{\partial}{\partial u} r_1(u,v)}{r_1(u,v)}$$

This gives,

$$r_1(u,v)a_1(u,v) = 1 - ua_1(u,0)\frac{\partial}{\partial u}r_1(u,v).$$

Proceeding on the similar lines we can also get

$$r_2(u,v)a_2(u,v) = 1 - va_2(0,v)\frac{\partial}{\partial v}r_2(u,v).$$

□

When the component variates are independent, the above mentioned bivariate properties will be reduce to the corresponding univariate concepts.
For example, when $C(u,v) = uv$,we have

$$\underset{\varepsilon_{--}}{\underline{a}}(u,v) = (a_1(u), a_2(v))$$

and

$$\underset{\varepsilon_{--}}{\underline{r}}(u,v) = (r_1(u), r_2(v)).$$

Theorem 6.7. *The relationship*

$$A_1(u,v)g_r\left[\phi(u),\psi(v)\right] = D_1(u,v) - \mu_1(v)$$

holds if

$$\frac{\partial}{\partial u}C(u,v) = \frac{b^*(v)\phi'(u)}{g_r\left[\phi(u),\psi(v)\right]\phi'(0)}e^{\int_0^u \frac{\phi'(p)[\phi(p)-\mu_1(v)]}{g_r[\phi(p),\psi(v)]}dp}$$

where $b^*(v) = g_r\left[0,\psi(v)\right]K(v)$ *in which* $K(v) = \frac{\partial}{\partial u}C(u,v)$ *given* $u = 0$ *provided* $\phi'(0) \neq 0$,$g_r\left[0,\psi(v)\right] \neq 0$ *and* $K(v) \neq 0$ *and* $D_1(u,v) = r_1(u,v) + \phi(u)$

The proof is similar to that of Theorem 6.4 and hence omitted.

Guidelines for further research

In the present work, we have examined the potential of Zenga curve as an alternate measure of inequality. In addition to examining the connection between the measure and other existing inequality measures, the relationship of the concept with certain reliability concepts are exploited to obtain characterization results for probability distributions. Further some results on a stochastic order using Zenga curve are also established. Instead of using the conventional distribution functional approach, the definitions and concepts are reformulated using quantiles.

During the course of present study, we are able to identify the following problems which require further investigation.

1. Since incomes are measured at specific points of time and a detailed study on the inequality measures in discrete time is to be undertaken.

2. Inference procedures such as estimation of Zenga index based on observed income data, formulation of tests for exponentiality using the truncated measures of inequality is yet to be studied.

3. Only very little work seems to have been done on bivariate copula in higher dimensions. Developing these ideas considering the same in four directions shall pave way for theoretical foundations in higher dimensions.

4. Several other quantile function based models for income data can be developed in varying situations and this may help to model income data.

5. The implication of stochastic orders based on inequality measures shall be studied in detail for other existing orders also.

We hope that the problems mentioned above shall be sorted out in a future work.

Bibliography

Aaberge, R. (2000) Characterizations of lorenz curves and income distributions. Social Choice and Welfare 17:639–653.

Abouammoh, A. & El-Neweihi, E. (1986) Closure of the NBUE and DRMRL classes under the formation of parallel systems. Statistics & probability letters 4:223–225.

Aggarwal, V. (1984) On optimum aggregation of income distribution data. Sankhyā: The Indian Journal of Statistics, Series B 343–355.

Ahmad, I. A. & Mugdadi, A. (2004) Further moments inequalities of life distributions with hypothesis testing applications: the IFRA, NBUC and DMRL classes. Journal of statistical planning and inference 120(1):1–12.

Alzaid, A. A. (1988) Mean residual life ordering. Statistical Papers 29:35–43.

Arnold, B. (1983) Pareto distributions. International Cooperative Publishing House, Fairland, MD .

Arnold, B. C. & Zahedi, H. (1988) On multivariate mean remaining life functions. Journal of multivariate analysis 25:1–9.

Atkinson, A. B. (1970) On the measurement of inequality. Journal of economic theory 2:244–263.

Averous, J. & Meste, M. (1997) Median balls: an extension of the interquantile intervals to multivariate distributions. Journal of Multivariate Analysis 63:222–241.

Barlow, R. E. & Proschan, F. (1975) Statistical theory of reliability and life testing: probability models. Holt, Rinehart and Winston Inc., New York.

Basu, A. (1971) Bivariate failure rate. Journal of the American Statistical Association 66:103–104.

Belzunce, F., Castano, A., Olvera-Cervantes, A. & Suárez-Llorens, A. (2007) Quantile curves and dependence structure for bivariate distributions. Computational Statistics and Data Analysis 51:5112–5129.

Bergman, B. (1979) On age replacement and the total time on test concept. Scandinavian Journal of Statistics 6:161–168.

Berti, P. & Rigo, P. (2006) Concentration curve and index, zenga's. Encyclopedia of Statistical Sciences. John Wiley and Sons .

Bhattacharjee, M. (1993) How rich are the rich? modeling affluence and inequality via reliability theory. Sankhyā: The Indian Journal of Statistics, Series B 55:1–26.

Bhattacharya, N. & Mahalanobis, B. (1967) Regional disparities in household consumption in india. Journal of the American Statistical Association 62:143–161.

Block, H. W., Savits, T. H. & Singh, H. (1998) The reversed hazard rate function. Probability in the Engineering and Informational Sciences 12:69–90.

Bonferroni, C. (1930) Elementi di Statistica Generale. Seeber - Firenze.

Bradford, S. C. (1985) Sources of information on specific subjects. Journal of information Science 10:173–180.

Bryson, M. C. & Siddiqui, M. (1969) Some criteria for aging. Journal of the American Statistical Association 64:1472–1483.

Burrell, Q. L. (1991) The bradford distribution and the gini index. Scientometrics 21:181–194.

Burrell, Q. L. (2005) Symmetry and other transformation features of lorenz/leimkuhler representations of informetric data. Information processing and management 41:1317–1329.

Chandra, M. & Singpurwalla, N. D. (1981) Relationships between some notions which are common to reliability theory and economics. Mathematics of Operations Research 6:113–121.

Chatterjee, G. (1976) Disparities in per capita household consumption in india: A note. Economic and Political Weekly 557–567.

Chen, L.-A. & Welsh, A. (2002) Distribution-function-based bivariate quantiles. Journal of multivariate analysis 83:208–231.

Chotikapanich, D. (1993) A comparison of alternative functional forms for the lorenz curve. Economics Letters 41:129–138.

Cox, D. R. (1962) Renewal theory,Science paperbacks. Chapman and Hall, London.

Cox, D. R. (1992) Regression models and life-tables. In: Breakthroughs in Statistics, Springer 527–541.

Dalton, H. (1920) The measurement of the inequality of incomes. The Economic Journal 30:348–361.

Dancelli, L. (1990) On the behaviour of the z_p concentration curve. In: Income and Wealth Distribution, Inequality and Poverty, Springer 111–127.

Deshpande, J. V., Kochar, S. C. & Singh, H. (1986) Aspects of positive ageing. Journal of Applied Probability 748–758.

Egghe, L. (2002) Construction of concentration measures for general lorenz curves using riemann-stieltjes integrals. Mathematical and Computer Modelling 35:1149–1163.

Egghe, L. (2005a) Power laws in the information production process: Lotkaian informetrics. Elsevier.

Egghe, L. (2005b) Zipfian and lotkaian continuous concentration theory. Journal of the American Society for Information Science and Technology 56:935–945.

Egghe, L. & Rousseau, R. (1988) Reflections on a deflection: A note on different causes of the groos droop. Scientometrics 14:493–511.

Einhorn, H. A. (1962) Changes in concentration of domestic manufacturing establishment output: 1939–1958. Journal of the American Statistical Association 57:797–803.

Esteban, J. (1986) Income-share elasticity and the size distribution of income. International Economic Review 27:439–444.

Fernández-Ponce, J. & Suarez-Llorens, A. (2003) A multivariate dispersion ordering based on quantiles more widely separated. Journal of Multivariate Analysis 85:40–53.

Finkelstein, M. (2003) On one class of bivariate distributions. Statistics & probability letters 65:1–6.

Foster, J., Greer, J. & Thorbecke, E. (1984) A class of decomposable poverty measures. Econometrica: Journal of the Econometric Society 761–766.

Gajdos, T. & Weymark, J. A. (2005) Multidimensional generalized gini indices. Economic Theory 26:471–496.

Galambos, J. & Kotz, S. (1978) Characterizations of probability distributions. Springer-Verlag, Newyork.

Gastwirth, J. L. (1971) A general definition of the lorenz curve. Econometrica: Journal of the Econometric Society 1037–1039.

Gilchrist, W. (2000) Statistical modelling with quantile functions. CRC Press, Florida.

Gini, C. (1912) Variabilita e Mutabilita. Bologna.

Giorgi, G. M. & Crescenzi, M. (2001a) A look at the bonferroni inequality measure in a reliability framework. Statistica LXL 4:571–583.

Giorgi, G. M. & Crescenzi, M. (2001b) A proposal of poverty measures based on the bonferroni inequality index. Metron 59:3–16.

Govindarajulu, Z. (1977) A class of distributions useful in life testing and reliability with applications to non-parametric testing. Theory and Applications of Reliability 1:109–130.

Greenwood, J. A., Landwehr, J. M., Matalas, N. C. & Wallis, J. R. (1979) Probability weighted moments: definition and relation to parameters of several distributions expressable in inverse form. Water Resources Research 15:1049–1054.

Greselin, F. & Pasquazzi, L. (2009) Asymptotic confidence intervals for a new inequality measure. Communications in Statistics-Simulation and Computation 38:1742–1756.

Greselin, F., Puri, M. L. & Zitikis, R. (2009) L-functions, processes, and statistics in measuring economic inequality and actuarial risks. Statistics and Its Interface 2:227–245.

Greselin, F., Pasquazzi, L. & Zitikis, R. (2010) Zenga's new index of economic inequality, its estimation, and an analysis of incomes in italy. Journal of Probability and Statistics 2010:1–26.

Gupta, M. R. (1984) Functional form for estimating the lorenz curve. Econometrica 52:1313–1314.

Gupta, R. C. (2007) Role of equilibrium distribution in reliability studies. Probability in the Engineering and Informational Sciences 21:315.

Gupta, R. C. & Kirmani, S. (1990) The role of weighted distributions in stochastic modeling. Communications in Statistics-Theory and methods 19:3147–3162.

Hankin, R. K. & Lee, A. (2006) A new family of non-negative distributions. Australian and New Zealand Journal of Statistics 48:67–78.

Hastings, C., Mosteller, F., Tukey, J. W. & Winsor, C. P. (1947) Low moments for small samples: a comparative study of order statistics. The Annals of Mathematical Statistics 18:413–426.

Hesselager, O., Wang, S. & Willmot, G. (1998) Exponential and scale mixtures and equilibrium distributions. Scandinavian Actuarial Journal 1998:125–142.

Hogben, D. (1963) Some properties of tukey's test for non-additivity. Ph.D. thesis, The State University of New Jersey. Unpublished .

Hosking, J. R. (1990) L-moments: analysis and estimation of distributions using linear combinations of order statistics. Journal of the Royal Statistical Society. Series B 52:105–124.

Johnson, N. L. & Kotz, S. (1975) A vector multivariate hazard rate. Journal of Multivariate Analysis 5:53–66.

Kakwani, N. C. & Podder, N. (1973) On the estimation of lorenz curves from grouped observations. International Economic Review 14:278–292.

Keilson, J. & Sumita, U. (1982) Uniform stochastic ordering and related inequalities. Canadian Journal of Statistics 10:181–198.

Kendall, M. G. (1940) Note on the distribution of quantiles for large samples. Supplement to the Journal of the Royal Statistical Society 7:83–85.

Klefsjö, B. (1982) The HNBUE and HNWUE classes of life distributions. Naval Research Logistics Quarterly 29:331–344.

Klefsjö, B. (1984) Reliability interpretations of some concepts from economics. Naval research logistics quarterly 31:301–308.

Kleiber, C. & Kotz, S. (2003) Statistical size distributions in economics and actuarial sciences. Wiley Interscience.

Kochar, S. & Xu, M. (2009) Connections between some concepts in reliability and economics. Modeling, computation and optimization 6:45.

Koltchinskii, V. (1997) M-estimation, convexity and quantiles. The annals of Statistics 435–477.

Koshevoy, G. & Mosler, K. (1996) The lorenz zonoid of a multivariate distribution. Journal of the American Statistical Association 91:873–882.

Koshevoy, G. & Mosler, K. (1997) Zonoid trimming for multivariate distributions. The Annals of Statistics 1998–2017.

Kupka, J. & Loo, S. (1989) The hazard and vitality measures of ageing. Journal of applied probability 532–542.

Lai, C. D. & Xie, M. (2006) Stochastic ageing and dependence for reliability. Springer New York.

Ma, C. (1997) A note on stochastic ordering of order statistics. Journal of Applied Probability 785–789.

Maffenini, W. & Polisicchio, M. (2010) How potential is the $I(p)$ inequality curve in the analysis of empirical distributions. Dipartimento di Metodi Quantitativi per le Scienze Economiche ed Aziendali; Università degli studi di Milano-Bicocca .

Marshall, A. W. (1975) Some comments on the hazard gradient. Stochastic processes and their Applications 3:293–300.

Massé, J.-C. & Theodorescu, R. (1994) Halfplane trimming for bivariate distributions. Journal of Multivariate Analysis 48:188–202.

Mosler, K. C. (2002) Multivariate dispersion, central regions and depth: the lift zonoid approach, vol. 165. Lecture notes in Statistics. Springer.

Müller, A. & Stoyan, D. (2002) Comparison methods for stochastic models and risks. John Wiley and Sons Ltd., Chichester.

Nair, K. R. M. & Rajesh, G. (2000) Geometric vitality function and its applications to reliability. IAPQR TRANSACTIONS 25:1–8.

Nair, K. R. M. & Sreelakshmi, N. (2012) The new Zenga curve in the context of reliability analysis. Communication in Statistics-Theory and methods(to appear) .

Nair, N. U. & Asha, G. (2004) Characterizations using failure and reversed failure rates. Journal of the Indian Society for Probability and Statistics 8:45–56.

Nair, N. U. & Asha, G. (2008) Some characterizations based on bivariate reversed mean residual life. In: ProbStat Forum. 1:1–14.

Nair, N. U. & Sankaran, P. G. (2009) Quantile-based reliability analysis. Communications in Statistics-Theory and Methods 38:222–232.

Nair, N. U. & Vineshkumar, B. (2010) L-moments of residual life. Journal of Statistical Planning and Inference 140:2618–2631.

Nair, N. U. & Vineshkumar, B. (2011) Ageing concepts: An approach based on quantile function. Statistics and Probability Letters 81:2016–2025.

Nair, N. U., Sankaran, P. G. & Vineshkumar, B. (2008) Total time on test transforms of order n and their implications in reliability analysis. Journal of Applied Probability 45:1126–1139.

Nair, N. U., Nair, K. R. M. & Sreelakshmi, N. (2012) Some properties of the new Zenga curve. Statistica and Applicazioni X:43–52.

Nair, N. U., Sankaran, P. G. & Balakrishnan, N. (2013) Quantile-based reliability concepts. In: Quantile-Based Reliability Analysis, Springer 29–58.

Nanda, A. K., Singh, H., Misra, N. & Paul, P. (2003) Reliability properties of reversed residual lifetime. Communications in Statistics-Theory and Methods 32:2031–2042.

Nelsen, R. B. (1999) An introduction to copulas. Springer.

Nolan, D. (1992) Asymptotics for multivariate trimming. Stochastic processes and their applications 42:157–169.

Ord, J., Patil, G. & Taillie, C. (1983) Truncated distributions and measures of income inequality. Sankhyā: The Indian Journal of Statistics, Series B 413–430.

Parzen, E. (1979) Nonparametric statistical data modeling. Journal of the American Statistical Association 74:105–121.

Polisicchio, M. (2008) The continuous random variable with uniform point inequality measure $I(p)$. Statistica and applicazioni 6:137–151.

Porro, F. (2008) Equivalence between partial order based on curve $L(p)$ and partial order based on curve $I(p)$ Proceedings of SIS .

Porro, F. (2011) The distribution model with linear inequality curve $I(p)$. Statistica and Applicazioni 9:47–61.

Pundir, S., Arora, S. & Jain, K. (2005) Bonferroni curve and the related statistical inference. Statistics and probability letters 75:140–150.

Radaelli, P. (2010) On the decomposition by subgroups of the gini index and zenga's uniformity and inequality indexes. International Statistical Review 78:81–101.

Rohde, N. (2008) Lorenz curves and generalised entropy inequality measures. In: Modeling Income Distributions and Lorenz Curves, Springer 271–283.

Rohde, N. (2009) An alternative functional form for estimating the lorenz curve. Economics Letters 105:61–63.

Rolski, T. (1975) Mean residual life. Bulletin of International Statistical Institute 46:266–270.

Rousseau, R. (1987) The nuclear zone of a leimkuhler curve. Journal of Documentation 43:322–333.

Roy, D. (2002) A characterization of model approach for generating bivariate life distributions using reversed hazard rates. Journal of Japan Statistical Society 32:239–245.

Sankaran, P. & Gleeja, V. (2006) On bivariate reversed hazard rates. Journal of the Japan Statistical Society 36:213–224.

Sarabia, J.-M. (1997) A hierarchy of lorenz curves based on the generalized tukey's lambda distribution. Econometric Reviews 16:305–320.

Sarabia, J.-M., Castillo, E. & Slottje, D. J. (2001) An exponential family of lorenz curves. Southern Economic Journal 748–756.

Sarabia, J. M., Castillo, E., Pascual, M. & Sarabia, M. (2005) Mixture lorenz curves. Economics Letters 89:89–94.

Sarabia, J. M., Gómez-Déniz, E., Sarabia, M. & Prieto, F. (2010a) A general method for generating parametric lorenz and leimkuhler curves. Journal of Informetrics 4:524–539.

Sarabia, J. M., Prieto, F. & Sarabia, M. (2010b) Revisiting a functional form for the lorenz curve. Economics Letters 107:249–252.

Sathar, A. E. I., Suresh, R. & Nair, K. R. M. (2007) A vector valued bivariate gini index for truncated distributions. Statistical Papers 48:543–557.

Scarsini, M. & Shaked, M. (1990) Some conditions for stochastic equality. Naval Research Logistics (NRL) 37:617–625.

Sen, A. (1976) Poverty: an ordinal approach to measurement. Econometrica: Journal of the Econometric Society 219–231.

Sen, P. K. (1986) The gini coefficient and poverty indexes: Some reconciliations. Journal of the American Statistical Association 81:1050–1057.

Serfling, R. (2002) Quantile functions for multivariate analysis: approaches and applications. Statistica Neerlandica 56:214–232.

Shaked, M. & Shanthikumar, J. G. (2007) Stochastic orders. Springer.

Shanbhag, D. & Kotz, S. (1987) Some new approaches to multivariate probability distributions. Journal of multivariate analysis 22:189–211.

Sillitto, G. (1969) Derivation of approximants to the inverse distribution function of a continuous univariate population from the order statistics of a sample. Biometrika 56:641–650.

Sklar, A. (1959) Fonctions de reparition a n dimensions et leurs marges. Publications de l'Institut de Statistique de l'Universite de Paris 8:229-231.

Sun, K. & Basu, A. P. (1995) A characterization of a bivariate geometric distribution. Statistics and Probability Letters 23:307–311.

Sunoj, S. M. & Maya, S. S. (2008) The role of lower partial moments in stochastic modeling. Metron 66:223–242.

Taguchi, T. (1972) On the Two-Dimensional Concentration Surface and Extensions of Concentration Coefficient and Pareto Distribution to the Two-Dimensional Case-I. Annals of the Institute of Statistical Mathematics 24:355–381.

Takayama, N. (1979) Poverty, income inequality, and their measures: Professor sen's axiomatic approach reconsidered. Econometrica: Journal of the Econometric Society 747–759.

Theil, H. (1967) Economics and information theory, vol. 7. North-Holland Amsterdam.

Tukey, J. W. (1962) The future of data analysis. The Annals of Mathematical Statistics 33:1–67.

Tukey, J. W. (1977) Exploratory data analysis. Addisson-Weslley.

Wang, Z., Ng, Y.-K. & Smyth, R. (2007) Revisiting the ordered family of lorenz curves. Discussion paper No. 19/07. Department of Economics, Monash University.

Willmot, G. E. & Cai, J. (2000) On classes of lifetime distributions with unknown age. Probability in the Engineering and Informational Sciences 14:473–484.

Yitzhaki, S. (2003) Gini's mean difference: A superior measure of variability for non-normal distributions. Metron 61:285–316.

Zenga, M. (1984) Proposta per un Indice di Concentrazione Basato sui Rapportitra Quantili di Popolazione e Quantili di Reddito. Giornale degli Economisti e Annali di Economia pp. 301–326.

Zenga, M. (1990) Concentration Curves and Concentration Indexes Derived from Them. In: Income and Wealth Distribution: Inequality and Poverty. Springer 94–110.

Zenga, M. (2007) Inequality curve and inequality index based on the ratios between lower and upper arithmetic means. Statistica and Applicazioni 5:3–27.

Zenga, M. (2008) An extension of the inequality index I and the $I(p)$ curve to non economic variables. In: XLIV Riunione Scientifica della SIS.