Stochastic Models

STUDIES ON INVENTORY WITH POSITIVE SERVICE TIME UNDER LOCAL PURCHASE DRIVEN BY N/T-POLICY

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by

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CERTIFICATE

This is to certify that the thesis entitled "**Studies on Inventory with Positive Service Time Under Local Purchase Driven by N/T-Policy**" submitted to the Cochin University of Science & Technology by Ms. Resmi Varghese for the award of degree of Doctor of Philosophy under the Faculty of Science is a bonafide record of research carried out by her under my supervision in the Department of Mathematics, Cochin University of Science & Technology. The results embodied in the thesis have not been included in any other thesis submitted previously for the award of any degree or diploma.

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DECLARATION

I, Resmi Varghese, hereby declare that this thesis entitled "**Studies on Inventory with Positive Service Time Under Local Purchase Driven by N/T-Policy**" is the outcome of the original work done by me and that, the work did not form part of any dissertation submitted for the award of any degree, diploma, associateship, or any other title or recognition from any University/Institution and that to the best of my knowledge and belief, it contains no material previously published by any person except where due reference are made in the text of the thesis.

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CERTIFICATE

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To

My Parents

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Resmi Varghese

STUDIES ON INVENTORY WITH POSITIVE SERVICE TIME UNDER LOCAL PURCHASE DRIVEN BY N/T-POLICY

Contents

List of Notations and Abbreviations

CTMC	:	Continuous-Time Markov Chain.
LIQBD	:	Level-Independent Quasi-Birth-Death process.
PH	:	Phase-Type.
$\bar{\mathbf{e}}$	:	Column vector of 1's of appropriate dimension.
O	:	Zero matrix of appropriate dimension.
$\mathbf{0}$	:	A vector consisting of 0's, with appropriate dimension.
I_r	:	Identity matrix of dimension r.

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Chapter 1

Introduction

The term 'inventory' usually refers to items or goods that are kept in a system for the purpose of business. Inventory management is primarily concerned with identifying the number of the item that are to be kept in stock and also with making decision about the exact level or situation at which an order is to be placed, and also about how much to order.

Inventory systems are usually managed by one of the three following replenishment policies: (1) order up to level S (2) random size (3) fixed ordering quantity. In order up to level S policy, the number of items ordered equals the number of items required to bring the level back to S at the time of replenishment. In random size order policy, the decision on number of the item to be ordered is based on a discrete probability function on the set of integers $\{1, 2, 3, \ldots, S\}$. In fixed order quantity, the number of items ordered is fixed and is equal to $Q = S - s$. In an (s, Q) inventory system, we take s as the reorder level and Q as the fixed ordering quantity. Usually Q is taken sufficiently large so that once the replenishment occurs, the inventory level in the system is greater than the reorder level s. Status of an (s, Q) inventory system with positive lead time can be observed from the inventory level; that is, the number of items in stock reflects whether the replenishment has occurred or not since the most recent replenishment order is placed.

In all the work reported on inventory systems before 1992, it was assumed that serving of items to the customers is instantaneous; that is, only negligible service time is required to deliver the item to the customer. We call inventory problems with negligible service time, as 'classical type', where the customer is supplied the required item instantaneously, provided that sufficient inventory is available on stock at the epoch of arrival of a demand. The results dealing with inventory with negligible service time is insufficient to handle many of the real-life situations which involve positive service time to give away the inventory on hand. When a positive amount of service time is needed to give away the on hand inventory, it may result in queueing up of the customers, and it is advisable to maintain a balance between waiting of customers and the number of items in the inventory as safety stock and also the quantity to be ordered at an order placement epoch. There comes the relevance of analyzing such problems.

If the items are assumed to have infinite life time, then they are said to be non-perishable. But practically this need not be the situation in many cases. We can find certain items having only fixed or finite lifetime in several cases. Such goods are refered to as perishable items. Food items having natural degradation, mobile recharge coupons with prespecified validity, medicines labelled with exact month of expiry etc. can be considered as good examples of such items.

Also there can be perishability independent of expiry, where items are always subject to decay, that is a fixed ratio of inventory is lost in a continuous manner during the course of inventory management. Radioactive substances, alcohol, petrol, diesel etc. are good examples of such inventory. The term perishability may also be viewed in another dimension—the inventory becomes outdated when new versions of the model arise in the market. In such a case, the old model may be left unsold and finally may forcefully get discarded. Electronic goods, ready made dresses, fashion accessories, motor cars etc. are good examples of such inventory. Physically such goods are not perished but they become obsolete.

In stochastic perishable inventory problems, the system state may be reviewed continuously or periodically, the first refers to review of inventory level in a continuous manner, and the second refers to review of inventory level at discrete points in time. During earlier times, periodic review was commonly practised but, now-a-days continuous review of stock is becoming more and more common owing to automation in inventory management.

In an (s, S) production inventory system, when the inventory level reaches s, production process is switched 'on' and is kept in the 'on' mode till the inventory level reaches S. In such inventory systems, information regarding the status of the production process that is, whether is in 'on' or 'off' mode, as well as the inventory position are to be observed to understand the system status. We consider situations where items produced in the production plant require a positive amount of time to get processed before it is served to the customer. The time taken to process a demand is referred to as the service time in production inventory.

Coxian distributions are usually applied in Computer science and telecommunication. This distribution is denoted by C_m or K_m. It is a special case of phase-type distribution.

Matrix Analytic Methods are considered as powerful tools that help us in analyzing complex stochastic problems. The methods can be used to measure the performance of various real life models, of which the most common example is that of telecommunication systems.

1.1 Preliminaries

1.1.1 Markov Chain

A stochastic process $\{X(t), t \in T\}$ is said to be a Markov chain if it satisfies the condition $P(X(t_n) = x_n | X(t_{n-1}) = x_{n-1}, X(t_{n-2}) = x_{n-2}, \ldots, X(t_0) = x_0) = P(X(t_n) = x_n | X(t_{n-1}) = x_{n-1})$, for $t_0 < t_1 < \cdots < t_n$ and $x_i \in S$ for all $i = 0, 1, \ldots, n$ where S is the state space of the process.

1.1.2 Poisson Process

Let $N(t)$ denote the number of occurrences of a specified event in an interval of length t. Let $P_n(t) = P(N(t) = n)$, $n = 0, 1, 2, \ldots$. Then $\{N(t), t \geq 0\}$

is said to be a Poisson process with parameter λ if it satisfies the following postulates:

(i) Independent increments: The number of events occuring in two disjoint intervals of time are independent.

i.e. if $t_0 < t_1 < \ldots$, then the increments $N(t_1) - N(t_0)$, $N(t_2) - N(t_1), \ldots$ are independent random variables.

(ii) Homogeneity in time: The random variable $\{N(t+s) - N(s)\}$ depends on the length of the interval t and not on s or on the value of $N(s)$.

(iii) Regularity or orderliness: Let h be an interval of infinitesimal length. Then the probability of exactly one occurrence in h is

$$P_1(h) = \lambda h + o(h)$$

and the probability of two or more occurrence in h is

$$\sum_{k=2}^{\infty} P_k(h) = o(h).$$

Then $N(t)$ will follow Poisson distribution with parameter λt.

1.1.3 Exponential Distribution

A continuous random variable X is said to follow exponential distribution with parameter λ if its probability density function is $f(x) = \lambda e^{-\lambda x}$, $x \geq 0$, $\lambda > 0$. Exponential distribution is the only univariate continuous distribution having memoryless property and so this distribution has great importance in applied probability.

Memoryless property. Let X be a random variable following exponential distribution. Then $P(X > t + x | X > t) = P(X > x)$ for any $t \geq 0$.

1.1.4 PH-Distribution

Consider a Markov chain on the states $\{1, 2, \ldots, m+1\}$ with infinitesimal generator

$$Q = \begin{bmatrix} T & \mathbf{T^0} \\ \mathbf{0} & 0 \end{bmatrix}$$

where the $m \times m$ matrix T satisfies $T_{ii} < 0$, for $1 \leq i \leq m$, and $T_{ij} \geq 0$, for $i \neq j$. Also $T\bar{\mathbf{e}} + \mathbf{T^0} = \mathbf{0}$, where $\bar{\mathbf{e}}$ is a column vector of 1's of appropriate order. Let the initial probability vector of Q be $(\boldsymbol{\alpha}, \alpha_{m+1})$, with $\boldsymbol{\alpha}\bar{\mathbf{e}} + \alpha_{m+1} = 1$. The states $1, 2, \ldots, m$ are assumed to be transient, so that absorption into the state $m+1$ from any initial state, is certain. The necessary and sufficient condition for the states $1, 2, \ldots, m$ to be transient is that the matrix T is non singular.

Definition. A probability distribution $F(\cdot)$ on $[0, \infty)$ is a distribution of phase type (PH-distribution) iff it is the distribution of the time until absorption in a finite Markov chain in the form of Q as defined above. It is given by

$$F(t) = 1 - \boldsymbol{\alpha} \exp(Tt)\bar{\mathbf{e}}.$$

The pair $(\boldsymbol{\alpha}, T)$ is called a representation of $F(\cdot)$ (Neuts [40]).

1.1.5 Level Independent Quasi-Birth-Death (LIQBD) Process

Definition.

Consider a Markov chain $\{X(t), t \geq 0\}$ with state space

$$\bigcup_{n=0}^{\infty} \{(n, j) : 1 \leq j \leq m\}$$

The first component n is called *level* of the chain and the second component j is called *phases* of the nth level. The MC is called a *QBD process* if one-step transitions from a state are restricted to states in the same level or in the two adjacent levels: it is possible to move in one step from (n, j) to (n_1, j_1) only if $n_1 = n, n+1$ or $n-1$ (provided that $n \geq 1$ in the last case).

The process is said to be LIQBD process if it is independent of the levels; else it is called level dependent QBD process.

1.1.6 Matrix Analytic Method

Assume that the QBD process with infinitesimal generator Q of the form

$$Q = \begin{bmatrix} B_1 & A_0 & & & \\ B_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

is irreducible and ergodic; therefore it has a unique steady-state solution. Let $\boldsymbol{x}$ be the steady-state probability vector of Q and is partitioned as $\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \dots)$. $\boldsymbol{x}_i$ has matrix-geometric form $\boldsymbol{x}_i = \boldsymbol{x}_0 R^i$ for $i \geq 1$. R is the minimal non-negative solution of the matrix equation $R^2A_2+RA_1+A_0 = O$, R is called rate matrix.

Theorem. The QBD process with infinitesimal generator Q of the above form is positive recurrent iff the minimal non-negative solution R of the matrix quadratic equation $R^2A_2 + RA_1 + A_0 = O$ has all its eigen values inside the unit disc and the finite system of equations $\boldsymbol{x}_0(B_1 + RB_2) = \mathbf{0}$ and the normalizing equation $\boldsymbol{x}_0(I - R^{-1})\bar{\mathbf{e}} = 1$ has a unique solution $\boldsymbol{x}_0$. If the matrix $A = A_0 + A_1 + A_2$ is irreducible, then $\text{Sp}(R) < 1$ iff $\boldsymbol{\pi} A_0\bar{\mathbf{e}} < \boldsymbol{\pi} A_2\bar{\mathbf{e}}$ where $\boldsymbol{\pi}$ is the stationary probability vector of A. The stationary probability vector $\boldsymbol{x} = (\boldsymbol{x}_0, \boldsymbol{x}_1, \dots)$ of Q is given by $\boldsymbol{x}_i = \boldsymbol{x}_0 R^i$ for $i \geq 1$ (Neuts [40]).

1.1.7 Coxian-2 Distribution

A random variable X is said to follow Coxian-2 distribution with parameter (b, μ_1, μ_2) if it can be represented as

$$X = \begin{cases} X_1 + X_2 & \text{with probability } b \\ X_1 & \text{with probability } 1 - b \end{cases}$$

where X_1 and X_2 are independent random variables having exponential distributions with means $1/\mu_1$ and $1/\mu_2$ respectively. Without loss of generality it can be assumed that $\mu_1 \geq \mu_2$.

The probability density of the Coxian-2 distributed random variable X is given by

$$f(t) = \begin{cases} p_1\mu_1 e^{-\mu_1 t} + (1-p_1)\mu_2 e^{-\mu_2 t} & \text{if } \mu_1 \neq \mu_2 \\ p_1\mu_1 e^{-\mu_1 t} + (1-p_1)\mu_1^2 t e^{-\mu_1 t} & \text{if } \mu_1 = \mu_2 \end{cases}$$

where $p_1 = 1 - b\mu_1/(\mu_1 - \mu_2)$ if $\mu_1 \neq \mu_2$
and $p_1 = 1 - b$ if $\mu_1 = \mu_2$.

1.2 Review of Literature

The well-known EOQ formula (Economic Order Quantity formula), introduced by Ford Harris in 1915 [20] forms the basis of inventory theory. EOQ formula is also called Harris-Wilson economic lot size formula, due to the popularization of the same by Wilson in 1918. EOQ formula points out the optimal ordering quantity so that the total cost which includes holding cost, reordering cost, order procurement cost, shortage cost etc. is a minimum.

Active research on stochastic inventory models started in 1950's, and progressed in a faster pace since 1960. Early works on stochastic inventory models include Arrow, Harris and Marschak [2], Churchman et al. [11], Arrow, Karlin and Scarf [3], Hadley and Whitin [19], Veinott [57] and Ryshikov [42]. Sivazlian and Stanfel [50] contains application of renewal theory to stochastic inventory system. Also certain realistic examples on inventory systems can be viewed in Tijms [53]. Srinivasan [51] considers (s, S) inventory policy with renewal demands and general lead time distribution. Sahin [44] considers an (s, S) inventory system in which lead time is a constant while demand quantity is a continuous random variable. In her paper that appeared in 1982 [45], she generalizes the work of Srinivasan by analyzing an (s, S) inventory system involving compound renewal demands and arbitrarily distributed lead time. Kalpakam and Arivarignan

[22] analyze an (s, S) inventory system involving different types of sources of demands.

The first work on inventory with positive service time was by Sigman and Simchi-Levi [48], involving $M/G/1$ queue with inventory. This work was closely followed by Berman et al. [6] in which both the demand and service rates are assumed to be deterministic and constant. Berman and Kim [7] use dynamic programing method for cost optimization of stochastic inventory models with positive service time. They consider a model where order replenishments are instantaneous, and the optimal policy is never to order when the system is empty, and to place an order only when the inventory level drops to zero (see also [35]) for the stochastic case). Berman and Sapna [8] consider an inventory system with service facilities where there is limited waiting space for customers. Sivakumar and Arivarignan [49] consider a perishable inventory system at service facilities with negative customers. Krishnamoorthy et al. [25] analyse an (s, S) inventory system with positive service time in which the effective utilization of server idle time is considered. In this model, if an item is available on stock at a service completion epoch, processing of it is done by the server, even in the absence of a customer. Deepak et al. [14] consider an inventory system with positive service time and obtain an explicit product-form solution because of the assumption of zero lead time. Lalitha [36] considers different (s, S) inventory systems with positive service time and lead time but fails to produce product form solution; even closed form solution could not be arrived at by her.

For more detailed reports on inventory systems with positive service time, one may refer the survey paper by Krishnamoorthy et al. [26], eventhough few of the important developments till then are missing in that paper. Schwarz et al.[46] are the first to produce product form solution in M/M/1 queueing inventory with positive lead time. Krishnamoorthy et al. [35] also could achieve this fete for the N-policy; however they restrict the lead time to be zero. Saffari et al. [43] consider an M/M/1 queue with associated inventory where the lead time for replenishment is arbitrarily distributed; they produce product form solution for the system state distribution. Anoop et al. [1] deal with a multi-server inventory model with zero lead time in which the servers are essentially the inventoried items. This paper is a class in itself for the simple reason that a server vanishes with a service completion. This means that the number of servers available in the service station depends on the number of inventoried items with the system. Though they are not able to arrive at a product form solution because of the heavy dependence of the number of customers and the number of inventoried items in the system, despite the fact that the lead time is zero, a number of conditional distributions are derived by the authors. Krishnamoorthy et al. [27] is one of most recent of the work giving product form solution when lead time is assumed to be positive. Their work generalizes some of the results in Schwarz et al.[46].

Deepak et al. [13] consider an inventory model with two parallel service facilities. Customers are transferred from longer to shorter queue whenever their difference reaches a prescribed quantity. Simultaneously, depending on the availability of items, a certain number of inventory is also transferred from one service facility to the other.

Perishable inventory models were analysed by dynamic programming technique in Fries [16] and Nahmias [38]. A continuous review perishable inventory model was studied in Kalpakam and Arivarignan [23]. Kalpakam and Sapna [24] consider continuous review (s, S) inventory system with random lifetimes and positive lead times. Nahmias [39] gives a detailed survey of the literature on periodic review models.

Control policies such as N-policy, T-policy, D-policy and their combinations discussed in queueing literature have been extended to inventory with negligible service time by several researchers (see [28], [29] and [41] for details). N-policy was first introduced in 1963 by Yadin and Naor [59] in queueing literature to minimize the total operational cost in a cycle. Balachandran [5], Teghem [52], Artalejo [4] and Gakis et al. [17] also considered N-policy in queueing problems. Krishnamoorthy and Raju [28, 29] used N-policy in (s, S) inventory system with lead time and negligible service time, involving perishable as well as non-perishable items. Krishnamoorthy and Raju [30] used N-policy for a production inventory system with random lifetimes and negligible service time. Krishnamoorthy et al. [35] used N-policy in (s, S) inventory system with positive service time, where the server, when becomes idle, waits till N customers accumulate to begin the next cycle of service. Krishnamoorthy et al. [32] considered N-policy in reliability where a k-out-of-n system with repair was analyzed. Ushakumari and Krishnamoorthy [55] considered k-out-of-n system with general repair under N-policy.

T-policy has been discussed by Artalejo [4] in $M/G/1$ queueing systems with removable server. T-policy in inventory systems with negligible service time was investigated by Krishnamoorthy and Rekha [31]. T-policy in reliability was considered by Rekha [41] in which k-out-of-n systems with repair under T-policy were analysed. Heyman [21] used T-policy in order to sense customer arrivals in an $M/G/1$ queueing model where the server cannot continuously monitor the queue. Here T-policy is introduced in such a way that the server is activated at T time units after the end of the last busy period. Ushakumari and Krishnamoorthy [56] considered a k-out-of-n system with repair under max (N, T) policy where max (N, T) policy is used in such a way that the repair facility is activated for repair of failed units whenever the maximum of an exponentially distributed time duration T and the sum of N $(1 \leq N \leq n - k)$ random variables is realized.

The concept of local purchase has been introduced in (s, S) inventory system by Krishnamoorthy and Raju in a series of research papers [28, 29] involving models with negligible service time. In their models, local purchase was based on N-policy. Krishnamoorthy and Rekha [31] considered T-policy in (s, S) inventory system with negligible service time, where local purchase was done based on T-policy. It is common practice that when an item is not available in a shop for which a demand arrives, the same is purchased from other shops locally and supplied to the customer.

Doshi et al. [15] consider a continuous review (s, S) production inventory system with a compound Poisson arrival of demands. Sharafali [47] considers an (s, S) production inventory system in which the machine is subject to failure and the repair time follows general distribution. William et al. [58] consider a periodic review production inventory system with non stationary demand process.

The first work on production inventory with positive service time was by Krishnamoorthy and Viswanath [33] in which MAP arrivals and MPP (Markovian Production Process) are involved. Krishnamoorthy and Viswanath [34] consider a production inventory model with positive service time in which stochastic decomposition is obtained because of the assumption that no customer joins the queue when the inventory level is zero.

Important works on Coxian distribution can be viewed in Cox [12], Gelenbe and Mitrani [18], Yao and Buzacott [60], Botta et al. [10] and Bertsimas and Papaconstantinou [9]. Tijms [54] compared an $M/\mathrm{Cox}_2/1$ and $M/D/1$ and showed that Cox_2 acts as a very good approximation to deterministic service. One can see certain Coxian-2 queueing models in Tijms [53].

Matrix analytic methods was introduced by Neuts [40], in which an algorithmic analysis of $M/G/1$ and $GI/M/1$ type stochastic queueing models were considered. Detailed discussion on matrix-analytic methods can also be viewed in Latouche and Ramaswami [37].

1.3 Present Work in a Nutshell

In this thesis, certain continuous time inventory problems with positive service time under local purchase guided by N/T-policy are analysed. In most of the cases analysed, we arrive at stochastic decomposition of system states, that is, the joint distribution of the system states is obtained as the product of marginal distributions of the components.

The thesis is divided into five chapters where chapter 1 is of introductory nature, which include the literature survey and certain prerequisites.

In chapter 2, (s, Q) inventory systems involving perishable as well as non-perishable items are considered. In both the models, arrivals are according to a Poisson process, service time and lead time follow independent exponential distributions. Ordering quantity is fixed and is $Q = S - s$ $(Q > s)$. N-policy is defined as follows: As and when the inventory level drops to $s - N$ (where $s \geq N$) during a lead time, an immediate local purchase of $Q + N$ units is made, by cancelling the order already placed. Cancellation of order is taken into account since, otherwise, the inventory level goes beyond S as making local purchase. Also it is assumed that supply of items is instantaneous in local purchase. In the model of perishable goods, we assume that the items are subject to decay with the decay time following exponential distribution. We arrive at product form solution in both the models. Several performance measures are derived. Explicit cost functions are obtained and are analysed numerically.

In chapter 3, (s, Q) inventory systems with exponential service time, involving perishable as well as non-perishable items, where T-Policy is adopted during lead time, are considered. Arrival of demands are according to a Poisson process and lead time follows exponential distribution. T is an exponentially distributed random variable with parameter α. In both the models considered, T-Policy is introduced as follows: As and when the inventory level drops to the reorder level s, an order is placed for $Q = S - s$. If the replenishment doesn't occur within a time of T units from the order placement epoch, then a local purchase is made to bring the inventory level to S, by cancelling the order that is already placed. If the inventory level reaches zero before the realization of T time units, and before the occurrence of replenishment, regardless of the number of customers present in the system, an immediate local purchase of S units is made, by cancelling the order that is already placed. Local purchase can be done when T is realized or when the inventory level reaches zero, whichever occurs first. It is assumed that supply of items is instantaneous in local purchase. We derive stability condition for both models. Also we establish stochastic decomposition of the system state. Certain performance measures are derived. Convexity of the total expected cost per unit time as a function of α is exhibited numerically.

In chapter 4 we consider two (s, S) production inventory models involving local purchase. In both the models, arrival of demands is according to a Poisson process. As and when the inventory level reaches s, the production process is switched 'on'. The production process is such that the items are produced one at a time, and the time taken to produce an item follows exponential distribution. The produced item requires a processing time before it is served to the customer, and the processing time is a random variable which follows exponential distribution. Once the production process is switched 'on', it will be kept in the 'on' mode till the inventory level reaches S. As soon as the inventory level reaches S, the production process is switched 'off'. In the first model, we assume that as and when the inventory level reaches zero, a local purchase of one unit is made at a higher cost, to avoid customer loss. In the second model, we assume that as and when the inventory level reaches zero, a local purchase of N units is made (where $2 \leq N < s$) at a higher cost, to avoid customer loss. Supply of items is instantaneous in local purchase. In both the models, stability conditions and certain performance measures are derived. Also we obtained stochastic decomposition of system state. Convexity of cost functions is verified numerically.

In chapter 5, we consider an (s, Q) inventory system with positive service time and lead time. The reorder level is s and ordering quantity is fixed as $Q = S - s$. We assume that there is only one server. The inter-arrival time has a Coxian-2 distribution with parameters $(b, \lambda_1, \lambda_2)$. The arrival mechanism may be considered as follows: An arriving customer first goes through phase 1 for an exponentially distributed time with parameter λ_1 and gets into the system with probability $1 - b$, or goes through a second phase with probability b. The sojourn time in phases are independent exponentials with means $1/\lambda_1$ and $1/\lambda_2$ respectively, that is, the arrival mechanism is consisting either of only one exponential stage with mean $1/\lambda_1$ (with probability $1 - b$) after which the arrival is admitted to the system, or of two successive independent exponential stages with means $1/\lambda_1$ and $1/\lambda_2$ respectively, after which absorption occurs. Also, we assume that the service time of a customer has a Coxian-2 distribution with parameters (θ, μ_1, μ_2). The service mechanism may be considered as follows: The customer first goes through phase 1 to get his service completed with probability $1 - \theta$, or goes through a second phase with probability θ. The sojourn time in the two phases are independent exponential random variables with means $1/\mu_1$ and $1/\mu_2$ respectively, that is, the service mechanism consists either of one exponential stage (with probability $1-\theta$) with mean $1/\mu_1$ after which the service is completed, or of two independent exponential stages with means $1/\mu_1$ and $1/\mu_2$ respectively, after which the service is completed, the probability of the second stage of service is θ. The model also involves exponentially distributed lead time during which N-policy is adopted as follows: As and when the inventory level drops to $s-N$ (where $s \geq N$) due to N service completions after placing a natural purchase order, an immediate local purchase of $Q+N$ units is made, by cancelling the order already placed. Also it is assumed that supply of items is instantaneous in local purchase. The problem is modelled as a continuous time Markov chain and is analyzed using matrix geometric method. Several performance measures of the model are obtained. Convexity of cost function is established numerically.

The thesis concludes with the publication details of the research papers/books cited in the text.

Chapter 2

(s, Q) Inventory Systems with Positive Lead Time and Service Time under N-Policy

2.1 Introduction

In this chapter[1], we consider two (s, Q) inventory models with positive service time involving perishable as well as non perishable items, where local purchase driven by N-policy is made during the lead time. In model I, items are assumed to be non-perishable. In model II, the items are subject to decay and the decaying time follows exponential distribution with parameter β. In both the models, the ordering quantity is fixed and is equal to $Q = S - s$ where s is the reorder level. The models involve replenishment lead time which follows exponential distribution with parameter γ. Arrival of demands is according to a Poisson process with parameter λ and service time follows exponential distribution with parameter μ. N-policy is adopted during a lead time and is as follows: As and when the inventory level drops to $s - N$ (where $s \geq N$) during a lead time, after placing a natural purchase order, an immediate local purchase of $Q + N$ units is made, by cancelling

[1]Part of this chapter is included in the following paper: Krishnamoorthy A., Resmi Varghese and Lakshmy B., An (s, Q) Inventory System with Positive Lead Time and Service Time under N-Policy, 2013 (Communicated).

the order already placed (Cancellation of order is necessary since, otherwise, the inventory level go beyond S). In otherwords, we will not permit the inventory level to reduce beyond $s-N+1$. We go for an immediate local purchase at the moment at which the inventory level drops to $s-N$, regardless of the number of customers present in the system. Also it is assumed that supply of items is instantaneous in local purchase, and at a higher cost.

2.2 Model I: Non-Perishable Items

2.2.1 Model Formulation and Analysis

Let $X(t)$ = Number of customers in the system at time t and
$I(t)$ = Inventory level at time t.
$\{(X(t),I(t)),t\geq 0\}$ is a CTMC with state space $E=E_1\times E_2$ where $E_1=\{0,1,2,\ldots,\}$ and $E_2=\{s-N+1,s-N+2,\ldots,S\}$ where $s-N+1>0$.
Therefore

$$E=\{(i,j)|i\in E_1,j\in E_2\}.$$

2.2.1.1 Infinitesimal Generator $\tilde{A}$

We write infinitesimal generator of the process as

$$\tilde{A}=(a((i,j),(m,n)))$$

where $(i,j),(m,n)\in E$.

The elements of $\tilde{A}$ can be obtained as

$$a((i,j),(m,n))=\begin{cases}\lambda, & m=i+1;\quad i=0,1,2,\ldots\\ & n=j;\quad j=s-N+1,\ldots,S\\ \mu, & m=i-1;\quad i=1,2,3,\ldots\\ & n=j-1;\quad j=s-N+2,\ldots,S\\ \mu, & m=i-1;\quad i=1,2,3,\ldots\\ & n=S;\quad j=s-N+1\end{cases}$$

and

$$a((i,j);(m,n)) = \begin{cases} \gamma, & m = i; \quad i = 0,1,2,\dots \\ & n = j+Q; \quad j = s-N+1,\dots,s \\ -(\lambda+\gamma), & m = i; \quad i = 0 \\ & n = j; \quad j = s-N+1,\dots,s \\ -\lambda, & m = i; \quad i = 0 \\ & n = j; \quad j = s+1,\dots,S \\ -(\lambda+\gamma+\mu), & m = i; \quad i = 1,2,3,\dots \\ & n = j; \quad j = s-N+1,\dots,s \\ -(\lambda+\mu), & m = i; \quad i = 1,2,3,\dots \\ & n = j; \quad j = s+1,\dots,S \\ 0, & \text{otherwise.} \end{cases}$$

$\tilde{A}$ can be written in terms of sub matrices as follows:

$$\tilde{A} = \begin{bmatrix} B_1 & A_0 & & \\ A_2 & A_1 & A_0 & \\ & A_2 & A_1 & A_0 \\ & \ddots & \ddots & \ddots \end{bmatrix}$$

where $A_0 = \lambda I_{Q+N}$,

$$A_1 = \begin{bmatrix} -(\lambda+\gamma+\mu)I_N & O_{N\times(Q-N)} & \gamma I_N \\ O_{(Q-N)\times N} & -(\lambda+\mu)I_{Q-N} & O_{(Q-N)\times N} \\ OI_N & O_{N\times(Q-N)} & -(\lambda+\mu)I_N \end{bmatrix},$$

$$A_2 = \begin{bmatrix} \mathbf{0}_{1\times(Q+N-1)} & \mu I_1 \\ \mu I_{Q+N-1} & \mathbf{0}_{(Q+N-1)\times 1} \end{bmatrix} \quad \text{and}$$

$$B_1 = \begin{bmatrix} -(\lambda+\gamma)I_N & O_{N\times(Q-N)} & \gamma I_N \\ O_{(Q-N)\times N} & -\lambda I_{Q-N} & O_{(Q-N)\times N} \\ OI_N & O_{N\times(Q-N)} & -\lambda I_N \end{bmatrix}.$$

A_1, A_2 and B_1 are square matrices of order $Q+N$.

2.2.1.2 Steady-State Analysis

Let $A = A_0 + A_1 + A_2$ be the generator matrix of order $Q + N$ and is obtained as

$$\begin{bmatrix}
-(\gamma+\mu) & & & & & & \gamma & & \mu \\
\mu & -(\gamma+\mu) & & & & & & & \\
& \mu & \ddots & & & & & \ddots & \\
& & & -(\gamma+\mu) & & & & & \gamma \\
& & & \mu & -\mu & & & & \\
& & & & \ddots & \ddots & & & \\
& & & & & & & \mu & -\mu
\end{bmatrix}$$

First we investigate the stationary distribution of A. This will help in analyzing the stability of the larger system, namely the CTMC $\{(X(t), I(t)), t \geq 0\}$ as $t \to \infty$.

Theorem 2.2.1. *The steady-state probability distribution* $\mathbf{\Phi} = (\phi_{s-N+1}, \phi_{s-N+2}, \dots, \phi_S)$ *corresponding to the matrix A is given by*

$$\phi_j = q_j \phi_{s-N+1}, \quad j = s - N + 1, \dots, S \tag{2.1}$$

where

$$q_j = \begin{cases} 1, & j = s - N + 1 \\ (\frac{\gamma+\mu}{\mu})^{j-s+N-1}, & j = s - N + 2, \dots, s + 1 \\ (\frac{\gamma+\mu}{\mu})^{N}, & j = s + 2, \dots, S - N + 1 \\ (\frac{\gamma+\mu}{\mu})^{N} + 1 - (\frac{\gamma+\mu}{\mu})^{j-S+N-1}, & j = S - N + 2, \dots, S \end{cases}$$

and ϕ_{s-N+1} is obtained by solving the equation $\mathbf{\Phi}\bar{\mathbf{e}} = 1$, as

$$\phi_{s-N+1} = \left[N + Q \left(\frac{\mu+\gamma}{\mu} \right)^N \right]^{-1}.$$

Proof. We have $\mathbf{\Phi}A = \mathbf{0}$ and $\mathbf{\Phi}\bar{\mathbf{e}} = 1$

$$\mathbf{\Phi}A = \mathbf{0} \Rightarrow$$

$$-\phi_l(\gamma+\mu)+\phi_{l+1}\mu = 0, \tag{2.2}$$

for $l = s-N+1, \ldots, s$.

$$-\phi_l\mu+\phi_{l+1}\mu = 0, \tag{2.3}$$

for $l = s+1, \ldots, S-N$.

$$-\phi_l\mu+\phi_{l+1}\mu+\phi_{l-Q}\gamma = 0, \tag{2.4}$$

for $l = S-N+1, \ldots, S-1$.

$$-\phi_S\mu+\phi_s\gamma+\phi_{s-N+1}\mu = 0. \tag{2.5}$$

Equation (2.2) gives

$$\phi_{s-N+x} = \left(\frac{\gamma+\mu}{\mu}\right)^{x-1}\phi_{s-N+1},$$

for $x = 1, 2, \ldots, N+1$,

$$\text{or} \quad \phi_j = \left(\frac{\gamma+\mu}{\mu}\right)^{j-s+N-1}\phi_{s-N+1}, \tag{2.6}$$

for $j = s-N+1, \ldots, s+1$.
Equation (2.3) gives

$$\phi_j = \left(\frac{\gamma+\mu}{\mu}\right)^{N}\phi_{s-N+1}, \tag{2.7}$$

for $j = s+1, \ldots, S-N+1$.
Equation (2.4) gives,

$$\phi_{S-N+x} = \left[\left(\frac{\gamma+\mu}{\mu}\right)^{N} + 1 - \left(\frac{\gamma+\mu}{\mu}\right)^{x-1}\right]\phi_{s-N+1},$$

for $x = 1, 2, 3, \ldots, N$,

$$\text{or} \quad \phi_j = \left[\left(\frac{\gamma+\mu}{\mu}\right)^{N} + 1 - \left(\frac{\gamma+\mu}{\mu}\right)^{j-S+N-1}\right]\phi_{s-N+1}, \tag{2.8}$$

for $j = S-N+1, S-N+2, \ldots, S$.

Hence equation (2.1) is obtained from the equations (2.6), (2.7) and (2.8).

Next, we proceed to find ϕ_{s-N+1}. Consider the normalizing condition $\mathbf{\Phi}\bar{\mathbf{e}} = 1$. This implies

$$\left[1 + \sum_{j=s-N+2}^{s+1} q_j + \sum_{j=s+2}^{S-N+1} q_j + \sum_{j=S-N+2}^{S} q_j\right] \phi_{s-N+1} = 1. \tag{2.9}$$

We can get

$$\sum_{j=s-N+2}^{s+1} q_j = \left(\frac{\gamma+\mu}{\gamma}\right)\left(\frac{\gamma+\mu}{\mu}\right)^N - \left(\frac{\gamma+\mu}{\mu}\right). \tag{2.10}$$

$$\sum_{j=s+2}^{S-N+1} q_j = (Q-N)\left(\frac{\gamma+\mu}{\mu}\right)^N. \tag{2.11}$$

Also,

$$\sum_{j=S-N+2}^{S} q_j = \left(\frac{\gamma+\mu}{\mu}\right)^N N - \left(\frac{\gamma+\mu}{\mu}\right)^N + N - 1 - \left(\frac{\gamma+\mu}{\gamma}\right)\left(\frac{\gamma+\mu}{\mu}\right)^{N-1} + \left(\frac{\gamma+\mu}{\gamma}\right). \tag{2.12}$$

Using (2.10), (2.11) and (2.12) in (2.9), we get

$$\left[1 + \left(\frac{\gamma+\mu}{\gamma}\right)\left(\frac{\gamma+\mu}{\mu}\right)^N - \left(\frac{\gamma+\mu}{\gamma}\right) + Q\left(\frac{\gamma+\mu}{\mu}\right)^N - N\left(\frac{\gamma+\mu}{\mu}\right)^N + N\left(\frac{\gamma+\mu}{\mu}\right)^N - \left(\frac{\gamma+\mu}{\mu}\right)^N + N - 1 - \left(\frac{\gamma+\mu}{\gamma}\right)\left(\frac{\gamma+\mu}{\mu}\right)^{N-1} + \left(\frac{\gamma+\mu}{\gamma}\right)\right] \phi_{s-N+1} = 1.$$

On simplification,

$$\left[N + Q\left(\frac{\gamma+\mu}{\mu}\right)^N\right] \phi_{s-N+1} = 1.$$

Hence we get

$$\phi_{s-N+1} = \left[N + Q \left(\frac{\gamma + \mu}{\mu} \right)^N \right]^{-1}.$$

Hence the theorem. □

2.2.1.3 Stability Condition

The result in Theorem 2.2.1 enables us to compute the stability of the CTMC $\{(X(t), I(t)), t \geq 0\}$.

Theorem 2.2.2. *The process* $\{(X(t), I(t)), t \geq 0\}$ *is stable iff* $\lambda < \mu$.

Proof. Since the process under consideration is an LIQBD process, it will be stable iff

$$\mathbf{\Phi} A_0 \bar{\mathbf{e}} < \mathbf{\Phi} A_2 \bar{\mathbf{e}} \tag{2.13}$$

(Neuts [40]), where $\mathbf{\Phi}$ is the steady-state distribution of the generator matrix $A = A_0 + A_1 + A_2$.

$$\mathbf{\Phi} A_0 \bar{\mathbf{e}} = (\phi_{s-N+1}\phi_{s-N+2}\ldots\phi_S)\lambda I_{Q+N} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{(Q+N)\times 1} = \lambda.$$

$$\mathbf{\Phi} A_2 \bar{\mathbf{e}} = (\phi_{s-N+1}\phi_{s-N+2}\ldots\phi_S)_{1\times(Q+N)} \begin{bmatrix} 0 & 0 & \ldots & \mu \\ \mu & 0 & \ldots & \\ & \mu & & \\ & & & \\ & & \mu & 0 \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{(Q+N)\times 1} = \mu.$$

Hence using (2.13) we get $\lambda < \mu$. □

Having obtained the condition for the system to stabilize, we turn to compute the long-run probability distribution of the system state. Infact we show that the joint distribution of the system can be written as the product of the marginal distributions of the component random variables.

2.2.2 The Steady-State Probability Distribution of $\tilde{A}$

2.2.2.1 Stochastic Decomposition of System States

Let $\bar{\boldsymbol{\pi}}$ be the steady-state probability vector of $\tilde{A}$.

$$\bar{\boldsymbol{\pi}} = (\boldsymbol{\pi}^{(0)}, \boldsymbol{\pi}^{(1)}, \boldsymbol{\pi}^{(2)}, \dots)$$

where $\boldsymbol{\pi}^{(i)} = (\pi^{(i,s-N+1)}, \pi^{(i,s-N+2)}, \dots, \pi^{(i,S)})$, $i = 0, 1, 2, \dots$ and

$$\pi^{(i,j)} = \lim_{t\to\infty} P(X(t) = i, I(t) = j)$$

where $(i, j) \in E$.
$\pi^{(i,j)}$ is the steady-state probability for the state (i, j).
We claim that

$$\boldsymbol{\pi}^{(i)} = K\rho^i \boldsymbol{\Delta}, \quad i \geq 0 \tag{2.14}$$

where $\boldsymbol{\Delta} = (r_{s-N+1}, r_{s-N+2}, \dots, r_S)$ is the steady-state probability vector when the service time is negligible, K is a constant to be determined and $\rho = \frac{\lambda}{\mu}$.

Proof. We have $\bar{\boldsymbol{\pi}}\tilde{A} = \mathbf{0}$ and $\bar{\boldsymbol{\pi}}\bar{\mathbf{e}} = 1$.

$$\bar{\boldsymbol{\pi}}\tilde{A} = \mathbf{0} \Rightarrow (\boldsymbol{\pi}^{(0)}, \boldsymbol{\pi}^{(1)}, \boldsymbol{\pi}^{(2)}, \dots) \begin{bmatrix} B_1 & A_0 & & & \\ A_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix} = \mathbf{0}$$

which gives

$$\boldsymbol{\pi}^{(0)} B_1 + \boldsymbol{\pi}^{(1)} A_2 = \mathbf{0} \tag{2.15}$$

$$\boldsymbol{\pi}^{(i+2)} A_2 + \boldsymbol{\pi}^{(i+1)} A_1 + \boldsymbol{\pi}^{(i)} A_0 = \mathbf{0} \tag{2.16}$$

$i = 0, 1, 2, \dots$.

When (2.14) is true, we get from (2.15),

$$K\rho^0 \boldsymbol{\Delta} B_1 + K\rho \boldsymbol{\Delta} A_2 = \mathbf{0}$$

$$\boldsymbol{\Delta}(B_1 + \rho A_2) = \mathbf{0}$$

$$\text{That is,} \quad \boldsymbol{\Delta}\tilde{Q} = \mathbf{0}, \text{ which is true,}$$

since $\mathbf{\Delta} = (r_{s-N+1}, r_{s-N+2}, \ldots, r_S)$ is the steady-state probability vector corresponding to the generator $\tilde{Q}$ associated with the Markovian chain of the inventory process under consideration when service time is negligible. $\tilde{Q}$ is given as

$$\begin{bmatrix}
-(\gamma+\lambda) & & & & & \gamma & & \lambda \\
\lambda & -(\gamma+\lambda) & & & & & & \\
 & \lambda & \ddots & & & & \ddots & \\
 & & & & & & & \\
 & & & -(\gamma+\lambda) & & & & \gamma \\
 & & & \lambda & -\lambda & & & \\
 & & & & \ddots & \ddots & & \\
 & & & & & & & \\
 & & & & & & \lambda & -\lambda
\end{bmatrix}$$

When (2.14) is true, we get from (2.16),

$$K\rho^{i+2}\mathbf{\Delta}A_2 + K\rho^{i+1}\mathbf{\Delta}A_1 + K\rho^{i}\mathbf{\Delta}A_0 = \mathbf{0}$$

where $i = 0, 1, 2, \ldots$

$$\text{That is,} \quad \mathbf{\Delta}(\rho A_2 + A_1 + \frac{1}{\rho}A_0) = \mathbf{0}$$

$$\text{That is,} \quad \mathbf{\Delta}\tilde{Q} = \mathbf{0},$$

which is true, by following the same argument given above.
Hence the stochastic decomposition of system state is verified. □

2.2.2.2 Determination of K

We have

$$\sum_{i=0}^{\infty} \sum_{j=s-N+1}^{S} \pi^{(i,j)} = 1$$

$$\text{That is,} \quad \sum_{i=0}^{\infty} \sum_{j=s-N+1}^{S} K\rho^i r_j = 1 \quad \text{(Using (2.14))}$$

$$\text{which gives} \quad K = 1 - \rho.$$

2.2.2.3 Explicit Solution

From the above discussions, we can write the steady-state probability vector explicitly as in the following theorem:

Theorem 2.2.3. *The steady-state probability vector* $\bar{\boldsymbol{\pi}}$ *of* $\tilde{A}$ *partitioned as* $\bar{\boldsymbol{\pi}} = (\boldsymbol{\pi}^{(0)}, \boldsymbol{\pi}^{(1)}, \boldsymbol{\pi}^{(2)}, \ldots)$*, where each* $\boldsymbol{\pi}^{(i)}$*,* $i = 0, 1, 2, \ldots$ *again partitioned as*

$$\boldsymbol{\pi}^{(i)} = \left(\pi^{(i,s-N+1)}, \pi^{(i,s-N+2)}, \ldots, \pi^{(i,S)}\right)$$

are obtained by

$$\pi^{(i,j)} = (1-\rho)\rho^i r_j \tag{2.17}$$

where $\rho = \frac{\lambda}{\mu}$ *and* r_j*;* $j = s-N+1, s-N+2, \ldots, S$ *represent the inventory level probabilities when service time is negligible and are given as*

$$r_j = \begin{cases} a\omega^{j-s+N-1}, & j = s-N+1, \ldots, s \\ a\omega^N, & j = s+1, \ldots, S-N+1 \\ a(\omega^N - \omega^{j-S+N-1} + 1) & j = S-N+2, \ldots, S \end{cases} \tag{2.18}$$

where $a = r_{s-N+1} = (N + Q\omega^N)^{-1}$ *and* $\omega = \frac{\lambda+\gamma}{\lambda}$.

The result indicated by (2.18) is obtained in Krishnamoorthy and Raju [29]. The result indicated by (2.17) not only tells us that the original system possess stochastic decomposition but also the important fact that the system state distribution is the product of the distribution of its marginals: one component is the classical $M/M/1$ whose long run distribution for i customers in the system is $(1-\rho)\rho^i$, $i \geq 0$ and the other factor is the probability of j items in the inventory. Next we turn to find out how the system performs. The measures given in the following section are pointers to the system performance.

2.2.3 System Performance Measures

(a) Expected inventory held in the system (Mean inventory level),

$$\begin{aligned}
E(I) &= \sum_{i=0}^{\infty} \sum_{j=s-N+1}^{S} j\pi^{(i,j)} \\
&= \sum_{i=0}^{\infty}(1-\rho)\rho^i \sum_{j=s-N+1}^{S} jr_j \\
&= \sum_{j=s-N+1}^{S} jr_j \\
&= [(s-N+1)a + (s-N+2)a\omega + (s-N+3)a\omega^2 + \dots + sa\omega^{N-1}] \\
&\quad + [(s+1)a\omega^N + (s+2)a\omega^N + \dots + (S-N+1)a\omega^N] \\
&\quad + [(S-N+2)a(\omega^N + 1 - \omega) + (S-N+3)a(\omega^N + 1 - \omega^2) + \dots \\
&\quad + Sa(\omega^N + 1 - \omega^{N-1})] \\
&= S_1 + S_2 + S_3 \qquad (2.19)
\end{aligned}$$

where $a = r_{s-N+1}$, $\omega = \frac{\lambda+\gamma}{\lambda}$,

$$\begin{aligned}
S_1 &= (s-N+1)a + (s-N+2)a\omega + (s-N+3)a\omega^2 + \dots + sa\omega^{N-1}, \\
S_2 &= (s+1)a\omega^N + (s+2)a\omega^N + \dots + (S-N+1)a\omega^N \quad \text{and} \\
S_3 &= (S-N+2)a(\omega^N + 1 - \omega) + (s-N+3)a(\omega^N + 1 - \omega^2) \\
&\quad + \dots + sa(\omega^N + 1 - \omega^{N-1}).
\end{aligned}$$

Let us write S_1 as

$$S_1 = a[x_1 + (x_1+1)\omega + (x_1+2)\omega^2 + \dots + (x_1+N-1)\omega^{N-1}]$$

where $x_1 = s - N + 1$. That is,

$$S_1 = aS_4 \qquad (2.20)$$

where $S_4 = x_1 + (x_1+1)\omega + (x_1+2)\omega^2 + \dots + (x_1+N-1)\omega^{N-1}$.

$$(1-\omega)S_4 = x_1 + \frac{\omega^N - \omega}{\omega - 1} - s\omega^N.$$

Therefore, we get

$$S_4 = \frac{s(1-\omega^N)(1-\omega) + (1-N)(1-\omega) - (\omega^N - \omega)}{(1-\omega)^2}. \quad (2.21)$$

Using (2.21) in (2.20), we get

$$S_1 = \frac{a[s(1-\omega^N)(1-\omega) + (1-N)(1-\omega) - (\omega^N - \omega)]}{(1-\omega)^2}. \quad (2.22)$$

On simplification, S_2 is obtained as

$$S_2 = \frac{1}{2}(Q - N + 1)(S + s - N + 2)a\omega^N. \quad (2.23)$$

S_3 can be written as

$$\begin{aligned} S_3 = a[&x_2(\omega^N + 1 - \omega) + (x_2+1)(\omega^N + 1 - \omega^2) + (x_2+2)(\omega^N + 1 - \omega^3) \\ &+ \cdots + (x_2 + N - 2)(\omega^N + 1 - \omega^{N-1})] \\ &\quad (\text{where } x_2 = S - N + 2) \end{aligned}$$

which gives

$$S_3 = a[(\omega^N + 1)(N-1)(x_2 + \frac{N-2}{2}) - x_2\frac{(\omega^N - \omega)}{\omega - 1} - \omega^2 S_5] \quad (2.24)$$

where $S_5 = 1 + 2\omega + 3\omega^2 + 4\omega^3 + \cdots + (N-2)\omega^{N-3}$ which can be obtained as

$$S_5 = \frac{1 - \omega^{N-2}[1 - (N-2)(\omega - 1)]}{(\omega - 1)^2}. \quad (2.25)$$

Substituting (2.25) in (2.24) and on simplification, we get

$$\begin{aligned} S_3 = a\Bigg[&(\omega^N + 1)(N-1)(S + \frac{N}{2} + 1) - (S - N + 2)\frac{(\omega^N - \omega)}{\omega - 1} \\ &- \omega^2\left[\frac{1 - \omega^{N-2}[1 - (N-2)(\omega - 1)]}{(\omega - 1)^2}\right]\Bigg]. \end{aligned} \quad (2.26)$$

Using (2.22), (2.23) and (2.26), in equation (2.19), we get

$$E(I) = \frac{a(s(1-\omega^N)(1-\omega) + (1-N)(1-\omega) - (\omega^N - \omega))}{(1-\omega)^2}$$

$$+\frac{1}{2}(Q-N+1)(S+s-N+2)a\omega^N$$
$$+a((\omega^N+1)(N-1)(S-\frac{N}{2}+1)-(S-N+2)\frac{(\omega^N-\omega)}{\omega-1}$$
$$-\frac{\omega^2}{(\omega-1)^2}\left(1-\omega^{N-2}(1-(N-2)(\omega-1))\right)\right) \tag{2.27}$$

where $a = r_{s-N+1} = (N+\omega^N Q)^{-1}$ and $\omega = \frac{\lambda+\gamma}{\lambda}$.

(b) Mean waiting time of customers in the system,

$$W_S = \frac{L}{\lambda}$$

where

$$\begin{aligned} L &= \text{Expected number of customers in the system} \\ &= \sum_{i=0}^{\infty}\sum_{j=s-N+1}^{S} i\pi^{(i,j)} \\ &= \sum_{i=0}^{\infty} i(1-\rho)\rho^i \sum_{j=s-N+1}^{S} r_j \\ &= \frac{\rho}{1-\rho}. \end{aligned}$$

Therefore,

$$\begin{aligned} W_S &= \frac{1}{\lambda}(\frac{\rho}{1-\rho}) \\ &= \frac{1}{\mu-\lambda}. \end{aligned} \tag{2.28}$$

(c) Mean reorder rate,

$$\begin{aligned} R_r &= \mu\sum_{i=1}^{\infty}\pi^{(i,s+1)} \\ &= \mu\sum_{i=1}^{\infty}(1-\rho)\rho^i r_{s+1} \\ &= \lambda\omega^N(N+\omega^N Q)^{-1} \end{aligned} \tag{2.29}$$

where $\omega = \frac{\lambda+\gamma}{\lambda}$.

(d) Mean local purchase rate (Mean order cancellation rate),

$$\begin{aligned} R_{\text{LP}} &= \mu \sum_{i=1}^{\infty} \pi^{(i,s-N+1)} \\ &= \mu \sum_{i=1}^{\infty} (1-\rho)\rho^i r_{s-N+1} \\ &= \lambda (N + \omega^N Q)^{-1}. \end{aligned} \tag{2.30}$$

2.2.3.1 Analysis of Inventory Cycle Length

In an (s, Q) inventory system with lead time we define a cycle length as the expected time elapsed between two consecutive order placement epochs. The inventory cycle length of the model under consideration is a random variable which follows Phase-Type distribution.

Let i be the number of customers in the system when the inventory level is s. (i.e. i is the number of customers in the starting state of the cycle). In order to analyze the inventory cycle length, let us consider the following cases:-

Case I: $0 \le i < Q + N$. Then

Theorem 2.2.4. *Let (i, s) be the starting state, where s is the reorder level.*

(i) If $N - 1 \le i < Q + N$, then the time till absorption to $\{\boldsymbol{\Delta}_1\}$, i.e., τ_{1i} follows Phase-Type distribution with representation $(\bar{\boldsymbol{\alpha}}_{1i}, T_{1i})$ where $\{\boldsymbol{\Delta}_1\}$ denotes the absorbing set of states and is given by

$$\begin{aligned} \{\boldsymbol{\Delta}_1\} = \{&(i,S),(i-1,S-1),(i-2,S-2),\ldots,(i-(N-1),S-N+1), \\ &(i-N,S),(i-(N+1),S-1),(i-(N+2),S-2),\ldots,(i-(2N-1),S-N+1), \\ &(i-2N,S),(i-(2N+1),S-1),(i-(2N+2),S-2),\ldots, \\ &\ldots,(i-(3N-1),S-N+1),\cdots \text{ upto } N \text{ terms when } i = N-1 \\ &\& \text{ upto } N+1 \text{ terms when } i > N-1\}. \end{aligned} \tag{2.31}$$

(ii) If $0 < i < N-1$, then the time till absorption to $\{\boldsymbol{\Delta}_2\}$ i.e., τ_{2i} follows Phase-Type distribution where $\{\boldsymbol{\Delta}_2\}$ denotes the absorbing state and

is given by

$$\begin{aligned}\{\mathbf{\Delta}_2\} = \{&(i,S),(i+1,S),\dots,(N-1,S)\}\\ &\cup\{(i-1,S-1),(i,S-1),\dots,(N-1,S-1)\}\\ &\cup\cdots\cup\{(0,S-i),(1,S-i),\dots,(N-1,S-i)\}\\ &\cup\{(0,S-(i+1)),(1,S-(i+1)),\dots,(N-1,S-(i+1))\}\\ &\cup\cdots\cup\{(0,S-N+1),(1,S-N+1),\dots,(N-1,S-N+1)\}\\ &\cup\{(0,S)(1,S),\dots,(N-2,S)\}.\end{aligned} \tag{2.32}$$

(iii) If $i = 0$*, then the time till absorption to* $\{\mathbf{\Delta}_3\}$*, i.e.,* τ_{00} *follows Phase-Type distribution where* $\{\mathbf{\Delta}_3\}$ *denotes the absorbing state and is given by*

$$\begin{aligned}\{\mathbf{\Delta}_3\} =&\{(0,S),(1,S),\dots,(N-1,S)\}\cup\{(0,S-1),(1,S-1),\dots,\\ &(N-1,S-1)\}\cup\{(0,S-2),(1,S-2),\dots,(N-1,S-2)\}\\ &\cup\cdots\cup\{(0,S-N+2),(1,S-N+2),\dots,(N-1,S-N+2)\}\\ &\cup\{(0,S-N+1),(1,S-N+1),\dots,(N-1,S-N+1)\}.\end{aligned} \tag{2.33}$$

Proof. Starting from a state with inventory level s, we need to find the time till absorption to the states with inventory levels $S, S-1, \dots, S-N+1$. Let us consider the Markov chain

$$Y_1(t) = \{(X_1(t), I_1(t))\},$$

where $X_1(t)$ denotes the number of customers in the system at time t and $I_1(t)$ denotes the inventory level at that instant.

(i) Let $N-1 \le i < Q+N$.

The state space of $Y_1(t)$ is $\{(m,n)\}\cup\{\mathbf{\Delta}_1\}$ where $i-(N-1) \le m \le i$, $s-N+1 \le n \le s$ and $\{\mathbf{\Delta}_1\}$ denotes the absorbing state which is given by (2.31).

Clearly $Y_1(t)$ is a finite state Markov chain. The possible transitions and the corresponding instantaneous rates are given in Table 2.1.

Table 2.1: Transitions and corresponding instantaneous rates for $Y_1(t)$ in part (i) of Theorem 2.2.4. (Also for $Z_1(t)$ in Theorem 2.2.7)

From	To	Rate	
(m, n)	$(m-1, n-1)$	μ	where (m, n) belongs to $\{(i-(N-2), s-N+2),$ $(i-(N-3), s-N+3),$ $\ldots, (i, s)\}$
(m, n)	$\{\mathbf{\Delta}_1\}$	γ	where (m, n) belongs to $\{(i-(N-1), s-N+1),$ $(i-(N-2), s-N+2),$ $\ldots, (i, s)\}$
$(m, s-N+1)$	$\{\mathbf{\Delta}_1\}$	μ	only when $i > N-1$ and $m = i-(N-1)$

When $N-1 \leq i < Q+N$, future arrivals need not be considered to reach $\{\mathbf{\Delta}_1\}$.

Hence the infinitesimal generator Q_{1i} of the Markov chain $Y_1(t)$ is of the form

$$\begin{bmatrix} T_{1i} & \mathbf{T^0_{1i}} \\ \mathbf{0} & 0 \end{bmatrix}$$

with initial probability vector $\bar{\boldsymbol{\alpha}}_{1i} = (0, 0, \ldots, 1, 0, \ldots, 0)$ where 1 is in the ith position. T_{1i} is a square matrix of order N. $\bar{\boldsymbol{\alpha}}_{1i}$ has N elements.

Therefore, when $N-1 \leq i < Q+N$, the time till absorption to $\{\mathbf{\Delta}_1\}$, denoted by τ_{1i}, follows Phase-Type distribution with representation $(\bar{\boldsymbol{\alpha}}_{1i}, T_{1i})$. It's mean value is given by

$$E(\tau_{1i}) = -\bar{\boldsymbol{\alpha}}_{1i}(T_{1i})^{-1}\bar{\mathbf{e}}.$$

(ii) Let $0 < i < N-1$.

The state space of $Y_1(t)$ is $\{(m, n)\} \cup \{\mathbf{\Delta}_2\}$ where $0 \leq m \leq N-1$, $s-N+1 \leq n \leq s$ and $\{\mathbf{\Delta}_2\}$ denotes the absorbing state which is given by (2.32).

Clearly $Y_1(t)$ is a finite state Markov chain. The possible transitions and the corresponding instantaneous rates are given in Table 2.2.

Table 2.2: Transitions and corresponding instantaneous rates for $Y_1(t)$ in part (ii) of Theorem 2.2.4

From	To	Rate	
$(m, s-N+1)$	$\{\boldsymbol{\Delta}_2\}$	μ	for $m = 1, 2, \ldots N-1$
(m, n)	$\{\boldsymbol{\Delta}_2\}$	γ	where $(m, n) \in \{(i, s), (i+1, s), \ldots,$ $(N-1, s), (i-1, s-1), (i, s-1), \ldots,$ $(N-1, s-1), \cdots\cdots\cdots,$ $(0, s-i), (1, s-i), \ldots, (N-1, s-i),$ $(0, s-(i+1)), (1, s-(i+1)), \ldots,$ $(N-1, s-(i+1)), \cdots\cdots\cdots,$ $(0, s-N+1), (1, s-N+1), \ldots,$ $(N-1, s-N+1)\}.$
(m, n)	$(m+1, n)$	λ	where $(m, n) \in \{(i, s), (i+1, s), \ldots,$ $(N-2, s),(i-1, s-1), (i, s-1), \ldots,$ $(N-2, s-1), \cdots\cdots\cdots,$ $(0, s-i), (1, s-i), \ldots, (N-2, s-i),$ $(0, s-(i+1)), (1, s-(i+1)), \ldots,$ $(N-2, s-(i+1)), \cdots\cdots\cdots,$ $(0, s-N+1), (1, s-N+1), \ldots,$ $(N-2, s-N+1)\}.$
(m, n)	$(m-1, n-1)$	μ	where $(m, n) \in \{(i, s), (i+1, s), \ldots,$ $(N-1, s), (i-1, s-1), (i, s-1), \ldots,$ $(N-1, s-1), \cdots\cdots\cdots,$ $(1, s-i), (2, s-i), \ldots, (N-1, s-i),$ $(1, s-(i+1)), (2, s-(i+1)), \ldots,$ $(N-1, s-(i+1)), \cdots\cdots\cdots,$ $(1, s-N+2), (2, s-N+2), \ldots,$ $(N-1, s-N+2)\}.$

When $0 < i < N-1$, future arrivals till the number of customers sufficient to take away the available inventory are to be considered to reach the absorbing state $\{\boldsymbol{\Delta}_2\}$.

Hence the infinitesimal generator Q_{2i} of the Markov chain $Y_1(t)$ is of the form

$$\begin{bmatrix} T_{2i} & \mathbf{T^0_{2i}} \\ \mathbf{0} & 0 \end{bmatrix}$$

with initial probability vector $\bar{\boldsymbol{\alpha}}_{2i} = (0, 0, \ldots, 1, 0, \ldots, 0)$ where 1 is in the ith position.

T_{2i} is a square matrix of order $N^2 - \frac{i}{2}(1+i)$, $\bar{\boldsymbol{\alpha}}_{2i}$ has $N^2 - \frac{i}{2}(1+i)$ elements.

Therefore, when $0 < i < N-1$, the time till absorption to $\{\boldsymbol{\Delta}_2\}$, denoted by τ_{2i}, follows Phase-Type distribution with representation $(\bar{\boldsymbol{\alpha}}_{2i}, T_{2i})$.

It's mean value is given by

$$E(\tau_{2i}) = -\bar{\boldsymbol{\alpha}}_{2i}(T_{2i})^{-1}\bar{\mathbf{e}}.$$

(iii) Let $i = 0$.

The state space of $Y_1(t)$ is $\{(m, n)\} \cup \{\boldsymbol{\Delta}_3\}$ where $0 \leq m \leq N-1$, $s - N + 1 \leq n \leq s$ and $\{\boldsymbol{\Delta}_3\}$ denotes the absorbing state which is given by (2.33).

Clearly $Y_1(t)$ is a finite state Markov chain. The possible transitions and the corresponding instantaneous rates are given in Table 2.3.

When $i = 0$, future arrivals that are sufficient to reach the absorbing state $\{\boldsymbol{\Delta}_3\}$ are to be considered.

Hence the infinitesimal generator Q_{00} of the Markov chain $Y_1(t)$ is of the form

$$\begin{bmatrix} T_{00} & \mathbf{T^0_{00}} \\ \mathbf{0} & 0 \end{bmatrix}$$

with initial probability vector $\bar{\boldsymbol{\alpha}}_{00} = (1, 0, 0, \ldots, 0)$.

T_{00} is a square matrix of order $N(Q+N)$. $\bar{\boldsymbol{\alpha}}_{00}$ has $N(Q+N)$ elements.

Therefore, when $i = 0$, the time till absorption to $\{\boldsymbol{\Delta}_3\}$, denoted by τ_{00}, follows Phase-Type distribution with representation $(\bar{\boldsymbol{\alpha}}_{00}, T_{00})$.

It's mean value is given by

$$E(\tau_{00}) = -\bar{\boldsymbol{\alpha}}_{00}(T_{00})^{-1}\bar{\mathbf{e}}.$$

□

Table 2.3: Transitions and corresponding instantaneous rates for $Y_1(t)$ in part (iii) of Theorem 2.2.4

From	To	Rate	
(m,n)	$\{\boldsymbol{\Delta}_3\}$	γ	where $(m,n) \in \{(0,s),(1,s),\dots,$ $(N-1,s),(0,s-1),(1,s-1),\dots,$ $(N-1,s-1),(0,s-2),(1,s-2),\dots,$ $(N-1,s-2),\cdots\cdots\cdots,$ $(0,s-N+2),(1,s-N+2),\dots,$ $(N-1,s-N+2),(0,s-N+1),$ $(1,s-N+1)),\dots$ $\dots,(N-1,s-N+1)\}$.
(m,n)	$(m+1,n)$	λ	where $(m,n) \in \{(0,s),(1,s),\dots,$ $(N-2,s),(0,s-1),(1,s-1),\dots,$ $(N-2,s-1),\cdots\cdots\cdots,$ $(0,s-N+2)(1,s-N+2),\dots,$ $(N-2,s-N+2),(0,s-N+1)$ $(1,s-N+1),\dots$ $\dots,(N-2,s-N+1)\}$.
(m,n)	$(m-1,n-1)$	μ	where $(m,n) \in \{(1,s),(2,s),\dots,$ $(N-1,s),(1,s-1),(2,s-1),\dots,$ $(N-1,s-1),(1,s-2),(2,s-2),\dots,$ $(N-1,s-2),\cdots\cdots\cdots,$ $(1,s-N+2),(2,s-N+2),\dots$ $\dots,(N-1,s-N+2)\}$.
$(m,s-N+1)$	$\{\boldsymbol{\Delta}_3\}$	μ	for $m = 1,2,\dots,N-1$.

Corollary 2.2.1. *The time till absorption to states with inventory levels $S, S-1, \dots, S-N+1$, starting from a state (i,s), where $0 \le i < Q+N$, is denoted by τ_{a_1}, and τ_{a_1} follows Phase-Type distribution with representation $(\bar{\boldsymbol{\alpha}}_{3i}, T_{3i})$, where*

$$\bar{\boldsymbol{\alpha}}_{3i} = \begin{cases} \bar{\boldsymbol{\alpha}}_{1i}, & \text{when } N-1 \le i < Q+N \\ \bar{\boldsymbol{\alpha}}_{2i}, & \text{when } 0 < i < N-1 \\ \bar{\boldsymbol{\alpha}}_{00}, & \text{when } i = 0 \end{cases}$$

and

$$T_{3i} = \begin{cases} T_{1i}, & \text{when } N-1 \le i < Q+N \\ T_{2i}, & \text{when } 0 < i < N-1 \\ T_{00}, & \text{when } i = 0. \end{cases}$$

Proof. The proof immediately follows from the above theorem. □

Corollary 2.2.2.

$$E(\tau_{a_1}) = (1-\rho)(N+Q\omega^N)^{-1}\omega^{N-1}\Big[E(\tau_{00}) + \sum_{i=1}^{N-2}\rho^i E(\tau_{2i}) + \sum_{i=N-1}^{Q+N-1}\rho^i E(\tau_{1i})\Big]$$

where $\omega = \frac{\lambda+\gamma}{\lambda}$.

Proof.

$$\begin{aligned} E(\tau_{a_1}) &= \pi^{(0,s)}E(\tau_{00}) + \sum_{i=1}^{N-2}\pi^{(i,s)}E(\tau_{2i}) + \sum_{i=N-1}^{Q+N-1}\pi^{(i,s)}E(\tau_{1i}) \\ &= (1-\rho)\rho^0 r_s E(\tau_{00}) + (1-\rho)r_s\sum_{i=1}^{N-2}\rho^i E(\tau_{2i}) + (1-\rho)r_s\sum_{i=N-1}^{Q+N-1}\rho^i E(\tau_{1i}) \\ &= (1-\rho)(N+Q\omega^N)^{-1}\omega^{N-1}\Big[E(\tau_{00}) + \sum_{i=1}^{N-2}\rho^i E(\tau_{2i}) + \sum_{i=N-1}^{Q+N-1}\rho^i E(\tau_{1i})\Big] \end{aligned}$$

where $\omega = \frac{\lambda+\gamma}{\lambda}$. □

Theorem 2.2.5. *Let $0 \le i < Q+N$. Then the time till absorption to s i.e., τ_{b_1}, starting from $\{\mathbf{\Delta}_1\}\cup\{\mathbf{\Delta}_2\}\cup\{\mathbf{\Delta}_3\}$ follows Phase-Type distribution with representation $(\bar{\mathbf{v}}, \tilde{T})$.*

Proof. Consider the Markov chain $Y_2(t) = \{(X_2(t), I_2(t))\}$ where $X_2(t)$ is the number of customers in the system at time t and $I_2(t)$ is the inventory level at time t.

The state space of $Y_1(t)$ is $\{(m,n)\}\cup\{\mathbf{\Delta}_4\}$ where $0 \le m \le Q+N$, $s+1 \le n \le S$, and $\{\mathbf{\Delta}_4\}$ denotes the absorbing state $\{(m,s)\}$ where $0 \le m \le Q+N-1$. Clearly $Y_2(t)$ is a finite state Markov chain. The possible transitions and the corresponding instantaneous rates are given in Table 2.4.

Hence the infinitesimal generator $\tilde{Q}$ of the Markov chain $Y_2(t)$ is of the form

$$\begin{pmatrix} \tilde{T} & \tilde{\mathbf{T}}^{\mathbf{0}} \\ \mathbf{0} & 0 \end{pmatrix}$$

with initial probability vector $\bar{\mathbf{v}} = (1, 0, 0, \ldots, 0)$, and of appropriate order which depends upon the starting state. $\tilde{T}$ is a square matrix whose order depends upon the starting state.

Therefore the time till absorption to s, denoted by τ_{b_1}, starting from $\{\mathbf{\Delta}_1\} \cup \{\mathbf{\Delta}_2\} \cup \{\mathbf{\Delta}_3\}$, follows Phase-Type distribution with representation $(\bar{\mathbf{v}}, \tilde{T})$.

It's mean value is given by

$$E(\tau_{b_1}) = -\bar{\mathbf{v}}(\tilde{T})^{-1}\bar{\mathbf{e}}. \qquad \square$$

Theorem 2.2.6. *Let $0 \leq i < Q + N$, where i = Number of customers in the initial state. Let τ_{c_1} be the time elapsed between two order placement epochs. Then τ_{c_1} follows Phase-Type distribution.*

Proof. Let τ_{a_1} be the time till absorption to $\{\mathbf{\Delta}_1\} \cup \{\mathbf{\Delta}_2\} \cup \{\mathbf{\Delta}_3\}$, starting from a state with inventory level s and τ_{b_1} be the time till absorption to s, starting from $\{\mathbf{\Delta}_1\} \cup \{\mathbf{\Delta}_2\} \cup \{\mathbf{\Delta}_3\}$. We have τ_{a_1} follows Phase-Type distribution with representation $(\bar{\boldsymbol{\alpha}}_{3i}, T_{3i})$ for each i where $0 \leq i < Q + N$ and τ_{b_1} follows Phase-Type distribution with representation $(\bar{\mathbf{v}}, \tilde{T})$.

$\tau_{c_1} = \tau_{a_1} + \tau_{b_1}$ is the time elapsed between two order placement epochs.

Let $F_1(\cdot)$ be the probability distribution of τ_{a_1} and $F_2(\cdot)$ be the probability distribution of τ_{b_1}. Let $F(\cdot)$ be the probability distribution of τ_{c_1}. The convolution $F_1 * F_2(\cdot)$ is a Phase-Type distribution with representation

Table 2.4: Transitions and corresponding instantaneous rates for $Y_2(t)$

From	To	Rate	
(m, n)	$(m+1, n)$	λ	for $m \in \{0, 1, 2, \ldots, Q+N-1\}$ $n \in \{s+1, \ldots, S\}$
(m, n)	$(m-1, n-1)$	μ	for $m \in \{1, 2, \ldots, Q+N\}$, $n \in \{s+2, \ldots, S\}$
$(m, s+1)$	$\{\mathbf{\Delta}_4\}$	μ	for $m > 0$, $m \in \{1, 2, \ldots, Q+N\}$

$(\bar{\mathbf{u}}_1, L_1)$ where

$$\bar{\mathbf{u}}_1 = \begin{cases} (\bar{\boldsymbol{\alpha}}_{1i}, \alpha_{N+1}\bar{\mathbf{v}}) & \text{when } N-1 \leq i < Q+N, \\ (\bar{\boldsymbol{\alpha}}_{2i}, \alpha_{N^2-\frac{i}{2}(1+i)+1}\bar{\mathbf{v}}) & \text{when } 0 < i < N-1, \\ (\bar{\boldsymbol{\alpha}}_{00}, \alpha_{N(Q+N)+1}\bar{\mathbf{v}}) & \text{when } i = 0 \end{cases}$$

and

$$L_1 = \begin{cases} \begin{bmatrix} T_{1i} & \mathbf{T^0_{1i}}\bar{\mathbf{v}} \\ \mathbf{O} & \tilde{T} \end{bmatrix} & \text{when } N-1 \leq i < Q+N \\ \begin{bmatrix} T_{2i} & \mathbf{T^0_{2i}}\bar{\mathbf{v}} \\ \mathbf{O} & \tilde{T} \end{bmatrix} & \text{when } 0 < i < N-1 \\ \begin{bmatrix} T_{00} & \mathbf{T^0_{00}}\bar{\mathbf{v}} \\ \mathbf{O} & \tilde{T} \end{bmatrix} & \text{when } i = 0 \end{cases}$$

We know, $F(\cdot) = F_1 * F_2(\cdot)$.

Hence we get τ_{c_1} follows PH-distribution. □

Corollary 2.2.3. *When $0 \leq i < Q+N$, the expected inventory cycle time $E(\tau_{c_1})$ is given by*

$$E(\tau_{c_1}) = (1-\rho)(N+Q\omega^N)^{-1}\omega^{N-1}[E(\tau_{00}) + \sum_{i=1}^{N-2}\rho^i E(\tau_{2i}) + \sum_{i=N-1}^{Q+N-1}\rho^i E(\tau_{1i})] - \bar{\mathbf{v}}(\tilde{T})^{-1}\bar{\mathbf{e}}$$

where $\omega = \frac{\lambda+\gamma}{\lambda}$.

Proof.

$$\begin{aligned} E(\tau_{c_1}) &= E(\tau_{a_1} + \tau_{b_1}) \\ &= (1-\rho)(N+Q\omega^N)^{-1}\omega^{N-1}[E(\tau_{00} + \sum_{i=1}^{N-2}\rho^i E(\tau_{2i})) \\ &\quad + \sum_{i=N-1}^{Q+N-1}\rho^i E(\tau_{1i})] - \bar{\mathbf{v}}(\tilde{T})^{-1}\bar{\mathbf{e}} \end{aligned}$$

where $\omega = \frac{\lambda+\gamma}{\lambda}$. □

Case II: $i \geq Q + N$. Then,

Theorem 2.2.7. *Let (i, s) be the starting state, where s is the reorder level. Then the time till absorption to $\{\boldsymbol{\Delta}_1\}$. i.e., τ_{a_2} follows Phase-Type distribution where $\{\boldsymbol{\Delta}_1\}$ denotes the absorbing state and is given by (2.31). Also*

$$E(\tau_{a_2}) = E(\tau_{1i}).$$

Proof. Staring from a state with inventory level s, we need to find the time till absorption to the states with inventory levels $S, S-1, \ldots S-N+1$. Let us consider the Markov chain $Z_1(t) = \{(X_3(t), I_3(t))\}$ where $X_3(t)$ denotes the number of customers in the system at time t and $I_3(t)$ denotes the inventory level at that time.

The state space of $Z_1(t)$ is the same as that of $Y_1(t)$ (given in part (i) of Theorem 2.2.4) and $\{\boldsymbol{\Delta}_1\}$ denotes the absorbing state which is given by (2.31).

Clearly $Z_1(t)$ is a finite state Markov chain. The possible transitions and the corresponding instantaneous rates are given in Table 2.1. When $i \geq Q + N$, further arrivals need not be considered. The infinitesimal generator of the Markov chain $Z_1(t)$ is Q_{1i} and is of the form

$$\begin{bmatrix} T_{1i} & \mathbf{T^0_{1i}} \\ \mathbf{0} & 0 \end{bmatrix}$$

with initial probability vector $\bar{\boldsymbol{\alpha}}_{1i} = (0, 0, \ldots, 1, 0, \ldots, 0)$ where 1 is in the ith position. T_{1i} is square matrix of order N. $\bar{\boldsymbol{\alpha}}_{1i}$ has N elements.

Therefore, $\tau_{a_2} = \tau_{1i}$, and hence $\tau_{a_2} \sim$ PH $(\bar{\boldsymbol{\alpha}}_{1i}, T_{1i})$ and $E(\tau_{a_2}) = E(\tau_{1i})$. □

Theorem 2.2.8. *Let $i \geq Q + N$. Then the time till absorption to s i.e., τ_{b_2} starting from $\{\boldsymbol{\Delta}_1\}$ follows Erlang distribution.*

Proof. Consider the Markov chain $Z_2(t) = \{(X_4(t), I_4(t))\}$ where $X_4(t) =$ Number of customers in the system at time t and $I_4(t) =$ Inventory level at time t. The state space of $Z_2(t)$ is $\{(m, n)\} \cup \{\boldsymbol{\Delta}_5\}$ where $i - Q + 1 \leq m \leq i$,

Table 2.5: Transitions and corresponding instantaneous rates for $Z_2(t)$

From	To	Rate	
(m, n)	$(m-1, n-1)$	μ	where (m, n) belongs to $\{(i-N-(Q-2), s+2), \ldots, (i-N, S)\}$, when starting from the state $(i-N, S)$ and (m, n) belongs to $\{(i-Q+2, s+2), (i-Q+3, s+3), \ldots, (i, S)\}$, when starting from the states $(i, S), (i-1, S-1), \ldots, (i-(N-1), S-N+1)$
$(i-Q+1, s+1)$	$\{\boldsymbol{\Delta}_5\}$	μ	when starting from the states $(i, S), (i-1, S-1), \ldots, (i-(N-1), S-N+1)$
$(i-N-(Q-1), s+1)$	$\{\boldsymbol{\Delta}_5\}$	μ	when starting from the state $(i-N, S)$

$s+1 \leq n \leq S$, and $\{\boldsymbol{\Delta}_5\}$ denotes the absorbing state $\{(m, s)\}$ where $m \geq i-N-Q$.

Clearly $Z_2(t)$ is a finite state Markov chain. The possible transitions and the corresponding instantaneous rates are given in Table 2.5.

Hence τ_{b_2} follows Erlang distribution with parameter μ, and is of order $Q-l$, $l = 0, 1, 2, \ldots, N-1$. □

Theorem 2.2.9. *Let $i \geq Q+N$, where i = Number of customers in the starting state. Let τ_{c_2} be the time elapsed between two order placement epochs. Then τ_{c_2} follows Phase-Type distribution.*

Proof. Let τ_{a_2} be the time till absorption to $\{\boldsymbol{\Delta}_1\}$, starting from (i, s).

Then $\tau_{a_2} \sim$ PH $(\bar{\boldsymbol{\alpha}}_{1i}, T_{1i})$.

Let τ_{b_2} be the time till absorption to s, starting from $\{\boldsymbol{\Delta}_1\}$. Then τ_{b_2} follows Erlang distribution. Hence, the time elapsed between two order placement epochs can be written as $\tau_{c_2} = \tau_{a_2} + \tau_{b_2}$.

Let $G_1(\cdot)$, $G_2(\cdot)$ and $G(\cdot)$ be the probability distributions of τ_{a_2}, τ_{b_2} and τ_{c_2} respectively.

We know $G(\cdot) = G_1 * G_2(\cdot)$ and $G_1 * G_2(\cdot)$ follows Phase-Type distribution. Hence we get τ_{c_2} follows Phase-Type distribution. □

2.2.4 Cost Analysis

Next, we find a cost function. Let the various costs involved in the model be as given below:
C_H: Inventory holding cost per unit item per unit time
C_S: Set up cost per unit order, under natural purchase
C_W: Waiting time cost per customer per unit time
C_{LP}: Local purchase cost per unit order
C_{NP}: Natural purchase cost per unit order
C_C: Cancellation cost per unit order cancelled.

The total expected cost per unit time,

$$\text{TEC} = C_H E(I) + (C_S + C_{\text{NP}} Q) R_r + C_{\text{LP}} R_{\text{LP}} (Q + N) + C_C R_{\text{LP}} + C_W W_S.$$

Using (2.27), (2.28), (2.29) and (2.30) we get TEC as

$$\begin{aligned}\text{TEC} = &\Big\{ C_H \Big[\frac{s(1-\omega^N)(1-\omega) + (1-N)(1-\omega) - (\omega^N - \omega)}{(1-\omega)^2} \\ &+ \frac{1}{2}(Q - N + 1)(S + s - N + 2)\omega^N \\ &+ (\omega^N + 1)(N-1)(S - \frac{N}{2} + 1) - (S - N + 2)\frac{(\omega^N - \omega)}{\omega - 1} \\ &- \frac{\omega^2}{(\omega-1)^2}[1 - \omega^{N-2}(1 - (N-2)(\omega - 1))]\Big] \\ &+ ((C_S + C_{\text{NP}} Q)\omega^N + C_{\text{LP}}(Q + N) + C_C)\lambda \Big\} (N + \omega^N Q)^{-1} \\ &+ C_W (\frac{1}{\mu - \lambda}). \end{aligned} \tag{2.34}$$

To verify the convexity of the above cost function with respect to N, the derivative with respect to N may be computed, then equate it to zero.

Nevertheless, solving it is a laborious task. This can be viewed from the following Lemma.

Lemma 2.2.1. *The necessary condition for cost function to be optimal with respect to N is*

$$\begin{aligned} & C_H \Bigg[\frac{1}{\omega-1}(s\omega^N \log_e \omega + 1) + S(1+\omega^N) + \frac{3}{2} - N - \frac{\omega}{\omega-1} \\ & \quad +\omega^N \log_e \omega[-(\omega-1)^{-2}(N-1) - S(\omega-1)^{-1} + \frac{Q}{2}(S+s+1)]\Bigg] \\ & \quad + \lambda[(c_S + C_{NP}Q)\omega^N \log_e \omega + C_{LP}] \\ & = F_1(N+\omega^N Q)^{-1}(1+Q\omega^N \log_e \omega) \end{aligned} \tag{2.35}$$

where

$$\begin{aligned} F_1 = C_H & \Bigg[\frac{s(1-\omega^N)(1-\omega) + (1-N)(1-\omega) - (\omega^N - \omega)}{(1-\omega)^2} \\ & + \frac{1}{2}(Q-N+1)(S+s-N+2)\omega^N + (\omega^N+1)(N-1)(S - \frac{N}{2} + 1) \\ & -(S-N+2)\left(\frac{\omega^N - \omega}{\omega - 1}\right) - \frac{\omega^2}{(\omega-1)^2}[1-\omega^{N-2}(1-(N-2)(\omega-1))]\Bigg] \\ & + (C_S + C_{NP}Q)\lambda\omega^N + C_{LP}(Q+N)\lambda + C_c\lambda \end{aligned}$$

and $\omega = \frac{\lambda+\gamma}{\lambda}$.

Proof. $\frac{\partial}{\partial N}(\text{TEC}) = 0$ gives the required condition on simplification. □

Remark. Note that while computing the zeros of the above equation, which are values of N, we could admit only a positive integer as the value of N, since N is the number of service completions after placing a natural order.

2.2.4.1 Numerical Analysis of TEC

Analysis of TEC as function of s, S or N is quite complex. Hence we give a few numerical illustrations:

Case 1: Analysis of TEC as function of N.

Here change in the values of cost function as N varies, is analyzed.

Input data: $S = 20$, $\lambda = 23$, $\mu = 25$, $\gamma = 20$, $C_{\text{NP}} = 30$, $C_{\text{LP}} = 35$, $C_H = 0.5$, $C_S = 1000$, $C_C = 16$, $C_W = 1200$.

Table 2.6: Effect of N on cost function TEC

N	TEC when $s = 8$	TEC when $s = 9$
1	3564.8	3724.3
2	3359.8	3518.5
3	3270.3	3432.0
4	3232.2	3397.5
5	3217.1	3385.3
6	3211.9	**3382.3**
7	**3210.7**	3382.6
8	3210.9	3383.9

In Table 2.6, the total expected costs per unit time against various values of N for a given set of input parameters are displayed.

When $s = 8$ and N varies from 1 to 8, the TEC values decrease, reach a minimum at $N = 7$, and then increase. When $s = 9$ and N varies from 1 to 8, the TEC values decrease, reach a minimum at $N = 6$, and then increase.

Hence the convexity of cost function is verified numerically.

Case 2: Analysis of TEC as a function of S.

Input data: $\lambda = 23$, $\mu = 25$, $\gamma = 20$, $N = 5$, $s = 8$, $C_{\text{NP}} = 30$, $C_{\text{LP}} = 35$, $C_H = 0.5$, $C_S = 1000$, $C_C = 16$, $C_W = 1200$.

In Table 2.7, the total expected costs per unit time against various values of S for a given set of input parameters are displayed.

Table 2.7: Effect of S on TEC

S	TEC
20	3217.1
21	3074.7
22	2952.4
23	2846.2
24	2753.2
25	2670.9
26	2597.8
27	2532.2

Table 2.7 shows that as S increases, TEC function is monotonically decreasing and hence convex.

Case 3: Analysis of TEC when s, S and N are varied simultaneously.

Here change in the values of cost function as s, S and N varies simultaneously, is analyzed.

Input data: $\lambda = 23$, $\mu = 25$, $\gamma = 20$, $C_{\text{NP}} = 30$, $C_{\text{LP}} = 35$, $C_H = 0.5$, $C_S = 1000$, $C_C = 16$, $C_W = 1200$.

In Table 2.8, total expected cost per unit time against various values of the triplet (s, S, N) are displayed. We can observe that TEC values decrease, reach a minimum at the values (11,30,10) of (s, S, N) and then increase. Hence convexity of cost function is verified numerically.

Table 2.8: Effect of (s, S, N) on cost function TEC

(s, S, N)	TEC
(9,28,8)	2511.8862
(10,29,9)	2511.0493
(11,30,10)	**2510.9289**
(12,31,11)	2511.1480
(13,32,12)	2511.5251
(14,33,13)	2511.9741
(15,34,14)	2512.4546

2.3 General Case-Model II: Perishable Items

2.3.1 Model Formulation and Analysis

Let $X(t)$ = Number of customers in the system at time t and
$I(t)$ = Inventory level at time t.
$\{(X(t), I(t)), t \geq 0\}$ is a CTMC with state space $E = E_1 \times E_2$ where $E_1 = \{0, 1, 2, \dots\}$ and $E_2 = \{s-N+1, s-N+2, \dots, S\}$ where $s-N+1 > 0$. Therefore,

$$E = \{(i, j) | i \in E_1, j \in E_2\}$$

2.3.1.1 Infinitesimal Generator $\tilde{A}$

We write the infinitesimal generator $\tilde{A}$ of the process as

$$\tilde{A} = (a((i,j),(m,n))), \text{ where } (i,j),(m,n) \in E.$$

The elements of $\tilde{A}$ can be obtained as

$$a((i,j),(m,n)) = \begin{cases} \lambda, & m = i+1; \quad i = 0,1,2,\ldots \\ & n = j; \quad j = s-N+1,\ldots,S \\ \mu, & m = i-1; \quad i = 1,2,3,\ldots \\ & n = j-1; \quad j = s-N+2,\ldots,S \\ \mu, & m = i-1; \quad i = 1,2,3,\ldots \\ & n = S; \quad j = s-N+1 \\ \gamma, & m = i; \quad i = 0,1,2,\ldots \\ & n = j+Q; \quad j = s-N+1,\ldots,s \\ j\beta, & m = i; \quad i = 0,1,2,\ldots \\ & n = j-1; \quad j = s-N+2,\ldots,S \\ j\beta, & m = i; \quad i = 0,1,2,\ldots \\ & n = S; \quad j = s-N+1 \\ -(\lambda+j\beta+\gamma), & m = i; \quad i = 0 \\ & n = j; \quad j = s-N+1,\ldots,s \\ -(\lambda+j\beta), & m = i; \quad i = 0 \\ & n = j; \quad j = s+1,\ldots,S \\ -(\lambda+j\beta+\gamma+\mu), & m = i; \quad i = 1,2,3,\ldots \\ & n = j; \quad j = s-N+1,\ldots,s \\ -(\lambda+j\beta+\mu), & m = i; \quad i = 1,2,3,\ldots \\ & n = j; \quad j = s+1,\ldots,S \\ 0, & \text{otherwise.} \end{cases}$$

We can write $\tilde{A}$ interms of submatrices as follows:

$$\tilde{A} = \begin{bmatrix} B_1 & A_0 & & & \\ A_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix}$$

where $A_0 = \lambda I_{Q+N}$ and

$$A_2 = \begin{bmatrix} \mathbf{0}_{1\times(Q+N-1)} & \mu I_1 \\ \mu I_{Q+N-1} & \mathbf{0}_{(Q+N-1)\times 1} \end{bmatrix}.$$

A_1, B_1 are given in page 45.

The elements d_j, $j = s - N + 1, \ldots, S$ in A_1 are given by

$$d_j = \begin{cases} \lambda + j\beta + \gamma + \mu, & j = s - N + 1, \ldots, s \\ \lambda + j\beta + \mu, & j = s + 1, \ldots, S. \end{cases}$$

and elements f_j, $j = s - N + 1, \ldots, S$ in B_1 are given by

$$f_j = \begin{cases} \lambda + j\beta + \gamma, & j = s - N + 1, \ldots, s \\ \lambda + j\beta, & j = s + 1, \ldots, S. \end{cases}$$

A_0, A_1, A_2 and B_1 are square matrices of order $Q + N$.

2.3.1.2 Steady-State Analysis

Let $A = A_0 + A_1 + A_2$ be the generator matrix and is given in page 47.

The elements h_j, $j = s - N + 1, \ldots, S$ in A are given by

$$h_j = \begin{cases} j\beta + \gamma + \mu, & j = s - N + 1, \ldots, s \\ j\beta + \mu, & j = s + 1, \ldots, S \end{cases}$$

and A is a square matrix of order $Q + N$.

First we investigate the stationary distribution of A. This will help in analysing the stability of the larger system, namely the CTMC $\{(X(t), I(t)), t \geq 0\}$ as $t \to \infty$.

$$A_1 = \begin{bmatrix}
-d_{s-N+1} & & & & & & & & \gamma & & & & (s-N+1)\beta \\
(s-N+2)\beta & -d_{s-N+2} & & & & & & & & \gamma & & & \\
& (s-N+3)\beta & -d_{s-N+3} & & & & & & & & \ddots & & \\
& & & \ddots & & & & & & & & & \\
& & & (s-1)\beta & -d_{s-1} & & & & & & & \gamma & \\
& & & & s\beta & -d_s & & & & & & & \gamma \\
& & & & & (s+1)\beta & -d_{s+1} & & & & & & 0 \\
& & & & & & & \ddots & & & & & \\
& & & & & & & & \ddots & & & & \\
& & & & & & & & & (S-1)\beta & -d_{S-1} & & \\
& & & & & & & & & & S\beta & & -d_S
\end{bmatrix}$$

$$B_1 = \begin{bmatrix}
-f_{s-N+1} & & & & & & & & \gamma & & & & (s-N+1)\beta \\
(s-N+2)\beta & -f_{s-N+2} & & & & & & & & \gamma & & & \\
& (s-N+3)\beta & -f_{s-N+3} & & & & & & & & \ddots & & \\
& & & \ddots & & & & & & & & & \\
& & & (s-1)\beta & -f_{s-1} & & & & & & & \gamma & \\
& & & & s\beta & -f_s & & & & & & & \gamma \\
& & & & & (s+1)\beta & -f_{s+1} & & & & & & 0 \\
& & & & & & & \ddots & & & & & \\
& & & & & & & & \ddots & & & & \\
& & & & & & & & & (S-1)\beta & -f_{S-1} & & \\
& & & & & & & & & & S\beta & & -f_S
\end{bmatrix}$$

Theorem 2.3.1. *The steady-state probability distribution* $\mathbf{\Phi} = (\phi_{s-N+1}, \phi_{s-N+2}, \ldots, \phi_S)$ *corresponding to the matrix A is given by*

$$\phi_j = q_j \phi_{s-N+1}, \quad j = s-N+2, \ldots, S \tag{2.36}$$

where

$$q_j = \begin{cases} \frac{\tilde{\eta}(s-N+1,j-1)}{\tilde{\psi}(s-N+2,j)} & j = s-N+2, \ldots, s+1 \\ \frac{1}{j\beta+\mu} \frac{\tilde{\eta}(s-N+1,s)}{\tilde{\psi}(s-N+2,s)}, & j = s+2, \ldots, S-N+1 \\ \frac{1}{j\beta+\mu} \left(\frac{\tilde{\eta}(s-N+1,s)}{\tilde{\psi}(s-N+2,s)} - \gamma \right), & j = S-N+2 \\ \frac{1}{j\beta+\mu} \left(\frac{\tilde{\eta}(s-N+1,s)}{\tilde{\psi}(s-N+2,s)} - \gamma \right. \\ \qquad \left. -\gamma \sum_{k_1=s-N+1}^{j-Q-2} \frac{\tilde{\eta}(s-N+1,k_1)}{\tilde{\psi}(s-N+2,k_1+1)} \right), \\ \qquad j = S-N+3, \ldots, S \end{cases}$$

$$\text{where} \quad \tilde{\eta}(j_1, j_2) = \prod_{\tilde{k}=j_1}^{j_2} (\tilde{k}\beta + \gamma + \mu) \text{ and}$$

$$\tilde{\psi}(j_1, j_2) = \prod_{\tilde{k}=j_1}^{j_2} (\tilde{k}\beta + \mu)$$

ϕ_{s-N+1} *can be obtained by solving the equation* $\mathbf{\Phi}\bar{\mathbf{e}} = 1$ *as*

$$\begin{aligned} \phi_{s-N+1} = \Big[1 + &\sum_{j=s-N+2}^{s+1} \frac{\tilde{\eta}(s-N+1, j-1)}{\tilde{\psi}(s-N+2, j)} \\ &+ \tilde{c} \sum_{j=s+2}^{S-N+1} \left(\frac{1}{\mu + j\beta} \right) + \frac{\tilde{c} - \gamma}{\mu + (S-N+2)\beta} \\ + \sum_{j=S-N+3}^{S} &\Big[\frac{1}{\mu + j\beta} \Big(\tilde{c} - \gamma - \gamma \sum_{k_1=s-N+1}^{j-Q-2} \frac{\tilde{\eta}(s-N+1, k_1)}{\tilde{\psi}(s-N+2, k_1+1)} \Big) \Big] \Big]^{-1} \end{aligned}$$

where

$$\tilde{c} = \frac{\tilde{\eta}(s-N+1, s)}{\tilde{\psi}(s-N+2, s)}.$$

$$\begin{bmatrix}
-h_{s\text{-}N+1} & & & & & & & \gamma & & & & (s\text{-}N+1)\beta+\mu \\
(s\text{-}N+2)\beta+\mu & -h_{s\text{-}N+2} & & & & & & & \gamma & & & \\
& (s\text{-}N+3)\beta+\mu & -h_{s\text{-}N+3} & & & & & & & \ddots & & \\
& & & \ddots & & & & & & & & \\
& & & (s\text{-}1)\beta+\mu & -h_{s\text{-}1} & & & & & & \gamma & \\
& & & & s\beta+\mu & -h_{s} & & & & & & \gamma \\
& & & & & (s+1)\beta+\mu & -h_{s+1} & & & & & 0 \\
& & & & & & \ddots & \ddots & & & & \\
& & & & & & & \ddots & \ddots & & & \\
& & & & & & & & & (S\text{-}1)\beta+\mu & -h_{S\text{-}1} & \\
& & & & & & & & & & S\beta+\mu & -h_{S}
\end{bmatrix}$$

Proof. We have $\mathbf{\Phi} A = \mathbf{0}$ and $\mathbf{\Phi}\bar{\mathbf{e}} = 1$

$$\begin{aligned}\mathbf{\Phi} A = \mathbf{0} &\Rightarrow \\ &-\phi_l(l\beta+\gamma+\mu)+\phi_{l+1}((l+1)\beta+\mu)=0,\end{aligned} \tag{2.37}$$

for $l = s-N+1,\ldots,s$.

$$-\phi_l(l\beta+\mu)+\phi_{l+1}((l+1)\beta+\mu)=0, \tag{2.38}$$

for $l = s+1,\ldots,S-N$.

$$-\phi_l(l\beta+\mu)+\phi_{l+1}((l+1)\beta+\mu)+\phi_{l-Q}\gamma=0, \tag{2.39}$$

for $l = S-N+1,\ldots,S-1$. Also we get

$$-\phi_S(S\beta+\mu)+\phi_s\gamma+\phi_{s-N+1}((s-N+1)\beta+\mu)=0. \tag{2.40}$$

Consider equation (2.37).
Equation (2.37) gives

$$\phi_j = \frac{\tilde{\eta}(s-N+1,j-1)}{\tilde{\psi}(s-N+2,j)}\phi_{s-N+1} \tag{2.41}$$

for $j = s-N+2,\ldots,s+1$.
Note that

$$\begin{aligned}\tilde{\eta}(j_1,j_2) &= \prod_{\tilde{k}=j_1}^{j_2}(\tilde{k}\beta+\gamma+\mu) \text{ and}\\ \tilde{\psi}(j_1,j_2) &= \prod_{\tilde{k}=j_1}^{j_2}(\tilde{k}\beta+\mu).\end{aligned}$$

Equation (2.38) gives

$$\phi_j = \frac{1}{(j\beta+\mu)}\frac{\tilde{\eta}(s-N+1,s)}{\tilde{\psi}(s-N+2,s)}\phi_{s-N+1} \tag{2.42}$$

for $j = s+2,\ldots,S-N+1$.

Using equation (2.39) we get

$$\phi_{S-N+2} = \frac{1}{(S-N+2)\beta+\mu}\left(\frac{\tilde{\eta}(s-N+1,s)}{\tilde{\psi}(s-N+2,s)} - \gamma\right)\phi_{s-N+1} \tag{2.43}$$

and

$$\phi_j = \frac{1}{(j\beta+\mu)}\left(\frac{\tilde{\eta}(s-N+1,s)}{\tilde{\psi}(s-N+2,s)} - \gamma - \gamma\sum_{k_1=s-N+1}^{j-Q-2}\frac{\tilde{\eta}(s-N+1,k_1)}{\tilde{\psi}(s-N+2,k_1+1)}\right) \times \phi_{s-N+1} \tag{2.44}$$

for $j = S-N+3,\ldots,S$.

Using equations (2.41), (2.42), (2.43) and (2.44), we get equation (2.36).

Now to find ϕ_{s-N+1}. Consider $\mathbf{\Phi}\bar{\mathbf{e}} = 1$.

$$\left(1 + \sum_{j=s-N+2}^{s+1} q_j + \sum_{j=s+2}^{S-N+1} q_j + q_{S-N+2} + \sum_{j=s-N+3}^{S} q_j\right)\phi_{s-N+1} = 1. \tag{2.45}$$

$$\sum_{j=s-N+2}^{s+1} q_j = \sum_{j=s-N+2}^{s+1}\left(\frac{\prod_{\tilde{k}=s-N+1}^{j-1}(\mu+\gamma+\tilde{k}\beta)}{\prod_{\tilde{k}=s-N+2}^{j}(\mu+\tilde{k}\beta)}\right). \tag{2.46}$$

$$\sum_{j=s+2}^{S-N+1} q_j = \sum_{j=s+2}^{S-N+1}\frac{1}{\mu+\beta j}\tilde{c} \tag{2.47}$$

where

$$\tilde{c} = \frac{\prod_{\tilde{k}=s-N+1}^{s}(\mu+\gamma+\tilde{k}\beta)}{\prod_{\tilde{k}=s-N+2}^{s}(\mu+\tilde{k}\beta)}.$$

$$q_{S-N+2} = \frac{\tilde{c}-\gamma}{\mu+(S-N+2)\beta} \tag{2.48}$$

$$\sum_{j=S-N+3}^{S} q_j = \sum_{j=S-N+3}^{S}\left(\frac{1}{\mu+j\beta}\left(\tilde{c}-\gamma-\gamma\sum_{k_1=s-N+1}^{j-Q-2}\frac{\tilde{\eta}(s-N+1,k_1)}{\tilde{\psi}(s-N+2,k_1+1)}\right)\right) \tag{2.49}$$

Using equations (2.46), (2.47), (2.48), (2.49) and on simplification, we get equation (2.45) as

$$\phi_{s-N+1} = \Bigg[1 + \sum_{j=s-N+2}^{s+1} \frac{\tilde{\eta}(s-N+1, j-1)}{\tilde{\psi}(s-N+2, j)} + \tilde{c} \sum_{j=s+2}^{S-N+1} (\frac{1}{\mu + j\beta}) + \frac{\tilde{c} - \gamma}{\mu + (S-N+2)\beta} + \sum_{j=S-N+3}^{S} \left(\frac{1}{\mu + j\beta} \left(\tilde{c} - \gamma - \gamma \sum_{k_1=s-N+1}^{j-Q-2} \frac{\tilde{\eta}(s-N+1, k_1)}{\tilde{\psi}(s-N+2, k_1+1)} \right) \right) \Bigg]^{-1}$$

where

$$\tilde{c} = \frac{\tilde{\eta}(s-N+1, s)}{\tilde{\psi}(s-N+2, s)},$$

$$\tilde{\eta}(j_1, j_2) = \prod_{\tilde{k}=j_1}^{j_2} (\mu + \gamma + \tilde{k}\beta) \text{ and}$$

$$\tilde{\psi}(j_1, j_2) = \prod_{\tilde{k}=j_1}^{j_2} (\mu + \tilde{k}\beta).$$

Hence the proof is completed. □

2.3.1.3 Stability Condition

The result in Theorem 2.3.1 enables us to compute the stability of the CTMC $\{(X(t), I(t)), t \geq 0\}$.

Theorem 2.3.2. *The process* $\{(X(t), I(t)), t \geq 0\}$ *is stable iff* $\lambda < \mu$.

Proof. Since the process under consideration is an LIQBD, it is stable iff

$$\mathbf{\Phi} A_0 \bar{\mathbf{e}} < \mathbf{\Phi} A_2 \bar{e} \tag{2.50}$$

(Neuts [40]), where $\mathbf{\Phi}$ is the steady-state distribution of the generator matrix $A = A_0 + A_1 + A_2$.

It can be shown that

$$\mathbf{\Phi} A_0 \bar{\mathbf{e}} = (\phi_{s-N+1}\phi_{s-N+2}\dots\phi_S)\lambda I_{Q+N}\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}_{(Q+N)\times 1} = \lambda.$$

$$\mathbf{\Phi} A_2 \bar{\mathbf{e}} = (\phi_{s-N+1}\phi_{s-N+2}\dots\phi_S)\begin{bmatrix}\mathbf{0}_{1\times(Q+N-1)} & \mu I_1\\ \mu I_{Q+N-1} & \mathbf{0}_{(Q+N-1)\times 1}\end{bmatrix}\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}_{(Q+N)\times 1} = \mu.$$

Using (2.50) we get $\lambda < \mu$. □

Having obtained the condition for the system to stabilize, we turn to compute the long-run probability distribution of the system states. Infact we show that the joint distribution of the system state can be written as the product of the marginal distributions of the components.

2.3.2 The Steady-State Probability Distribution of $\tilde{A}$

2.3.2.1 Stationary Distribution when Service Time is Negligible

The generator matrix corresponding to the process when service time is negligible is denoted by $\tilde{Q}$ and is given in page 53.

The elements $\tilde{h}_j$, $j = s-N+1,\dots,S$ in $\tilde{Q}$ is given by

$$\tilde{h}_j = \begin{cases} j\beta+\gamma+\lambda, & j = s-N+1,\dots,s \\ j\beta+\lambda, & j = s+1,\dots,S. \end{cases}$$

Theorem 2.3.3. *The steady-state probability distribution* $\mathbf{\Delta} = (r_{s-N+1}, r_{s-N+2},\dots,r_S)$ *corresponding to the matrix* $\tilde{Q}$ *is given by*

$$r_j = \tilde{q}_j r_{s-N+1}, \quad j = s-N+2,\dots,S \tag{2.51}$$

where

$$\tilde{q}_j = \begin{cases} \frac{\hat{\eta}(s-N+1,j-1)}{\hat{\psi}(s-N+2,j)} & j = s-N+2,\dots,s+1 \\ \frac{1}{j\beta+\lambda}\frac{\hat{\eta}(s-N+1,s)}{\hat{\psi}(s-N+2,s)}, & j = s+2,\dots,S-N+1 \\ \frac{1}{j\beta+\lambda}\left(\frac{\hat{\eta}(s-N+1,s)}{\hat{\psi}(s-N+2,s)} - \gamma\right), & j = S-N+2 \\ \frac{1}{j\beta+\lambda}\left(\frac{\hat{\eta}(s-N+1,s)}{\hat{\psi}(s-N+2,s)} - \gamma \right. \\ \qquad \left. -\gamma\sum_{k_1=s-N+1}^{j-Q-2}\frac{\hat{\eta}(s-N+1,k_1)}{\hat{\psi}(s-N+2,k_1+1)}\right), \\ \qquad j = S-N+3,\dots,S \end{cases}$$

where

$$\hat{\eta}(j_1,j_2) = \prod_{\tilde{k}=j_1}^{j_2}(\tilde{k}\beta+\gamma+\lambda) \text{ and}$$

$$\hat{\psi}(j_1,j_2) = \prod_{\tilde{k}=j_1}^{j_2}(\tilde{k}\beta+\lambda).$$

r_{s-N+1} *can be obtained by solving the equation* $\mathbf{\Delta}\bar{\mathbf{e}} = 1$ *as*

$$\begin{aligned} r_{s-N+1} = \Big[1 + &\sum_{j=s-N+2}^{s+1}\frac{\hat{\eta}(s-N+1,j-1)}{\hat{\psi}(s-N+2,j)} \\ &+ \hat{c}\sum_{j=s+2}^{S-N+1}\Big(\frac{1}{\lambda+j\beta}\Big) + \frac{\hat{c}-\gamma}{\lambda+(S-N+2)\beta} \\ + \sum_{j=S-N+3}^{S}\Big[\frac{1}{\lambda+j\beta}&\Big(\hat{c}-\gamma-\gamma\sum_{k_1=s-N+1}^{j-Q-2}\frac{\hat{\eta}(s-N+1,k_1)}{\hat{\psi}(s-N+2,k_1+1)}\Big)\Big]\Big]^{-1} \end{aligned}$$

where

$$\hat{c} = \frac{\hat{\eta}(s-N+1,s)}{\hat{\psi}(s-N+2,s)}.$$

Proof. We have $\mathbf{\Delta}\tilde{Q} = \mathbf{0}$ and $\mathbf{\Delta}\bar{\mathbf{e}} = 1$.

$$\begin{aligned} &\mathbf{\Delta}\tilde{Q} = \mathbf{0} \Rightarrow \\ &\quad - r_l(l\beta+\gamma+\lambda) + r_{l+1}((l+1)\beta+\lambda) = 0, \end{aligned} \tag{2.52}$$

$$
\begin{bmatrix}
-\tilde{h}_{s\text{-}N+1} & & & & & & & & \gamma & & & & & (s\text{-}N+1)\beta+\lambda \\
(s\text{-}N+2)\beta+\lambda & -\tilde{h}_{s\text{-}N+2} & & & & & & & & \gamma & & & & \\
 & (s\text{-}N+3)\beta+\lambda & -\tilde{h}_{s\text{-}N+3} & & & & & & & \ddots & & & & \\
 & & & \ddots & & & & & & & & & & \\
 & & & (s\text{-}1)\beta+\lambda & -\tilde{h}_{s\text{-}1} & & & & & & & \gamma & & \\
 & & & & s\beta+\lambda & -\tilde{h}_{s} & & & & & & & \gamma & \\
 & & & & & (s+1)\beta+\lambda & -\tilde{h}_{s+1} & & & & & & 0 & \\
 & & & & & & \ddots & \ddots & & & & & & \\
 & & & & & & & & \ddots & \ddots & & & & \\
 & & & & & & & & & & (S\text{-}1)\beta+\lambda & -\tilde{h}_{S\text{-}1} & & \\
 & & & & & & & & & & & S\beta+\lambda & -\tilde{h}_{S} &
\end{bmatrix}
$$

for $l = s - N + 1, \ldots, s$.

$$-r_l(l\beta + \lambda) + r_{l+1}((l+1)\beta + \lambda) = 0, \tag{2.53}$$

for $l = s + 1, \ldots, S - N$.

$$-r_l(l\beta + \lambda) + r_{l+1}((l+1)\beta + \lambda) + r_{l-Q}\gamma = 0, \tag{2.54}$$

for $l = S - N + 1, \ldots, S - 1$. Also,

$$-r_S(S\beta + \lambda) + r_s\gamma + r_{s-N+1}((s - N + 1)\beta + \lambda) = 0. \tag{2.55}$$

Equation (2.52) gives

$$r_j = \frac{\hat{\eta}(s - N + 1, j - 1)}{\hat{\psi}(s - N + 2, j)} r_{s-N+1} \tag{2.56}$$

for $j = s - N + 2, \ldots, s + 1$. Note that

$$\hat{\eta}(j_1, j_2) = \prod_{\tilde{k}=j_1}^{j_2} (\tilde{k}\beta + \gamma + \lambda) \text{ and}$$

$$\hat{\psi}(j_1, j_2) = \prod_{\tilde{k}=j_1}^{j_2} (\tilde{k}\beta + \lambda).$$

Equation (2.53) gives

$$r_j = \frac{1}{(j\beta + \lambda)} \frac{\hat{\eta}(s - N + 1, s)}{\hat{\psi}(s - N + 2, s)} r_{s-N+1} \tag{2.57}$$

for $j = s + 2, \ldots, S - N + 1$.

Equation (2.54) gives

$$r_{S-N+2} = \frac{1}{(S - N + 2)\beta + \lambda} \left(\frac{\hat{\eta}(s - N + 1, s)}{\hat{\psi}(s - N + 2, s)} - \gamma \right) r_{s-N+1} \tag{2.58}$$

and

$$r_j = \frac{1}{(j\beta + \lambda)} \left(\frac{\hat{\eta}(s - N + 1, s)}{\hat{\psi}(s - N + 2, s)} - \gamma - \gamma \sum_{k_1=s-N+1}^{j-Q-2} \frac{\hat{\eta}(s - N + 1, k_1)}{\hat{\psi}(s - N + 2, k_1 + 1)} \right) \times r_{s-N+1} \tag{2.59}$$

for $j = S - N + 3, \ldots, S$.

Using equations (2.56), (2.57), (2.58) and (2.59), we get equation (2.51).

Now to find r_{s-N+1}. Consider $\mathbf{\Delta \bar{e}} = 1$.

$$\left(1 + \sum_{j=s-N+2}^{s+1} \tilde{q}_j + \sum_{j=s+2}^{S-N+1} \tilde{q}_j + \tilde{q}_{S-N+2} + \sum_{j=S-N+3}^{S} \tilde{q}_j\right) r_{s-N+1} = 1 \quad (2.60)$$

$$\sum_{j=s-N+2}^{s+1} \tilde{q}_j = \sum_{j=s-N+2}^{s+1} \left(\frac{\prod_{\tilde{k}=s-N+1}^{j-1}(\lambda + \gamma + \tilde{k}\beta)}{\prod_{\tilde{k}=s-N+2}^{j}(\lambda + \tilde{k}\beta)}\right) \quad (2.61)$$

$$\sum_{j=s+2}^{S-N+1} \tilde{q}_j = \sum_{j=s+2}^{S-N+1} \frac{1}{\lambda + \beta j}\hat{c} \quad (2.62)$$

where

$$\hat{c} = \frac{\prod_{\tilde{k}=s-N+1}^{s}(\lambda + \gamma + \tilde{k}\beta)}{\prod_{\tilde{k}=s-N+2}^{s}(\lambda + \tilde{k}\beta)}.$$

$$\tilde{q}_{S-N+2} = \frac{\hat{c} - \gamma}{\lambda + (S - N + 2)\beta} \quad (2.63)$$

$$\sum_{j=S-N+3}^{S} \tilde{q}_j = \sum_{j=S-N+3}^{S} \left(\frac{1}{\lambda + j\beta}\left(\hat{c} - \gamma - \gamma \sum_{k_1=s-N+1}^{j-Q-2} \frac{\hat{\eta}(s - N + 1, k_1)}{\hat{\psi}(s - N + 2, k_1 + 1)}\right)\right). \quad (2.64)$$

Using equations (2.61), (2.62), (2.63) and (2.64), and on simplification, we get equation (2.60) as

$$r_{s-N+1} = \left[1 + \sum_{j=s-N+2}^{s+1} \frac{\hat{\eta}(s - N + 1, j - 1)}{\hat{\psi}(s - N + 2, j)} + \hat{c} \sum_{j=s+2}^{S-N+1} (\frac{1}{\lambda + j\beta}) + \frac{\hat{c} - \gamma}{\lambda + (S - N + 2)\beta} + \sum_{j=S-N+3}^{S} \left(\frac{1}{\lambda + j\beta}\left(\hat{c} - \gamma - \gamma \sum_{k_1=s-N+1}^{j-Q-2} \frac{\hat{\eta}(s - N + 1, k_1)}{\hat{\psi}(s - N + 2, k_1 + 1)}\right)\right)\right]^{-1},$$

where

$$\hat{c} = \frac{\hat{\eta}(s-N+1,s)}{\hat{\psi}(s-N+2,s)},$$

$$\hat{\eta}(j_1,j_2) = \prod_{\tilde{k}=j_1}^{j_2} (\lambda + \gamma + \tilde{k}\beta) \text{ and}$$

$$\hat{\psi}(j_1,j_2) = \prod_{\tilde{k}=j_1}^{j_2} (\lambda + \tilde{k}\beta).$$

Hence the proof is completed. □

Remark. The results given by equation (2.51) is obtained in an equivalent form by Krishnamoorthy and Raju [28] using balance equations.

2.3.2.2 Stochastic Decomposition of System State

Let $\bar{\boldsymbol{\pi}}$ be the steady-state probability vector of $\tilde{A}$.

$$\bar{\boldsymbol{\pi}} = (\boldsymbol{\pi}^{(0)}, \boldsymbol{\pi}^{(1)}, \boldsymbol{\pi}^{(2)}, \dots)$$

where $\boldsymbol{\pi}^{(i)} = (\pi^{(i,s-N+1)}, \pi^{(i,s-N+2)}, \dots, \pi^{(i,S)})$, $i = 0, 1, 2, \dots$ and

$$\pi^{(i,j)} = \lim_{t\to\infty} P(X(t) = i, I(t) = j), \quad (i,j) \in E.$$

$\pi^{(i,j)}$ is the steady-state probability for the state (i, j). We claim that

$$\boldsymbol{\pi}^{(i)} = K\rho^i \boldsymbol{\Delta}, \quad i \geq 0 \tag{2.65}$$

where $\boldsymbol{\Delta} = (r_{s-N+1}, r_{s-N+2}, \dots, r_S)$ is the steady-state probability vector when the service time is negligible, K is a constant to be determined and $\rho = \frac{\lambda}{\mu}$.

Proof. We have $\bar{\boldsymbol{\pi}}\tilde{A} = \mathbf{0}$ and $\boldsymbol{\pi}\bar{\mathbf{e}} = 1$.

$$\bar{\boldsymbol{\pi}}\tilde{A} = \mathbf{0} \Rightarrow (\boldsymbol{\pi}^{(0)}, \boldsymbol{\pi}^{(1)}, \boldsymbol{\pi}^{(2)}, \dots) \begin{bmatrix} B_1 & A_0 & & & \\ A_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & & \ddots & \ddots & \ddots \end{bmatrix} = \mathbf{0}$$

which gives

$$\boldsymbol{\pi}^{(0)}B_1 + \boldsymbol{\pi}^{(1)}A_2 = \mathbf{0} \tag{2.66}$$

$$\boldsymbol{\pi}^{(i+2)}A_2 + \boldsymbol{\pi}^{(i+1)}A_1 + \boldsymbol{\pi}^{(i)}A_0 = \mathbf{0} \tag{2.67}$$

$i = 0, 1, 2, \ldots$.
When (2.65) is true, we get from (2.66),

$$K\rho^0\boldsymbol{\Delta}B_1 + K\rho\boldsymbol{\Delta}A_2 = \mathbf{0}$$
$$\text{i.e.,}\quad \boldsymbol{\Delta}(B_1 + \rho A_2) = \mathbf{0}$$
$$\text{i.e.,}\quad \boldsymbol{\Delta}\tilde{Q} = \mathbf{0},$$

which is true since $\boldsymbol{\Delta} = (r_{s-N+1}, r_{s-N+2}, \ldots, r_S)$ is the steady-state probability vector corresponding to the generator $\tilde{Q}$ associated with the Markovian chain of the inventory process under consideration when service time is negligible.

When (2.65) is true, we get from (2.67),

$$K\rho^{i+2}\boldsymbol{\Delta}A_2 + K\rho^{i+1}\boldsymbol{\Delta}A_1 + K\rho^{i}\boldsymbol{\Delta}A_0 = \mathbf{0}$$

where $i = 0, 1, 2, \ldots$

$$\text{i.e.,}\ \boldsymbol{\Delta}(\rho A_2 + A_1 + \frac{1}{\rho}A_0) = \mathbf{0}$$
$$\text{i.e.,}\ \boldsymbol{\Delta}\tilde{Q} = \mathbf{0},$$

which is true, by following the same argument given above.

Hence the stochastic decomposition of system state is verified. □

2.3.2.3 Determination of K

We have

$$\sum_{i=0}^{\infty}\sum_{j=s-N+1}^{S}\pi^{(i,j)} = 1$$
$$K\left(\sum_{i=0}^{\infty}\rho^i\right)\left(\sum_{j=s-N+1}^{S} r_j\right) = 1 \quad (\text{using (2.65)})$$
$$\text{Therefore,}\quad K = 1 - \rho.$$

2.3.2.4 Explicit Solution

From the above discussions, we can write the steady-state probability vector explicitly as in the following theorem:

Theorem 2.3.4. *The steady-state probability vector* $\bar{\boldsymbol{\pi}}$ *of* $\tilde{A}$ *partitioned as* $\bar{\boldsymbol{\pi}} = (\boldsymbol{\pi}^{(0)}, \boldsymbol{\pi}^{(1)}, \boldsymbol{\pi}^{(2)}, \dots)$*, where each* $\boldsymbol{\pi}^{(i)}$*,* $i = 0, 1, 2, \dots$ *again partitioned as*

$$\boldsymbol{\pi}^{(i)} = (\pi^{(i,s-N+1)}, \pi^{(i,s-N+2)}, \dots, \pi^{(i,S)})$$

are obtained by

$$\pi^{(i,j)} = (1-\rho)\rho^i r_j \tag{2.68}$$

where $\rho = \frac{\lambda}{\mu}$ *and* r_j*;* $j = s-N+1, \dots, S$*, represent the inventory level probabilities when service time is negligible and are given as*

$$r_j = \begin{cases} \frac{\hat{\eta}(s-N+1,j-1)}{\hat{\psi}(s-N+2,j)} r_{s-N+1}, & j = s-N+2, \dots, s+1 \\ \frac{1}{j\beta+\lambda} \frac{\hat{\eta}(s-N+1,s)}{\hat{\psi}(s-N+2,s)} r_{s-N+1}, & j = s+2, \dots, S-N+1 \\ \frac{1}{j\beta+\lambda} \left(\frac{\hat{\eta}(s-N+1,s)}{\hat{\psi}(s-N+2,s)} - \gamma \right) r_{s-N+1}, & j = S-N+2 \\ \frac{1}{j\beta+\lambda} \left(\frac{\hat{\eta}(s-N+1,s)}{\hat{\psi}(s-N+2,s)} - \gamma \right. & \\ \qquad \left. -\gamma \sum_{k_1=s-N+1}^{j-Q-2} \frac{\hat{\eta}(s-N+1,k_1)}{\hat{\psi}(s-N+2,k_1+1)} \right), & j = S-N+3, \dots, S \end{cases}$$

where

$$\hat{\eta}(j_1, j_2) = \prod_{\tilde{k}=j_1}^{j_2} (\tilde{k}\beta + \gamma + \lambda),$$

$$\hat{\psi}(j_1, j_2) = \prod_{\tilde{k}=j_1}^{j_2} (\tilde{k}\beta + \lambda) \text{ and}$$

$$r_{s-N+1} = \Bigg[1 + \sum_{j=s-N+2}^{s+1} \frac{\hat{\eta}(s-N+1, j-1)}{\hat{\psi}(s-N+2, j)} + \hat{c} \sum_{j=s+2}^{S-N+1} \Big(\frac{1}{\lambda + j\beta}\Big) + \frac{\hat{c} - \gamma}{\lambda + (S-N+2)\beta} + \sum_{j=S-N+3}^{S} \Big[\frac{1}{\lambda + j\beta}\Big(\hat{c} - \gamma - \gamma \sum_{k_1=s-N+1}^{j-Q-2} \frac{\hat{\eta}(s-N+1, k_1)}{\hat{\psi}(s-N+2, k_1+1)}\Big)\Big]\Bigg]^{-1}$$

where

$$\hat{c} = \frac{\hat{\eta}(s-N+1,s)}{\hat{\psi}(s-N+2,s)}.$$

The above result not only tells us that the original system possesses stochastic decomposition but also the important fact that the system state distribution is the product of the distribution of its marginals: one component is the classical $M|M|1$ whose long run distribution for i customers in the system is $(1-\rho)\rho^i$, $i \geq 0$ and the other factor is the probability of j items in the inventory. Next we turn to find out how the system performs. The measures given in the following section are pointers to the system performance.

2.3.3 System Performance Measures

(a) Expected inventory held in the system (Mean inventory level),

$$\begin{aligned}
E(I) &= \sum_{i=0}^{\infty} \sum_{j=s-N+1}^{S} j\pi^{(i,j)} \\
&= \left(\sum_{i=0}^{\infty}(1-\rho)\rho^i\right)\left(\sum_{j=s-N+1}^{S} jr_j\right) \\
&= \sum_{j=s-N+1}^{S} jr_j \\
&= \Bigg\{ s-N+1+\sum_{j=s-N+2}^{s+1}\left(j\frac{\hat{\eta}(s-N+1,j-1)}{\hat{\psi}(s-N+2,j)}\right) \\
&+\hat{c}\sum_{j=s+2}^{S-N+1}\left(\frac{j}{\lambda+j\beta}\right)+\frac{(S-N+2)(\hat{c}-\gamma)}{\lambda+(S-N+2)\beta} \\
&.+\sum_{j=S-N+3}^{S}\Bigg(\frac{j}{\lambda+j\beta} \\
&\quad\times\left(\hat{c}-\gamma-\gamma\sum_{k_1=s-N+1}^{j-Q-2}\left(\frac{\hat{\eta}(s-N+1,k_1)}{\hat{\psi}(s-N+2,k_1+1)}\right)\right)\Bigg)\Bigg\} \\
&\qquad\times r_{s-N+1}.
\end{aligned}$$

(b) Mean decay rate, $R_D = \beta E(I)$

(c) Mean waiting time of customers, $W_s = \frac{L}{\lambda}$ where
$L =$ Expected number of customers in the system

$$\begin{aligned}
L &= \sum_{j=s-N+1}^{S} \sum_{i=0}^{\infty} i\pi^{(i,j)} \\
&= \left(\sum_{i=0}^{\infty} i(1-\rho)\rho^i\right)\left(\sum_{j=s-N+1}^{S} r_j\right) \\
&= \frac{\rho}{1-\rho} \quad \text{where } \rho = \lambda/\mu.
\end{aligned}$$

Therefore

$$W_S = \frac{1}{\mu - \lambda}.$$

(d) Mean reorder rate,

$$\begin{aligned}
R_r &= \mu \sum_{i=1}^{\infty} \pi^{(i,s+1)} + (s+1)\beta \sum_{i=0}^{\infty} \pi^{(i,s+1)} \\
&= \mu \sum_{i=1}^{\infty} (1-\rho)\rho^i r_{s+1} + (s+1)\beta \sum_{i=0}^{\infty} (1-\rho)\rho^i r_{s+1} \\
&= \frac{\hat{\eta}(s-N+1, s)}{\hat{\psi}(s-N+2, s)} r_{s-N+1}.
\end{aligned}$$

(e) Mean local purchase rate,

$$\begin{aligned}
R_{\text{LP}} &= \mu \sum_{i=1}^{\infty} (1-\rho)\rho^i r_{s-N+1} + (s-N+1)\beta \sum_{i=0}^{\infty} (1-\rho)\rho^i r_{s-N+1} \\
&= (\lambda + (s-N+1)\beta) r_{s-N+1}.
\end{aligned}$$

Remark. If we put $\beta = 0$ the above measures will get reduced to those of model I.

2.3.4 Cost Analysis

Next, we find a cost function. Let the various costs involved in the model be as given below:

C_H: Inventory holding cost per unit item per unit time
C_S: Set up cost per unit order, under natural purchase
C_W: Waiting time cost per customer per unit time
C_{LP}: Local purchase cost per unit order
C_{NP}: Natural purchase cost per unit order
C_C: Cancellation cost per unit order cancelled
C_D: Decay cost per unit item per unit time.

The total expected cost per unit time,

$$\begin{aligned}
\mathrm{TEC} &= (C_S + C_{\mathrm{NP}}Q)R_r + C_{\mathrm{LP}}(Q+N)R_{\mathrm{LP}} + C_C R_{\mathrm{LP}} \\
&\quad + C_H E(I) + C_D R_D + C_W W_S \\
&= \Bigg[(C_S + C_{\mathrm{NP}}Q)\frac{\hat{\eta}(s-N+1,s)}{\hat{\psi}(s-N+2,s)} \\
&\quad + (C_{\mathrm{LP}}(Q+N) + C_C)(\lambda + (s-N+1)\beta) \\
&\quad + (C_H + C_D\beta)\Bigg\{s-N+1+\sum_{j=s-N+2}^{s+1}\left(j\frac{\hat{\eta}(s-N+1,j-1)}{\hat{\psi}(s-N+2,j)}\right) \\
&\quad + \hat{c}\sum_{j=s+2}^{S-N+1}\left(\frac{j}{\lambda+j\beta}\right) + \frac{(S-N+2)(\hat{c}-\gamma)}{\lambda+(S-N+2)\beta} \\
&\quad + \sum_{j=S-N+3}^{S}\left(\frac{j}{\lambda+j\beta}\left(\hat{c}-\gamma-\gamma\sum_{k_1=s-N+1}^{j-Q-2}\left(\frac{\hat{\eta}(s-N+1,k_1)}{\hat{\psi}(s-N+2,k_1+1)}\right)\right)\right)\Bigg\}\Bigg] \\
&\quad \times r_{s-N+1} + C_W\left(\frac{1}{\mu-\lambda}\right)
\end{aligned}$$

where

$$\begin{aligned}
r_{s-N+1} = \Bigg[1 &+ \sum_{j=s-N+2}^{s+1}\frac{\hat{\eta}(s-N+1,j-1)}{\hat{\psi}(s-N+2,j)} + \hat{c}\sum_{j=s+2}^{S-N+1}\left(\frac{1}{\lambda+j\beta}\right) \\
&+ \frac{\hat{c}-\gamma}{\lambda+(S-N+2)\beta} \\
&+ \sum_{j=S-N+3}^{S}\left(\frac{1}{\lambda+j\beta}\left(\hat{c}-\gamma-\gamma\sum_{k_1=s-N+1}^{j-Q-2}\frac{\hat{\eta}(s-N+1,k_1)}{\hat{\psi}(s-N+2,k_1+1)}\right)\right)\Bigg]^{-1}
\end{aligned}$$

where

$$\hat{c} = \frac{\hat{\eta}(s-N+1,s)}{\hat{\psi}(s-N+2,s)}, \hat{\eta}(j_1,j_2) = \prod_{\tilde{k}=j_1}^{j_2}(\lambda+\gamma+\tilde{k}\beta) \text{ and}$$

$$\hat{\psi}(j_1,j_2) = \prod_{\tilde{k}=j_1}^{j_2}(\lambda+\tilde{k}\beta).$$

To verify the convexity of the above cost function with respect to s, S or N, the corresponding derivative may be computed, then equate it to zero. Nevertheless, solving it is a laborious task. Since analysis of TEC as function of s, S or N is complex, we give a few numerical illustrations.

2.3.5 Numerical Analysis

Case 1: Analysis of TEC as a function of S.

Input Data:
$C_{\text{NP}} = 12$, $C_{\text{LP}} = 15$, $C_H = 0.5$, $C_S = 15$, $C_C = 8$, $C_W = 30$, $C_D = 7$, $\lambda = 23$, $\mu = 25$, $\gamma = 16$, $\beta = 0.2$, $s = 9$, $N = 2$

Table 2.9: Effect of S on TEC

S	TEC ($\beta = 0.2$)	TEC ($\beta = 0$)
19	531.1766	452.8553
20	530.6603	450.1204
21	**530.6305**	447.8557
22	530.9799	445.9583
23	531.6299	444.3531
24	532.5222	442.9843
25	533.6123	441.8095

Table 2.9 shows that as S increases from 19 to 25, TEC values decrease, reach a minimum at $S = 21$ and then increase (when $\beta = 0.2$). Hence convexity of TEC with respect to S is verified numerically.

Case 2: Analysis of TEC as function of N.

Input Data:
$C_{\text{NP}} = 12$, $C_{\text{LP}} = 15$, $C_H = 0.5$, $C_S = 15$, $C_C = 8$, $C_W = 30$, $C_D = 7$, $\lambda = 23$, $\mu = 25$, $\gamma = 16$, $\beta = 0.2$, $s = 9$, $S = 20$

Table 2.10 shows that TEC function is monotonically decreasing, as N increases from 1 to 9 (when $\beta = 0.2$).

Table 2.10: Effect of N on TEC

N	TEC ($\beta = 0.2$)	TEC ($\beta = 0$)
1	599.001	534.0973
2	530.6603	450.1204
3	471.0809	401.6862
4	433.7624	372.8300
5	409.9854	355.4405
6	394.6799	344.9396
7	384.7420	338.6094
8	378.2326	334.8041
9	373.9357	332.5230

Comparison of Perishable and Non-Perishable Inventory Models

In the above input data, if we put $\beta = 0$ and $C_D = 0$, we get the TEC values of non-perishable inventory model and are given in Tables 2.9 and 2.10. We can see from Table 2.9 that as S increases, TEC function is monotonically decreasing (when $\beta = 0$). Also on comparing the TEC values in Table 2.9 we get that, TEC as function of S is higher in the case of perishable items. From Table 2.10 we get that as N increases, TEC function is monotonically decreasing (when $\beta = 0$). Also on comparing the TEC values in Table 2.10 we get that, TEC as function of N is higher in the case of perishable items.

Chapter 3

(s, Q) inventory systems with positive lead time and service time under T-policy

3.1 Introduction

In the previous chapter we investigated the effect of N-policy on local purchase for both non-perishable and perishable items. Next we proceed to analyze the effect of the T-policy.

In this chapter[1], we consider two (s, Q) inventory models with service time, in which T-policy is adopted during lead time. In model I, items are assumed to be non-perishable. In Model II, items are subject to decay and the time to decay follows exponential distribution with parameter β. In both the models, arrival of demands are according to a Poisson process with parameter λ and lead time follows exponential distribution with parameter γ. Service time follows exponential distribution with parameter μ. The reorder level is s. T-policy is brought into it as follows. T is assumed to be a random variable whose distribution will be specified in the course of

[1]Part of this chapter is included in the paper: Krishnamoorthy A., Resmi Varghese and Lakshmy B. "(s, Q) inventory systems with positive lead time and service time under T-policy" (Accepted for presentation in 23$^{\text{rd}}$ Swadeshi Science Congress, a national seminar organized by SSM, 6–8 Nov, 2013 at M. G. University, Kottayam, Kerala).

discussion. As and when the inventory level drops to s, an order is placed for $Q = S - s$ units. After placing an order, if the replenishment does not occur within a time of T units from the order placement epoch, then a local purchase is made to bring the inventory level to S, by cancelling the order that is already placed. (Cancellation of order is necessary since, otherwise the inventory level may go beyond S as and when the replenishment against the most recent order placed takes place). If the inventory level reaches zero before the realization of T time units and also before the occurrence of replenishment, regardless of the number of customers present in the system, an immediate local purchase of S units is made, by cancelling the order that is already placed. Local purchase can be done when T is realized or when the inventory level reaches zero, which ever occurs first. T is assumed as a random variable which follows exponential distribution with parameter α.

3.2 Model I: Non-Perishable Items

3.2.1 Model Formulation and Analysis

Let $X(t)$ = Number of customers in the system at time t and
$I(t)$ = Inventory level at time t.

$\{(X(t), I(t)), t \geq 0\}$ is a CTMC with state space $E = E_1 \times E_2$ where $E_1 = \{0, 1, 2, \ldots,\}$ and $E_2 = \{1, 2, \ldots, S\}$. Therefore

$$E = \{(i, j) | i \in E_1, j \in E_2\}.$$

3.2.1.1 Infinitesimal Generator $\tilde{A}$

We write the infinitesimal generator of the process as

$$\tilde{A} = (a((i, j), (m, n)))$$

where $(i, j), (m, n) \in E$.

The elements of $\tilde{A}$ can be obtained as

$$a((i,j),(m,n)) = \begin{cases} \lambda, & m = i+1; \quad i = 0,1,2,\dots \\ & n = j; \quad j = 1,2,\dots,S \\ \mu, & m = i-1; \quad i = 1,2,3,\dots \\ & n = j-1; \quad j = 2,\dots,S \\ \mu, & m = i-1; \quad i = 1,2,3,\dots \\ & n = S; \quad j = 1 \\ \gamma, & m = i; \quad i = 0,1,2,\dots \\ & n = j+Q; \quad j = 1,2,\dots,s \\ -(\lambda+\gamma+\alpha), & m = i; \quad i = 0 \\ & n = j; \quad j = 1,2,\dots,s \\ -\lambda, & m = i; \quad i = 0 \\ & n = j; \quad j = s+1,\dots,S \\ -(\lambda+\gamma+\mu+\alpha), & m = i; \quad i = 1,2,3,\dots \\ & n = j; \quad j = 1,2,\dots,s \\ -(\lambda+\mu), & m = i; \quad i = 1,2,3,\dots \\ & n = j; \quad j = s+1,\dots,S \\ 0, & \text{otherwise.} \end{cases}$$

$\tilde{A}$ can be written in terms of sub matrices as follows:

$$\tilde{A} = \begin{bmatrix} B_1 & A_0 & & \\ A_2 & A_1 & A_0 & \\ & A_2 & A_1 & A_0 \\ & \ddots & \ddots & \ddots \end{bmatrix}$$

where $A_0 = \lambda I_S$. A_1 is

$$\begin{bmatrix} -(\lambda+\gamma+\alpha+\mu)I_{s-1} & \mathbf{0}_{(s-1)\times 1} & O_{(s-1)\times(S-2s)} & \gamma I_{s-1} & \alpha\bar{\mathbf{e}}_{(s-1)\times 1} \\ \mathbf{0}_{1\times(s-1)} & -(\lambda+\gamma+\alpha+\mu)I_1 & \mathbf{0}_{1\times(S-2s)} & \mathbf{0}_{1\times(s-1)} & (\gamma+\alpha)I_1 \\ O_{(S-2s)\times(s-1)} & \mathbf{0}_{(S-2s)\times 1} & -(\lambda+\mu)I_{S-2s} & O_{(S-2s)\times(s-1)} & \mathbf{0}_{(S-2s)\times 1} \\ O_{(s-1)\times(s-1)} & \mathbf{0}_{(s-1)\times 1} & O_{(s-1)\times(S-2s)} & -(\lambda+\mu)I_{s-1} & \mathbf{0}_{(s-1)\times 1} \\ \mathbf{0}_{1\times(s-1)} & \mathbf{0}_{1\times 1} & \mathbf{0}_{1\times(S-2s)} & \mathbf{0}_{1\times(s-1)} & -(\lambda+\mu)I_1 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} \mathbf{0}_{1\times(S-1)} & \mu I_1 \\ \mu I_{S-1} & \mathbf{0}_{(S-1)\times 1} \end{bmatrix} \text{ and}$$

$$B_1 = \begin{bmatrix} -(\lambda+\gamma+\alpha)I_{s-1} & \mathbf{0}_{(s-1)\times 1} & O_{(s-1)\times(S-2s)} & \gamma I_{s-1} & \alpha\bar{\mathbf{e}}_{(s-1)\times 1} \\ \mathbf{0}_{1\times(s-1)} & -(\lambda+\gamma+\alpha)I_1 & \mathbf{0}_{1\times(S-2s)} & \mathbf{0}_{1\times(s-1)} & (\gamma+\alpha)I_1 \\ O_{(S-2s)\times(s-1)} & \mathbf{0}_{(S-2s)\times 1} & -\lambda I_{S-2s} & O_{(S-2s)\times(s-1)} & \mathbf{0}_{(S-2s)\times 1} \\ O_{(s-1)\times(s-1)} & \mathbf{0}_{(s-1)\times 1} & O_{(s-1)\times(S-2s)} & -\lambda I_{s-1} & \mathbf{0}_{(s-1)\times 1} \\ \mathbf{0}_{1\times(s-1)} & \mathbf{0}_{1\times 1} & \mathbf{0}_{1\times(S-2s)} & \mathbf{0}_{1\times(s-1)} & -\lambda I_1 \end{bmatrix}$$

A_0, A_1, A_2 and B_1 are square matrices of order S.

3.2.1.2 Steady-State Analysis

Let $A = A_0 + A_1 + A_2$ be the generator matrix of a MC on the finite state space $\{1, \ldots, S\}$ and is obtained as

$$\begin{bmatrix}
-(\gamma+\alpha+\mu) & & & & & & \gamma & & & \alpha+\mu \\
\mu & -(\gamma+\alpha+\mu) & & & & & & & & \alpha \\
 & \mu & \ddots & & & & & \ddots & & \vdots \\
 & & & \ddots & & & & & \gamma & \alpha \\
 & & & & -(\gamma+\alpha+\mu) & & & & & \gamma+\alpha \\
 & & & & \mu & -\mu & & & & \\
 & & & & & \ddots & \ddots & & & \\
 & & & & & & & & \mu & -\mu
\end{bmatrix}$$

A is a square matrix of order S.

First we investigate the stationary distribution of A. This will help us in analyzing the stability of the larger system, namely the CTMC $\{(X(t), I(t)), t \geq 0\}$ as $t \to \infty$.

Theorem 3.2.1. *The steady-state probability distribution* $\mathbf{\Phi} = (\phi_1, \phi_2, \ldots, \phi_S)$ *corresponding to the matrix* A *is given by*

$$\phi_j = q_j \phi_1, \quad j = 1, 2, \ldots, S \tag{3.1}$$

where

$$q_j = \begin{cases} \left(\frac{\mu+\gamma+\alpha}{\mu}\right)^{j-1}, & j = 1, 2, \ldots, s+1 \\ \left(\frac{\mu+\gamma+\alpha}{\mu}\right)^{s}, & j = s+2, \ldots, S-s+1 \\ \left(\frac{\mu+\gamma+\alpha}{\mu}\right)^{s} - \frac{\gamma}{\gamma+\alpha}\left(\left(\frac{\mu+\gamma+\alpha}{\mu}\right)^{j-Q-1} - 1\right), & \\ & j = S-s+2, S-s+3, \ldots, S \end{cases}$$

and ϕ_1 *can be obtained by solving the equation* $\mathbf{\Phi}\bar{\mathbf{e}} = 1$*, as*

$$\phi_1 = \frac{\mu(B-1)^2}{(\mu(B-1)^2 Q + \alpha)B^s + s\gamma(B-1) - \alpha} \tag{3.2}$$

where $B = \frac{\mu+\gamma+\alpha}{\mu}$.

Proof. We have $\mathbf{\Phi}A = \mathbf{0}$ and $\mathbf{\Phi}\bar{\mathbf{e}} = 1$.
$\mathbf{\Phi}A = \mathbf{0} \Rightarrow$
For $l = 1, 2, \ldots, s$,

$$-\phi_l(\mu+\gamma+\alpha) + \phi_{l+1}\mu = 0 \tag{3.3}$$

For $l = s+1, \ldots, S-s$, we have

$$-\phi_l\mu + \phi_{l+1}\mu = 0. \tag{3.4}$$

Next for $l = S-s+1, \ldots, S-1$, we have

$$\phi_{l-Q}\gamma - \phi_l\mu + \phi_{l+1}\mu = 0. \tag{3.5}$$

Also,

$$\phi_1(\alpha+\mu) + (\phi_2+\phi_3+\cdots+\phi_{s-1})\alpha + \phi_s(\gamma+\alpha) - \phi_S\mu = 0. \tag{3.6}$$

Equation (3.3) gives

$$\phi_j = \left(\frac{\mu+\gamma+\alpha}{\mu}\right)^{j-1}\phi_1 \tag{3.7}$$

for $j = 2, 3, \ldots, s+1$. From (3.4), we have

$$\phi_j = \left(\frac{\mu+\gamma+\alpha}{\mu}\right)^{s}\phi_1 \tag{3.8}$$

for $j = s+2, s+3, \ldots, S-s+1$.
Now pass on to (3.5) to get

$$\phi_{S-s+x} = \left[\left(\frac{\mu+\gamma+\alpha}{\mu}\right)^s - \frac{\gamma}{\gamma+\alpha}\left(\left(\frac{\mu+\gamma+\alpha}{\mu}\right)^{x-1} - 1\right)\right]\phi_1,$$

where $x = 2, 3, 4, \ldots, s$.

The above equation can be rewritten as

$$\phi_j = \left[\left(\frac{\mu+\gamma+\alpha}{\mu}\right)^s - \frac{\gamma}{\gamma+\alpha}\left(\left(\frac{\mu+\gamma+\alpha}{\mu}\right)^{j-Q-1} - 1\right)\right]\phi_1 \qquad (3.9)$$

where $j = S-s+2, S-s+3, \ldots, S$.

From equations (3.7), (3.8) and (3.9), we get the relation (3.1).

Now to get ϕ_1, consider the normalizing condition $\mathbf{\Phi\bar{e}} = 1$. That is,

$$\phi_1\left[1 + \sum_{j=2}^{s+1} B^{j-1} + \sum_{j=s+2}^{S-s+1} B^s + \sum_{j=S-s+2}^{S}(B^s - \frac{\gamma}{\gamma+\alpha}(B^{j-Q-1} - 1))\right] = 1 \qquad (3.10)$$

where $B = \frac{\mu+\gamma+\alpha}{\mu}$.

$$\sum_{j=2}^{s+1} B^{j-1} = \frac{B(B^s-1)}{B-1}. \qquad (3.11)$$

$$\sum_{j=s+2}^{S-s+1} B^s = B^s(S-2s). \qquad (3.12)$$

$$\sum_{j=S-s+2}^{S}\Big(B^s - \frac{\gamma}{\gamma+\alpha}(B^{j-Q-1} - 1)\Big)$$
$$= (s-1)(B^s + \frac{\gamma}{\gamma+\alpha}) - (\frac{\gamma}{\gamma+\alpha})B\frac{(B^{s-1}-1)}{B-1}. \qquad (3.13)$$

Using equations (3.11), (3.12) and (3.13), we get equation (3.10) as

$$\phi_1\left[1 + \frac{B(B^s-1)}{B-1} + B^s(S-2s) + (s-1)\left(B^s + \frac{\gamma}{\gamma+\alpha}\right) - (\frac{\gamma}{\gamma+\alpha})B\frac{(B^{s-1}-1)}{B-1}\right] = 1.$$

$$\frac{\phi_1}{(B-1)(\gamma+\alpha)}[(\gamma+\alpha)(B^{s+1}-1+(B^{s+1}-B^s)(S-s-1)) \\ -\gamma(B^s-B-(B-1)(s-1))]=1.$$

$$\frac{\phi_1}{(B-1)(\gamma+\alpha)}[B^{s+1}(\gamma+\alpha)Q-B^s((\gamma+\alpha)Q-\alpha+s\gamma(B-1)-\alpha]=1.$$

This gives

$$\phi_1=\frac{\mu(B-1)^2}{B^s(\mu(B-1)^2Q+\alpha)+s\gamma(B-1)-\alpha}.$$

Hence the theorem. □

3.2.1.3 Stability Condition

The result in Theorem 3.2.1 enables us to compute the stability of the CTMC $\{(X(t),I(t)),t\geq 0\}$.

Theorem 3.2.2. *The process* $\{(X(t),I(t)),t\geq 0\}$ *is stable iff* $\lambda<\mu$.

Proof. For an LIQBD process it is well known that

$$\mathbf{\Phi}A_0\bar{\mathbf{e}}<\mathbf{\Phi}A_2\bar{\mathbf{e}} \tag{3.14}$$

(Neuts [40]), where $\mathbf{\Phi}$ is the steady-state distribution of the generator matrix $A=A_0+A_1+A_2$.

It can be shown that

$$\mathbf{\Phi}A_0\bar{\mathbf{e}}=(\phi_1\phi_2\ldots\phi_S)\lambda I_S\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}_{S\times 1}$$

$$=(\lambda(\phi_1+\phi_2+\cdots+\phi_S))=\lambda.$$

$$\mathbf{\Phi}A_2\bar{\mathbf{e}}=(\phi_1\phi_2\ldots\phi_S)\begin{bmatrix}\mathbf{0}_{1\times(S-1)} & \mu I_1\\ \mu I_{S-1} & \mathbf{0}_{(S-1)\times 1}\end{bmatrix}\begin{pmatrix}1\\1\\\vdots\\1\end{pmatrix}_{S\times 1}$$

$$=(\mu(\phi_1+\phi_2+\cdots+\phi_S))=\mu.$$

Using equation (3.14), we get $\lambda<\mu$. □

Having obtained the condition for the system to stabilize, we turn to compute the long-run probability distribution of the system states. Infact we show that the joint distribution of the system state can be written as the product of the marginal distributions of the components.

3.2.2 The Steady-State Probability Distribution of $\tilde{A}$

3.2.2.1 Stationary Distribution when Service Time is Negligible

Let $\tilde{Q}$ be the generator matrix associated with the Markovian chain of the inventory process under consideration when service time is negligible, and $\mathbf{\Delta} = (r_1, r_2, \ldots, r_S)$ be the stationary probability vector corresponding to $\tilde{Q}$. We get $\tilde{Q}$ as

$$\begin{bmatrix}
-(\gamma+\alpha+\lambda) & & & & & & & \gamma & & \alpha+\lambda \\
\lambda & -(\gamma+\alpha+\lambda) & & & & & & & & \alpha \\
 & \lambda & \ddots & & & & & & \ddots & \vdots \\
 & & & \ddots & & & & & \gamma & \alpha \\
 & & & & -(\gamma+\alpha+\lambda) & & & & & \gamma+\alpha \\
 & & & & \lambda & -\lambda & & & & \\
 & & & & & \ddots & \ddots & & & \\
 & & & & & & & & \lambda & -\lambda
\end{bmatrix}$$

Theorem 3.2.3. *The steady-state probability distribution* $\mathbf{\Delta} = (r_1, r_2, \ldots, r_S)$ *corresponding to the matrix* $\tilde{Q}$ *is given by*

$$r_j = \tilde{q}_j r_1, \quad j = 1, 2, \ldots, S \tag{3.15}$$

where

$$\tilde{q}_j = \begin{cases} \left(\frac{\lambda+\gamma+\alpha}{\lambda}\right)^{j-1}, & j = 1, 2, \ldots, s+1 \\ \left(\frac{\lambda+\gamma+\alpha}{\lambda}\right)^{s}, & j = s+2, \ldots, S-s+1 \\ \left(\frac{\lambda+\gamma+\alpha}{\lambda}\right)^{s} - \frac{\gamma}{\gamma+\alpha}\left(\left(\frac{\lambda+\gamma+\alpha}{\lambda}\right)^{j-Q-1} - 1\right), & j = S-s+2, \ldots, S \end{cases}$$

and r_1 *can be obtained by solving the equation* $\mathbf{\Delta \bar{e}} = 1$*, as*

$$r_1 = \frac{\lambda F^2}{(\lambda F^2 Q + \alpha)M^s + s\gamma F - \alpha} \tag{3.16}$$

where $F = M - 1$ *and* $M = \frac{\lambda+\gamma+\alpha}{\lambda}$*.*

Proof. We have $\mathbf{\Delta}\tilde{Q} = \mathbf{0}$ and $\mathbf{\Delta \bar{e}} = 1$.
$\mathbf{\Delta}\tilde{Q} = \mathbf{0} \quad \Rightarrow$

$$-r_l(\lambda + \gamma + \alpha) + r_{l+1}\lambda = 0 \tag{3.17}$$

for $l = 1, 2, \ldots, s$;

$$-r_l\lambda + r_{l+1}\lambda = 0 \tag{3.18}$$

for $l = s + 1, \ldots, S - s$ and

$$r_{l-Q}\gamma - r_l\lambda + r_{l+1}\lambda = 0 \tag{3.19}$$

where $l = S - s + 1, \ldots, S - 1$.
Finally we have

$$r_1(\alpha + \lambda) + (r_2 + r_3 + \cdots + r_{s-1})\alpha + r_s(\gamma + \alpha) - r_S\lambda = 0. \tag{3.20}$$

Consider the equation (3.17).

We have

$$r_j = \left(\frac{\lambda + \gamma + \alpha}{\lambda}\right)^{j-1} r_1 \tag{3.21}$$

for $j = 2, 3, \ldots, s + 1$.

Consider the equation (3.18). From this we have

$$r_j = \left(\frac{\lambda + \gamma + \alpha}{\lambda}\right)^{s} r_1 \tag{3.22}$$

for $j = s + 2, s + 3, \ldots, S - s + 1$.
Next with equation (3.19) we have

$$r_{S-s+x} = \left[\left(\frac{\lambda + \gamma + \alpha}{\lambda}\right)^{s} - \frac{\gamma}{\gamma + \alpha}\left(\left(\frac{\lambda + \gamma + \alpha}{\lambda}\right)^{x-1} - 1\right)\right] r_1$$

for $x = 2, 3, \ldots, s$. The above equation can be rewritten as

$$r_j = \left[\left(\frac{\lambda+\gamma+\alpha}{\lambda} \right)^s - \frac{\gamma}{\gamma+\alpha} \left(\left(\frac{\lambda+\gamma+\alpha}{\lambda} \right)^{j-Q-1} - 1 \right) \right] r_1 \tag{3.23}$$

where $j = S-s+2, S-s+3, \ldots, S$.

Hence from equations (3.21), (3.22) and (3.23) we get equation (3.15). Now to get r_1, consider the normalizing condition $\mathbf{\Delta}\bar{\mathbf{e}} = 1$. We have

$$r_1 \left[1 + \sum_{j=2}^{s+1} M^{j-1} + \sum_{j=s+2}^{S-s+1} M^s + \sum_{j=S-s+2}^{S} \left(M^s - \frac{\gamma}{\gamma+\alpha}(M^{j-Q-1} - 1) \right) \right] = 1 \tag{3.24}$$

where $M = \frac{\lambda+\gamma+\alpha}{\lambda}$.

$$\sum_{j=2}^{s+1} M^{j-1} = \frac{M(M^s-1)}{M-1}. \tag{3.25}$$

$$\sum_{j=s+2}^{S-s+1} M^s = M^s(S-2s). \tag{3.26}$$

$$\sum_{S-s+2}^{S} (M^s - \frac{\gamma}{\gamma+\alpha}(M^{j-Q-1}-1)) = (s-1)(M^s + \frac{\gamma}{\gamma+\alpha}) - \left(\frac{\gamma}{\gamma+\alpha} \right) \frac{M(M^{s-1}-1)}{M-1}. \tag{3.27}$$

Using equations (3.25), (3.26) and (3.27), we get equation (3.24) as

$$r_1 \left[1 + M\frac{(M^s-1)}{M-1} + M^s(S-2s) + (s-1)\left(M^s + \frac{\gamma}{\gamma+\alpha} \right) - \frac{\gamma}{\gamma+\alpha} \frac{M(M^{s-1}-1)}{M-1} \right] = 1.$$

$$\frac{r_1}{(M-1)(\gamma+\alpha)} [(\gamma+\alpha)(M^{s+1} - 1 + (M^{s+1} - M^s)(S-s-1)) - \gamma(M^s - M - (M-1)(s-1))] = 1.$$

$$\frac{r_1}{(M-1)(\gamma+\alpha)}[M^{s+1}(\gamma+\alpha)Q - M^s((\gamma+\alpha)Q-\alpha) + s\gamma(M-1) - \alpha] = 1.$$

$$\begin{aligned} r_1 &= \frac{(M-1)(\gamma+\alpha)}{M^s((\gamma+\alpha)Q(M-1)+\alpha) + s\gamma(M-1) - \alpha} \\ &= \frac{\lambda(M-1)^2}{M^s(\lambda(M-1)^2Q+\alpha) + s\gamma(M-1) - \alpha}. \end{aligned}$$

Hence the theorem. □

3.2.2.2 Stochastic Decomposition of System State

Let $\bar{\boldsymbol{\pi}}$ be the steady-state probability vector of $\tilde{A}$.

$$\bar{\boldsymbol{\pi}} = (\boldsymbol{\pi}^{(0)}, \boldsymbol{\pi}^{(1)}, \boldsymbol{\pi}^{(2)}, \dots)$$

where $\boldsymbol{\pi}^{(i)} = (\pi^{(i,1)}, \pi^{(i,2)}, \dots, \pi^{(i,S)})$, where $i = 0, 1, 2, \dots$ and

$$\pi^{(i,j)} = \lim_{t\to\infty} P(X(t) = i, I(t) = j),\ (i,j) \in E.$$

$\pi^{(i,j)}$ is the steady-state probability for the state (i, j).

We claim that

$$\boldsymbol{\pi}^{(i)} = K\rho^i\boldsymbol{\Delta}, \quad i \geq 0 \tag{3.28}$$

where $\boldsymbol{\Delta} = (r_1, r_2, \dots r_S)$ is the steady-state probability vector when the service time is negligible, K is a constant to be determined and $\rho = \frac{\lambda}{\mu}$.

Proof. We have $\bar{\boldsymbol{\pi}}\tilde{A} = \mathbf{0}$ and $\bar{\boldsymbol{\pi}}\bar{\mathbf{e}} = 1$.

$$\bar{\boldsymbol{\pi}}\tilde{A} = \mathbf{0} \Rightarrow (\boldsymbol{\pi}^{(0)}, \boldsymbol{\pi}^{(1)}, \boldsymbol{\pi}^{(2)}, \dots) \begin{bmatrix} B_1 & A_0 & & \\ A_2 & A_1 & A_0 & \\ & A_2 & A_1 & A_0 \\ & \ddots & \ddots & \ddots \end{bmatrix} = \mathbf{0}$$

which gives

$$\boldsymbol{\pi}^{(0)}B_1 + \boldsymbol{\pi}^{(1)}A_2 = \mathbf{0} \tag{3.29}$$

$$\text{and } \boldsymbol{\pi}^{(i+2)}A_2 + \boldsymbol{\pi}^{(i+1)}A_1 + \boldsymbol{\pi}^{(i)}A_0 = \mathbf{0}, \tag{3.30}$$

$i = 0, 1, \ldots$.
When (3.28) is true, we get from (3.29),

$$
\begin{aligned}
& K\rho^0 \mathbf{\Delta} B_1 + K\rho \mathbf{\Delta} A_2 = \mathbf{0}. \\
\text{That is} \quad & K\mathbf{\Delta}(B_1 + \rho A_2) = \mathbf{0}. \\
\text{That is} \quad & \mathbf{\Delta}\tilde{Q} = \mathbf{0}.
\end{aligned}
$$

which is true since $\mathbf{\Delta} = (r_1, r_2, \ldots, r_S)$ is the steady-state probability vector corresponding to the generator $\tilde{Q}$ associated with the Markovian chain of the inventory process under consideration when service time is negligible.
When (3.28) is true, we get from (3.30),

$$
\begin{aligned}
& K\rho^{i+2}\mathbf{\Delta} A_2 + K\rho^{i+1}\mathbf{\Delta} A_1 + K\rho^{i}\mathbf{\Delta} A_0 = \mathbf{0}, \quad i = 0, 1, 2, \ldots \\
\text{That is} \quad & \mathbf{\Delta}(\rho A_2 + A_1 + \frac{1}{\rho} A_0) = \mathbf{0}. \\
\text{That is} \quad & \mathbf{\Delta}\tilde{Q} = \mathbf{0},
\end{aligned}
$$

which is true, by following the same argument given above.

Hence, the stochastic decomposition of system states is verified. □

3.2.2.3 Determination of K

We have

$$
\begin{aligned}
\sum_{i=0}^{\infty}\sum_{j=1}^{S} \pi^{(i,j)} &= 1. \\
K\left(\sum_{i=0}^{\infty}\rho^i\right)\left(\sum_{j=1}^{S} r_j\right) &= 1. \\
\text{That is, } K\sum_{i=0}^{\infty}\rho^i &= 1.
\end{aligned}
$$

Therefore, $K = 1 - \rho$.

3.2.2.4 Explicit Solution

From the above discussions, we can write the steady-state probability vector explicitly as in the following theorem:

Theorem 3.2.4. *The steady-state probability vector* $\bar{\boldsymbol{\pi}}$ *of* $\tilde{A}$ *partitioned as* $\bar{\boldsymbol{\pi}} = (\boldsymbol{\pi}^{(0)}, \boldsymbol{\pi}^{(1)}, \ldots)$ *where each* $\boldsymbol{\pi}^{(i)}$, $i = 0, 1, 2, \ldots$ *again partitioned as* $\boldsymbol{\pi}^{(i)} = (\pi^{(i,1)}, \pi^{(i,2)}, \ldots, \pi^{(i,S)},)$ *are obtained by*

$$\pi^{(i,j)} = (1-\rho)\rho^i r_j \tag{3.31}$$

where $\rho = \frac{\lambda}{\mu}$ *and* r_j, $j = 1, 2, \ldots, S$, *represent the inventory level probabilities when service time is negligible and are given as*

$$r_j = \begin{cases} \left(\frac{\lambda+\gamma+\alpha}{\lambda}\right)^{j-1} r_1, & j = 2, 3, \ldots, s+1 \\ \left(\frac{\lambda+\gamma+\alpha}{\lambda}\right)^{s} r_1, & j = s+2, \ldots, S-s+1 \\ \left(\frac{\lambda+\gamma+\alpha}{\lambda}\right)^{s} - \frac{\gamma}{\gamma+\alpha}\left[\left(\frac{\lambda+\gamma+\alpha}{\lambda}\right)^{j-Q-1} - 1\right] r_1, & i = S-s+2, \ldots, S \end{cases}$$

and

$$r_1 = \frac{\lambda F^2}{(\lambda F^2 Q + \alpha)M^s + s\gamma F - \alpha}$$

where $F = M - 1$ *and* $M = \frac{\lambda+\gamma+\alpha}{\lambda}$.

The result indicated by (3.31) not only tells us that the original system possess stochastic decomposition but also the important fact that the system state distribution is the product of the distribution of its marginals: one component is the distribution of the classical $M/M/1$ queue whose long run distribution for i customers in the system is $(1-\rho)\rho^i$, $i \geq 0$, and the other factor is the probability of j items in the inventory. Next we turn to find out how the system performs. The measures given in the following are pointers to the system performance.

3.2.3 System Performance Measures

(a) Expected inventory held in the system (Mean inventory level),

$$\begin{aligned} E(I) &= \sum_{i=0}^{\infty}\sum_{j=1}^{S} j\pi^{(i,j)} \\ &= \Big(\sum_{i=0}^{\infty}(1-\rho)\rho^i\Big)\Big(\sum_{j=1}^{S} jr_j\Big) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{S} j r_j \\
&= r_1\Big(1 + \sum_{j=2}^{s+1} j M^{j-1} + \sum_{j=s+2}^{S-s+1} j M^s \\
&\quad + \sum_{j=S-s+2}^{S} j(M^s - \frac{\gamma}{\gamma+\alpha}(M^{j-Q-1} - 1))\Big). \qquad (3.32)
\end{aligned}
$$

$$\sum_{j=2}^{s+1} j M^{j-1} = 2M + 3M^2 + 4M^3 + \cdots + (s+1)M^s.$$

Let $S_M = 2M + 3M^2 + 4M^3 + \cdots + (s+1)M^s$.

$$
\begin{aligned}
(1-M)S_M &= 2M + \frac{M^2(M^{s-1}-1)}{M-1} - (s+1)M^{s+1} \\
&= \frac{M^2 - 2M - (s+1)M^{s+2} + M^{s+1}(s+2)}{M-1}. \\
\text{Therefore, } S_M &= \frac{M^2 - 2M - (s+1)M^{s+2} + M^{s+1}(s+2)}{(M-1)(1-M)}.
\end{aligned}
$$

Therefore,

$$\sum_{j=2}^{s+1} j M^{j-1} = \frac{2M - M^2 - M^{s+1}(s+2) + M^{s+2}(s+1)}{(M-1)^2}. \qquad (3.33)$$

$$
\begin{aligned}
\sum_{j=s+2}^{S-s+1} j M^s &= M^s[(s+2) + (s+3) + \cdots + (S-s+1)] \\
&= M^s \left(\frac{S-2s}{2}\right)(s+3). \qquad (3.34)
\end{aligned}
$$

$$
\begin{aligned}
&\sum_{j=S-s+2}^{S} j(M^s - \frac{\gamma}{\gamma+\alpha}(M^{j-Q-1} - 1)) \\
&= M^s \sum_{j=S-s+2}^{S} j - \frac{\gamma}{\gamma+\alpha} \sum_{j=S-s+2}^{S} j M^{j-Q-1} + \frac{\gamma}{\gamma+\alpha} \sum_{j=S-s+2}^{S} j. \qquad (3.35)
\end{aligned}
$$

$$\sum_{j=S-s+2}^{S} j = \frac{(s-1)}{2}(2S - s + 2). \tag{3.36}$$

$$\sum_{j=S-s+2}^{S} jM^{j-Q-1} = (S-s+2)M + (S-s+3)M^2 + (S-s+4)M^3 + \cdots + (S-s+s)M^{s-1}. \tag{3.37}$$

Let $\hat{S}_M = (S-s+2)M + (S-s+3)M^2 + (S-s+4)M^3 + \ldots$
$+(S-s+s)M^{s-1}$.

$$\begin{aligned}(1-M)\hat{S}_M &= (S-s+2)M + M^2\left(\frac{M^{s-2}-1}{M-1}\right) - SM^s \\ &= \frac{-SM^{s+1} + (1+s)M^s + M^2(S-s+1) - M(S-s+2)}{M-1}. \\ \hat{S}_M &= \frac{SM^{s+1} - (s+1)M^s - M^2(Q+1) + M(Q+2)}{(M-1)^2}.\end{aligned}$$

Therefore, equation (3.37) becomes

$$\sum_{j=S-s+2}^{S} jM^{j-Q-1} = \frac{SM^{s+1} - (S+1)M^s - M^2(Q+1) + M(Q+2)}{(M-1)^2}. \tag{3.38}$$

Hence using (3.36) and (3.38), we get (3.35) as

$$\begin{aligned}&\sum_{j=S-s+2}^{S} j(M^s - \frac{\gamma}{\gamma+\alpha}(M^{j-Q-1}-1)) \\ &\quad = \frac{(s-1)}{2}(2S-s+2)\left(M^s + \frac{\gamma}{\gamma+\alpha}\right) \\ &\qquad - \frac{\gamma}{\gamma+\alpha}\left(\frac{SM^{s+1} - (S+1)M^s - M^2(Q+1) + M(Q+2)}{(M-1)^2}\right).\end{aligned} \tag{3.39}$$

Now, using (3.33), (3.34) and (3.39), we get (3.32) as

$$\begin{aligned} E(I) &= r_1\Bigg[1 + \frac{2M - M^2 - M^{s+1}(s+2) + M^{s+2}(s+1)}{(M-1)^2} \\ &+ M^s\left(\frac{S-2s}{2}\right)(S+3) + \frac{(s-1)}{2}(2S-s+2)\left(M^s + \frac{\gamma}{\gamma+\alpha}\right) \\ &- \frac{\gamma}{\gamma+\alpha}\left(\frac{SM^{s+1} - (S+1)M^s - M^2(Q+1) + M(Q+2)}{(M-1)^2}\right)\Bigg] \\ &= r_1(1 + \hat{R} + \hat{P} + \hat{Q}). \end{aligned} \tag{3.40}$$

where

$$\begin{aligned} \hat{R} &= \frac{(2M - M^2 - M^{s+1}(s+2) + (s+1)M^{s+2})}{(M-1)^2} \\ &\qquad - \frac{\gamma(M(Q+2) - M^2(Q+1) - (S+1)M^s + SM^{s+1})}{(\gamma+\alpha)(M-1)^2}, \end{aligned}$$

$$\hat{P} = M^s\left[\left(\tfrac{S-2s}{2}\right)(S+3) + \left(\tfrac{s-1}{2}\right)(2S-s+2)\right]$$

$$\text{and} \quad \hat{Q} = \frac{\gamma}{\gamma+\alpha}\left(\frac{s-1}{2}\right)(2S-s+2). \tag{3.41}$$

On simplification, we get

$$\hat{P} = \frac{M^s}{2}[S(S+1) - s(s+3) - 2] \tag{3.42}$$

and

$$\begin{aligned} \hat{R} &= \{M[2\alpha - \gamma Q] - M^2[\alpha - \gamma Q] + \gamma(S+1)M^s \\ &\quad - M^{s+1}[(s+2)(\gamma+\alpha) + \gamma S] + M^{s+2}(s+1)(\gamma+\alpha)\} \\ &\quad \times \frac{1}{(\gamma+\alpha)(M-1)^2}. \end{aligned} \tag{3.43}$$

Hence using equations (3.41), (3.42) and (3.43) we get equation (3.40) as

$$\begin{aligned} E(I) = r_1\Bigg[&1 + \frac{M(2\alpha-\gamma Q)}{(\gamma+\alpha)(M-1)^2} - \frac{M^2(\alpha-\gamma Q)}{(\gamma+\alpha)(M-1)^2} + \frac{\gamma(S+1)M^s}{(\gamma+\alpha)(M-1)^2} \\ &- M^{s+1}\frac{((s+2)(\gamma+\alpha)+\gamma S)}{(\gamma+\alpha)(M-1)^2} + \frac{M^{s+2}(s+1)}{(M-1)^2} \\ &+ \frac{M^s}{2}(S(S+1) - s(s+3) - 2) + \frac{\gamma}{\gamma+\alpha}(\frac{s-1}{2})(2S-s+2)\Bigg] \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{V}[\lambda F^2 + \frac{M}{F}(2\alpha - \gamma Q) - \frac{M^2}{F}(\alpha - \gamma Q) \\
&+ M^s(\frac{1}{2}(S(S+1) - s(s+3) - 2)\lambda F^2 + \frac{\gamma}{F}(S+1)) \\
&- M^{s+1}((s+2)\lambda + \frac{\gamma S}{F}) + M^{s+2}(s+1)\lambda + \gamma F(s-1)(S - \frac{s}{2} + 1)]
\end{aligned} \tag{3.44}
$$

where $V = (\lambda F^2 Q + \alpha)M^s + \gamma s F - \alpha$,

$$F = M - 1 \text{ and } M = \frac{\lambda + \gamma + \alpha}{\lambda}.$$

(b) Mean waiting time of customers in the system,

$$W_S = \frac{L}{\lambda}$$

where $L =$ Expected number of customers in the system

$$
\begin{aligned}
&= \sum_{i=0}^{\infty}\sum_{j=1}^{S} i\pi^{(i,j)} \\
&= \left(\sum_{i=0}^{\infty} i(1-\rho)\rho^i\right)\left(\sum_{j=1}^{S} r_j\right) \\
&= \frac{\rho}{1-\rho}.
\end{aligned}
$$

$$
\begin{aligned}
\text{Therefore} \quad W_s &= \frac{L}{\lambda} \\
&= \frac{1}{\mu - \lambda}.
\end{aligned} \tag{3.45}
$$

(c) Mean reorder rate,

$$
\begin{aligned}
R_r &= \mu \sum_{i=1}^{\infty} \pi^{(i,s+1)} \\
&= \mu \sum_{i=1}^{\infty} (1-\rho)\rho^i r_{s+1} \\
&= \lambda r_{s+1} \\
&= \frac{\lambda^2 M^s F^2}{V}.
\end{aligned} \tag{3.46}
$$

(d) Mean local purchase rate (Mean order cancellation rate), R_{LP}

Let R_{LP_1} = Local purchase rate due to T-realization and R_{LP_2} = Local purchase rate resulting from zero inventory level.

$$\begin{aligned} R_{\mathrm{LP}_1} &= \alpha \sum_{i=0}^{\infty} \sum_{j=1}^{s} \pi^{(i,j)} \\ &= \alpha(1-\rho)\left(\sum_{i=0}^{\infty} \rho^i\right)\left(\sum_{j=1}^{s} r_j\right). \end{aligned} \tag{3.47}$$

$$\sum_{i=0}^{\infty} \rho^i = (1-\rho)^{-1}. \tag{3.48}$$

$$\begin{aligned} \sum_{j=1}^{s} r_j &= \sum_{j=1}^{s} M^{j-1} r_1 \\ &= \frac{(M^s-1)}{M-1} r_1. \end{aligned} \tag{3.49}$$

Using equations (3.48) and (3.49) we get the equation (3.47) as

$$\begin{aligned} R_{\mathrm{LP}_1} &= \alpha(1-\rho) \times \frac{1}{(1-\rho)} \frac{(M^s-1)}{(M-1)} r_1 \\ &= \alpha(M^s-1)\frac{\lambda F}{V}. \end{aligned} \tag{3.50}$$

$$\begin{aligned} R_{\mathrm{LP}_2} &= \mu \sum_{i=1}^{\infty} \pi^{(i,1)} \\ &= \mu \sum_{i=1}^{\infty} (1-\rho)\rho^i r_1 \\ &= \frac{\lambda^2 F^2}{V}. \end{aligned} \tag{3.51}$$

$$\begin{aligned} R_{\mathrm{LP}} &= R_{\mathrm{LP}_1} + R_{\mathrm{LP}_2} \\ &= (\lambda F + \alpha(M^s-1))\frac{\lambda F}{V}, \end{aligned}$$

by using (3.50) and (3.51)

(e) Expected (average) number of items locally purchased on T-realization,

$$N_{\text{LP}} = \sum_{x=Q}^{Q+s} xP(x)$$

where $P(x)$ is the probability that x units are locally purchased.

$$N_{\text{LP}} = Q\sum_{j=1}^{s}\phi_j + \sum_{j=0}^{s-1} j\phi_{s-j}. \tag{3.52}$$

$$\sum_{j=1}^{s}\phi_j = \frac{(B^s-1)}{B-1}\phi_1. \tag{3.53}$$

$$\sum_{j=0}^{s-1} j\phi_{s-j} = B^s S_{\frac{1}{B}}\phi_1.$$

$$\text{where } S_{\frac{1}{B}} = \Big(\frac{1}{B}\Big)^2 + 2\Big(\frac{1}{B}\Big)^3 + \cdots + (s-1)\Big(\frac{1}{B}\Big)^s.$$

$$\Big(1-\frac{1}{B}\Big)S_{\frac{1}{B}} = \Big(\frac{1}{B}\Big)^2 \frac{\Big(1-\big(\frac{1}{B}\big)^{s-2}\Big)}{1-\big(\frac{1}{B}\big)} - (s-1)\Big(\frac{1}{B}\Big)^{s+1}.$$

$$S_{\frac{1}{B}} = \frac{\big(\frac{1}{B}\big)^2 - \big(\frac{1}{B}\big)^s - (s-1)\big(\frac{1}{B}\big)^{s+1} + (s-1)\big(\frac{1}{B}\big)^{s+2}}{(1-\frac{1}{B})^2}.$$

$$\sum_{j=0}^{s-1} j\phi_{s-j} = \frac{(B^s - B^2 - (s-1)(B-1))\phi_1}{(B-1)^2} \tag{3.54}$$

where

$$\phi_1 = \frac{\mu(B-1)^2}{(\mu(B-1)^2Q+\alpha)B^s + \gamma s(B-1) - \alpha}.$$

Using (3.53) and (3.54) in (3.52) we get

$$N_{\text{LP}} = (Q(B^s-1)(B-1) + B^s - B^2 - (s-1)(B-1))\frac{\mu}{W} \tag{3.55}$$

where $W = (\mu(B-1)^2Q+\alpha)B^s + \gamma s(B-1) - \alpha$.

3.2.4 Cost Analysis

Now we introduce a cost function. Let the various costs involved in the model be as given below:

C_H : Inventory holding cost per unit item per unit time

C_S : Set up cost per unit order, under natural purchase
C_W : Waiting time cost per customer per unit time
C_{LP} : Local purchase cost per unit order
C_{NP} : Natural purchase cost per unit order
C_C : Cancellation cost per unit order cancelled.

The total expected cost per unit time,

$$\begin{aligned}\mathrm{TEC} = {} & C_H E(I) + (C_S + C_{\mathrm{NP}} Q) R_r + C_C R_{\mathrm{LP}} + C_W W_S \\ & + C_{\mathrm{LP}}(R_{\mathrm{LP}_1} N_{\mathrm{LP}} + R_{\mathrm{LP}_2} S).\end{aligned}$$

$$\begin{aligned}R_{\mathrm{LP}_1} N_{\mathrm{LP}} + R_{\mathrm{LP}_2} S = \Big[& \lambda S + \frac{\alpha(M^s - 1)}{F}(Q(B^s - 1)(B - 1) \\ & + B^s - B^2 - (s-1)(B-1))\frac{\mu}{W}\Big]\frac{\lambda F^2}{V}\end{aligned}$$

where $V = (\lambda F^2 Q + \alpha) M^s + \gamma s F - \alpha$.

$$\begin{aligned}\mathrm{TEC} = {} & \frac{C_H}{V}[\lambda F^2 + \frac{M}{F}(2\alpha - \gamma Q) \\ & - \frac{M^2}{F}(\alpha - \gamma Q) + M^s(\frac{1}{2}(S(S+1) - s(s+3) - 2)\lambda F^2 + \frac{\gamma}{F}(S+1)) \\ & - M^{s+1}\Big((s+2)\lambda + \frac{\gamma S}{F}\Big) + M^{s+2}(s+1)\lambda + \gamma F(s-1)(S - \frac{s}{2} + 1)] \\ & + (C_S + C_{\mathrm{NP}} Q)\frac{\lambda^2 M^s F^2}{V} + \frac{C_W}{\mu - \lambda} \\ & + C_C \frac{\lambda F}{V}(\lambda F + \alpha(M^s - 1)) \\ & + C_{\mathrm{LP}}[\lambda S + \frac{\alpha}{F}(M^s - 1)(Q(B^s - 1)(B - 1) \\ & + B^s - B^2 - (s-1)(B-1))\frac{\mu}{W}]\frac{\lambda F^2}{V}.\end{aligned}$$

3.2.4.1 Numerical Analysis

To verify the convexity of the above cost function, the derivative with respect to α may be computed, then equate it to zero. Nevertheless, solving it is a laborious task. Since analysis of TEC as a function of α is quite complex, we give a few numerical illustrations.

Case 1: Analysis of TEC as a function of α.

Input data: $S = 21$, $s = 9$, $\lambda = 23$, $\mu = 25$, $\gamma = 16$, $C_{\mathrm{NP}} = 30$, $C_{\mathrm{LP}} = 35$, $C_H = 0.5$, $C_S = 1000$, $C_C = 16$, $C_W = 1250$.

Table 3.1: Effect of α on TEC

α	TEC
7	3206.7
8	3205.5
9	3204.7
10	3204.1
11	3203.8
12	**3203.7**
13	3203.8
14	3204.0
15	3204.3
16	3204.6
17	3205.1

Table 3.1 shows that as α varies from 7 to 17, TEC values decrease, reach a minimum at $\alpha = 12$, and then increase. Hence convexity of TEC as a function of α is verified numerically.

Case 2: Analysis of TEC during simultaneous variation of (s, S, α).

Input data: $\lambda = 23$, $\mu = 25$, $\gamma = 16$, $C_{\mathrm{NP}} = 30$, $C_{\mathrm{LP}} = 35$, $C_H = 0.5$, $C_S = 1000$, $C_C = 16$, $C_W = 1250$.

Table 3.2 shows that as values of (s, S, α) increases simultaneously, the TEC values decrease, reach a minimum at the values (11,23,10) of (s, S, α)

Table 3.2: Effect of simultaneous variation of (s, S, α) on TEC

(s, S, α)	TEC
(8,20,7)	3206.6
(9,21,8)	3205.5
(10,22,9)	3205.0
(11,23,10)	**3204.9**
(12,24,11)	3205.1
(13,25,12)	3205.5

and then increase. Hence convexity of TEC, as (s, S, α) varies simultaneously, has been verified numerically.

3.3 Model II: Perishable Items

3.3.1 Model Formulation and Analysis

Let $X(t)$ = Number of customers in the system at time t and
$I(t)$ = Inventory level at time t.
$\{(X(t), I(t)), t \geq 0\}$ is a CTMC with state space $E = E_1 \times E_2$ where $E_1 = \{0, 1, 2, \dots\}$ and $E_2 = \{1, 2, \dots, S\}$.

$$E = \{(i, j) | i \in E_1, j \in E_2\}.$$

3.3.1.1 Infinitesimal Generator $\tilde{A}$

We write the infinitesimal generator $\tilde{A}$ of the process as $\tilde{A} = (a((i, j), (m, n)))$ where $(i, j), (m, n) \in E$.

$$a((i,j),(m,n)) = \begin{cases} \lambda, & m = i+1; \quad i = 0,1,2,\dots \\ & n = j; \quad j = 1,2,\dots,S \\ \mu, & m = i-1; \quad i = 1,2,3,\dots \\ & n = j-1; \quad j = 2,3,\dots,S \\ \mu, & m = i-1; \quad i = 1,2,3,\dots \\ & n = S; \quad j = 1, \\ \gamma, & m = i; \quad i = 0,1,2,\dots \\ & n = j+Q; \quad j = 1,2,\dots,s \\ \alpha, & m = i; \quad i = 0,1,2,\dots \\ & n = S; \quad j = 1,2,\dots,s \end{cases}$$

and

$$a((i,j),(m,n)) = \begin{cases} j\beta, & m=i; \quad i=0,1,2,\ldots \\ & n=j-1; \quad j=2,3,\ldots,S \\ \beta, & m=i; \quad i=0,1,2,\ldots \\ & n=S; \quad j=1 \\ -(\lambda+j\beta+\gamma+\alpha), & m=i; \quad i=0 \\ & n=j; \quad j=1,2,\ldots,s \\ -(\lambda+j\beta), & m=i; \quad i=0 \\ & n=j; \quad j=s+1,\ldots,S \\ -(\lambda+\mu+\gamma+\alpha+j\beta), & m=i; \quad i=1,2,\ldots \\ & n=j; \quad j=1,2,\ldots,s \\ -(\lambda+\mu+j\beta), & m=i; \quad i=1,2,\ldots \\ & n=j; \quad j=s+1,\ldots,S \\ 0, & \text{otherwise.} \end{cases}$$

We can write $\tilde{A}$ interms of submatrices as follows:

$$\tilde{A} = \begin{bmatrix} B_1 & A_0 & & & \\ A_2 & A_1 & A_0 & & \\ & A_2 & A_1 & A_0 & \\ & \ddots & \ddots & \ddots & \end{bmatrix}$$

where $A_0 = \lambda I_S$ and

$$A_1 = \begin{bmatrix} -h_1 & & & & & & \gamma & & & \alpha+\beta \\ 2\beta & -h_2 & & & & & & & & \alpha \\ & 3\beta & & & & & & & & \\ & & \ddots & & & & & \ddots & & \vdots \\ & & & -h_{s-1} & & & & & & \alpha \\ & & & s\beta & -h_s & & & & & \gamma+\alpha \\ & & & & (s+1)\beta & -h_{s+1} & & & & \\ & & & & & (s+2)\beta & & & & \\ & & & & & & \ddots & \ddots & & \\ & & & & & & & & -h_{S-1} & \\ & & & & & & & & S\beta & -h_S \end{bmatrix}$$

where

$$h_i = \begin{cases} \lambda + \mu + \gamma + \alpha + i\beta, & i = 1, 2, \ldots, s \\ \lambda + \mu + i\beta, & i = s+1, \ldots, S. \end{cases}$$

Also

$$A_2 = \begin{bmatrix} \mathbf{0}_{1\times(S-1)} & \mu I_1 \\ \mu I_{S-1} & \mathbf{0}_{(S-1)\times 1} \end{bmatrix}$$

and

$$B_1 = \begin{bmatrix} -g_1 & & & & & & & \gamma & & & \alpha+\beta \\ 2\beta & -g_2 & & & & & & & & & \alpha \\ & 3\beta & & & & & & & & & \\ & & \ddots & & & & & & \ddots & & \vdots \\ & & & -g_{s-1} & & & & & & & \alpha \\ & & & s\beta & -g_s & & & & & & \gamma+\alpha \\ & & & & (s+1)\beta & -g_{s+1} & & & & & \\ & & & & & (s+2)\beta & & & & & \\ & & & & & & \ddots & \ddots & & & \\ & & & & & & & & & -g_{S-1} & \\ & & & & & & & & & S\beta & -g_S \end{bmatrix}$$

where

$$g_i = \begin{cases} \lambda + \gamma + \alpha + i\beta, & i = 1, 2, \ldots, s \\ \lambda + i\beta, & i = s+1, \ldots, S. \end{cases}$$

A_0, A_1, A_2 and B_1 are square matrices of order S.

3.3.1.2 Steady-State Analysis

Let $A = A_0 + A_1 + A_2$ be the generator matrix and is given in page 89.

A is a square matrix of order S.

First we investigate the stationary distribution of A. This will help in analyzing the stability of the larger system, namely the CTMC $\{(X(t), I(t)), t \geq 0\}$ as $t \to \infty$.

Theorem 3.3.1. *The steady-state probability distribution*

$$\mathbf{\Phi} = (\phi_1, \phi_2, \ldots, \phi_S)$$

$$
A = \begin{bmatrix}
\lambda\text{-}h_1 & & & & & & & & & \gamma & & & \alpha+\beta+\mu \\
\mu+2\beta & \lambda\text{-}h_2 & & & & & & & & & & & \alpha \\
 & \mu+3\beta & \lambda\text{-}h_3 & & & & & & & & \ddots & & \alpha \\
 & & & \ddots & & & & & & & & & \vdots \\
 & & & & \lambda\text{-}h_{s\text{-}1} & & & & & & & \gamma & \alpha \\
 & & & & \mu+s\beta & \lambda\text{-}h_s & & & & & & & \gamma+\alpha \\
 & & & & & \mu+(s+1)\beta & \lambda\text{-}h_{s+1} & & & & & & \\
 & & & & & & & \ddots & & & & & \\
 & & & & & & & & \lambda\text{-}h_{S\text{-}s} & & & & \\
 & & & & & & & & \mu+(S\text{-}s+1)\beta & \lambda\text{-}h_{S\text{-}s+1} & & & \\
 & & & & & & & & & & \ddots & & \\
 & & & & & & & & & & & \lambda\text{-}h_{S\text{-}1} & \\
 & & & & & & & & & & & \mu+S\beta & \lambda\text{-}h_S
\end{bmatrix}
$$

corresponding to the matrix A is given by

$$\phi_j = q_j \phi_1, \quad j = 1, 2, \ldots, S \tag{3.56}$$

where

$$q_j = \begin{cases} 1, & j = 1 \\ \frac{\eta^*(1,j-1)}{\psi^*(2,j)}, & j = 2, 3, \ldots, s+1 \\ \frac{1}{\mu+j\beta}\frac{\eta^*(1,s)}{\psi^*(2,s)}, & j = s+2, \ldots, Q+1 \\ \frac{1}{\mu+j\beta}\left[\frac{\eta^*(1,s)}{\psi^*(2,s)} - \gamma\right], & j = Q+2 \\ \frac{1}{\mu+j\beta}\left[\frac{\eta^*(1,s)}{\psi^*(2,s)} - \gamma - \gamma\sum_{k_1=1}^{j-Q-2}\frac{\eta^*(1,k_1)}{\psi^*(2,k_1+1)}\right], \\ \qquad j = Q+3, \ldots, S. \end{cases}$$

where

$$\eta^*(j_1, j_2) = \prod_{\tilde{k}=j_1}^{j_2} (\tilde{k}\beta + \mu + \gamma + \alpha)$$

and

$$\psi^*(j_1, j_2) = \prod_{\tilde{k}=j_1}^{j_2} (\tilde{k}\beta + \mu).$$

ϕ_1 can be obtained by solving the equation $\mathbf{\Phi}\bar{\mathbf{e}} = 1$ as

$$\phi_1 = \left[1 + \sum_{j=2}^{s+1}\frac{\eta^*(1,j-1)}{\psi^*(2,j)} + \frac{\eta^*(1,s)}{\psi^*(2,s)}\sum_{j=s+2}^{Q+1}\frac{1}{\mu+j\beta} + \frac{1}{\mu+(Q+2)\beta}\left[\frac{\eta^*(1,s)}{\psi^*(2,s)} - \gamma\right] + \sum_{j=Q+3}^{S}\frac{1}{\mu+j\beta}\left[\frac{\eta^*(1,s)}{\psi^*(2,s)} - \gamma - \gamma\sum_{k_1=1}^{j-Q-2}\frac{\eta^*(1,k_1)}{\psi^*(2,k_1+1)}\right]\right]^{-1}.$$

Proof. We have $\mathbf{\Phi}A = \mathbf{0}$ and $\mathbf{\Phi}\bar{\mathbf{e}} = 1$ for $l = 1, 2, \ldots, s$.
$\mathbf{\Phi}A = \mathbf{0} \Rightarrow$
For $l = 1, 2, \ldots, s$,

$$\phi_l(\lambda - h_l) + \phi_{l+1}(\mu + (l+1)\beta) = 0. \tag{3.57}$$

For $l = s+1, \ldots, S-s$,

$$\phi_l(\lambda - h_l) + \phi_{l+1}(\mu + (l+1)\beta) = 0. \tag{3.58}$$

For $l = 1, 2, \ldots, s-1$,

$$\phi_l \gamma + \phi_{Q+l}(\lambda - h_{Q+l}) + \phi_{Q+l+1}(\mu + (Q+l+1)\beta) = 0. \tag{3.59}$$

Finally we have

$$\phi_1(\alpha+\beta+\mu)+(\phi_2+\phi_3+\cdots+\phi_{s-1})\alpha+\phi_s(\alpha+\gamma)+\phi_S(\lambda-h_S) = 0. \tag{3.60}$$

Equation (3.57) gives for $l = 1, 2, \ldots, s$,

$$\begin{aligned} \phi_{l+1} &= -\frac{(\lambda - h_l)}{\mu + (l+1)\beta}\phi_l, \\ &= \left(\frac{\mu + \gamma + \alpha + l\beta}{\mu + (l+1)\beta}\right)\phi_l \end{aligned}$$

from which we get

$$\phi_j = \frac{\eta^*(1, j-1)}{\psi^*(2, j)}\phi_1 \tag{3.61}$$

for $j = 2, 3, \ldots, s+1$.
Equation (3.58) gives

$$\begin{aligned} \phi_{l+1} &= \frac{(h_l - \lambda)\phi_l}{\mu + (l+1)\beta} \\ &= \left(\frac{\mu + l\beta}{\mu + (l+1)\beta}\right)\phi_l \end{aligned} \tag{3.62}$$

for $l = s+1, \ldots, S-s$, from which we get

$$\phi_j = \left(\frac{1}{\mu + j\beta}\right)\frac{\eta^*(1, s)}{\psi^*(2, s)}\phi_1 \tag{3.63}$$

for $j = s+2, \ldots, Q+1$.
Equation (3.59) gives for $l = 1, 2, \ldots, s-1$,

$$\phi_{Q+l+1} = \frac{\phi_{Q+l}(h_{Q+l} - \lambda) - \phi_l\gamma}{\mu + (Q+l+1)\beta} \tag{3.64}$$

from which we get

$$\begin{aligned} \phi_{Q+2} &= \frac{(h_{Q+1} - \lambda)\phi_{Q+1} - \gamma\phi_1}{\mu + (Q+2)\beta} \\ &= \left(\frac{1}{\mu + (Q+2)\beta}\right)\left(\frac{\eta^*(1, s)}{\psi^*(2, s)} - \gamma\right)\phi_1 \end{aligned} \tag{3.65}$$

and, for $j = Q+3, \ldots, S$,

$$\phi_j = \left(\frac{1}{\mu + j\beta}\right)\left[\frac{\eta^*(1,s)}{\psi^*(2,s)} - \gamma - \gamma\sum_{k_1=1}^{j-Q-2}\frac{\eta^*(1,k_1)}{\psi^*(2,k_1+1)}\right]\phi_1. \tag{3.66}$$

Hence from equations (3.61), (3.63) (3.65) and (3.66), we get (3.56).

Next, to find ϕ_1. Consider the normalizing equation $\mathbf{\Phi}\bar{\mathbf{e}} = 1$

$$\text{That is} \quad \Bigg\{1 + \sum_{j=2}^{s+1}\frac{\eta^*(1,j-1)}{\psi^*(2,j)} + \sum_{j=s+2}^{Q+1}\left(\frac{1}{\mu + j\beta}\right)\frac{\eta^*(1,s)}{\psi^*(2,s)} + \left(\frac{1}{\mu + (Q+2)\beta}\right)\left(\frac{\eta^*(1,s)}{\psi^*(2,s)} - \gamma\right) + \sum_{j=Q+3}^{S}\left(\frac{1}{\mu + j\beta}\right)\left[\frac{\eta^*(1,s)}{\psi^*(2,s)} - \gamma - \gamma\sum_{k_1=1}^{j-Q-2}\frac{\eta^*(1,k_1)}{\psi^*(2,k_1+1)}\right]\Bigg\}\phi_1 = 1.$$

Hence

$$\phi_1 = \Bigg[1 + \sum_{j=2}^{s+1}\frac{\eta^*(1,j-1)}{\psi^*(2,j)} + \frac{\eta^*(1,s)}{\psi^*(2,s)}\sum_{j=s+2}^{Q+1}\frac{1}{\mu + j\beta} + \frac{1}{\mu + (Q+2)\beta}\left[\frac{\eta^*(1,s)}{\psi^*(2,s)} - \gamma\right] + \sum_{j=Q+3}^{S}\frac{1}{\mu + j\beta}\left[\frac{\eta^*(1,s)}{\psi^*(2,s)} - \gamma - \gamma\sum_{k_1=1}^{j-Q-2}\frac{\eta^*(1,k_1)}{\psi^*(2,k_1+1)}\right]\Bigg]^{-1}$$

where

$$\eta^*(j_1, j_2) = \prod_{\tilde{k}=j_1}^{j_2}(\tilde{k}\beta + \mu + \gamma + \alpha)$$

and

$$\psi^*(j_1, j_2) = \prod_{\tilde{k}=j_1}^{j_2}(\tilde{k}\beta + \mu).$$

Hence the theorem. □

3.3.1.3 Stability Condition

The result in Theorem 3.3.1 enables us to compute the stability of the CTMC

$$\{(X(t), I(t)), t \geq 0\}.$$

Theorem 3.3.2. *The process* $\{(X(t), I(t)), t \geq 0\}$ *is stable iff* $\lambda < \mu$.

Proof. Since the process under consideration is an LIQBD, it is stable iff

$$\mathbf{\Phi} A_0 \bar{\mathbf{e}} < \mathbf{\Phi} A_2 \bar{\mathbf{e}} \tag{3.67}$$

(Neuts [40]), where $\mathbf{\Phi}$ is the steady-state distribution of the generator matrix $A = A_0 + A_1 + A_2$ and $\bar{\mathbf{e}}$ is a column vector of 1's of appropriate order.

It can be shown that

$$\mathbf{\Phi} A_0 \bar{\mathbf{e}} = (\phi_1 \phi_2 \ldots \phi_S) \lambda I_S \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{S \times 1} = \lambda.$$

$$\mathbf{\Phi} A_2 \bar{\mathbf{e}} = (\phi_1 \phi_2 \ldots \phi_S) \begin{bmatrix} \mathbf{0}_{1 \times (S-1)} & \mu I_1 \\ \mu I_{S-1} & \mathbf{0}_{(S-1) \times 1} \end{bmatrix} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}_{S \times 1} = \mu.$$

Using equation (3.67), we get $\lambda < \mu$. □

It may be noted that decay of items and positive lead time do not have a bearing on the stability condition, despite the strong correlation between the number of customers joining during a lead time and the decay time. This could be attributed to the local purchase which keeps the inventory level always positive.

Having obtained the condition for the system to stabilize, we turn to compute the long-run probability distribution of the system states. Infact we show that the joint distribution of the system state can be written as the product of the marginal distributions of the components.

3.3.2 The Steady-State Probability Distribution of $\tilde{A}$

3.3.2.1 Stationary Distribution when Service Time is Negligible

Let $\tilde{Q}$ be the generator matrix associated with the Markovian chain of the inventory process under consideration when service time is negligible, and

$\mathbf{\Delta} = (r_1, r_2, \ldots, r_S)$ be the stationary probability vector corresponding to $\tilde{Q}$. We get $\tilde{Q}$ as in page 95.

The elements h_i in $\tilde{Q}$ are given as

$$h_i = \begin{cases} \lambda + \mu + \gamma + \alpha + i\beta, & i = 1, 2, \ldots, s \\ \lambda + \mu + i\beta, & i = s+1, \ldots, S. \end{cases}$$

Theorem 3.3.3. *The steady-state probability distribution $\mathbf{\Delta} = (r_1, r_2, \ldots, r_S)$ corresponding to the matrix $\tilde{Q}$ is given by*

$$r_j = \tilde{q}_j r_1 \tag{3.68}$$

where $j = 1, 2, \ldots, S$ and

$$\tilde{q}_j = \begin{cases} 1, & j = 1 \\ \frac{\eta_1^*(1,j-1)}{\psi_1^*(2,j)}, & j = 2, 3, \ldots, s+1 \\ \frac{1}{\lambda+j\beta}\frac{\eta_1^*(1,s)}{\psi_1^*(2,s)}, & j = s+2, \ldots, Q+1 \\ \frac{1}{\lambda+j\beta}\left[\frac{\eta_1^*(1,s)}{\psi_1^*(2,s)} - \gamma\right], & j = Q+2 \\ \frac{1}{\lambda+j\beta}\left[\frac{\eta_1^*(1,s)}{\psi_1^*(2,s)} - \gamma - \gamma\sum_{k_1=1}^{j-Q-2}\frac{\eta_1^*(1,k_1)}{\psi_1^*(2,k_1+1)}\right], \\ \qquad j = Q+3, \ldots, S \end{cases}$$

where

$$\eta_1^*(j_1, j_2) = \prod_{\tilde{k}=j_1}^{j_2} (\tilde{k}\beta + \lambda + \gamma + \alpha)$$

and

$$\psi_1^*(j_1, j_2) = \prod_{\tilde{k}=j_1}^{j_2} (\tilde{k}\beta + \lambda)$$

r_1 is obtained by solving the equation $\mathbf{\Delta}\bar{\mathbf{e}} = 1$ as

$$r_1 = \Bigg[1 + \sum_{j=2}^{s+1}\frac{\eta_1^*(1,j-1)}{\psi_1^*(2,j)} + \frac{\eta_1^*(1,s)}{\psi_1^*(2,s)}\sum_{j=s+2}^{Q+1}\frac{1}{\lambda+j\beta} + \frac{1}{\lambda+(Q+2)\beta}\left[\frac{\eta_1^*(1,s)}{\psi_1^*(2,s)} - \gamma\right] + \sum_{j=Q+3}^{S}\frac{1}{\lambda+j\beta}\left[\frac{\eta_1^*(1,s)}{\psi_1^*(2,s)} - \gamma - \gamma\sum_{k_1=1}^{j-Q-2}\frac{\eta_1^*(1,k_1)}{\psi_1^*(2,k_1+1)}\right]\Bigg]^{-1}.$$

$$\tilde{Q} = \begin{bmatrix}
\mu\text{-}h_1 & & & & & & & \gamma & & & \alpha+\beta+\lambda \\
\lambda+2\beta & \mu\text{-}h_2 & & & & & & & & & \alpha \\
 & \lambda+3\beta & \mu\text{-}h_3 & & & & & & \ddots & & \alpha \\
 & & \ddots & & & & & & & & \vdots \\
 & & & \mu\text{-}h_{s\text{-}1} & & & & & & \gamma & \alpha \\
 & & & \lambda+s\beta & \mu\text{-}h_s & & & & & & \gamma+\alpha \\
 & & & & \lambda+(s+1)\beta & \mu\text{-}h_{s+1} & & & & & 0 \\
 & & & & & \ddots & & & & & \\
 & & & & & & \mu\text{-}h_{S\text{-}s} & & & & \\
 & & & & & & \lambda+(S\text{-}s+1)\beta & \mu\text{-}h_{S\text{-}s+1} & & & \\
 & & & & & & & & \ddots & & \\
 & & & & & & & & & \mu\text{-}h_{S\text{-}1} & \\
 & & & & & & & & & \lambda+S\beta & \mu\text{-}h_S
\end{bmatrix}$$

Proof. We have $\boldsymbol{\Delta}\tilde{Q} = \mathbf{0}$ and $\boldsymbol{\Delta}\bar{\mathbf{e}} = 1$.
$\boldsymbol{\Delta}\tilde{Q} = \mathbf{0} \Rightarrow$
For $l = 1, 2, \ldots, s$,

$$r_l(\mu - h_l) + r_{l+1}(\lambda + (l+1)\beta) = 0. \tag{3.69}$$

For $l = s+1, \ldots, S-s$,

$$r_l(\mu - h_l) + r_{l+1}(\lambda + (l+1)\beta) = 0. \tag{3.70}$$

For $l = 1, 2, \ldots, s-1$,

$$r_l\gamma + r_{Q+l}(\mu - h_{Q+l}) + r_{Q+l+1}(\lambda + (Q+l+1)\beta) = 0 \tag{3.71}$$

and finally

$$r_1(\alpha+\beta+\lambda) + (r_2+\gamma_3+\cdots+r_{s-1})\alpha + r_s(\alpha+\gamma) + r_S(\mu - h_S) = 0. \tag{3.72}$$

Equation (3.69) $\Rightarrow$

$$\begin{aligned} r_{l+1} &= -\left(\frac{(\mu - h_l)}{\lambda + (l+1)\beta}\right) r_l \\ &= \left(\frac{\lambda+\gamma+\alpha+l\beta}{\lambda+(l+1)\beta}\right) r_l \end{aligned}$$

for $l = 1, 2, \ldots, s$, from which we get

$$r_j = \frac{\eta_1^*(1, j-1)}{\psi_1^*(2, j)} r_1, \tag{3.73}$$

for $j = 2, 3, \ldots, s+1$.
Equation (3.70) gives

$$\begin{aligned} r_{l+1} &= \frac{(h_l - \mu) r_l}{\lambda + (l+1)\beta} \\ &= \left(\frac{\lambda + l\beta}{\lambda + (l+1)\beta}\right) r_l \end{aligned}$$

for $l = s+1, \ldots, S-s$, from which we get

$$r_j = \left(\frac{1}{\lambda + j\beta}\right) \frac{\eta_1^*(1, s)}{\psi_1^*(2, s)} r_1 \tag{3.74}$$

for $j = s+2, \ldots, Q+1$.
Equation (3.71) $\Rightarrow$

$$r_{Q+l+1} = \frac{r_{Q+l}(h_{Q+l} - \mu) - r_l\gamma}{\lambda + (Q+l+1)\beta}$$

for $l = 1, 2, \ldots, s-1$, from which we get

$$\begin{aligned} r_{Q+2} &= \frac{(h_{Q+1} - \mu) r_{Q+1} - \gamma r_1}{\lambda + (Q+2)\beta} \\ &= \left(\frac{1}{\lambda + (Q+2)\beta}\right)\left(\frac{\eta_1^*(1,s)}{\psi_1^*(2,s)} - \gamma\right) r_1 \end{aligned} \tag{3.75}$$

and

$$r_j = \left(\frac{1}{\lambda + j\beta}\right)\left[\frac{\eta_1^*(1,s)}{\psi_1^*(2,s)} - \gamma - \gamma \sum_{k_1=1}^{j-Q-2} \frac{\eta_1^*(1,k_1)}{\psi_1^*(2,k_1+1)}\right] r_1, \tag{3.76}$$

for $j = Q+3, \ldots, S$.

Hence from equations (3.73), (3.74),(3.75) and (3.76), we get (3.68).

Next, to find r_1. Consider the normalizing equation

$$\mathbf{\Delta}\bar{\mathbf{e}} = 1.$$

$$\begin{aligned} \text{That is} \quad & \Bigg\{1 + \sum_{j=2}^{s+1} \frac{\eta_1^*(1,j-1)}{\psi_1^*(2,j)} + \sum_{j=s+2}^{Q+1}\left(\frac{1}{\lambda + j\beta}\right)\frac{\eta_1^*(1,s)}{\psi_1^*(2,s)} \\ & + \left(\frac{1}{\lambda + (Q+2)\beta}\right)\left(\frac{\eta_1^*(1,s)}{\psi_1^*(2,s)} - \gamma\right) \\ + \sum_{j=Q+3}^{S} & \left(\frac{1}{\lambda + j\beta}\right)\left[\frac{\eta_1^*(1,s)}{\psi_1^*(2,s)} - \gamma - \gamma \sum_{k_1=1}^{j-Q-2} \frac{\eta_1^*(1,k_1)}{\psi_1^*(2,k_1+1)}\right]\Bigg\} r_1 = 1. \end{aligned}$$

$$\begin{aligned} r_1 = \Bigg[1 &+ \sum_{j=2}^{s+1} \frac{\eta_1^*(1,j-1)}{\psi_1^*(2,j)} + \frac{\eta_1^*(1,s)}{\psi_1^*(2,s)} \sum_{j=s+2}^{Q+1} \frac{1}{\lambda + j\beta} \\ & + \frac{1}{\lambda + (Q+2)\beta}\left[\frac{\eta_1^*(1,s)}{\psi_1^*(2,s)} - \gamma\right] \\ & + \sum_{j=Q+3}^{S} \frac{1}{\lambda + j\beta}\left[\frac{\eta_1^*(1,s)}{\psi_1^*(2,s)} - \gamma - \gamma \sum_{k_1=1}^{j-Q-2} \frac{\eta_1^*(1,k_1)}{\psi_1^*(2,k_1+1)}\right]\Bigg]^{-1} \end{aligned}$$

where

$$\eta_1^*(j_1,j_2) = \prod_{\tilde{k}=j_1}^{j_2} (\tilde{k}\beta + \lambda + \gamma + \alpha)$$

and

$$\psi_1^*(j_1,j_2) = \prod_{\tilde{k}=j_1}^{j_2} (\tilde{k}\beta + \lambda).$$

Hence the theorem. □

3.3.2.2 Stochastic Decomposition of System State

Let $\bar{\boldsymbol{\pi}}$ be the steady-state probability vector of $\tilde{A}$.

$$\bar{\boldsymbol{\pi}} = (\boldsymbol{\pi}^{(0)}, \boldsymbol{\pi}^{(1)}, \boldsymbol{\pi}^{(2)}, \dots)$$

where $\boldsymbol{\pi}^{(i)} = (\pi^{(i,1)}, \pi^{(i,2)}, \dots, \pi^{(i,S)})$, where $i = 0, 1, 2, \dots$ and

$$\pi^{(i,j)} = \lim_{t\to\infty} P(X(t) = i, I(t) = j), \ (i,j) \in E.$$

$\pi^{(i,j)}$ is the steady-state probability for the state (i,j).

We claim that

$$\boldsymbol{\pi}^{(i)} = K\rho^i \boldsymbol{\Delta}, \quad i \geq 0 \tag{3.77}$$

where $\boldsymbol{\Delta} = (r_1, r_2, \dots r_S)$ is the steady-state probability vector when the service time is negligible, K is a constant to be determined and $\rho = \frac{\lambda}{\mu}$.

Proof. We have $\bar{\boldsymbol{\pi}}\tilde{A} = \mathbf{0}$ and $\bar{\boldsymbol{\pi}}\bar{\mathbf{e}} = 1$.

$$\bar{\boldsymbol{\pi}}\tilde{A} = \mathbf{0} \Rightarrow (\boldsymbol{\pi}^{(0)}, \boldsymbol{\pi}^{(1)}, \boldsymbol{\pi}^{(2)}, \dots) \begin{bmatrix} B_1 & A_0 & & \\ A_2 & A_1 & A_0 & \\ & A_2 & A_1 & A_0 \\ & \ddots & \ddots & \ddots \end{bmatrix} = \mathbf{0}$$

which gives

$$\boldsymbol{\pi}^{(0)} B_1 + \boldsymbol{\pi}^{(1)} A_2 = \mathbf{0} \tag{3.78}$$

$$\text{and} \quad \boldsymbol{\pi}^{(i+2)} A_2 + \boldsymbol{\pi}^{(i+1)} A_1 + \boldsymbol{\pi}^{(i)} A_0 = \mathbf{0} \tag{3.79}$$

$i = 0, 1, \ldots$.
When (3.77) is true, we get from (3.78),

$$K\rho^0 \mathbf{\Delta} B_1 + K\rho \mathbf{\Delta} A_2 = \mathbf{0}.$$
$$\text{That is} \quad K\mathbf{\Delta}(B_1 + \rho A_2) = \mathbf{0}.$$
$$\text{That is} \quad \mathbf{\Delta}\tilde{Q} = \mathbf{0},$$

which is true since $\mathbf{\Delta} = (r_1, r_2, \ldots, r_S)$ is the steady-state probability vector corresponding to the generator $\tilde{Q}$ associated with the Markovian chain of the inventory process under consideration when service time is negligible.
When (3.77) is true, we get from (3.79),

$$K\rho^{i+2}\mathbf{\Delta} A_2 + K\rho^{i+1}\mathbf{\Delta} A_1 + K\rho^i \mathbf{\Delta} A_0 = \mathbf{0}, \quad i = 0, 1, 2, \ldots$$
$$\text{That is} \quad \mathbf{\Delta}(\rho A_2 + A_1 + \frac{1}{\rho} A_0) = \mathbf{0}.$$
$$\text{That is} \quad \mathbf{\Delta}\tilde{Q} = \mathbf{0},$$

which is true, by following the same argument given above.

Hence, the stochastic decomposition of system states is verified. □

3.3.2.3 Determination of K

We have

$$\sum_{i=0}^{\infty}\sum_{j=1}^{S} \pi^{(i,j)} = 1.$$
$$K\left(\sum_{i=0}^{\infty}\rho^i\right)\left(\sum_{j=1}^{S} r_j\right) = 1 \text{ using (3.77)}.$$
$$\text{That is, } K\sum_{i=0}^{\infty}\rho^i = 1.$$

Therefore, $K = 1 - \rho$.

3.3.2.4 Explicit Solution

From the above discussions, we can write the steady-state probability vector explicitly as in the following theorem:

Theorem 3.3.4. *The steady-state probability vector* $\bar{\boldsymbol{\pi}}$ *of* $\tilde{A}$ *partitioned as* $\bar{\boldsymbol{\pi}} = (\boldsymbol{\pi}^{(0)}, \boldsymbol{\pi}^{(1)}, \ldots)$ *where each* $\boldsymbol{\pi}^{(i)}$, $i = 0, 1, 2, \ldots$ *again partitioned as* $\boldsymbol{\pi}^{(i)} = (\pi^{(i,1)}, \pi^{(i,2)}, \ldots, \pi^{(i,S)},)$ *are obtained by*

$$\pi^{(i,j)} = (1-\rho)\rho^i r_j \tag{3.80}$$

where $\rho = \frac{\lambda}{\mu}$ *and* r_j, $j = 1, 2, \ldots, S$, *represent the inventory level probabilities when service time is negligible and are given as*

$$r_j = \tilde{q}_j r_1$$

where $j = 1, 2, \ldots, S$ *and*

$$\tilde{q}_j = \begin{cases} 1, & j = 1 \\ \frac{\eta_1^*(1,j-1)}{\psi_1^*(2,j)}, & j = 2, 3, \ldots, s+1 \\ \frac{1}{\lambda+j\beta}\frac{\eta_1^*(1,s)}{\psi_1^*(2,s)}, & j = s+2, \ldots, Q+1 \\ \frac{1}{\lambda+j\beta}\left[\frac{\eta_1^*(1,s)}{\psi_1^*(2,s)} - \gamma\right], & j = Q+2 \\ \frac{1}{\lambda+j\beta}\left[\frac{\eta_1^*(1,s)}{\psi_1^*(2,s)} - \gamma - \gamma\sum_{k_1=1}^{j-Q-2}\frac{\eta_1^*(1,k_1)}{\psi_1^*(2,k_1+1)}\right], & j = Q+3, \ldots, S \end{cases}$$

where

$$\eta_1^*(j_1, j_2) = \prod_{\tilde{k}=j_1}^{j_2}(\tilde{k}\beta + \lambda + \gamma + \alpha)$$

and

$$\psi_1^*(j_1, j_2) = \prod_{\tilde{k}=j_1}^{j_2}(\tilde{k}\beta + \lambda).$$

r_1 *is obtained by solving the equation* $\boldsymbol{\Delta}\bar{\mathbf{e}} = 1$ *as*

$$r_1 = \left[1 + \sum_{j=2}^{s+1}\frac{\eta_1^*(1,j-1)}{\psi_1^*(2,j)} + \frac{\eta_1^*(1,s)}{\psi_1^*(2,s)}\sum_{j=s+2}^{Q+1}\frac{1}{\lambda+j\beta} + \frac{1}{\lambda+(Q+2)\beta}\left[\frac{\eta_1^*(1,s)}{\psi_1^*(2,s)} - \gamma\right] + \sum_{j=Q+3}^{S}\frac{1}{\lambda+j\beta}\left[\frac{\eta_1^*(1,s)}{\psi_1^*(2,s)} - \gamma - \gamma\sum_{k_1=1}^{j-Q-2}\frac{\eta_1^*(1,k_1)}{\psi_1^*(2,k_1+1)}\right]\right]^{-1}.$$

The result indicated by (3.80) not only tells us that the original system possess stochastic decomposition but also the important fact that the system state distribution is the product of the distribution of its marginals: one component is the distribution of the classical $M/M/1$ queue whose long run distribution for i customers in the system is $(1-\rho)\rho^i$, $i \geq 0$ and the other factor is the probability of j items in the inventory. Next we turn to find out how the system performs. The measures given in the following are pointers to the system performance.

3.3.3 System Performance Measures

(a) Expected inventory held in the system (Mean inventory level),

$$
\begin{aligned}
E(I) &= \sum_{i=0}^{\infty}\sum_{j=1}^{S} j\pi^{(i,j)} \\
&= \Big(\sum_{i=0}^{\infty}(1-\rho)\rho^i\Big)\Big(\sum_{j=0}^{S} jr_j\Big) \\
&= \sum_{j=1}^{S} jr_j \\
&= \Bigg[1 + \sum_{j=2}^{s+1} j\frac{\eta_1^*(1,j-1)}{\psi_1^*(2,j)} + \frac{\eta_1^*(1,s)}{\psi_1^*(2,s)} \sum_{j=s+2}^{Q+1}\Big(\frac{j}{\lambda+j\beta}\Big) \\
&+ \frac{(Q+2)}{\lambda+(Q+2)\beta}\left[\frac{\eta_1^*(1,s)}{\psi_1^*(2,s)} - \gamma\right] \\
&+ \sum_{j=Q+3}^{S}\Big(\frac{j}{\lambda+j\beta}\Big)\left(\frac{\eta_1^*(1,s)}{\psi_1^*(2,s)} - \gamma - \gamma\sum_{k_1=1}^{j-Q-2}\frac{\eta_1^*(1,k_1)}{\psi_1^*(2,k_1+1)}\right)\Bigg] r_1.
\end{aligned}
$$

(b) Mean waiting time of customers in the system,

$$
W_S = \frac{L}{\lambda}
$$

where

$$\begin{aligned}
L &= \text{ Expected number of customers in the system} \\
&= \sum_{i=0}^{\infty}\sum_{j=1}^{S} i\pi^{(i,j)} \\
&= \left(\sum_{i=0}^{\infty} i(1-\rho)\rho^i\right)\left(\sum_{j=1}^{S} r_j\right) \\
&= \frac{\rho}{1-\rho}.
\end{aligned}$$

$$\begin{aligned}
\text{Therefore} \quad W_s &= \frac{L}{\lambda} \\
&= \frac{1}{\mu-\lambda}. \qquad (3.81)
\end{aligned}$$

(c) Mean reorder rate,

$$\begin{aligned}
R_r &= \mu\sum_{i=1}^{\infty}\pi^{(i,s+1)} + (s+1)\beta\sum_{i=0}^{\infty}\pi^{(i,s+1)} \\
&= \mu(\sum_{i=1}^{\infty}(1-\rho)\rho^i)r_{s+1} + (s+1)\beta(\sum_{i=0}^{\infty}(1-\rho)\rho^i)r_{s+1} \\
&= (\lambda+(s+1)\beta)\frac{\eta_1^*(1,s)}{\psi_1^*(2,s+1)}r_1 \\
&= \frac{\eta_1^*(1,s)}{\psi_1^*(2,s)}r_1.
\end{aligned}$$

(d) Mean local purchase rate (Mean order cancellation rate), R_{LP}

Let R_{LP_1} = Local purchase rate due to T-realization

$$\begin{aligned}
&= \alpha\sum_{i=0}^{\infty}\sum_{j=1}^{s}\pi^{(i,j)} \\
&= \alpha\sum_{i=0}^{\infty}\sum_{j=1}^{s}(1-\rho)\rho^i r_j \\
&= \alpha\left(\sum_{j=1}^{s} r_j\right).
\end{aligned}$$

Let R_{LP_2} = Local purchase rate resulting from zero inventory level

$$\begin{aligned}
&= \mu \sum_{i=1}^{\infty} \pi^{(i,1)} + \beta \sum_{i=0}^{\infty} \pi^{(i,1)} \\
&= \mu(1-\rho) \left(\sum_{i=1}^{\infty} \rho^i \right) r_1 + \beta \left(\sum_{i=0}^{\infty} (1-\rho)\rho^i \right) r_1 \\
&= (\lambda + \beta) r_1. \\
R_{\mathrm{LP}} &= R_{\mathrm{LP}_1} + R_{\mathrm{LP}_2} \\
&= \alpha \left(\sum_{j=1}^{s} r_j \right) + (\lambda + \beta) r_1 \\
&= \left[\alpha \left(1 + \sum_{j=2}^{s} \frac{\eta_1^*(1, j-1)}{\psi_1^*(2, j)} \right) + \lambda + \beta \right] r_1.
\end{aligned}$$

(e) Mean decay rate,

$$R_D = \beta E(I).$$

(f) Expected number of units locally purchased due to T-realization,

$$N_{\mathrm{LP}} = \sum_{x=Q}^{Q+s} x P(x)$$

where $P(x)$ = Probability that x units are locally purchased

$$\begin{aligned}
&= Q\phi_s + (Q+1)\phi_{s-1} + \cdots + (Q + (s-1))\phi_1 \\
&= Q(\sum_{j=1}^{s} \phi_j) + \sum_{j=0}^{s-1} j\phi_{s-j}.
\end{aligned}$$

Remark. If we put $\beta = 0$ the above measures will get reduced to those of model I.

3.3.4 Cost Analysis

Now, we introduce a cost function. Let the various costs involved in the model be as given below:

C_H : Inventory holding cost per unit item per unit time.
C_D : Decay cost per unit item per unit time.
C_S : Set up cost per unit order, under natural purchase.
C_W : Waiting time cost per customer per unit time.
C_{LP} : Local purchase cost per unit order.
C_{NP} : Natural purchase cost per unit order.
C_C : Cancellation cost per unit order cancelled.

$$
\begin{aligned}
\text{TEC} &= C_H E(I) + (C_S + C_{\text{NP}} Q) R_r + C_C R_{\text{LP}} + C_D R_D \\
&\quad + C_W W_S + + C_{\text{LP}} (R_{\text{LP}_1} N_{\text{LP}} + R_{\text{LP}_2} S) \\
&= \Bigg\{ (C_H + C_D \beta) \left[1 + \sum_{j=2}^{s+1} j \frac{\eta_1^*(1, j-1)}{\psi_1^*(2, j)} + (G + \gamma) \sum_{j=s+2}^{Q+1} \frac{j}{\lambda + j\beta} \right. \\
&\quad \left. + U + \sum_{j=Q+3}^{S} \left(\frac{j}{\lambda + j\beta} \right) \left(G - \gamma \sum_{k_1=1}^{j-Q-2} \frac{\eta_1^*(1, k_1)}{\psi_1^*(2, k_1 + 1)} \right) \right] \\
&\quad + (C_S + C_{\text{NP}} Q)(G + \gamma) \ + C_C \left[\lambda + \beta + \alpha \left(1 + \sum_{j=2}^{s} \frac{\eta_1^*(1, j-1)}{\psi_1^*(2, j)} \right) \right] \\
&\quad + C_{\text{LP}} \left[\left(\alpha \sum_{j=1}^{s} \tilde{q}_j \right) \left(Q \sum_{j=1}^{s} q_j + \sum_{j=0}^{s-1} j q_{s-j} \right) \phi_1 \right. \\
&\quad \left. + (\lambda + \beta) S \right] \Bigg\} r_1 + C_W \left(\frac{1}{\mu - \lambda} \right)
\end{aligned}
$$

where

$$
G = \frac{\eta_1^*(1, s)}{\psi_1^*(2, s)} - \gamma \text{ and } U = \frac{(Q+2)G}{\lambda + (Q+2)\beta}
$$

and

$$
\begin{aligned}
r_1 = & \left[1 + \sum_{j=2}^{s+1} \frac{\eta_1^*(1, j-1)}{\psi_1^*(2, j)} + (G + \gamma) \sum_{j=s+2}^{Q+1} \left(\frac{1}{\lambda + j\beta} \right) \right. \\
& \left. + \frac{G}{\lambda + (Q+2)\beta} + \sum_{j=Q+3}^{S} \left(\frac{1}{\lambda + j\beta} \right) \left(G - \gamma \sum_{k_1=1}^{j-Q-2} \frac{\eta_1^*(1, k_1)}{\psi_1^*(2, k_1 + 1)} \right) \right]^{-1} .
\end{aligned}
$$

To verify the convexity of the above cost function, with respect to s, S or α the corresponding derivative may be computed, then equate it to zero.

Nevertheless, solving it is a laborious task. Since analysis of TEC as a function of s, S or α is quite complex, we give a few numerical illustrations.

3.3.4.1 Numerical Analysis

Case 1. Analysis of TEC as a function of s (when $\beta = 0.2$).

Input Data:
$C_H = 0.5$, $C_D = 7$, $C_S = 15$, $C_{\text{NP}} = 12$, $C_C = 8$, $C_{\text{LP}} = 15$, $C_W = 30$, $\alpha = 20$, $\lambda = 23$, $\mu = 25$, $\gamma = 16$, $\beta = 0.2$, $S = 17$.

Table 3.3: Effect of s on TEC

s	TEC ($\beta = 0.2$)	TEC ($\beta = 0$)
2	425.8603	381.3717
3	395.2335	351.6098
4	385.7502	341.4396
5	**385.1315**	**339.4069**
6	388.4944	341.0169
7	393.9350	344.5680
8	400.8004	349.5299

Table 3.3 shows that as s increases, TEC values decrease, reach a minimum at $s = 5$ and then increase (when $\beta = 0.2$). Hence it is numerically verified that TEC function is convex with respect to s.

Case 2. Analysis of TEC as a function of S (when $\beta = 0.2$).

Input Data:
$C_H = 0.5$, $C_D = 7$, $C_S = 15$, $C_{\text{NP}} = 12$, $C_C = 8$, $C_{\text{LP}} = 15$, $C_W = 30$, $\alpha = 20$, $\lambda = 23$, $\mu = 25$, $\gamma = 16$, $\beta = 0.2$, $s = 2$.

Table 3.4 shows that as S increases, TEC values decrease, reach a minimum at $S = 17$ and then increase (when $\beta = 0.2$). Hence it is numerically verified that TEC function is convex with respect to S.

Case 3. Analysis of TEC as a function of α (when $\beta = 0.2$).

Input Data:
$C_H = 0.5$, $C_D = 7$, $C_S = 15$, $C_{\text{NP}} = 12$, $C_C = 8$, $C_{\text{LP}} = 15$, $C_W = 30$, $\lambda = 23$, $\mu = 25$, $\gamma = 16$, $\beta = 0.2$, $S = 17$, $s = 2$.

Table 3.4: Effect of S on TEC

S	TEC ($\beta = 0.2$)	TEC ($\beta = 0$)
10	437.4388	406.6912
11	433.3057	400.6476
12	430.3947	395.7957
13	428.3858	391.8263
14	427.0613	388.5287
15	426.2681	385.7543
16	425.8952	383.3953
17	**425.8603**	381.3717
18	426.1010	379.6226
19	426.5691	378.1012
20	427.2271	376.7707

Table 3.5: Effect of α on TEC

α	TEC ($\beta = 0.2$)	TEC ($\beta = 0$)
21	424.1680	379.9000
22	422.5521	378.3900
23	421.0081	376.9480
24	419.5318	375.5700
25	418.1194	374.2522
26	416.7672	372.9913
27	415.4720	371.7840
28	414.2307	370.6275
29	413.0402	369.5188
30	411.8979	368.4555
31	410.8013	367.4351

Table 3.5 shows that as α increases, TEC function is monotonically decreasing and hence convex with respect to α (when $\beta = 0.2$).

Comparison of Perishable and Non-Perishable Inventory Models

Finally we compare the optimal values of TEC as function of s, S or α in non-perishable and perishable cases.

In the above set of input data, if we put $\beta = 0$ and $C_D = 0$, we get the TEC values of non-perishable items which are given in Tables 3.3, 3.4 and 3.5.

When $\beta = 0$, Table 3.3 shows that as s increases, TEC values decrease, reach a minimum at $s = 5$ and then increase. Hence it is numerically verified that TEC function is convex. Also, on comparing the TEC values in Table 3.3 we get that the total expected cost per unit time as a function of s is highest in the case of perishable items.

When $\beta = 0$, Table 3.4 shows that as S increases, TEC function is monotonically decreasing, and hence convex. Also, on comparing the TEC values in Table 3.4 we get that the total expected cost per unit time as a function of S is highest in the case of perishable items.

When $\beta = 0$, Table 3.5 shows that as α increases, TEC function is monotonically decreasing, and hence convex. Also, on comparing the TEC values in Table 3.5 we get that the total expected cost per unit time as a function of α is highest in the case of perishable items.

Chapter 4

(s, S) Production Inventory Systems with Positive Service Time

4.1 Introduction

In the previous two chapters we considered replenishment in bulk against orders placed on the inventory level reaching s. In this chapter we consider the case of addition taking place to stock one at a time through a production process. As in the case of bulk replenishment, in the present case also production is switched on when inventory level depletes to s from S. Thereafter the production process remains on until the inventory level is back at S. The time to produce an item (inter-production time/lead time) is assumed to be exponentially distributed.

In this chapter, we consider two different (s, S) production inventory models involving positive service time. In both the models, it is assumed that there is only one server. As soon as the inventory level drops to zero due to demand, a local purchase is made to prevent customer loss, thereby ensuring customer satisfaction, and thus their goodwill. Also in both the models it is assumed that arrivals are according to Poisson process with parameter λ. As and when the inventory level reaches s, the production process is switched 'on'. Time taken to produce an item is exponentially distributed with parameter η. The produced item requires a processing time before it is served to the customer, and the processing time is a random variable which follows exponential distribution with parameter μ. Once the production process is switched on, it will be kept in the 'on' mode till the inventory level reaches S. As soon as the inventory level reaches S, the production process is switched 'off'.

In model I, we assume that as and when the inventory level reaches zero, a local purchase of one unit at a time is made at a higher cost, and in model II, a local purchase of N units is made (where $2 \leq N < s$) at a time at higher cost. Also it is assumed that supply of items is instantaneous in local purchase. The above situations can be modelled as continuous time Markov chains as follows.

4.2 Model I: Local Purchase of One Unit

4.2.1 Model Formulation and Analysis

Let $X(t)$ = Number of customers in the system at time t,
$I(t)$ = Inventory level at time t and
$K(t)$ = Status of the production process:

$$K(t) = \begin{cases} 1, & \text{if production process is 'on' at time } t \\ 0, & \text{if production process is 'off' at time } t \end{cases}$$

$\{(X(t), I(t), K(t)), t \geq 0\}$ is a CTMC with state space

$$\{(i,j) | i \geq 0, 1 \leq j \leq s\} \cup \{(i,j,k) | i \geq 0; s+1 \leq j \leq S-1; k = 0,1\}$$
$$\cup \{(i,S) | i \geq 0\}.$$

Note that $K(t) = 1$, whenever $1 \leq I(t) \leq s$ and so in the above description of state space we have excluded the third element k from (i,j,k) for $j = 1, 2, \ldots, s$.
Also $K(t) = 0$, when $I(t) = S$ and $K(t)$ is either 0 or 1 when $s+1 \leq I(t) \leq S-1$.

4.2.1.1 Infinitesimal Generator $\tilde{A}$

The infinitesimal generator of the process under consideration is obtained interms of submatrices as follows:

$$\tilde{A} = \begin{bmatrix} B_1 & A_0 & & \\ A_2 & A_1 & A_0 & \\ & A_2 & A_1 & A_0 \\ & \ddots & \ddots & \ddots \end{bmatrix}$$

where $A_0 = \lambda I_{2S-s-1}$

and

$$A_1 = \begin{bmatrix} \tilde{G}_1 & \eta & & & & & & & & & \\ & \tilde{G}_1 & \eta & & & & & & & & \\ & & \ddots & \ddots & & & & & & & \\ & & & \tilde{G}_1 & \eta & & & & & & \\ & & & & \tilde{G}_1 & P_1 & & & & & \\ & & & & & \tilde{G}_2 & P_2 & & & & \\ & & & & & & \ddots & \ddots & & & \\ & & & & & & & \tilde{G}_2 & P_2 & & \\ & & & & & & & & \tilde{G}_2 & P_3 \\ & & & & & & & & & \tilde{G}_3 \end{bmatrix}$$

where $\tilde{G}_1 = [-(\lambda+\eta+\mu)]$, $\tilde{G}_2 = \begin{bmatrix} -(\lambda+\mu) & \\ & -(\lambda+\eta+\mu) \end{bmatrix}$ and $\tilde{G}_3 = [-(\lambda+\mu)]$.

$$A_2 = \begin{bmatrix} \mu & & & & & & & & & \\ \mu & & & & & & & & & \\ & \ddots & & & & & & & & \\ & & \mu & & & & & & & \\ & & & \mu & & & & & & \\ & & & & H_1 & & & & & \\ & & & & & H_2 & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & H_2 & & \\ & & & & & & & & H_3 & 0 \end{bmatrix}$$

where $H_1 = \begin{bmatrix} \mu \\ \mu \end{bmatrix}$, $H_2 = \mu I_2$, $H_3 = [\mu \quad 0]$.

$$B_1 = \begin{bmatrix} G_1 & \eta & & & & & & & & & \\ & G_1 & \eta & & & & & & & & \\ & & \ddots & \ddots & & & & & & & \\ & & & G_1 & \eta & & & & & & \\ & & & & G_1 & P_1 & & & & & \\ & & & & & G_2 & P_2 & & & & \\ & & & & & & \ddots & \ddots & & & \\ & & & & & & & G_2 & P_2 & & \\ & & & & & & & & G_2 & P_3 \\ & & & & & & & & & G_3 \end{bmatrix}$$

where $G_1 = [-(\lambda+\eta)]$, $G_2 = \begin{bmatrix} -\lambda & \\ & -(\lambda+\eta) \end{bmatrix}$,

$G_3 = [-\lambda]$, $P_1 = [0 \quad \eta]$, $P_2 = \begin{bmatrix} 0 & 0 \\ 0 & \eta \end{bmatrix}$ and $P_3 = \begin{bmatrix} 0 \\ \eta \end{bmatrix}$.

A_0, A_1, A_2 and B_1 are square matrices of order $2S - s - 1$.

4.2.1.2 Steady-State Analysis

Let $A = A_0 + A_1 + A_2$ be the generator matrix and is obtained as

$$A = \begin{bmatrix} -\eta & \eta & & & & & & & & & \\ \mu & \hat{G}_1 & \eta & & & & & & & & \\ & \mu & \hat{G}_1 & \eta & & & & & & & \\ & \ddots & \ddots & \ddots & & & & & & & \\ & & \mu & \hat{G}_1 & \eta & & & & & & \\ & & & \mu & \hat{G}_1 & P_1 & & & & & \\ & & & & H_1 & \hat{G}_2 & P_2 & & & & \\ & & & & & H_2 & \hat{G}_2 & P_2 & & & \\ & & & & & \ddots & \ddots & \ddots & & & \\ & & & & & & H_2 & \hat{G}_2 & P_2 & & \\ & & & & & & & H_2 & \hat{G}_2 & P_3 \\ & & & & & & & & H_3 & \hat{G}_3 \end{bmatrix}$$

where $\hat{G}_1 = [-(\mu+\eta)]$, $\hat{G}_2 = \begin{bmatrix} -\mu & 0 \\ 0 & -(\mu+\eta) \end{bmatrix}$ and $\hat{G}_3 = [-\mu]$.

A is a square matrix of order $2S - s - 1$.

First we investigate the stationary distribution of A. This will help us in analyzing the stability of the larger system namely, the CTMC $\{(X(t), I(t), K(t)), t \geq 0\}$ as $t \to \infty$.

Theorem 4.2.1. *The steady-state probability vector* $\mathbf{\Phi}$ *of* A *is*

$$\mathbf{\Phi} = (\phi_1, \phi_2, \ldots, \phi_{s-1}, \phi_s, \tilde{\phi}_{s+1}, \tilde{\phi}_{s+2}, \ldots, \tilde{\phi}_{S-1}, \phi_S)$$

where

$$\tilde{\phi}_{l_1} = (\phi_{l_1,0}, \phi_{l_1,1}), \quad l_1 = s+1, \ldots, S-1$$

and

$$\phi_l = (\frac{\mu}{\eta})^{s-l}[\frac{\mu}{\eta} + (\frac{\mu}{\eta})^2 + (\frac{\mu}{\eta})^3 + \cdots + (\frac{\mu}{\eta})^Q]\phi_S$$

where $l = 1, 2, \ldots, s$.

$$\phi_{l,0} = \phi_S, \ \textit{where}\ l = s+1, s+2, \ldots, S-1.$$

$$\phi_{l,1} = [\frac{\mu}{\eta} + (\frac{\mu}{\eta})^2 + (\frac{\mu}{\eta})^3 + \cdots + (\frac{\mu}{\eta})^{S-l}]\phi_S,$$

where $l = s+1, \ldots, S-1$.

When $\mu \neq \eta$, ϕ_S *can be obtained by solving* $\mathbf{\Phi}\bar{\mathbf{e}} = 1$ *as*

$$\phi_S = \frac{(1-\frac{\mu}{\eta})^2}{\frac{\mu}{\eta}[(\frac{\mu}{\eta})^S - (\frac{\mu}{\eta})^s - Q] + Q - (1-\frac{\mu}{\eta})^2}.$$

Proof. We have $\mathbf{\Phi}A = \mathbf{0}$ and $\mathbf{\Phi}\bar{\mathbf{e}} = 1$.

$$\mathbf{\Phi}A = \mathbf{0} \Rightarrow$$

$$\phi_1 \times -\eta + \phi_2\mu = 0. \tag{4.1}$$

$$\phi_j\eta - \phi_{j+1}(\mu+\eta) + \phi_{j+2}\mu = 0 \tag{4.2}$$

for $j = 1, 2, \ldots, s-2$.

$$\phi_{s-1}\eta + \phi_s \times -(\mu+\eta) + \tilde{\phi}_{s+1}H_1 = 0. \tag{4.3}$$

$$\phi_s P_1 + \tilde{\phi}_{s+1}\hat{G}_2 + \tilde{\phi}_{s+2}H_2 = 0. \tag{4.4}$$

$$\tilde{\phi}_j P_2 + \tilde{\phi}_{j+1}\hat{G}_2 + \tilde{\phi}_{j+2}H_2 = 0. \tag{4.5}$$

for $j = s+1, s+2, \ldots, S-3$.

$$\tilde{\phi}_{S-2}P_2 + \tilde{\phi}_{S-1}\hat{G}_2 + \phi_S H_3 = 0. \tag{4.6}$$

$$\tilde{\phi}_{S-1}P_3 + \phi_S\hat{G}_3 = 0. \tag{4.7}$$

Equation (4.1) gives

$$\phi_2 = \frac{\eta}{\mu}\phi_1.$$

Equation (4.2) gives

$$\phi_{j+2} = \left(\frac{\mu+\eta}{\mu}\right)\phi_{j+1} - \left(\frac{\eta}{\mu}\right)\phi_j \tag{4.8}$$

for $j = 1, 2, \ldots, s-2$, from which we get

$$\phi_l = \left(\frac{\eta}{\mu}\right)^{l-1}\phi_1 \tag{4.9}$$

for $l = 2, 3, \ldots, s$.
Equation (4.3) gives

$$\phi_{s+1,0} + \phi_{s+1,1} = \left(\frac{\eta}{\mu}\right)^{s}\phi_1. \tag{4.10}$$

Equation (4.4) gives

$$\phi_{s+1,0} = \phi_{s+2,0} \tag{4.11}$$

and

$$\eta\phi_s - (\mu+\eta)\phi_{s+1,1} + \mu\phi_{s+2,1} = 0. \tag{4.12}$$

Substituting (4.11) in (4.10), we get

$$\phi_{s+2,0} + \phi_{s+1,1} = \left(\frac{\eta}{\mu}\right)^{s}\phi_1. \tag{4.13}$$

Equation (4.7) gives

$$\phi_{S-1,1} = \frac{\mu}{\eta}\phi_S. \tag{4.14}$$

Equation (4.6) gives

$$\phi_{S-1,0} = \phi_S \tag{4.15}$$

$$\text{and } \eta\phi_{S-2,1} - (\mu+\eta)\phi_{S-1,1} = 0. \tag{4.16}$$

Equations (4.16) and (4.14) give

$$\eta\phi_{S-2,1} = \mu(\phi_S + \frac{\mu}{\eta}\phi_S).$$

$$\phi_{S-2,1} = \left[\frac{\mu}{\eta} + \left(\frac{\mu}{\eta}\right)^2\right]\phi_S. \tag{4.17}$$

Equation (4.5) gives

$$-\mu\phi_{j+1,0} + \mu\phi_{j+2,0} = 0 \tag{4.18}$$

for $j = s+1, s+2, \dots, S-3$, and

$$\eta\phi_{j,1} - (\mu+\eta)\phi_{j+1,1} + \mu\phi_{j+2,1} = 0 \tag{4.19}$$

for $j = s+1, s+2, \dots, S-3$.

Using (4.11), (4.15) and (4.18) and after some steps, we get

$$\phi_{l,0} = \phi_S \tag{4.20}$$

for $l = s+1, s+2, \dots, S-1$.
Using equations (4.19), (4.14) and (4.17), and after some steps, we get

$$\phi_{l,1} = \left[\frac{\mu}{\eta} + \left(\frac{\mu}{\eta}\right)^2 + \left(\frac{\mu}{\eta}\right)^3 + \cdots + \left(\frac{\mu}{\eta}\right)^{S-l}\right]\phi_S \tag{4.21}$$

for $l = s+1, s+2, \dots, S-3$.
Using (4.14), (4.17) and (4.21) we get

$$\phi_{l,1} = \left[\frac{\mu}{\eta} + \left(\frac{\mu}{\eta}\right)^2 + \left(\frac{\mu}{\eta}\right)^3 + \cdots + \left(\frac{\mu}{\eta}\right)^{S-l}\right]\phi_S \tag{4.22}$$

for $l = s+1, s+2, \dots, S-1$.
Equations (4.12) and (4.22) give

$$\phi_s = \left[\left(\frac{\mu}{\eta}\right) + \left(\frac{\mu}{\eta}\right)^2 + \left(\frac{\mu}{\eta}\right)^3 + \cdots + \left(\frac{\mu}{\eta}\right)^{Q}\right]\phi_S. \tag{4.23}$$

Equations (4.9) and (4.23) give

$$\phi_l = \left(\frac{\mu}{\eta}\right)^{s-l}\left[\left(\frac{\mu}{\eta}\right) + \left(\frac{\mu}{\eta}\right)^2 + \left(\frac{\mu}{\eta}\right)^3 + \cdots + \left(\frac{\mu}{\eta}\right)^{Q}\right]\phi_S \tag{4.24}$$

for $l = 1, 2, \dots, s$.

Using equation (4.20), (4.22) and (4.24), we get the required result.
Next, to find ϕ_S. Consider the normalizing equation

$$\mathbf{\Phi}\bar{\mathbf{e}} = 1.$$

$$\sum_{l=1}^{s}\phi_l + \sum_{l=s+1}^{S-1}\phi_{l,0} + \sum_{l=s+1}^{S-1}\phi_{l,1} + \phi_S = 1. \tag{4.25}$$

Let $W_1 = \frac{\mu}{\eta} + (\frac{\mu}{\eta})^2 + (\frac{\mu}{\eta})^3 + \cdots + (\frac{\mu}{\eta})^Q$.
Using (4.20), (4.22) and (4.24), we get (4.25) as

$$\phi_S \Bigg[W_1 \left(\sum_{l=1}^{s} \left(\frac{\mu}{\eta}\right)^{s-l} \right) + Q - 1 + \sum_{l=s+1}^{S-1} \left(\frac{\mu}{\eta} + \left(\frac{\mu}{\eta}\right)^2 + \left(\frac{\mu}{\eta}\right)^3 + \cdots + \left(\frac{\mu}{\eta}\right)^{S-l} \right) + 1 \Bigg] = 0. \tag{4.26}$$

When $\mu \neq \eta$, we get

$$\sum_{l=s+1}^{S-1} \left(\frac{\mu}{\eta} + \left(\frac{\mu}{\eta}\right)^2 + \cdots + \left(\frac{\mu}{\eta}\right)^{S-l} \right) = \frac{\mu}{\eta}\left(1 - \frac{\mu}{\eta}\right)^{-1} \left[Q - 1 - \frac{\mu}{\eta} \frac{\left(1 - \frac{\mu}{\eta}\right)^{Q-1}}{1 - \frac{\mu}{\eta}} \right] \tag{4.27}$$

$$\sum_{l=1}^{s} \left(\frac{\mu}{\eta}\right)^{s-l} = \frac{1 - \left(\frac{\mu}{\eta}\right)^s}{1 - \frac{\mu}{\eta}}. \tag{4.28}$$

Also

$$W_1 = \frac{\frac{\mu}{\eta}\left(1 - \left(\frac{\mu}{\eta}\right)^Q\right)}{1 - \frac{\mu}{\eta}}. \tag{4.29}$$

Hence by using equations (4.27), (4.28) and (4.29), we get equation (4.26) as

$$\frac{\phi_S}{\left(1 - \frac{\mu}{\eta}\right)^2} \left\{ \left(\frac{\mu}{\eta}\right)^{Q+s+1} - \left(\frac{\mu}{\eta}\right)^{s+1} + Q - Q\frac{\mu}{\eta} - \left(1 - \frac{\mu}{\eta}\right)^2 \right\} = 1.$$

Therefore,

$$\phi_S = \frac{\left(1 - \frac{\mu}{\eta}\right)^2}{\frac{\mu}{\eta}\left[\left(\frac{\mu}{\eta}\right)^S - \left(\frac{\mu}{\eta}\right)^s - Q\right] + Q - \left(1 - \frac{\mu}{\eta}\right)^2}.$$

Hence the theorem. □

Remark. When $\mu = \eta$, we get the following:

$$\phi_l = Q\phi_S, \quad \text{where } l = 1, 2, \ldots, s.$$
$$\phi_{l,0} = \phi_S, \quad \text{where } l = s+1, \ldots, S-1.$$
$$\phi_{l,1} = (S-l)\phi_S, \quad \text{where } l = s+1, \ldots, S-1$$
$$\text{and } \phi_S = \left[sQ + (Q-1)\left(1 + \frac{Q}{2}\right) + 1\right]^{-1}.$$

4.2.1.3 Stability Condition

The result in Theorem 4.2.1 enables us to compute the stability of the CTMC $\{(X(t), I(t), K(t)), t \geq 0\}$.

Theorem 4.2.2. *The process* $\{(X(t), I(t), K(t)), t \geq 0\}$ *is stable iff* $\lambda < \mu$.

Proof. Since the process under consideration is an LIQBD, it will be stable iff

$$\mathbf{\Phi} A_0 \bar{\mathbf{e}} < \mathbf{\Phi} A_2 \bar{\mathbf{e}} \quad \text{(Neuts [40])} \tag{4.30}$$

where $\mathbf{\Phi}$ represents the steady-state probability vector of the generator matrix $A = A_0 + A_1 + A_2$ and $\bar{\mathbf{e}}$ is a column vector of 1's of appropriate order.

$$\begin{aligned}\mathbf{\Phi} A_0 \bar{\mathbf{e}} &= [\lambda(\phi_1 + \phi_2 + \cdots + \phi_s + \phi_{s+1,0} + \phi_{s+1,1} + \ldots \\ &\quad + \phi_{S-1,0} + \phi_{S-1,1} + \phi_S)] \\ &= \lambda. \\ \mathbf{\Phi} A_2 \bar{\mathbf{e}} &= [\mu(\phi_1 + \phi_2 + \cdots + \phi_s + \phi_{s+1,0} + \phi_{s+1,1} + \ldots \\ &\quad + \phi_{S-1,0} + \phi_{S-1,1} + \phi_S)] \\ &= \mu.\end{aligned}$$

Hence the using (4.30) we get $\lambda < \mu$. Hence the theorem. □

Having obtained the condition for the system to stabilize, we turn to compute the long-run probability distribution of the system states. Infact we show that the joint distribution of the system state can be written as the product of the marginal distribution of the components.

4.2.2 The Steady-State Probability Distribution of $\tilde{A}$

4.2.2.1 Stationary Distribution when Service Time is Negligible

Let $\tilde{Q}$ be the generator matrix associated with the Markovian chain of the inventory process under consideration when service time is negligible, and $\mathbf{\Delta} = (r_1, r_2, \ldots, r_S)$ be the stationary probability vector corresponding to $\tilde{Q}$.

$\tilde{Q}$ is obtained as

$$\tilde{Q} = \begin{bmatrix} -\eta & \eta & & & & & & & & & & \\ \lambda & G_1 & \eta & & & & & & & & & \\ & \lambda & G_1 & \eta & & & & & & & & \\ & \ddots & \ddots & \ddots & & & & & & & & \\ & & \lambda & G_1 & \eta & & & & & & & \\ & & & \lambda & G_1 & P_1 & & & & & & \\ & & & & \hat{H}_1 & G_2 & P_2 & & & & & \\ & & & & & \hat{H}_2 & G_2 & P_2 & & & & \\ & & & & & \ddots & \ddots & \ddots & & & & \\ & & & & & & \hat{H}_2 & G_2 & P_2 & & & \\ & & & & & & & \hat{H}_2 & G_2 & P_3 & & \\ & & & & & & & & \hat{H}_3 & G_3 & & \end{bmatrix}$$

where $G_1 = [-(\lambda+\eta)]$, $G_2 = \begin{bmatrix} -\lambda & 0 \\ 0 & -(\lambda+\eta) \end{bmatrix}$, $G_3 = [-\lambda]$, $P_1 = [0 \quad \eta]$, $P_2 = \begin{bmatrix} 0 & 0 \\ 0 & \eta \end{bmatrix}$, $P_3 = \begin{bmatrix} 0 \\ \eta \end{bmatrix}$, $\hat{H}_1 = \begin{bmatrix} \lambda \\ \lambda \end{bmatrix}$, $\hat{H}_2 = \lambda I_2$ and $\hat{H}_3 = [\lambda \ 0]$.

Theorem 4.2.3. *The steady-state probability vector $\mathbf{\Delta}$ of $\tilde{Q}$ is*

$$\mathbf{\Delta} = (r_1, r_2, \ldots, r_{s-1}, r_s, \tilde{r}_{s+1}, \tilde{r}_{s+2}, \ldots, \tilde{r}_{S-1}, r_S)$$

where

$$\tilde{r}_{l_1} = (r_{l_1,0}, r_{l_1,1}), \quad l_1 = s+1, \ldots, S-1$$

and

$$r_l = (\frac{\lambda}{\eta})^{s-l}[\frac{\lambda}{\eta} + (\frac{\lambda}{\eta})^2 + (\frac{\lambda}{\eta})^3 + \cdots + (\frac{\lambda}{\eta})^Q]r_S,$$

where $l = 1, 2, \ldots, s$.

$$r_{l,0} = r_S,$$

where $l = s+1, s+2, \ldots, S-1$.

$$r_{l,1} = [\frac{\lambda}{\eta} + (\frac{\lambda}{\eta})^2 + (\frac{\lambda}{\eta})^3 + \cdots + (\frac{\lambda}{\eta})^{S-l}]r_S,$$

where $l = s+1, \ldots, S-1$.

When $\lambda \neq \eta$, r_S *can be obtained by solving* $\mathbf{\Delta}\bar{\mathbf{e}} = 1$ *as*

$$r_S = \frac{(1-\frac{\lambda}{\eta})^2}{\frac{\lambda}{\eta}[(\frac{\lambda}{\eta})^S - (\frac{\lambda}{\eta})^s - Q] + Q - (1-\frac{\lambda}{\eta})^2}.$$

Proof. We have $\mathbf{\Delta}\tilde{Q} = \mathbf{0}$ and $\mathbf{\Delta}\bar{\mathbf{e}} = 1$.

$$\mathbf{\Delta}\tilde{Q} = \mathbf{0} \Rightarrow$$

$$r_1 \times -\eta + r_2\lambda = 0. \tag{4.31}$$

$$r_j\eta - r_{j+1}(\lambda+\eta) + r_{j+2}\lambda = 0 \tag{4.32}$$

for $j = 1, 2, \ldots, s-2$.

$$r_{s-1}\eta + r_s \times -(\lambda+\eta) + \tilde{r}_{s+1}\hat{H}_1 = 0. \tag{4.33}$$

$$r_sP_1 + \tilde{r}_{s+1}G_2 + \tilde{r}_{s+2}\hat{H}_2 = 0. \tag{4.34}$$

$$\tilde{r}_jP_2 + \tilde{r}_{j+1}G_2 + \tilde{r}_{j+2}\hat{H}_2 = 0 \tag{4.35}$$

for $j = s+1, s+2, \ldots, S-3$.

$$\tilde{r}_{S-2}P_2 + \tilde{r}_{S-1}G_2 + r_S\hat{H}_3 = 0. \tag{4.36}$$

$$\tilde{r}_{S-1}P_3 + r_SG_3 = 0. \tag{4.37}$$

Equation (4.31) $\Rightarrow$

$$r_2 = \frac{\eta}{\lambda}r_1.$$

Equation (4.32) $\Rightarrow$

$$r_{j+2} = \left(\frac{\lambda+\eta}{\lambda}\right)r_{j+1} - \left(\frac{\eta}{\lambda}\right)r_j \tag{4.38}$$

for $j = 1, 2, \ldots, s-2$.
Hence equation (4.38) gives

$$r_l = \left(\frac{\eta}{\lambda}\right)^{l-1} r_1 \tag{4.39}$$

for $l = 2, 3, \ldots, s$.
Equation (4.33) gives

$$r_{s+1,0} + r_{s+1,1} = \left(\frac{\eta}{\lambda}\right)^{s} r_1. \tag{4.40}$$

Equation (4.34) gives

$$r_{s+1,0} = r_{s+2,0} \tag{4.41}$$

and

$$\eta r_s - (\lambda + \eta) r_{s+1,1} + \lambda r_{s+2,1} = 0. \tag{4.42}$$

Substituting (4.41) in (4.40), we get

$$r_{s+2,0} + r_{s+1,1} = \left(\frac{\eta}{\lambda}\right)^{s} r_1. \tag{4.43}$$

Equation (4.37) gives

$$r_{S-1,1} = \frac{\lambda}{\eta} r_S. \tag{4.44}$$

Equation (4.36) gives

$$r_{S-1,0} = r_S \tag{4.45}$$

$$\text{and } \eta r_{S-2,1} - (\lambda + \eta) r_{S-1,1} = 0. \tag{4.46}$$

Using (4.44) and (4.46) and after some steps we get

$$r_{S-2,1} = \left[\frac{\lambda}{\eta} + \left(\frac{\lambda}{\eta}\right)^2\right] r_S. \tag{4.47}$$

Equation (4.35) gives

$$-\lambda r_{j+1,0} + \lambda r_{j+2,0} = 0 \tag{4.48}$$

for $j = s+1, s+2, \ldots, S-3$, and

$$\eta r_{j,1} - (\lambda + \eta) r_{j+1,1} + \lambda r_{j+2,1} = 0 \tag{4.49}$$

for $j = s+1, s+2, \ldots, S-3$.
Using (4.41), (4.45) and (4.48) and after some steps, we get

$$r_{l,0} = r_S \tag{4.50}$$

for $l = s+1, s+2, \ldots, S-1$.
Now, using (4.44) and (4.47) in (4.49) we get

$$r_{l,1} = \left[\frac{\lambda}{\eta} + \left(\frac{\lambda}{\eta}\right)^2 + \left(\frac{\lambda}{\eta}\right)^3 + \cdots + \left(\frac{\lambda}{\eta}\right)^{S-l}\right] r_S. \tag{4.51}$$

for $l = s+1, s+2, \ldots, S-3$.
Using (4.44), (4.47) and (4.51) we get

$$r_{l,1} = \left[\frac{\lambda}{\eta} + \left(\frac{\lambda}{\eta}\right)^2 + \left(\frac{\lambda}{\eta}\right)^3 + \cdots + \left(\frac{\lambda}{\eta}\right)^{S-l}\right] r_S \tag{4.52}$$

for $l = s+1, s+2, \ldots, S-1$.
Equations (4.42) and (4.52) give

$$r_s = \left[\left(\frac{\lambda}{\eta}\right) + \left(\frac{\lambda}{\eta}\right)^2 + \left(\frac{\lambda}{\eta}\right)^3 + \cdots + \left(\frac{\lambda}{\eta}\right)^{Q}\right] r_S. \tag{4.53}$$

Hence equation (4.42) gives

$$r_l = \left(\frac{\lambda}{\eta}\right)^{s-l} \left[\left(\frac{\lambda}{\eta}\right) + \left(\frac{\lambda}{\eta}\right)^2 + \left(\frac{\lambda}{\eta}\right)^3 + \cdots + \left(\frac{\lambda}{\eta}\right)^{Q}\right] r_S \tag{4.54}$$

for $l = 1, 2, \ldots, s$.
Using equations (4.50), (4.52) and (4.54), we get the required result.
Next, to find r_S.
Consider the normalizing equation

$$\mathbf{\Delta}\bar{\mathbf{e}} = 1.$$

$$\sum_{l=1}^{s} r_l + \sum_{l=s+1}^{S-1} r_{l,0} + \sum_{l=s+1}^{S-1} r_{l,1} + r_S = 1. \tag{4.55}$$

Let $W_1 = \frac{\lambda}{\eta} + (\frac{\lambda}{\eta})^2 + (\frac{\lambda}{\eta})^3 + \cdots + (\frac{\lambda}{\eta})^Q$.
Also using equations (4.50), (4.52) and (4.54) we get (4.55) as

$$r_S \left[W_1 \left(\sum_{l=1}^{s} \left(\frac{\lambda}{\eta}\right)^{s-l} \right) + Q - 1 + \sum_{l=s+1}^{S-1} \left(\frac{\lambda}{\eta} + \left(\frac{\lambda}{\eta}\right)^2 + \left(\frac{\lambda}{\eta}\right)^3 + \cdots + \left(\frac{\lambda}{\eta}\right)^{S-l} \right) + 1 \right] = 0. \quad (4.56)$$

When $\lambda \neq \eta$ we get,

$$\sum_{l=s+1}^{S-1} \left(\frac{\lambda}{\eta} + \left(\frac{\lambda}{\eta}\right)^2 + \cdots + \left(\frac{\lambda}{\eta}\right)^{S-l} \right) = \frac{\lambda}{\eta} \left(1 - \frac{\lambda}{\eta}\right)^{-1} \left[Q - 1 - \frac{\lambda}{\eta} \frac{\left(1 - \frac{\lambda}{\eta}\right)^{Q-1}}{1 - \frac{\lambda}{\eta}} \right] \quad (4.57)$$

$$\sum_{l=1}^{s} \left(\frac{\lambda}{\eta}\right)^{s-l} = \frac{1 - \left(\frac{\lambda}{\eta}\right)^s}{1 - \frac{\lambda}{\eta}}. \quad (4.58)$$

Also

$$W_1 = \frac{\frac{\lambda}{\eta}\left(1 - \left(\frac{\lambda}{\eta}\right)^Q\right)}{1 - \frac{\lambda}{\eta}}. \quad (4.59)$$

Hence by using equations (4.57), (4.58) and (4.59), we get equation (4.56) as

$$\frac{r_S}{\left(1 - \frac{\lambda}{\eta}\right)^2} \left\{ \left(\frac{\lambda}{\eta}\right)^{Q+s+1} - \left(\frac{\lambda}{\eta}\right)^{s+1} + Q - Q\frac{\lambda}{\eta} - \left(1 - \frac{\lambda}{\eta}\right)^2 \right\} = 1.$$

Therefore,

$$r_S = \frac{\left(1 - \frac{\lambda}{\eta}\right)^2}{\frac{\lambda}{\eta}\left[\left(\frac{\lambda}{\eta}\right)^S - \left(\frac{\lambda}{\eta}\right)^s - Q\right] + Q - \left(1 - \frac{\lambda}{\eta}\right)^2}.$$

Hence the theorem. □

Remark. When $\lambda = \eta$, we get the following:

$$r_l = Qr_S, \quad \text{where } l = 1, 2, \ldots, s.$$
$$r_{l,0} = r_S, \quad \text{where } l = s+1, \ldots, S-1.$$
$$r_{l,1} = (S-l)r_S, \quad \text{where } l = s+1, \ldots, S-1$$
$$\text{and } r_S = [sQ + (Q-1)\left(1 + \frac{Q}{2}\right) + 1]^{-1}.$$

4.2.2.2 Stochastic Decomposition of System States

Let $\bar{\boldsymbol{\pi}}$ be the steady-state probability vector of $\tilde{A}$.

$$\bar{\boldsymbol{\pi}} = (\boldsymbol{\pi}^{(0)}, \boldsymbol{\pi}^{(1)}, \boldsymbol{\pi}^{(2)}, \ldots)$$

where $\boldsymbol{\pi}^{(i)} = (\pi^{(i,1)}, \pi^{(i,2)}, \ldots, \pi^{(i,s)}, \pi^{(i,s+1,0)}, \pi^{(i,s+1,1)}, \ldots, \pi^{(i,S-1,0)}, \pi^{(i,S-1,1)}, \pi^{(i,S)})$ where $i = 0, 1, 2, \ldots$ and $\pi^{(i,j)} = \lim_{t\to\infty} P(X(t) = i, I(t) = j)$ and $\pi^{(i,j,k)} = \lim_{t\to\infty} P(X(t) = i, I(t) = j, K(t) = k)$.

$\pi^{(i,j)}$ is the steady-state probability for the state (i, j) and $\pi^{(i,j,k)}$ is the steady-state probability for the state (i, j, k).

We claim that

$$\boldsymbol{\pi}^{(i)} = K\rho^i \boldsymbol{\Delta}, i \geq 0 \tag{4.60}$$

where $\boldsymbol{\Delta} = (r_1, r_2, \ldots, r_s, r_{s+1,0}, r_{s+1,1}, \ldots, r_{S-1,0}, r_{S-1,1}, r_S)$ is the steady-state probability vector when the service time is negligible, K is a constant to be determined and $\rho = \frac{\lambda}{\mu}$.

Proof. We have $\bar{\boldsymbol{\pi}}\tilde{A} = \mathbf{0}$ and $\bar{\boldsymbol{\pi}}\bar{\mathbf{e}} = 1$.

$$\bar{\boldsymbol{\pi}}\tilde{A} = \mathbf{0} \Rightarrow (\boldsymbol{\pi}^{(0)}, \boldsymbol{\pi}^{(1)}, \boldsymbol{\pi}^{(2)}, \ldots) \begin{bmatrix} B_1 & A_0 & & \\ A_2 & A_1 & A_0 & \\ & A_2 & A_1 & A_0 \\ & \ddots & \ddots & \ddots \end{bmatrix} = \mathbf{0}$$

which gives

$$\boldsymbol{\pi}^{(0)} B_1 + \boldsymbol{\pi}^{(1)} A_2 = \mathbf{0}. \tag{4.61}$$
$$\boldsymbol{\pi}^{(i+2)} A_2 + \boldsymbol{\pi}^{(i+1)} A_1 + \boldsymbol{\pi}^{(i)} A_0 = \mathbf{0}. \tag{4.62}$$

$i = 0, 1, \ldots$.

When (4.60) is true, we get from (4.61),

$$K\rho^0 \mathbf{\Delta} B_1 + K\rho \mathbf{\Delta} A_2 = \mathbf{0}.$$

$$\text{That is} \quad K\mathbf{\Delta}(B_1 + \rho A_2) = \mathbf{0}.$$

$$\text{That is} \quad \mathbf{\Delta}\tilde{Q} = \mathbf{0},$$

which is true since $\mathbf{\Delta} = (r_1, r_2, \ldots, r_S)$ is the steady-state probability vector corresponding to the generator $\tilde{Q}$ associated with the Markovian chain of the inventory process under consideration when service time is negligible. When (4.60) is true, we get from (4.62),

$$K\rho^{i+2}\mathbf{\Delta} A_2 + K\rho^{i+1}\mathbf{\Delta} A_1 + K\rho^i \mathbf{\Delta} A_0 = \mathbf{0}, \quad i = 0, 1, 2, \ldots$$

$$\text{That is} \quad \mathbf{\Delta}(\rho A_2 + A_1 + \frac{1}{\rho} A_0) = \mathbf{0}.$$

$$\text{That is} \quad \mathbf{\Delta}\tilde{Q} = \mathbf{0},$$

which is true, by following the same argument given above.

Hence, the stochastic decomposition of system states is verified. □

4.2.2.3 Determination of K

We have $\bar{\boldsymbol{\pi}}\bar{\mathbf{e}} = 1$. That is

$$\sum_{i=0}^{\infty}\sum_{j=1}^{s} \pi^{(i,j)} + \sum_{i=0}^{\infty}\sum_{j=s+1}^{S-1} \pi^{(i,j,0)} + \sum_{i=0}^{\infty}\sum_{j=s+1}^{S-1} \pi^{(i,j,1)} + \sum_{i=0}^{\infty} \pi^{(i,S)} = 1.$$

That is,

$$K\left[\sum_{i=0}^{\infty}\rho^i\sum_{j=1}^{s} r_j + \sum_{i=0}^{\infty}\rho^i\sum_{j=s+1}^{S-1}(r_{j,0} + r_{j,1}) + \sum_{i=0}^{\infty}\rho^i r_S\right] = 1.$$

Therefore $K\sum_{i=0}^{\infty}\rho^i = 1$.

Therefore $K = 1 - \rho$ where $\rho = \frac{\lambda}{\mu}$.

4.2.2.4 Explicit Solution

From the above discussions, we can write the steady-state probability vector explicitly as given in the following theorem:

Theorem 4.2.4. *The steady-state probability vector* $\bar{\boldsymbol{\pi}}$ *of* $\tilde{A}$ *partitioned as*

$$\bar{\boldsymbol{\pi}} = (\boldsymbol{\pi}^{(0)}, \boldsymbol{\pi}^{(1)}, \boldsymbol{\pi}^{(2)}, \dots),$$

where each $\boldsymbol{\pi}^{(i)}$, $i = 0, 1, 2, \dots$ *again partitioned as* $\boldsymbol{\pi}^{(i)} = (\pi^{(i,1)}, \pi^{(i,2)}, \dots,$ $\pi^{(i,s)}$, $\pi^{(i,s+1,0)}$, $\pi^{(i,s+1,1)} \dots$, $\pi^{(i,S-1,0)}$, $\pi^{(i,S-1,1)}$, $\pi^{(i,S)})$ *are obtained by*

$$\boldsymbol{\pi}^{(i)} = (1-\rho)\rho^i \boldsymbol{\Delta}, \quad i \geq 0 \tag{4.63}$$

where $\rho = \lambda/\mu$ *and* $\boldsymbol{\Delta} = (r_1, r_2, \dots, r_s, r_{s+1,0}, r_{s+1,1}, \dots, r_{S-1,0},\ r_{S-1,1}, r_S)$ *be the steady state probability vector when the service time is negligible.*

$\boldsymbol{\Delta}$ *can be obtained from:*

$$r_l = \left(\frac{\lambda}{\eta}\right)^{s-l} \left(\frac{\lambda}{\eta} + \left(\frac{\lambda}{\eta}\right)^2 + \left(\frac{\lambda}{\eta}\right)^3 + \cdots + \left(\frac{\lambda}{\eta}\right)^Q\right) r_S$$

where $l = 1, 2, \dots, s$.

$$r_{l,0} = r_S$$

where $l = s+1, s+2, \dots, S-1$.

$$r_{l,1} = \left(\frac{\lambda}{\eta} + \left(\frac{\lambda}{\eta}\right)^2 + \left(\frac{\lambda}{\eta}\right)^3 + \cdots + \left(\frac{\lambda}{\eta}\right)^{S-l}\right) r_S$$

where $l = s+1, s+2, \dots, S-1$ *and when* $\lambda \neq \eta$,

$$r_S = \frac{\left(1 - \frac{\lambda}{\eta}\right)^2}{\frac{\lambda}{\eta}\left[\left(\frac{\lambda}{\eta}\right)^S - \left(\frac{\lambda}{\eta}\right)^s - Q\right] + Q - \left(1 - \frac{\lambda}{\eta}\right)^2}.$$

The result indicated by (4.63) not only tells us that the original system possess stochastic decomposition but also the important fact that the system state distribution is the product of the distribution of its marginals: one component is the classical $M/M/1$ whose long run distribution for i customers in the system is $(1-\rho)\rho^i$, $i \geq 0$ and the other factor is the probability of j items in the inventory. Next we turn to find out how the system performs. The measures given in the following are pointers to the system performance.

4.2.3 System Performance Measures

When $\lambda \neq \eta$, we get the following:

(a) Expected number of customers in the system,

$$\begin{aligned} E_i &= \sum_{i=0}^{\infty} i(1-\rho)\rho^i \\ &= \frac{\rho}{1-\rho} \\ &= \frac{\lambda}{\mu-\lambda}. \end{aligned}$$

(b) Expected inventory held in the system,

$$\begin{aligned} E(I) = &\sum_{i=0}^{\infty}\sum_{j=1}^{s} j\pi^{(i,j)} + \sum_{i=0}^{\infty}\sum_{j=s+1}^{S-1} j(\pi^{(i,j,0)} + \pi^{(i,j,1)}) \\ &+ \sum_{i=0}^{\infty} S\pi^{(i,S)} \\ = &\sum_{i=0}^{\infty}(1-\rho)\rho^i \left[\sum_{j=1}^{s} jr_j + \sum_{j=s+1}^{S-1} j(r_{j,0}+r_{j,1}) + Sr_S\right] \\ = &\sum_{j=1}^{s} jr_j + \sum_{j=s+1}^{S-1} j(r_{j,0}+r_{j,1}) + Sr_S. \end{aligned} \tag{4.64}$$

$$\begin{aligned} \sum_{j=1}^{s} jr_j &= \sum_{j=1}^{s} j\left(\frac{\lambda}{\eta}\right)^{s-j}\left(\frac{\lambda}{\eta} + \left(\frac{\lambda}{\eta}\right)^2 + \left(\frac{\lambda}{\eta}\right)^3 + \cdots + \left(\frac{\lambda}{\eta}\right)^Q\right) r_S \\ &= \frac{\frac{\lambda}{\eta}\left(1-\left(\frac{\lambda}{\eta}\right)^Q\right)}{1-\frac{\lambda}{\eta}} \sum_{j=1}^{s} j\left(\frac{\lambda}{\eta}\right)^{s-j} r_S. \end{aligned} \tag{4.65}$$

$$\begin{aligned} \sum_{j=1}^{s} j\left(\frac{\lambda}{\eta}\right)^{s-j} = &\left(\frac{\lambda}{\eta}\right)^{s-1} + 2\left(\frac{\lambda}{\eta}\right)^{s-2} + 3\left(\frac{\lambda}{\eta}\right)^{s-3} + \ldots \\ &+ (s-1)\frac{\lambda}{\eta} + s. \end{aligned}$$

Let

$$S_1 = \left(\frac{\lambda}{\eta}\right)^{s-1} + 2\left(\frac{\lambda}{\eta}\right)^{s-2} + 3\left(\frac{\lambda}{\eta}\right)^{s-3} + \cdots + (s-1)\frac{\lambda}{\eta} + s.$$

$$\left(\frac{\lambda}{\eta}-1\right)S_1=\left(\frac{\lambda}{\eta}\right)^s+\left(\frac{\lambda}{\eta}\right)^{s-1}+\left(\frac{\lambda}{\eta}\right)^{s-2}+\cdots+\left(\frac{\lambda}{\eta}\right)^2+\left(\frac{\lambda}{\eta}\right)-s.$$

$$S_1=\frac{\frac{\lambda}{\eta}\left(1-\left(\frac{\lambda}{\eta}\right)^s\right)}{\left(1-\frac{\lambda}{\eta}\right)\left(\frac{\lambda}{\eta}-1\right)}-\frac{s}{\frac{\lambda}{\eta}-1}.$$

Therefore, equation (4.65) becomes

$$\sum_{j=1}^{s} jr_j=\frac{\frac{\lambda}{\eta}}{\left(1-\frac{\lambda}{\eta}\right)^3}\left[1-\left(\frac{\lambda}{\eta}\right)^Q\right]\left[s\left(1-\frac{\lambda}{\eta}\right)-\frac{\lambda}{\eta}\left[1-\left(\frac{\lambda}{\eta}\right)^s\right]r_S\right].$$

By expanding and then simplifying, we get

$$\sum_{j=1}^{s} jr_j=\frac{\left[s-(s+1)\frac{\lambda}{\eta}+\left(\frac{\lambda}{\eta}\right)^{s+1}-\left(\frac{\lambda}{\eta}\right)^{S+1}-s\left(\frac{\lambda}{\eta}\right)^Q+(s+1)\left(\frac{\lambda}{\eta}\right)^{Q+1}\right]\frac{\lambda}{\eta}}{\left(1-\frac{\lambda}{\eta}\right)\left\{\frac{\lambda}{\eta}\left[\left(\frac{\lambda}{\eta}\right)^S-\left(\frac{\lambda}{\eta}\right)^s-Q\right]+Q-\left(1-\frac{\lambda}{\eta}\right)^2\right\}}. \tag{4.66}$$

$$\begin{aligned}\sum_{j=s+1}^{S-1} & j(r_{j,0}+r_{j,1})\\ &=\sum_{j=s+1}^{S-1} j\left(r_S+\left[\frac{\lambda}{\eta}+\left(\frac{\lambda}{\eta}\right)^2+\left(\frac{\lambda}{\eta}\right)^3+\cdots+\left(\frac{\lambda}{\eta}\right)^{S-j}\right]r_S\right)\\ &=\frac{r_S}{1-\frac{\lambda}{\eta}}[(Q-1)\left(s+\frac{Q}{2}\right)-S_2]\end{aligned} \tag{4.67}$$

where

$$S_2=(s+1)\left(\frac{\lambda}{\eta}\right)^Q+(s+2)\left(\frac{\lambda}{\eta}\right)^{Q-1}+(s+3)\left(\frac{\lambda}{\eta}\right)^{Q-2}+\cdots+(S-1)\left(\frac{\lambda}{\eta}\right)^2.$$

$$\begin{aligned}(1-\frac{\eta}{\lambda})S_2=(s+1)\left(\frac{\lambda}{\eta}\right)^Q&\left[\left(\frac{\lambda}{\eta}\right)^{Q-1}+\left(\frac{\lambda}{\eta}\right)^{Q-2}+\cdots+\left(\frac{\lambda}{\eta}\right)^2+\frac{\lambda}{\eta}\right]\\ &-(s+Q)\frac{\lambda}{\eta}.\end{aligned}$$

$$S_2=\frac{(s+1)\left(\frac{\lambda}{\eta}\right)^Q}{1-\frac{\eta}{\lambda}}+\frac{\frac{\lambda}{\eta}\left(1-\left(\frac{\lambda}{\eta}\right)^{Q-1}\right)}{\left(1-\frac{\lambda}{\eta}\right)(1-\frac{\eta}{\lambda})}-\frac{(s+Q)\frac{\lambda}{\eta}}{1-\frac{\eta}{\lambda}}.$$

Therefore, equation (4.67) becomes

$$\sum_{j=s+1}^{S-1} j(r_{j,0}+r_{j,1})$$
$$= \frac{\left(1-\frac{\lambda}{\eta}\right)}{\frac{\lambda}{\eta}\left[\left(\frac{\lambda}{\eta}\right)^S-\left(\frac{\lambda}{\eta}\right)^s-Q\right]+Q-\left(1-\frac{\lambda}{\eta}\right)^2}\left[(Q-1)(s+\frac{Q}{2})\right.$$
$$\left.+\frac{\frac{\lambda}{\eta}}{1-\frac{\eta}{\lambda}}\left(\left(\frac{\lambda}{\eta}\right)^{Q-1}\left(\frac{1}{1-\frac{\lambda}{\eta}}-(s+1)\right)+s+Q-\frac{1}{1-\frac{\lambda}{\eta}}\right)\right]. \tag{4.68}$$

Using equations (4.66) and (4.68), we get equation (4.64) as

$$E(I)=\left\{\frac{\frac{\lambda}{\eta}}{1-\frac{\lambda}{\eta}}[s-(s+1)\frac{\lambda}{\eta}+(\frac{\lambda}{\eta})^{s+1}-(\frac{\lambda}{\eta})^{S+1}-s(\frac{\lambda}{\eta})^Q\right.$$
$$+(s+1)(\frac{\lambda}{\eta})^{Q+1}]+(1-\frac{\lambda}{\eta})[(Q-1)(s+\frac{Q}{2})$$
$$+\frac{\frac{\lambda}{\eta}}{1-\frac{\eta}{\lambda}}[(\frac{\lambda}{\eta})^{Q-1}((1-\frac{\lambda}{\eta})^{-1}-(s+1))+(s+Q)-(1-\frac{\lambda}{\eta})^{-1}]]$$
$$\left.+S(1-\frac{\lambda}{\eta})^2\right\}\times[\frac{\lambda}{\eta}[(\frac{\lambda}{\eta})^S-(\frac{\lambda}{\eta})^s-Q]+Q-(1-\frac{\lambda}{\eta})^2]^{-1}.$$

(c) Expected rate at which production process is switched 'on',

$$\begin{aligned}R_{\text{ON}} &= \mu\sum_{i=1}^{\infty}\pi^{(i,s+1,0)}\\ &= \mu(1-\rho)\sum_{i=1}^{\infty}\rho^i r_{s+1,0}\\ &= \frac{\lambda(1-\frac{\lambda}{\eta})^2}{\frac{\lambda}{\eta}[(\frac{\lambda}{\eta})^S-(\frac{\lambda}{\eta})^s-Q]+Q-(1-\frac{\lambda}{\eta})^2}.\end{aligned}$$

(d) Expected production rate,

$$R_P=\Big(\sum_{i=0}^{\infty}\sum_{j=1}^{s}\pi^{(i,j)}+\sum_{i=0}^{\infty}\sum_{j=s+1}^{S-1}\pi^{(i,j,1)}\Big)\eta$$

$$= \sum_{i=0}^{\infty}(1-\rho)\rho^i \left[\sum_{j=1}^{s} r_j + \sum_{j=s+1}^{S-1} r_{j,1}\right]\eta$$

$$= \Big[\sum_{j=1}^{s} r_j + \sum_{j=s+1}^{S-1} r_{j,1}\Big]\eta. \tag{4.69}$$

$$\sum_{j=1}^{s} r_j = \sum_{j=1}^{s}\left(\frac{\lambda}{\eta}\right)^{s-j}\left(\frac{\lambda}{\eta}+\left(\frac{\lambda}{\eta}\right)^2+\left(\frac{\lambda}{\eta}\right)^3+\cdots+\left(\frac{\lambda}{\eta}\right)^Q\right) r_S$$

$$= \frac{\left(\frac{\lambda}{\eta}\right)\left[1-\left(\frac{\lambda}{\eta}\right)^s-\left(\frac{\lambda}{\eta}\right)^Q+\left(\frac{\lambda}{\eta}\right)^S\right]}{\frac{\lambda}{\eta}\left[\left(\frac{\lambda}{\eta}\right)^S-\left(\frac{\lambda}{\eta}\right)^s-Q\right]+Q-\left(1-\frac{\lambda}{\eta}\right)^2}. \tag{4.70}$$

$$\sum_{j=s+1}^{S-1} r_{j,1} = r_S\sum_{j=s+1}^{S-1}\left[\frac{\lambda}{\eta}+\left(\frac{\lambda}{\eta}\right)^2+\left(\frac{\lambda}{\eta}\right)^3+\cdots+\left(\frac{\lambda}{\eta}\right)^{S-j}\right]$$

$$= \frac{(Q-1)\frac{\lambda}{\eta}-\left(\frac{\lambda}{\eta}\right)^2+\left(\frac{\lambda}{\eta}\right)^{Q+1}}{\frac{\lambda}{\eta}\left[\left(\frac{\lambda}{\eta}\right)^S-\left(\frac{\lambda}{\eta}\right)^s-Q\right]+Q-\left(1-\frac{\lambda}{\eta}\right)^2}. \tag{4.71}$$

Using equation (4.70) and (4.71), we get equation (4.69) as

$$R_P = \frac{\eta[(\frac{\lambda}{\eta})^{S+1}-(\frac{\lambda}{\eta})^{s+1}+Q(\frac{\lambda}{\eta})-(\frac{\lambda}{\eta})^2]}{\frac{\lambda}{\eta}[(\frac{\lambda}{\eta})^S-(\frac{\lambda}{\eta})^s-Q]+Q-(1-\frac{\lambda}{\eta})^2}.$$

(e) Expected local purchase rate,

$$R_{\mathrm{LP}} = \mu\sum_{i=1}^{\infty}\pi^{(i,1)}$$

$$= \mu(1-\rho)\left(\sum_{i=1}^{\infty}\rho^i\right) r_1$$

$$= \frac{\lambda[(\frac{\lambda}{\eta})^s-(\frac{\lambda}{\eta})^S+(\frac{\lambda}{\eta})^{s+1}+(\frac{\lambda}{\eta})^{S+1}]}{\frac{\lambda}{\eta}[(\frac{\lambda}{\eta})^S-(\frac{\lambda}{\eta})^s-Q]+Q-(1-\frac{\lambda}{\eta})^2}.$$

(f) Mean waiting time of customers in the system

$$W_S = \frac{L}{\lambda}$$

where $L =$ Expected number of customers in the system

$$
\begin{aligned}
&= \sum_{i=0}^{\infty}\sum_{j=1}^{s} i\pi^{(i,j)} + \sum_{i=0}^{\infty}\sum_{j=s+1}^{S-1} i(\pi^{(i,j,0)} + \pi^{(i,j,1)}) + \sum_{i=0}^{\infty} i\pi^{(i,S)} \\
&= \sum_{i=0}^{\infty} i(1-\rho)\rho^i \sum_{j=1}^{s} r_j + \sum_{i=0}^{\infty} i(1-\rho)\rho^i \sum_{j=s+1}^{S-1} (r_{j,0} + r_{j,1}) \\
&\quad + \sum_{i=0}^{\infty} i(1-\rho)\rho^i r_S \\
&= \frac{\rho}{1-\rho}.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
W_S &= \frac{L}{\lambda} \\
&= \frac{1}{\lambda}\left(\frac{\rho}{1-\rho}\right) \\
&= \frac{1}{\mu - \lambda}.
\end{aligned}
$$

Remark. When $\lambda = \eta$, we get the following performance measures:

(a) $E(I) = [\frac{Qs(s+1)+(S+1)(Q-1)(S+s)}{2} - \sum_{j=s+1}^{S-1} j^2 + S]r_S$.

(b) $R_P = \frac{Q}{2}[s(s+1) + Q - 1]\eta r_S$.

(c) $R_{\text{ON}} = \lambda r_S$.

(d) $R_{\text{LP}} = \lambda Q r_S$.

4.2.4 Cost Analysis

Next, we find a cost function. Let the various costs involved in the model be as given below:

C_H : Holding cost per unit time per unit inventory
C_P : Cost of production per unit time per unit inventory
C_S : Fixed cost for starting the production
C_{LP} : Cost per unit item per unit time under local purchase

C_W : Waiting time cost per customer per unit time.

The total expected cost per unit time,

$$\text{TEC} = C_H E(I) + C_P R_P + C_S R_{\text{ON}} + C_{\text{LP}} R_{\text{LP}} + C_W W_S.$$

When $\lambda \neq \eta$, we get,

$$\begin{aligned}
\text{TEC} = & \left[C_H \left\{ \frac{\lambda}{\eta}\left(1-\frac{\lambda}{\eta}\right)^{-1} \left[s-(s+1)\frac{\lambda}{\eta} + \left(\frac{\lambda}{\eta}\right)^{s+1} - \left(\frac{\lambda}{\eta}\right)^{S+1} \right.\right.\right. \\
& \left. -s\left(\frac{\lambda}{\eta}\right)^{Q} + (s+1)\left(\frac{\lambda}{\eta}\right)^{Q+1} \right] \\
& + \left(1-\frac{\lambda}{\eta}\right) \left[(Q-1)(s+\frac{Q}{2}) + \frac{\lambda}{\eta}\left(1-\frac{\eta}{\lambda}\right)^{-1} \right. \\
& \left.\left.\left[\left(\frac{\lambda}{\eta}\right)^{Q-1} \left(\left(1-\frac{\lambda}{\eta}\right)^{-1} - (s+1)\right) + S - \left(1-\frac{\lambda}{\eta}\right)^{-1} \right]\right] \right\} \\
& + (C_H S + C_S\lambda)\left(1-\frac{\lambda}{\eta}\right)^{2} + (C_P\eta + C_{\text{LP}}\lambda)\left[\left(\frac{\lambda}{\eta}\right)^{S+1} - \left(\frac{\lambda}{\eta}\right)^{s+1}\right] \\
& \left. + C_P\eta\left(Q\frac{\lambda}{\eta} - \left(\frac{\lambda}{\eta}\right)^{2}\right) + C_{\text{LP}}\lambda\left(\left(\frac{\lambda}{\eta}\right)^{s} - \left(\frac{\lambda}{\eta}\right)^{S}\right)\right] \\
& \times \left[\frac{\lambda}{\eta}\left[\left(\frac{\lambda}{\eta}\right)^{S} - \left(\frac{\lambda}{\eta}\right)^{s} - Q\right] + Q - \left(1-\frac{\lambda}{\eta}\right)^{2}\right]^{-1}.
\end{aligned} \tag{4.72}$$

Remark. When $\lambda = \eta$, we get the cost function as

$$\begin{aligned}
\text{TEC} = & \left\{ C_H \left[Q\frac{s(s+1)}{2} + (S+1)(Q-1)\frac{(S+s)}{2} - \sum_{j=s+1}^{S-1} j^2 + S \right] \right. \\
& \left. + C_P\frac{Q\eta}{2}[s(s+1)+Q-1] + C_S\lambda + C_{\text{LP}}\lambda Q \right\} r_S + C_W\left(\frac{1}{\mu-\lambda}\right).
\end{aligned}$$

To verify the convexity of the above cost functions, the derivative with respect to s or S may be computed, and then equate it to zero. Nevertheless, solving it is a laborious task. Since analysis of TEC as a function of s or S is quite complex, we give a few numerical illustrations.

4.2.4.1 Numerical Analysis

Case 1: Analysis of TEC as function of s (when $\lambda \neq \eta$).

Input Data:

$C_H = 50$, $C_P = 200$, $C_S = 2000$, $C_{\text{LP}} = 220$, $C_W = 2250$, $S = 25$, $\lambda = 2$, $\mu = 3$, $\eta = 2.5$.

Table 4.1: Effect of s on TEC ($\lambda \neq \eta$)

s	TEC
2	5182.2
3	5160.6
4	5149.6
5	**5147.0**
6	5151.3
7	5161.2
8	5175.9
9	5194.6
10	5216.9

Table 4.1 shows that as s increases from 2 to 10, the TEC values decrease, reach a minimum at $s = 5$ and then increase. Hence it is numerically verified that TEC function is convex with respect to s.

Case 2: Analysis of TEC as a function of S (when $\lambda \neq \eta$).

Input Data:

$C_H = 50$, $C_P = 200$, $C_S = 2000$, $C_{\text{LP}} = 220$, $C_W = 2250$, $s = 2$, $\lambda = 2$, $\mu = 3$, $\eta = 2.5$.

Table 4.2 shows that as S increases from 10 to 18, the TEC values decrease, reach a minimum at $S = 15$ and then increase. Hence it is numerically verified that TEC function is convex with respect to S.

Case 3: Analysis of TEC when s and S are varied simultaneously (when $\lambda \neq \eta$).

Input Data:

$C_H = 50$, $C_P = 200$, $C_S = 2000$, $C_{\text{LP}} = 220$, $C_W = 2250$, $\lambda = 2$, $\eta = 2.5$, $\mu = 3$.

Table 4.2: Effect of S on TEC ($\lambda \neq \eta$)

S	TEC
10	5147.5
11	5125.0
12	5109.1
13	5098.5
14	5092.3
15	**5089.9**
16	5090.6
17	5094.0
18	5099.6

Table 4.3: Effect of simultaneous variation of (s, S) on TEC ($\lambda \neq \eta$)

(s, S)	TEC
(2,25)	5182.2
(3,26)	**5178.1**
(4,27)	5187.2
(5, 28)	5206.5
(6,29)	5233.5
(7,30)	5266.6

Table 4.3 shows that as (s, S) values increase simultaneously, TEC values decrease, reach a minimum, at the values $(3, 26)$ of (s, S) and then increase. Hence it is numerically verified that TEC function is convex.

Case 4: Analysis of TEC as a function of S when $\lambda = \eta$.

Input Data:

$C_H = 50$, $C_P = 200$, $C_S = 2000$, $C_{\text{LP}} = 220$, $C_W = 2250$, $\lambda = \eta = 1.5$, $\mu = 6$, $s = 5$.

Table 4.4: Effect of S on TEC ($\lambda = \eta$)

S	TEC
28	432.9668
29	394.0952
30	355.5556
31	317.3181
32	279.3567
33	241.6484

Table 4.4 shows that as S increases, TEC function is monotonically decreasing and hence convex.

4.3 Model II: Local Purchase of N Units (where $2 \leq N < s$)

4.3.1 Model Formulation and Analysis

Let $X(t)$ = Number of customers in the system at time t,
$I(t)$ = Inventory level at time t and
$K(t)$ = Status of the production process.

$$K(t) = \begin{cases} 1, & \text{if production process is 'on' at time } t \\ 0, & \text{if production process is 'off' at time } t. \end{cases}$$

$\{(X(t), I(t), K(t)), t \geq 0\}$ is a CTMC with state space

$$\{(i,j) | i \geq 0; 1 \leq j \leq s\} \\ \cup \{(i,j,k) | i \geq 0; s+1 \leq j \leq S-1; k = 0,1\} \\ \cup \{(i,S) | i \geq 0\}.$$

$K(t) = 1$, when $1 \leq I(t) \leq s$, $K(t) = 0$, when $I(t) = S$ and $K(t)$ is either 0 or 1, when $s+1 \leq I(t) \leq S-1$.

4.3.1.1 Infinitesimal Generator $\tilde{A}$

The infinitesimal generator of the process under consideration is obtained in terms of submatrices as follows:

$$\tilde{A} = \begin{bmatrix} B_1 & A_0 & & \\ A_2 & A_1 & A_0 & \\ & A_2 & A_1 & A_0 \\ & \ddots & \ddots & \ddots \end{bmatrix}$$

where $B_1 = \begin{bmatrix} G_1 & \eta & & & & & & & & & \\ & G_1 & \eta & & & & & & & & \\ & & \ddots & \ddots & & & & & & & \\ & & & G_1 & \eta & & & & & & \\ & & & & G_1 & P_1 & & & & & \\ & & & & & G_2 & P_2 & & & & \\ & & & & & & \ddots & \ddots & & & \\ & & & & & & & G_2 & P_2 & & \\ & & & & & & & & G_2 & P_3 \\ & & & & & & & & & G_3 \end{bmatrix}$

where $G_1 = [-(\lambda + \eta)]$,

$G_2 = \begin{bmatrix} -\lambda & \\ & -(\lambda + \eta) \end{bmatrix}$,

$G_3 = [-\lambda]$, $P_1 = [0 \quad \eta]$,

$P_2 = \begin{bmatrix} 0 & 0 \\ 0 & \eta \end{bmatrix}$ and $P_3 = \begin{bmatrix} 0 \\ \eta \end{bmatrix}$.

B is a square matrix of order $2S - s - 1$.

$A_0 = \lambda I_{2S-s-1}$,

$$A_1 = \begin{bmatrix} \tilde{G}_1 & \eta & & & & & & & & & \\ & \tilde{G}_1 & \eta & & & & & & & & \\ & & \ddots & \ddots & & & & & & & \\ & & & \tilde{G}_1 & \eta & & & & & & \\ & & & & \tilde{G}_1 & P_1 & & & & & \\ & & & & & \tilde{G}_2 & P_2 & & & & \\ & & & & & & \ddots & \ddots & & & \\ & & & & & & & \tilde{G}_2 & P_2 & & \\ & & & & & & & & \tilde{G}_2 & P_3 \\ & & & & & & & & & \tilde{G}_3 \end{bmatrix}$$

where $\tilde{G}_1 = [-(\lambda + \eta + \mu)]$, $\tilde{G}_2 = \begin{bmatrix} -(\lambda + \mu) & \\ & -(\lambda + \eta + \mu) \end{bmatrix}$ and $\tilde{G}_3 = [-(\lambda + \mu)]$.

A_1 is a square matrix of order $2S - s - 1$.

$$A_2 = \begin{bmatrix} 0 & & & \mu & & & & & & \\ \mu & & & & & & & & & \\ & \mu & & & & & & & & \\ & & \ddots & & & & & & & \\ & & & \mu & & & & & & \\ & & & & H_1 & & & & & \\ & & & & & H_2 & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & H_2 & & \\ & & & & & & & & H_3 & 0 \end{bmatrix}.$$

A_2 is a square matrix of order $2S - s - 1$.

4.3.1.2 Steady-State Analysis

Let $A = A_0 + A_1 + A_2$ be the generator matrix and is obtained as

$$A = \begin{bmatrix} \hat{G}_1 & \eta & & & \mu & & & & & & \\ \mu & \hat{G}_1 & \eta & & & & & & & & \\ & \mu & \hat{G}_1 & \eta & & & & & & & \\ & & \ddots & \ddots & \ddots & & & & & & \\ & & & \mu & \hat{G}_1 & \eta & & & & & \\ & & & & \mu & \hat{G}_1 & P_1 & & & & \\ & & & & & H_1 & \hat{G}_2 & P_2 & & & \\ & & & & & & H_2 & \hat{G}_2 & P_2 & & \\ & & & & & & \ddots & \ddots & \ddots & & \\ & & & & & & & H_2 & \hat{G}_2 & P_2 & \\ & & & & & & & & H_2 & \hat{G}_2 & P_3 \\ & & & & & & & & & H_3 & \hat{G}_3 \end{bmatrix}$$

where $\hat{G}_1 = [-(\mu + \eta)]$, $\hat{G}_2 = \begin{bmatrix} -\mu & 0 \\ 0 & -(\mu + \eta) \end{bmatrix}$, $\hat{G}_3 = [-\mu]$, $H_1 = \begin{bmatrix} \mu \\ \mu \end{bmatrix}$, $H_2 = \mu I_2$ and $H_3 = [\mu \quad 0]$.

A is a square matrix of order $2S - s - 1$.

Theorem 4.3.1. *The steady-state probability vector* $\mathbf{\Phi}$ *of* A *partitioned as*

$$\mathbf{\Phi} = (\phi_1, \phi_2, \ldots, \phi_N, \ldots, \phi_{s-1}, \phi_s, \tilde{\phi}_{s+1}, \tilde{\phi}_{s+2}, \ldots, \tilde{\phi}_{S-1}, \phi_S)$$

where

$$\tilde{\phi}_{l_1} = (\phi_{l_1,0}, \phi_{l_1,1}); l_1 = s+1, s+2, \ldots, S-1$$

and when $\mu \neq \eta$, $\tilde{\phi}_l$ *is given by*

$$\begin{aligned}
\phi_l &= \frac{(\mu^l - \eta^l)}{\mu^{l-1}(\mu-\eta)} \frac{\mu^s(\mu^Q - \eta^Q)}{\eta^Q(\mu^s - \eta^s)} \phi_S, \\
&\quad l = 1, 2, \ldots, s \text{ (and } l \neq N+1) \\
\phi_{N+1} &= \frac{\eta}{\mu^N} \frac{(\mu^N - \eta^N)}{\mu - \eta} \mu^s \frac{(\mu^Q - \eta^Q)}{\eta^Q(\mu^s - \eta^s)} \phi_S \\
\phi_{l,0} &= \phi_S, \quad l = s+1, \ldots, S-1 \\
\phi_{l,1} &= \frac{\mu(\mu^{S-l} - \eta^{S-l})}{\eta^{S-l}(\mu-\eta)} \phi_S, \\
&\quad l = s+1, \ldots, S-1
\end{aligned}$$

and ϕ_S *can be obtained by using* $\mathbf{\Phi}\bar{\mathbf{e}} = 1$ *as*

$$\phi_S = \left\{ \frac{1}{\mu-\eta} \left[\frac{(\chi_1^Q - 1)\chi_1^s}{(\chi_1^s - 1)} (\mu(s-1) + \eta) - \mu + (1-Q)\eta \right] + 1 \right\}^{-1}$$

where $\chi_1 = \mu/\eta$.

Proof. We have $\mathbf{\Phi}A = \mathbf{0}$ and $\mathbf{\Phi}\bar{\mathbf{e}} = 1$.

$$\mathbf{\Phi}A = \mathbf{0} \Rightarrow$$

$$\phi_1 \times -(\mu+\eta) + \phi_2\mu = 0. \tag{4.73}$$

$$\phi_j\eta + \phi_{j+1} \times -(\mu+\eta) + \phi_{j+2}\mu = 0 \tag{4.74}$$

for $j = 1, 2, \ldots, s-2$ and $j \neq N-1$.

$$\phi_1\mu + \phi_{N-1}\eta + \phi_N \times -(\mu+\eta) + \phi_{N+1}\mu = 0. \tag{4.75}$$

$$\phi_{s-1}\eta + \phi_s \times -(\mu+\eta) + \tilde{\phi}_{s+1}H_1 = 0. \tag{4.76}$$

$$\phi_s P_1 + \tilde{\phi}_{s+1}\hat{G}_2 + \tilde{\phi}_{s+2}H_2 = 0. \tag{4.77}$$

$$\tilde{\phi}_j P_2 + \tilde{\phi}_{j+1}\hat{G}_2 + \tilde{\phi}_{j+2}H_2 = 0. \tag{4.78}$$

for $j = s+1, s+2, \ldots, S-3$.

$$\tilde{\phi}_{S-2}P_2 + \tilde{\phi}_{S-1}\hat{G}_2 + \phi_S H_3 = 0. \tag{4.79}$$

$$\tilde{\phi}_{S-1}P_3 + \phi_S\hat{G}_3 = 0. \tag{4.80}$$

Equation (4.73) gives

$$\phi_2 = \left(1 + \frac{\eta}{\mu}\right)\phi_1 \tag{4.81}$$

Equation (4.74) gives

$$\phi_{j+2} = \left(1 + \frac{\eta}{\mu}\right)\phi_{j+1} - \frac{\eta}{\mu}\phi_j \tag{4.82}$$

for $j = 1, 2, \ldots, s-2$ and $j \neq N-1$.
Equation (4.82) gives

$$\phi_l = \frac{(\mu^l - \eta^l)\phi_1}{\mu^{l-1}(\mu - \eta)} \tag{4.83}$$

for $l = 1, 2, \ldots, s$ and $l \neq N+1$.
Equation (4.75) gives

$$\phi_{N+1} = \left(1 + \frac{\eta}{\mu}\right)\phi_N - \phi_1 - \frac{\eta}{\mu}\phi_{N-1}$$

which reduces to

$$\phi_{N+1} = \frac{\eta}{\mu^N}\frac{\left(\mu^N - \eta^N\right)}{(\mu - \eta)}\phi_1. \tag{4.84}$$

Equation (4.76) gives

$$\phi_{s+1,0} + \phi_{s+1,1} = \left[1 + \frac{\eta}{\mu} + \left(\frac{\eta}{\mu}\right)^2 + \left(\frac{\eta}{\mu}\right)^3 + \cdots + \left(\frac{\eta}{\mu}\right)^s\right]\phi_1. \tag{4.85}$$

Equation (4.77) gives

$$\phi_{s+1,0} = \phi_{s+2,0} \tag{4.86}$$

$$\text{and } \eta\phi_s - (\mu + \eta)\phi_{s+1,1} + \mu\phi_{s+2,1} = 0. \tag{4.87}$$

Using (4.86) in (4.85) we get

$$\phi_{s+2,0} + \phi_{s+1,1} = \left[1 + \frac{\eta}{\mu} + \left(\frac{\eta}{\mu}\right)^2 + \left(\frac{\eta}{\mu}\right)^3 + \cdots + \left(\frac{\eta}{\mu}\right)^s\right]\phi_1. \quad (4.88)$$

Equation (4.80) gives

$$\phi_{S-1,1} = \left(\frac{\mu}{\eta}\right)\phi_S \quad (4.89)$$

Equation (4.79) gives

$$\phi_{S-1,0} = \phi_S \quad (4.90)$$

$$\text{and} \quad \eta\phi_{S-2,1} - (\mu+\eta)\phi_{S-1,1} = 0. \quad (4.91)$$

Equation (4.91) and (4.89) gives

$$\phi_{S-2,1} = \left[\frac{\mu}{\eta} + \left(\frac{\mu}{\eta}\right)^2\right]\phi_S. \quad (4.92)$$

Equation (4.78) gives

$$\mu\phi_{j+1,0} = \mu\phi_{j+2,0} \quad (4.93)$$

for $j = s+1, s+2, \ldots, S-3$ and

$$\eta\phi_{j,1} - (\mu+\eta)\phi_{j+1,1} + \mu\phi_{j+2,1} = 0 \quad (4.94)$$

for $j = s+1, s+2, \ldots, S-3$.
Using (4.93), (4.90) and (4.86) we get

$$\phi_{l,0} = \phi_S \quad (4.95)$$

for $l = s+1, s+2, \ldots, S-1$.
Using (4.89) and (4.92) in (4.94) we get

$$\phi_{l,1} = \left(\frac{\mu}{\eta} + \left(\frac{\mu}{\eta}\right)^2 + \left(\frac{\mu}{\eta}\right)^3 + \cdots + \left(\frac{\mu}{\eta}\right)^{S-l}\right)\phi_S \quad (4.96)$$

for $l = s+1, \ldots, S-3$.

Using (4.89), (4.92) and (4.96), and after some steps, we get

$$\phi_{l,1} = \left(\frac{\mu}{\eta} + \left(\frac{\mu}{\eta}\right)^2 + \left(\frac{\mu}{\eta}\right)^3 + \cdots + \left(\frac{\mu}{\eta}\right)^{S-l}\right)\phi_S \quad (4.97)$$

$$\text{That is} \quad \phi_{l,1} = \frac{\mu}{\eta^{S-1}}\frac{(\mu^{S-l} - \eta^{S-l})}{(\mu-\eta)}\phi_S \quad (4.98)$$

for $l = s+1, \ldots, S-1$.
Equation (4.87) gives

$$\phi_s = \left(1 + \frac{\mu}{\eta}\right)\phi_{s+1,1} - \frac{\mu}{\eta}\phi_{s+2,1}.$$

Also using (4.97) in the above equation, we get

$$\phi_s = \left[\frac{\mu}{\eta} + \left(\frac{\mu}{\eta}\right)^2 + \left(\frac{\mu}{\eta}\right)^3 + \cdots + \left(\frac{\mu}{\eta}\right)^Q\right]\phi_S \tag{4.99}$$

Equation (4.83) and (4.99) gives

$$\phi_1 = \frac{\mu^s(\mu^Q - \eta^Q)}{\eta^Q(\mu^s - \eta^s)}\phi_S. \tag{4.100}$$

Using (4.83) and (4.100), we get

$$\phi_l = \frac{(\mu^l - \eta^l)}{\mu^{l-1}(\mu - \eta)}\frac{\mu^s(\mu^Q - \eta^Q)}{\eta^Q(\mu^s - \eta^s)}\phi_S \tag{4.101}$$

for $l = 1, 2, 3, \ldots, s$ and $l \neq N+1$.

Using (4.100) we get (4.84) as

$$\phi_{N+1} = \frac{\eta}{\mu^N}\frac{(\mu^N - \eta^N)}{\mu - \eta}\frac{\mu^s(\mu^Q - \eta^Q)}{\eta^Q(\mu^s - \eta^s)}\phi_S. \tag{4.102}$$

Hence equations (4.101), (4.102), (4.95) and (4.98) gives the required result when $l = 1, 2, 3, \ldots, S-1$.

Next, to find ϕ_S. Consider the normalizing equation $\mathbf{\Phi}\bar{\mathbf{e}} = 1$. That is,

$$\sum_{\substack{l=1\\(l\neq N+1)}}^{s}\phi_l + \phi_{N+1} + \sum_{l=s+1}^{S-1}\phi_{l,0} + \sum_{l=s+1}^{S-1}\phi_{l,1} + \phi_S = 1. \tag{4.103}$$

$$\sum_{\substack{l=1\\(l\neq N+1)}}^{s}\phi_l = \frac{\mu^s}{\eta^Q}\frac{(\mu^Q - \eta^Q)}{(\mu^s - \eta^s)(\mu - \eta)}\sum_{\substack{l=1\\(l\neq N+1)}}^{s}\frac{(\mu^l - \eta^l)}{\mu^{l-1}}\phi_S. \tag{4.104}$$

$$\sum_{\substack{l=1\\(l\neq N+1)}}^{s}\frac{(\mu^l - \eta^l)}{\mu^{l-1}}\phi_S = \left[\mu - \eta + \frac{\mu^2 - \eta^2}{\mu} + \frac{\mu^3 - \eta^3}{\mu^2} + \cdots + \frac{\mu^N - \eta^N}{\mu^{N-1}}\right.$$

$$+\frac{\mu^{N+2}-\eta^{N+2}}{\mu^{N+1}}+\cdots+\frac{\mu^{s}-\eta^{s}}{\mu^{s-1}}\Bigg]\phi_S$$
$$=\left[(s-1)\mu-\frac{\eta\mu(\mu^s-\eta^s)}{\mu^s(\mu-\eta)}+\eta\left(\frac{\eta}{\mu}\right)^N\right]\phi_S.$$

Therefore (4.104) becomes

$$\sum_{\substack{l=1\\(l\neq N+1)}}^{s}\phi_l=\frac{\mu^s}{\eta^Q}\frac{(\mu^Q-\eta^Q)}{(\mu^s-\eta^s)(\mu-\eta)}\left[(s-1)\mu-\frac{\eta\mu}{\mu^s}\frac{(\mu^s-\eta^s)}{(\mu-\eta)}+\eta\left(\frac{\eta}{\mu}\right)^N\right]\phi_S. \tag{4.105}$$

$$\sum_{l=s+1}^{S-1}\phi_{l,0}=(Q-1)\phi_S. \tag{4.106}$$

$$\sum_{l=s+1}^{S-1}\phi_{l,1}=\sum_{l=s+1}^{S-1}\frac{\mu(\mu^{S-l}-\eta^{S-l})}{\eta^{S-l}(\mu-\eta)}\phi_S$$
$$=\frac{\mu}{\eta^S(\mu-\eta)}\left[\sum_{l=s+1}^{S-1}\frac{(\mu^{S-l}-\eta^{S-l})}{\eta^{-l}}\right]\phi_S. \tag{4.107}$$

$$\sum_{l=s+1}^{S-1}\frac{\mu^{S-l}-\eta^{S-l}}{\eta^{-l}}=\sum_{l=s+1}^{S-1}(\eta^l\mu^{S-l}-\eta^S)$$
$$=\eta^{s+1}\mu\frac{(\mu^{Q-1}-\eta^{Q-1})}{\mu-\eta}-(Q-1)\eta^S. \tag{4.108}$$

Using (4.108) in (4.107) we get

$$\sum_{l=s+1}^{S-1}\phi_{l,1}=\left[\frac{\mu^2}{\eta^{Q-1}}\frac{(\mu^{Q-1}-\eta^{Q-1})}{(\mu-\eta)^2}-\frac{(Q-1)\mu}{\mu-\eta}\right]\phi_S. \tag{4.109}$$

Using (4.102), (4.105), (4.106) and (4.109) and on simplification, we get (4.103) as

$$\left\{\frac{1}{\mu-\eta}\left[\frac{(\chi_1^Q-1)\chi_1^s}{\chi_1^s-1}(\mu(s-1)+\eta)-\mu+(1-Q)\eta\right]+1\right\}\phi_S=1$$

where $\chi_1=\mu/\eta$. Therefore

$$\phi_S=\left\{\frac{1}{\mu-\eta}\left[\frac{(\chi_1^Q-1)\chi_1^s}{\chi_1^s-1}(\mu(s-1)+\eta)-\mu+(1-Q)\eta\right]+1\right\}^{-1}.$$

Hence the theorem. □

Remark. When $\mu = \eta$, we get the following:

$$\phi_l = (l+1)\frac{Q}{s}\phi_S, \quad \text{where } l = 1, 2, \ldots, s \text{ (and } l \neq N+1).$$

$$\phi_{N+1} = N\frac{Q}{s}\phi_S.$$

$$\phi_{l,0} = \phi_S, \quad \text{where } l = s+1, \ldots, S-1.$$

$$\phi_{l,1} = (S-l)\phi_S, \quad \text{where } l = s+1, \ldots, S-1.$$

$$\phi_S = \Big[\frac{Q}{s}(\frac{s^2}{2} + \frac{3}{2}s - 2 + N) + (Q-1)\left(\frac{Q}{2}+1\right) + 1]^{-1}.$$

4.3.1.3 Stability Condition

The result in the Theorem 4.3.1 enables us to compute the stability of the CTMC $\{(X(t), I(t), K(t)), t \geq 0\}$.

Theorem 4.3.2. *The process* $\{(X(t), I(t), K(t)), t \geq 0\}$ *is stable iff* $\lambda < \mu$.

Proof. Since the process under consideration is an LIQBD, it will be stable iff

$$\mathbf{\Phi} A_0 \bar{\mathbf{e}} < \mathbf{\Phi} A_2 \bar{\mathbf{e}} \quad \text{(Neuts [40])} \tag{4.110}$$

where $\mathbf{\Phi}$ represents the steady-state probability vector of the generator matrix $A = A_0 + A_1 + A_2$.

$$\begin{aligned} \mathbf{\Phi} A_0 \bar{\mathbf{e}} &= [\lambda(\phi_1 + \phi_2 + \cdots + \phi_s + \phi_{s+1,0} + \phi_{s+1,1} + \ldots \\ &\quad + \phi_{s-1,0} + \phi_{s-1,1} + \phi_S)] \\ &= \lambda. \\ \mathbf{\Phi} A_2 \bar{\mathbf{e}} &= [\mu(\phi_1 + \phi_2 + \cdots + \phi_s + \phi_{s+1,0} + \phi_{s+1,1} + \ldots \\ &\quad + \phi_{S-1,0} + \phi_{S-1,1} + \phi_S)] \\ &= \mu. \end{aligned}$$

Hence the using (4.110) we get $\lambda < \mu$. Hence the theorem. □

Having obtained the condition for the system to stabilize, we turn to compute the long-run probability distribution of the system states. Infact we show that the joint distribution of the system state can be written as the product of the marginal distribution of the components.

4.3.2 The Steady-State Probability Distribution of $\tilde{A}$

4.3.2.1 Stationary Distribution when Service Time is Negligible

Let $\tilde{Q}$ be the generator matrix associated with the Markovian chain of the inventory process under consideration when service time is negligible, and $\mathbf{\Delta} = (r_1, r_2, \ldots, r_S)$ be the stationary probability vector corresponding to $\tilde{Q}$.

$\tilde{Q}$ is obtained as

$$\tilde{Q} = \begin{bmatrix} G_1 & \eta & & & \lambda & & & & & & & \\ \lambda & G_1 & \eta & & & & & & & & & \\ & \lambda & G_1 & \eta & & & & & & & & \\ & & \ddots & \ddots & \ddots & & & & & & & \\ & & & \lambda & G_1 & \eta & & & & & & \\ & & & & \lambda & G_1 & P_1 & & & & & \\ & & & & & \hat{H}_1 & G_2 & P_2 & & & & \\ & & & & & & \hat{H}_2 & G_2 & P_2 & & & \\ & & & & & & \ddots & \ddots & \ddots & & & \\ & & & & & & & \hat{H}_2 & G_2 & P_2 & & \\ & & & & & & & & \hat{H}_2 & G_2 & P_3 & \\ & & & & & & & & & \hat{H}_3 & G_3 & \end{bmatrix}$$

where $G_1 = [-(\lambda+\eta)]$, $G_2 = \begin{bmatrix} -\lambda & 0 \\ 0 & -(\lambda+\eta) \end{bmatrix}$, $G_3 = [-\lambda]$, $P_1 = [0 \quad \eta]$, $P_2 = \begin{bmatrix} 0 & 0 \\ 0 & \eta \end{bmatrix}$, $P_3 = \begin{bmatrix} 0 \\ \eta \end{bmatrix}$, $\hat{H}_1 = \begin{bmatrix} \lambda \\ \lambda \end{bmatrix}$, $\hat{H}_2 = \lambda I_2$ and $\hat{H}_3 = [\lambda \ 0]$.

Theorem 4.3.3. *The steady-state probability vector* $\mathbf{\Delta}$ *of* A *is*

$$\mathbf{\Delta} = (r_1, r_2, \ldots, r_N, \ldots, r_{s-1}, r_s, \tilde{r}_{s+1}, \tilde{r}_{s+2}, \ldots, \tilde{r}_{S-1}, r_S)$$

where

$$\tilde{r}_{l_1} = (r_{l_1,0}, r_{l_1,1}); l_1 = s+1, s+2, \ldots, S-1$$

and when $\lambda \neq \eta$, $\mathbf{\Delta}$ *is given by*

$$r_l = \frac{(\lambda^l - \eta^l)}{\lambda^{l-1}(\lambda-\eta)} \frac{\lambda^s(\lambda^Q - \eta^Q)}{\eta^Q(\lambda^s - \eta^s)} r_S,$$

$$l = 1, 2, \ldots, s \text{ (and } l \neq N+1\text{)}$$

$$r_{N+1} = \frac{\eta}{\lambda^N}\frac{(\lambda^N - \eta^N)}{\lambda - \eta}\lambda^s\frac{(\lambda^Q - \eta^Q)}{\eta^Q(\lambda^s - \eta^s)}r_S$$

$$r_{l,0} = r_S, \quad l = s+1, \ldots, S-1$$

$$r_{l,1} = \frac{\lambda(\lambda^{S-l} - \eta^{S-l})}{\eta^{S-l}(\lambda - \eta)}r_S, \quad l = s+1, \ldots, S-1$$

and r_S *can be obtained by using* $\mathbf{\Delta\bar{e}} = 1$ *as*

$$r_S = \left\{\frac{1}{\lambda - \eta}\left[\frac{(\chi^Q - 1)\chi^s}{(\chi^s - 1)}(\lambda(s-1) + \eta) - \lambda + (1-Q)\eta\right] + 1\right\}^{-1}$$

where $\chi = \lambda/\eta$.

Proof. We have $\mathbf{\Delta}\tilde{Q} = \mathbf{0}$ and $\mathbf{\Delta\bar{e}} = 1$

$$\mathbf{\Delta}\tilde{Q} = \mathbf{0} \Rightarrow$$

$$r_1 \times -(\lambda + \eta) + r_2\lambda = 0 \tag{4.111}$$

$$r_j\eta + r_{j+1} \times -(\lambda + \eta) + r_{j+2}\lambda = 0 \tag{4.112}$$

for $j = 1, 2, \ldots, s-2$ and $j \neq N-1$.

$$r_1\lambda + r_{N-1}\eta + r_N \times -(\lambda + \eta) + r_{N+1}\lambda = 0 \tag{4.113}$$

$$r_{s-1}\eta + r_s \times -(\lambda + \eta) + \tilde{r}_{s+1}\hat{H}_1 = 0 \tag{4.114}$$

$$r_sP_1 + \tilde{r}_{s+1}G_2 + \tilde{r}_{s+2}\hat{H}_2 = 0 \tag{4.115}$$

$$\tilde{r}_jP_2 + \tilde{r}_{j+1}G_2 + \tilde{r}_{j+2}\hat{H}_2 = 0 \tag{4.116}$$

where $j = s+1, s+2, \ldots, S-3$.

$$\tilde{r}_{S-2}P_2 + \tilde{r}_{S-1}G_2 + r_S\hat{H}_3 = 0. \tag{4.117}$$

$$\tilde{r}_{S-1}P_3 + r_SG_3 = 0. \tag{4.118}$$

Equation (4.111) gives

$$r_2 = \left(1 + \frac{\eta}{\lambda}\right)r_1. \tag{4.119}$$

Equation (4.112) gives

$$r_{j+2} = \left(1 + \frac{\eta}{\lambda}\right) r_{j+1} - \frac{\eta}{\lambda} r_j \tag{4.120}$$

where $j = 1, 2, \ldots, s - 2$ and $j \neq N - 1$.
Equation (4.120) gives

$$r_l = \frac{(\lambda^l - \eta^l) r_1}{\lambda^{l-1}(\lambda - \eta)} \tag{4.121}$$

where $l = 1, 2, \ldots, s$ and $l \neq N + 1$.
Equation (4.113) and (4.121) gives

$$r_{N+1} = \frac{\eta}{\lambda^N} \frac{\left(\lambda^N - \eta^N\right)}{(\lambda - \eta)} r_1. \tag{4.122}$$

Equation (4.114) gives

$$r_{s+1,0} + r_{s+1,1} = \left[1 + \frac{\eta}{\lambda} + \left(\frac{\eta}{\lambda}\right)^2 + \left(\frac{\eta}{\lambda}\right)^3 + \cdots + \left(\frac{\eta}{\lambda}\right)^s\right] r_1. \tag{4.123}$$

This equation (4.115) gives

$$r_{s+1,0} = r_{s+2,0} \tag{4.124}$$

$$\text{and } \eta r_s - (\lambda + \eta) r_{s+1,1} + \lambda r_{s+2,1} = 0 \tag{4.125}$$

Using (4.124) in (4.123) we get

$$r_{s+2,0} + r_{s+1,1} = \left[1 + \frac{\eta}{\lambda} + \left(\frac{\eta}{\lambda}\right)^2 + \left(\frac{\eta}{\lambda}\right)^3 + \cdots + \left(\frac{\eta}{\lambda}\right)^s\right] r_1. \tag{4.126}$$

Equation (4.118) gives

$$r_{S-1,1} = \left(\frac{\lambda}{\eta}\right) r_S. \tag{4.127}$$

Equation (4.117) gives

$$r_{S-1,0} = r_S \tag{4.128}$$

$$\text{and} \quad \eta r_{S-2,1} - (\lambda + \eta) r_{S-1,1} = 0. \tag{4.129}$$

Equation (4.129) and (4.127) gives

$$r_{S-2,1} = \left[\frac{\lambda}{\eta} + \left(\frac{\lambda}{\eta}\right)^2\right] r_S. \tag{4.130}$$

Equation (4.116) gives

$$\lambda r_{j+1,0} = \lambda r_{j+2,0} \tag{4.131}$$

where $j = s+1, s+2, \ldots, S-3$ and

$$\eta r_{j,1} - (\lambda+\eta) r_{j+1,1} + \lambda r_{j+2,1} = 0 \tag{4.132}$$

where $j = s+1, s+2, \ldots, S-3$.

Hence by using (4.128), (4.124) and (4.131) we get

$$r_{l,0} = r_S \tag{4.133}$$

where $l = s+1, s+2, \ldots, S-1$.

Using (4.127) and (4.130) in (4.132), we get

$$r_{l,1} = \left(\frac{\lambda}{\eta} + \left(\frac{\lambda}{\eta}\right)^2 + \left(\frac{\lambda}{\eta}\right)^3 + \cdots + \left(\frac{\lambda}{\eta}\right)^{S-l}\right) r_S \tag{4.134}$$

where $l = s+1, \ldots, S-3$.

Using (4.127), (4.130) and (4.134), we get

$$r_{l,1} = \left[\frac{\lambda}{\eta} + \left(\frac{\lambda}{\eta}\right)^2 + \left(\frac{\lambda}{\eta}\right)^3 + \cdots + \left(\frac{\lambda}{\eta}\right)^{S-l}\right] r_S \tag{4.135}$$

That is, $r_{l,1} = \dfrac{\lambda}{\eta^{S-1}} \dfrac{(\lambda^{S-l} - \eta^{S-l})}{(\lambda - \eta)} r_S$ (4.136)

where $l = s+1, \ldots, S-1$.

Equation (4.125) gives

$$r_s = \left(1 + \frac{\lambda}{\eta}\right) r_{s+1,1} - \frac{\lambda}{\eta} r_{s+2,1}.$$

Also using (4.135) in the above equation, we get

$$r_s = \left[\frac{\lambda}{\eta} + \left(\frac{\lambda}{\eta}\right)^2 + \left(\frac{\lambda}{\eta}\right)^3 + \cdots + \left(\frac{\lambda}{\eta}\right)^{Q}\right] r_S. \tag{4.137}$$

Equations (4.121) and (4.137) give

$$r_1 = \frac{\lambda^s(\lambda^Q - \eta^Q)}{\eta^Q(\lambda^s - \eta^s)} r_S. \tag{4.138}$$

Using (4.121) and (4.138), we get

$$r_l = \frac{(\lambda^l - \eta^l)}{\lambda^{l-1}(\lambda - \eta)} \frac{\lambda^s(\lambda^Q - \eta^Q)}{\eta^Q(\lambda^s - \eta^s)} r_S \tag{4.139}$$

where $l = 1, 2, 3, \ldots, s$ and $l \neq N + 1$.

Using (4.138) we get (4.122) as

$$r_{N+1} = \frac{\eta}{\lambda^N} \frac{(\lambda^N - \eta^N)}{\lambda - \eta} \frac{\lambda^s(\lambda^Q - \eta^Q)}{\eta^Q(\lambda^s - \eta^s)} r_S. \tag{4.140}$$

Hence equations (4.139), (4.140), (4.133) and (4.136) gives the required result when $l = 1, 2, 3, \ldots, S - 1$.

Next, to find r_S.

Consider the normalizing equation $\mathbf{\Delta}\bar{\mathbf{e}} = 1$. That is,

$$\sum_{\substack{l=1 \\ (l \neq N+1)}}^{s} r_l + r_{N+1} + \sum_{l=s+1}^{S-1} r_{l,0} + \sum_{l=s+1}^{S-1} r_{l,1} + r_S = 1. \tag{4.141}$$

$$\sum_{\substack{l=1 \\ (l \neq N+1)}}^{s} r_l = \frac{\lambda^s}{\eta^Q} \frac{(\lambda^Q - \eta^Q)}{(\lambda^s - \eta^s)(\lambda - \eta)} \sum_{\substack{l=1 \\ (l \neq N+1)}}^{s} \frac{(\lambda^l - \eta^l)}{\lambda^{l-1}} r_S. \tag{4.142}$$

$$\begin{aligned}
\sum_{\substack{l=1 \\ (l \neq N+1)}}^{s} \frac{(\lambda^l - \eta^l)}{\lambda^{l-1}} r_S &= \left[\lambda - \eta + \frac{\lambda^2 - \eta^2}{\lambda} + \frac{\lambda^3 - \eta^3}{\lambda^2} + \cdots + \frac{\lambda^N - \eta^N}{\lambda^{N-1}}\right. \\
&\quad \left. + \frac{\lambda^{N+2} - \eta^{N+2}}{\lambda^{N+1}} + \cdots + \frac{\lambda^s - \eta^s}{\lambda^{s-1}}\right] r_S \\
&= \left[(s-1)\lambda - \frac{\eta\lambda(\lambda^s - \eta^s)}{\lambda^s(\lambda - \eta)} + \eta\left(\frac{\eta}{\lambda}\right)^N\right] r_S.
\end{aligned}$$

Therefore (4.142) becomes

$$\sum_{\substack{l=1 \\ (l \neq N+1)}}^{s} r_l = \frac{\lambda^s}{\eta^Q} \frac{(\lambda^Q - \eta^Q)}{(\lambda^s - \eta^s)(\lambda - \eta)} \left[(s-1)\lambda - \frac{\eta\lambda}{\lambda^s} \frac{(\lambda^s - \eta^s)}{(\lambda - \eta)} + \eta\left(\frac{\eta}{\lambda}\right)^N\right] r_S. \tag{4.143}$$

$$\sum_{l=s+1}^{S-1} r_{l,0} = (Q - 1) r_S. \tag{4.144}$$

$$\sum_{l=s+1}^{S-1} r_{l,1} = \frac{\lambda}{\eta^S(\lambda-\eta)} \left[\sum_{l=s+1}^{S-1} \frac{(\lambda^{S-l}-\eta^{S-l})}{\eta^{-l}} \right] r_S. \tag{4.145}$$

$$\begin{aligned} \sum_{l=s+1}^{S-1} \frac{\lambda^{S-l}-\eta^{S-l}}{\eta^{-l}} &= \sum_{l=s+1}^{S-1} (\eta^l \lambda^{S-l} - \eta^S) \\ &= \eta^{s+1}\lambda \frac{(\lambda^{Q-1}-\eta^{Q-1})}{\lambda-\eta} - (Q-1)\eta^S. \end{aligned} \tag{4.146}$$

Using (4.146) in (4.145) we get

$$\sum_{l=s+1}^{S-1} r_{l,1} = \left[\frac{\lambda^2}{\eta^{Q-1}} \frac{(\lambda^{Q-1}-\eta^{Q-1})}{(\lambda-\eta)^2} - \frac{(Q-1)\lambda}{\lambda-\eta} \right] r_S. \tag{4.147}$$

Using (4.140), (4.143), (4.144) and (4.147) and on simplification, we get (4.141) as

$$\left\{ \frac{1}{\lambda-\eta} \left[\frac{(\chi^Q-1)\chi^s}{\chi^s-1} (\lambda(s-1)+\eta) - \lambda + (1-Q)\eta \right] + 1 \right\} r_S = 1$$

where $\chi = \lambda/\eta$. Therefore

$$r_S = \left\{ \frac{1}{\lambda-\eta} \left[\frac{(\chi^Q-1)\chi^s}{\chi^s-1} (\lambda(s-1)+\eta) - \lambda + (1-Q)\eta \right] + 1 \right\}^{-1}.$$

Hence the theorem. □

Remark. When $\lambda = \eta$, we get the following:

$$\begin{aligned} r_l &= (l+1)\frac{Q}{s} r_S, \quad \text{where } l = 1, 2, \ldots, s \text{ (and } l \neq N+1) \\ r_{N+1} &= N\frac{Q}{s} r_S \\ r_{l,0} &= r_S, \quad \text{where } l = s+1, \ldots, S-1 \\ r_{l,1} &= (S-l) r_S, \quad \text{where } l = s+1, \ldots, S-1 \\ \text{and } r_S &= \left[\frac{Q}{s} \left(\frac{s^2}{2} + \frac{3}{2}s - 2 + N \right) + (Q-1)\left(\frac{Q}{2} + 1 \right) + 1 \right]^{-1}. \end{aligned}$$

4.3.2.2 Stochastic Decomposition of System States

Let $\bar{\boldsymbol{\pi}}$ be the steady-state probability vector of $\tilde{A}$.

$$\bar{\boldsymbol{\pi}} = (\boldsymbol{\pi}^{(0)}, \boldsymbol{\pi}^{(1)}, \boldsymbol{\pi}^{(2)}, \ldots)$$

where $\boldsymbol{\pi}^{(i)} = (\pi^{(i,1)}, \pi^{(i,2)}, \ldots, \pi^{(i,N)}, \ldots, \pi^{(i,s)}, \pi^{(i,s+1,0)}, \pi^{(i,s+1,1)}, \ldots, \pi^{(i,S-1,0)}, \pi^{(i,S-1,1)}, \pi^{(i,S)})$ where $i = 0, 1, 2, \ldots$ and $\pi^{(i,j)} = \lim_{t\to\infty} P(X(t) = i, I(t) = j)$ and $\pi^{(i,j,k)} = \lim_{t\to\infty} P(X(t) = i, I(t) = j, K(t) = k)$.

$\pi^{(i,j)}$ is the steady-state probability for the state (i, j) and $\pi^{(i,j,k)}$ is the steady-state probability for the state (i, j, k).

We claim that

$$\boldsymbol{\pi}^{(i)} = K\rho^i \boldsymbol{\Delta}, i \geq 0 \tag{4.148}$$

where $\boldsymbol{\Delta} = (r_1, r_2, \ldots, r_N, \ldots, r_s, r_{s+1,0}, r_{s+1,1}, \ldots, r_{S-1,0}, r_{S-1,1}, r_S)$ is the steady-state probability vector when the service time is negligible, K is a constant to be determined and $\rho = \frac{\lambda}{\mu}$.

Proof. We have $\bar{\boldsymbol{\pi}}\tilde{A} = \mathbf{0}$ and $\bar{\boldsymbol{\pi}}\bar{\mathbf{e}} = 1$.

$$\bar{\boldsymbol{\pi}}\tilde{A} = \mathbf{0} \Rightarrow (\boldsymbol{\pi}^{(0)}, \boldsymbol{\pi}^{(1)}, \boldsymbol{\pi}^{(2)}, \ldots) \begin{bmatrix} B_1 & A_0 & & \\ A_2 & A_1 & A_0 & \\ & A_2 & A_1 & A_0 \\ & \ddots & \ddots & \ddots \end{bmatrix} = \mathbf{0}$$

which gives

$$\boldsymbol{\pi}^{(0)} B_1 + \boldsymbol{\pi}^{(1)} A_2 = \mathbf{0} \tag{4.149}$$

$$\boldsymbol{\pi}^{(i+2)} A_2 + \boldsymbol{\pi}^{(i+1)} A_1 + \boldsymbol{\pi}^{(i)} A_0 = \mathbf{0} \tag{4.150}$$

$i = 0, 1, \ldots$.

When (4.148) is true, we get from (4.149),

$$K\rho^0 \boldsymbol{\Delta} B_1 + K\rho \boldsymbol{\Delta} A_2 = \mathbf{0}$$

$$\text{That is} \quad K\boldsymbol{\Delta}(B_1 + \rho A_2) = \mathbf{0}.$$

$$\text{That is} \quad \boldsymbol{\Delta}\tilde{Q} = \mathbf{0}.$$

which is true since $\boldsymbol{\Delta} = (r_1, r_2, \ldots, r_S)$ is the steady-state probability vector corresponding to the generator $\tilde{Q}$ associated with the Markovian chain of the inventory process under consideration when service time is negligible.

When (4.148) is true, we get from (4.150),

$$K\rho^{i+2}\boldsymbol{\Delta}A_2 + K\rho^{i+1}\boldsymbol{\Delta}A_1 + K\rho^{i}\boldsymbol{\Delta}A_0 = \mathbf{0}, \quad i = 0, 1, 2, \ldots$$

$$\text{That is} \quad \boldsymbol{\Delta}(\rho A_2 + A_1 + \frac{1}{\rho}A_0) = \mathbf{0}.$$

$$\text{That is} \quad \boldsymbol{\Delta}\tilde{Q} = \mathbf{0},$$

which is true, by following the same argument given above.

Hence, the stochastic decomposition of system states is verified. □

4.3.2.3 Determination of K

We have $\bar{\boldsymbol{\pi}}\bar{\mathbf{e}} = 1$. That is

$$\sum_{i=0}^{\infty} \sum_{\substack{j=1 \\ (j \neq N+1)}}^{s} \pi^{(i,j)} + \sum_{i=0}^{\infty} \pi^{(i,N+1)} + \sum_{i=0}^{\infty} \sum_{j=s+1}^{S-1} (\pi^{(i,j,0)} + \pi^{(i,j-1)}) + \sum_{i=0}^{\infty} \pi^{(i,S)} = 1.$$

That is,

$$K\left[\sum_{i=0}^{\infty} \rho^i\right]\left[\sum_{\substack{j=1 \\ (j \neq N+1)}}^{s} r_j + r_{N+1} + \sum_{j=s+1}^{S-1}(r_{j,0} + r_{j,1}) + r_S\right] = 1.$$

That is, $K\sum_{i=0}^{\infty} \rho^i = 1$.
Therefore $K = 1 - \rho$ where $\rho = \frac{\lambda}{\mu}$.

4.3.2.4 Explicit Solution

From the above discussions, we can write the steady-state probability vector explicitly as in the following theorem:

Theorem 4.3.4. *The steady-state probability vector* $\bar{\boldsymbol{\pi}}$ *of* $\tilde{A}$ *partitioned as*

$$\bar{\boldsymbol{\pi}} = (\boldsymbol{\pi}^{(0)}, \boldsymbol{\pi}^{(1)}, \boldsymbol{\pi}^{(2)}, \ldots),$$

where each $\boldsymbol{\pi}^{(i)}$, $i = 0, 1, 2, \ldots$ *again partitioned as* $\boldsymbol{\pi}^{(i)} = (\pi^{(i,1)}, \pi^{(i,2)}, \ldots,$ $\pi^{(i,N)}, \ldots, \pi^{(i,s)}, \pi^{(i,s+1,0)}, \pi^{(i,s+1,1)} \ldots, \pi^{(i,S-1,0)}, \pi^{(i,S-1,1)}, \pi^{(i,S)})$ *are obtained by*

$$\boldsymbol{\pi}^{(i)} = (1 - \rho)\rho^i\boldsymbol{\Delta}, \quad i \geq 0 \tag{4.151}$$

where $\rho = \lambda/\mu$ *and*
$\mathbf{\Delta} = (r_1, r_2, \ldots, r_N, \ldots, r_s, r_{s+1,0}, r_{s+1,1}, \ldots, r_{S-1,0},\ r_{S-1,1}, r_S)$ *be the steady state probability vector when the service time is negligible.*

$\mathbf{\Delta}$ *can be obtained from:*

$$\begin{aligned} r_l &= \frac{(\lambda^l - \eta^l)}{\lambda^{l-1}(\lambda-\eta)} \frac{\lambda^s(\lambda^Q - \eta^Q)}{\eta^Q(\lambda^s - \eta^s)} r_S, \\ &\quad l = 1, 2, \ldots, s \text{ (and } l \neq N+1\text{)}. \\ r_{N+1} &= \frac{\eta}{\lambda^N} \frac{(\lambda^N - \eta^N)}{\lambda - \eta} \lambda^s \frac{(\lambda^Q - \eta^Q)}{\eta^Q(\lambda^s - \eta^s)} r_S. \\ r_{l,0} &= r_S, \quad l = s+1, \ldots, S-1. \\ r_{l,1} &= \frac{\lambda(\lambda^{S-l} - \eta^{S-l})}{\eta^{S-l}(\lambda - \eta)} r_S, \quad l = s+1, \ldots, S-1 \end{aligned}$$

and r_S *can be obtained by using* $\mathbf{\Delta}\bar{\mathbf{e}} = 1$ *as*

$$r_S = \left\{ \frac{1}{\lambda - \eta} \left[\frac{(\chi^Q - 1)\chi^s}{(\chi^s - 1)} (\lambda(s-1) + \eta) - \lambda + (1-Q)\eta \right] + 1 \right\}^{-1}$$

where $\chi = \lambda/\eta$.

The result indicated by (4.151) not only tells us that the original system possess stochastic decomposition but also the important fact that the system state distribution is the product of the distribution of its marginals: one component is the classical $M/M/1$ whose long run distribution for i customers in the system is $(1-\rho)\rho^i$, $i \geq 0$ and the other factor is the probability of j items in the inventory. Next we turn to find out how the system performs. The measures given in the following are pointers to the system performance.

4.3.3 System Performance Measures

(a) Expected number of customers in the system,

$$\begin{aligned} E_i &= \sum_{i=0}^{\infty} i(1-\rho)\rho^i \\ &= \frac{\rho}{1-\rho} \\ &= \frac{\lambda}{\mu-\lambda}. \end{aligned}$$

(b) Expected inventory held in the system,

$$\begin{aligned} E(I) &= \sum_{i=0}^{\infty} \sum_{\substack{j=1 \\ (j\neq N+1)}}^{s} j\pi^{(i,j)} + (N+1)\sum_{i=0}^{\infty} \pi^{(i,N+1)} \\ &\quad + \sum_{i=0}^{\infty}\sum_{j=s+1}^{S-1} j(\pi^{(i,j,0)} + \pi^{(i,j,1)}) + \sum_{i=0}^{\infty} S\pi^{(i,S)} \\ &= \left(\sum_{i=0}^{\infty}(1-\rho)\rho^i\right)\left[\sum_{\substack{j=1 \\ (j\neq N+1)}}^{s} jr_j + (N+1)r_{N+1} \right. \\ &\quad \left. + \sum_{j=s+1}^{S-1} j(r_{j,0}+r_{j,1}) + Sr_S\right] \\ &= \sum_{\substack{j=1 \\ (j\neq N+1)}}^{s} jr_j + (N+1)r_{N+1} + \sum_{j=s+1}^{S-1} j(r_{j,0}+r_{j,1}) + Sr_S \\ &= \left[\frac{(\chi^Q-1)}{(1-\chi^{-1})(1-\chi^{-s})}\left(\sum_{\substack{j=1 \\ (j\neq N+1)}}^{s} j(1-\chi^{-j})\right)\right. \\ &\quad + \frac{(N+1)(1-\chi^{-N})(\chi^Q-1)}{(\chi-1)(1-\chi^{-s})} \\ &\quad \left. + \left(\sum_{j=s+1}^{S-1} j\right) + \left(\sum_{j=s+1}^{S-1} j(\chi^{S-j}-1)\right)\frac{1}{1-\chi^{-1}} + S\right] r_S \qquad (4.152) \end{aligned}$$

$$\sum_{\substack{j=1 \\ (j \neq N+1)}}^{s} j(1-\chi^{-j}) = \sum_{\substack{j=1 \\ (j \neq N+1)}}^{s} j - \sum_{\substack{j=1 \\ (j \neq N+1)}}^{s} j\chi^{-j}. \tag{4.153}$$

Put $\delta = \chi^{-1}$

$$\sum_{\substack{j=1 \\ (j \neq N+1)}}^{s} j = \frac{s(s+1)}{2} - (N+1) \tag{4.154}$$

$$\sum_{\substack{j=1 \\ (j \neq N+1)}}^{s} j\chi^{-j} = \psi_1 + \psi_2 \tag{4.155}$$

where

$$\psi_1 = \delta + 2\delta^2 + 3\delta^3 + \cdots + N\delta^N \quad \text{and}$$
$$\psi_2 = (N+2)\delta^{N+2} + (N+3)\delta^{N+3} + \cdots + s\delta^s$$

ψ_1 can be obtained as

$$\psi_1 = \frac{\delta(1-\delta^N)}{(1-\delta)^2} - N\frac{\delta^{N+1}}{1-\delta} \tag{4.156}$$

and ψ_2 can be obtained as

$$\psi_2 = \frac{(N+2)\delta^{N+2} - s\delta^{s+1})}{1-\delta} - \delta^{N+3}\frac{1-\delta^{s-N-2}}{(1-\delta)^2} \tag{4.157}$$

$$\sum_{j=s+1}^{S-1} j = (Q-1)\left(\frac{Q}{2} + s\right). \tag{4.158}$$

Also we get on simplification

$$\sum_{j=s+1}^{S-1} j(\chi^{S-j} - 1) = \chi^{Q-1}\frac{(1-\chi^{-(Q-1)})}{1-\chi^{-1}}\left(s + \frac{1}{1-\chi^{-1}}\right) - (Q-1)\left[(1-\chi^{-1})^{-1} + s + \frac{Q}{2}\right]. \tag{4.159}$$

Using (4.153), (4.154), (4.155), (4.156), (4.157), (4.158) and (4.159) in (4.152) and on simplification, we get,

$$\begin{aligned}
E(I) = & \Bigg[\frac{(\chi^Q - 1)}{(1-\chi^{-1})(1-\chi^{-s})} \Bigg[\frac{s(1+s)}{2} - (N+1) \\
& - \frac{\chi^{-1}(1-\chi^{-N})}{(1-\chi^{-1})^2} + \frac{N\chi^{-(N+1)}}{1-\chi^{-1}} \\
& - \frac{[(N+2)\chi^{-(N+2)} - s\chi^{-(s+1)}]}{1-\chi^{-1}} - \chi^{-(N+3)} \frac{(1-\chi^{-(s-N-2)})}{(1-\chi^{-1})^2} \Bigg] \\
& + \frac{(N+1)(1-\chi^{-N})(\chi^Q - 1)}{(\chi - 1)(1-\chi^{-s})} + (Q-1)(s + \frac{Q}{2}) \\
& + \Bigg[\chi^{Q-1} \left(\frac{1-\chi^{-(Q-1)}}{1-\chi^{-1}} \right) \left(s + \frac{1}{1-\chi^{-1}} \right) \\
& -(Q-1) \left(\frac{1}{1-\chi^{-1}} + s + \frac{Q}{2} \right) \Bigg] \left(\frac{1}{1-\chi^{-1}} \right) + S \Bigg] r_S.
\end{aligned}$$

(c) Expected rate at which production process is switched 'on'

$$\begin{aligned}
R_{\text{ON}} &= \mu \sum_{i=1}^{\infty} \pi^{(i,s+1,0)} \\
&= \mu(1-\rho) \sum_{i=1}^{\infty} \rho^i r_{s+1,0} \\
&= \lambda r_S.
\end{aligned}$$

(d) Expected production rate

$$\begin{aligned}
R_P &= \Big(\sum_{i=0}^{\infty} \sum_{j=1}^{s} \pi^{(i,j)} + \sum_{i=0}^{\infty} \sum_{j=s+1}^{S-1} \pi^{(i,j,1)} \Big) \eta \\
&= \left(\sum_{i=0}^{\infty} (1-\rho)\rho^i \right) \Big(\sum_{j=1}^{s} r_j + \sum_{j=s+1}^{S-1} r_{j,1} \Big) \eta \\
&= \Big[\sum_{j=1}^{s} r_j + \sum_{j=s+1}^{S-1} r_{j,1} \Big] \eta. \qquad (4.160)
\end{aligned}$$

Using Theorem 4.3.3 and on simplification, we get

$$\sum_{j=1}^{s} r_j = \frac{(\chi^Q - 1)\chi^s}{\chi^s - 1} \Bigg[\left(\frac{\chi}{\chi - 1} \right)^2 ((s-1)(1-\chi^{-1})$$

$$-\chi^{-1}(1-\chi^{-s}-\chi^{-N}(1-\chi^{-1}))) + \frac{\chi^N-1}{\chi^N(\chi-1)}\Bigg] r_S \qquad (4.161)$$

and

$$\sum_{j=s+1}^{S-1} r_{j,1} = \frac{1}{(1-\chi^{-1})^2}[Q(\chi^{-1}-1)+\chi^{-1}(\chi^Q-1)]r_S. \qquad (4.162)$$

Using (4.161) and (4.162) in (4.160), we get

$$R_P = \Bigg\{ \frac{(\chi^Q-1)\chi^s}{\chi^s-1}\Bigg[\left(\frac{\chi}{\chi-1}\right)^2((s-1)(1-\chi^{-1})-\chi^{-1} \times(1-\chi^{-s}-\chi^{-N}(1-\chi^{-1}))) + \frac{\chi^N-1}{\chi^N(\chi-1)}\Bigg] + \frac{1}{(1-\chi^{-1})^2}[Q(\chi^{-1}-1)+\chi^{-1}(\chi^Q-1)]\Bigg\}\eta r_S.$$

(e) Expected local purchase rate,

$$\begin{aligned} R_{\text{LP}} &= \mu\sum_{i=1}^{\infty}\pi^{(i,1)} \\ &= \mu\sum_{i=1}^{\infty}(1-\rho)\rho^i r_1 \\ &= \lambda\frac{(\chi^Q-1)}{(1-\chi^{-s})}r_S. \end{aligned}$$

(f) Expected waiting time of customers in the system,

$$W_s = \frac{L}{\lambda} \quad \text{where}$$

$$\begin{aligned}
L =& \text{ Expected number of customers in the system} \\
=& \sum_{i=0}^{\infty}\sum_{j=1}^{s} i\pi^{(i,j)} + \sum_{i=0}^{\infty} i\pi^{(i,N+1)} \\
&+ \sum_{i=0}^{\infty}\sum_{j=s+1}^{S-1} i\pi^{(i,j,0)} + \sum_{i=0}^{\infty}\sum_{j=s+1}^{S-1} i\pi^{(i,j,1)} + \sum_{i=0}^{\infty} i\pi^{(i,S)} \\
=& \left[\sum_{i=0}^{\infty} i(1-\rho)\rho^i\right]\left[\sum_{\substack{j=1\\(j\neq N+1)}} r_j + r_{N+1} + \sum_{j=s+1}^{S-1}(r_{j,0}+r_{j,1}) + r_S\right] \\
=& \frac{\rho}{1-\rho}.
\end{aligned}$$

Therefore

$$\begin{aligned}
W_S &= \frac{1}{\lambda}\left(\frac{\rho}{1-\rho}\right) \\
&= \frac{1}{\mu-\lambda}.
\end{aligned}$$

Remark. When $\lambda = \eta$, we get the following performance measures:

(a) $E(I) = [\frac{Q}{6}(s+1)(2s+4) + (N+1)\frac{NQ}{s}$
$+\frac{1}{2}[(S+1)(Q-1)(S+s) - \sum_{j=s+1}^{S-1} j^2 + S]r_S$

(b) $R_P = [\frac{Q}{s}(\frac{s(s+1)}{2} + s - 2) + \frac{NQ}{s} + \frac{Q(Q-1)}{2}]\eta r_S$

(c) $R_{\mathrm{ON}} = \lambda r_S$

(d) $R_{\mathrm{LP}} = \frac{2\lambda Q}{s} r_S$

4.3.4 Cost Analysis

Next, we find a cost function. Let the various costs involved in the model be as given below.

C_H : Holding cost per unit time per unit inventory
C_P : Cost of production per unit time per unit inventory
C_S : Fixed cost for starting the production

C_{LP} : Cost per unit item per unit time under local purchase
C_W : Waiting time cost per customer per unit time.

The total expected cost per unit time is

$$\text{TEC} = C_H E(I) + C_P R_P + C_S R_{\text{ON}} + C_{\text{LP}} R_{\text{LP}} N + C_W W_S.$$

When $\lambda \neq \eta$, we get,

$$\begin{aligned}
\text{TEC} = \Bigg[C_H \Bigg\{ & \frac{(\chi^Q - 1)}{(1-\chi^{-1})(1-\chi^{-s})} \left[\frac{s(1+s)}{2} - (N+1) - \frac{\chi^{-1}(1-\chi^{-N})}{(1-\chi^{-1})^2} \right. \\
& \left. + \frac{N\chi^{-(N+1)}}{1-\chi^{-1}} - \frac{[(N+2)\chi^{-(N+2)} - s\chi^{-(s+1)}]}{1-\chi^{-1}} - \chi^{-(N+3)} \frac{(1-\chi^{-(s-N-2)})}{(1-\chi^{-1})^2} \right] \\
& + \frac{(N+1)(1-\chi^{-N})(\chi^Q - 1)}{(\chi - 1)(1-\chi^{-s})} + (Q-1)\left(s + \frac{Q}{2}\right) \\
& + \left[\chi^{Q-1} \frac{(1-\chi^{-(Q-1)})}{1-\chi^{-1}} \left(s + \frac{1}{1-\chi^{-1}} \right) \right. \\
& \left. -(Q-1)\left(\frac{1}{1-\chi^{-1}} + s + \frac{Q}{2} \right) \right] \left(\frac{1}{1-\chi^{-1}} \right) + S \Bigg\} \\
& + C_P \left\{ \frac{(\chi^Q - 1)\chi^s}{\chi^s - 1} \left[\left(\frac{\chi}{\chi - 1} \right)^2 ((s-1)(1-\chi^{-1}) \right. \right. \\
& \left. -\chi^{-1}(1 - \chi^{-s} - \chi^{-N}(1-\chi^{-1}))) + \frac{\chi^N - 1}{\chi^N(\chi - 1)} \right] \\
& \left. + \frac{1}{(1-\chi^{-1})^2} [Q(\chi^{-1} - 1) + \chi^{-1}(\chi^Q - 1)] \right\} \eta \\
& + C_S \lambda + C_{\text{LP}} \left\{ \lambda \frac{(\chi^Q - 1)}{(1-\chi^{-s})} \right\} N \Bigg] r_S + C_W \times \frac{1}{\mu - \lambda}
\end{aligned}$$

where $\chi = \frac{\lambda}{\eta}$ and

$$r_S = \left[\left(\frac{1}{\lambda - \eta} \right) \left(\frac{(\chi^Q - 1)\chi^s}{(\chi^s - 1)} (\lambda(s-1) + \eta) - \lambda + (1-Q)\eta \right) + 1 \right]^{-1}.$$

Remark. When $\lambda = \eta$, we get the cost function as

$$\text{TEC} = \left\{ C_H \left[\frac{Q(s+1)(2s+4)}{6} + (N+1)N\frac{Q}{s} + (S+1)(Q-1)\frac{(S+s)}{2} - \sum_{j=s+1}^{S-1} j^2 + S \right] + C_P \left[\frac{Q}{2}\left(\frac{s(s+1)}{2} + s - 2 \right) + \frac{NQ}{s} + \frac{Q(Q-1)}{2} \right] \eta + \lambda(C_S + C_{\text{LP}}\frac{2Q}{s}) \right\} r_S + C_W \left(\frac{1}{\mu - \lambda} \right).$$

To verify the convexity of the above cost functions with respect to s, S or N, the corresponding derivative is to be computed, then equate it to zero. Nevertheless, solving it is a laborious task. Since analysis of TEC as a function of s, S or N is quite complex, we give a few numerical illustrations.

4.3.4.1 Numerical Analysis

Case 1: Analysis of TEC as a function of S.

Input Data:

$C_H = 25$, $C_P = 200$, $C_S = 3000$, $C_{\text{LP}} = 220$, $C_W = 3500$, $s = 9$, $N = 5$, $\lambda = 1.5$, $\mu = 5$, $\eta = 2.5$.

Table 4.5 shows that as S increases, TEC values decrease, reach a minimum at $S = 25$ and then increase. Hence it is numerically verified that TEC function is convex in S.

Table 4.5: Effect of S on TEC ($\lambda \neq \eta$)

S	TEC
19	1960.1
20	1942.4
21	1929.6
22	1920.6
23	1914.7
24	1911.3
25	**1909.8**
26	1910.0
27	1911.5
28	1914.2

Table 4.6: Effect of N on TEC ($\lambda \neq \eta$)

N	TEC
2	1916.7
3	1915.4
4	1913.3
5	1909.8
6	1903.9
7	1893.9
8	1877.3

Table 4.7: Effect of N on TEC ($\lambda = \eta$)

N	TEC
2	1588.0
3	1586.9
4	1586.3
5	**1586.0**
6	1586.1
7	1586.6
8	1587.4

Case 2: Analysis of TEC as a function of N.
Input Data:
$C_H = 25$, $C_P = 200$, $C_S = 3000$, $C_{\text{LP}} = 220$, $C_W = 3500$, $s = 9$, $S = 25$ $\lambda = 1.5$, $\mu = 5$, $\eta = 2.5$.

Table 4.6 shows that as N increases, TEC function is monotonically decreasing, and hence convex.

Case 3: Analysis of TEC as function of N (when $\lambda = \eta$)
Input Data: $C_H = 25$, $C_P = 200$, $C_S = 3000$, $C_{\text{LP}} = 220$, $C_W = 3500$, $\lambda = \eta = 1.5$, $\mu = 5$, $s = 9$, $S = 25$.

Table 4.7 shows that as N increases, TEC values decrease, reach a minimum at $N = 5$ and then increase. Hence it is numerically verified that TEC function is convex in N.

Chapter 5

(s, Q) Inventory Systems with Positive Lead Time and Service Time under N-Policy with Coxian-2 Arrivals and Services

5.1 Introduction

So far we have restricted the distributions involved to be exponential. Though mathematically it is a very nice function, its applicability is limited. However our purpose of concentrating to exponential distribution as underlying distribution starts from our primary objective of deriving stochastic decomposition of the systems under study. Now we go beyond exponential distribution. A nice object of the next stage is Coxian distribution of order 2. Tijms [54] compared an $M/\text{Cox}_2/1$ and $M/D/1$ and showed that Cox_2 acts as a very good approximation to deterministic service.

In this chapter an (s, Q) inventory system with service time, in which N-policy is adopted during lead time, is considered. The reorder level is s and ordering quantity is fixed at $Q = S - s$. A replenishment lead time which follows exponential distribution with parameter γ is assumed. It is assumed that there is only one server. Also suppose that the inter-arrival time has a Coxian-2 distribution with parameters $(b, \lambda_1, \lambda_2)$. Without loss of generality we may assume that $\lambda_1 \geq \lambda_2$. The arrival mechanism may be described as follows: An arriving customer first goes through phase 1 for an exponentially distributed time with parameter λ_1 and gets into the system with probability $1 - b$, or goes through a second phase with probability b. The sojourn time in phases are independent exponentials with means $1/\lambda_1$ and $1/\lambda_2$ respectively, that is, the arrival mechanism is consisting either of only one exponential stage with mean $1/\lambda_1$ (with probability $1 - b$) after which the arrival is admitted to the system, or of two successive independent exponential stages with means $1/\lambda_1$ and $1/\lambda_2$ respectively, after which absorption occurs.

Also, suppose that the service time of a customer has a Coxian-2 distribution with parameters (θ, μ_1, μ_2). Without loss of generality we may assume that $\mu_1 \geq \mu_2$. The service mechanism may be considered as follows: The customer first goes through phase 1 to get his service completed with probability $1 - \theta$, or goes through a second phase with probability θ. The sojourn time in the two phases are independent exponential random variables with means $1/\mu_1$ and $1/\mu_2$, respectively, that is, the service mechanism consists either of one exponential stage (with probability $1 - \theta$) with mean $1/\mu_1$ after which the service is completed, or of two independent exponential stages with means $1/\mu_1$ and $1/\mu_2$ respectively, after which the service is completed, the probability of the second stage of service being θ.

In this model, N-policy is adopted during a lead time, and is as follows:-

As and when the inventory level drops to $s - N$ (where $s \geq N$) during a lead time, due to N service completions after placing a natural purchase order at level s, an immediate local purchase of $Q + N$ units is made, by cancelling the order that is already placed. Cancellation of order is necessary, since otherwise the inventory level may go beyond S at the time of the replenishment against that order. In otherwords, we will not permit the inventory level to reduce beyond $s - N + 1$. We go for an immediate local purchase at the moment at which the inventory level drops to $s - N$, regardless of the number of customers present in the system. Also it is assumed that supply of items is instantaneous in local purchase but at a much higher cost. This idea of local purchase is used in Saffari et al. [43] to obtain product form solution for arbitrary distributed replenishment time.

5.2 Model Formulation and Analysis

Let $X(t)$ = Number of customers in the system at time t,
$I(t)$ = Inventory level at time t,
$Z_1(t)$ = Phase of the inter-arrival time in progress at time t and
$Z_2(t)$ = Phase of the service time in progress at time t.

$$\tilde{Y} = \{(X(t), Z_1(t), I(t), Z_2(t)), t \geq 0\}$$

is a continuous-time stochastic process with state space

$$\{(i, z_1, j, z_2) | i \geq 1; z_1 = 1, 2; j = s - N + 1, \ldots, S; z_2 = 1, 2\} \\ \cup \{(0, z_1, j) | z_1 = 1, 2; j = s - N + 1, \ldots, S\}.$$

Let $X(0) = 0$, $I(0) = S$, $Z_1(0) = 1$.

5.2.1 Infinitesimal Generator $\tilde{A}$

$\tilde{A}$ can be obtained in terms of submatrices as follows:

$$\tilde{A} = \begin{bmatrix} B_1 & B_0 & & \\ B_2 & A_1 & A_0 & \\ & A_2 & A_1 & A_0 \\ & \ddots & \ddots & \ddots \end{bmatrix}$$

where

$$A_0 = \begin{bmatrix} \lambda_1(1-b)I_{2(Q+N)} & O_{2(Q+N)} \\ O_{2(Q+N)} & \lambda_2 I_{2(Q+N)} \end{bmatrix} \quad \text{and}$$

$$
A_1 = \left[\begin{array}{cccccccccccccccc}
J_1 & & & & & D_1 & & & D_2 & & & & & & & \\
& \ddots & & & & & \ddots & & & \ddots & & & & & & \\
& & J_1 & & & & & D_1 & & & & & & & & \\
& & & T_1 & & & & & & & & & & & & \\
& & & & \ddots & & & & & & & & & & & \\
& & & & & \ddots & & & & & & & & & & \\
& & & & & & & T_1 & & & & & & & & D_2 \\
& & & & & & & & J_2 & & & & & D_1 & & \\
& & & & & & & & & \ddots & & & & & \ddots & \\
& & & & & & & & & & J_2 & & & & & D_1 \\
& & & & & & & & & & & T_2 & & & & \\
& & & & & & & & & & & & \ddots & & & \\
& & & & & & & & & & & & & & & T_2
\end{array}\right]
$$

where $J_1 = \begin{bmatrix} -(\gamma + \lambda_1 + \mu_1) & \theta\mu_1 \\ 0 & -(\gamma + \lambda_1 + \mu_2) \end{bmatrix}$,
$J_2 = \begin{bmatrix} -(\gamma + \lambda_2 + \mu_1) & \theta\mu_1 \\ 0 & -(\gamma + \lambda_2 + \mu_2) \end{bmatrix}$,

$$
T_1 = \begin{bmatrix} -(\lambda_1 + \mu_1) & \theta\mu_1 \\ 0 & -(\lambda_1 + \mu_2) \end{bmatrix}, \quad T_2 = \begin{bmatrix} -(\lambda_2 + \mu_1) & \theta\mu_1 \\ 0 & -(\lambda_2 + \mu_2) \end{bmatrix},
$$

$D_1 = \gamma I_2$ and $D_2 = \lambda_1 b I_2$.
B_1 is obtained as given in page 165.

$$
B_0 = \begin{bmatrix}
J_4 & & & & & & \\
& J_4 & & & & & \\
& & \ddots & & & & \\
& & & J_4 & & & \\
& & & & J_5 & & \\
& & & & & \ddots & \\
& & & & & & J_5
\end{bmatrix}
$$

where $J_4 = [(1-b)\lambda_1 \quad 0]$ and $J_5 = [\lambda_2 \quad 0]$.

$$
B_1 = \begin{bmatrix}
-(\gamma+\lambda_1) & & & & \gamma & & & b\lambda_1 & & & & & & \\
 & \ddots & & & & \ddots & & & & & & & & \\
 & & -(\gamma+\lambda_1) & & & & \gamma & & & & & & & \\
 & & & -\lambda_1 & & & & & & & & & & \\
 & & & & & & & & & & \ddots & & & \\
 & & & & \ddots & & & & & & & & & \\
 & & & & & & -\lambda_1 & & & & & & & b\lambda_1 \\
 & & & & & & & -(\gamma+\lambda_2) & & \gamma & & & & \\
 & & & & & & & & \ddots & & & & & \\
 & & & & & & & & & & & & \ddots & \\
 & & & & & & & & & & -(\gamma+\lambda_2) & & & \gamma \\
 & & & & & & & & & & & -\lambda_2 & & \\
 & & & & & & & & & & & & \ddots & \\
 & & & & & & & & & & & & & -\lambda_2
\end{bmatrix}
$$

$$B_2 = \begin{bmatrix} \mathbf{0}_{2\times 1} & & & J_6 & & & & \\ J_6 & & & & & & & \\ & \ddots & & & & & & \\ & & J_6 & & & & & \\ & & & \mathbf{0}_{2\times 1} & & & & J_6 \\ & & & & J_6 & & & \\ & & & & & \ddots & & \\ & & & & & & J_6 & \mathbf{0}_{2\times 1} \end{bmatrix}$$

where $J_6 = \begin{bmatrix} (1-\theta)\mu_1 \\ \mu_2 \end{bmatrix}$.

$$A_2 = \begin{bmatrix} & & & J_3 & & & \\ J_3 & & & & & & \\ & \ddots & & & & & \\ & & J_3 & & & & \\ & & & & & & J_3 \\ & & & J_3 & & & \\ & & & & \ddots & & \\ & & & & & J_3 & \end{bmatrix}$$

where $J_3 = \begin{bmatrix} (1-\theta)\mu_1 & \\ & \mu_2 \end{bmatrix}$.

A_0, A_1, A_2 are square matrices of order $4(Q+N)$, B_0 is a matrix of order $2(Q+N) \times 4(Q+N)$, B_1 is a square matrix of order $2(Q+N)$ and B_2 is a matrix of order $4(Q+N) \times 2(Q+N)$.

5.2.2 Steady-State Analysis

Let $A = A_0 + A_1 + A_2$ be the generator matrix and is obtained as

$$
\begin{bmatrix}
T_7 & & & & & D_1 & & J_3 & D_2 & & & & & & & \\
J_3 & \ddots & & & & & \ddots & & & \ddots & & & & & & \\
& \ddots & T_7 & & & & & D_1 & & & & & & & & \\
& & & T_8 & & & & & & & & & & & & \\
& & & & \ddots & & & & & & & & & & & \\
& & & & & \ddots & & & & & & & & & & \\
& & & & & & & T_8 & & & & & & & & D_2 \\
& & & & & & & & T_9 & & & & & D_1 & & J_3 \\
& & & & & & & & & \ddots & & & & & \ddots & \\
& & & & & & & & & & T_9 & & & & & D_1 \\
& & & & & & & & & & & T_{10} & & & & \\
& & & & & & & & & & & & \ddots & & & \\
& & & & & & & & & & & & & & & \\
& & & & & & & & & & & & & & J_3 & T_{10}
\end{bmatrix}
$$

Note that A is a square matrix of order $4(Q+N)$.

5.2.3 Stability Condition

Theorem 5.2.1. *The process under study is stable iff*

$$
\lambda_1(1-b)\sum_{j=s-N+1}^{S}\sum_{z_2=1}^{2}\phi_{1,j,z_2}+\lambda_2\sum_{j=s-N+1}^{S}\sum_{z_2=1}^{2}\phi_{2,j,z_2}
< \mu_1(1-\theta)\sum_{j=s-N+1}^{S}\sum_{z_1=1}^{2}\phi_{z_1,j,1}+\mu_2\sum_{j=s-N+1}^{S}\sum_{z_1=1}^{2}\phi_{z_1,j,2}. \quad (5.1)
$$

Proof. Since the process under consideration is a level-independent quasi-birth-death process, it is stable iff

$$
\mathbf{\Phi} A_0\bar{\mathbf{e}} < \mathbf{\Phi} A_2\bar{\mathbf{e}} \quad (5.2)
$$

(Neuts [40]), where $\mathbf{\Phi}$ is the steady-state distribution of the generator matrix $A = A_0 + A_1 + A_2$. Write

$$
\mathbf{\Phi} = (\phi_{1,s-N+1,1}, \phi_{1,s-N+1,2}, \phi_{1,s-N+2,1}, , \phi_{1,s-N+2,2}, \ldots, \phi_{1,S,1}, \phi_{1,S,2}, \phi_{2,s-N+1,1}, \phi_{2,s-N+1,2}, \ldots, \phi_{2,s-N+2,1}, \phi_{2,s-N+2,2}, \ldots, \phi_{2,S,1}, \phi_{2,S,2}).
$$

Then after some algebra we get

$$\mathbf{\Phi} A_0\bar{\mathbf{e}} = \lambda_1(1-b)\sum_{j=s-N+1}^{S}\sum_{z_2=1}^{2}\phi_{1,j,z_2} + \lambda_2\sum_{j=s-N+1}^{S}\sum_{z_2=1}^{2}\phi_{2,j,z_2}$$

and

$$\mathbf{\Phi} A_2\bar{\mathbf{e}} = (1-\theta)\mu_1\sum_{j=s-N+1}^{S}\sum_{z_1=1}^{2}\phi_{z_1,j,1} + \mu_2\sum_{j=s-N+1}^{S}\sum_{z_1=1}^{2}\phi_{z_1,j,2}.$$

Hence by using equation (5.2), we get the stability condition as

$$\begin{aligned}\lambda_1(1-b)&\sum_{j=s-N+1}^{S}\sum_{z_2=1}^{2}\phi_{1,j,z_2} + \lambda_2\sum_{j=s-N+1}^{S}\sum_{z_2=1}^{2}\phi_{2,j,z_2}\\ &< \mu_1(1-\theta)\sum_{j=s-N+1}^{S}\sum_{z_1=1}^{2}\phi_{z_1,j,1} + \mu_2\sum_{j=s-N+1}^{S}\sum_{z_1=1}^{2}\phi_{z_1,j,2}.\end{aligned}$$

□

Theorem 5.2.2. *When the stability condition holds, the steady-state probability vector* $\bar{\boldsymbol{\pi}}$ *which is partitioned as* $\bar{\boldsymbol{\pi}} = (\boldsymbol{\pi}^{(0)}, \boldsymbol{\pi}^{(1)}, \boldsymbol{\pi}^{(2)}, \ldots)$ *where each* $\boldsymbol{\pi}^{(i)} = \big(\pi^{(i,1,s-N+1,1)}, \pi^{(i,1,s-N+1,2)}, \pi^{(i,1,s-N+2,1)}, \pi^{(i,1,s-N+2,2)}, \ldots,$
$\pi^{(i,1,S,1)}, \pi^{(i,1,S,2)}, \pi^{(i,2,s-N+1,1)}, \pi^{(i,2,s-N+1,2)}, \ldots, \pi^{(i,2,S,1)}, \pi^{(i,2,S,2)}\big)$,
where $i \geq 1$ *and*

$$\begin{aligned}\boldsymbol{\pi}^{(0)} = \big(\pi^{(0,1,s-N+1)}, \pi^{(0,1,s-N+2)}, \ldots, \pi^{(0,1,S)},\\ \pi^{(0,2,s-N+1)}, \pi^{(0,2,s-N+2)}, \ldots, \pi^{(0,2,S)}\big)\end{aligned}$$

is given by

$$\boldsymbol{\pi}^{(i)} = \boldsymbol{\pi}^{(0)}R^i, \quad i = 0,1,2,\ldots$$

The matrix R *is the minimal non-negative solution of the matrix-quadratic equation* $R^2A_2 + RA_1 + A_0 = O$ *and the vector* $\boldsymbol{\pi}^{(0)}$ *is obtained by solving* $\boldsymbol{\pi}^{(0)}(B_1 + RA_2) = \mathbf{0}$ *and* $\boldsymbol{\pi}^{(0)}(B_0 + RA_1 + R^2A_2) = \mathbf{0}$ *subject to the normalizing condition* $\boldsymbol{\pi}^{(0)}(I-R)^{-1}\bar{\mathbf{e}} = 1$ *(Neuts [40]).*

Proof. Since Markov process is regular, the stationary probability distribution exists and is given by

$$\bar{\boldsymbol{\pi}}\tilde{A} = \mathbf{0} \text{ and } \bar{\boldsymbol{\pi}}\bar{\mathbf{e}} = 1$$

where

$$\tilde{A} = \begin{bmatrix} B_1 & B_0 & & \\ B_2 & A_1 & A_0 & \\ & A_2 & A_1 & A_0 \\ & \ddots & \ddots & \ddots \end{bmatrix}.$$

$$\bar{\boldsymbol{\pi}}\tilde{A} = \mathbf{0} \Rightarrow$$

$$\boldsymbol{\pi}^{(0)}B_1 + \boldsymbol{\pi}^{(1)}B_2 = \mathbf{0} \tag{5.3}$$

$$\boldsymbol{\pi}^{(0)}B_0 + \boldsymbol{\pi}^{(1)}A_1 + \boldsymbol{\pi}^{(2)}A_2 = \mathbf{0} \tag{5.4}$$

$$\boldsymbol{\pi}^{(i)}A_0 + \boldsymbol{\pi}^{(i+1)}A_1 + \boldsymbol{\pi}^{(i+2)}A_2 = \mathbf{0} \tag{5.5}$$

where $i = 1, 2, \ldots$.

In order to express the solution in a recursive form, we assume that

$$\boldsymbol{\pi}^{(i)} = \boldsymbol{\pi}^{(0)}R^i, \quad i = 0, 1, 2, 3, \ldots \tag{5.6}$$

where the spectral radius of R is less than 1, which is ensured by the stability condition given by (5.1).
(5.6) in (5.3) $\Rightarrow$

$$\boldsymbol{\pi}^{(0)}(B_1 + RB_2) = \mathbf{0}. \tag{5.7}$$

(5.6) in (5.4) $\Rightarrow$

$$\boldsymbol{\pi}^{(0)}(B_0 + RA_1 + R^2A_2) = \mathbf{0}. \tag{5.8}$$

(5.6) in (5.5) $\Rightarrow$

$$\boldsymbol{\pi}^{(0)}R^i(A_0 + RA_1 + R^2A_2) = \mathbf{0},$$

where $i = 1, 2, 3, \ldots$.

$$\text{That is,} \quad \boldsymbol{\pi}^{(i)}(A_0 + RA_1 + R^2A_2) = \mathbf{0}, \tag{5.9}$$

where $i = 1, 2, 3, \ldots$.

Since (5.9) in true for $i = 1, 2, 3, \ldots$ we get

$$R^2 A_2 + R A_1 + A_0 = O \tag{5.10}$$

where A_0, A_1 and A_2 are known.

Hence R is a solution of the matrix-quadratic equation (5.10).

Also, we have $\bar{\boldsymbol{\pi}}\bar{\mathbf{e}} = 1$ which is the normalizing condition.

$$\sum_{i=0}^{\infty} \boldsymbol{\pi}^{(i)} \bar{\boldsymbol{e}} = 1.$$

$$\sum_{i=0}^{\infty} (\boldsymbol{\pi}^{(0)} R^i) \bar{\boldsymbol{e}} = 1.$$

$$\boldsymbol{\pi}^{(0)} (I - R)^{-1} \bar{\boldsymbol{e}} = 1. \tag{5.11}$$

$\boldsymbol{\pi}^{(0)}$ is obtained by solving (5.7) and (5.8) subject to the normalizing condition (5.11).

Hence the theorem. □

5.3 System Performance Measures

(a) Expected inventory held in the system,

$$E(I) = \sum_{i=1}^{\infty} \sum_{z_1=1}^{2} \sum_{j=s-N+1}^{S} \sum_{z_2=1}^{2} j\pi^{(i,z_1,j,z_2)} + \sum_{z_1=1}^{2} \sum_{j=s-N+1}^{S} j\pi^{(0,z_1,j)}.$$

(b) Mean waiting time of customers in the system,

$$W_S = \left(\frac{1}{\lambda_1} + b\frac{1}{\lambda_2} \right) L$$

where

$$L = \sum_{i=1}^{\infty} \sum_{z_1=1}^{2} \sum_{j=s-N+1}^{S} \sum_{z_2=1}^{2} i\pi^{(i,z_1,j,z_2)},$$

is the expected number of customers in the system.

(c) Mean reorder rate

$$R_r = (1-\theta)\mu_1 \sum_{i=1}^{\infty}\sum_{z_1=1}^{2} \pi^{(i,z_1,s+1,1)} + \mu_2 \sum_{i=1}^{\infty}\sum_{z_1=1}^{2} \pi^{(i,z_1,s+1,2)}.$$

(d) Mean local purchase rate

$$R_{\text{LP}} = (1-\theta)\mu_1 \sum_{i=1}^{\infty}\sum_{z_1=1}^{2} \pi^{(i,z_1,s-N+1,1)} + \mu_2 \sum_{i=1}^{\infty}\sum_{z_1=1}^{2} \pi^{(i,z_1,s-N+1,2)}.$$

5.4 Cost Analysis

Now we obtain a cost function. Let the various costs involved in the model be as given below:
C_H : Inventory holding cost per unit item per unit time
C_S : Fixed setup cost per unit order, under natural purchase
C_C : Cost per unit order cancelled
C_W : Cost of waiting time per customer per unit time
C_{LP} : Cost per unit order under local purchase
C_{NP} : Cost per unit order under natural purchase.

$$\text{TEC} = C_H E(I) + (C_S + C_{\text{NP}} Q) R_r + C_{\text{LP}}(Q+N) R_{\text{LP}} + C_C R_{\text{LP}} + C_W W_S.$$

5.4.1 Numerical Analysis

Case 1. Analysis of TEC and certain performance measures as functions of N.

Input Data:
$s = 8$, $S = 20$, $b = 0.3$, $\theta = 0.6$, $\lambda_1 = 23$, $\lambda_2 = 20$, $\mu_1 = 25$, $\mu_2 = 24$, $\gamma = 20$, $C_H = 0.5$, $C_S = 1000$, $C_C = 16$, $C_W = 1200$, $C_{\text{LP}} = 35$, $C_{\text{NP}} = 30$.

Table 5.1 shows that as N increases, TEC values decrease, reach a minimum at $N = 7$ and then increase. Hence it is numerically verified that TEC function is convex with respect to N.

Table 5.2 shows that mean reorder rate R_r is a convex function in N. Also we get that as N increases, both $E(I)$ and R_{LP} are monotonically decreasing.

Table 5.1: Effect of N on TEC

N	TEC
1	2902.7886
2	2727.6583
3	2655.2768
4	2626.3241
5	2615.6430
6	2612.3043
7	**2611.6755**
8	2611.8947

Table 5.2: Effect of N on $E(I)$, R_r and R_{LP}

N	$E(I)$	R_r	R_{LP}
1	14.2486	1.5993	0.8086
2	13.9919	**1.5985**	0.4089
3	13.7986	1.6145	0.2086
4	13.6711	1.6313	0.1062
5	13.5937	1.6442	0.0539
6	13.5495	1.6530	0.0273
7	13.5253	1.6587	0.0137
8	13.5126	1.6621	0.0069

Case 2. Analysis of TEC and certain performance measures as functions of S.

Input Data:

$N = 5$, $s = 8$, $b = 0.3$, $\theta = 0.6$, $\lambda_1 = 23$, $\lambda_2 = 20$, $\mu_1 = 25$, $\mu_2 = 24$, $\gamma = 20$, $C_H = 0.5$, $C_S = 1000$, $C_C = 16$, $C_W = 1200$, $C_{\text{LP}} = 35$, $C_{\text{NP}} = 30$.

Table 5.3: Effect of S on TEC

S	TEC
20	2615.6430
21	2490.8471
22	2383.7080
23	2290.7326
24	2209.2922
25	2137.3705
26	2073.3955
27	2016.1232

Table 5.3 shows that as S increases, TEC function is monotonically decreasing.

Table 5.4: Effect of S on $E(I)$, R_r and R_{LP}

S	$E(I)$	R_r	R_{LP}
20	13.5937	1.6442	0.0539
21	14.0954	1.5193	0.0498
22	14.5968	1.4120	0.0463
23	15.0980	1.3189	0.0432
24	15.5991	1.2373	0.0406
25	16.1001	1.1652	0.0382
26	16.6009	1.1011	0.0361
27	17.1016	1.0436	0.0342

Table 5.4 shows that as S increases, R_r and R_{LP} are monotonically decreasing and $E(I)$ is monotonically increasing.

Case 3. Analysis of TEC and certain performance measures when (N, s, S) varies simultaneously.

Input Data: $b = 0.3$, $\theta = 0.6$, $\lambda_1 = 23$, $\lambda_2 = 20$, $\mu_1 = 25$, $\mu_2 = 24$, $\gamma = 20$, $C_H = 0.5$, $C_S = 1000$, $C_C = 16$, $C_W = 1200$, $C_{\text{LP}} = 35$, $C_{\text{NP}} = 30$.

Table 5.5: Effect of simultaneous variation of (N, s, S) on TEC

(N, s, S)	TEC
(5,11,28)	2138.8705
(6,12,29)	2131.8418
(7,13,30)	2129.0491
(8,14,31)	2128.1484
(9,15,32)	**2128.0742**
(10,16,33)	2128.3515
(11,17,34)	2128.7727

Table 5.5 shows that as (N, s, S) values increase simultaneously, TEC values decrease first, reach a minimum at the values $(9, 15, 32)$ of (N, s, S) and then start increasing. Hence it is numerically verified that TEC function is convex.

Table 5.6: Effect of simultaneous variation of (N, s, S) on $E(I)$, R_r and R_{LP}

(N, s, S)	$E(I)$	R_r	R_{LP}
(5,11,28)	19.1001	1.1652	0.0382
(6,12,29)	20.0545	1.1697	0.0193
(7,13,30)	21.0289	1.1725	0.0097
(8,14,31)	22.0151	1.1742	0.0049
(9,15,32)	23.0077	1.1752	0.0025
(10,16,33)	24.0039	1.1757	0.0012
(11,17,34)	25.0019	1.1761	0.0006

Table 5.6 shows that as (N, s, S) values increase simultaneously, R_{LP} is monotonically decreasing but $E(I)$ and R_r are monotonically increasing.

Concluding Remarks

In this thesis we have considered certain queuing-inventory models with positive service time and lead time. In all the models considered, the concept of local purchase has been introduced to prevent customer loss, thereby ensuring customer satisfaction and goodwill. It is common practice that when an item is not available in a shop for which a demand arrives, the same is purchased from other shops locally and supplied to the customer. It is assumed that local purchase of items is made at a larger cost, and is done during a lead time. For the models considered in the thesis, local purchase is driven by N/T-policy. Also supply of items is assumed to be instantaneous in local purchase. These are aimed at getting stochastic decomposition of system state and also product form solution. Hence our assumptions turn out to be sharper. The only exception to this is the last chapter where we have brought in Coxian distribution of order 2. We are encouraged to do this in the light of the paper by Tijms [54].

In chapters 2 and 3, we have considered perishable as well as non-perishable (s, Q) inventory systems with Poisson arrivals, exponential service time and lead time. When the inventory level depletes to s from S, a replenishment order of $Q = S - s$ units is placed. In both the models given in chapter 2, local purchase is guided by N-policy and in both the models given in chapter 3, local purchase is guided by T-policy. Also, for the inventory models in these two chapters, we considered replenishment in bulk against orders placed on the inventory level reaching s.

In chapter 4, we have considered (s, S) production inventory models, with positive service time that follows exponential distribution. Arrivals of demands are according to Poisson process. Here additions of items take place to stock one at a time through the production process. As in the case of bulk replenishment, in this case also, production is switched on when inventory depletes from s to S. Therefore, the production process is on till the inventory level reaches S. The time to produce an item (inter-production time/lead time) is assumed to be exponentially distributed. The produced item requires a processing time before it is supplied to the customer, and the processing time (service time) is a random variable following exponential distribution. In all models in chapters 2, 3 and 4, we have arrived at product-form solution for the state probabilities. Explicit cost functions are analyzed encouraged by the stochastic decomposition property enjoyed by the models. We have established convexity of cost function numerically. In chapter 5, we considered an (s, Q) queueing inventory model with positive service time and lead time in which both inter-arrival time and service time are assumed to follow Coxian-2 distribution. Lead time is assumed to follow exponential distribution and whenever the inventory level drops to $s - N$ during a lead time, a local purchase of $Q + N$ units is made to raise the inventory level to S. In this model, we are not able to reach at product-form solution, and the situation is analyzed using matrix-geometric method. Nevertheless, convexity of this cost function is established numerically.

The results in this thesis can be extended to more general situations, such as MAP arrivals and phase type service time. The results in chapters 4 and 5 can be extended to the case of perishable items with or without common life time. Also we aim at analyzing inventory models with positive service time driven by D-policy and also by dyadic policies like Max (N, T), Min (N, T), Min (N, D) etc.

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APPENDIX

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