Characterizations of Bivariate Models Using Some Dynamic Conditional Information Divergence Measures

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In this article, we study some relevant information divergence measures viz. Renyi divergence and Kerridge's inaccuracy measures. These measures are extended to conditionally specified models and they are used to characterize some bivariate distributions using the concepts of weighted and proportional hazard rate models. Moreover, some bounds are obtained for these measures using the likelihood ratio order.

1. Introduction

Measures of divergence are used as a way to evaluate the distance (divergence) between two populations or functions. They have a very long history initiated by the pioneer works of Pearson, Mahalanobis, Levy, and Kolmogorov. There are many discrimination measures available in the literature. Renyi's information divergence of order α is one of the most popular discrimination measures available in the literature (see, for example, Renyi, 1961; Asadi et al., 2005a, b; Abbasnejad and Arghami, 2010; and the references therein) and plays an important role in information theory, reliability and other related fields.

Let X and Y be two absolutely continuous, non negative random variables with common support (l, ∞) for $l \geq 0$ that describe the lifetimes of two items. Denote by f, F, and $\overline{F}$ the probability density function (pdf), cumulative distribution function (cdf), and survival (or reliability) function (SF) of X, respectively, and by g, G, and $\overline{G}$, the corresponding functions of Y. Then Renyi's information divergence of order α between X and Y is defined

(see Renyi, 1961) by

$$I_{X,Y} = \frac{1}{1-\alpha} \ln \int_l^\infty f^\alpha(x) g^{1-\alpha}(x) dx$$

for α such that $0 < \alpha \neq 1$.

However, in many applied problems viz., reliability, survival analysis, economics, business, actuary, etc., one might have information about the current age of the systems, and thus the available information is dynamic. In particular, consider an item under study with lifetime X, then information about the residual (or past) lifetime $(X - t|X > t)$ (resp. $(t - X|X < t)$) is an important task in many applied problems. Then the discrimination information function between two residual lifetime distributions based on Renyi's information divergence of order α (see Asadi et al., 2005a) is given by

$$I_{X,Y}(t) = \frac{1}{1-\alpha} \ln \int_t^\infty \frac{f^\alpha(x)}{\overline{F}^\alpha(t)} \frac{g^{1-\alpha}(x)}{\overline{F}^{1-\alpha}(t)} dx$$

for $0 < \alpha \neq 1$. Note that $I_{X,Y}(t) = I_{X_t,Y_t}$, where $X_t = (X-t|X > t)$ and $Y_t = (Y-t|Y > t)$ are the residual lifetimes associated with X and Y. A similar function obtained in terms of the inactivity times $(t - X|X < t)$ and $(t - Y|Y < t)$ is available in Asadi et al. (2005b) (see also Maya and Sunoj, 2008).

Recently, the inaccuracy measure due to Kerridge (1961) is also widely used as a useful tool to measure the inaccuracy between two random variables X and Y. It is given by

$$K_{X,Y} = -\int_l^\infty f(x) \ln g(x) dx.$$

It can be expressed as

$$K_{X,Y} = H(X, Y) + H(X),$$

where $H(X, Y) = \int_l^\infty f(x) \ln(f(x)/g(x)) dx$ is the Kullback-Leibler divergence between X and Y and $H(X) = -\int_l^\infty f(x) \ln f(x) dx$ is the measure of entropy of X.

Taneja et al. (2009) introduced a dynamic version of Kerridge's measure, given by

$$K_{X,Y}(t) = -\int_t^\infty \frac{f(x)}{\overline{F}(t)} \ln \frac{g(x)}{\overline{G}(t)} dx.$$

Note that $K_{X,Y}(t) = K_{X_t,Y_t}$. Clearly, when $X = Y$, $K_{X,X}(t)$ becomes the popular dynamic measure of uncertainty (residual entropy) due to Ebrahimi (1996). Some characterization results based on $K_{X,X}(t)$ were given in Belzunce et al. (2004). A similar expression for the inactivity times is available in Kumar et al. (2011).

Cox's Proportional Hazards Rate (PHR) model is the most widely used semi-parametric model in survival studies. Two random variables X and Y and with common support (l, ∞) satisfy the PHR model when

$$h_Y(t) = \theta h_X(t)$$

for all $t \geq l$, where $\theta > 0$ and $h_Y = g/\overline{G}$ and $h_X = f/\overline{F}$ are the respective failure (or hazard) rate functions. Equivalently, X and Y satisfy the PHR model when $\overline{G}(t) = \overline{F}^\theta(t)$ for all $t \geq l$ (see Cox, 1959).

The concept of weighted distributions was introduced by Rao (1965) in connection with modeling statistical data in situations where the usual practice of employing standard distributions was not found appropriate. Associated to a random variable X with pdf f and to a nonnegative real function w, the pdf of the weighted random variable X^w can be defined as

$$f^w(x) = \frac{w(x)f(x)}{E(w(X))}$$

whenever $0 < E(w(X)) < \infty$. For recent works in this area we refer to Navarro et al. (2006), Blazej (2008), Bartoszewicz (2009), Maya and Sunoj (2008), Navarro and Sarabia (2010), Sunoj and Sreejith (2012) and the references therein.

Specification of the joint distribution through conditional densities has been an important problem considered by many researchers. This approach of identifying a bivariate density using the conditionals is called the conditional specification of joint distribution (see Arnold et al., 1999). These models are often useful in two component reliability systems where the operational status of one component is known. For more recent works on conditionally specified models, we refer to Sunoj and Sankaran (2005), Kotz et al. (2007), Navarro and Sarabia (2010), Sunoj and Linu (2012), and the references therein.

In Navarro et al. (2011), some characterizations of bivariate models using extensions of the dynamic Kullback-Leibler discrimination measures to conditionally specified models are obtained. In the present article, Renyi divergence and Kerridge's inaccuracy measure for residual (past) lifetimes are extended to conditionally specified models and they are used to characterize some bivariate distributions. The concept of proportional hazard rate models and weighted distributions are also used to characterize some bivariate distributions. Moreover bounds for these measures are obtained by using the likelihood ratio order. The results for Renyi divergence are given in Sec. 2 and that for Kerridge's inaccuracy measure in Sec. 3.

2. Characterizations Using Renyi Divergence Measure

Let (X_1, X_2) and (Y_1, Y_2) be two bivariate random vectors with common support $(l, \infty) \times (l, \infty)$ for $l \geq 0$. The joint pdf and sf of (X_1, X_2) are denoted by f and $\overline{F}$ and that of (Y_1, Y_2) by g and $\overline{G}$, respectively. Consider the conditionally specified random variables $(X_i|X_j = t)$ and $(Y_i|Y_j = t)$ for $i, j = 1, 2, i \neq j$. Their pdf and sf are denoted by $f_i(s|t)$, $\overline{F}_i(s|t)$, $g_i(s|t)$, $\overline{G}_i(s|t)$, respectively, for $i = 1, 2$. Then we define the conditional Renyi's discrimination information (CRDI) functions as

$$I_{X_i, Y_i}(s|t) = \frac{1}{1-\alpha} \ln \int_s^\infty \frac{f_i^\alpha(x|t)}{\overline{F}_i^\alpha(s|t)} \frac{g_i^{1-\alpha}(x|t)}{\overline{G}_i^{1-\alpha}(s|t)} dx \tag{1}$$

for $i = 1, 2$ and $s, t \geq l$. Note that $I_{X_i, Y_i}(s|t) = I_{(X_i|X_{3-i}=t),(Y_i|Y_{3-i}=t)}(s)$ for $i = 1, 2$. Hence, (1) provides dynamic information on the distance between the conditionally specified random variables.

The PHR model is extended to conditional models as follows. The random vectors (X_1, X_2) and (Y_1, Y_2) satisfy the conditional proportional hazard rate (CPHR) model (see Sankaran and Sreeja, 2007) when the corresponding conditional hazard rate functions of

$(X_i|X_{3-i} = t)$ and $(Y_i|Y_{3-i} = t)$ satisfy

$$h_{(Y_i|Y_{3-i}=t)}(s|t) = \theta_i(t)h_{(X_i|X_{3-i}=t)}(s|t) \tag{2}$$

for $i = 1, 2$ and $s, t \geq l$, where $\theta_1(t)$ and $\theta_2(t)$ are positive functions of t. Then we have the following result.

Theorem 2.1. *For $i = 1, 2$ and $0 < \alpha \neq 1$, the function $I_{X_i, Y_i}(s|t)$ only depends on* t *if, and only if, $(X_i|X_{3-i} = t)$ and $(Y_i|Y_{3-i} = t)$ satisfy (2).*

Proof.   For $i = 1$, let us suppose that $(X_1|X_2 = t)$ and $(Y_1|Y_2 = t)$ satisfy (2). Then their survival functions satisfy $\overline{G}_1(s|t) = \overline{F}_1^{\theta_1(t)}(s|t)$. Hence, from (1), we get

$$I_{X_1, Y_1}(s|t) = \frac{1}{1-\alpha} \ln \frac{\theta_1^{1-\alpha}(t)}{(1-\alpha)\theta_1(t) + \alpha}$$

whenever $(1 - \alpha)\theta_1(t) + \alpha > 0$. Then $I_{X_1, Y_1}(s|t)$ only depends on t. The proof for $i = 2$ is similar.

Conversely, if $i = 1$, let us suppose that $I_{X_1, Y_1}(s|t)$ only depends on t for some $0 < \alpha \neq 1$. Then,

$$\frac{1}{1-\alpha} \ln \int_s^\infty \frac{f_1^\alpha(x|t)}{\overline{F}_1^\alpha(s|t)} \frac{g_1^{1-\alpha}(x|t)}{\overline{G}_1^{1-\alpha}(s|t)} dx = A_1(t),$$

say or, equivalently,

$$\int_s^\infty f_1^\alpha(x|t)g_1^{1-\alpha}(x|t)dx = B_1(t)\overline{F}_1^\alpha(s|t)\overline{G}_1^{1-\alpha}(s|t),$$

where $B_1(t) = \exp((1 - \alpha)A_1(t))$. Differentiating with respect to s we get

$$\frac{1}{B_1(t)} = \alpha\phi_1^{\alpha-1}(s|t) + (1 - \alpha)\phi_1^\alpha(s|t),$$

where $\phi_1(s, t) = h_{(Y_1|Y_2=t)}(s|t)/h_{(X_1|X_2=t)}(s|t)$. Differentiating again with respect to s we get

$$0 = \alpha(\alpha - 1)\phi_1^{\alpha-2}(s, t)(1 - \phi_1(s, t))\frac{\partial}{\partial s}\phi_1(s, t).$$

As $\alpha(\alpha - 1) \neq 0$, we have $\frac{\partial}{\partial s}\phi_1(s, t) = 0$ and hence $\phi_1(s, t) = \theta_1(t)$. Therefore, (2) holds for $i = 1$. The proof for the case $i = 2$ is similar.   □

Now we consider a random vector (X_1^w, X_2^w) which has a bivariate weighted distribution associated with (X_1, X_2) and to two nonnegative real functions w_1 and w_2, that is, its joint pdf is given by

$$f^w(x_1, x_2) = \frac{w_1(x_1)w_2(x_2)f(x_1, x_2)}{E(w_1(X_1)w_2(X_2))},$$

where f is the joint pdf of (X_1, X_2) and $0 < E(w_1(X_1)w_2(X_2)) < \infty$. With this definition it is easy to see that the marginal random variable X_i^w has the (univariate) weighted distribution

associated with X_i and

$$w_i^*(x_i) = w_i(x_i)E(w_{3-i}(X_{3-i})|X_i = x_i)$$

for $i = 1, 2$ (note in passing that there is a typo in this comment in Navarro et al., 2011). Analogously, it is easy to prove that $(X_i^w|X_{3-i}^w = t)$ has the (univariate) weighted distribution associated with $(X_i|X_{3-i} = t)$ and $w_i(x_i)$ for $i = 1, 2$. Particularly, when $w_1(x_1) = x_1$ and $w_2(x_2) = x_2$, the random vector (X_1^w, X_2^w) is called the length biased random vector. The details and other ways of defining the bivariate weighted distributions can be found in Navarro et al. (2006).

Now we can state the main result of this section.

Theorem 2.2. *Let (X_1, X_2) be an absolutely continuous random vector with support $(l, \infty) \times (l, \infty)$ for $l \geq 0$. Let (X_1^w, X_2^w) be a random vector which has the bivariate weighted distribution associated with (X_1, X_2) and to two non negative, monotone, and differentiable functions w_1 and w_2 in (l, ∞). Then the following conditions are equivalent:*

(a) (X_1^w, X_2^w) and (X_1, X_2) satisfy the CPHR model (2) for $i = 1, 2$;
(b) $I_{X_i, X_i^w}(s|t)$ only depends on t for $i = 1, 2$ and $0 < \alpha \neq 1$.
(c) the conditional survival functions of (X_1, X_2) satisfy

$$\ln \overline{F}_i(s|t) = \frac{\ln(w_i(s)/w_i(l))}{\theta_i(t) - 1}$$

for $i = 1, 2$;
(d) (X_1, X_2) has the following joint PDF

$$f(x_1, x_2) = \frac{ca_1a_2w_1'(x_1)w_2'(x_2)}{w_1^{a_1+1}(x_1)w_2^{a_2+1}(x_2)} \exp\left(-\phi a_1 a_2 \left(\ln \frac{w_1(x_1)}{w_1(l)}\right)\left(\ln \frac{w_2(x_2)}{w_2(l)}\right)\right)$$

for $x_1, x_2 \geq l$, where $c > 0$, $\phi \geq 0$ and $a_i > 1$ or $a_i < 0$ for $i = 1, 2$.

Proof. The equivalence of (a) and (b) is a direct consequence of Theorem 2.1.

The equivalences of (a), (c), and (d) were proved in Navarro et al. (2011). □

The comments given after Theorem 3 in Navarro et al. (2011) also hold for the present theorem. In particular, note that the model given in (d) is a truncated version of the conditional proportional hazard rate model considered by Arnold and Strauss (1988) and that Arnold and Strauss' model is obtained when $l = 0$. The reliability properties of this semiparametric model can be seen in Navarro and Sarabia (2013). Some particular parametric models can be obtained from the general model given in (d). For example, if $l = 1$ and $w_i(t) = t$ for $i = 1, 2$, then we get a bivariate Pareto model (see Navarro et al., 2011).

We end this section obtaining bounds for the CRDI functions by using the likelihood ratio (LR) order. The results are similar to that given in Di Crescenzo and Longobardi (2004) for the Kullback-Leibler divergence. First we give the definition of the LR order.

If X and Y have pdf f and g, respectively, then X is said to be less than Y in the likelihood ratio order (denoted by $X \leq_{LR} Y$) if g/f is increasing in the union of their supports.

Then we have the following results.

Theorem 2.3. *For $i = 1, 2$, if $(X_i|X_{3-i} = t) \leq_{LR} (Y_i|Y_{3-i} = t)$ $(\geq_{LR})$, then*

$$I_{X_i,Y_i}(s|t) \geq \ln \frac{h_{(Y_i|Y_{3-i}=t)}(s|t)}{h_{(X_i|X_{3-i}=t)}(s|t)} \quad (\leq).$$

Proof. From the definition, we have

$$I_{X_i,Y_i}(s|t) = \frac{1}{1-\alpha} \ln \int_s^\infty \frac{f_i(x|t)}{\overline{F}_i(s|t)} \frac{g_i^{1-\alpha}(x|t)}{f_i^{1-\alpha}(x|t)} \frac{\overline{F}_i^{1-\alpha}(s|t)}{\overline{G}_i^{1-\alpha}(s|t)} dx.$$

Now as $(X_i|X_{3-i} = t) \leq_{LR} (Y_i|Y_{3-i} = t)$, then $g_i(x|t)/f_i(x|t)$ increases and we get

$$I_{X_i,Y_i}(s|t) \geq \frac{1}{1-\alpha} \ln \left(\frac{g_i^{1-\alpha}(s|t)}{f_i^{1-\alpha}(s|t)} \frac{\overline{F}_i^{1-\alpha}(s|t)}{\overline{G}_i^{1-\alpha}(s|t)} \right)$$

and hence the stated result holds. The proof of the other case is similar. $\qquad\square$

Theorem 2.4. *For $i = 1, 2$ and $0 < \alpha \neq 1$, if w_i is increasing (decreasing), then*

$$I_{X_i,X_i^w}(s|t) \geq \ln \frac{w_i(s)}{E(w_i(X_i)|X_i > s, X_{3-i} = t)} \quad (\leq).$$

Proof. The proof is immediate from the preceding theorem since if w_i is increasing (decreasing), then $(X_i|X_{3-i} = t) \leq_{LR} (X_i^w|X_{3-i}^w = t)$ $(\geq_{LR})$ holds. $\qquad\square$

Theorem 2.5 *For $i = 1, 2$ and $0 < \alpha \neq 1$, if $(Y_i|Y_{3-i} = t) \leq_{LR} (Z_i|Z_{3-i} = t)$ $(\geq_{LR})$, then*

$$I_{X_i,Y_i}(s|t) \leq I_{X_i,Z_i}(s|t) + \ln \frac{h_{(Y_i|Y_{3-i}=t)}(s|t)}{h_{(Z_i|Z_{3-i}=t)}(s|t)} \quad (\geq).$$

Proof. From the definition, we have

$$I_{X_i,Y_i}(s|t) = \frac{1}{1-\alpha} \ln \int_s^\infty \frac{f_i^\alpha(x|t)}{\overline{F}_i^\alpha(s|t)} \frac{k_i^{1-\alpha}(x|t)}{\overline{K}_i^{1-\alpha}(s|t)} \frac{g_i^{1-\alpha}(x|t)}{k_i^{1-\alpha}(x|t)} \frac{\overline{K}_i^{1-\alpha}(s|t)}{\overline{G}_i^{1-\alpha}(s|t)} dx,$$

where k_i and $\overline{K}_i$ are the pdf and sf of $(Z_i|Z_{3-i} = t)$.

Now as $(Y_i|Y_{3-i} = t) \leq_{LR} (Z_i|Z_{3-i} = t)$, then $g_i(x|t)/k_i(x|t)$ decreases and we get

$$I_{X_i,Y_i}(s|t) \leq I_{X_i,Z_i}(s|t) + \frac{1}{1-\alpha} \ln \left(\frac{g_i^{1-\alpha}(s|t)}{k_i^{1-\alpha}(s|t)} \frac{\overline{K}_i^{1-\alpha}(s|t)}{\overline{G}_i^{1-\alpha}(s|t)} \right)$$

and hence the stated result holds. The proof of the other case is similar. $\qquad\square$

3. Characterizations Using Kerridge's Inaccuracy Measure

In this section we obtain properties similar to that given in Section 2 for the Kerridge's inaccuracy measures. The conditional Kerridge's inaccuracy measures (CKIM) of (X_1, X_2)

and (Y_1, Y_2) are defined by

$$K_{X_i, Y_i}(s|t) = -\int_s^\infty \frac{f_i(x|t)}{\overline{F}_i(s|t)} \ln \frac{g_i(x|t)}{\overline{G}_i(s|t)} dx \qquad (3)$$

for $i = 1, 2$ and $s, t \geq l$. Note that $K_{X_i, Y_i}(s|t) = K_{(X_i|X_{3-i}=t),(Y_i|Y_{3-i}=t)}(s)$ for $i = 1, 2$.

Then we have the following characterization result for the Arnold and Strauss's bivariate exponential distribution obtained in Arnold and Strauss (1988).

Theorem 3.1. *Let (X_1, X_2) and (Y_1, Y_2) be two random vectors which common support $(0, \infty) \times (0, \infty)$ and that satisfy the CPHR model given in (2) for $i = 1, 2$. Then the following conditions are equivalent:*

(a) $K_{X_i, Y_i}(s|t)$ only depends on t *for $i = 1, 2$;*
(b) (X_1, X_2) has the following joint pdf

$$f(x_1, x_2) = c \exp(-\lambda_1 x_1 - \lambda_2 x_2 - \theta x_1 x_2)$$

for $x_1, x_2 \geq 0$, where $c > 0, \theta \geq 0$ and $\lambda_i > 0$ for $i = 1, 2$.

Proof. If (X_1, X_2) and (Y_1, Y_2) satisfy the CPHR model given in (2), then

$$\overline{G}_i(s|t) = \overline{F}_i^{\theta_i(t)}(s|t)$$

for $i = 1, 2$.

To prove that (a) implies (b), let us assume that $K_{X_i, Y_i}(s|t)$ only depends on t for $i = 1, 2$. Then for $i = 1$ we have

$$-\int_s^\infty f_1(x|t) \ln\left(\theta_1(t)\overline{F}_1^{\theta_1(t)-1}(x|t)f_1(x|t)\right)dx$$
$$= C_1(t)\overline{F}_1(s|t) - \theta_1(t)\overline{F}_1(s|t)\ln\overline{F}_1(s|t).$$

Differentiating with respect to s we get

$$f_1(s|t) \ln\left(\theta_1(t)\overline{F}_1^{\theta_1(t)-1}(s|t)f_1(s|t)\right)$$
$$= -C_1(t)f_1(s|t) + \theta_1(t)f_1(s|t)(1 + \ln\overline{F}_1(s|t)).$$

Hence,

$$\ln\left(\theta_1(t)\overline{F}_1^{\theta_1(t)-1}(s|t)f_1(s|t)\right) = -C_1(t) + \theta_1(t) + \ln\overline{F}_1^{\theta_1(t)}(s|t)$$

and

$$\ln(\theta_1(t)h_1(s|t)) = \theta_1(t) - C_1(t),$$

where $h_1(s|t) = f_1(s|t)/\overline{F}_1(s|t)$. Therefore $h_1(s|t)$ only depends on t. Analogously, it can be proved that $h_2(s|t)$ only depends on t. Hence, both conditional distributions are exponential and, from Arnold and Strauss (1988), we obtain the pdf given in (b).

The converse proof is straightforward. □

Finally, we obtain bounds for the CKIM functions by using the LR order.

Theorem 3.2. *For $i = 1, 2$, if $g_i(x|t)$ is decreasing in x, then*

$$K_{X_i, Y_i}(s|t) \geq -\ln h_{(Y_i|Y_{3-i}=t)}(s|t).$$

Proof. From the definition, we have

$$K_{X_i, Y_i}(s|t) = -\int_s^\infty \frac{f_i(x|t)}{\overline{F}_i(s|t)} \ln \frac{g_i(x|t)}{\overline{G}_i(s|t)} dx$$

$$\geq -\ln \frac{g_i(s|t)}{\overline{G}_i(s|t)} \int_s^\infty \frac{f_i(x|t)}{\overline{F}_i(s|t)} dx$$

$$= -\ln \frac{g_i(s|t)}{\overline{G}_i(s|t)}$$

where the inequality is obtained by using that $g_i(x|t)$ is decreasing. $\square$

Theorem 3.3. *For $i = 1, 2$, if $(X_i|X_{3-i} = t) \leq_{LR} (Y_i|Y_{3-i} = t)$ $(\geq_{LR})$, then*

$$K_{X_i, Y_i}(s|t) \leq H_{(X_i|X_{3-i}=t)}(s|t) + \ln \frac{h_{(X_i|X_{3-i}=t)}(s|t)}{h_{(Y_i|Y_{3-i}=t)}(s|t)} \quad (\geq),$$

where

$$H_{(X_i|X_{3-i}=t)}(s|t) = -\int_s^\infty \frac{f_i(x|t)}{\overline{F}_i(s|t)} \ln \frac{f_i(x|t)}{\overline{F}_i(s|t)} dx$$

is the residual entropy of $(X_i|X_{3-i} = t)$.

Proof. From the definition, we have

$$K_{X_i, Y_i}(s|t) = -\int_s^\infty \frac{f_i(x|t)}{\overline{F}_i(s|t)} \ln \left(\frac{f_i(x|t)}{\overline{F}_i(s|t)} \frac{g_i(x|t)}{f_i(x|t)} \frac{\overline{F}_i(s|t)}{\overline{G}_i(s|t)} \right) dx$$

$$= H_{(X_i|X_{3-i}=t)}(s|t) - \int_s^\infty \frac{f_i(x|t)}{\overline{F}_i(s|t)} \ln \left(\frac{g_i(x|t)}{f_i(x|t)} \frac{\overline{F}_i(s|t)}{\overline{G}_i(s|t)} \right) dx$$

$$\leq H_{(X_i|X_{3-i}=t)}(s|t) - \int_s^\infty \frac{f_i(x|t)}{\overline{F}_i(s|t)} \ln \left(\frac{g_i(s|t)}{f_i(s|t)} \frac{\overline{F}_i(s|t)}{\overline{G}_i(s|t)} \right) dx$$

$$= H_{(X_i|X_{3-i}=t)}(s|t) - \ln \frac{g_i(s|t)/\overline{G}_i(s|t)}{f_i(s|t)/\overline{F}_i(s|t)},$$

where the inequality is obtained by using that $g_i(x|t)/f_i(x|t)$ is increasing. $\square$

Theorem 3.4. *For $i = 1, 2$, if w_i is increasing (decreasing), then*

$$K_{X_i, X_i^w}(s|t) \geq H_{(X_i|X_{3-i}=t)}(s|t) + \ln \frac{E(w_i(X_i)|X_i > s, X_{3-i} = t)}{w_i(s)} \quad (\leq),$$

where $H_{(X_i|X_{3-i}=t)}(s|t)$ is the residual entropy of $(X_i|X_{3-i} = t)$ (see Theorem 3.3).

Proof. The proof is immediate from the preceding theorem by using that if w_i is increasing (decreasing), then $(X_i|X_{3-i} = t) \leq_{LR} (X_i^w|X_{3-i}^w = t)$ ($\geq$). $\qquad\square$

Theorem 3.5. *For $i = 1, 2$, if $(Y_i|Y_{3-i} = t) \leq_{LR} (Z_i|Z_{3-i} = t)$ ($\geq_{LR}$), then*

$$K_{X_i,Y_i}(s|t) \geq K_{X_i,Z_i}(s|t) + \ln \frac{h_{(Z_i|Z_{3-i}=t)}(s|t)}{h_{(Y_i|Y_{3-i}=t)}(s|t)} \quad (\leq).$$

Proof. From the definition, we have

$$K_{X_i,Y_i}(s|t) = -\int_s^\infty \frac{f_i(x|t)}{\overline{F}_i(s|t)} \ln \left(\frac{k_i(x|t)}{\overline{K}_i(s|t)} \frac{g_i(x|t)}{k_i(x|t)} \frac{\overline{K}_i(s|t)}{\overline{G}_i(s|t)} \right) dx$$

$$= K_{X_i,Z_i}(s|t) - \int_s^\infty \frac{f_i(x|t)}{\overline{F}_i(s|t)} \ln \left(\frac{g_i(x|t)}{k_i(x|t)} \frac{\overline{K}_i(s|t)}{\overline{G}_i(s|t)} \right) dx,$$

where $k_i(s|t)$ and $\overline{K}_i(s|t)$ are the pdf and sf of $(Z_i|Z_{3-i} = t)$. Now using that $g_i(x|t)/k_i(x|t)$ is decreasing, we have

$$K_{X_i,Y_i}(s|t) \geq K_{X_i,Z_i}(s|t) - \int_s^\infty \frac{f_i(x|t)}{\overline{F}_i(s|t)} \ln \left(\frac{g_i(s|t)}{k_i(s|t)} \frac{\overline{K}_i(s|t)}{\overline{G}_i(s|t)} \right) dx$$

$$= K_{X_i,Z_i}(s|t) - \ln \left(\frac{g_i(s|t)}{k_i(s|t)} \frac{\overline{K}_i(s|t)}{\overline{G}_i(s|t)} \right) dx$$

and hence the stated result holds. The proof of the other case is similar. $\qquad\square$

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References

Abbasnejad, M., Arghami, N. R. (2010). Renyi entropy properties of order statistics. *Commun. Statist. Theory Meth.* 40:40–52.

Arnold, B. C., Castillo, E., Sarabia, J. M. (1999). *Conditional Specification of Statistical Models.* New York: Springer Verlag.

Arnold, B. C., Strauss, D. (1988). Bivariate distributions with conditionals in prescribed exponential families. *J. Royal Statist. Soc. Ser. B* 53:365–375.

Asadi, M., Ebrahimi, N., Hamedani, G. H., Soofi, E. (2005a). Minimum dynamic discrimination information models. *J. Appl. Probab.* 42:643–660.

Asadi, M., Ebrahimi, N., Soofi, E. (2005b). Dynamic generalized information measures. *Statist. Probab. Lett.* 71:89–98.

Balakrishnan, N., Castillo, E., Sarabia, J. M. (2010). Bivariate continuous distributions with specified conditional hazard functions. *Commun. Statist. Theory Meth.* 39:2473–2484.

Bartoszewicz, J. (2009). On a representation of weighted distributions. *Statist. Probab. Lett.* 79:1690–1694.

Belzunce, F., Navarro, J., Ruiz, J. M., del Aguila, Y. (2004). Some results on residual entropy function. *Metrika* 59:147–161.

Blazej, P. (2008). Preservation of classes of life distributions under weighting with a general weight function. *Statist. Probab. Lett.* 78:3056–3061.

Cox, D. R. (1959). The analysis of exponentially distributed lifetimes with two types of failure. *J. Royal Statist. Soc. Ser. B* 21:411–421.

Di Crescenzo, A., Longobardi, M. (2004). A measure of discrimination between past lifetime distributions. *Statist. Probab. Lett.* 67:173–182.

Ebrahimi, N. (1996). How to measure uncertainty about residual life time. *Sankhyā A* 58:48–57.

Kerridge, D. F. (1961). Inaccuracy and inference. *J. Roy. Statist. Soc. Ser. B.* 23:184–194.

Kotz, S., Navarro, J., Ruiz, J. M. (2007). Characterizations of Arnold and Strauss' and related bivariate exponential models. *J. Multivariate Anal.* 98:1494–1507.

Kumar, V., Taneja, H. C., Srivastava, R. (2011). A dynamic measure of inaccuracy between two past lifetime distributions. *Metrika* 74:1–10.

Maya, S. S., Sunoj, S. M. (2008). Some generalized information measures in the context of weighted models. *Statistica* LXVIII(1):71–84.

Navarro, J., Ruiz, J. M., del Aguila, Y. (2006). Multivariate weighted distributions. *Statistics* 40:51–54.

Navarro, J., Sarabia, J. M. (2010). Alternative definitions of bivariate equilibrium distributions. *J. Statist. Plann. Infer.* 140:2046–2056.

Navarro, J., Sarabia, J. M. (2013). Reliability properties of bivariate conditional proportional hazard rates models. *J. Multivariate Anal.* 113:116–127.

Navarro, J., Sunoj, S. M., Linu, M. N. (2011). Characterizations of bivariate models using dynamic Kullback-Leibler discrimination measures. *Statist. Probab. Lett.* 81:1594–1598.

Rao, C. R. (1965). On discrete distributions arising out of methods of ascertainment. *Sankhyā A.* 27:311–324.

Renyi, A. (1961). On measures of entropy and information. Proceeding of the Fourth Berkeley Symposium on Mathematical Statistics and Probability 1: University of California Press, Berkeley, 547–561.

Sankaran, P. G., Sreeja, V. N. (2007). Proportional hazard model for multivariate failure time data. *Commun. Statist. Theory Meth.* 36:1627–1641.

Sunoj, S. M., Linu, M. N. (2011). Dynamic cumulative residual Renyi's entropy. *Statistics* (in press). Published online first. DOI: 10.1080/02331888.2010.494730.

Sunoj, S. M., Sankaran, P. G. (2005). Bivariate weighted distributions in the context of reliability modeling. *Calcutta Statist. Assoc. Bull.* 57:227–228.

Sunoj, S. M., Sreejith, T. B. (2012). Some results of reciprocal subtangent in the context of weighted models. *Commun. Statist. Theory. Meth.* 41:1397–1410.

Taneja, H. C., Kumar, V., Srivastava, R. (2009). A dynamic measure of inaccuracy between two residual lifetime distributions. *Int. Mathemat. Forum* 25:1213–1220.