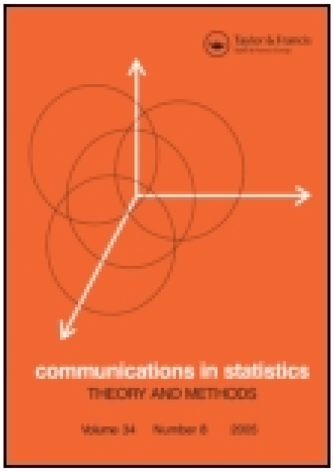

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Some Results on Reciprocal Subtangent in the Context of Weighted Models

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Some Results on Reciprocal Subtangent in the Context of Weighted Models

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Recently, reciprocal subtangent has been used as a useful tool to describe the behaviour of a density curve. Motivated by this, in the present article we extend the concept to the weighted models. Characterization results are proved for models viz. gamma, Rayleigh, equilibrium, residual lifetime, and proportional hazards. An identity under weighted distribution is also obtained when the reciprocal subtangent takes the form of a general class of distributions. Finally, an extension of reciprocal subtangent for the weighted models in the bivariate and multivariate cases are introduced and proved some useful results.

Keywords Characterization; Coordinate subtangent; Reliability measures; Weighted model.

Mathematics Subject Classification 62E10; 62N05; 62H05.

1. Introduction

The concept of reciprocal coordinate subtangent (RCST) has been used in the statistical literature as a useful tool to describe the behaviour of a density curve. It is considered as a measure for strongly unimodal property (see Hajek and Sidak, 1967). Let X be an absolutely continuous random variable (rv) having a probability density function (pdf) $f(x)$ such that $f'(x)$ exists. Then RCST to a curve $y = f(x)$ of the rv X is given by

$$T(x) = -\frac{f'(x)}{f(x)}. \tag{1.1}$$

RCST also plays very important role in reliability analysis, however, used it rather unknowingly. For example, failure rate or hazard rate is the CST measured on the curve $y = \overline{F}(x)$, where $\overline{F}(x)$ is the survival function. Also, since many of the failure rate functions have complex expressions, Glaser (1980) identified (1.1) (but

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not called as "RCST") as an easy statistical tool to determine the shape of the failure rate function. $T(x)$ can also be represent in terms of failure rate $h(x) = -\frac{\overline{F}'(x)}{\overline{F}(x)}$ by

$$T(x) = h(x) - \frac{h'(x)}{h(x)}, \tag{1.2}$$

provided $h'(x)$ exist. However, Mukherjee and Roy (1989) identified RCST as a measure to characterize various models by a unique determination of $f(x)$ from $T(x)$ by

$$f(x) = k \exp\left[-\int_0^x T(u)du\right],$$

where k is a normalizing constant. Mukherjee and Roy (1989) also studied some properties and applications of $T(x)$ and proved characterization results to certain important life distributions viz. exponential, Lomax, and finite range models. For a comprehensive literature review about RCST $T(x)$ and its applications, we refer to Pearson (1901), Ibragimov (1956), Gupta (2001), Gupta and Warren (2001), Block et al. (2002), Ghitany (2004), Mi (2004), and Lai and Xie (2006). Recently, Roy and Roy (2010) further extended the concept of RCST in the multivariate setup and proved some characterization theorems useful in reliability modeling.

In view of the importance and usefulness of RCST in various fields, in the present article, we further explore it in the context of weighted models. The article is organized as follows. In Sec. 2, we introduce RCST for weighted models. In Sec. 3, we prove some univariate characterizations to distributions such as gamma and Rayleigh, under the inversed length biased model. We also introduce characterizations to equilibrium, residual lifetime (reversed residual lifetime), and proportional hazard models in the context of weighted distributions. An identity for weighted distribution is also obtained when the reciprocal subtangent takes the form of a general class of distributions. Finally, in Secs. 4 and 5, we further extend RCST for weighted distributions to bivariate and multivariate setup and examine some characterization theorems arising out of it.

2. Weighted Models

The concept of weighted distributions was first formulated in a unified way by Rao (1965) in connection with modeling statistical data and in situations where the usual practice of employing standard distributions for the purpose was not found appropriate. If $f(x)$ is the pdf of a non negative rv X and $w(x)$ is a non negative weight function satisfying $\mu_w = E(w(X)) < \infty$, then the pdf $f_w(x)$ of the weighted rv X^w corresponding to X is given by

$$f_w(x) = \frac{w(x)f(x)}{\mu_w}.$$

Various weight functions are used in literature. For example, when the weight is proportional to length (size) of unit of interest, we use the weight function $w(x) = x(w(x) = x^\alpha, \alpha > 0)$ (see Gupta and Kirmani, 1990; Pakes et al., 2003; and references therein), whereas when $w(x) = \frac{1}{x}$, it is called inversed length biased distribution (see Barmi and Simonoff, 2000). For more details of applications and recent works of weighted distributions, we refer to Gupta and Kirmani (1990), Jones (1991), Navarro et al. (2001), Sunoj and Maya (2006), Di Crescenzo and Longobardi (2006), and Maya and Sunoj (2008).

Some of the known and important distributions in statistics and applied probability may be expressed as weighted distributions. Equilibrium distributions, distribution of order statistics, residual life distribution, and distribution in proportional hazard models (see Bartoszewicz and Skolimowska, 2004; Gupta and Kirmani, 1990) are some of the examples. Thus, the theory of weighted distributions is appropriate whenever these distributions are applied.

The wide applicability of weighted distributions in the univariate case has prompted many researchers to extend the concept to higher dimensions (see Mahfoud and Patil, 1982). Let $(X_1, X_2)'$ be a non negative bivariate random vector with pdf $f(x_1, x_2)$ and $w(x_1, x_2)$ be a non negative weight function such that $E(w(X_1, X_2)) < \infty$. Then the pdf of the bivariate weighted random vector $(X_1^w, X_2^w)'$, corresponding to $(X_1, X_2)'$ is given by

$$f_w(x_1, x_2) = \frac{w(x_1, x_2)f(x_1, x_2)}{E(w(X_1, X_2))}. \tag{2.1}$$

Jain and Nanda (1995) extended the definition to the p-variate case. Let $\boldsymbol{X} = (X_1, X_2, \ldots, X_P)'$ be a p-dimensional non negative random vector with pdf $f(\boldsymbol{x})$ and $\boldsymbol{X}^w = (X_1^w, X_2^w, \ldots, X_p^w)'$ be the multivariate weighted version of $\boldsymbol{X}$ such that the weight function $w(\boldsymbol{x})$ [$w : \boldsymbol{X} \to A \subseteq R^+$, where R^+ denotes the positive real line] is non negative with finite and non zero expectation. Then multivariate weighted density corresponding to $f(\boldsymbol{x})$ is given by

$$f_w(\boldsymbol{x}) = \frac{w(\boldsymbol{x})f(\boldsymbol{x})}{E(w(\boldsymbol{X}))}. \tag{2.2}$$

3. Univariate RCST for Weighted Models

By virtue of the definition of RCST in (1.1), the RCST function for the weighted rv X^w is given by

$$T^w(x) = -\frac{f_w'(x)}{f_w(x)},$$

where $f_w'(x)$ is the derivative of $f_w(x)$. Equivalently,

$$T^w(x) = T(x) - \frac{w'(x)}{w(x)}, \tag{3.1}$$

provided $w'(x)$ exist.

Remark 3.1. If $w(x)$ is monotonically increasing (decreasing) and $T(x)$ is monotonically increasing or decreasing, then $T^w(x) \leq (\geq) T(x)$.

Theorem 3.1. *For a non negative rv X, the RCST function of X^w, $T^w(x)$ uniquely determines the pdf $f_w(x)$ by*

$$f_w(x) = C\exp\left[-\int_0^x T^w(u)du\right], \tag{3.2}$$

where C is a constant to be determined from the identity $\int_0^\infty f_w(x)dx = 1$.

Proof. Proof follows from Theorem 3.1 of Mukherjee and Roy (1989).

Corollary 3.1. *For a non negative rv X, the RCST function of X^w, $T^w(x)$ uniquely determines the pdf $f(x)$ by*

$$f(x) = \frac{K}{w(x)}\exp\left[-\int_0^x T^w(u)du\right], \tag{3.3}$$

where K is a constant to be determined from the identity $\int_0^\infty f(x)dx = 1$.

Now we prove some characterization theorems to certain well-known univariate models viz. gamma, Rayleigh, and some applied models such as equilibrium, proportional hazard, and residual lifetime models using weighted RCST function.

Theorem 3.2. *For a non negative rv X and weight function $w(x) = \frac{1}{x}$, then $T^w(x) = cx + d$ if and only if X follows a gamma distribution with pdf*

$$f(x) = a^2xe^{-ax}, \quad x > 0, \ a > 0 \tag{3.4}$$

according as $c = 0$ and $d > 0$, and Rayleigh distribution with pdf

$$f(x) = 2axe^{-ax^2}, \quad x > 0, \ a > 0 \tag{3.5}$$

according as $c > 0$ and $d = 0$.

Proof. Suppose X follows a gamma distribution with pdf (3.4), then $T(x) = \frac{ax-1}{x}$ and hence $T^w(x) = d$, which is of the form $T^w(x) = cx + d$ where $c = 0$ and $d > 0$. On the other hand, when X follows a Rayleigh distribution with pdf (3.5), then $T(x) = \frac{2ax^2-1}{x}$ and therefore $T^w(x) = 2ax$, which is of the form $T^w(x) = cx + d$ with $c > 0$ and $d = 0$.

Conversely, assume that $T^w(x) = d$, a constant, then from (3.3) we have $f(x) = Kxe^{-\int_0^x d\,du} = Kxe^{-dx}$, where K is a constant. Now using the identity $\int_0^\infty f(x)dx = 1$, we get $K = d^2$ and hence $f(x) = d^2xe^{-dx}$, $x > 0$, $d > 0$, the gamma model (3.4). Similarly, if we assume that $T^w(x) = cx$, then from (3.3), we get $f(x) = Kxe^{-\int_0^x cu\,du} = Kxe^{-\frac{c}{2}x^2}$. Using the identity $\int_0^\infty f(x)dx = 1$, we have $K = c$ and therefore $f(x) = cxe^{-\frac{c}{2}x^2}$, $x > 0$, $c > 0$, which is the Rayleigh distribution (3.5).

Theorem 3.3. *For a non negative rv X, $T^w(x) = h(x)$ if and only if X^w follows an equilibrium distribution.*

Proof. Suppose X^w follows an equilibrium distribution, i.e., $w(x) = \frac{1}{h(x)}$ (see Gupta and Kirmani, 1990), then $\frac{w'(x)}{w(x)} = -\frac{h'(x)}{h(x)}$. Using (1.2) in (3.1), we get $T^w(x) = h(x)$.

Conversely, suppose that $T^w(x) = h(x)$, then from (3.2) we get $f_w(x) = Ce^{-\int_0^x h(u)du}$. Equivalently,

$$f_w(x) = C\overline{F}(x). \tag{3.6}$$

Now using the identity $\int_0^\infty f_w(x)dx = 1$, we have $C = \frac{1}{E(X)}$, where $E(X) = \int_0^\infty \overline{F}(x)dx$ and therefore (3.6) becomes $f_w(x) = \frac{\overline{F}(x)}{E(X)}$. i.e., X^w follows an equilibrium distribution.

Theorem 3.4. *For a non negative rv X, then* $T^w(x) = \theta h(x) - \frac{h'(x)}{h(x)}$, $\theta > 0$ *if and only if* X^w *follows proportional hazard model.*

Proof. Suppose X^w follows proportional hazard model. i.e., $w(x) = [\overline{F}(x)]^{\theta-1}$, $\theta > 0$ (see Bartoszewicz and Skolimowska, 2004), then $\frac{w'(x)}{w(x)} = (1-\theta)h(x)$. Using (1.2) in (3.1) we get, $T^w(x) = \theta h(x) - \frac{h'(x)}{h(x)}$.

Conversely, assume that $T^w(x) = \theta h(x) - \frac{h'(x)}{h(x)}$ holds, then from (3.2)

$$\begin{aligned} f_w(x) &= Ce^{-\int_0^x \left[\theta h(u) - \frac{h'(u)}{h(u)}\right]du}, \\ &= Ce^{\theta \int_0^x \left[\frac{d}{du}\log \overline{F}(u)\right]du + \int_0^x \left[\frac{d}{du}\log h(u)\right]du}, \\ &= A\left[\overline{F}(x)\right]^\theta h(x), \end{aligned}$$

$$f_w(x) = A\left[\overline{F}(x)\right]^{\theta-1} f(x). \tag{3.7}$$

Using the identity $\int_0^\infty f_w(x)dx = 1$, we get $A = \theta$ and therefore (3.7) becomes $f_w(x) = \theta[\overline{F}(x)]^{\theta-1}f(x)$, i.e., X^w follows proportional hazard model.

Theorem 3.5. *For a non negative rv X,* $T^w(x) = T(x+t)$ *if and only if* X^w *follows a residual life distribution.*

Proof. Suppose X^w follows a residual life distribution, i.e., $w(x) = \frac{f(x+t)}{f(x)}$ (see Gupta and Kirmani, 1990), then $\frac{w'(x)}{w(x)} = T(x) - T(x+t)$. From (3.1), we have $T^w(x) = T(x+t)$.

Conversely, suppose that $T^w(x) = T(x+t)$ holds, then from (3.2)

$$\begin{aligned} f_w(x) &= Ce^{-\int_0^x T(u+t)du}, \\ &= Ce^{\int_0^x \frac{d}{du}\log f(u+t)du}, \end{aligned}$$

$$f_w(x) = A(t)f(x+t). \tag{3.8}$$

Using the identity $\int_0^\infty f_w(x)dx = 1$, we get $A(t) = \frac{1}{\overline{F}(t)}$ and therefore (3.8) becomes $f_w(x) = \frac{f(x+t)}{\overline{F}(t)}$, which is the residual life distribution.

Corollary 3.2. $T^w(x) = T(t-x)$ *if and only if* X^w *follows a reversed residual life distribution* (*where* $w(x) = \frac{f(t-x)}{f(x)}$).

Theorem 3.6. *Let $T^w(x)$ be the RCST function of X^w and let $T(x)$, $h(x)$, and $\lambda(x) = \frac{f(x)}{F(x)}$, respectively, be the RCST function, failure rate, and reversed failure rate of X. Then $T^w(x) = T(x) + (1-j)\lambda(x) + (n-j)h(x)$; $j = 1, 2, \ldots, n$, if X^w follows the distribution of a jth order statistic.*

Proof. Let X^w follows the distribution of a jth order statistic, i.e., $w_j(x) = [F(x)]^{j-1}[\overline{F}(x)]^{n-j}$ (see Bartoszewicz and Skolimowska, 2004), then $\frac{w_j'(x)}{w_j(x)} = -[(1-j)\lambda(x) + (n-j)h(x)]$. From (3.1), we have $T^w(x) = T(x) + (1-j)\lambda(x) + (n-j)h(x)$; $j = 1, 2, \ldots, n$.

Corollary 3.3. *If X^w follows the distribution of a first order statistic, then $T^w(x) = T(x) + (n-1)h(x)$.*

Corollary 3.4. *If X^w follows the distribution of a nth order statistic, then $T^w(x) = T(x) + (1-n)\lambda(x)$.*

In the next theorem, we consider a general class of distributions by defining a rv X in the support of (a, b), a subset of the real line, $-\infty \leq a < x < b \leq \infty$ with $a = \inf\{x : F(x) > 0\}$ and $b = \sup\{x : F(x) < 1\}$. We say X belongs to the general class of distributions if the RCST $T(x)$ is of the form

$$T(x) = -\frac{k - B(x) - g'(x)}{g(x)}, \tag{3.9}$$

where k is a constant, $B(x)$ is a suitably chosen function of X, $g(x)$ is a real function defined on (a, b), and the derivative of $g(x)$ exist. The identity (3.9) is equivalent to

$$E\,[B(X) \mid X > x] = k + g(x)h(x) \tag{3.10}$$

and if $\lim_{x\to a}[g(x)h(x)] = 0$, then (3.10) reduces to $E[B(X) \mid X > x] = \mu + g(x)h(x)$, where $\mu = E[B(X)]$ (see Nair and Sankaran, 2008).

Theorem 3.7. *A rv X belongs to the general class of distributions* (3.9) *if and only if it satisfies the weighted identity corresponding to* (3.10) *as*

$$E\,[B(X)w(X) \mid X > x] = kE\,[w(X) \mid X > x] + w(x)g(x)h(x) + E\,[w'(X)g(X) \mid X > x] \tag{3.11}$$

under the regularity condition $\lim_{x\to b}[w(x)g(x)f(x)] = 0$.

Proof. When the rv X belongs to the general class of distributions (3.9), then from (3.1), we have

$$T^w(x) = -\frac{k - B(x) - g'(x)}{g(x)} - \frac{w'(x)}{w(x)}.$$

Equivalently,

$$\frac{f_w'(x)}{f_w(x)} - \frac{k - B(x) - g'(x)}{g(x)} = \frac{w'(x)}{w(x)},$$

$$g(x)f_w'(x) - kf_w(x) + B(x)f_w(x) + g'(x)f_w(x) = \frac{w'(x)}{w(x)}g(x)f_w(x),$$

which gives

$$B(x)f_w(x) = kf_w(x) - \frac{d}{dx}[g(x)f_w(x)] + \frac{w'(x)}{w(x)}g(x)f_w(x),$$

or

$$B(x)w(x)f(x) = kw(x)f(x) - \frac{d}{dx}[w(x)g(x)f(x)] + w'(x)g(x)f(x). \tag{3.12}$$

Integrating (3.12) and applying regularity condition, we get

$$\int_x^b B(t)w(t)f(t)dt = k\int_x^b w(t)f(t)dt + w(x)g(x)f(x) + \int_x^b w'(t)g(t)f(t)dt. \tag{3.13}$$

Dividing (3.13) by $\overline{F}(x)$, we get (3.11).

The converse part can be proved by retracing the above steps.

Corollary 3.5. *If $B(x) = x$ and $w(x) = x$,* (3.11) *reduces to*

$$E(X^2 \mid X > x) = kE(X \mid X > x) + xg(x)h(x) + E(g(X) \mid X > x). \tag{3.14}$$

Using (3.10), (3.14) *can be written as*

$$E(X^2 \mid X > x) = k^2 + g(x)h(x)l(x) + E(g(X) \mid X > x), \tag{3.15}$$

where $l(x)$ is a linear function in x and if $\lim_{x\to a}(g(x)h(x)l(x)) = 0$, then $k^2 = E(X^2) - E(g(X))$.

Corollary 3.6. *If the random variable X follows Pearson family, i.e., $g(x) = a_0 + a_1x + a_2x^2$ and $B(x) = x$, $w(x) = x$, then*

$$E(X^2 \mid X > x) = l_1(x)E(X \mid X > x) + l_2(x), \tag{3.16}$$

where $l_1(x)$ and $l_2(x)$ are linear functions in x, which is the same form as given in Glänzel (1991). *Using* (3.10), (3.16) *can also be expressed in terms of failure rate $h(x)$ as*

$$E(X^2 \mid X > x) = A + (A_0 + A_1x + A_2x^2)h(x)l_3(x),$$

where A is a constant, $A_i = \frac{a_i}{1-a_2}$, $a_2 \neq 1$, $i = 0, 1, 2$ and $l_3(x)$ is a linear function in x. If $\lim_{x\to a}(g(x)h(x)l_3(x)) = 0$, then $A = E(X^2)$.

Examples

1. Exponential: $k = 0$, $g(x) = \frac{ax+1}{a^2}$, $f(x) = ae^{-ax}$, $x > 0$, $a > 0$, then $E(X) = \frac{1}{a}$, $V(X) = \frac{1}{a^2}$. Therefore, (3.15) becomes

$$E(X^2 \mid X > x) = V(X) + h(x)q_1(x),$$

where $q_1(x)$ is a quadratic function in x.

2. Gamma: $k=\mu$, $g(x)=\frac{x}{a}$, $f(x)=\frac{a^{a\mu}}{\lceil a\mu}x^{a\mu-1}e^{-ax}$, $x>0$, $a>0$, $\mu>0$, then $E(X)=\mu$, $V(X)=\frac{\mu}{a}$, $E(X^2)=\frac{\mu}{a}+\mu^2$. The identity (3.15) becomes

$$E(X^2\mid X>x)=E(X^2)+h(x)q_2(x),$$

where $q_2(x)$ is a quadratic function in x.
3. Beta: $k=\mu$, $g(x)=\frac{x(1-x)}{a+b}$, $f(x)=\frac{1}{B(a,b)}x^{a-1}(1-x)^{b-1}$, $0<x<1$, $a>0, b>0$, then $E(X)=\frac{a}{a+b}$, $V(X)=\frac{ab}{(a+b)^2(a+b+1)}$, $E(X^2)=\frac{a(1+a)}{(a+b)(a+b+1)}$. Therefore, (3.15) becomes

$$E(X^2\mid X>x)=E(X^2)+h(x)c_1(x),$$

where $c_1(x)$ is a cubic function in x.
4. Pareto: $k=\mu$, $g(x)=\frac{x(x-a)}{c-1}$, $c>1$, $f(x)=\frac{c}{a}\left(\frac{x}{a}\right)^{-c-1}$, $a<x<\infty$, $a>0$, then $E(X)=\frac{ac}{c-1}$, $V(X)=\frac{a^2c}{(c-1)^2(c-2)}$, $E(X^2)=\frac{a^2c^3-2a^2c^2+a^2c}{(c-1)^2(c-2)}$. The identity (3.15) becomes

$$E(X^2\mid X>x)=E(X^2)+h(x)c_2(x),$$

where $c_2(x)$ is a cubic function in x.
5. Normal: $k=\mu$, $g(x)=\sigma^2$, $f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$, $-\infty<x<\infty$, $-\infty<\mu<\infty$, $\sigma>0$, then $E(X)=\mu$, $V(X)=\sigma^2$, $E(X^2)=\sigma^2+\mu^2$. Therefore, (3.15) becomes

$$E(X^2\mid X>x)=E(X^2)+h(x)l_4(x),$$

where $l_4(x)$ is a linear function in x.
6. Student – t: $k=0$, $g(x)=\frac{n+x^2}{n-1}$, $f(x)=\frac{1}{\sqrt{n}B\left(\frac{1}{2},\frac{n}{2}\right)}\frac{1}{\left(1+\frac{x^2}{n}\right)^{\frac{n+1}{2}}}$, $-\infty<x<\infty$, then $V(X)=\frac{n}{n-2}$, $n>2$. Therefore, (3.15) becomes

$$E(X^2\mid X>x)=V(X)+h(x)c_3(x),$$

where $c_3(x)$ is a cubic function in x.

4. Bivariate RCST for Weighted Models

For a non negative vector variable $\boldsymbol{X}=(X_1,X_2)'$ with pdf $f(x_1,x_2)$, the vector valued bivariate RCST (see Roy and Roy, 2010) is given by $T_i(x_1,x_2)=-\frac{\partial\log f(x_1,x_2)}{\partial x_i}$, $i=1,2$. Denoting $\boldsymbol{X}^w=(X_1^w,X_2^w)'$ be the bivariate weighted version of $\boldsymbol{X}$, then the vector valued bivariate RCST for $\boldsymbol{X}^w$ is defined as $T_i^w(x_1,x_2)=-\frac{\partial\log f_w(x_1,x_2)}{\partial x_i}$, $i=1,2$. Using (2.1), $T_i^w(x_1,x_2)$ can be written as

$$T_i^w(x_1,x_2)=T_i(x_1,x_2)-w_i(x_1,x_2), \tag{4.1}$$

where $w_i(x_1,x_2)=\frac{\partial\log w(x_1,x_2)}{\partial x_i}$.

Theorem 4.1. *For a bivariate setup, if the ith RCST of $\boldsymbol{X}^w$ is $T_i^w(x_1,x_2)$, $i=1,2$ and is continuous, then the weighted density curve can be uniquely determined in terms of* *the following two alternative forms*:

$$f_w(x_1, x_2) = C\exp\left[-\int_0^{x_1} T_1^w(u, 0)du - \int_0^{x_2} T_2^w(x_1, v)dv\right] \tag{4.2}$$

$$f_w(x_1, x_2) = C\exp\left[-\int_0^{x_2} T_2^w(0, v)dv - \int_0^{x_1} T_1^w(u, x_2)du\right]. \tag{4.3}$$

Corollary 4.1. *For a bivariate setup, if the ith RCST of* $\boldsymbol{X}^w$ *is* $T_i^w(x_1, x_2)$, $i = 1, 2$ *and is continuous, then the density curve can be uniquely determined in terms of the following two alternative forms*:

$$f(x_1, x_2) = \frac{K}{w(x_1, x_2)}\exp\left[-\int_0^{x_1} T_1^w(u, 0)du - \int_0^{x_2} T_2^w(x_1, v)dv\right]$$

$$f(x_1, x_2) = \frac{K}{w(x_1, x_2)}\exp\left[-\int_0^{x_2} T_2^w(0, v)dv - \int_0^{x_1} T_1^w(u, x_2)du\right].$$

Theorem 4.2. *Let* $h(x_1, x_2) = (h_1(x_1, x_2), h_2(x_1, x_2))$ *be the vector valued bivariate failure rate of a non negative random vector* $\boldsymbol{X} = (X_1, X_2)'$. *Then* $T_i^w(x_1, x_2) = h_i(x_1, x_2)$, $i = 1, 2$, *where* $h_i(x_1, x_2) = -\frac{\partial \log \overline{F}(x_1, x_2)}{\partial x_i}$ *if and only if* $\boldsymbol{X}^w$ *follows a bivariate equilibrium distribution.*

Proof. If $\boldsymbol{X}^w$ follows a bivariate equilibrium distribution, then $w(x_1, x_2) = \frac{1}{h(x_1, x_2)}$ (see Navarro et al., 2006), where $h(x_1, x_2) = \frac{f(x_1, x_2)}{\overline{F}(x_1, x_2)}$. It follows that $w_i(x_1, x_2) = T_i(x_1, x_2) - h_i(x_1, x_2)$, $i = 1, 2$ and therefore (4.1) becomes $T_i^w(x_1, x_2) = h_i(x_1, x_2)$, $i = 1, 2$.

Conversely, assume that $T_i^w(x_1, x_2) = h_i(x_1, x_2)$, $i = 1, 2$ holds. Now using (4.2) and (4.3), we have

$$f_w(x_1, x_2) = C\exp\left[-\int_0^{x_1} h_1(u, 0)du - \int_0^{x_2} h_2(x_1, v)dv\right] \tag{4.4}$$

$$f_w(x_1, x_2) = C\exp\left[-\int_0^{x_2} h_2(0, v)dv - \int_0^{x_1} h_1(u, x_2)du\right]. \tag{4.5}$$

Equations (4.4) and (4.5) together implies

$$f_w(x_1, x_2) = C\overline{F}(x_1, x_2). \tag{4.6}$$

Applying the condition of total probability, obtain $C = \frac{1}{E(X_1 X_2)}$, therefore (4.6) becomes $f_w(x_1, x_2) = \frac{\overline{F}(x_1, x_2)}{E(X_1 X_2)}$, the bivariate equilibrium distribution.

Theorem 4.3. *For a non negative random vector* $\boldsymbol{X} = (X_1, X_2)'$, *the relationship* $T_i^w(x_1, x_2) = T_i(x_1 + t_1, x_2 + t_2)$, $i = 1, 2$ *satisfies if and only if* $\boldsymbol{X}^w$ *follows a bivariate residual life distribution.*

Proof. If $\boldsymbol{X}^w$ follows a bivariate residual life distribution, then $w(x_1, x_2) = \frac{f(x_1 + t_1, x_2 + t_2)}{f(x_1, x_2)}$, then $w_i(x_1, x_2) = T_i(x_1, x_2) - T_i(x_1 + t_1, x_2 + t_2)$, $i = 1, 2$ and therefore, $T_i^w(x_1, x_2) = T_i(x_1 + t_1, x_2 + t_2)$, $i = 1, 2$.

Conversely, assume that $T_i^w(x_1, x_2) = T_i(x_1 + t_1, x_2 + t_2)$, $i = 1, 2$ holds, then using (4.2) and (4.3), we get

$$f_w(x_1, x_2) = C \exp\left[-\int_0^{x_1} T_1(u + t_1, t_2)du - \int_0^{x_2} T_2(x_1 + t_1, v + t_2)dv\right] \quad (4.7)$$

$$f_w(x_1, x_2) = C \exp\left[-\int_0^{x_2} T_2(t_1, v + t_2)dv - \int_0^{x_1} T_1(u + t_1, x_2 + t_2)du\right]. \quad (4.8)$$

Equations (4.7) and (4.8) can be rewritten as

$$f_w(x_1, x_2) = C \exp\left[\int_0^{x_1}\left[\frac{\partial}{\partial u}\log f(u + t_1, t_2)\right] du + \int_0^{x_2}\left[\frac{\partial}{\partial v}\log f(x_1 + t_1, v + t_2)\right] dv\right] \quad (4.9)$$

$$f_w(x_1, x_2) = C \exp\left[\int_0^{x_2}\left[\frac{\partial}{\partial v}\log f(t_1, v + t_2)\right] dv + \int_0^{x_1}\left[\frac{\partial}{\partial u}\log f(u + t_1, x_2 + t_2)\right] du\right]. \quad (4.10)$$

Equations (4.9) and (4.10) together give

$$f_w(x_1, x_2) = A(t_1, t_2) f(x_1 + t_1, x_2 + t_2). \quad (4.11)$$

Applying the condition of total probability, yield $A(t_1, t_2) = \frac{1}{\overline{F}(t_1, t_2)}$ and therefore (4.11) becomes $f_w(x_1, x_2) = \frac{f(x_1+t_1, x_2+t_2)}{\overline{F}(t_1, t_2)}$, proves the result.

Corollary 4.2. $T_i^w(x_1, x_2) = T_i(t_1 - x_1, t_2 - x_2)$, $i = 1, 2$ *if and only if* $\boldsymbol{X}^w$ *follows a reversed residual life distribution.* (*where* $w(x_1, x_2) = \frac{f(t_1 - x_1, t_2 - x_2)}{f(x_1, x_2)}$).

Theorem 4.4. *Let* X_1 *and* X_2 *be independent and identically distributed non negative rv's and let* $\lambda(x_1)$ *be the reversed failure rate of* X_1, $h(x_2)$ *be the failure rate of* X_2 *and* $r_1(x_1, x_2)$, $r_2(x_1, x_2)$ *be the generalized failure rates, where* $r_i(x_1, x_2) = \frac{f(x_i)}{F(x_2) - F(x_1)}$, $i = 1, 2$, *then the identities*

$$T_1^w(x_1, x_2) = T_1(x_1, x_2) + (1 - j)\,\lambda(x_1) + (k - j - 1)\,r_1(x_1, x_2)$$

and

$$T_2^w(x_1, x_2) = T_2(x_1, x_2) + (n - k)\,h(x_2) - (k - j - 1)\,r_2(x_1, x_2)$$

holds if $\boldsymbol{X}^w$ *follows the joint pdf of jth and kth order statistics,* $1 \le j < k \le n$.

Proof. Let $\boldsymbol{X}^w$ follows the joint pdf of jth and kth order statistics. i.e., $w_{jk}(x_1, x_2) = [F(x_1)]^{j-1}[F(x_2) - F(x_1)]^{k-j-1}[\overline{F}(x_2)]^{n-k}$, then

$$w_1(x_1, x_2) = \frac{\partial}{\partial x_1}\log w_{jk}(x_1, x_2) = -\left[(1 - j)\,\lambda(x_1) + (k - j - 1)\,r_1(x_1, x_2)\right]$$

and

$$w_2(x_1, x_2) = \frac{\partial}{\partial x_2}\log w_{jk}(x_1, x_2) = -\left[(n - k)\,h(x_2) - (k - j - 1)\,r_2(x_1, x_2)\right].$$

Clearly, (4.1) becomes

$$T_1^w(x_1, x_2) = T_1(x_1, x_2) + (1-j)\,\lambda(x_1) + (k-j-1)\,r_1(x_1, x_2)$$

and

$$T_2^w(x_1, x_2) = T_2(x_1, x_2) + (n-k)\,h(x_2) - (k-j-1)\,r_2(x_1, x_2), \quad 1 \le j < k \le n.$$

Corollary 4.3. *If X^w follows the joint pdf of jth and $(j+1)$th order statistics then $T_1^w(x_1, x_2) = T_1(x_1, x_2) + (1-j)\lambda(x_1)$ and $T_2^w(x_1, x_2) = T_2(x_1, x_2) + (n-j-1)h(x_2)$.*

Corollary 4.4. *If X^w follows the joint pdf of first and second order statistics then $T_1^w(x_1, x_2) = T_1(x_1, x_2)$ and $T_2^w(x_1, x_2) = T_2(x_1, x_2) + (n-2)h(x_2)$.*

Corollary 4.5. *If X^w follows the joint pdf of $(n-1)$th and nth order statistics then $T_1^w(x_1, x_2) = T_1(x_1, x_2) - (n-2)\lambda(x_1)$ and $T_2^w(x_1, x_2) = T_2(x_1, x_2)$.*

Corollary 4.6. *If X^w follows the joint pdf of first and nth order statistics then $T_1^w(x_1, x_2) = T_1(x_1, x_2) + (n-2)r_1(x_1, x_2)$ and $T_2^w(x_1, x_2) = T_2(x_1, x_2) - (n-2)r_2(x_1, x_2)$.*

Examples

1. Exponential: $f(x) = be^{-bx}$, $0 < x < \infty$, $b > 0$ and $f(x_1, x_2) = b^2 e^{-b(x_1+x_2)}$, then

$$T_1^w(x_1, x_2) = b + (1-j)b\frac{1}{e^{bx_1}-1} + (k-j-1)b\frac{1}{1-e^{b(x_1-x_2)}}$$

and

$$T_2^w(x_1, x_2) = b + (n-k)b + (k-j-1)b\frac{1}{1-e^{b(x_2-x_1)}}.$$

2. Power: $f(x) = c\, x^{c-1}$, $0 \le x \le 1$, $c > 0$ and $f(x_1, x_2) = c^2 x_1^{c-1} x_2^{c-1}$, then

$$T_1^w(x_1, x_2) = (1-j)\frac{1}{x_1} + (k-j-1)\frac{cx_1^{c-1}}{x_2^c - x_1^c}$$

and

$$T_2^w(x_1, x_2) = \frac{1-c}{x_2} + (n-k)\frac{cx_2^{c-1}}{1-x_2^c} - (k-j-1)\frac{cx_2^{c-1}}{x_2^c - x_1^c}.$$

3. Pareto: $f(x) = \frac{c}{a}\left(\frac{x}{a}\right)^{-c-1}$, $a \le x < \infty$, $a > 0$, $c > 0$ and

$$f(x_1, x_2) = \left(\frac{c}{a}\right)^2 \left(\frac{x_1}{a}\right)^{-c-1} \left(\frac{x_2}{a}\right)^{-c-1},$$

then

$$T_1^w(x_1, x_2) = \frac{c+1}{x_1} + (1-j)\frac{ca^c}{x_1(x_1^c - a^c)} + (k-j-1)\frac{cx_2^c}{x_1(x_2^c - x_1^c)}$$

and

$$T_2^w(x_1, x_2) = \frac{1}{x_2} + (n-k+1)\frac{c}{x_2} - (k-j-1)\frac{cx_1^c}{x_2(x_2^c - x_1^c)}.$$

5. Multivariate RCST for Weighted Models

The vector valued multivariate RCST of $\boldsymbol{X}$ is given by $\boldsymbol{T}(\boldsymbol{x}) = (T_1(\boldsymbol{x}), T_2(\boldsymbol{x}), \ldots, T_p(\boldsymbol{x}))'$, where $T_i(\boldsymbol{x}) = -\frac{\partial \log f(\boldsymbol{x})}{\partial x_i}$; $i = 1, 2, \ldots, p$. The corresponding vector valued multivariate RCST of $\boldsymbol{X}^w$ is given by

$$\boldsymbol{T}^w(\boldsymbol{x}) = \left(T_1^w(\boldsymbol{x}), T_2^w(\boldsymbol{x}), \ldots, T_p^w(\boldsymbol{x})\right)',$$

where $T_i^w(\boldsymbol{x}) = -\frac{\partial \log f_w(\boldsymbol{x})}{\partial x_i}$; $i = 1, 2, \ldots, p$. Using (2.2), $T_i^w(\boldsymbol{x})$ can be written as

$$T_i^w(\boldsymbol{x}) = T_i(\boldsymbol{x}) - w_i(\boldsymbol{x}),$$

where $w_i(\boldsymbol{x}) = \frac{\partial \log w(\boldsymbol{x})}{\partial x_i}$.

Remark 5.1. For $p = 1$ and $p = 2$ the above definition reduces to the corresponding univariate and bivariate RCST of weighted distribution, respectively.

Remark 5.2. If $w(\boldsymbol{x})$ is monotonically increasing (decreasing) and $T_i(\boldsymbol{x})$ is monotonically increasing or decreasing, then $T_i^w(\boldsymbol{x}) \leq (\geq) T_i(\boldsymbol{x})$, $i = 1, 2, \ldots, p$.

Theorem 5.1. *For a multivariate setup if the ith RCST of $\boldsymbol{X}^w$ is $T_i^w(x_1, x_2, \ldots, x_p)$, $i = 1, 2, \ldots, p$ and is continuous, then the weighted density curve can be uniquely determined as*

$$f_w(\boldsymbol{x}) = C \exp\left[-\int_\Gamma T^w(\boldsymbol{u}) d\boldsymbol{u}\right],$$

where the integration is a line integration with respect to $\boldsymbol{u} = (u_1, u_2, \ldots, u_p)$ over a piecewise smooth curve Γ, joining the points $(0, 0, \ldots, 0)$ and $\boldsymbol{x} = (x_1, x_2, \ldots, x_p)$ and where C is the normalizing constant such that the total probability is one (see Roy and Roy, 2010).

Corollary 5.1. *For a multivariate setup if the ith RCST of $\boldsymbol{X}^w$ is $T_i^w(x_1, x_2, \ldots, x_p)$, $i = 1, 2, \ldots, p$ and is continuous, then the density curve can be uniquely determined as*

$$f(\boldsymbol{x}) = \frac{K}{w(\boldsymbol{x})} \exp\left[-\int_\Gamma T^w(\boldsymbol{u}) d\boldsymbol{u}\right],$$

where K is the normalizing constant such that the total probability is one.

Similar to Roy and Roy (2010), we have the following theorem for weighted random variables.

Theorem 5.2. *If $X_1^w, X_2^w, \ldots, X_p^w$ are independent rv's then*

$$\boldsymbol{T}^w(\boldsymbol{x}) = \left(T_1^w(x_1), T_2^w(x_2), \ldots, T_p^w(x_p)\right)',$$

where $T_i^w(x_i) = T_i(x_i) - w_i(x_i)$ is the univariate RCST of X_i^w, $i = 1, 2, \ldots, p$.

We can define strictly constant vector valued multivariate RCST of $\boldsymbol{X}^w$ as $\boldsymbol{T}^w(\boldsymbol{x}) = (a_1, a_2, \ldots, a_p)'$, where $\boldsymbol{a} = (a_1, a_2, \ldots, a_p)'$ is an absolute constant with respect to all the variables.

Theorem 5.3. *If* $w(\boldsymbol{x}) = \prod_{i=1}^{p} \frac{1}{x_i}$ *then the vector valued multivariate RCST of* $\boldsymbol{X}^w$ *is an absolute constant if and only if the underlying distribution is a joint collection of independent univariate gamma distribution with pdf*

$$f(x) = a^2 x e^{-ax}, \quad x > 0, \; a > 0. \tag{5.1}$$

Proof. Suppose X_i's are follows independent univariate gamma distributions with pdf given by (5.1), under the weight function $w(\boldsymbol{x}) = \prod_{i=1}^{p} \frac{1}{x_i}$, X_i^w's are independent (see Arnold and Nagaraja, 1991) for the given pdf (5.1) with $w_i(x_i) = -\frac{1}{x_i}$, $i = 1, 2, \ldots, p$ we have $\boldsymbol{T}^w(\boldsymbol{x}) = (a_1, a_2, \ldots, a_p)'$.

Conversely, suppose $\boldsymbol{T}^w(\boldsymbol{x})$ is constant, i.e., $T_i^w(x_i) = a_i$, $i = 1, 2, \ldots, p$, or $\frac{\partial \log f_w(x)}{\partial x_i} = -a_i$. Equivalently, $\frac{\partial \log f(x)}{\partial x_i} + \frac{\partial \log w(x)}{\partial x_i} = -a_i$. Integrating both sides with respect to x_i, we get

$$f(\boldsymbol{x}) w(\boldsymbol{x}) = \exp(-a_i x_i) g_i(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_p), \quad i = 1, 2, \ldots, p$$

or

$$f(\boldsymbol{x}) = \frac{1}{w(\boldsymbol{x})} \exp(-a_i x_i) g_i(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_p), \quad i = 1, 2, \ldots, p.$$

Combining $f(\boldsymbol{x})$, $i = 1, 2, \ldots, p$, we have $f(\boldsymbol{x}) \propto \left[\prod_{i=1}^{p} x_i\right]\left[\exp\left(-\sum_{i=1}^{p} a_i x_i\right)\right]$. Applying the condition of total probability, we obtain $f(\boldsymbol{x}) = \prod_{i=1}^{p} a_i^2 x_i \exp[-a_i x_i]$, which proves the result.

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