Statistics and Probability Letters 83 (2013) 366–372

Contents lists available at SciVerse ScienceDirect

Statistics and Probability Letters

journal homepage: www.elsevier.com/locate/stapro

Quantile based entropy function in past lifetime

S.M. Sunoj [a,*], P.G. Sankaran [a], Asok K. Nanda [b]

[a] Department of Statistics, Cochin University of Science and Technology, Cochin 682022, Kerala, India

[b] Department of Mathematics and Statistics, Indian Institute of Science Education and Research, Kolkata, P.O. BCKV Campus Main Office, Mohanpur 741252, Nadia, West Bengal, India

ARTICLE INFO

Article history:
Received 30 April 2012
Received in revised form 5 September 2012
Accepted 14 September 2012
Available online 4 October 2012

MSC:
94A17
62E10
62N05

Keywords:
Shannon entropy
Past lifetime
Quantile function
Reliability measures
Characterizations

ABSTRACT

Di Crescenzo and Longobardi (2002) introduced a measure of uncertainty in past lifetime distributions and studied its relationship with residual entropy function. In the present paper, we introduce a quantile version of the entropy function in past lifetime and study its properties. Unlike the measure of uncertainty given in Di Crescenzo and Longobardi (2002) the proposed measure uniquely determines the underlying probability distribution. The measure is used to study two nonparametric classes of distributions. We prove characterizations theorems for some well known quantile lifetime distributions.

1. Introduction

Let X be an absolutely continuous random variable (rv) representing the lifetime of a component with cumulative distribution function (CDF) $F(t) = P(X \leq t)$ and survival function (SF) $\bar{F}(t) = P(X > t) = 1 - F(t)$. The measure of uncertainty (Shannon, 1948) is defined by

$$\xi(X) = \xi(f) = -\int_0^\infty (\log f(x)) f(x) dx = -E(\log f(X)), \quad (1)$$

where $f(t)$ is the probability density function (PDF) of X. Eq. (1) gives the expected uncertainty contained in $f(t)$ about the predictability of an outcome of X, which is known as the Shannon information measure. The length of time during a study period has been considered as a prime variable of interest in many fields such as reliability, survival analysis, economics, business, etc. In such cases, the information measures are functions of time and thus they are dynamic. Based on this idea, Ebrahimi (1996) defined the residual Shannon entropy of X at time t by

$$\xi(X; t) = \xi(f; t) = -\int_t^\infty \left(\frac{f(x)}{\bar{F}(t)}\right) \log \left(\frac{f(x)}{\bar{F}(t)}\right) dx. \quad (2)$$

* Corresponding author. Tel.: +91 484 2575893.
E-mail addresses: smsunoj@cusat.ac.in (S.M. Sunoj), sankaranpg@yahoo.com (P.G. Sankaran), asoknanda@yahoo.com (A.K. Nanda).

0167-7152/$ – see front matter © 2012 Elsevier B.V. All rights reserved.
doi:10.1016/j.spl.2012.09.016

Note that $\xi(X;t) = \xi(X_t)$, where $X_t = (X - t \mid X > t)$ is the residual time associated to X (see also Ebrahimi and Pellerey, 1995).

However, in many realistic situations, uncertainty is related to the past lifetime rather than the residual lifetime. For example, if we consider a system which had already failed at time t, then its uncertainty is related to the past, *i.e.*, on which instant $(0, t)$ it has failed. Motivated by this, Di Crescenzo and Longobardi (2002) introduced the past entropy as

$$\begin{aligned}\bar{\xi}(X;t) = \bar{\xi}(f;t) &= -\int_0^t \left(\frac{f(x)}{F(t)}\right)\log\left(\frac{f(x)}{F(t)}\right)dx,\\ &= \log F(t) - \frac{1}{F(t)}\int_0^t (\log f(x))f(x)dx,\end{aligned} \tag{3}$$

where $\bar{\xi}(X;t) = \bar{\xi}(X^t)$ and $X^t = (t - X \mid X \le t)$ is the past lifetime associated to X. Denoting $a(x) = f(x)/F(x)$ the reversed hazard rate (see Block et al., 1998), Eq. (3) can be rewritten as

$$\bar{\xi}(X;t) = 1 - \frac{1}{F(t)}\int_0^t \log(a(x))f(x)dx. \tag{4}$$

Given that at time t, a unit is found to be down, $\bar{\xi}(X;t)$ measures the uncertainty about its past lifetime. Interesting extensions and multivariate forms of measures of uncertainty are also available in literature. For recent works on entropy in past lifetime we refer to Di Crescenzo and Longobardi (2002, 2006), Nanda and Paul (2006) and Kundu et al. (2010).

All these theoretical results and applications thereof are based on the distribution function. A probability distribution can be specified either in terms of the distribution function or by the quantile functions (QF). Recently, it has been shown by many authors that quantile functions

$$Q(u) = F^{-1}(u) = \inf\{t \mid F(t) \ge u\}, \quad 0 \le u \le 1 \tag{5}$$

are efficient and equivalent alternatives to the distribution function in modeling and analysis of statistical data (see Gilchrist, 2000; Nair and Sankaran, 2009). In many cases, QF's are more convenient as they are less influenced by extreme observations and thus provide a straightforward analysis with a limited amount of information. For detailed and recent studies on QF, its properties and usefulness in the identification of models we refer to Nair et al. (2008, 2011), Nair and Sankaran (2009), Sankaran and Nair (2009), Sankaran et al. (2010) and the references therein.

Many of the quantile functions used in applied work like various forms of lambda distributions (Ramberg and Schmeiser, 1974; Freimer et al., 1988; van Staden and Loots, 2009; Gilchrist, 2000), the power-Pareto distribution (Gilchrist, 2000, Hankin and Lee, 2006), Govindarajulu distribution (Nair et al., 2011) etc. do not have tractable distributions. This makes the analytical study of the properties of $\xi(X)$ of these distributions by means of (1) difficult. Thus a formulation of the definition and properties of entropy function in terms of quantile functions is called for. Such a discussion has several advantages. Analytical properties of the entropy function obtained in this approach can be used as alternative tools in modeling data. Sometimes, the quantile-based approach is better in terms of tractability. New models and characterizations that are unresolvable in the distribution function approach can be resolved with the aid of QF's. In view of these, the objective of the present work is to initiate a discussion of entropy function in terms of QF's. The present paper introduces the Shannon entropy function for past lifetime using the QF's and proved some useful characterization theorems arising out of it. Unlike the entropy function in past lifetime due to Di Crescenzo and Longobardi (2002), an explicit expression is obtained for quantile based entropy function in past lifetime.

The rest of the paper is organized as follows. In Section 2, we consider some useful reliability measures in terms of quantile function and introduce the quantile based Shannon entropy in past lifetime. Various properties of the measure are discussed. Finally, Section 3 proves some characterization results based on the measures considered in Section 2.

2. Quantile based entropy in past lifetime

When F is continuous, we have from (5), $FQ(u) = u$, where $FQ(u)$ represents the composite function $F(Q(u))$. Defining the density quantile function by $fQ(u) = f(Q(u))$ (see Parzen, 1979) and quantile density function by $q(u) = Q'(u)$, where the prime denotes the differentiation, we have

$$q(u)fQ(u) = 1. \tag{6}$$

The reversed hazard rate function $a(x)$ can be expressed in terms of the quantile function, is given by

$$A(u) = aQ(u) = a(Q(u)) = u^{-1}fQ(u) = [uq(u)]^{-1}.$$

The function $A(u)$ is referred to as reversed hazard quantile function. Following Nair and Sankaran (2009), $A(u)$ explains the conditional probability of failure in the past small interval of time given that the failure occurred prior to the $100u\%$ point of distribution. Like $a(x)$ that determines the CDF or SF uniquely, $A(u)$ also uniquely determines the QF by the identity

$$Q(u) = \int_0^u \frac{dt}{tA(t)}.$$

Using the QF defined in (5) and (6), the Shannon entropy in (1) can be written as

$$\xi = \xi(X) = -\int_0^1 (\log fQ(p)) fQ(p) dQ(p),$$
$$= \int_0^1 (\log q(p)) dp. \tag{7}$$

Clearly, by knowing either $Q(u)$ or $q(u)$, the expression for $\xi(X)$ is quite simple to compute. Recently, Sunoj and Sankaran (2012) obtained a quantile version of the residual entropy $\xi(X;t)$, given by

$$\psi(u) = \xi(X; Q(u)) = \log(1-u) + (1-u)^{-1} \int_u^1 \log q(p) dp.$$

Unlike the measure (2), $\psi(u)$ determine the QF uniquely. For more properties of $\psi(u)$, one may refer to Sunoj and Sankaran (2012). The entropy function in past lifetime (3) in terms of QF is defined by

$$\overline{\psi}(u) = \overline{\xi}(X; Q(u)) = \log u + u^{-1} \int_0^u (\log q(p)) dp. \tag{8}$$

The measure (8) gives the expected uncertainty contained in the conditional density about the predictability of an outcome of X until 100u% point of distribution. From (4), we can write (8) as

$$\overline{\psi}(u) = 1 - u^{-1} \int_0^u \log A(p) dp. \tag{9}$$

Differentiating Eq. (8) with respect to u, we get

$$\overline{\psi}'(u) = \frac{1}{u} - \frac{1}{u^2} \int_0^u \log q(p) dp + \frac{1}{u} \log q(u),$$

equivalently,

$$u\overline{\psi}'(u) = 1 - [\overline{\psi}(u) - \log u] + \log q(u).$$

Thus,

$$q(u) = \exp\{u\overline{\psi}'(u) + \overline{\psi}(u) - \log u - 1\}. \tag{10}$$

The Eq. (10) provides a simple relationship that determines the quantile density function uniquely from $\overline{\psi}(u)$. However, the past entropy $\overline{\xi}(X;t)$ in (3) or (4) does not provide explicit relationship between $\overline{\xi}(X;t)$ and $f(t)$ and hence does not determine the PDF uniquely. The relationship (10) enables us to characterize probability distributions which will be discussed in Section 3.

Following Di Crescenzo and Longobardi (2002), we can easily show that (7) can be expressed in terms of $\psi(u)$ and $\overline{\psi}(u)$ as given below.

Proposition 1. *For all* $u > 0$

$$\xi(X) = \psi[u, 1-u] + u\overline{\psi}(u) + (1-u)\psi(u) \tag{11}$$

where $\psi[p, 1-p] = -p \log p - (1-p)\log(1-p)$ *is the entropy of a Bernoulli distribution.*

The identity (11) can be explained in the following manner (see Di Crescenzo and Longobardi, 2002). The quantile based uncertainty about the failure time of an item can be decomposed into three parts: (i) the uncertainty of whether the item has failed until or after 100u% point of distribution, (ii) the uncertainty until 100u% point of distribution, and (iii) the uncertainty after 100u% point of distribution.

Now on the basis of past quantile entropy (PQE), we define the following nonparametric classes of life distributions.

Definition 1. X is said to have decreasing (increasing) quantile entropy in past lifetime (DPQE (IPQE)) if $\overline{\psi}(u)$ is decreasing (increasing) in $u \geq 0$.

It is easy to show from the relationship (8) that if X is DPQE (IPQE), then $\overline{\psi}(u) \geq (\leq) 1 + \log(uq(u)) = 1 - \log A(u)$.

For the exponential distribution in the support of $(-\infty, 0)$ with $F(t) = \exp(\lambda t)$, $\lambda > 0$ (see Block et al., 1998) we have $Q(u) = \frac{1}{\lambda}\log u$, $q(u) = \frac{1}{\lambda u}$ and $A(u) = \lambda$ so that $\overline{\psi}(u) = 1 + \log(uq(u)) = 1 - \log A(u) = 1 - \log\lambda$. Thus exponential distribution with negative support is the boundary of IPQE and DPQE classes. Table 1 provides the QF's, $\overline{\psi}(u)$ and its monotone nature of it for certain families of distributions.

Table 1
QF, $\overline{\psi}(u)$ and monotone behavior of lifetime distributions.

Distribution	Quantile function	$\overline{\psi}(u)$	Monotone nature
Exponential	$\lambda^{-1}(-\log(1-u))$	$1-\log\lambda+\log u+\left(\frac{1-u}{u}\right)\log(1-u)$	IPQE
Uniform	$a+(b-a)u$	$\log u+\log(b-a)$	IPQE
Pareto II	$\alpha[(1-u)^{-\frac{1}{c}}-1]$	$\ln\left(\frac{\alpha}{c}\right)+\left(\frac{c+1}{c}\right)+\log u+\left(\frac{c+1}{c}\right)\left(\frac{1-u}{u}\right)\log(1-u)$	IPQE
Rescaled beta	$R[1-(1-u)^{\frac{1}{c}}]$	$\log\left(\frac{R}{c}\right)+\left(\frac{c-1}{c}\right)+\log u+\left(\frac{c-1}{c}\right)\left(\frac{1-u}{u}\right)\log(1-u)$	IPQE for $c>1$
Half logistic	$\sigma\log\left(\frac{1+u}{1-u}\right)$	$2+\log(2\sigma)+\log u-\left(\frac{1+u}{u}\right)\log(1+u)+\left(\frac{1-u}{u}\right)\log(1-u)$	IPQE
Power function	$\alpha u^{\frac{1}{\beta}}$	$\log\left(\frac{\alpha}{\beta}\right)+\left(\frac{\beta-1}{\beta}\right)+\frac{1}{\beta}\log u$	IPQE
Pareto I	$\sigma(1-u)^{-\frac{1}{\alpha}}$	$\log\left(\frac{\sigma}{\alpha}\right)+\left(\frac{\alpha+1}{\alpha}\right)+\log u+\left(\frac{\alpha+1}{\alpha}\right)\left(\frac{1-u}{u}\right)\log(1-u)$	IPQE
Generalized Pareto	$\frac{b}{a}[(1-u)^{-\frac{a}{(a+1)}}-1]$	$\log\left(\frac{b}{a+1}\right)+\left(\frac{2a+1}{a+1}\right)+\log u+\left(\frac{2a+1}{a+1}\right)\left(\frac{1-u}{u}\right)\log(1-u)$	IPQE for $a>0$
Log logistic	$\alpha^{-1}\left(\frac{u}{1-u}\right)^{\frac{1}{\beta}}$	$2-\log(\alpha\beta)+\frac{1}{\beta}\log u+\left(\frac{\beta+1}{\beta}\right)\left(\frac{1-u}{u}\right)\log(1-u)$	IPQE
Exponential geometric	$\frac{1}{\lambda}\log\left(\frac{1-pu}{1-u}\right)$	$2+\log\left(\frac{1-p}{\lambda}\right)+\log u+\left(\frac{1-u}{u}\right)\log(1-u)+\left(\frac{1-pu}{pu}\right)\log(1-pu)$	IPQE
Inverted exponential	$-\frac{\lambda}{\log u}$	$1-\frac{1}{u}\int_0^u\log(\log p)^2dp+\log\lambda$	DPQE
Linear hazard rate	$(a+b)^{-1}\log\left(\frac{a+bu}{a(1+u)}\right)$	$2+\log\left(\frac{b-a}{a+b}\right)+\log u-\left(\frac{1+u}{u}\right)\log(1+u)+\frac{a}{bu}\log a-\left(\frac{a+bu}{bu}\right)\log(a+bu)$	
Davies	$Cu^{\lambda_1}(1-u)^{-\lambda_2}$	$\log C+\lambda_2-\lambda_1+1+\lambda_1\log u+(\lambda_2+1)\left(\frac{1-u}{u}\right)\log(1-u)+\frac{(\lambda_1(1-u)+\lambda_2u)}{(\lambda_2-\lambda_1)u}\log(\lambda_1(1-u)+\lambda_2u)-\frac{\lambda_1\log\lambda_1}{(\lambda_2-\lambda_1)u}$	

Proposition 2. *If $A(u)$ is decreasing for all $u\geq 0$, then $\overline{\psi}(u)$ is increasing for all $u\geq 0$.*

The proof follows from the identity (10).

Theorem 1. (a) *If X is IPQE and if ϕ is nonnegative, increasing and convex, then $\phi(X)$ is also IPQE.* (b) *If X is DPQE and if ϕ is nonnegative, increasing and concave, then $\phi(X)$ is also DPQE.*

Proof. If $g(y)$ is the pdf of $Y=\phi(X)$, then $g(y)=\frac{f(\phi^{-1}(y))}{\phi'(\phi^{-1}(y))}$; hence, $g(Q_Y(u))=\frac{1}{q_Y(u)}=\frac{fQ(u)}{\phi'Q(u)}=\frac{1}{q_X(u)\phi'Q(u)}$. We have

$$\begin{aligned}\overline{\psi}_Y(u)&=\log u+u^{-1}\int_0^u(\log q_Y(p))dp,\\&=\log u+u^{-1}\int_0^u\log(q_X(p)\phi'Q(p)t)dp,\\&=\log u+u^{-1}\int_0^u\log q_X(p)dp+u^{-1}\int_0^u\log\phi'Q(p)dp\\&=\overline{\psi}_X(u)+u^{-1}\int_0^u\log\phi'Q(p)dp,\end{aligned}\tag{12}$$

where $\overline{\psi}_X(u)$ and $\overline{\psi}_Y(u)$ are the PQE's of X and Y respectively. Now differentiating (12) with respect to u, we get

$$\begin{aligned}\overline{\psi}'_Y(u)&=\overline{\psi}'_X(u)-\frac{1}{u^2}\int_0^u\log\phi'Q(p)dp+\frac{1}{u}\log\phi'Q(u),\\&=\overline{\psi}'_X(u)+\frac{1}{u^2}\left[u\log\phi'Q(u)-\int_0^u\log\phi'Q(p)dp\right].\end{aligned}$$

But since the QF $Q(u)$ is a nondecreasing function and if ϕ is nonnegative, increasing and convex, we have $\log\phi'Q(u)>\log\phi'Q(p)$, $0<p<u$ and on integration between the limits 0 to u, we get $u\log\phi'Q(u)>\int_0^u\log\phi'Q(p)dp$, thus the second term in the expression of $\overline{\psi}'_Y(u)$ is positive. Now if X is IPQE, $\overline{\psi}'_X(u)$ is positive so that $Y=\phi(X)$ is also IPQE. The proof of (b) is similar. □

Example 1. Let X have the inverted exponential distribution with CDF $F(t)=e^{-\lambda/t}$; $t>0$ and QF $Q(u)=-\frac{\lambda}{\log u}$. When $Y=X^{\alpha}$, $\alpha>0$, the distribution of Y is an inverted Weibull distribution with $F(t)=e^{-\lambda/t^{\alpha}}$; $t>0$. For the inverted exponential, we have $q(u)=\frac{\lambda}{u(\log u)^2}$ and $A(u)=\frac{(\log u)^2}{\lambda}$. Then from (9) we can get

$$u\overline{\psi}(u)=u-\int_0^u\log A(p)dp.\tag{13}$$

Differentiating (13) with respect to u and substituting $A(u) = \frac{(\log u)^2}{\lambda}$, we obtain

$$u\overline{\psi}'(u) + \overline{\psi}(u) = 1 - \log(\log u)^2 + \log\lambda. \tag{14}$$

For the inverted exponential, (9) provides

$$\overline{\psi}(u) = 1 - \frac{1}{u}\int_0^u \log(\log p)^2 dp + \log\lambda,$$

and (14) becomes

$$\overline{\psi}'(u) = \frac{1}{u}\left[\frac{1}{u}\int_0^u \log(\log p)^2 dp - \log(\log u)\right].$$

When $u > p$, we have $\log u > \log p$ or $\log(\log u)^2 > \log(\log p)^2$. Integrating both sides over $(0, u)$, we get

$$u(\log(\log u)^2) > \frac{1}{u}\int_0^u \log(\log p)^2 dp.$$

Therefore $\overline{\psi}'(u) < 0$, and thus X is DPQE. The nonnegative function $\phi(x) = x^\alpha$, $x > 0$, is concave if $0 < \alpha < 1$. Hence due to Theorem 1, the inverted Weibull is DPQE for $0 < \alpha < 1$.

The concept of weighted distributions is usually considered in connection with modeling statistical data, where the usual practice of employing standard distributions is not found appropriate in some cases (see Rao, 1965). In recent years, this concept has been applied in many areas of statistics, such as analysis of family size, human heredity, wildlife population study, renewal theory, biomedical, statistical ecology, reliability modeling, etc. Associated to a random variable X with PDF $f(t)$ and to a nonnegative real function $w(t)$, we can define the weighted random variable X_w with density function $f_w(t) = \frac{w(t)f(t)}{Ew(X)}$, where we assume $0 < Ew(X) < \infty$. When $w(t) = t$, X_w is called length (size) biased rv. For recent works on weighted distributions, we refer the reader to Navarro et al. (2006); Bartoszewicz (2009); Navarro et al. (2011) and the references therein. Using $f_w(t)$, the corresponding density quantile function is given by

$$f_w(Q(u)) = w(Q(u))f(Q(u))/\mu,$$

where $\mu = \int_0^1 w(Q(p))f(Q(p))d(Q(p)) = \int_0^1 w(Q(p))dp$. An equivalent version in the quantile density form is given by $\frac{1}{q_w(u)} = \frac{w(Q(u))}{\mu q(u)}$. Therefore, the PQE of X_w denoted by $\overline{\psi}_w(u)$ is of the form

$$\overline{\psi}_w(u) = \overline{\psi}_X(u) + \log\mu - u^{-1}\int_0^u \log w(Q(p))dp.$$

Then the following preservation theorem is immediate.

Theorem 2. (a) *If X is IPQE and if $w(X)$ is nonnegative, increasing and concave, then X_w is also IPQE.* (b) *If X is DPQE and if $w(X)$ is nonnegative, increasing and convex, then X_w is also DPQE.*

The proof is similar to Theorem 1.

Definition 2. X is said to have less PQE than Y if $\overline{\psi}_X(u) \le \overline{\psi}_Y(u)$ for all $u \ge 0$. We write $X \le_{LPQE} Y$.

It is interesting to note that if X and Y are two uniform $U(0, b_1)$ and $U(0, b_2)$ rv's, and if $b_1 \le b_2$, then $X \le_{LPQE} Y$. Let $q_Z(u)$ the quantile density function of the variable $Z = aX + b$, where $a > 0$ and $b \ge 0$. Then, using (8) we get

$$\overline{\psi}_Z(u) = \overline{\psi}_X(u) + \log a, \tag{15}$$

for all$u \ge 0$. Now we have the following theorem.

Theorem 3. *Let $Z_1 = a_1X + b_1$ and $Z_2 = a_2Y + b_2$, $a_1, a_2 > 0$ and $b_1, b_2 \ge 0$. If $X \le_{LPQE} Y$ and $a_1 \le a_2$, then $Z_1 \le_{LPQE} Z_2$.*

The proof follows directly from (15).

Definition 3. X is said to be larger than Y in quantile reversed failure rate ordering ($X \ge_{QRFR} Y$) if $A_X(u) \le A_Y(u)$ for all $u \ge 0$.

Theorem 4. *If $X \le_{QRFR} Y$, then $X \le_{LPQE} Y$.*

The proof follows from (9).

Theorem 5. *If $X \le_{QRFR} Y$, and if ϕ is nonnegative, increasing and convex, then $\phi(X) \le_{LPQE} \phi(Y)$.*

Proof. Let $q_X(u)$, $q_X^*(u)$, $q_Y(u)$ and $q_Y^*(u)$ denote the quantile density function of X, $\phi(X)$, Y and $\phi(Y)$ respectively. Then by Eq. (12), for all $u \geq 0$,

$$\overline{\psi}_{\phi(X)}(u) - \overline{\psi}_{\phi(Y)}(u) = \overline{\psi}_X(u) - \overline{\psi}_Y(u) + u^{-1}\int_0^u \log \phi_X' Q(p)dp - u^{-1}\int_0^u \log \phi_Y' Q(p)dp.$$

Now, when $X \leq_{QRFR} Y$ using Theorem 4 we have $X \leq_{LPQE} Y$, and since ϕ is nonnegative, increasing and convex, we obtain $\phi(X) \leq_{LPQE} \phi(Y)$. □

Remark 1. For equilibrium distribution with density function

$$f_E(t) = \overline{F}(t)/\mu, \tag{16}$$

we have $\frac{1}{q_E(u)} = \frac{(1-u)}{\mu}$, or $\log q_E(u) = \log\mu - \log(1-u)$, then using (8) the PQE is given by $\overline{\psi}_E(u) = 1 + \log\mu + \log u + \frac{(1-u)}{u}\log(1-u) = \psi_E(u) + \log u + \frac{(1-u)}{u}\log(1-u)$. It is interesting to note that $\psi_E(u)$ for equilibrium model is a constant (see Sunoj and Sankaran, 2012) while $\overline{\psi}_E(u)$ is not.

Theorem 6. *If $X \leq_{LPQE} Y$ and $X \geq_{QRFR} Y$, then $\overline{\psi}_X(u) - \overline{\psi}_Y(u)$ is nondecreasing in u.*

Proof. Using (8), we have

$$\overline{\psi}_X'(u) = \frac{1}{u}[1 - \overline{\psi}_X(u) - \log A_X(u)] \geq \frac{1}{u}[1 - \overline{\psi}_Y(u) - \log A_Y(u)] = \overline{\psi}_Y'(u),$$

where the inequality holds from $\overline{\psi}_X(u) \leq \overline{\psi}_Y(u)$ and $A_X(u) \leq A_Y(u)$. □

3. Characterizations

In this section, we prove some useful characterizations of important families of distributions which are commonly used in lifetime data analysis.

Theorem 7. *The rv X is distributed as generalized Pareto with*

$$Q(u) = \frac{b}{a}\left[(1-u)^{-\frac{a}{(a+1)}} - 1\right], \quad a > -1,\ b > 0 \tag{17}$$

if and only if for a real constant c,

$$\overline{\psi}(u) = c - \log A(u) + \frac{c}{u}\log(1-u). \tag{18}$$

Proof. Suppose that Eq. (18) holds. Then, using Eq. (9), we have

$$1 - \frac{1}{u}\int_0^u \log A(p)dp = c - \log A(u) + \frac{c}{u}\log(1-u).$$

Equivalently,

$$u - \int_0^u \log A(p)dp = cu - u\log A(u) + c\log(1-u). \tag{19}$$

Differentiating both sides of (19) with respect to u, we get.

$$1 - \log A(u) = c - \log A(u) - u\frac{d}{du}\log A(u) - \frac{c}{(1-u)},$$

simplifying, we have

$$\frac{d}{du}\log A(u) = \frac{(c-1)}{u} - \frac{c}{u} - \frac{c}{(1-u)}.$$

Integrating both sides we get the $A(u)$ of the generalized Pareto with $c = \left(\frac{2a+1}{a+1}\right)$ and thus the model (17). The converse part follows directly from Table 1. □

Remark 2. When $c = 1$, we have the exponential distribution and $c > 1 (0 < c < 1)$ provides the Pareto II (rescaled beta) distributions, respectively. When $c = 0$, (18) becomes a characterization for uniform distribution $U(a, b)$.

The following theorem characterizes power function distribution and half logistic distributions using $\overline{\psi}(u)$. The proofs of the results are direct.

Theorem 8. *The rv X is distributed as power function with* $Q(u) = \alpha u^{1/\beta}$; $\alpha, \beta > 0$, *holds for all u if and only if it satisfies the relationship* $\overline{\psi}(u) = C - \log A(u)$ *where* $0 < C < 1$.

Theorem 9. *The relationship* $\overline{\psi}(u) = 2 - \log A(u) - \frac{1}{u}\log\left(\frac{1+u}{1-u}\right)$, *holds for all u if and only if X follows a half-logistic distribution with* $Q(u) = \sigma \log\frac{(1+u)}{(1-u)}$, $\sigma > 0$.

Theorem 10. *The relationship* $\overline{\psi}_E(u) = 1 - \log A_E(u) + \frac{1}{u}\log(1-u)$ *holds for u if and only if X follows equilibrium distributions (16).*

Next we consider a family of distributions that has nonmonotone hazard quantile function.

Theorem 11. *The relationship*

$$\overline{\psi}(u) = A - \log A(u) + \frac{(A+\alpha)}{u}\log(1-u)$$

for all u if and only if

$$q(u) = Ku^{\alpha}(1-u)^{-(A+\alpha)}, \tag{20}$$

where α and A are real constants.

Remark 3. The family of distributions (20) contains several well-known distributions which include the exponential ($\alpha = 0, A = 1$), Pareto ($\alpha = 0, A < 1$), rescaled beta ($\alpha = 0, A > 1$), the loglogistic distribution ($\alpha = \lambda - 1, A = 2$) and the life distribution proposed by Govindarajulu (1977) ($\alpha = \beta - 1, A = -\beta$), with QF $Q(u) = \theta + \sigma((\beta+1)u^{\beta} - \beta u^{\beta+1})$. In terms of distribution functions (20) has the form

$$f(x) = K[F(x)]^{-\alpha}[1 - F(x)]^{A+\alpha},$$

belongs to the class of distributions defined by the relationship between their density and distribution functions of Jones (2007) (for more details, see Nair et al., 2011).

Acknowledgments

The authors are grateful to the editor and referees for their constructive comments. The authors would like to thank the support of the University Grants Commission, India, under the Special Assistance Programme.

References

Bartoszewicz, J., 2009. On a representation of weighted distributions. Statist. Probab. Lett. 79, 1690–1694.
Block, H.W., Savits, T.H., Singh, H., 1998. The reversed hazard rate function. Probab. Engrg. Inform. Sci. 12, 69–90.
Di Crescenzo, A., Longobardi, M., 2002. Entropy-based measure of uncertainty in past lifetime distributions. J. Appl. Probab. 39, 434–440.
Di Crescenzo, A., Longobardi, M., 2006. On weighted residual and past entropies. Sci. Math. Jpn. 64 (2), 255–266.
Ebrahimi, N., 1996. How to measure uncertainty in the residual lifetime distribution. Sankhyā Ser. A 58, 48–56.
Ebrahimi, N., Pellerey, F., 1995. New partial ordering of survival functions based on the notion of uncertainty. J. Appl. Probab. 32, 202–211.
Freimer, M., Mudholkar, G.S., Kollia, G., Lin, C.J., 1988. A study of the generalized Tukey lambda family. Comm. Statist. Theory Methods 17, 3547–3567.
Gilchrist, W., 2000. Statistical Modelling with Quantile Functions. Chapman and Hall/CRC, Boca Raton, FL.
Govindarajulu, A., 1977. A class of distributions useful in lifetesting and reliability with applications to nonparametric testing. In: Tsokos, C.P., Shimi, I.N. (Eds.), Theory and Applications of Reliability, Vol.1. Academic Press, New York, pp. 109–130.
Hankin, R.K.S., Lee, A., 2006. A new family of non-negative distributions. Aust. N. Z. J. Stat. 48, 67–78.
Jones, M.C., 2007. On a class of distributions defined by the relationship between their density and distribution functions. Comm. Statist. Theory Methods 36, 1835–1843.
Kundu, C., Nanda, A.K., Maiti, S.S., 2010. Some distributional results through past entropy. J. Statist. Plann. Inference 140 (5), 1280–1291.
Nair, N.U., Sankaran, P.G., 2009. Quantile based reliability analysis. Comm. Statist. Theory Methods 38, 222–232.
Nair, N.U., Sankaran, P.G., Vinesh Kumar, B., 2008. Total time on test transforms of order *n* and its implications in reliability analysis. J. Appl. Probab. 45, 1126–1139.
Nair, N.U., Sankaran, P.G., Vinesh Kumar, B., 2011. Modelling lifetimes by quantile functions using Parzen's score function. Statistics 1, 1–13.
Nanda, A.K., Paul, P., 2006. Some properties of past entropy and their applications. Metrika 64, 47–61.
Navarro, J., Ruiz, J.M., del Aguila, Y., 2006. Multivariate weighted distributions. Statistics 40, 51–54.
Navarro, J., Sunoj, S.M., Linu, M.N., 2011. Characterizations of bivariate models using dynamic Kullback–Leibler discrimination measures. Statist. Probab. Lett. 81, 1594–1598.
Parzen, E., 1979. Nonparametric statistical data modeling. J. Amer. Statist. Assoc. 74, 105–122.
Ramberg, J.S., Schmeiser, B.W., 1974. An approximate method for generating asymmetric random variables. Comm. ACM. 17, 78–82.
Rao, C.R., 1965. On discrete distributions arising out of methods of ascertainment. Sankhyā Ser. A 27, 311–324.
Sankaran, P.G., Nair, N.U., 2009. Nonparametric estimation of the hazard quantile function. J. Nonparametr. Stat. 21, 757–767.
Sankaran, P.G., Nair, N.U., Sreedevi, E.P., 2010. A quantile based test for comparing cumulative incidence functions of competing risks models. Statist. Probab. Lett. 80, 886–891.
Shannon, C.E., 1948. A mathematical theory of communication. Bell Syst. Tech. J. 27, 379–423.
Sunoj, S.M., Sankaran, P.G., 2012. Quantile based entropy function. Statist. Probab. Lett. 82, 1049–1053.
van Staden, P.J., Loots, M.T., 2009. *L*-moment estimation for the generalized lambda distribution. In: Third Annual ASEARC Conference, New Castle, Australia.