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Characterizations of some continuous distributions using partial moments

Summary - In this paper, we define partial moments for a univariate continuous random variable. A recurrence relationship for the Pearson curve using the partial moments is established. The interrelationship between the partial moments and other reliability measures such as failure rate, mean residual life function are proved. We also prove some characterization theorems using the partial moments in the context of length biased models and equilibrium distributions.

Key Words - Partial moments; Characterization; Failure rate; Mean residual life function; Length biased models; Equilibrium distribution.

1. Introduction

Let $(\Omega, \Im, P)$ be a probability space and $X : \Omega \to H$ be a random variable, where $H = (a, b)$ is a subset of the real line with $-\infty \leq a < b \leq \infty$, the interval of support of X. When the distribution function $F(x)$ of X is absolutely continuous with probability density function $f(x)$, if $E(|X|^r) < \infty$, the r th partial moment about a point t is defined as

$$p_r(t) = E[(X - t)^+]^r,\ r = 1, 2, \ldots\ ;\ t \geq 0 \tag{1.1}$$

where

$$(X - t)^+ = \begin{cases} (X - t), & X \geq t \\ 0, & X < t\,. \end{cases}$$

The random variable $(X - t)^+$ is interpreted as residual life in the context of life length studies (see for example Lin (2003)) and the moments (1.1) are extensively used in actuarial sciences in the analysis of risks (see Denuit (2002)). When X represents the income of an individual and t is the tax exemption level, $(X - t)^+$ represents the taxable income, therefore it is quite meaningful in the assessment of income tax.

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Earlier, Chong (1977) has established a characterization of the exponential and geometric laws based on the property of partial moments. Later, Gupta and Gupta (1983) have proved that the r th partial moment of a continuous random variable determines the underlying distribution uniquely for any positive real r (see also Hurlimann (2000)). When r is a positive integer, there exists a recurrence relation between two consecutive partial moments, which then determines all the remaining partial moments and the distribution itself. Recently, Nair *et al.* (2000) derived the general properties of descending factorial moment and characterizations based on them. They investigated the applications of partial moments in characterizing discrete probability distributions and in reliability modeling. A similar investigation using ascending factorial moment is available in Priya *et al.* (2000). For other properties and applications of partial moments to reliability analysis we refer to Hitha (1991) and Priya *et al.* (2000).

In the present paper, we focus attention on the partial moments in the continuous set up. We present recurrence relations satisfying the partial moments for the Pearson distributions and exponential family of distributions. Finally, we characterize some life distributions in the context of length biased models and equilibrium distributions using partial moments.

2. Properties of partial moments

By virtue of the relationship (1.1), we have

$$p_r(t) = \int_t^b (x-t)^r f(x)dx \,. \tag{2.1}$$

Let X be a non negative random variable and $R(x) = P(X > x)$ be the survival function of X with $E(X^r) < \infty$, (2.1) is equivalent to

$$p_r(t) = r \int_t^b (x-t)^{r-1} R(x)dx \,. \tag{2.2}$$

Directly from the definition it follows that

$$p_1'(t) = -R(t) \Rightarrow f(t) = p_1''(t) \,.$$

Further

$$p_1(t) = r(t)R(t) \tag{2.3}$$

where $r(x) = E(X - x|X > x)$ is the mean residual life function (MRLF). From (2.3), we obtain

$$r(t) = -\frac{p_1(t)}{p_1'(t)} \tag{2.4}$$

and

$$h(t) = -\frac{p_1''(t)}{p_1'(t)}\,. \tag{2.5}$$

Theorem 2.1. *For any positive integer r, $p_r(t)$ determines the distribution uniquely.*

Proof. For proof we refer to Gupta and Gupta (1983). The survival function can be obtained using the relationship

$$\frac{d^r p_r(t)}{dt^r} = (-1)^r r! R(t)$$

or

$$R(t) = \frac{\dfrac{d^r p_r(t)}{dt^r}}{(-1)^r r!} \tag{2.6}$$

(see also Hurlimann (2000)). For a more general result we refer to Lin (2003).

A list of various distributions with its pdf and the corresponding forms of $p_r(t)$ are given in Table 2.1.

TABLE 2.1.

Distribution	$f(x)$	$p_r(t)$
Exponential	$ae^{-ax};\ x > 0$	$f(t)\dfrac{\Gamma(r+1)}{a^{r+1}}$
Pareto II	$ac(1+ax)^{-c-1};\ x > 0,\ a, c > 0$	$f(t)\dfrac{r(1+at)^{r+1}}{ca^{r+1}}B(c-r, r)$
Finite range	$ac(1-ax)^c;\ 0 < x < \dfrac{1}{a},\ c > 0$	$f(t)\dfrac{r(1-at)^{r+1}}{ca^{r+1}}B(c+1, r)$
Translated Pareto	$\left(\dfrac{a}{c}\right)^c;\ x > a > 0,\ c > 0$	$f(t)rt^r B(c-r, r)$
Uniform	$\dfrac{1}{b-a};\ a < x < b$	$f(t)\dfrac{(b-t)^{r+1}}{(r+1)}$
Gamma	$\dfrac{a^n}{\Gamma(n)}x^{n+1}e^{-ax};\ x > 0$	$f(t)\displaystyle\sum_{k=0}^{n-1}\binom{n-1}{n-k-1}\frac{\Gamma(r+k+1)}{a^{r+k+1}t^k}$ provided n is integer
Power function	$nx^{n-1};\ 0 \le x \le 1$	$f(t)\displaystyle\sum_{k=0}^{n-1}\binom{n-1}{n-k-1}\frac{(1-t)^{r+k+1}}{(r+k+1)t^k}$ provided n is integer
Beta	$\dfrac{x^{m-1}(1-x)^{n-1}}{B(m,n)};$ $0 < x < 1$	$f(t)\displaystyle\sum_{k=0}^{m-1}\binom{m-1}{m-k-1}\frac{(1-t)^{r+k}}{t^k}B(r+k+1, n)$ provided m is integer

3. Partial moments of Pearson and exponential family

For every member of Pearson family the probability density function satisfies a differential equation of the form

$$\frac{1}{f(x)}\frac{df(x)}{dx} = \frac{-(x+a)}{c_0 + c_1 x + c_2 x^2}\,. \tag{3.1}$$

The shape of the distribution depends on the values of the parameters a, c_0, c_1 and c_2.

Theorem 3.1. *If X is a random variable in the support of R with $E(|X|^r) < \infty$, has the distribution belonging to the Pearson family, then it satisfies the recurrence relationship*

$$\begin{aligned} -r(c_0 + c_1 t + c_2 t^2)p_{r-1}(t) + [-(r+1)(c_1 + 2c_2 t) + t + a]p_r(t) \\ + [-(r+2)c_2 + 1]p_{r+1}(t) = 0\,. \end{aligned} \tag{3.2}$$

Proof. When the distribution belongs to Pearson family

$$(c_0 + c_1 x + c_2 x^2)\frac{df(x)}{dx} = -(x+a)f(x)\,.$$

Multiplying both sides by $(x-t)^r$ and using

$$x^2 = (x-t)^2 + 2t(x-t) + t^2$$

and integrating from t to b, we have the required result.

Theorem 3.2. *The distribution of X belongs to the exponential family with pdf*

$$f(x) = \exp[\theta x + c(x) + D(\theta)] \tag{3.3}$$

if and only if the partial moments satisfy the recurrence relationship

$$p_{r+1}(t) = \frac{dp_r(t)}{d\theta} - (t + D'(\theta))p_r(t)\,. \tag{3.4}$$

Proof. Suppose that (3.4) holds. Then we get

$$\int_t^b (x-t)^{r+1} f(x)dx + (t + D'(\theta))\int_t^b (x-t)^r f(x)dx = \int_t^b (x-t)^r \frac{df}{d\theta}dx$$

which is equivalent to

$$\int_t^b (x-t)^r \left[(x + D'(\theta))f(x) - \frac{df}{d\theta}\right] dx = 0\,.$$

This is true for every $t > 0$ only when

$$\begin{aligned}(x + D'(\theta)) f(x) &= \frac{df}{d\theta} \\ \frac{d \log f}{d\theta} &= x + D'(\theta)\,.\end{aligned} \tag{3.5}$$

Differentiating (3.5) with respect to θ, we obtain

$$\log f = \theta x + D(\theta) + k(x)$$

which proves the result. The 'only if' part is straightforward.

4. Lenght biased models

The statistical interpretation of the length biased distribution was originally identified by Cox (1962) in the context to renewal theory. Assume that the population of failure times is distributed according to $f(x)$, the probability of selection of any individual in the population proportional to its life length x, then the density function of the life length for the sampled component of random variable Y has the form

$$g(x) = \frac{x f(x)}{\mu}, \; x > 0 \tag{4.1}$$

where $\mu = E(X) < \infty$, which is the length biased form (see also Blumenthal (1967), Scheaffer (1972)). A detailed survey of literature on applications of length biased models we refer to Rao (1965), Patil and Rao (1977), Gupta and Kirmani (1990) and Sunoj (2000).

In the present section, we examine the structural relationships between the random variables of X and Y in the context of partial moments. Let $p_r^L(t)$ denotes the partial moment of the random variable Y, which is defined as

$$p_r^L(t) = E[(Y - t)^+]^r; \; r = 1, 2, \ldots \tag{4.2}$$

where

$$(Y - t)^+ = \begin{cases} Y - t, & Y \geq t \\ 0, & Y < t\,. \end{cases}$$

Then

$$\begin{aligned} p_r^L(t) &= \int_t^b (u - t)^r g(u) du \\ &= \int_t^b (u - t)^r \frac{u f(u)}{\mu} du \\ &= \frac{1}{\mu} \int_t^b (u - t)^r (u - t + t) f(u) du \end{aligned}$$

which yields

$$p_r^L(t) = \frac{p_{r+1}(t) + tp_r(t)}{\mu}\,. \tag{4.3}$$

Remark. It can be easily shown that differentiating (4.3) with respect to t successively $(r+1)$ times, it remains the pdf of the length biased models of form (4.1).

5. Equilibrium distribution

The equilibrium distribution arises naturally in renewal theory (see e.g., Cox (1962), Blumenthal (1967), Despande *et al.* (1986), Singh (1989) or Nair and Hitha (1990), for a discussion). It is the distribution of the backward or forward recurrence time in the limiting case. A formal definition of the equilibrium distribution is as follows. Let X be a random variable admitting absolutely continuous distribution function $F(x)$ with respect to Lebesgue measure in the support of the set of non-negative real numbers. Associated with X a random variable Y can be defined with probability density function

$$g(x) = \frac{R(x)}{\mu},\ \ x > 0 \tag{5.1}$$

with $F(0) = 0$ and $\mu = E(X) < \infty$.

The partial moment of the random variable Y obtained from (5.1) is denoted by $p_r^E(t)$ and is given by

$$p_r^E(t) = \frac{p_{r+1}(t)}{(r+1)\mu}\,. \tag{5.2}$$

For a recent discussion of equilibrium distributions and partial moments we refer to Hesselager *et al.* (1998) and Willmese and Koppelaar (2000).

Remark. It can be easily shown that differentiating (5.2) with respect to t successively $(r+1)$ times, it remains the pdf of the equilibrium distribution (5.1).

Table 5.1 gives the pdf's of different distributions and its corresponding length biased and equilibrium models.

TABLE 5.1

Distribution	$R(x)$	$p_r(t)$	$p_r^L(t)$	$p_r^E(t)$
Exponential	$e^{-ax},\ x>0$	$\frac{r!e^{-at}}{a^r}$	$(1+r+at)p_r(t)$	$p_r(t)$
Pareto II	$(1+ax)^{-c}$, $x>0,\ a>0,\ c>1$	$\frac{r(1+at)^{r-c}}{a^r}$ $B(c-r,r)$	$\frac{(c-1)}{(c-r-1)}$ $(1+r+cat)p_r(t)$	$\frac{(c-1)}{(c-r-1)}$ $(1+at)p_r(t)$
Power	$(1-ax)^c$, $0<x<\frac{1}{a},\ c>0$	$\frac{r(1-at)^{r+c}}{a^r}$ $B(c+1,r)$	$\frac{(c+1)}{(c+r+1)}$ $(1+r+cat)p_r(t)$	$\frac{(c+1)}{(c+r+1)}$ $(1-at)p_r(t)$
Translated Pareto	$\left(\frac{a}{x}\right)^c$, $x>a>0$	$ra^ct^{r-c}B(c-r,r)$	$\frac{ct}{(c-r-1)\mu}p_r(t)$	$\frac{t}{(c-r-1)\mu}p_r(t)$
Uniform	$\frac{b-x}{b-a}$	$\frac{(b-t)^{r+1}}{(r+1)(b-a)}$	$\frac{(t+(r+1)b)}{(r+2)\mu}p_r(t)$	$\frac{(b-t)}{(r+2)\mu}p_r(t)$

6. CHARACTERIZATIONS OF DISTRIBUTIONS USING LENGHT BIASED AND EQUILIBRIUM DISTRIBUTIONS

In this section, we prove some characterization theorems for uniform, exponential, Pareto II, beta and translated Pareto distributions using partial moments in the context of length biased and equilibrium distributions.

Theorem 6.1. *The partial moments of length biased models and original models satisfy the relationship*

$$\frac{p_r^L(t)}{p_r(t)} = \frac{(A+Bt)}{\mu} \tag{6.1}$$

if and only if X has Pareto II with pdf

$$f(x) = (1+ax)^{-c},\ x>0,\ a>0,\ c>1 \tag{6.2}$$

for $B>1$, *the exponential distribution with pdf*

$$f(x) = ae^{-ax},\ x>0 \tag{6.3}$$

for $B=1$, *Power distribution with pdf*

$$f(x) = (1-ax)^c,\ 0<x<\frac{1}{a},\ c>0 \tag{6.4}$$

for $0<B<1$ *and Translated Pareto distribution with pdf*

$$f(x) = \frac{c}{x}\left(\frac{a}{x}\right),\ x>a>0 \tag{6.5}$$

for $A=0$ *and* $B>1$.

Proof. For Pareto II distribution (6.2), we obtain

$$p_r(t) = \frac{r(1+at)^{r-c}}{a^c} B(c-r, r) \tag{6.6}$$

and using (4.3), we get

$$p_r^L(t) = \frac{(c-1)}{(c-r-1)}(1+r+cat)p_r(t)$$

which is of the form (6.1) with $A = \frac{(r+1)}{a(c-r-1)}$, $B = \frac{c}{c-r-1} > 1$ and $\mu = \frac{1}{a(c-1)}$. For exponential distribution (6.3), we have

$$p_r(t) = \frac{r!e^{-at}}{a^r}, \tag{6.7}$$

then

$$p_r^L(t) = (1+r+at)p_r(t) \tag{6.8}$$

with $A = \frac{(r+1)}{a}$, $B = 1$ and $\mu = \frac{1}{a}$. For Power distribution (6.4), we get

$$p_r(t) = \frac{r(1-at)^{r+c}}{a^c} B(c+1, r) \tag{6.9}$$

and accordingly

$$p_r^L(t) = \frac{(c+1)}{(c+r+1)}(1+r+cat)p_r(t) \tag{6.10}$$

which is of the form (6.1) with $A = \frac{(r+1)}{a(c+r+1)}$, $B = \frac{c}{c+r+1} < 1$ and $\mu = \frac{1}{a(c+1)}$. Finally, for Translated Pareto distribution (6.5), we've

$$p_r^L(t) = \frac{(c-1)t}{(c-r-1)a} p_r(t) \tag{6.11}$$

which is also the form of (6.1) with $A = 0$ and $B = \frac{c}{c-r-1} > 1$ and $\mu = \frac{ac}{(c-1)}$.

Conversely, if (6.3) holds and comparing with (4.3), we obtain

$$\frac{p_{r+1}(t) + tp_r(t)}{\mu p_r(t)} = \frac{(A+Bt)}{\mu} \tag{6.12}$$

which is equivalent to

$$p_{r+1}(t) = Ap_r(t) + (B-1)tp_r(t)$$

or

$$p_{r+1}(t) = Ap_r(t) + Ctp_r(t) \tag{6.13}$$

where $C = B - 1$. Differentiating (6.13) with respect to t successively $(r+1)$ times and using Theorem 2.1 we obtain the required result.

Remark. The ratio (6.1) takes the form

$$\frac{p_r^L(t)}{p_r(t)} = \frac{t + (r+1)b}{(r+2)\mu} \tag{6.14}$$

if and only if X has Uniform distribution with pdf

$$f(x) = \frac{1}{b-a}, \; a < x < b\,. \tag{6.15}$$

We next prove some similar characterization theorems for the equilibrium distribution.

Theorem 6.2. *The partial moments satisfy the relationship*

$$\frac{p_r^E(t)}{p_r(t)} = \frac{(A + Bt)}{\mu} \tag{6.16}$$

if and only if X has Pareto II distribution (6.2) *for $B > 0$, exponential distribution* (6.3) *for $B = 0$, Power distribution* (6.4) *for $B < 0$ and Translated Pareto distribution* (6.5) *for $A = 0$, $B > 0$.*

Proof. The proof is similar to Theorem 6.1.

Corollary. *The relationship*

$$p_r^L(t) = cp_r^E(t) \tag{6.17}$$

if and only if X has translated Pareto distribution (6.5).

Remark. The ratio (6.16) takes the form

$$\frac{p_r^E(t)}{p_r(t)} = \frac{b-t}{(r+2)\mu} \tag{6.18}$$

if and only if X has a Uniform distribution (6.15).

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