Morphological filtering on hypergraphs

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Abstract

The focus of this article is to develop computationally efficient mathematical morphology operators on hypergraphs. To this aim we consider lattice structures on hypergraphs on which we build morphological operators. We develop a pair of dual adjunctions between the vertex set and the hyper edge set of a hypergraph H, by defining a vertex-hyperedge correspondence. This allows us to recover the classical notion of a dilation/erosion of a subset of vertices and to extend it to subhypergraphs of H. Afterward, we propose several new openings, closings, granulometries and alternate sequential filters acting (i) on the subsets of the vertex and hyperedge set of H and (ii) on the subhypergraphs of a hypergraph.

Keywords: Hypergraphs, Mathematical Morphology, Complete Lattices, Adjunctions, Granulometries, Alternating Sequential Filters.

1   Introduction

Mathematical morphology, appeared in 1960s is a theory of nonlinear information processing [10], [17], [18], [19]. It is a branch of image analysis based on algebraic, set-theoretic and geometric principles [13], [16]. Originally, it is devoleped for binary images by Matheron and Serra. They are the first to observe that a general theory of mathematical morphology is based on the assumption that the underlying image space is a complete lattice. Most of the morphological theory at this abstract level was developed and presented without making references to the properties of the underlying space. Considering digital objects carrying structural information, mathematical morphology has

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been developed on graphs [6], [7], [11], [21] and simplicial complexes [9], but little work has been done on hypergraphs [2], [3], [4], [20].

When dealing with a hypergraph H, we need to consider the hypergraph induced by the subset $X^\bullet$ of vertices of H (see figure 1(a) and (b), where the blue vertices and edges in (b) represents X). We associate with $X^\bullet$ the largest subset of hyperedges of H such that the obtained pair is a hypergraph. We denote it by $H(X^\bullet)$ (see section 2.1 and figure 1(b)). We also consider a hypergraph induced by a subset $X^\times$ of the edges of H, namely $H(X^\times)$.

Here we propose a systematic study of the basic operators that are used to derive a set of hyper edges from a set of vertices and a set of vertices from a set of hyperedges. These operators are the hypergraph extension to the operators defined by Cousty [6], [7] for graphs. Since a hypergraph becomes a graph when $|v(e)| = 2$ for every hyperedge e, all the properties of these operators are satisfied for graphs also. We emphasis that the input and output of these operators are both hypergraphs. The blue subhypergraph in figure 1(c) is the result of the dilation $[\triangle, \delta](X)$ of the blue subhypergraph X in figure 1(b) proposed in this paper. Here the resultant subhypergrah in figure 1(c) is not induced by its vertex set.

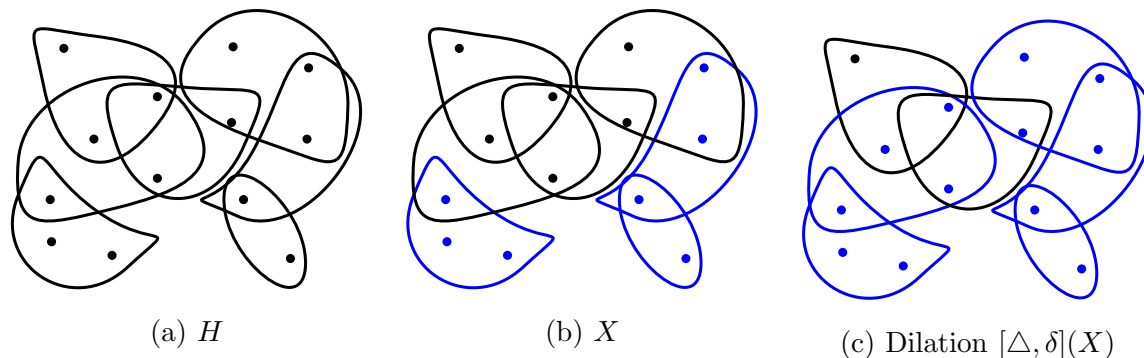(a) H (b) X (c) Dilation $[\triangle, \delta](X)$

Figure 1: Illustration of hypergraph dilation

This paper is organized as follows. In Section 2 we recall some related works on graphs and hypergraphs. In Section 3, we describe some preliminary definitions and results on mathematical morphology and hypergraphs. In section 4, we define the vertex-hyperedge correspondence along with various dilations, erosions and adjunctions on hypergraphs. The properties of these morphological operators are studied in this section. In section 5, we propose several new openings, closings, granulometries and alternate sequential filters acting (i) on the subsets of vertices and hyperedges and (ii) on the subhypergraphs. Section 6 concludes the paper.

2 Related works

Graph theoretic methods have found increasing applications in image analysis. Morphological operators are well studied on graphs. Vincent [21] defined morphological operators on a graph $G = (V, E)$,

where V represents a set of weighted vertices and E, a set of edges between vertices. The dilation (resp. erosion) replace the value of each vertex with the maximum (resp. minimum) value of its neighbors. Cousty et. al. [6], [7] considered a graph as a pair $G = (G^\bullet, G^\times)$, where $G^\bullet$ is the set of vertices and $G^\times$ is the edge set of the graph G. They define morphological operators on various lattices formed by the graph G by defining an edge-vertex correspondence. This powerful tool allows them to recover the clasical notion of a dilation/erosion of a subset of vertices of G. This lead them to propose several new openings, closings, granulometries and alternate sequential filters acting on the subsets of the edge sets, subsets of vertex sets and the lattice of subgraphs of G. These operators are further extended to functions that weight the vertices and edges of G [14] and are found to be useful in image filtering. In this work we aim to develop morphological operators on hypergraphs by defining a vertex-hyperedge correspondence.

The theory of hypergraphs originated as a natural generalisation of graphs in 1960s. In a hypergraph, edges can connect any number of vertices and are called hyperedges. Considering the topological and geometrical aspects of an image, Bretto [5] has proposed a hypergraph model to represent an image. The theory of hypergraphs became an active area of research in image analysis [3], [8]. The study of mathematical morphology operators on hypergraphs started recently, and little work being reported in this regard. Properties of morphological operators on hypergraphs are studied in [20], in which subhypergraphs are considered as relations on hypergraphs. Recently, Bloch and Bretto [2] introduced mathematical morphology on hypergraphs by forming various lattices on hypergraphs. Similarity and pseudo-metrics based on mathematical morphology are defined and illustrated in [4]. Based on these morphological operators, similarity measures are used for classification of data represented as hypergraphs [3].

3   Preliminaries

3.1   Hypergraphs

We define a *hypergraph* [1], [2] as a pair $H = (H^\bullet, H^\times)$ where $H^\bullet$ is a set of points called *vertices* and $H^\times$ is composed of a family of subsets of $H^\bullet$ called *hyperedges*. We denote $H^\times$ by $H^\times = (e_i)_{i \in I}$ where I is a finite set of indices. The set of vertices forming the hyperedge e is denoted by $v(e)$. A vertex x in $H^\bullet$ is called an *isolated vertex* of H if $x \notin \underset{i \in I}{\cup} v(e_i)$. The *empty hypergraph* is the hypergraph H_ϕ such that $H^\bullet = \phi$ and $H^\times = \phi$. The *partial hypergraph* H' of H generated by $J \subseteq I$ is the hypergraph $H' = (H'^\bullet, H'^\times)$ where $H'^\bullet = H^\bullet$ and $H'^\times = (e_j)_{j \in J}$. A hypergraph $X = (X^\bullet, X^\times)$ is called a *subhypergraph* of H, denoted by $X \subseteq H$, if $X^\bullet \subseteq H^\bullet$ and $X^\times \subseteq H^\times$.

Let $X^\bullet \subseteq H^\bullet$ and $X^\times \subseteq H^\times$ where $X^\times = (e_j), j \in J$ such that $J \subseteq I$. We denote by $\overline{X^\bullet}$ (resp. $\overline{X^\times}$) by the complementary set of $X^\bullet$ (resp. $X^\times$). Let $H(X^\bullet)$ and $H(X^\times)$ respectively denote the hypergraphs $(X^\bullet, \{e_i, i \in I | v(e_i) \subseteq X^\bullet\})$ and $(\underset{j \in J}{\cup} v(e_j), (e_j)_{j \in J})$.

While dealing with a hypergraph H, we consider the subhypergraph induced by a subset $X^\bullet$ of vertices of H namely $H(X^\bullet)$, and the subhypergraph induced by a subset $X^\times$ of hyperedges namely $H(X^\times)$. $H(X^\bullet)$ is the largest subhypergraph of H with $X^\bullet$ as vertex set and $H(X^\times)$ is the smallest subhypergraph of H with $X^\times$ as its hyperedge set.

3.2   Mathematical Morphology

Now let us briefly recall some algebraic tools that are fundamental in mathematical morphology [6], [12], [13], [16]. Given two lattices $\mathcal{L}_1$ and $\mathcal{L}_2$, any operator $\delta : \mathcal{L}_1 \to \mathcal{L}_2$ that distributes over the supremum and preserves the least element is called a *dilation* (*i.e.* $\forall \varepsilon \subseteq \mathcal{L}_1, \delta(\vee_1 \varepsilon) = \vee_2 \{\delta(X) | X \in \varepsilon\}$). Similarly an operator that distributes over the infimum and preserves the greatest element is called an *erosion*.

Two operators $\epsilon : \mathcal{L}_1 \to \mathcal{L}_2$ and $\delta : \mathcal{L}_2 \to \mathcal{L}_1$ form an *adjunction* (ϵ, δ), if for any $X \in \mathcal{L}_1$ and any $Y \in \mathcal{L}_2$, we have $\delta(X) \leq_1 Y \Leftrightarrow X \leq_2 \epsilon(Y)$, where $\leq_1$ and $\leq_2$ denote the order relations in $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively [12]. Given two operators ϵ and δ, if the pair (ϵ, δ) is an adjunction, then ϵ is an erosion and δ is a dilation. If $\leq_1$, $\leq_2$ and $\leq_3$ are three lattices and if $\delta : \mathcal{L}_1 \to \mathcal{L}_2$, $\delta' : \mathcal{L}_2 \to \mathcal{L}_3$, $\epsilon : \mathcal{L}_2 \to \mathcal{L}_1$ and $\epsilon' : \mathcal{L}_2 \to \mathcal{L}_2$ are four operators such that (ϵ, δ) and (ϵ', δ') are adjunctions, then the pair $(\epsilon \circ \epsilon', \delta \circ \delta')$ is also an adjunction.

Given two complemented lattices, $\mathcal{L}_1$ and $\mathcal{L}_2$, two operators α and β are *dual* with respect to the complement of each other, if for each $X \in \mathcal{L}_1$, we have $\beta(X) = \alpha(\overline{X})$. If α and β are dual of each other, then β is an erosion whenever α is a dilation.

4   Hypergraph morphology: dilations, erosions and adjunctions

In a hypergraph H, we can consider sets of points as well as sets of hyperedges. Therefore it is convenient to consider operators that go from one kind of sets to the other one. In this section we define such operators and study their morphological properties. Based on these operators, we propose several dilations, erosions and adjunctions on various lattices formed by H.

Hereafter the workspace (see [6] and [7] for a similar structure defined for graphs) is a hypergraph $H = (H^\bullet, H^\times)$ and we consider the sets $\mathcal{H}^\bullet$, $\mathcal{H}^\times$ and $\mathcal{H}$ of respectively all subsets of $H^\bullet$, all subsets of $H^\times$ and all subhypergraphs of $\mathcal{H}$.

The set $\mathcal{H}$ of all subhypergraphs of a hypergraph H form a complete lattice [20]. $\mathcal{H}$ is not a Boolean algebra as the complement of a subhypergraph of H need not be a subhypergraph of H. But $\mathcal{H}^\bullet$ and $\mathcal{H}^\times$ are Boolean algebras. We define morphological operators on these lattices. For, we establish a correspondence between the vertex set and the hyperedge set of H. Composing these mappings produces morphological operators on the lattices $\mathcal{H}^\bullet$, $\mathcal{H}^\times$ and $\mathcal{H}$.

Definition 1. (**Vertex-Hyperedge Correspondence**) We define the operators $\delta^\bullet$, $\epsilon^\bullet$ from $\mathcal{H}^\times$ into $\mathcal{H}^\bullet$ and the operators $\delta^\times$, $\epsilon^\times$ from $\mathcal{H}^\bullet$ into $\mathcal{H}^\times$ as follows.

	$\mathcal{H}^\times \to \mathcal{H}^\bullet$	$\mathcal{H}^\bullet \to \mathcal{H}^\times$
Provide the object with a hypergraph structure	$X^\times \to \delta^\bullet(X^\times)$ such that $(\delta^\bullet(X^\times), X^\times) = H(X^\times)$	$X^\bullet \to \epsilon^\times(X^\bullet)$ such that $(X^\bullet, \epsilon^\times(X^\bullet)) = H(X^\bullet)$
Provide its complement with a hypergraph structure	$X^\times \to \epsilon^\bullet(X^\times)$ such that $(\epsilon^\bullet(X^\times), \overline{X^\times}) = H(\overline{X^\times})$	$X^\bullet \to \delta^\times(X^\bullet)$ such that $(\overline{X^\bullet}, \delta^\times(X^\bullet)) = H(\overline{X^\bullet})$

These operators are illustrated in figures 2(a)-(f). The choice of H is in such a way that every

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hyperedge of H is incident with exactly four vertices, and the choice of X is made to present a representative sample of the different possible configurations on subhypergraphs.
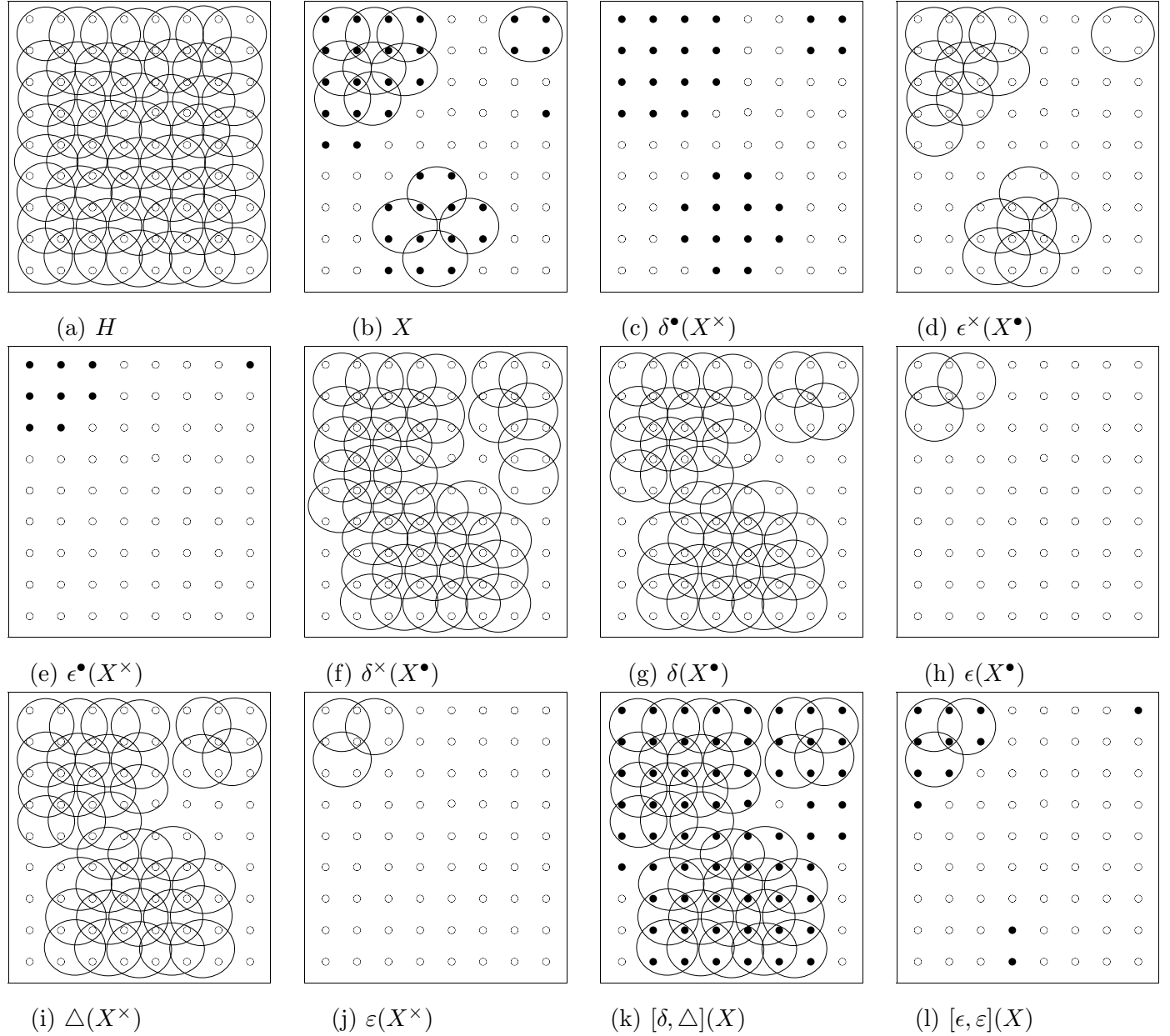

Figure 2: Illustration of dilations and erosions

Property 1. For any $X^\bullet \subseteq H^\bullet$ and any $X^\times \subseteq H^\times$, where $X^\times = (e_j), j \in J$ such that $J \subseteq I$

1. $\delta^\bullet \colon H^\times \to H^\bullet$ is such that $\delta^\bullet(X^\times) = \underset{j \in J}{\cup} v(e_j)$;

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2. $\epsilon^{\times}\colon H^{\bullet} \to H^{\times}$ is such that $\epsilon^{\times}(X^{\bullet}) = \{e_i, i \in I | v(e_i) \subseteq X^{\bullet}\}$;

3. $\epsilon^{\bullet}\colon H^{\times} \to H^{\bullet}$ is such that $\epsilon^{\bullet}(X^{\times}) = \underset{j \notin J}{\cap}\, \overline{v(e_j)}$;

4. $\delta^{\times}\colon H^{\bullet} \to H^{\times}$ is such that $\delta^{\times}(X^{\bullet}) = \{e_i, i \in I | v(e_i) \cap X^{\bullet} \neq \phi\}$.

Proof. 1. and 2. follows from the definition of $\delta^{\bullet}$ and $\epsilon^{\times}$.

3. $H(\overline{X^{\times}}) = (\underset{j \notin J}{\cup} v(e_j), (e_j)_{j \notin J})$. Thus

$$\begin{aligned} \epsilon^{\bullet}(X^{\times}) &= \overline{\underset{j \notin J}{\cup} v(e_j)} \\ &= \underset{j \notin J}{\cap} \overline{v(e_j)} \qquad\qquad\qquad \text{(By De Morgan's Law)} \end{aligned}$$

4. $\overline{\delta^{\times}(X^{\bullet})} = \{e_i, i \in I | v(e_i) \subseteq \overline{X^{\bullet}}\}$). Thus $\delta^{\times}(X^{\bullet}) = \{e_i, i \in I | v(e_i) \cap X^{\bullet} \neq \phi\}$.

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Note that $\delta^{\bullet}(X^{\times}) = \{x \in H^{\bullet} | \exists e_j \in X^{\times} \text{ such that } x \in v(e_j) \text{ for some } j \in J\}$. This property states that $\delta^{\bullet}(X^{\times})$ is the set of all vertices which belong to a hyperedge of $X^{\times}$. $\epsilon^{\times}(X^{\bullet})$ is the set of all hyperedges whose vertices are composed of vertices of $X^{\bullet}$. $\epsilon^{\bullet}(X^{\times})$ is the set of all vertices which do not belong to any edge of $\overline{X^{\times}}$, and $\delta^{\times}(X^{\bullet})$ is the set of all hyperedges in $H^{\times}$ with atleast one vertex in $X^{\bullet}$. Therefore the previous property locally characterizes the operators defined in vertex-hyperedge correspondence. This property leads to simple linear time algorithms (with respect to $|H^{\bullet}|$ and $|H^{\times}|$) to compute $\delta^{\bullet}$, $\delta^{\times}$, $\epsilon^{\bullet}$ and $\epsilon^{\times}$.

Property 2. (dilation, erosion, adjunction, duality)

1. Operators $\epsilon^{\times}$ and $\delta^{\times}$ (resp. $\epsilon^{\bullet}$ and $\delta^{\bullet}$) are dual of each other.

2. Both $(\epsilon^{\times}, \delta^{\bullet})$ and $(\epsilon^{\bullet}, \delta^{\times})$ are adjunctions.

3. Operators $\epsilon^{\bullet}$ and $\epsilon^{\times}$ are erosions.

4. Operators $\delta^{\bullet}$ and $\delta^{\times}$ are dilations.

Proof. 1. We will prove that $\overline{\delta^{\times}(\overline{X^{\bullet}})} = \epsilon^{\times}(X^{\bullet})$ and $\overline{\delta^{\bullet}(\overline{X^{\times}})} = \epsilon^{\bullet}(X^{\times})$

$$\begin{aligned} \delta^{\times}(\overline{X^{\bullet}}) &= \{e_i, i \in I | v(e_i) \cap \overline{X^{\bullet}} \neq \phi\} \qquad\qquad \text{(By property 1 of } \delta^{\times}) \\ \overline{\delta^{\times}(\overline{X^{\bullet}})} &= \{e_i, i \in I | v(e_i) \subseteq X^{\bullet}\} \\ &= \epsilon^{\times}(X^{\bullet}). \end{aligned}$$

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Thus $\epsilon^{\times}$ and $\delta^{\times}$ are duals.

$$\begin{aligned}
\delta^{\bullet}(X^{\times}) &= \underset{j \in J}{\cup}\, v(e_j) \\
\delta^{\bullet}(\overline{X^{\times}}) &= \underset{j \notin J}{\cup}\, v(e_j) \\
\overline{\delta^{\bullet}(\overline{X^{\times}})} &= \overline{\underset{j \notin J}{\cup}\, v(e_j)} \\
&= \underset{j \notin J}{\cap}\, \overline{v(e_j)} && \text{(By De Morgan's Law)} \\
&= \epsilon^{\bullet}(X^{\times}).
\end{aligned}$$

Therefore $\epsilon^{\bullet}$ and $\delta^{\bullet}$ are duals.

2. Suppose that $X^{\times} \subseteq \epsilon^{\times}(Y^{\bullet})$. Then

$$\begin{aligned}
x \in \delta^{\bullet}(X^{\times}) \Rightarrow\;\; & x \in \underset{j \in J}{\cup}\, v(e_j) \\
\Rightarrow\;\; & x \in v(e_j) \text{ for some } j \in J \\
\Rightarrow\;\; & \exists\, e \in X^{\times} \text{ such that } x \in v(e) \\
\Rightarrow\;\; & e \in \epsilon^{\times}(Y^{\bullet}) && (\because X^{\times} \subseteq \epsilon^{\times}(Y^{\bullet})) \\
\Rightarrow\;\; & e \in \{e_i, i \in I | v(e_i) \subseteq Y^{\bullet}\} \\
\Rightarrow\;\; & v(e) \subseteq Y^{\bullet} \\
\Rightarrow\;\; & x \in Y^{\bullet} && (\because x \in v(e))
\end{aligned}$$

Therefore $\delta^{\bullet}(X^{\times}) \subseteq Y^{\bullet}$.
Conversly, if $\delta^{\bullet}(X^{\times}) \subseteq Y^{\bullet}$. Then

$$\begin{aligned}
e \in X^{\times} \Rightarrow\;\; & v(e) \subseteq \delta^{\bullet}(X^{\times}) \\
\Rightarrow\;\; & v(e) \subseteq Y^{\bullet} && (\because \delta^{\bullet}(X^{\times}) \subseteq Y^{\bullet}) \\
\Rightarrow\;\; & e \in \epsilon^{\times}(Y^{\bullet})
\end{aligned}$$

Thus $X^{\times} \subseteq \epsilon^{\times}(Y^{\bullet})$. Therefore $(\epsilon^{\times}, \delta^{\bullet})$ is an adjunction.

$$\begin{aligned}
\delta^{\times}(X^{\bullet}) \subseteq Y^{\times} \;\Leftrightarrow\;\; & \overline{\epsilon^{\times}(\overline{X^{\bullet}})} \subseteq Y^{\times} && \text{(By duality of } \epsilon^{\times} \text{ and } \delta^{\times}) \\
\Leftrightarrow\;\; & \overline{Y^{\times}} \subseteq \epsilon^{\times}(\overline{X^{\bullet}}) \\
\Leftrightarrow\;\; & \delta^{\bullet}(\overline{Y^{\times}}) \subseteq \overline{X^{\bullet}} && \text{(By adjunction property of } (\epsilon^{\times}, \delta^{\bullet})) \\
\Leftrightarrow\;\; & X^{\bullet} \subseteq \overline{\delta^{\bullet}(\overline{Y^{\times}})} \\
\Leftrightarrow\;\; & X^{\bullet} \subseteq \epsilon^{\bullet}(Y^{\times}) && \text{(By duality of } \epsilon^{\bullet} \text{ and } \delta^{\bullet})
\end{aligned}$$

Therefore $(\epsilon^{\bullet}, \delta^{\times})$ is an adjunction.

Properties 3. and 4. follows from the dilation / erosion property of adjunctions. $\square$

Definition 2. (vertex dilation, vertex erosion). We define δ and ϵ that act on $\mathcal{H}^\bullet$ by $\delta = \delta^\bullet \circ \delta^\times$ and $\epsilon = \epsilon^\bullet \circ \epsilon^\times$.

Property 3. For any $X^\bullet \subseteq H^\bullet$.

1. $\delta(X^\bullet) = \{x \in H^\bullet | \ \exists \ e_i, i \in I \text{ such that } x \in v(e_i) \text{ and } v(e_i) \cap X^\bullet \neq \phi\}$.

2. $\epsilon(X^\bullet) = \{x \in H^\bullet | \ \forall \ e_i, i \in I \text{ such that } x \in v(e_i), v(e_i) \subseteq X^\bullet\}$.

Proof. 1.

$$
\begin{aligned}
\delta(X^\bullet) &= \delta^\bullet(\delta^\times(X^\bullet)) \\
&= \delta^\bullet[\{e_i, i \in I | v(e_i) \cap X^\bullet \neq \phi\}] && \text{(By property 1 of } \delta^\times) \\
&= \bigcup_{i \in I, v(e_i) \cap X^\bullet \neq \phi} v(e_i) && \text{(By property 1 of } \delta^\bullet) \\
&= \{x \in H^\bullet | \ \exists e_i, i \in I \text{ such that } x \in v(e_i) \text{ and } v(e_i) \cap X^\bullet \neq \phi\}.
\end{aligned}
$$

2.

$$
\begin{aligned}
\epsilon(X^\bullet) &= \epsilon^\bullet(\epsilon^\times(X^\bullet)) \\
&= \epsilon^\bullet[\{e_i, i \in I | v(e_i) \subseteq X^\bullet)\}] && \text{(By property 1 of } \epsilon^\times) \\
&= \bigcap_{i \in I, v(e_i) \not\subseteq X^\bullet} \overline{v(e_i)} && \text{(By property 1 of } \epsilon^\bullet) \\
&= \{x \in X^\bullet | \ \forall e_i \in H^\times \text{ with } x \in v(e_i), v(e_i) \subseteq X^\bullet\}
\end{aligned}
$$

$\square$

Definition 3. (hyper-edge dilation, hyper-edge erosion) We define $\triangle$ and ε that act on $\mathcal{H}^\times$ by $\triangle = \delta^\times \circ \delta^\bullet$ and $\varepsilon = \epsilon^\times \circ \epsilon^\bullet$.

Property 4. For any $X^\times \subseteq H^\times$, $X^\times = (e_j)_{j \in J}$.

1. $\triangle(X^\times) = \{e_i, i \in I | \ \exists e_j, j \in J \text{ such that } v(e_i) \cap v(e_j) \neq \phi\}$.

2. $\varepsilon(X^\times) = \{e_j, j \in J | v(e_j) \cap v(e_i) = \phi, \forall i \in I \setminus J\}$.

Proof. 1.

$$
\begin{aligned}
\triangle(X^\times) &= \delta^\times \circ \delta^\bullet(X^\times) \\
&= \delta^\times[\bigcup_{j \in J} v(e_j)] && \text{(By property 1 of } \delta^\times) \\
&= \{e_i, i \in I | v(e_i) \cap [\bigcup_{j \in J} v(e_j)] \neq \phi\}. && \text{(By property 1 of } \delta^\bullet) \\
&= \{e_i, i \in I | \exists e_j, j \in J \text{ such that } v(e_i) \cap v(e_j) \neq \phi\}.
\end{aligned}
$$

2.

$$\begin{aligned}
\varepsilon(X^\times) &= \epsilon^\times \circ \epsilon^\bullet(X^\times) \\
&= \epsilon^\times[\underset{i\in I\setminus J}{\cap}\,\overline{v(e_i)}] & \text{(By property 1 of } \epsilon^\bullet) \\
&= \{e_j, j\in J | v(e_j) \subseteq [\underset{i\in I\setminus J}{\cap}\,\overline{v(e_i)}] \\
&= \{e_j, j\in J | v(e_j) \subseteq [\overline{\underset{i\in I\setminus J}{\cup}\,v(e_i)}]\} & \text{(By De Morgan's Law)} \\
&= \{e_j, j\in J | v(e_j)\cap v(e_i) = \phi, \forall i\in I\setminus J\}
\end{aligned}$$

$\square$

Remark 1. Being the compositions of respectively dilations and erosions, δ and ϵ are respectively a dilation and an erosion [12]. Moreover by composition of adjunctions and dual operators, δ and ϵ are dual and (ϵ, δ) is an adjunction. In a similar manner $(\varepsilon, \triangle)$ is also an adjunction.

Definition 4. (**hypergraph dilation, hypergraph erosion**) We define the operators $[\delta, \triangle]$ and $[\epsilon, \varepsilon]$ by respectively $[\delta, \triangle](X) = (\delta(X^\bullet), \triangle(X^\times))$ and $[\epsilon, \varepsilon](X) = (\epsilon(X^\bullet), \varepsilon(X^\times))$, for any $X \in \mathcal{H}$.

Theorem 1. *The operators $[\delta, \triangle]$ and $[\epsilon, \varepsilon]$ are respectively a dilation and an erosion acting on the lattice $(\mathcal{H}, \subseteq)$.*

Proof. We will prove that for every $e \in \triangle(X^\times)$, $v(e) \subseteq \delta(X^\bullet)$. $e \in \triangle(X^\times)$ implies, there exists some $j \in J$ such that $v(e) \cap v(e_j) \neq \phi$. But $v(e_j) \subseteq X^\bullet$, since $j \in J$. Thus $v(e) \cap X^\bullet \neq \phi$. Therefore $v(e) \subseteq \underset{j\in J, v(e_i)\cap X^\bullet \neq \phi}{\cup} v(e_i) = \delta(X^\bullet)$. This implies $[\delta, \triangle](X) \in \mathcal{H}$.
If $e \in \varepsilon(X^\times)$, then $v(e)\cap v(e_i) = \phi$ for every $i \in I\setminus J$, and so $v(e)\cap[\underset{i\in I\setminus J}{\cup} v(e_i)] = \phi$.

$$\begin{aligned}
v(e) &\subseteq \overline{\underset{i\in I\setminus J}{\cup}\,v(e_i)} \\
&= \underset{i\in I\setminus J}{\cap}\,\overline{v(e_i)} \\
&\subseteq \underset{v(e_i)\nsubseteq X^\bullet}{\cap}\,\overline{v(e_i)} & \text{(Since } v(e_i) \subseteq X^\bullet, \forall i\in J) \\
&= \varepsilon(X^\bullet) & \text{(By Property 3 of } \varepsilon)
\end{aligned}$$

Therefore $[\epsilon, \varepsilon](X) \in \mathcal{H}$. $\square$

Theorem 2. $([\epsilon, \varepsilon], [\delta, \triangle])$ *is an adjunction.*

Proof. Let X and Y are two hypergraphs in $\mathcal{H}$. The following statements are equivalent.

$[\delta, \triangle](X) \subseteq Y$
$\delta(X^\bullet) \subseteq Y^\bullet$ and $\triangle(X^\times) \subseteq Y^\times$
$X^\bullet \subseteq \epsilon(Y^\bullet)$ and $X^\times \subseteq \varepsilon(Y^\times)$ $\qquad$ (since $[\epsilon, \varepsilon]$ and $[\delta, \triangle]$ are adjunctions on $\mathcal{H}$)
$X \subseteq [\epsilon, \varepsilon](Y)$

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Thus the pair $([\epsilon, \varepsilon], [\delta, \triangle])$ is an adjunction, which implies that $[\epsilon, \varepsilon]$ is an erosion and $[\delta, \triangle]$ is a dilation. $\qquad\square$

5 Filters

In mathematical morphology, a *filter* [6], [15] is an operator α acting on a lattice $\mathcal{L}$, which is increasing (*i.e.*$\forall X, Y \in \mathcal{L}, X \leq Y \implies \alpha(X) \leq \alpha(Y)$) and idempotent (*i.e.*$\forall X \in \mathcal{L}, \alpha(\alpha(X)) = \alpha(X)$). A filter on $\mathcal{L}$ which is extensive (*i.e.*$\forall X \in \mathcal{L}, X \leq \alpha(X)$) is called a *closing* on $\mathcal{L}$ and a filter on $\mathcal{L}$ which is anti-extensive (*i.e.*$\forall X \in \mathcal{L}, \alpha(X) \leq X$) is called an *opening*. If (α, β) is an adjunction then α is an erosion, β is a dilation, $\beta \circ \alpha$ is an opening and $\alpha \circ \beta$ is a closing on $\mathcal{L}$.

Definition 5. (opening, closing).

1. We define γ_1 and ϕ_1, that act on $\mathcal{H}^\bullet$, by $\gamma_1 = \delta \circ \epsilon$ and $\phi_1 = \epsilon \circ \delta$.

2. We define Γ_1 and Φ_1, that act on $\mathcal{H}^\times$, by $\Gamma_1 = \triangle \circ \varepsilon$ and $\phi_1 = \varepsilon \circ \triangle$.

3. We define $[\gamma, \Gamma]_1$ and $[\phi, \Phi]_1$, that act on $\mathcal{H}$ by respectively $[\gamma, \Gamma]_1(X) = (\gamma_1(X^\bullet), \Gamma_1(X^\times))$ and $[\phi, \Phi]_1(X) = (\phi_1(X^\bullet), \Phi_1(X^\times))$ for any $X \in \mathcal{H}$.

Since (ϵ, δ) and $(\varepsilon, \triangle)$ are adjunctions, γ_1, Γ_1 are openings and ϕ_1, Φ_1 are closings on the respective lattices. Now we will prove that $[\gamma, \Gamma]_1$ and $[\phi, \Phi]_1$ are respectively an opening and a closing on $\mathcal{H}$.

Proposition 1. *The following statements are true.*

1. $[\gamma, \Gamma]_1 = [\delta, \triangle] \circ [\epsilon, \varepsilon]$

2. $[\phi, \Phi]_1 = [\epsilon, \varepsilon] \circ [\delta, \triangle]$

Proof. Let X be any hypergraph in $\mathcal{H}$. Then

$$
\begin{aligned}
[\gamma, \Gamma]_1(X) &= (\gamma_1(X^\bullet), \Gamma_1(X^\times)) \\
&= ((\delta \circ \epsilon)(X^\bullet), (\triangle \circ \varepsilon)(X^\times)) \\
&= (\delta(X^\bullet), \triangle(X^\times)) \circ (\epsilon(X^\bullet), \varepsilon(X^\times)) \\
&= [\delta, \triangle] \circ [\epsilon, \varepsilon](X)
\end{aligned}
$$

This proves 1. A similar line of arguments will prove 2. $\qquad\square$

$([\epsilon, \varepsilon], [\delta, \triangle])$ is an adjunction on $\mathcal{H}$ implies that $[\delta, \triangle] \circ [\epsilon, \varepsilon]$ is an opening and $[\epsilon, \varepsilon] \circ [\delta, \triangle]$ is a closing on $\mathcal{H}$.

Definition 6. (half-opening, half-closing).

1. We define $\gamma_{1/2}$ and $\phi_{1/2}$, that act on $\mathcal{H}^\bullet$, by $\gamma_{1/2} = \delta^\bullet \circ \epsilon^\times$ and $\phi_{1/2} = \epsilon^\bullet \circ \delta^\times$.

2. We define $\Gamma_{1/2}$ and $\Phi_{1/2}$, that act on $\mathcal{H}^\times$, by $\Gamma_{1/2} = \delta^\times \circ \epsilon^\bullet$ and $\phi_{1/2} = \epsilon^\times \circ \delta^\bullet$.

3. We define $[\gamma, \Gamma]_{1/2}$ and $[\phi, \Phi]_{1/2}$, that act on $\mathcal{H}$ by respectively $[\gamma, \Gamma]_{1/2}(X) = (\gamma_{1/2}(X^\bullet), \Gamma_{1/2}(X^\times))$ and $[\phi, \Phi]_{1/2}(X) = (\phi_{1/2}(X^\bullet), \Phi_{1/2}(X^\times))$ for any $X \in \mathcal{H}$.

Property 5. Let $X^\bullet \subseteq H^\bullet$ and $X^\times \subseteq H^\times$. The following properties are true.

1. $\gamma_{1/2}(X^\bullet) = \underset{i \in I, v(e_i) \subseteq X^\bullet}{\cup} v(e_i)$

2. $\phi_{1/2}(X^\bullet) = \{x \in H^\bullet | \forall e_i, i \in I \text{ with } x \in v(e_i) \text{ and } v(e_i) \cap X^\bullet \neq \phi\}$.

3. $\Gamma_{1/2}(X^\times) = \{e_i, i \in I | \exists \ x \in v(e_i) \text{ with } \{e_i \in H^\times \text{ with } x \in v(e_i)\} \subseteq X^\times\}$.

4. $\phi_{1/2}(X^\times) = \{e_i, i \in I | v(e_i) \subseteq \underset{j \in J}{\cup} v(e_j)\}$.

Proof.   1. $\gamma_{1/2}(X^\bullet) = \delta^\bullet \circ \epsilon^\times(X^\bullet)$

$$= \ \delta^\bullet \circ \{e_i, i \in I | v(e_i) \subseteq X^\bullet\} \text{ (By property of } \epsilon^\times)$$
$$= \ \underset{i \in I, v(e_i) \subseteq X^\bullet}{\cup} v(e_i) \text{ (By property of } \delta^\bullet)$$

2. $\phi_{1/2}(X^\bullet) = \epsilon^\bullet \circ \delta^\times(X^\bullet)$

$$= \ \epsilon^\bullet \circ \{e_i, i \in I | v(e_i) \cap X^\bullet \neq \phi\}$$
$$= \ \epsilon^\bullet \circ \{e_k, k \in K\}; \text{ where } K \subseteq I \text{ is some index set and } e_k \text{ is such that } v(e_k) \cap X^\bullet \neq \phi$$
$$\text{(By property of } \delta^\times(X^\bullet))$$
$$= \ \underset{k \notin K}{\cap} \overline{v(e_k)}$$

3. $\Gamma_{1/2}(X^\times) = \delta^\times \circ \epsilon^\bullet(X^\times)$

$$= \ \{e_i, i \in I | v(e_i) \cap \epsilon^\bullet(X^\times) \neq \phi\} \text{ (By property of } \delta^\times)$$
$$= \ \text{ set of all edges in } H^\times \text{ which do not belong to any edge of } \overline{X^\times}$$
$$= \ \{e_i, i \in I | \exists x \in v(e_i) \text{ with } \{e_i \in H^\times \text{ with } x \in v(e_i)\} \subseteq X^\times\}$$

4. $\phi_{1/2}(X^\times) = \epsilon^\times \circ \delta^\bullet(X^\times)$

$$= \ \epsilon^\times \circ \underset{j \in J}{\cup} v(e_j)$$
$$= \ \{e_i, i \in I | v(e_i) \subseteq \underset{j \in J}{\cup} v(e_j)\}$$

$\square$

Remark 2. The following statements are true about $\gamma_{1/2}$.

(m) $\gamma 1$

(n) $\phi 1$

(o) $\Gamma 1$

(p) $\varphi 1$

(q) $[\gamma, \Gamma]1$

(r) $[\phi, \varphi]1$

(s) $\gamma 1/2$

(t) $\phi 1/2$

(u) $\Gamma 1/2$

(v) $\varphi 1/2$

(w) $[\gamma, \Gamma]1/2$
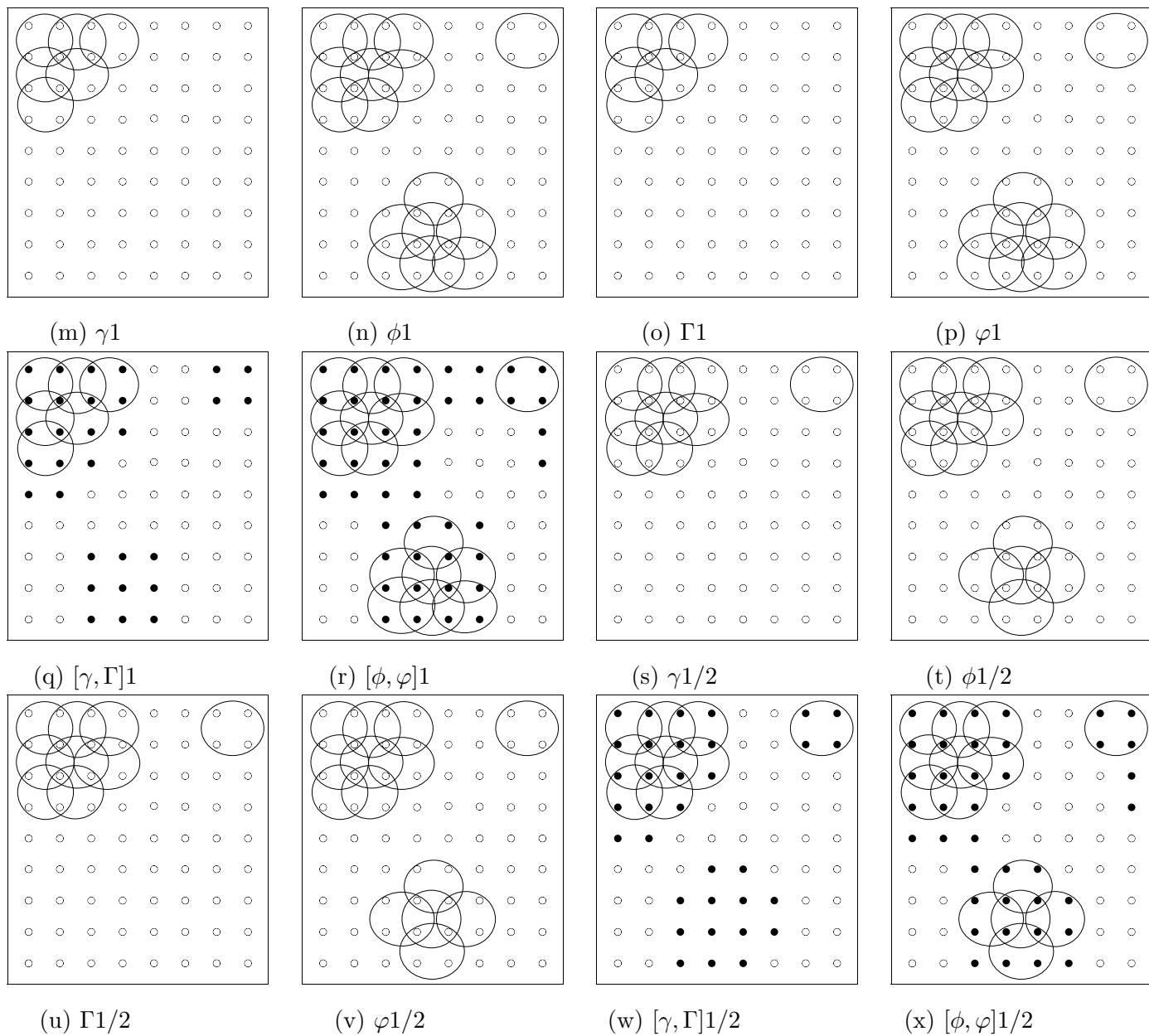
(x) $[\phi, \varphi]1/2$

Figure 3: Illustration of openings and closings

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1. $\gamma_{1/2}(X^\bullet) = \{x \in X^\bullet | \exists e_i, i \in I \text{ with } x \in v(e_i) \text{ and } v(e_i) \subseteq X^\bullet\}$.

2. $\gamma_{1/2}(X^\bullet) = X^\bullet \setminus \{x \in X^\bullet | \forall e_i, i \in I \text{ with } x \in v(e_i) \text{ and } v(e_i) \nsubseteq X^\bullet\}$.

Now we will enumerate some properties of these operators on the respective lattices based on the partial order relations defined on them.

Property 6. Let $X^\bullet \subseteq H^\bullet$ and $X^\times \subseteq H^\times$. The following properties hold true.

1. $\gamma_1(X^\bullet) \subseteq \gamma_{1/2}(X^\bullet) \subseteq X^\bullet \subseteq \phi_{1/2}(X^\bullet) \subseteq \phi_1(X^\bullet)$.

2. $\Gamma_1(X^\times) \subseteq \Gamma_{1/2}(X^\times) \subseteq X^\times \subseteq \Phi_{1/2}(X^\times) \subseteq \Phi_1(X^\times)$.

3. $[\gamma, \Gamma]_1(X) \subseteq [\gamma, \Gamma]_{1/2}(X) \subseteq X \subseteq [\phi, \Phi]_{1/2}(X) \subseteq [\phi, \Phi]_1(X)$.

Proof. 1.

$$
\begin{aligned}
\gamma_1(X^\bullet) &= \delta \circ \epsilon(X^\bullet) \\
&= \{x \in H^\bullet | \exists e_i, i \in I \text{ such that } x \in v(e_i) \text{ and } v(e_i) \cap \epsilon(X^\bullet) \neq \phi\} \text{ (By property of } \delta \text{)} \\
&\subseteq \{x \in H^\bullet | \exists e_i, i \in I \text{ such that } x \in v(e_i) \text{ and } v(e_i) \subseteq X^\bullet\} \\
&= \bigcup_{v(e_i) \subseteq X^\bullet} v(e_i) \\
&= \gamma_{1/2}(X^\bullet).
\end{aligned}
$$

Now $\gamma_{1/2}(X^\bullet) = \bigcup_{v(e_i) \subseteq X^\bullet} v(e_i) \subseteq X^\bullet$. Also $X^\bullet \subseteq \phi_{1/2}(X^\bullet)$, because $\phi_{1/2}$ is a closing on $H^\bullet$.

$$
\begin{aligned}
\phi_{1/2}(X^\bullet) &= \epsilon^\bullet \circ \delta^\times(X^\bullet) \\
&= \epsilon^\bullet \circ I \circ \delta^\times(X^\bullet), \text{ (where } I \text{ is the identity on } H^\times) \\
&\subseteq \epsilon^\bullet \circ (\epsilon^\times \circ \delta^\bullet) \circ \delta^\times(X^\bullet) \ (\because I(X^\times) = X^\times \subseteq \epsilon^\times \circ \delta^\bullet(X^\times) \text{ and } \epsilon^\times \circ \delta^\bullet \text{ is a closing}) \\
&= (\epsilon^\bullet \circ \epsilon^\times) \circ (\delta^\bullet \circ \delta^\times)(X^\bullet) \\
&= \epsilon \circ \delta(X^\bullet) \\
&= \phi_1(X^\bullet).
\end{aligned}
$$

This proves 1.

Properties 2 and 3 can be proved in a similar manner. $\square$

Property 7. [hypergraph opening, hypergraph closing]

1. The operators $\gamma_{1/2}$ and γ_1 (resp. $\Gamma_{1/2}$ and Γ_1) are openings on $\mathcal{H}^\bullet$ (resp. $\mathcal{H}^\times$) and $\phi_{1/2}$ and ϕ_1 (resp. $\Phi_{1/2}$ and Φ_1) are closings on $\mathcal{H}^\bullet$ (resp. $\mathcal{H}^\times$).

2. The family $\mathcal{H}$ is closed under $[\gamma, \Gamma]_{1/2}$, $[\phi, \Phi]_{1/2}$, $[\gamma, \Gamma]_1$ and $[\phi, \Phi]_1$.

3. $[\gamma, \Gamma]_{1/2}$ and $[\gamma, \Gamma]_1$ are openings on $\mathcal{H}$ and $[\phi, \Phi]_{1/2}$ and $[\phi, \Phi]_1$ are closings on $\mathcal{H}$.

Proof.   1. If (α, β) is an adjunction on a lattice $\mathcal{L}$ then $\beta \circ \alpha$ is an opening and $\alpha \circ \beta$ is a closing on $\mathcal{L}$. The result follows from the fact that $(\epsilon^\bullet, \delta^\times)$, $(\epsilon^\times, \delta^\bullet)$, (ϵ, δ) and (ε, Δ) are adjunctions.

2. We will prove that $[\gamma, \Gamma]_{1/2}(X) \in \mathcal{H}$, whenever $X \in \mathcal{H}$. Since $[\gamma, \Gamma]_{1/2}(X) = (\gamma_{1/2}(X^\bullet), \Gamma_{1/2}(X^\times))$, it is enough to show that if $e \in \Gamma_{1/2}(X^\times)$, then $v(e) \subseteq \gamma_{1/2}(X^\bullet)$.

$$
\begin{aligned}
e \in \Gamma_{1/2}(X^\times) &\Rightarrow e \in \delta^\times \circ \epsilon^\bullet(X^\times) \\
&\Rightarrow e \in X^\times \text{ (since } \delta^\times \circ \epsilon^\bullet \text{ is an opening } \delta^\times \circ \epsilon^\bullet(X^\times) \subseteq X^\times) \\
&\Rightarrow e \in \epsilon^\times(X^\bullet) \text{ (since } X^\times \subseteq \epsilon^\times(X^\bullet))
\end{aligned}
$$

Since $e_j \in X^\times \Rightarrow v(e_j) \subseteq \delta^\bullet(X^\times)$, we have $v(e) \subseteq \delta^\bullet \circ \epsilon^\times(X^\bullet)$. Thus $v(e) \subseteq \gamma_{1/2}(X^\bullet)$. Hence if X is a hypergraph, then $[\gamma, \Gamma]_{1/2}(X)$ is a hypergraph and so $\mathcal{H}$ is closed under $[\gamma, \Gamma]_{1/2}$. $\mathcal{H}$ is closed under $[\gamma, \Gamma]_1$ and $[\phi, \Phi]_1$ due to proposition 1.

Now we will prove that $\mathcal{H}$ is closed under $[\phi, \Phi]_{1/2}$. It is enough to show that if $X \in \mathcal{H}$, then $[\phi, \Phi]_{1/2}(X) \in \mathcal{H}$. We know that $[\phi, \Phi]_{1/2}(X) = (\phi_{1/2}(X^\bullet), \Phi_{1/2}(X^\times))$. If $e \in \Phi_{1/2}(X^\times)$, we need to prove that $v(e) \subseteq \phi_{1/2}(X^\bullet)$.

$$
\begin{aligned}
e \in \Phi_{1/2}(X^\times) &\Rightarrow e \in \epsilon^\times \circ \delta^\bullet(X^\times) \\
&\Rightarrow e \in \{e_i, i \in I | v(e_i) \subseteq \delta^\bullet(X^\times)\} \\
&\Rightarrow v(e) \subseteq \delta^\bullet(X^\times)
\end{aligned}
$$

But $\delta^\bullet(X^\times) = \bigcup_{e \in X^\times} v(e) \subseteq X^\bullet$. Since $\epsilon^\bullet \circ \delta^\times$ is a closing, $X^\bullet \subseteq \epsilon^\bullet \circ \delta^\times(X^\bullet)$. Therefore $\delta^\bullet(X^\times) \subseteq \epsilon^\bullet \circ \delta^\times(X^\bullet)$. Thus

$$
\begin{aligned}
e \in \Phi_{1/2}(X^\times) &\Rightarrow v(e) \subseteq \epsilon^\bullet \circ \delta^\times(X^\bullet) \\
&\Rightarrow v(e) \subseteq \phi_{1/2}(X^\bullet)
\end{aligned}
$$

Thus $[\phi, \Phi]_{1/2}(X) \in \mathcal{H}$. Hence $\mathcal{H}$ is closed under $[\phi, \Phi]_{1/2}$.

3. In order to prove that $[\gamma, \Gamma]_{1/2}$ is an opening, it is enough to show that $[\gamma, \Gamma]_{1/2}$ is increasing and idempotent on $\mathcal{H}$. Let $X, Y \in \mathcal{H}$ be such that $X \subseteq Y \subseteq H$. Thus $X^\bullet \subseteq Y^\bullet \subseteq H^\bullet$ and $X^\times \subseteq Y^\times \subseteq H^\times$. We have $[\gamma, \Gamma]_{1/2}(X) = (\gamma_{1/2}(X^\bullet), \Gamma_{1/2}(X^\times))$. Therefore

$$
\begin{aligned}
[\gamma, \Gamma]_{1/2} \circ [\gamma, \Gamma]_{1/2}(X) &= [\gamma, \Gamma]_{1/2} \circ (\gamma_{1/2}(X^\bullet), \Gamma_{1/2}(X^\times)) \\
&= (\gamma_{1/2} \circ \gamma_{1/2}(X^\bullet), \Gamma_{1/2} \circ \Gamma_{1/2}(X^\times)) \\
&= (\gamma_{1/2}(X^\bullet), \Gamma_{1/2}(X^\times)) \text{ (since } \gamma_{1/2} \text{ and } \Gamma_{1/2} \text{ are openings)} \\
&= [\gamma, \Gamma]_{1/2}(X)
\end{aligned}
$$

Therefore $[\gamma, \Gamma]_{1/2}$ is idempotent.
Since $\gamma_{1/2}$ and $\Gamma_{1/2}$ are openings, $\gamma_{1/2}(X^\bullet) \subseteq \gamma_{1/2}(Y^\bullet)$ and $\Gamma_{1/2}(X^\times) \subseteq \Gamma_{1/2}(Y^\times)$. Thus $[\gamma, \Gamma]_{1/2}(X) \subseteq [\gamma, \Gamma]_{1/2}(Y)$ so that $[\gamma, \Gamma]_{1/2}$ is increasing on $\mathcal{H}$. Hence $[\gamma, \Gamma]_{1/2}$ is an opening. Similarly, we can prove that $[\gamma, \Gamma]_1$ is an opening, and that $[\phi, \Phi]_{1/2}$ and $[\phi, \Phi]_1$ are closings.

$\square$

5.1 Granulometries

Granulometries [6], [15] deal with families of openings and closings that are parametrized by a positive number. A family $\{\gamma_\lambda\}$ of mappings from a lattice $\mathcal{L} \to \mathcal{L}$, depending on a positive parameter λ is a granulometry when

(i) γ_λ is an opening $\forall \lambda \geq 0$

(ii) $\lambda \geq \mu \geq 0 \Rightarrow \gamma_\lambda \geq \gamma_\mu$

These conditions are called Matheron's axioms for granulometry.

Definition 7. Let $\lambda \in N$. We define $[\gamma, \Gamma]_{\lambda/2}$ (resp. $[\phi, \Phi]_{\lambda/2}$) as follows. $[\gamma, \Gamma]_{\lambda/2} = [\delta, \Delta]^i \circ ([\gamma, \Gamma]_{1/2})^j \circ [\epsilon, \varepsilon]^i$, where i and j are respectively the quotient and remainder when λ is divided by 2.

Theorem 3. *The families $\{[\gamma, \Gamma]_{\lambda/2} | \lambda \in N\}$ and $\{[\phi, \Phi]_{\lambda/2} | \lambda \in N\}$ are granulometries.*

Proof. We know that if (α, β) is an adjunction, then $(\alpha \circ \alpha, \beta \circ \beta)$ is an adjunction and so (α^i, β^i) is also an adjunction for every $i \in N$ [13]. Now $([\epsilon, \varepsilon]^i, [\delta, \Delta]^i)$ is an adjunction for every $i \in N$, since $([\epsilon, \varepsilon], [\delta, \Delta])$ is an adjunction on $\mathcal{H}$. This implies $[\delta, \Delta]^i \circ [\epsilon, \varepsilon]^i$ is an opening. But $[\delta, \Delta]^i \circ [\epsilon, \varepsilon]^i = [\gamma, \Gamma]_{2i/2}$. This implies $[\gamma, \Gamma]_{\lambda/2}$ is an opening if λ is even. If λ is odd (*i.e.* if $\lambda = (2i + 1)/2$) then $[\gamma, \Gamma]_{\lambda/2} = [\gamma, \Gamma]_{(2i+1)/2} = [\delta, \Delta]^i \circ [\gamma, \Gamma]_{1/2} \circ [\epsilon, \varepsilon]^i$. If (α, β) is an adjunction and γ is an opening on a lattice $\mathcal{L}$, then $\beta \circ \gamma \circ \alpha$ is an opening [12]. Since $([\epsilon, \varepsilon]^i, [\delta, \Delta]^i)$ is an adjunction and $[\gamma, \Gamma]_{1/2}$ is an opening, we have $[\delta, \Delta]^i \circ [\gamma, \Gamma]_{1/2} \circ [\epsilon, \varepsilon]^i$ is an opening.

Now we need to prove that $[\gamma, \Gamma]_{\mu/2}(X) \subseteq [\gamma, \Gamma]_{\lambda/2}(X)$, for $\lambda \leq \mu, \lambda, \mu \in N$ and $X \in \mathcal{H}$. We have $[\gamma, \Gamma]_1(X) \subseteq [\gamma, \Gamma]_{1/2}(X) \subseteq X$, for every $X \in \mathcal{H}$. (By property 6). Replacing X with $[\epsilon, \varepsilon]^i(X)$, we have $[\gamma, \Gamma]_1 \circ [\epsilon, \varepsilon]^i(X) \subseteq [\gamma, \Gamma]_{1/2} \circ [\epsilon, \varepsilon]^i(X) \subseteq [\epsilon, \varepsilon]^i(X)$. Now $[\delta, \Delta]^i$ is a dilation, because $([\epsilon, \varepsilon]^i, [\delta, \Delta]^i)$ is an adjunction. This implies $[\delta, \Delta]^i \circ [\gamma, \Gamma]_1 \circ [\epsilon, \varepsilon]^i(X) \subseteq [\delta, \Delta]^i \circ [\gamma, \Gamma]_{1/2} \circ [\epsilon, \varepsilon]^i(X) \subseteq [\delta, \Delta]^i \circ [\epsilon, \varepsilon]^i(X)$. That is $[\delta, \Delta]^{i+1} \circ [\epsilon, \varepsilon]^{i+1}(X) \subseteq [\delta, \Delta]^i \circ [\gamma, \Gamma]_{1/2} \circ [\epsilon, \varepsilon]^i(X) \subseteq [\delta, \Delta]^i \circ [\epsilon, \varepsilon]^i(X)$. Hence $[\gamma, \Gamma]_{(2i+2)/2}(X) \subseteq [\gamma, \Gamma]_{(2i+1)/2}(X) \subseteq [\gamma, \Gamma]_{2i/2}(X)$ for every $i \in N$. This implies $[\gamma, \Gamma]_{\mu/2}(X) \subseteq [\gamma, \Gamma]_{\lambda/2}(X)$ for every $\mu \geq \lambda, \mu, \lambda \in N$. Therefore the family $\{[\gamma, \Gamma]_{\lambda/2} | \lambda \in N\}$ is a granulometry. Similar line of arguments can be used to prove that $\{[\phi, \Phi]_{\lambda/2} | \lambda \in N\}$ is a granulometry. $\square$

Definition 8. Let $\lambda \in N$ and $X \in \mathcal{H}$. We define the operator $ASF_{\lambda/2}$ by

$$ASF_{\lambda/2}(X) = \begin{cases} X & \text{if } \lambda = 0 \\ [\gamma, \Gamma]_{\lambda/2} \circ [\phi, \Phi]_{\lambda/2} \circ ASF_{(\lambda-1)/2}(X) & \text{if } \lambda \neq 0 \end{cases}$$

Corollary 1. *The family $\{ASF_{\lambda/2} | \lambda \in N\}$ is a family of alternate sequential filters.*

6 Conclusion

This paper investigates the lattice of all subhypergraphs of a hypergraph H and provides it with morphological operators. We created new dilations, erosions, openings, closings, granulometries and

alternate sequential filters whose input and output are both hypergraphs. The proposed framework is then extended from subgraphs to functions that weight the vertices and hyperedges of a hypergraph. The proposed framework of hypergraphs presented in this paper encompasses mathematical morphology on graphs, simplicial complexes and morphology based on discrete spatially variant structuring elements.

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