SIMULTANEOUS EMBEDDINGS OF GRAPHS AS MEDIAN AND ANTIMEDIAN SUBGRAPHS

K. BALAKRISHNAN, B. BREŠAR, M. CHANGAT, S. KLAVŽAR, M. KOVŠE, AND A. R. SUBHAMATHI

ABSTRACT. The distance $D_G(v)$ of a vertex v in an undirected graph G is the sum of the distances between v and all other vertices of G. The set of vertices in G with maximum (minimum) distance is the antimedian (median) set of a graph G. It is proved that for arbitrary graphs G and J and a positive integer $r \geq 2$, there exists a connected graph H such that G is the antimedian and J the median subgraphs of H, respectively, and that $d_H(G, J) = r$. When both G and J are connected, G and J can in addition be made convex subgraphs of H.

Keywords: facility location problems; median sets; antimedian sets; convex subgraphs

1. Introduction

Location theory has grown into a vast area of research; see the collected references at [7] and the survey paper on obnoxious facility location problems [4]. Here we just mention algorithmic studies [2, 14], applications of location theory in biological networks [18], and studies of Steiner centers and Steiner medians [13].

In the design of a network one often needs to take care of both desired facilities and undesired facilities [5, 10, 17]. A natural model to distribute such facilities in a network is to put them at medians and antimedians of the network, respectively. For the mutual placement of desired and undesired facilities we can also have certain requirements that can be modeled by respective graphs. For instance, imagine that in a certain region there is a

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real scarcity for land, energy, etc. Suppose that the local government decided to construct a certain number of (nuclear) power plants and has identified some vacant lands. Then the problem is to locate the waste disposal sites at "antimedians" and the township of the plants at "medians". Moreover, in the network we wish to have desired and undesired facilities separated by a prescribed distance. We hence pose the following question: Given arbitrary graphs J and G and a positive integer r, does there exist a connected graph H such that J is the median of H, that G is the antimedian of H, and that the distance between G and J is r?

Let us now formalize the above model. The *distance* $d_G(v,x)$ between vertices v and x in a connected graph G is the length of a shortest v,x-path in G, that is, its number of edges. For a vertex v of G, the sum

$$D_G(v) = \sum_{x\in V(G)} d_G(v,x)$$

is called the *distance* of v. The distance between the subgraphs H_1 and H_2 of a connected graph G is

$$d_G(H_1,H_2) = \min_{\substack{u\in V(H_1)\\ v\in V(H_2)}} d_G(u,v)\,.$$

The vertex v is called a *median vertex* of G if $D_G(v)$ is minimum. The *median set* $M(G)$ of G is the set of all median vertices of G. The subgraph induced by $M(G)$ is called the *median subgraph* of G. If in these definitions we change minimum to maximum, we obtain *antimedian vertices*, the *antimedian set* $AM(G)$, and the *antimedian subgraph*. There are many graphs in which the median and the antimedian subgraphs lie far apart. For example, in trees the median always consists of either a single vertex or two adjacent vertices and lies in "the middle" of the graph, whereas the antimedian will occur at peripheral vertices. In general the structure of the median and antimedian subgraphs can be arbitrary. In a certain way we establish this arbitrariness with the following affirmative answer to the above question.

Theorem 1. *For any graphs G and J and any integer $r\geq 2$, there exists a connected graph H such that $AM(H)=V(G)$, $M(H)=V(J)$, and $d_H(G,J)=r$.*

This theorem relates and extends several previous results. The first such theorem is due to Slater [15] who proved that for every graph G there exists a connected graph H such that G is a subgraph of H with $M(H) = V(G)$. Recently, Dankelmann and Sabidussi [6] extended Slater's result by obtaining the median as an isometric subgraph of the host graph. Bielak and Sysło [3] followed with an analogue of Slater's theorem for the antimedian case. (For related studies see also [12].) Hence Theorem 1 can be viewed as a unification of their theorems.

We also mention the following related works. For a given graph G, Miller [11] and Hendry [8, 9] studied the minimum order of a graph H such that $M(H) = G$. A result parallel to our theorem is due to Smart and Slater [16]. They proved that the center, the median, and the so-called centroid can be arbitrarily far apart in a connected graph in the sense that given any three graphs H, J, K and a positive integer $k \geq 4$, there exists a connected graph G with the center, the median, and the centroid subgraphs isomorphic to H, J, and K, respectively and the distance between any two of these subgraphs is at least k.

Theorem 1 is proved in the next section. Then, in Section 3, we consider the special case when G and/or J are connected and show that in these cases the corresponding constructions lead to convex embeddings of G and/or J into H. In the concluding remarks the order of the constructed host graph is discussed.

2. Proof of Theorem 1

Let H be a subgraph of a graph G. Then for a vertex $v \in V(G)$ we will write

$$D_G(v, H) = \sum_{x \in V(H)} d_G(v, x)\,.$$

Note that the previously introduced notation $D_G(v)$ is an abbreviation for $D_G(v, G)$.

In the first part of this section we construct the graph H. Let $V(G) = \{u_1, \ldots, u_{n_1}\}$ and $V(J) = \{v_1, \ldots, v_{n_2}\}$. Let c_1 and c_2 be non-negative integers such that $c_1 + c_2 = r - 2$ and let K, K' and K'' be copies of the complete graph K_N, where the value of N will be determined later.

We first construct the graph H_0 as follows. Start with the disjoint union of graphs G, J, K, K', and K''. Consider disjoint paths with end vertices w_i of length c_1 from each vertex u_i in G and join each w_i to a unique vertex of K, say x. In this way, there are n_1 internally disjoint paths of length c_1+1 between each vertex of G and the vertex x in K. Note that it is possible that $w_i = u_i$ for all i. Similarly connect each vertex v_i of J by disjoint paths of length c_2 with end vertices x_i and join each vertex x_i to the vertex x in K. Note again that it is possible that $v_i = x_i$ for all i. Similar construction of paths is effected from J to K' and K'' of length c_2+1 and let the corresponding end vertices of the paths in K' and K'' respectively be x' and x''. The construction of H_0 is schematically shown in Figure 1.

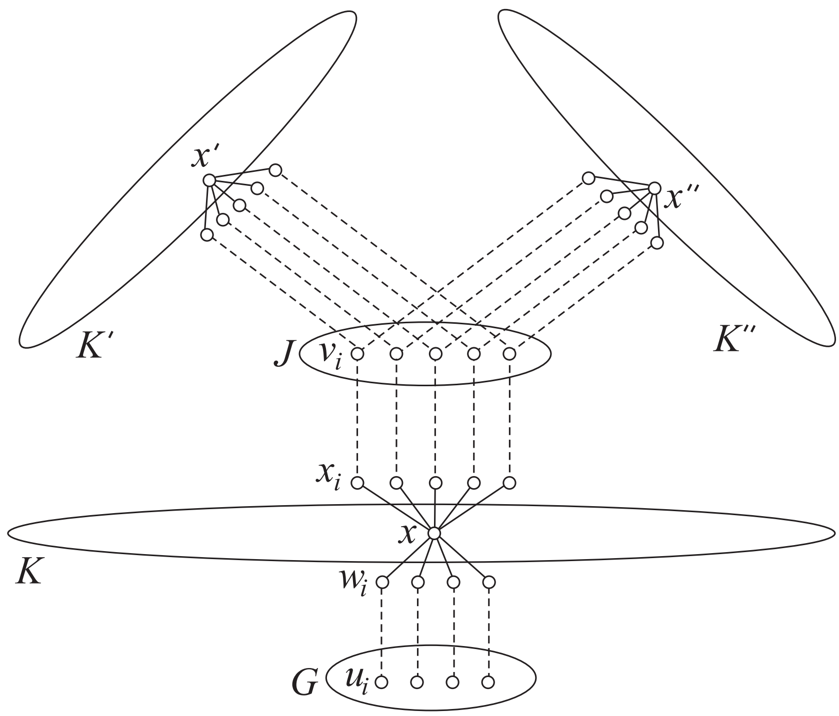

FIGURE 1. Graph H_0 from the construction.

Setting

$$
\begin{aligned}
\epsilon(G) &= \min\{D_{H_0}(u) \mid u \in V(G)\}\,,\\
\epsilon(J) &= \min\{D_{H_0}(u) \mid u \in V(J)\}\,,\\
a_i &= D_{H_0}(u_i) - \epsilon(G),\ 1 \le i \le n_1\,,\\
b_i &= D_{H_0}(v_i) - \epsilon(J),\ 1 \le i \le n_2\,,
\end{aligned}
$$

the graph H is constructed from H_0 by connecting w_i, $1 \le i \le n_1$, to a_i additional (not necessarily distinct) neighbors in K, and by connecting x_i, $1 \le i \le n_2$, to b_i additional (not necessarily distinct) neighbors in K. Hence, w_i and x_i are in H adjacent to $a_i + 1$ and $b_i + 1$ neighbors of K, respectively.

We next claim that all the vertices of G have the same distance in H and that also all the vertices of J have the same distance in H. Note that $D_H(u_i)$ and $D_H(v_i)$ are reduced by a_i and b_i compared to $D_{H_0}(u_i)$ and $D_{H_0}(v_i)$, respectively, because w_i and x_i have a_i+1 and $b_i + 1$ neighbors of K in H and a single neighbor in H_0. Therefore

$$
\begin{aligned}
D_H(u_i) &= D_{H_0}(u_i) - a_i\\
&= D_{H_0}(u_i) - (D_{H_0}(u_i) - \epsilon(G))\\
&= \epsilon(G)
\end{aligned}
$$

holds for any vertex u_i in G, $i = 1, \dots, n_1$. Similarly $D_H(v_i) = \epsilon(J)$ holds for any vertex v_i in J, $i = 1, \dots, n_2$, and the claim is proved.

Let U, W, W', W'' be the sets of all vertices of H that lie in the interior of the paths between G and K, between J and K, between J and K', and between J and K'', respectively. Note that by the construction itself it follows that the subsets $V(J)$, $V(G)$, U, $V(K)$, $V(K')$, $V(K'')$, W, W', and W'' are pairwise disjoint and hence determine a partition of $V(H)$. Note also that $|U| = c_1 n_1$ and $|W| = |W'| = |W''| = c_2 n_2$. Then for any vertex v,

$$
\begin{aligned}
(1) \qquad D_H(v) &= D_H(v, K') + D_H(v, K'') + D_H(v, W') + D_H(v, W'') +\\
&\quad\ D_H(v, J) + D_H(v, W) + D_H(v, K) + D_H(v, U) + D_H(v, G)\,.
\end{aligned}
$$

We next need to make N large enough in order to have enough neighbors in K for the vertices w_i and x_i. Note that, $D_{H_0}(u_i, K')$, $D_{H_0}(u_i, K'')$, $D_{H_0}(u_i, W')$, $D_{H_0}(u_i, W'')$, $D_{H_0}(u_i, J)$, $D_{H_0}(u_i, W)$, and $D_{H_0}(u_i, K)$ are constant for any vertex $u_i \in G$ and that $\max(d_H(u_i, x) - d_H(u_j, x)) \leq 2c_1 + 2$ for any x in $G \cup U$. Therefore, $a_i \leq (2c_1+2)(c_1+1)n_1$, since there are $(c_1 + 1)n_1$ vertices in $G \cup U$. Similarly we get $b_i \leq (2c_2 + 2)(3c_2 + 1)n_2$. Thus N must satisfy the inequality

$$N > \max\{(2c_1 + 2)(c_1 + 1)n_1, (2c_2 + 2)(3c_2 + 1)n_2\}\,. \tag{2}$$

Lemma 2. *For any vertex $x \in V(H) \setminus V(J)$,*

$$D_H(x) > D_H(v_i), \quad \text{for all } v_i \in V(J)\,.$$

Proof. We distinguish two cases.

Case 1: $x \in W \cup V(K) \cup U \cup V(G)$.
Assume first that x is a vertex of W that is adjacent to a vertex v_j in J. Then $d(x, .) = d(v_j, .) + 1$ for the vertices from K', K'', W', and W'', hence

$$D_H(x, K') = D_H(v_j, K') + N, \quad D_H(x, K'') = D_H(v_j, K'') + N\,, \tag{3}$$

and

$$D_H(x, W') = D_H(v_j, W') + c_2n_2, \quad D_H(x, W'') = D_H(v_j, W'') + c_2n_2\,. \tag{4}$$

On the other hand, for the vertices in K, U, and G, $d(x, .) = d(v_j, .) - 1$, hence

$$D_H(x, K) = D_H(v_j, K) - N, \quad D_H(x, U) = D_H(v_j, U) - c_1n_1\,, \tag{5}$$

and

$$D_H(x, G) = D_H(v_j, G) - n_1\,. \tag{6}$$

Finally, if a shortest path from v_j to J or W contains vertices of K, then $d(x, .) = d(v_j, .) - 1$, otherwise $d(v_j, .) \leq d(x, .) \leq d(v_j, .) + 1$. Hence for the vertices from W and J we have $d(x, .) \geq d(v_j, .) - 1$ and consequently

$$D_H(x, W) \geq D_H(v_j, W) - c_2n_2, \quad D_H(x, J) \geq D_H(v_j, J) - n_2\,. \tag{7}$$

Inserting (3)-(7) into (1) we get

$$D_H(x) - D_H(v_j) \geq N + (c_2 - 1)n_2 - (c_1 + 1)n_1 > 0\,,$$

where the last inequality clearly holds because we have assumed that $N > \max((2c_1 + 2)(c_1 + 1)n_1, (2c_2 + 2)(3c_2 + 1)n_2)$.

From the construction of H it follows that the distance increases when $d_H(x, J) > 1$ and $x \in W \cup V(K) \cup U \cup V(G)$.

Case 2: $x \in W' \cup W'' \cup V(K') \cup V(K'')$.

By symmetry it suffices to consider only vertices in $W' \cup V(K')$. Moreover, by the construction of H, it suffices to show that $D_H(x) - D_H(v_i) > 0$, where x is a vertex of W' that is adjacent to the vertex v_i of J. Similar to the first case we infer the following relations:

$$\begin{aligned}
D_H(x, K) &= D_H(v_i, K) + N, & D_H(x, K'') &= D_H(v_i, K'') + N,\\
D_H(x, U) &= D_H(v_i, U) + c_1 n_1, & D_H(x, G) &= D_H(v_i, G) + n_1,\\
D_H(x, J) &\geq D_H(v_i, J) - n_2, & D_H(x, K') &= D_H(v_i, K') - N,\\
D_H(x, W) &\geq D_H(v_i, W) + c_2 n_2, & D_H(x, W'') &= D_H(v_i, W'') + c_2 n_2,\\
D_H(x, W') &\geq D_H(v_i, W') - c_2 n_2\,.
\end{aligned}$$

Therefore, inserting these relations into (1) we get

$$D_H(x) - D_H(v_i) \geq N + (c_1 + 1)n_1 + (c_2 - 1)n_2 > 0\,,$$

where we again used the assumption $N > \max((2c_1+2)(c_1+1)n_1, (2c_2+2)(3c_2+1)n_2)$. $\square$

By Lemma 2 and the fact that $D_H(v_i) = \epsilon(J)$ holds for any $v_i \in V(J)$, we conclude that $M(H) = V(J)$.

Lemma 3. *Let* $N > (2c_1 + 2)(c_1 + 1)n_1 + 2c_2 n_1 + 2(c_2 + 2)(3c_2 + 1)n_2$. *Then for any vertex* $x \in V(H) \setminus V(G)$, $D_H(x) < D_H(u_i)$, for all $u_i \in V(G)$.

Proof. From the proof of Lemma 2 we know that the distance of a vertex increases when we move from a vertex in J to a vertex in K', K'' or G. Hence we only need to compare the distance of vertices in G with the vertices of K' and K'' and by symmetry we can reduce the comparison to the vertices of K'. Now, for any vertex u_i in G,

$$\begin{aligned}
D_H(u_i, K') &= (c_1 + 2c_2 + 4)N - 1, \\
D_H(u_i, K'') &= (c_1 + 2c_2 + 4)N - 1, \\
D_H(u_i, W') &= (c_2(c_2 + 1)/2)n_2 + (c_1 + c_2 + 2)c_2n_2, \\
D_H(u_i, W'') &= (c_2(c_2 + 1)/2)n_2 + (c_1 + c_2 + 2)c_2n_2, \\
D_H(u_i, J) &= (c_1 + c_2 + 2)n_2, \\
D_H(u_i, W) &= (c_2(c_2 + 1)/2)n_2 + (c_1 + 1)c_2n_2, \\
D_H(u_i, K) &\geq (c_1 + 1)N, \\
D_H(u_i, U) &\geq (c_1(c_1 + 1)/2)n_1, \\
D_H(u_i, G) &\geq n_1 - 1.
\end{aligned}$$

and for any vertex x in K',

$$\begin{aligned}
D_H(x, K') &= N - 1, \\
D_H(x, K'') &\leq (2c_2 + 4)N - 1, \\
D_H(x, W') &\leq (c_2(c_2 + 1)/2)n_2 + c_2n_2, \\
D_H(x, W'') &\leq (c_2(c_2 + 1)/2)n_2 + (c_2 + 2)c_2n_2, \\
D_H(x, J) &\leq (c_2 + 2)n_2, \\
D_H(x, W) &\leq (c_2(c_2 + 1)/2)n_2 + (c_2 + 2)c_2n_2, \\
D_H(x, K) &\leq (2c_2 + 4)N - 1, \\
D_H(x, U) &\leq (c_1(c_1 + 1)/2)n_1 + (2c_2 + 4)c_1n_1, \\
D_H(x, G) &\leq (2c_2 + 4 + c_1)n_1.
\end{aligned}$$

Inserting the above relations into (1) we get

$$\begin{aligned}
D_H(u_i) - D_H(x) \geq\ & (3c_1)N + 3c_1c_2n_2 + c_1n_2 + n_1 \\
& -((2c_2 + 5)c_1n_1 + (2c_2 + 4)n_1).
\end{aligned}$$

Since $N > (2c_1+2)(c_1+1)n_1 + 2c_2n_1 + 2(c_2+2)(3c_2+1)n_2$ we conclude that $D_H(u_i) - D_H(x) > 0$. $\square$

By Lemma 3 and the fact that $D_H(u_i) = \epsilon(G)$ for any $u_i \in V(G)$, we conclude that $AM(H) = V(G)$.

By the assumption on N from Lemma 3 and from (2) we conclude that any

$$N \geq (2c_1+2)(c_1+1)n_1 + 2c_2n_1 + 2(c_2+2)(3c_2+1)n_2$$

will do the job.

Finally, observe that $d_H(G, J) = c_1 + c_2 + 2 = r$. This completes the proof of the theorem.

3. Convex embeddings

Recall that a subgraph G of a graph H is *convex* if for any vertices u and v of G, every shortest u, v-path from H lies completely in G. (Recall also that a convex subgraph is isometric but the converse need not hold.)

If the graphs G and J from Theorem 1 are not connected then they clearly cannot be embedded as convex subgraphs. However, for connected G and J we have:

Theorem 4. *Let G and J be connected graphs with diameters d_1 and d_2, respectively. Then for any $r \geq \lfloor d_1/2 \rfloor + \lfloor d_2/2 \rfloor + 2$ there exists a connected graph H such that G and J are convex subgraphs of H, $AM(H) = V(G)$, $M(H) = V(J)$, and $d_H(G, J) = r$.*

Proof. In the construction of the graph H from the proof of Theorem 1, set $c_1 \geq \lfloor d_1/2 \rfloor$ and $c_2 \geq \lfloor d_2/2 \rfloor$. Note that then any path P in H between two vertices of G which contains a vertex not from G, is of length at least $2(c_1+1) > d_1$. Therefore, G is a convex subgraph of H. Similarly, any path in H between two vertices of J which contains a vertex not from J, is of length at least $2(c_2+1) > d_2$, so J is also a convex subgraph of H. $\square$

If only one of the graphs G and J is connected, then by an analogous approach as in Theorem 4 we can assure that the connected graph is embedded as a convex subgraph (and that all the remaining conclusions hold).

Suppose next that we want to embed only one graph, say G (so that J can be considered as the empty graph). In this case, consider the subgraph of H induced by the vertex set $V(G) \cup U \cup V(K)$. Then by a similar approach as in the general case (for instance, set $c_1 = r-2$ and $N > (2c_1+2)(c_1+1)n_1$), we obtain that G is the antimedian of H. Moreover, if $r \geq \lfloor d_1/2 \rfloor + 2$, then G is a convex subgraph of H. This construction is the construction of [1]. Hence Theorem 1 generalizes this construction that in turn strengthens the result of Bielak and Sysło [3].

Assume next that we want to embed only J (so that G is considered as the empty graph). In this case, consider the subgraph of H (from the main construction) induced by the vertex set $V(H) \setminus (U \cup V(G))$. Now we select $N > (2(c_2+2)(3c_2+1)n_2)$ and using similar arguments as in the proof of Theorem 1, we get $M(H) = V(J)$. Moreover, if $c_2 \geq \lfloor d_2/2 \rfloor + 2$, then J is a convex subgraph. Thus we have the following result strengthening the result of Dankelmann and Sabidussi [6].

Corollary 5. *Let J be a connected graph. Then there exists a connected graph H such that J is a convex subgraph of H and $V(J)$ is the median set of H.*

4. Concluding remarks

The order of the graph H from Theorem 1 is $3N + (3c_2+1)n_2 + (c_1+1)n_1$, where $N = (2c_1+2)(c_1+1)n_1 + 2c_2n_1 + 2(c_2+2)(3c_2+1)n_2$ which is $O(n_1c_1^2 + n_2c_2^2)$. The construction of Bielak and Sysło [3] uses this construction for the special case when J is the empty graph and $c_1 = c_2 = 0$. In the construction of Dankelmann and Sabidussi [6], the number of vertices of the host graph is of the order $O((2r)^n)$, where n is the number of vertices and r the diameter of the given graph. On the other hand, their host graphs are vertex-transitive.

In the construction of the graph H that gave the main result of this paper, the vertices x, x', and x'' could be of larger degree than might be desirable for applications. If this is the case, the construction can be modified as follows. As before, start with the disjoint union of graphs G, J, K, K', and K''. Consider a subset $S \subset V(K)$, with $|S| = \max(n_1, n_2)$. Take n_1 vertices from S and connect each of these vertices to its own vertex of G with a path of length c_1+1 and select n_2 vertices from S and connect each of these vertices to its own vertex of J with a path of length c_2+1 in the same way. Similarly select n_2 vertices of K' and connect them to J with paths of length c_2+1 and select n_2 vertices of K'' and connect them to J in the same way. The rest of the construction then goes as before. This modified construction gives the same conclusion as in Theorem 1. Note that the order N of the complete graphs involved needs to be selected as $N = (2c_1+3)(c_1+c_2+1)n_1 + 2c_2n_1 + (2c_2+3)(3c_2+1)n_2 + \max(n_1, n_2)$, but still the order of the constructed graph in this modified construction is $O(n_1c_1^2 + n_2c_2^2)$.

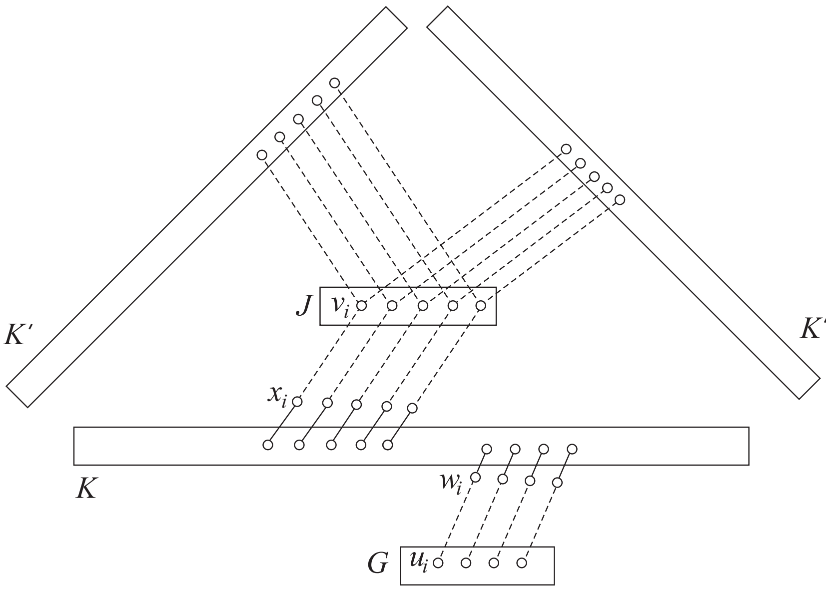

FIGURE 2. Graph H_0 from the construction by connecting vertices of G and J to distinct vertices.

We can also modify the construction by connecting the vertices of G to distinct vertices of K and vertices of J to distinct vertices of K, K', and K'' as shown in Figure 2. In addition, connect w_i, $1 \le i \le n_1$, to a_i additional *private* neighbors in K, and connect x_i, $1 \le i \le n_2$, to b_i additional *private* neighbors in K. This modified construction also gives the same conclusion as in Theorem 1. However, the order N of the complete graphs involved needs to be selected as $N = (2c_1+3)(c_1+c_2+1)n_1^2 + (2c_2+3)(3c_2 +$

$1)n_2^2$, and therefore the order of the constructed graph in this modified construction is $O((n_1c_1)^2 + (n_2c_2)^2)$ and the distance between the given graphs G and J in the host graph H, $d(G, J) = r \geq 3$, where as in the previous constructions $d(G, J) = r \geq 2$.

As mentioned above, slight modifications are possible in the construction of H_0 and H without affecting the conclusion of the main Theorem 1, but such modifications change the order N of the complete graphs K, K', and K''.

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Department of Computer Applications, Cochin University of Science and Technology, Cochin-22, India
E-mail address: bkannan@cusat.ac.in

Faculty of Natural Sciences and Mathematics, University of Maribor, Koroška 160, 2000 Maribor, Slovenia
E-mail address: bostjan.bresar@uni-mb.si

Department of Futures Studies, University of Kerala, Trivandrum-695034, India
E-mail address: mchangat@gmail.com

Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, 1000 Ljubljana, Slovenia

Faculty of Natural Sciences and Mathematics, University of Maribor, Koroška 160, 2000 Maribor, Slovenia
E-mail address: sandi.klavzar@fmf.uni-lj.si

Faculty of Natural Sciences and Mathematics, University of Maribor, Koroška 160, 2000 Maribor, Slovenia
E-mail address: matjaz.kovse@gmail.com

Department of Futures Studies, University of Kerala, Trivandrum-695034, India
E-mail address: ajithars@gmail.com