A STUDY ON ULTRA L-TOPOLOGIES AND LATTICES OF L-TOPOLOGIES

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Certificate

This is to certify that the work reported in the thesis entitled **'A Study on Ultra L-topologies and Lattices of L-topologies'** that is being submitted by Smt. Raji George for the award of Doctor of Philosophy to Cochin University of Science and Technology is based on bona fide research work carried out by her under my supervision in the Department of Mathematics, Cochin University of Science and Technology. The results embodied in this thesis have not been included in any other thesis submitted previously for the award of any degree or diploma.

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Declaration

I, Raji George, hereby declare that this thesis entitled **'A Study on Ultra L-topologies and Lattices of L-topologies'** contains no material which had been accepted for any other Degree or Diploma in any University and to the best of my knowledge and belief, it contains no material previously published by any person except where due references are made in the text of the thesis.

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To

The loving memory of

my Parents

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"I can do everything through the LORD, who strengthens me."

(Philippians 4:13)

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Raji George

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Contents

ABSTRACT AND KEY WORDS

The thesis is divided into nine chapters including introduction.

Mainly

- we determine ultra L-topologies in the lattice of L- topologies and study their properties.
- We find some sublattices in the lattice of L-topologies and study their properties.
- Also we study the lattice structure of the set of all L-closure operators on a set X.

Keywords: Ultra L-topology, principal ultra L-topology, non principal ultra L-topology, scott continuous function, T_1-L topology, weakly induced T_1-L topology, stratified T_1-L topology, principal L-topology, weakly induced principal L-topology, stratified principal L-topology, L-closure operator, infra L-closure operature, ultra L-closure operator, join complement, meet complement, complement, F-lattice, atom, dual atom.

Chapter 1

Introduction

1.1 Introduction

"One should study Mathematics because it is only through Mathematics that nature can be conceived in harmonious form"

Birkhoff

"In most sciences one generation tears down what another has built and what one has established another undoes. In Mathematics alone each generation builds a new story to the old structure"

Herman Hankel

In the first half of the nineteenth century, George Boole's attempt to formalize propositional logic led to the concept of Boolean algebra. While investigating the axiomatics of Boolean algebra at the end of the nineteenth century, Charles S. Peirce and Earnst Schröder found it useful to introduce the lattice concept. Independently, Richard Dedekind's research on ideals of algebraic numbers led to the same discovery. Dedekind also introduced modularity, a weakened form of distributivity. The hostility towards lattice theory began when Dedekind published two fundamental papers that brought the theory to life.

Lattices are partially ordered sets in which least upper bounds and greatest lower bounds of any two elements exist. Dedekind discovered that this property may be axiomatized by identities. A lattice is a set on which two operations are defined called join and meet, denoted by $\vee$ and $\wedge$, which satisfy the idempotent, commutative and associative laws, as well as absorption laws $a \vee (b \wedge a) = a, a \wedge (b \vee a) = a$. Lattices are better behaved than partially ordered sets lacking upper or lower bounds. The contrast is evident in the example of the lattice of partitions of a set and the partially ordered set of partitions of a number. The family of all partitions of a set(equivalence relations) is a lattice when partitions are ordered by refinement. Although some of the early results of these mathematicians and of Edward V. Huntington are very elegant and far from trivial, they did not attract the attention of the mathematical community.

It was Garret Birkhoff's work in the mid-thirties that started the general development of lattice theory. In a brilliant series of papers he demonstrated the importance of lattice theory and showed that it provides a unifying framework for unrelated developments in many mathematical disciplines. Birkhoff himself, Valere Glivenko, Karl Menger, John Von Neumann, Oystein Ore and others had developed enough of this new field for Birkhoff to attempt to "sell" it to the general mathematical community, which he did with astonishing success in the first edition of his book Lattice Theory. The further development of the subject is evident from the first, second and third editions of his book (G. Birkhoff 1940 [8], 1948 [9] and 1967 [10]).

In George Grätzer's view, distributive lattices have played a many faceted role in the development of lattice theory. Historically lattice theory started with (Boolean) disributive lattices and the theory of distributive lattices is one of the most extensive and most satisfying chapters of lattice theory. Many conditions on lattices and on elements and ideals of lattices are weakened forms of disributivity. Also, in many applications the condition of disributivity is imposed on lattices arising in various areas of mathematics in topology and related topics.

General topology and lattice theory are two related branches of mathematics, each influencing the other. Many mathematicians obtained a lot of excellent results combining topology and lattice theory ([17], [22], [53], [61], [69], [71], [70]). Correspondence between order and topology was investigated by many mathematicians in different contexts. Perhaps Birkhoff [11] and Vaidyanathaswamy [66] are the fore runners in this direction.

Zadeh's pioneering paper "Fuzzy Sets" in 1965 opened a new discipline in mathematics. Only in twentieth century, mathematicians defined the concept of sets and functions to represent problems. This way of representing problems is more rigid. In many circumstances the solutions using this concept are meaningless. The difficulty was overcome by the fuzzy concept. But the origin of fuzzy sets dates back to the well known controversy between Cantor and Kronecker regarding the mathematical meaning of infinite sets that took place during the later half of the nineteenth century. Cantor was in favour of infinite sets where as Kronecker refused to accept the concept of infinite sets. The mathematician Dedekind came in support of Cantor. A compromise between Kronecker's and Dedekind's point of view was reached which could be given as follows. A set S is completely determined if and only if there is a decision procedure satisfying whether an element is a member of S or not. Using the ideas of naive set theory such an approach leads to characteristic functions in the context of binary logic whereas in the case of many-valued logic the approach leads to the concept of membership functions introduced by Zadeh. Thus the rejection of infinite sets by Kronecker and the defence of Cantor's notion of infinite sets by Dedekind paved the way for the advent of fuzzy set theory. According to S. Mac Lane-"Math Intelligencer Vol.5 no.4, 1983"

" ....The case of fuzzy sets is even more striking. The original idea was an attractive one..... Someone then recalled (Pace Lowere) that all Mathematics can be based on set theory; it followed at once that all mathematics could be rewritten so as to be based on fuzzy sets. Moreover, it could be based on fuzzy sets in more than one way, so this turned out to be a fine blue print for the publication of lots and lots of newly based mathematics." Hence it is a must to popularise these ideas for our future generation.

In order to study the central problems of complicated systems and dealing with fuzzy information, American Cyberneticist Zadeh [77] introdued fuzzy set theory, describing fuzziness mathematically. Following the study on certainity and randomness, the study of mathematics began to explore the previously restricted zone-fuzziness. Fuzziness is a kind of uncertainity. Since the sixteenth century, probability theory has been studying a kind of uncertainity-randomness, i.e., the uncertainity of the occurrance of an event; but in this case, the event itself is completely certain, the only uncertain thing is whether the event will occur or not. However, there exists another kind of uncertainity-fuzziness, i.e. for some events, it cannot be completely determined that which cases these events should be subordinated to, they are in a nonblack nonwhite state; that is to say, the law of excluded midddle in logic cannot be applied any more. In mathematics, a set A can be equivalently represented by its characteristic function- a mapping χ_A from the universe X of discourse containing A to the 2-valued set $\{0,1\}$; that is to say x belongs to A if and only if $\chi_A(x) = 1$. But in "fuzzy" case "belonging to" relation $\chi_A(x)$ between x and A is no longer "0 or otherwise 1", it has a degree of "belonging to", i.e., membership degree such as α, where α lies between 0 and 1. Therefore the range has to be extended from $\{0,1\}$ to $[0,1]$; or more generally, a lattice L because all the membership degrees, in mathematical view, form an ordered structure, a lattice.

A mapping from X to $[0,1]$ or a lattice L called a generalized characteristic function describes the fuzziness of "set" in general. A fuzzy set on a universe X is simply just a mapping from X to $[0,1]$. Such a set is characterized by a membership function which assigns to each object a grade of membership ranging btween zero and one. When compared with ordinary set theory, fuzzy set theory has greater applications and it enables researchers to review various concepts and theorems of mathematics in the broader frame work of fuzzy setting. The notion of inclusion, union, intersection, complement relation, convexity etc. are extended to such sets and various properties of this notions in the context of fuzzy sets are established.

Thus fuzzy set extended the basic mathematical concept-'set'. In view of the fact that set theory is the cornerstone of modern mathematics, a new and more general framework of mathematics was established. Fuzzy mathematics is just a kind of mathematics developed in this framework and fuzzy topology is just a kind of topology developed on fuzzy sets. Hence fuzzy mathematics is a kind of mathematical theory which contains wider content than the classical theory.

Denote the family of all fuzzy sets on the universe X which takes $I = [0, 1]$ as the range, by I^X. After introducing the fuzzy set, Chang [13], in 1968, introduced fuzzy topology on a set X as a family $\tau \subset I^X$, satisfying the arbitrary union condition and finite inersection condition, substituting inclusive relation by the order relation in I^X. Now a days it is called I-topology rather than I-fuzzy topology. He introduced a topological structure naturally into I^X so that fuzzy topology is a common carrier of ordered structure and topological structure. According to the point of view of Bourbakian School, there are mainly three kinds of structures in mathematics-topological structure, algebraic structure and ordered stucture. Fuzzy topology fuses just two large structures-ordered structure and topological structure. Therefore even if we consider only its mathematical significance but not its practical background, fuzzy topology do has important value to research.

Fuzzy topology is a generalization of topology in classical mathematics. But it also has its own marked characteristics. Also, it can deepen the understanding of basic structure of clssical mathematics, offer new methods and results, and obtain significant results of classical mathematics. Moreover it also has applications in some important aspects of science and technology.

In Chang's definition of fuzzy topology some authors notice fuzziness in the concept of openness of a fuzzy set has not been considered. Keeping this in view, Shostak [52] began the study of fuzzy structure of topological type. Chattopadhyay et. al. [14] rediscovered the Shostak's fuzzy topology concept and called gradation of openness. After this, a fuzzy topology in Shostak's sense will be called gradation of openness and define a fuzzy topoloical space or fts for short, as a pair (X, τ) where τ is a fuzzy topology in Chang's sense on X. A set is called open if it is in τ and closed if its complement is in τ. The interior of a fuzzy set f is the largest open fuzzy set contained in f. The closure of a fuzzy set f is the smallest closed fuzzy set containing f. A fuzzy set which is both open and closed is said to be clopen.

In 1973 Goguen [23] generalized the concept of fuzzy sets with L-fuzzy sets, where L is a lattice. He considered different order structures for the membership set. The ordinary set theory is a special case of L-fuzzy set theory where the membership set is $\{0, 1\}$. The theory of general topology is based on the set operations union, intersection and complementation. L-fuzzy sets do have the same kind of operations. It is therefore natural to extend the concept of point set topology to L-fuzzy subsets resulting in a theory of L-fuzzy topology. The study of general topology can be regarded as a special case of L-fuzzy topology, where all fuzzy subsets in questions take 0 and 1 only.

The definitions, theorems and proofs of L-fuzzy set theory always hold for non fuzzy sets. The theory of L-fuzzy sets has a wider scope of applicability than classical set theory in solving problems. L-fuzzy set theory has now become a major area of research and finds applications in various fields like lattice theory, algebra, topology, functional analysis, operational research, artificial intelligence, image processing, biological and medical sciences, economics, geography and many related topoics. Our interest of L-fuzzy set theory is in its application to theory of general topology and lattice theory. The concept of L-fuzzy sets and fuzzy topology led to the discussion of various aspects of L-fuzzy topology by Lowen([35], [36]), Warren [74], Hutton [26], Rodabaugh [46], Ulrich H$\ddot{o}$hle [65] and many others. Lowen obtained a fuzzy version of Tychonoff theorem. Here we call L-fuzzy subsets as L-subsets and L-fuzzy topology as L-topology. We take the definition of L-topology in the sense of Chang [13] as in [34]. While developing the theory of L-topology, Mathematicians have used different order structure like (i) complete chain (ii) complete Heyting Algebra (iii) complete and distributive lattice (iv) complete Boolean Algebra and many other related structures.

Let (X, τ) be a topological space. A function $f : X \to [0, 1]$ is lower semi continuous(l. s. c) if $f^{-1}(\alpha, 1]$ is open in X for every $0 \leqslant \alpha < 1$. Let $\omega(\tau)$ be the set of all l.s.c. functions on X. Then clearly $\omega(\tau)$ is a fuzzy topology on X. Conversely let (X, F) be a given fuzzy topological space. Then the smallest topology on X which makes every $f \in F$ l.s.c. is called the associated topology for F and is denoted by $i(F)$. The concept of induced fuzzy topological space was introduced by Weiss [75]. Lowen [35] called these spaces as topologically generated spaces. A fuzzy topology F on X is called topologically generated if there exist a topology τ on X such that $F = \omega(\tau)$. Martin [38] introduced a generalized concept weakly induced fuzzy topolgical space, which was called semi induced space by Mashhour, Ghanim, Wakeil and Morsi [40]. The notion of l.s.c. function plays an important tool in defining the above concept in ([5], [7]). Bhaumik and Mukherjee introduced two new classes of fuzzy topological spaces using the tool completely l.s.c. functions [6]. These are defined with the generalized concept of completely continuous functins introduced by Arya and Gupta [2].

In [24] Ayg*ü*n, Warner and Kudri introduced a new class of functions from a topological space (X, τ) to a F-lattice(fuzzy lattice) L with its scott topology called completely scott continuous function as a generalization of completely l.s.c. functions from (X, τ) to L is an L-topology which is a generalization of the fuzzy topology of completely lower semicontinuous functions presented in ([5], [7]). The L-topology $\omega(\tau)$ obtained from a given ordinary topology is called completely induced L-topology. Completely Scott continuous functions turn out to be the natural tool for studying completely induced L-topological space.

In this thesis we take X as a nonempty ordinary set and $L = (<, \leqslant, \vee, \wedge, ')$ be a F-lattice. That is a completely distributive lattice with smallest element 0 and largest element $1 (0 \neq 1)$ and with an order reversig involuton $a \to a' (a \in L)$. In 1936, Birkhoff [11] described the comparison of two topologies on a set and proved that the collection of all topologies on a set X forms a complete lattice. In 1947, Vaidyanathaswamy [66] proved that this lattice is atomic and determined a class of dual atoms. In 1964, Fr*ö*hlich [18] determined a class of dual atoms (ultra topologies) and proved that the lattice is dually atomic.

In 1958, Juris Hartmanis [25] proved that the lattice of topologies on a finite set is complemented and raised the question about the complementation in the lattice of topologies on an arbitrary set. Gaifman [19] proved that the lattice of topologies on a countable set is complemented. In 1968, Steiner [58] proved that the lattice of topologies on an arbitrary set is complemented. In 1968, Van Rooji [68] gave a simpler proof independently that the lattice of topologies is complemented. Hartmanis noted that even in the lattice of topologies on a set with three elements only, the least and the greatest elements have unique complements. Paul S. Schnare [41] proved that every element in the lattice of topologies on a set except the least and the greatest element have atleast n-1 complements when X is finite such that $|X| = n \geqslant 2$ and have infinitely many complements when X is infinite.

In 1989, Babu Sundar [3] proved that the collection of all fuzzy topologies on a fixed set forms a complete lattice with the natural order of set inclusion. He introduced t-irreducible subsets in the membership lattice and solved comlementation problem in the negative. Lattice structure of the set of all fuzzy topologies on a fixed set X was further explored by Johnson. For a given topology τ on X, he studied properties of the lattice F_τ of fuzzy topologies defined by families of lower semi continuous functions with reference to a topology τ on X. He deduced from the lattice F_τ that the set of all fuzzy topologies on a fixed set forms a complete atomic lattice and the lattice is not complemented [29]. In 2002, Sunil C. Mathew [62] introduced the concept of immediate predicessor and of immediate successor or cover in the lattice of fuzzy topologies. He defined simple extensions of fuzzy topologies and studied some of its properties and consequently that of adjacent fuzzy topologies. In 2004, Johnson [30] studied the lattice structure of the set of all L-topologies on a fixed set X and proved that the lattice of L-topologies is not complemented. In 2008, Jose investigated the lattice stucture of the set of all stratified L-topologies [32] and weakly induced L-topologies on a fixed set X [31].

The concept of a topological space is generally introduced in terms of the axioms for the open sets. However alternate methods to describe a topology in the set X are often used in terms of neighbourhood systems, the family of closed sets, the closure operator, the interior operator etc. Of these, the closure operator was axiomatised by Kuratowski and he associated a topology from a closure space by taking closed sets as sets A such that $clA = A$, where clA is the topological closure of a subset A of X. It is also found that clA is the smallest closed set containing A.

$\check{C}$ech introduced the concept of $\check{C}$ech closure spaces. In $\check{C}$ech's approach the condition $ccA = cA$ among Kuratowski axioms need not hold for every subset A of X. When this condition is also true, c is called a topological closure operator. The concept of closure space is thus a generalization of that of toplogical spaces. We studied the definitions and theorems in the topological context from [60].

The concept of a fuzzy closure space has been introduced and studied by Mashhour and Ghanim in [39] and Srivasthava et. al. in [54]. The definitions of Mashhour and Ghanim is an analogue of $\check{C}$ech closure spaces and Srivasthava et. al. have introduced it as an analogue of the definition of closure space given by Dikranjan et. al. [16]. In 1985, Ramachandran [43] studied the properties of the lattice of closure operators. In 1992 Johnson [28] determined completely homogenous fuzzy closure spaces and proved that the set $L(X)$ of all fuzzy closure operators on a fixed set X forms a complete lattice. Some other properties of the lattice including complementation are also discussed. In 1994, Sunitha [63] introduced and studied T_0 and T_1-closure spaces in topological context. In 1994, Srivasthava et. al [54] introduced the concept of T_0-fuzzy closure spaces. The notion of T_1-fuzzy closure space was introduced by Rekha Srivasthava and Manjari Srivasthava [45]. They have studied T_0 and T_1 separation axioms in a fuzzy closure spaces. Also they observed that T_0 and T_1 satisfied the hereditary, productive and projective properties and in addition both were "good extensions" of the corresponding concepts in a closure space. In 2005, Wu-Neng Zhou [76] introduced the concept of L-closure spaces and the convergence in L-closure spaces. In 2012, Madhavan Namboothiri [37]discussed the properties of L-fuzzy $\check{C}$ech closure operators on a set in relation with associated c-reflexive relation on the set of all L-fuzzy points.

Many results and theorems in L-topological spaces can further be extended to L-closure spaces. Mashhour and Ghanim studied $\check{C}$ech fuzzy closure spaces and extended many results to $\check{C}$ech fuzzy closure spaces [39]. So it is quite natural to search for validity of our results and theorems in L-closure spaces. With this in view, we introduce the concept of T_1-L closure space.

A related problem in the lattice of L-topologies is
(i) to determine ultra L-topologies in the lattice of L-topologies and to study their properties.
(ii) to find sublattices in the lattice of L-topologies and study their properties.
(iii) to study the lattice structure of set of all L-closure operators on a set X.
In this thesis we have attempted to present our studies on these problems.

1.2 Basic Concepts and Definitions

In this section we include certain definitions and known results needed for the susequent development. Throughout our discussions X always denote a non empty ordinary set and L, a F-lattice

Definition 1.2.1. [34] Let L be a lattice. L is called distributive, if L satisfies the following two conidtions

(i) $\forall a, b, c \in L, a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

(ii) $\forall a, b, c \in L, a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

A distributive lattice L is also called finitely distributive.

Definition 1.2.2. [34] Let L be a poset. L is called a complete join-semilattice if every join for an arbitrary subset of L exists; partiuclarly the smallest element exists as a join of empty subset. L is called a complete meet-semilattice if every meet for an arbitrary subset of L exists; particularly the largest element exists as the meet of empty subset. L is called a complete lattice if it is both a complete join-semilattice and a complete meet-semilattice.

Definition 1.2.3. [34] Let L be a complete lattice. L is called infinitely distributive, if L satisfies the following two conditions :

(i) $\forall a \in L, \forall B \subset L, a \wedge \bigvee B = \bigvee\limits_{b \in B} (a \wedge b)$,

(ii) $\forall a \in L, \forall B \subset L, a \vee \bigwedge B = \bigwedge_{b \in B} (a \vee b)$,

Definition 1.2.4. [34] Let L be a complete lattice. L is called completely distributive if L satisfies the following two conditions :
$\forall \{\{a_{i,j} : j \in J_i\} : i \in I\} \subset \wp(L) \backslash \{\phi\}, I \neq \phi$

(i) $\bigwedge_{i \in I} (\bigvee_{j \in Ji} a_{i,j}) = \bigvee_{\varphi \in \Pi Ji} (\bigwedge_{i \in I} a_{i,\varphi(i)})$,

(ii) $\bigvee_{i \in I} (\bigwedge_{j \in Ji} a_{i,j}) = \bigwedge_{\varphi \in \Pi Ji} (\bigvee_{i \in I} a_{i,\varphi(i)})$

Definition 1.2.5. [20] A lattice L is called modular if it satisfies the condition $x \geqslant z$ implies that $(x \wedge y) \vee z = x \wedge (y \vee z), \forall x, y, z \in L$.

Theorem 1.2.1. *[20] A Lattice L is modular iff it does not contain a pentagon.*

Definition 1.2.6. [34] Let L be a lattice. A mapping $' : L \to L$ is called order reversing if for all $a, b \in L, a \leqslant b \Rightarrow a' \geqslant b'$, called an involution on L if $''$ = identity mapping(id_L) : $L \to L$

Definition 1.2.7. [34] A completely distributive lattice L is called a F-lattice, if L has an order reversing involution $' : L \to L$

Definition 1.2.8. [34] Let X be a non empty set, L a F-lattice. An L-fuzzy subset of X is characterized by a mapping $f : X \to L$. We call it L-subset rather than L-fuzzy set. Hence the family of all L-subsets on X is just L^X consisting of all mappings from X to L.

Let c be an order reversing involution on L. For any $f \in L^X$, we use the order-reversing involution c to define an operation c on X by

$[c(f)](x) = c(f(x))$ for all x in X. We call $c : L^X \to L^X$ the pseudo-complementary operation on L^X and $c(f)$ the pseudo complementary L-subset of f in L^X. Then c is an order reversing involution on L^X.

For each point x in X, f(x) is called the membership value of x in the L-subset f. Let f and g be L-subsets in X. Then we define

$$f = g \Leftrightarrow f(x) = g(x), \forall x \in X$$

$$f \leqslant g \Leftrightarrow f(x) \leqslant g(x), \forall x \in X$$

$$h = f \vee g \Leftrightarrow h(x) = \max\{f(x), g(x)\}, \forall x \in X$$

$$i = f \wedge g \Leftrightarrow i(x) = \min\{f(x), g(x)\}, \forall x \in X$$

$$g = c(f) \Leftrightarrow g(x) = c[f(x)], \forall x \in X$$

Also for $\{f_\alpha\}_{\alpha \in A}$, we define

$$h = \bigvee_{\alpha \in A} f_\alpha = \sup_{\alpha \in A} f_\alpha \Leftrightarrow h(x) = \sup\{f_\alpha(x) : \alpha \in A\}, \forall x \in X,$$

$$k = \bigwedge_{\alpha \in A} f_\alpha = \inf_{\alpha \in A} f_\alpha \Leftrightarrow k(x) = \inf\{f_\alpha(x) : \alpha \in A\}, \forall x \in X.$$

An L-subset of X with membership value α for all elements in X is denoted by $\underline{\alpha}, \alpha \in L$.

Definition 1.2.9. Let X be a non empty ordinary set, L a F-lattice. A subset F of L^X is called an L-topology on X if

(i) $\underline{0}, \underline{1} \in F$

(ii) $f, g \in F \Rightarrow f \wedge g \in F$

(iii) $f_\alpha \in F, \forall \alpha \in A \Rightarrow \bigvee_{\alpha \in A} f_\alpha \in F$, where A is some index set.

The set X together with F is called L-topological space denoted by (X, F). The element of F are called open L-subsets. An L-subset $f \in L^X$ is called closed if $c(f)$ is open L-subset in X. This definition of L-topological space is in the sense of Chang [13] as in [34]. Particularly when $L = [0,1], (X, F)$ is called an I-topological space.

Definition 1.2.10. [34] Let X be a non empty ordinary set, L a F-lattice, δ_0, δ_1 two L-fuzzy topologies on X. Then δ_0 is coarser than δ_1 or δ_1 is finer than $\delta 0$ if $\delta_0 \subset \delta_1$.

Example 1.2.1. Let X be a non empty ordinary set, L a F-lattice. Then

(i) $\delta = \{\underline{0}, \underline{1}\} \subset L^X$ is the trivial L- topology on X and is the coarsest one.

(ii) $\delta = L^X$ is the discrete L- topology on X and is the finest one.

(iii $\delta = \{\underline{\alpha} : \alpha \in L\} \subset L^X$, is an L- topology on X.

(iv) Suppose τ is an ordinary topology on X, then $\delta = \{\chi_U : U \in \tau\} \subset L^X$ is an L- topology on X.

The set $\{x \in X | f(x) > 0\}$is called the support of f and is denoted by

supp.f. If f takes only the values 0 and 1 then f is called a crisp subset of X.
The fuzzy subset x_λ of X, with $x \in X$ and $0 < \lambda \leqslant 1$ defined by

$$x_\lambda(y) = \begin{cases} \lambda & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases} \quad \text{where } 0 < \lambda \leqslant 1$$

is called a fuzzy point in X with support x and value λ. Two fuzzy points with different supports are called distinct. Note that a fuzzy point x_λ is a fuzzy subset of a fuzzy set f, that is $x_\lambda \in f$, if and only if $\lambda \leqslant f(x)$.

Clearly any fuzzy set f on X can be decomposed in terms of fuzzy points contained in it. Thus $f = \vee\{x_\lambda | x \in X, x_\lambda \in f\}$. We know that the set of all fuzzy topologies on X forms a lattice under the operation of ordinary set inclusion. We denote by F_X, the lattice of all fuzzy topologies on a set X.

Definition 1.2.11. [21] A fuzzy topological space (X, F) is called normal if for any two closed fuzzy sets f_1 and f_2 in X such that $f_1 \leqslant c(f_2)$, there exists $g, h \in F$ such that $f_1 \leqslant g$ and $f_2 \leqslant h$ with $g \leqslant c(h)$.

Definition 1.2.12. [55] A fuzzy topological space (X, F) is said to be Hausdorff if for any two fuzzy points x_λ and y_γ, there exists $f, g \in F$ such that $x_\lambda \in f$ and $y_\gamma \in g$ with $f \wedge g = \underline{0}$.

Definition 1.2.13. [12] A function c from a power set of X to itself is called a closure operator for X provided that the following conditions are satisfied.

(i) $c\phi = \phi$

(ii) $A \subset c(A)$

(iii) $c(A \cup B) = c(A) \cup c(B)$

A structure (X, c) where X is a set and c is a closure operation for X will be called closure space or Čech space. A Čech space which satisfies the condition $c(cA) = cA$ for every $A \subset X$, is called Kuratowski(topological)space [12].

Definition 1.2.14. [12] A closure c is said to be coarser than a closure c' on the same set X if $c'(A) \subset c(A)$ for each $A \subset X$. In this case we say $c < c'$.

Definition 1.2.15. [12] The identity relation on the powerset of X is the finest closure for X and it will be called the discrete closure for X. Setting $c\phi = \phi$ and $c(A) = X$ for every $A \subset X$ we get the coarsest closure for X and it will be called the indiscrete closure for X.

Definition 1.2.16. [12] A subset A of a closure space (X, c) will be called closed if $c(A) = A$ and open if its complement is closed. That is if $c(X - A) = X - A$.

Example 1.2.2. Let $X = \{x, y, z\}$, c be defined on X such that $c\{x\} = \{x\}, c\{y\} = \{y, z\}, c\{z\} = \{x, z\}, c\{x, z\} = \{x, z\}, c\{x, y\} = X, c\{y, z\} = X, cX = X, c\phi = \phi$. Then c is a closure operator on X.

If (X, c) is a closure space, we denote the associated topology on X by τ. That is $\tau = \{A' : cA = A\}$, where A' denotes the complement of

A. Members of τ are the open sets of (X, c) and their complements the closed sets.

Let τ be a topology on a set X. Then a function c from $\wp(X)$ in to $\wp(X)$ defined by $c(A) = \bar{A}$ for every A in $\wp(X)$, where $\bar{A}$ is the closure of A in (X, τ), is a closure operator on X called the closure operator associated with the topology τ. Note that a closure operator on a set X is topological if and only if it is in the closure operator associated with a topology on X. Also the different closure operators can have the same associated topology. The topology associated with the discrete closure operator is the discrete topology and the topology associated with the indiscrete closure operator is the indiscrete topology.

Definition 1.2.17. [63] A closure space (X, c) is said to be T_0 if for every $x \neq y$ in X, either $x \notin c\{y\}$ or $y \notin c\{x\}$.

Theorem 1.2.2. *[63] If (X, τ) is T_0, then (X, c) is T_0.*

Converse of this result is not true.

Example 1.2.3. Let $X = \{x, y, z\}$ and c be defined on X such that $c\{x\} = \{x, y\}, c\{y\} = \{y, z\}, c\{z\} = \{x, z\}, c\{x, y\} = c\{y, z\} = c\{x, z\} = cX = X, c\phi = \phi$. Then c is a closure operation on X and (X, c) is T_0. But (X, τ) is the indiscrete topology, which is not T_0.

Definition 1.2.18. [63] A closure space (X, c) is said to be T_1 if for $x \neq y$, we have $x \notin c\{y\}$ and $y \notin c\{x\}$.

Theorem 1.2.3. *[63] Every T_1 space is also T_0.*

But the converse need not be true.

Example 1.2.4. Let $X = \{x, y, z\}$ and c be defined on X such that $c\{x\} = \{x, y\}, c\{y\} = \{y, z\}, c\{z\} = \{x, z\}, c\{x, y\} = c\{y, z\} = c\{x, z\} = cX = X, c\phi = \phi$. Then c is a closure operation on X and (X, c) is T_0 but it is not T_1.

Definition 1.2.19. [39] A Čech fuzzy closure operator on a set X is a function $\chi : I^X \to I^X$, satisfying the following three axioms:

(i) $\chi(\underline{0}) = \underline{0}$

(ii) $f \leqslant \chi(f), \forall f \in I^X$

(iii) $\chi(f \vee g) = \chi(f) \vee \chi(g), I = [0, 1]$

For convenience it is called fuzzy closure operator on X and (X, χ) is called fuzzy closure space.

Definition 1.2.20. In a fuzzy closure space (X, χ), a fuzzy subset f of X is said to be closed if $\chi(f) = f$. A fuzzy subset f of X is open if its complement is closed in (X, χ). The set of all open fuzzy subsets of (X, χ) forms a fuzzy topology on X called the fuzzy topology associated with the fuzzy closure operator χ.

Let F be a fuzzy topology on a set X. Then a function χ from I^X in to I^X defined by $\chi(f) = \bar{f}$ for every f in I^X, where $\bar{f}$ is the fuzzy closure of f in (X, F), is a fuzzy closure operator on X called the fuzzy closure operator associated with the fuzzy topology F.

A fuzzy closure operator on a set X is called fuzzy topological if it is the fuzzy closure operator associated with a fuzzy topology on X. Note that different fuzzy closure operators can have the same associated fuzzy topology.

Example 1.2.5. Let $X = \{a, b, c\}, I = [0, 1]$. Let $\psi_1 : I^X \to I^X$ defined by

$$\psi_1(f) = \begin{cases} \underline{0} & \text{if } f = 0 \\ \underline{\beta} & \text{if} f(x) < \beta, \forall x \\ \underline{1} & \text{otherwise} \end{cases}$$

Then ψ_1 is a fuzzy closure operator.
$\psi_2 : L^X \to L^X$ defined by

$$\psi_2(f) = \begin{cases} \underline{0} & \text{if } f = 0 \\ \underline{1} & \text{otherwise} \end{cases}$$

Then ψ_2 is a fuzzy closure operator
Associated fuzzy topologies of ψ_1 and ψ_2 are same, which is the indiscrete fuzzy topology.

Definition 1.2.21. Let χ_1 and χ_2 be fuzzy closure operators on X. Then $\chi_1 \leqslant \chi_2$ if and only if $\chi_2(f) \leqslant \chi_1(f)$ for every f in I^X. The set $L(X)$ of all fuzzy closure operators forms a lattice with the relation $\leqslant$.

Definition 1.2.22. The fuzzy closure operator D on X defined by $D(f) = f$ for every f in I^X is called the discrete fuzzy closure operator. The fuzzy closure operator I on X defined by $I(f) = \begin{cases} 0 & \text{if } f = 0 \\ 1 & \text{otherwise} \end{cases}$ is called the indiscrete fuzzy closure operator.

Remark 1.2.1. D and I are the fuzzy closure operators associated with the discrete and indiscrete fuzzy topologies on X respectively. More over D is the unique fuzzy closure operator whose associated fuzzy topology is discrete. Also I and D are smallest and the largest elements of $L(X)$ respectively.

Definition 1.2.23. [16] A map $c : 2^X \to 2^X$ is said to be a closure operation on X if the following conditions hold for any $M, N \in 2^X$;

(i) $c(\phi) = \phi$,

(ii) $M \subseteq c(M)$,

(iii) $M \subseteq N \Rightarrow c(M) \subseteq c(N)$,

(iv) $c(c(M)) = c(M)$.

The pair (X, c) is called a closure space and subsets $M \subseteq X$ with $c(M) = M$ are called C-closed sets in X. Analogue of this has been given in the following definition.

Definition 1.2.24. [54] A function $c : I^X \to I^X$ is called a fuzzy closure operation on X if it satisfies the following conditions for any $A, B \in I^X, \alpha \in [0, 1]$:

(i) $c(\underline{\alpha}) = \underline{\alpha}$,

(ii) $A \subseteq c(A)$,

(iii) $A \subseteq B \Rightarrow c(A) \subseteq c(B)$,

(iv) $c(c(A)) = c(A)$.

The pair (X, c) is called a fuzzy closure space and $U \in I^X$ is called a C-closed fuzzy set if $c(U) = U$.

Definition 1.2.25. [54] A fuzzy closure space (X, c) is said to be T_0 if for all $x, y \in X, x \neq y$, there exists a c-closed fuzzy set U such that $U(x) \neq U(y)$.

Definition 1.2.26. [45] A fuzzy closure space (X, c) is said to be T_1 if $\{x\}$ is c-closed $\forall x \in X$.

Remark 1.2.2. [17] In a fuzzy closure space, obviously T_1-ness $\Rightarrow$ T_0-ness but not conversely as can be seen in the following counter example.

Example 1.2.6. Let $X = \{x, y\}$ and $\Im$ denote the family of all possible intersections of the members of $\{x_\alpha\} \cup \{\underline{\alpha} : \alpha \in [0, 1]\}$. Let $c : I^X \rightarrow I^X$ be defined as $c(A) = \bigwedge\{U \in \Im; U \supseteq A\}$. Then (X, c) is a fuzzy closure space which is obviously T_0 but not T_1 since $\{y_\alpha\}$ is not c-closed in X.

1.3 Summary

The thesis entitled with "A STUDY ON ULTRA L-TOPOLOGIES AND LATTICES OF L-TOPOLOGIES" is arranged into nine chapters. The thesis starts with an introduction to the topoic of research. In the second chapter we determine ultra L-topologies and it is classified into principal and non principal ultra L-topologies in the lattice of L-topologies under certain conditions of the membership lattice L. Also we determine the number of ultra L-topologies and study some topological properties of them.

The lattice structure of the set of all T_1-L topologies on a given set X is investigated in the third chapter. It is a complete sublattice of the lattice of L-topologies on X. Here we prove that the lattice of T_1-L topologies on a given set X has dual atoms if and only if membership lattice L has dual atoms. It is also proved that this lattice is not atomic, not modular, not complemented and not dually atomic in general.

In the fourth chapter we generalize the concept weakly induced space introduced by Martin using the tool Scott continuous functions and study the lattice structure of the set of all weakly induced T_1-L topologies defined by families of (completely)Scott continuous functions on X. It is proved that this lattice is complete, not atomic, not distributive, not complemented and not dually atomic. From this we deduce the properties of the lattice of all weakly induced T_1-L topologies on a given set X.

In the fifth chapter, we study the lattice structure of the set of all stratified T_1-L topologies on X. Here we prove that the lattice of stratified T_1-L topologies is complete and not complemented and this has atoms and dual atoms if and only if L has atoms and dual atoms respectively. It is also proved that this lattice is not atomic and dually atomic in general.

The lattice structure of the set of all principal L-topologies on a given set X is investigated in the sixth chapter. We prove that the lattice of principal ultra L-topologies is atomic and not even modular. It is also proved that this lattice is complete and not complemented. Again we prove that if this lattice has dual atoms, then L has dual atoms and atoms. Also if L is a finite pseudo complemented chain or a Boolean lattice, then the lattice of principal ultra L-topologies has dual atoms.

In the seventh chapter we study the properties of the lattice of weakly induced principal L-topologies defined by families of (completely) Scott continuous functions with reference to the principal topology τ on X. This lattice is complete, not atomic, not complemented and not distributive. From this lattice we deduce properties of the lattice of all weakly induced principal L-topologies on X. It is also proved that this lattice is join complemented.

In the eighth chapter we investigate the lattice structure of the set of all stratified principal L-topologies on a given set X. We prove that this lattice has atoms if and only if L has atoms. If the lattice of stratified principal L-topologies $S_P(X)$ on a set X has dual atoms, then L has dual atoms and atoms. Also if L is a finite pseudo complemented chain or a Boolean lattice, then $S_P(X)$ has dual atoms. It is proved that this lattice is complete, semi complemented and not dually atomic in general.

In the last chapter we study the lattice structure of the set of all L-closure operators on a fixed set X. We prove that this lattice is not modular. We identify the infra L-closure operators and ultra L-closure operators. It is also established the relation between ultra L-topologies and ultra L-closure operators. Again we characterize T_0 and T_1 L-closure spaces.

Chapter 2

Ultra L-Topologies in the Lattice of L-Topologies

2.1 Introduction

In the paper 'On the combination of topologies' [11], G.Birkhoff proved that the collection of all topologies on a given set X forms a complete lattice. Birkhoff's ordering was the natural one of set inclusion; that is, if τ and τ' are topologies on a given set X, τ is less than or equal to τ' if and only if τ is a subset of τ'. The least element is the indiscrete topology and the greatest element is the discrete topology. In the above lattice, the least upperbound of a collection of topologies is the topology generated by

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Some results of this chapter are included in the following paper.
Raji George and T. P. Johnson : Ultra L-Topologies in the Lattice of L-Topologies. InternationalJournal of Engineering Research and Technology, Vol.2, no.1, 2013

their union and the greatest lower bound is their intersection. Since 1936, many topologists, Vaidynathaswamy [66],Otto Fr$\ddot{o}$hlich [18], Hartmanis [25], Steiner [58], Van Rooji [68] have investigated several properties of this lattice.

In [30] Johnson studied the lattice structure of the set of all L-topologies on a given set X. The least upper bound of a collection of L-topologies is the L-topology generated by their union and the greatest lower bound is their intersection. In this paper Johnson proved that this lattice is complete, atomic and not complemented. Also he showed that it is neither modular nor dually atomic in general. In [18] Fr$\ddot{o}$hlich determined the ultra spaces(ultra topologies) on a set X, and he proved that if $|X| = n$, there are $n(n-1)$ principal ultra topologies in the lattice of topologies on a set X. In [59] Steiner studied some topological properties of the ultra spaces. A related problem in the lattice of L-topologies is to identify the ultra L-topologies in the lattice of L-topologies. In this chapter we show that if $|X| = n$ and L is a finite pseudocomplemented chain or a Boolean lattice, there are $n(n-1)mk$ principle ultra L- topologies, where m and k are the number of dual atoms and atoms in L respectively. If X is infinite, there are $|X|$ principal ultra L- topologies and $|X|$ nonprincipal ultra L-topologies. Also we study some topological properties of the ultra L topologies and characterise T_0, T_1, T_2 L-topologies.

2.2 Preliminaries

Let X be a non empty ordinary set and $L = L(\leqslant, \vee, \wedge, ')$ be a completely distributive lattice with the smallest element 0 and the largest element

$1((0 \neq 1)$ and with an order reversing involution $a \to a^{'}$ called F-lattice [34](which is also called Hutton algebra in e.g., [47]). We denote the constant function in L^X taking the value $\alpha \in L$ by $\underline{\alpha}$. Here we call L-fuzzy subsets as L-subsets and a subset F of L^X is called an L-topology in the sense of Chang [13] and Goguen [23] as in [34] if

(i) $\underline{0}, \underline{1} \in F$

(ii) $f, g \in F \Rightarrow f \wedge g \in F$

(iii) $f_i \in F$ for each $i \in I \Rightarrow \bigvee\limits_{i \in I} f_i \in F$.

In this chapter, L-filter on X are defined according to the definition given by Katsaras [33] and Srivastava and Gupta[56] by taking a F-lattice L to be the membership lattice, instead of the closed unit interval $[0, 1]$.

Definition 2.2.1. A non empty subset $\mathscr{U}$ of L^X is said to be an L-filter if

(i) $\underline{0} \notin \mathscr{U}$

(ii) $f, g \in \mathscr{U}$ implies $f \wedge g \in \mathscr{U}$ and

(iii) $f \in \mathscr{U}, g \in L^X$ and $g \geqslant f$ implies $g \in \mathscr{U}$.

An L-filter is said to be an ultra L-filter if it is not properly contained in any other L-filter.

Definition 2.2.2. Let $x \in X, \lambda \in L$ An L-point x_λ is defined by

$$x_\lambda(y) = \begin{cases} \lambda & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases} \quad \text{where } 0 < \lambda \leqslant 1$$

Definition 2.2.3. In a filter $\mathscr{U}$, if there is an L-subset with finite support, then $\mathscr{U}$ is called a principal L-filter.

Example 2.2.1. Let $\mathscr{U} = \{f \in L^X | f \geqslant x_\lambda, \text{where } x_\lambda \text{ is an } L\text{-point}\}$. Then $\mathscr{U}$ is a principal L-filter.

Definition 2.2.4. In a filter $\mathscr{U}$, if there is no L-subset with finite support, then $\mathscr{U}$ is called a non principal L-filter.

Example 2.2.2. Let $\mathscr{U}$=$\{f \in L^X | f > 0$ for all but finite number of points$\}$. Then $\mathscr{U}$ is a nonprincipal L-filter.

Let f be a nonzero L-subset with finite support. Then $\mathscr{U}(f) \subset L^X$ defined by $\mathscr{U}(f) = \{g \in L^X | g \geqslant f\}$ is an L-filter on X, called the principal L-filter at f. Every L-filter is contained in an ultra L-filter. From the definition it follows that on a finite set X, there are only principal ultra L-filters.

2.3 Ultra L-topologies

An L-topology F on X is an ultra L-topology if the only L-topology on X strictly finer than F is the discrete L-topology.

Definition 2.3.1. [62] Let (X, F) be an L-topological space and suppose that $g \in L^X$ and $g \notin F$. Then the collection $F(g) = \{g_1 \vee (g_2 \wedge g) | g_1, g_2 \in F\}$ is called the simple extension of F determined by g.

Theorem 2.3.1. *[62] Let (X, F) be an L-topological space and suppose that $F(g)$ be the simple extension of F determined by g. Then $F(g)$ is an L-topology on X.*

Theorem 2.3.2. *[62] Let F and G be two L-topologies on a set X such that G is a cover of F. Then G is a simple extension of F.*

Theorem 2.3.3. *[18] The ultraspaces on a set E are exactly the topologies of the form $\mathfrak{S}(x, \mathscr{U}) = \wp(E - \{x\}) \cup \mathscr{U}$ where $x \in E$ and $\mathscr{U}$ is an ultrafilter on E not containing $\{x\}$.*

Analogously we can define ultra L-topologies in the lattice of L-topologies according to the nature of lattices. If it contains principal ultra L-filter, then it is called principal ultra L-topology and if it contains non principal ultra L-filter, it is called non principal ultra L-topology.

Theorem 2.3.4. *[3] A principal L-filter at x_λ on X is an ultra L-filter iff λ is an atom in L.*

Theorem 2.3.5. *Let a be a fixed point in X and $\mathscr{U}$ be an ultra L-filter not containing $a_\alpha, 0 \neq \alpha \in L$. Define $\mathscr{F}_a = \{f \in L^X | f(a) = 0\}$. Then $\mathfrak{S} = \mathfrak{S}(a, \mathscr{U}) = \mathscr{F}_a \cup \mathscr{U}$ is an L-topology.*

Proof. Can be easily proved. □

Theorem 2.3.6. *If X is a finite set having n elements and L is a finite pseudo complemented chain or a Boolean lattice, there are $n(n-1)mk$*

principal ultra L-topologies, where m and k are the number of dual atoms and atoms in L respectively. If $k = m$ there are $n(n-1)m^2$ ultra L-topologies.

Illustration:

1. Let $X = \{a, b, c\}, L = \{0, \alpha, \beta, 1\}$, a pseudo complemented chain. Here α is the atom and β is the dual atom. (Refer figure 2.1)

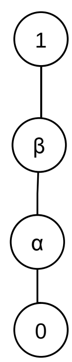

Figure 2.1:

Let $\mathfrak{S} = \mathfrak{S}(a, \mathscr{U}(b_\alpha)) = \{f | f(a) = 0\} \cup \{f | f \geqslant b_\alpha\}$, $\mathfrak{S}$ does not contain the L-points a_α, a_β, a_1. Then $\mathfrak{S}(a, \mathscr{U}(b_\alpha), a_\beta) = \mathfrak{S}(a_\beta) =$ simple extension of $\mathfrak{S}$ by $a_\beta = \{f \vee (g \wedge a_\beta) | f, g \in \mathfrak{S}, a_\beta \notin \mathfrak{S}\}$ is an ultra L-topology, since $\mathfrak{S}(a_1)$ is the discrete L-topology. Similarly

if $\mathfrak{S} = \mathfrak{S}(a, \mathscr{U}(c_\alpha))$, then $\mathfrak{S}(a_\beta)$ is an ultra L-topology.

if $\mathfrak{S} = \mathfrak{S}(b, \mathscr{U}(a_\alpha))$, then $\mathfrak{S}(b_\beta)$,, if $\mathfrak{S} = \mathfrak{S}(b, \mathscr{U}(c_\alpha))$, then $\mathfrak{S}(b_\beta)$,,

if $\mathfrak{S} = \mathfrak{S}(c, \mathscr{U}(a_\alpha))$, then $\mathfrak{S}(c_\beta)$,,

if $\mathfrak{S} = \mathfrak{S}(c, \mathscr{U}(b_\alpha))$, then $\mathfrak{S}c_\beta)$,,

Number of ultra L-topologies $= 6 = 3 * 2 * 1 * 1 = n(n-1)m^2$, where $n = 3, k = m = 1$.

2. Let $X = \{a, b, c\}, L =$Diamond lattice $\{0, \beta_1, \beta_2, 1\}$. (Refer figure 2.2)

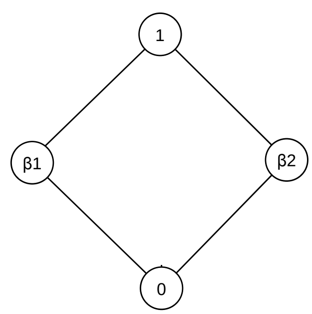

Figure 2.2:

Here β_1 and β_2 are the atoms as well as the dual atoms. Let $\mathfrak{S} =$

$\mathfrak{S}(a, \mathscr{U}(b_{\beta 1})) = \{f|f(a) = 0\} \cup \{f|f \geqslant b_{\beta 1}\}$, does not contain the L-points $a_{\beta 1}, a_{\beta 2}, a_1$. Then the simple extension $\mathfrak{S}(a_{\beta 1})$ contains the L-point $a_{\beta 1}$ also. Let $\mathfrak{S}_1 = \mathfrak{S}(a_{\beta 1})$. Then the simple extension $\mathfrak{S}_1(a_{\beta 2})$ contains all L-points and hence it is discrete. So $\mathfrak{S}(a_{\beta 1}) = \mathfrak{S}(a, \mathscr{U}(b_{\beta 1}), a_{\beta 1})$ is an ultra L-topology. Similarly the simple extension $\mathfrak{S}(a_{\beta 2}) = \mathfrak{S}(a, \mathscr{U}(b_{\beta 1}), a_{\beta 2})$ is an ultra L-topology. If $\mathfrak{S} = \mathfrak{S}(a, \mathscr{U}(b_{\beta 2}) = \{f|f(a) = 0\} \cup \{f|f \geqslant b_{\beta 2}\}$, Then the simple extensions $\mathfrak{S}(a_{\beta 1})$ and $\mathfrak{S}(a_{\beta 2})$ are ultra L-topologies. That is corresponding to the elements a and b there are 4 ultra L-topologies. Similarly corresponding to the elements a and c , there are 4 ultra L-topologies. So there are 8 ultra L-topologies corresponding to a. Similarly there are 8 ultra L-topologies corresponding to b and 8 ultra L-topologies corresponding to c. Hence total number of ultra L-topologies $= 8 + 8 + 8 = 24 = 3 * 2 * 2 * 2 = n(n-1)m^2$, where $n = 3, k = m = 2$.

3. Let $X = \{a, b, c\}, L = \wp(X) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. $\alpha_1 = \{a\}, \alpha_2 = \{b\}, \alpha_3 = \{c\}, \beta_1 = \{a, b\}, \beta_2 = \{a, c\}, \beta_3 = \{b, c\}$. Atoms are $\alpha_1, \alpha_2, \alpha_3$ and dual atoms are $\beta_1, \beta_2, \beta_3$. (Refer figure 2.3)

Let $\mathfrak{S} = \mathfrak{S}(a, \mathscr{U}(b_{\alpha 1})) = \{f|f(a) = 0\} \cup \{f|f \geqslant b_{\alpha 1}\}$, does not contain the L-points $a_{\alpha 1}, a_{\alpha 2}, a_{\alpha 3}, a_{\beta 1}, a_{\beta 2}, a_{\beta 3}, a_1$. Let $\mathfrak{S}_1 =$ Simple extension of $\mathfrak{S}$ by $a_{\beta 1}$ denoted by $\mathfrak{S}(a_{\beta 1})$. Then $\mathfrak{S}_1$ contains more L-subsets than $\mathfrak{S}$, but not discrete L-topology. Let $\mathfrak{S}_2 = \mathfrak{S}_1(a_{\beta 2})$, simple extension of $\mathfrak{S}_1$ by $a_{\beta 2}$. Then $\mathfrak{S}_2$ contain more L subsets than $\mathfrak{S}_1$ but not discrete L-topology. Let $\mathfrak{S}_3 = \mathfrak{S}_2(a_{\beta 3})$, simple extension of $\mathfrak{S}_2$ by $a_{\beta 3}$, which is a discrete L-topology. Hence $\mathfrak{S}_2 = \mathfrak{S}_1(a_{\beta 2})$ is an ultra L-topology, which is the L-topology generated by $\mathfrak{S}(a_{\beta 1})$ and $\mathfrak{S}(a_{\beta 2})$. Also L-topology generated by $\mathfrak{S}(a_{\beta 1})$ and $\mathfrak{S}(a_{\beta 3})$ and L-topology generated by $\mathfrak{S}(a_{\beta 2})$ and $\mathfrak{S}(a_{\beta 3})$ are ultra L-topologies. That is if $\mathfrak{S} = \mathfrak{S}(a, \mathscr{U}(b_{\alpha 1}))$, there are 3 ultra L-topologies. Similarly if $\mathfrak{S} = \mathfrak{S}(a, \mathscr{U}(b_{\alpha 2}))$, there are 3 ultra L-topologies

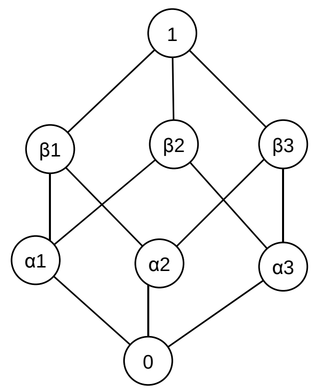

Figure 2.3:

and if $\mathfrak{S} = \mathfrak{S}(a, \mathscr{U}(b_{\alpha 3}))$, there are 3 ultra L-topologies. So corresponding to the elements a, b there are 9 ultra L-topologies. Similarly corresponding to the elements a, c there are 9 ultra L-topologies. Hence there are 18 ultra L-topologies corresponding to the element a. Similarly corresponding to each element b and c there are 18 ultra L-topologies. So total number of ultra L-topologies $= 54 = 3{*}2{*}3{*}3 = n(n-1)m^2, n = 3, k = m = 3$.
4. Let $X = \{a, b, c, d\}, L = \wp(X) = \{\phi, \{a\}, \{b\}, \{c\}, \{d\}, \{a, b\}, \{a, c\}, \{a, d\}, \{b, c\}, \{b, d\}, \{c, d\}, \{a, b, c\}, \{a, b, d\}, \{b, c, d\}, \{c, d, a\}, X\}$. Let $\{a\} = \alpha_1, \{b\} = \alpha_2, \{c\} = \alpha_3, \{d\} = \alpha_4, \{a, b\} = \gamma_1, \{a, c\} = \gamma_2, \{a, d\} = \gamma_3, \{b, c\} = \gamma_4, \{b, d\} = \gamma_5, \{c, d\} = \gamma_6, \{a, b, c\} = \beta_1, \{a, b, d\} = \beta_2, \{b, c, d\} = \beta_3, \{c, d, a\} = \beta_4$. (Refer figure 2.4)
If $\mathfrak{S} = \mathfrak{S}(a, \mathscr{U}(b_{\alpha 1}))$, there are 4 ultra L -topologies.

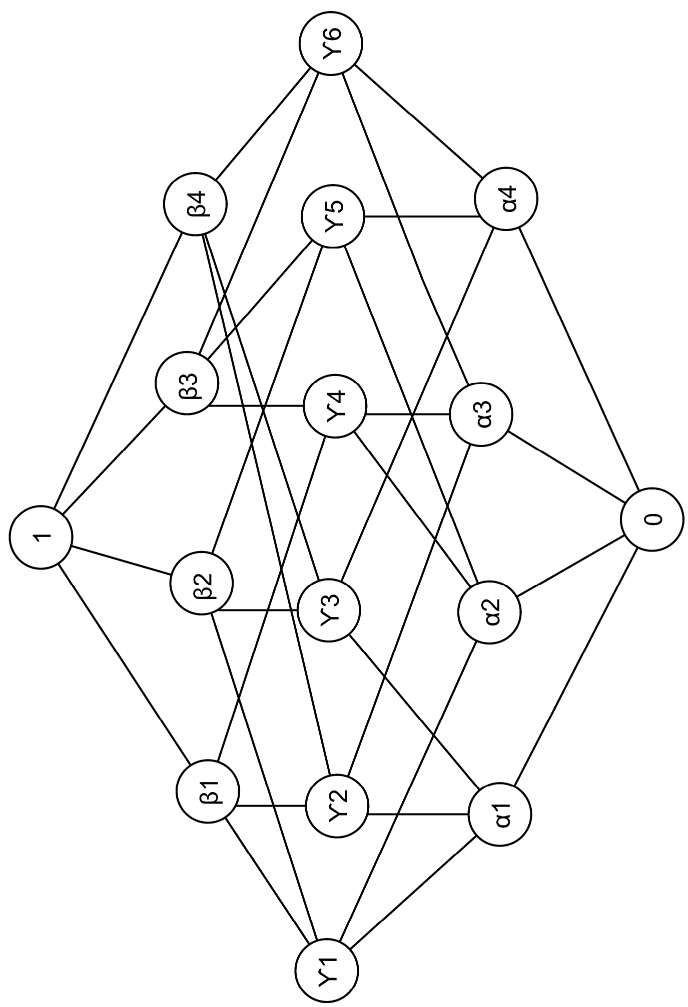

Figure 2.4:

If $\mathfrak{S} = \mathfrak{S}(a, \mathscr{U}(b_{\alpha 2}))$,,

If $\mathfrak{S} = \mathfrak{S}(a, \mathscr{U}(b_{\alpha 3}))$,,

If $\mathfrak{S} = \mathfrak{S}(a, \mathscr{U}(b_{\alpha 4}))$,,

So corresponding to the elements a, b, there are 16 ultra L-topologies. Similarly corresponding to the elements a, c, there are 16 ultra L-topologies and corresponding to the elements a, d, there are 16 ultra L-topologies. Hence there are 48 ultra L-topologies corresponding to the element a. Similarly corresponding to each elements b, c and d, there are 48 ultra L-topologies. So total number of ultra L-topologies $= 48 * 4 = 192 = 4 * 3 * 4 * 4 = n(n-1)m^2, n = 4, k = m = 4$. In general if $|X| = n$ and L is a finite pseudo complemented chain or a Boolean lattice, there are $n(n-1)mk$ ultra L-topologies where m and k are the number of dual atoms and number of atoms respectively. If $k = m$, it is equal to $n(n-1)m^2$.

Remark 2.3.1. If L is neither a finite pseudo complemented chain nor a Boolean lattice, we cannot identify the principal ultra L-topologies in this way. But we can identify ultra L-topology in certain cases.

Example 2.3.1. Let $X = \{a, b, c\}, L = D_{12} = \{1, 2, 3, 4, 6, 12\}$

Here the atoms are $\alpha_1 = 2, \alpha_2 = 3$ and dual atoms are $\beta_1 = 4, \beta_2 = 6$. If $\mathfrak{S} = \mathfrak{S}(a, \mathscr{U}(b_{\alpha 1})) = \{f | f(a) = 0\} \cup \{f | f \geqslant b_{\alpha 1}\}$, L-topology generated by $\mathfrak{S}(a_{\beta 1})$ and $\mathfrak{S}(a_{\beta 2})$ does not contain the L-point$a_{\alpha 2}$. It is not a discrete L-topology. So we cannot say that $\mathfrak{S}(a_{\beta 1})$ is a principal ultra L-topology. But L-topology generated by $\mathfrak{S}(a_{\beta 1})$ and $\mathfrak{S}(a_{\beta 2})$ is a principal ultra L-topology.

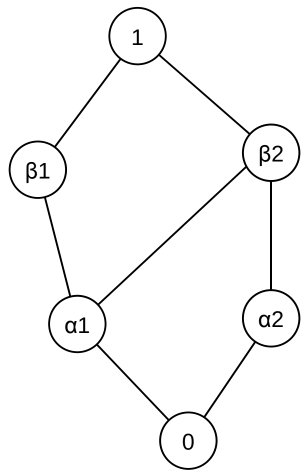

Figure 2.5:

Theorem 2.3.7. *If X is infinite and L is a finite pseudo complemented chain or a Boolean lattice, there are $|X|$ principal ultra L-topologies and $|X|$ non principal ultra L-topologies.*

Illustration:

If X is countably infinite, we have $|X|$, cardinality of $X = \aleph_0$ and If X is uncountable, we have $|X| > \aleph_0$

Case 1.
X is infinite and L is finite

Let $X = \{a, b, ...\}, L = \{0, \alpha, \beta, 1\}$ a pseudo complemented chain.

Let$\mathfrak{S} = \mathfrak{S}(a, \mathscr{U}(b_\alpha)) = \{f|f(a) = 0\} \cup \{f|f \geqslant b_\alpha\}$. $\mathfrak{S}$ does not contain the L points a_α, a_β, a_1. Here $\mathfrak{S}(a_\beta) = \mathfrak{S}(a, \mathscr{U}(b_\alpha), a_\beta)$ is a principal ultra L-topology since $\mathfrak{S}(a_1)$ is the discrete L-topology, where $\mathfrak{S}(a_\beta)$ is the simple extension of $\mathfrak{S}$ by a_β. Similarly we can identify other ultra L-topologies. Hence corresponding to the element a, there are $|X| - 1 = |X|$ principal ultra L-topologies. Similarly corresponding to each element $b, c, d, ...$ there are $|X|$ principal ultra L-topologies. So total number of principal ultra L-topologies $= |X||X| = |X|$. If $\mathfrak{S} = \mathfrak{S}(a, \mathscr{U}) = \{f|f(a) = 0\} \cup \mathscr{U}$, where $\mathscr{U}$ is a nonprincipal ultra L-filter not containing $a_\lambda, 0 \neq \lambda \in L$. Then the simple extension of $\mathfrak{S}$ by $a_\beta = \mathfrak{S}(a_\beta) = \mathfrak{S}(a, \mathscr{U}, a_\beta)$ is a non-principal ultra L-topology since $\mathfrak{S}(a_1)$ is discrete L-topology. So there are $|X|$ non principal ultra L topologies.

Case 2.

X and L are infinite

Let $X = \{a, b, c, ...\}, L = \wp(X)$. There are $|X|$ atoms and $|X|$ dual atoms. Number of principal ultra L-topologies corresponding to one element $= |X||X|(|X| - 1) = |X|$. Hence total number of principal ultra L-topologies $= |X||X| = |X|$. Let $\mathfrak{S} = \mathfrak{S}(a, \mathscr{U}) = \{f|f(a) = 0\} \cup \mathscr{U}$, where $\mathscr{U}$ is a nonprincipal ultra filter not containing $a_\lambda, 0 \neq \lambda \in L$. There are $|X|$ nonprincipal ultra L-filters not containing a_λ so that corresponding to a there are $|X||X| = |X|$ nonprincipal ultra L-topologies. So total number of nonprincipal ultra L-topologies $= |X||X| = |X|$.

2.4 Topological Properties

(a). Principal Ultra L-topologies

Let X be a non empty set and L is a finite pseudo complemented chain. If $\mathfrak{S} = \mathfrak{S}(a, \mathscr{U}(b_\lambda)) = \{f|f(a) = 0\} \cup \{f|f \geqslant b_\lambda\}$, then a principal ultra L-topology $= \mathfrak{S}(a, \mathscr{U}(b_\lambda), a_\beta) = \mathfrak{S}(a_\beta)$, which is the simple extension of $\mathfrak{S}$ by a_β i.e., $\mathfrak{S}(a_\beta) = \{f \vee (g \wedge a_\beta), f, g, \in \mathfrak{S}, a_\beta \notin \mathfrak{S}\}$, where $a, b \in X, \lambda$ and β are the atom and dual atom in L respectively.

Let X be a non empty set and L is a finite Boolean lattice. If $\mathfrak{S} = \mathfrak{S}(a, \mathscr{U}(b_\lambda)) = \{f|f(a) = 0\} \cup \{f|f \geqslant b_\lambda\}$ where $a, b \in X, \lambda$ is an atom, then a principal ultra L-topology denoted by $\mathfrak{S}_{\beta j} = \mathfrak{S}_{\beta j}(a, \mathscr{U}(b_\lambda)) = L$-topology generated by any $(m{-}1)$ $\mathfrak{S}(a_{\beta i})$ among m $\mathfrak{S}(a_{\beta i}), i = 1, 2, ..., m, j = 1, 2, ..., m, i \neq j$ if there are m dual atoms $\beta_1, \beta_2, ...\beta_m$, where $\mathfrak{S}(a_{\beta i}) = \mathfrak{S}(a, \mathscr{U}(b_\lambda), a_{\beta i})$.

Definition 2.4.1. An L-topology F is said to be a T_0-L topology if for every two distinct L-points x_λ and y_γ with distinct support, there is an open L subset containing one and not the other.

Definition 2.4.2. An L-topology F is said to be a T_1-L topology if for every two distinct L-points x_λ and y_γ, with distinct support, there exists an $f \in F$ such that $x_\lambda \in f$ and $y_\gamma \notin f$ and another $g \in F$ such that $y_\gamma \in g$ and $x_\lambda \notin g$ $\forall \lambda, \gamma \in L \backslash \{0\}$.

Definition 2.4.3. An L-topology F is said to be a T_2-L topology if for every two distinct L-points x_λ and y_γ, with distinct support, there exists $f, g \in F$ such that $x_\lambda \in f$ and $y_\gamma \in g$ with $f \wedge g = \underline{0}$.

Theorem 2.4.1. *Let X be a non empty set and L is a finite pseudo complemented chain or a Boolean lattice. Then every principal ultra L-topology $\mathfrak{S}_{\beta j} = \mathfrak{S}_{\beta j}(a, \mathscr{U}(b_\lambda))$ is T_0-L topology but not T_1-L topology.*

Example 2.4.1. Let X be a non empty set

Suppose that L is a finite pseudo complemented chain and $a, b \in X, \lambda, \beta$ are atom and dual atom in L respectively. Take two distinct L points a_1, b_λ. b_λ is an open L subset contain b_λ but not a_1. Since$\mathscr{U}(b_\lambda) = \{f | f \geqslant b_\lambda\}$, any open set contains a_1 must contain b_λ. So $\mathfrak{S}_{\beta j} = \mathfrak{S}_{\beta j}(a, \mathscr{U}(b_\lambda))$ is a T_0-L topology but not T_1-L topology.

Suppose that L is a finite Boolean lattice and $a, b \in X, \lambda$ is an atom and $\beta_1, \beta_2, ...$ are dual atoms in L. Take two distinct L-ponints a_1, b_λ. b_λ is an open L-subset that contains b_λ but not a_1 . Since $\mathscr{U}(b_\lambda) = \{f | f \geqslant b_\lambda\}$, any open set contains a_1 must contain b_λ. So the principal ultra L-topology $\mathfrak{S}_{\beta j} = \mathfrak{S}_{\beta j}(a, \mathscr{U}(b_\lambda))$ is T_0-L topology but not T_1-L topology.

Definition 2.4.4. An L-topological space $(X, F), F \subseteq L^X$ is called door L-space if every L-subset g of X is either L-open or L-closed in F.

Example 2.4.2. Let $X = \{a, b\}$ and $L = \{o, .5, 1\}$. Define $f_1(a) = 0, f_1(b) = 0, f_2(a) = 0, f_2(b) = .5, f_3(a) = 0, f_3(b) = 1, f_4(a) = .5, f_4(b) = 0, f_5(a) = .5, f_5(b) = .5, f_6(a) = .5, f_6(b) = 1, f_7(a) = 1, f_7(b) = 0, f_8(a) = 1, f_8(b) = .5, f_9(a) = 1, f_9(b) = 1$. Let $F = \{f_1, f_9, f_2, f_3, f_4, f_5, f_6\}$. Then$f_7$ and f_8 are closed L-subsets. So (X, F) is a door L-space.

Let $X = \{a, b, c\}$, $L = [0, 1]$ and the the L-topology $F = \{\underline{0}, \mu_{\{a\}}, \mu_{\{b,c\}}, \underline{1}\}$. Then (X, F) is not a door L-space since $\mu_{\{b\}}$ is neither an L-open set nor an L-closed set.

In a principal ultra L-topology $\mathfrak{S}_{\beta j} = \mathfrak{S}_{\beta j}(a, \mathscr{U}(b_\lambda))$ every L-subset of X is either open or closed if L is a finite pseudo complemented chain or a Boolean lattice . So every principal ultra L-topological space $\mathfrak{S}_{\beta j}$ is a door L space.

Definition 2.4.5. An L-topological space (X, F) is said to be regular at an L-point a_λ if for every closed L subset h of X not containing a_λ, there exists disjoint open sets f, g such that $a_\lambda \in f$and $h \in g$. (X, F) is said to be regular L-topology if it is regular at each of its L-points.

Theorem 2.4.2. *Let X be a non empty set and $L = \wp(X)$. Then the principal ultra L-topology $\mathfrak{S}_{\beta j} = \mathfrak{S}_{\beta j}(a, \mathscr{U}(b_\lambda))$ is not regular if $|X| \geqslant 3$.*

Example 2.4.3. Let $X = \{a, b, c\}, L = \wp(X) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. $\alpha_1 = \{a\}, \alpha_2 = \{b\}, \alpha_3 = \{c\}, \beta_1 = \{a, b\}, \beta_2 = \{a, c\}, \beta_3 = \{b, c\}$. Atoms are $\alpha_1, \alpha_2, \alpha_3$ and dual atoms are $\beta_1, \beta_2, \beta_3$. Take $\lambda = \alpha_1$ in the principal ultra L-topology $\mathfrak{S}_{\beta 3}$, which is an L-topology generated by $\mathfrak{S}(a, \mathscr{U}(b_\lambda), a_{\beta 1})$ and $\mathfrak{S}(a, \mathscr{U}(b_\lambda), a_{\beta 2})$. Consider the point $a_{\beta 1}$ and then $a_{\alpha 3}$ is a closed L subset not containing $a_{\beta 1}$. Consider the open sets f, g such that $f(a) = \beta_1, f(b) = \alpha_1, f(c) = \alpha_1, g(a) = \beta_2, g(b) = 0, g(c) = 0$. f is an open set containing $a_{\beta 1}$ and g is an open set containing $a_{\alpha 3}$ but $f \wedge g \neq \underline{0}$. That is f and g are not disjoint.

Definition 2.4.6. An L topological space (X, F) is said to be normal if for every two disjoint closed L subsets h and k , there exists two disjoint open L subsets f, g such that $h \in f$ and $k \in g$.

Theorem 2.4.3. *Let X be a non empty set and $L = \wp(X)$. Then the principal ultra L-topology $\mathfrak{S}_{\beta j} = \mathfrak{S}_{\beta j}(a, \mathscr{U}(b_\lambda))$ is not a normal L-topology if $|X| \geqslant 3$.*

Example 2.4.4. Let $X = \{a, b, c\}, L = \wp(X) = \{\phi, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}$. $\alpha_1 = \{a\}, \alpha_2 = \{b\}, \alpha_3 = \{c\}, \beta_1 = \{a, b\}, \beta_2 = \{a, c\}, \beta_3 = \{b, c\}$. Atoms are $\alpha_1, \alpha_2, \alpha_3$ and dual atoms are $\beta_1, \beta_2, \beta_3$. Take $\lambda = \alpha_1$ in the principal ultra L-topology $\mathfrak{S}_{\beta 3}$. Then $a_{\alpha 2}$ and $a_{\alpha 3}$ are disjoint closed L subsets. There is no disjoint open L subsets containing $a_{\alpha 2}$ and $a_{\alpha 3}$.

(b). Non Principal Ultra L-topology

Let X be an infinite set and L is a finite pseudo complemented chain. If $\mathfrak{S} = \mathfrak{S}(a, \mathscr{U}) = \{f | f(a) = 0\} \cup \mathscr{U}$ where $\mathscr{U}$ is a non principal ultra L-filter not containing $a_\lambda, 0 \neq \lambda \in L$. Then the non principal ultra L-topology $= \mathfrak{S}(a, \mathscr{U}, a_\beta) = \mathfrak{S}(a_\beta)$, is the simple extension of $\mathfrak{S}$ by a_β,i.e., $\mathfrak{S}(a_\beta) = \{f \vee (g \wedge a_\beta), f, g, \in \mathfrak{S}, a_\beta \notin \mathfrak{S}\}$, where $a \in X, \beta$ is the dual atom in L.

Let X be an infinite set and L be a Boolean lattice. If $\mathfrak{S} = \mathfrak{S}(a, \mathscr{U})$, then a non principal ultra L-topology denoted by $\mathfrak{S}_{\beta j} = \mathfrak{S}_{\beta j}(a, \mathscr{U})$ is the L-topology generated by any $(m - 1)$ $\mathfrak{S}(a_{\beta i})$ among m $\mathfrak{S}(a_{\beta i}), i = 1, 2, ..., m, j = 1, 2, ..., m, i \neq j$ if there are m dual atoms $\beta_1, \beta_2, ...\beta_m$ where $\mathfrak{S}(a_{\beta i}) = \mathfrak{S}(a, \mathscr{U}, a_{\beta i})$.

Theorem 2.4.4. *Every non principal ultra L-topology $\mathfrak{S}_{\beta j}(a, \mathscr{U})$ is a T_1-L topology.*

Proof. Let X be an infinite set and $\mathfrak{S}_{\beta j} = \mathfrak{S}_{\beta j}(a, \mathscr{U})$ be a non principal ultra L-topology. Let a_α, b_β be any two distinct L-points, $a, b \in X, \alpha, \beta \in L$. Since $\mathscr{U}$ is a non principal ultra L-filter, there exists L open sets containing each L-points but not the other. $\square$

Theorem 2.4.5. *Every non principal ultra L topology $\mathfrak{S}_{\beta j}(a, \mathscr{U})$ is a T_2-L topology.*

Proof. Let X be an infinite set and $\mathfrak{S}_{\beta j}(a, \mathscr{U})$ be a non principal ultra L-topology. Take two distinct L-points a_α, b_β, where $a, b, \in X, \alpha, \beta \in L$. Since $\mathscr{U}$ is a non principal ultra L-filter, we can find disjoint open sets f and g such that $a_\alpha \in f, b_\beta \notin f$ and $b_\beta \in g, a_\alpha \notin g$. □

Theorem 2.4.6. *Suppose that X is an infinite set and L is a Boolean lattice. Then every non principal ultra L-topology is a door L-space.*

Proof. Let X be an infinite set, L be a Boolean lattice and $\mathfrak{S}_{\beta j}(a, \mathscr{U})$ be a non principal ultra L-topology. Since L is a Boolean Lattice, it is complemented. So every L-subset of X is either L-closed or L-open in $\mathfrak{S}_{\beta j}(a, \mathscr{U})$. Since a and β_j are arbitrary, every non principal ultra L-topology is door L-space. □

Remark 2.4.1. If L is not a complemented F-lattice except a finite pseudo complemented chain, $\mathfrak{S}_{\beta j}(a, \mathscr{U})$ is not a door L-space.

Example 2.4.5. Let X be a nonempty set and $L = D_{12} = \{1, 2, 3, 4, 6, 12\}$. Here the atoms are $\alpha_1 = 2, \alpha_2 = 3$ and dual atoms are $\beta_1 = 4, \beta_2 = 6$ (Refer figure 2.5). Take the non principal ultra L topology $\mathfrak{S}_{\beta 1}(a, \mathscr{U})$. The L point $a_{\beta 2}$ is not open in $\mathfrak{S}_{\beta 1}(a, \mathscr{U})$. Since L is not complemented, the L point $a_{\beta 2}$ is not closed also in $\mathfrak{S}_{\beta 1}(a, \mathscr{U})$.

Theorem 2.4.7. *If X is an infinite set and L is a finite pseudo complemented chain or a diamond lattice, then the non principal ultra L-topology $\mathfrak{S}_{\beta j}(a, \mathscr{U})$ is a regular L-topology.*

Proof. It is trivial. □

Theorem 2.4.8. *Let X be an infinite set and $L = \wp(X)$. Then the non principal ultra L-topology $\mathfrak{S}_{\beta j}(a, \mathscr{U})$ is not a regular L-topology.*

Proof. Let $X = \{a, b, c....\}, L = \wp(X)$. Let $\alpha_1, \alpha_2, ...$ be atoms and $\beta_1, \beta_2, ...$ are dual atoms in L. Consider $a_{\beta 1}$. Then there exists a closed L subset $a_{\alpha i}$ for some i not containing $a_{\beta 1}$. But we cannot find disjoint open L subsets f and g such that f contains $a_{\beta 1}$ and g contains $a_{\alpha i}$. □

Theorem 2.4.9. *If X is an infinite set and L is a finite pseudo complemented chain or a diamond lattice, the non principal ultra L-topology $\mathfrak{S}(a, \mathscr{U}, a_\beta)$ is a normal L-topology.*

Proof. It is trivial □

Theorem 2.4.10. *If X is an infinite set and $L = \wp(X)$ having dual atoms $\beta_1, \beta_2, ...$, then the non principal ultra L topology $\mathfrak{S}_{\beta j}(a, \mathscr{U}), a \in X$ is not a normal L-topology.*

Proof. Let $X = \{a, b, c, ...\}, L = \wp(X)$. Let $\alpha_1, \alpha_2, ...$ be atoms and $\beta_1, \beta_2, ...$ are dual atoms in L. Then there exists two closed L subsets $a_{\alpha i}$ and $a_{\alpha j}$ for some i and j . But there does not exists disjoint open L subsets f and g such that f contains $a_{\alpha i}$ and g contains $a_{\alpha j}$. □

Theorem 2.4.11. *Let X is an infinite set and L is a finite pseudo complemented chain or a Boolean lattice. An ultra L-topology F is a T_1-L topology if and only if it is a non principal ultra L-topology.*

Proof. Suppose that the ultra L-topology F is a T_1-L topology. We have to show that F is a non principal ultra L-topology. F is a principal ultra L-topology implies F is not a T_1-L topology. So we can say that F is a T_1-L topology implies F is a non principal ultra L-topology.

Next assume that F is a non principal ultra L-topology. Then by theorem 2.4.4 F is a T_1-L topology. □

Theorem 2.4.12. *An L-topology F on X is a T_1-L topology if and only if it is the infimum of non principal ultra L-topologies.*

Proof. Necessary part

Any L-topology finer than a T_1-L topology must also be a T_1-L topology. So a T_1-L topology can be the infimum of only non principal ultra L topologies.
Sufficient part

Each non principal ultra L-topology on X contains non principal ultra L- filter. So there exists distinct L-points a_λ, b_γ where $a, b \in X; \lambda, \gamma \in L$ and L-open sets f, g such that $a_\lambda \in f, b_\gamma \notin f$ and $a_\lambda \notin g, b_\gamma \in g$. This is also true in the infimum of any family of non principal ultra L-topologies since every L-points are closed in non principal ultra L-filters. So infimum of any family of non principal ultra L-topologies is a T_1-L topology. □

Theorem 2.4.13. *Let X is an infinite set and L is a finite pseudo complemented chain or a Boolean lattice . Then an ultra L-topology is a T_2-L topology if and only if it is a non principal ultra L-topology.*

Proof. Suppose that an ultra L-topology is a T_2-L topology. This implies that the ultra L-topology is a T_1-L topology. Hence it is a non principal ultra L-topology.

Conversely suppose that the ultra L-topology is a non principal ultra L-topology. Since a non principal ultra L-topology contains a non principal ultra L-filter, for any two distinct L-points in the non principal ultra L-topology there exists disjoint L-open sets contains each L-point but not the other. So it is a T_2-Ltopology. $\square$

2.5 Mixed L- topologies

In [59] Steiner studied the mixed topologies. Analogously we can say that a mixed L-topology on X is not a T_1-L topology and does not have a principal representation. Thus a mixed L-topology is the intersection of a T_1-L topology and a principal L-topology.

The representation of a mixed L-topology as the infimum of a T_1-L topology and a principal L-topology need not be unique.

Example 2.5.1. Let $\mathscr{C} = \{\mu_A | X - A \text{ is finite}\}$ together with $\underline{0}$, is a T_1-L topology and δ and δ' be the principal L-topologies given by $\delta = \bigwedge_{a \in X - \{b,c\}} \mathfrak{S}_{\beta j}$, $\delta' = \bigwedge_{a \in X - \{b\}} \mathfrak{S}_{\beta j}$, if $\mathfrak{S} = \mathfrak{S}(a, \mathscr{U}(b_\lambda))$, λ is an atom and β_j's are dual atoms in L.
$\mathscr{C} \wedge \delta = \{\mu_A | b \in A, X - A \text{ is finite or } f = \underline{0}\} = \mathscr{C} \wedge \delta'$ is a mixed L-topology. Here $c_\lambda \in \delta$ and $c_\lambda \notin \delta'$. That is the representation of a mixed topology as the infimum of T_1-L topology and principal L-topology need not be unique.

Chapter 3

Lattice of T_1-L topologies

3.1 Introduction

In this chapter we investigate the lattice structure of the collection of all T_1-L topologies on a given set X. In [30], Johnson studied the lattice structure of the set of all L-topologies on a given set X. It is quite natural to find sublattices in the lattice of L-topologies and study their properties. The collection of all T_1-L topologies on a given set X forms one of the sublattice of the lattice of L-topologies on X. One distinguishing feature between these two lattices is that the lattice of L-topologies is atomic while the collection of all T_1-L topologies is not. Lattice of T_1-L topologies is a complete sublattice of lattice of L-topologies. Also, the collection of

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Some results of this chapter are included in the following paper.
Raji George and T.P. Johnson : Lattice Properties of T_1-L Topologies. Missouri Journal of Mathematical Sciences, USA, Volume 24, Number 2, 2012

all T_1-L topologies is not modular. In [64] Liu determined dual atoms in the lattice of T_1 topologies and Frolich [18] proved this lattice is dually atomic. However, we prove that the collection of all T_1-L topologies has dual atoms if and only if L has dual atoms and that the collection of all T_1-L topologies is not dually atomic in general.

3.2 Preliminaries

Let X be a non empty ordinary set and $L = L(\leqslant, \vee, \wedge, ')$ be a F-lattice, i.e, a completely distributive lattice with a smallest element 0 and a largest element $1((0 \neq 1)$ and with an order-reversing involution $a \to a'(a \in L)$ [34]. Assume L has more than two elements. An L-fuzzy subset on X is a mapping $f : X \to L$. The family of all L-fuzzy subsets on X is denoted by L^X. We denote the constant function in L^X taking the value $\alpha \in L$ by $\underline{\alpha}$. Here we call L-fuzzy subsets as L-subsets and $F \subseteq L^X$ is called an L-topology in the sense of Chang [13] and Goguen [23] as in [34], if

(i) $\underline{0}, \underline{1} \in F$,

(ii) $f, g \in F \Rightarrow f \wedge g \in F$,

(iii) $f_i \in F$ for each $i \in I \Rightarrow \bigvee_{i \in I} f_i \in F$.

Definition 3.2.1. [44] A fuzzy point x_λ in a set X is a fuzzy set in

X defined by

$$x_\lambda(y) = \begin{cases} \lambda & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases} \quad \text{where } 0 < \lambda \leqslant 1$$

In an L-topological space x_λ is called an L-point.

Definition 3.2.2. [44] An L-topological space (X, F) is said to be a $T_1 - L$ topological space if for every two distinct fuzzy points x_p and y_q, with distinct support, there exists an $f \in F$ such that $x_p \in f$ and $y_q \notin f$ and another $g \in F$ such that $y_q \in g$ and $x_p \notin g, \forall p, q \in L \backslash \{0\}$.

Remark 3.2.1. We take the definition of L-points $x_\lambda, 0 < \lambda \leqslant 1$ so as to include all crisp singletons. Hence every crisp T_1 topology is a T_1-L topology by identifying it with its characteristic function. If τ is any topology on a finite set, then τ is T_1, if and only if it is discrete. However, the same is not true in L-topology.

Example 3.2.1. Let $X = \{a, b, c\}$ and $L = \{0, \alpha, \beta, 1\}$ be the diamond lattice, then $F = \{\underline{0}, \mu_{\{a\}}, \mu_{\{b\}}, \mu_{\{c\}}, \mu_{\{a,b\}}, \mu_{\{a,c\}}, \mu_{\{b,c\}}, \underline{1}\}$ is a $T_1 - L$ topology. Let $a_\lambda, b_\lambda, c_\lambda, 0 \neq \lambda \in L$ are L-points. The complements of $a_\lambda, b_\lambda, c_\lambda$ are not open in F so that $a_\lambda, b_\lambda, c_\lambda$ are not closed.

Definition 3.2.3. [22] An element $p \in L$ is called prime if $p \neq 1$ and whenever $a, b \in L$ with $a \wedge b \leqslant p$, then $a \leqslant p$ or $b \leqslant p$. The set of all prime elements of L will be denoted by $P_r(L)$.

Definition 3.2.4. [73] Scott topology on L is the topology generated by the sets of the form $\{t \in L : t \nleqslant p\}$, where $p \in P_r(L)$. Let (X, τ) be a topological space and $f : (X, \tau) \to L$ be a function where L has its Scott topology, we say that f is Scott continuous if for every $p \in P_r(L), f^{-1}(t \in L : t \nleq p) \in \tau$. (Some authors used the notation $f^{\leftarrow}$ instead of f^{-1}, for example in [48], [49], [50], [51]).

Remark 3.2.2. When $L = [0,1]$, the Scott topology coincides with the topology of topologically generated spaces of Lowen [35]. Every Scott continuous function need not be lower semi continuous.

Example 3.2.2. Suppose k is a large positive integer. Let D_k be the set of all devisors of k. Give the order a/b in $D_k; a, b \in D_k$ such that $a \wedge b = gcd(a,b)$, $a \vee b = lcm(a,b)$ and the corresponding Scott topology. Consider $X = D_k$ with the Scott topology, $L = D_k$ Then $f : X \to L$ defined as $f(x) = x$ is Scott continuous since $f^{\leftarrow}(p, \infty) = (p, \infty)$, which is open in X for any prime p. But not lower semi continuous since $f^{\leftarrow}(n, \infty) = (n, \infty)$, where n is not a prime is not open in X.

Remark 3.2.3. The set $\omega_L(\tau) = \{f \in L^X ; f : (X, \tau) \to L$ is scott continuous $\}$ is an L-topology. An L-topology F on X is called an induced L-topology if there exists a topology τ on X such that $F = \omega_L(\tau)$. If τ is a T_1 topology, $\omega_L(\tau)$ is a T_1-L topology.

Note 1.
A lattice L is modular if and only if, it has no sublattice isomorphic to N_5, where N_5 is a standard non modular lattice [20].

Definition 3.2.5. [34] An element of a lattice L is called an atom if it is the minimal element of $L \backslash \{0\}$.

Definition 3.2.6. [34] An element of a lattice L is called a dual atom if it is the maximal element of $L \backslash \{1\}$.

3.3 Lattice of T_1-L topologies

For any set X, the set $\Omega(X)$ of all T_1-L topologies on X forms a lattice with natural order of set inclusion. The least upper bound of a collection of T_1-L topologies belonging to $\Omega(X)$ is the T_1-L topology generated by their union and the greatest lower bound is their intersection. The smallest T_1-L topology is the cofinite topology denoted by 0 and largest T_1-L topology is the discrete L-topology denoted by 1.

Theorem 3.3.1. *[18] The Ultra spaces on a set E are exactly the topologies of the form $\mathfrak{S}(x, \mathscr{U}) = \wp(E - x) \cup \mathscr{U}$, where $x \in E, \mathscr{U}$ is an ultrafilter on E not containing $\{x\}$.*

Theorem 3.3.2. *[62] Let (X, F) and (X, G) be two fuzzy topological spaces on X. Then G covers F if and only if $G = F(g)$ for every $g \in G - F$, where $F(g)$ is the simple extension of F by g.*

Theorem 3.3.3. *The lattice $\Omega(X)$ is complete.*

Proof. Let S be a subset of $\Omega(X)$ and $G = \bigcap\limits_{\delta \in S} \delta$. Then G is a T_1-L topology and G is the greatest lower bound of S. Since any join(resp. meet) complete lattice with a smallest (resp.largest) element is complete, $\Omega(X)$ is complete. □

Note 2.

Let CFT denote the crisp cofinite topology, where CFT $= \{\chi_A | A$ is a subset of X whose complement is finite $\}$ together with $\underline{0}$, χ_A is the characteristic function of A.

Theorem 3.3.4. *$\Omega(X)$ is not atomic.*

Proof. Atoms in $\Omega(X)$ are the T_1-L topologies generated by CFT $\cup\{x_\lambda\}$, $0 < \lambda \leqslant 1$, or CFT $\cup\underline{\lambda}$, $0 < \lambda < 1$, where x_λ is an L-point. Let $P = \{f \in L^X : f(x) > 0$ for all but finite number of points of $X\}$ together with $\underline{0}$. Then P is a T_1-L topology and P cannot be expressed as join of atoms. Hence $\Omega(X)$ is not atomic. □

Theorem 3.3.5. *$\Omega(X)$ is not modular.*

Proof. Let $x_1, x_2, x_3 \in X$and $\alpha, \beta, \gamma \in (0, 1)$.
Let F be the T_1-L topology generated by CFT $\cup\{f_1, f_2, f_3\}$ where f_1, f_2, f_3 are L subsets defined by

$$f_1(y) = \begin{cases} \alpha & \text{when } y = x_1 \\ 0 & \text{when } y \neq x_1 \end{cases}$$

$$f_2(y) = \begin{cases} \alpha & \text{when } y = x_1 \\ \beta & \text{when } y = x_2 \\ \gamma & \text{when } y = x_3 \\ 0 & \text{when } y \neq x_1, x_2, x_3 \end{cases}$$

$$f_3(y) = \begin{cases} \beta & \text{when } y = x_2 \\ \gamma & \text{when } y = x_3 \\ 0 & \text{when } y \neq x_2, x_3 \end{cases}$$

Let F_1 be the T_1-L topology generated by CFT $\cup\{f_1\}$.
Let F_2 be the T_1-L topology generated by CFT $\cup\{f_1, f_2\}$.

Let F_3 be the T_1-L topology generated by CFT $\cup\{f_3\}$.
Then, we notice that $F_2 \vee F_3 = F$ and $F_1 \vee F_3 = F$ so that {CFT, $F_1, F_2, F_3, F\}$ forms a sublattice of $\Omega(X)$ isomorphic to N_5, where N_5 is the standard non-modular lattice. Hence $\Omega(X)$ is not modular. □

Theorem 3.3.6. *$\Omega(X)$ is not complemented.*

Proof. Let F be the T_1-L topology generated by CFT $\cup\{x_\lambda\}$. Then 1 is not a complement of F since $F \wedge 1 \neq 0$. Let H be any T_1-L topology other than 1, the discrete L-topology. If $F \subset H$, then H cannot be the complement of F. Suppose that $F \not\subseteq H$, then H cannot contain simultaneously all characteristic functions of open sets in τ and all constant L-subsets. Then the set $K = \{k : k$ is a function from (X, τ) to L and $k \notin H\}$ is non empty. Let $F \vee H = G$ and G has the subbase $\{f \wedge h | f \in F, h \in H\}$. Then G cannot be equal to the discrete L-topology, since there exists at least one subset of K which is not contained in G. Hence H is not a complement of F. □

Theorem 3.3.7. *If L has dual atoms, then $\Omega(X)$ has dual atoms.*

Proof. **Case 1.**
Let X be a non empty set and L be a finite pseudo complemented chain.

Let τ be a dual atom in the lattice of T_1 topologies on X. Then by theorem 3.3.1, τ must be of the form $\mathfrak{S}(a, \mathscr{U}) = \wp(X - a) \cup \mathscr{U}$, where $a \in X, \mathscr{U}$ is non principal ultrafilter not containing $\{a\}$. Then $\omega_{1L}(\tau) = \{f | f : (X, \tau) \to L$ is a scott continuous function$\}$. Then $a_\lambda \notin \omega_{1L}(\tau), \lambda \in L$. Let β be the dual atom in L and $F = \omega_{1L}(\tau) \vee a_\beta$ and then F is the ultra L-topology $\mathfrak{S}(a_\beta)$ in $\Omega(X)$ since the simple extension of F

by a_1 is the discrete L-topology.

Case 2.

Let X be a non empty set and L is not a finite pseudo complemented chain.

Let τ be a dual atom in the lattice of T_1 topologies on X. Then by theorem 3.3.1, τ must be of the form $\mathfrak{S}(a, \mathscr{U}) = \wp(X-a) \cup \mathscr{U}$, where $a \in X, \mathscr{U}$ is non principal ultrafilter not containing $\{a\}$. Then $\omega_{1L}(\tau) = \{f | f : (X, \tau) \to L$ is a scott continuous function$\}$. Then $a_\lambda \notin \omega_{1L}(\tau), \lambda \in L$. Let $\beta_1, \beta_2, ..., \beta_m$ are dual atoms in L and $F(a_{\beta 1}) = \omega_{1L}(\tau) \vee a_{\beta 1}, F(a_{\beta 2}) = \omega_{1L}(\tau) \vee a_{\beta 2}, ..., F(a_{\beta m}) = \omega_{1L}(\tau) \vee a_{\beta m}$. Let $F_{\beta j}$ is the L-topology generated by $(m-1)$ $F(a_{\beta i})$ from m $F(a_{\beta i}), i = 1, 2, ...m, j = 1, 2, ...m, i \neq j$. Then as in case 1. $F_{\beta j}$ is the ultra L-topology $\mathfrak{S}_{\beta j}$ in $\Omega(X)$ since the simple extension of $F_{\beta j}$ by $a_{\beta j}$ is the discrete L-topology.

In both cases since L has dual atoms, $\Omega(X)$ has dual atoms. Hence the theorem. □

Note 3.

Let τ be a dual atom in the lattice of T_1 topologies on X, β be the dual atom in L and $A \subset X$ not in τ. Then $\omega_{1L}(\tau) \vee a_\beta = \omega_{1L}(\tau) \vee \mu_A^\beta$, μ_A^β is defined by $\mu_A^\beta(x) = \begin{cases} \beta & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$

Theorem 3.3.8. *If $\Omega(X)$ has dual atoms, then L has dual atoms.*

Proof. **Case 1.**

Let X be a nonempty set and L, a finite pseudocomplemented chain.

Suppose that F is a dual atom in $\Omega(X)$. Then F is of the form $\mathfrak{S}(a_\beta)$ and β must be the dual atom in L. Otherwise there exists an element G greater than F and less than 1. Which is a contradiction to the hypothesis.

Case 2.
Let X be a non empty set and L is not a finite pseudo complemented chain.

Suppose that F is a dual atom in $\Omega(X)$. Then F is of the form $\mathfrak{S}_{\beta j}$ and $\beta_1, \beta_2, ...$ must be dual atoms in L. Otherwise there exists an element G greater than F and less than 1. Which is a contradiction to the hypothesis.

So in either case if $\Omega(X)$ has dual atoms, then L has dual atoms. Hence the proof of the theorem is completed. □

Combining theorem 3.3.7 and theorem 3.3.8, we have

Theorem 3.3.9. *The lattice of T_1-L topologies $\Omega(X)$ has dual atoms if and only if L has dual atoms.*

Theorem 3.3.10. *$\Omega(X)$ is not dually atomic in general.*

Proof. This follows from Theorem 3.3.7. □

Chapter 4

Lattice of Weakly Induced T_1-L topologies

4.1 Introduction

In [11] Birkhoff proved that the set of all T_1 topologies, $\Lambda(X)$ is a complete sublattice of $\Sigma(X)$, the lattice of all topologies. $\Lambda(X)$ possess atoms, but is not atomic [67]. Also $\Lambda(X)$ is not a complemented lattice [57]. The concept of induced fuzzy topological space was introduced by Weiss [75]. Lowen called these spaces a topologically generated spaces. Martin [38] introduced a generalized concept, weakly induced spaces, which was called semi induced space by Mashhour et al.[40]. The notion of lower

Some results of this chapter are included in the following paper.
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semicontinuous functions plays an important tool in defining the above concepts. In [24] Aygun et al. introduced a new class of functions from a topological space (X, τ) to a fuzzy lattice L with its scott topology called (completely) scott continuous functions as a generalization of (completely) lower-semi continuous functions from (X, τ) to $[0, 1]$. It is known that [30] lattice of L-topologies is complete, atomic and not complemented. In [31] Jose and Johnson genralized weakly induced spaces introduced by Martin [38] using the tool (completely) scott continuous functions and studied the lattice structure of the set $W(X)$ of all weakly induced L-topologies on a given set X. A related problem is to find subfamilies in $W(X)$ having certain properties. The collection of all weakly induced T_1-L topologies $W_1(X)$ form a lattice with natural order of set inclusion. In [64] Liu determined dual atoms in the lattice of T_1 topologies and Frolich [18] proved this lattice is dually atomic. Here we study properties of the lattice $W_{1\tau}$ of weakly induced T_1-L topologies defined by families of (completely) scott continuous functions with reference to a T_1 topology τ on X. It has dual atoms if and only if the membership lattice L has dual atoms and it is not dually atomic in general. From the lattice $W_{1\tau}$ we deduce the lattice $W_1(X)$ of all weakly induced T_1-L topologies on a given set X.

4.2 Preliminaries

Let X be a non empty ordinary set and $L = L(\leqslant, \vee, \wedge, ')$ be a completely distributive lattice with smallest element 0 and largest element $1, 0 \neq 1$ and with an order reversing involution $a \to a'(a \in L)$. We identify the constant function from X to L with value α by $\underline{\alpha}$. The fundamental definition of L-fuzzy set theory and L-topology are assumed to be familiar to the reader in the sense of Chang [13].

Definition 4.2.1. [44] A fuzzy point x_λ in a set X is a fuzzy set in X defined by $x_\lambda(y) = \begin{cases} \lambda & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases}$ where $0 < \lambda \leqslant 1$

In an L-topological space x_λ is called an L-point.

Definition 4.2.2. [44] An L-topological space (X, F) is said to be a T_1-L topological space if for every two distinct L-points x_p and y_q, with distinct support, there exists an $f \in F$ such that $x_p \in f$ and $y_q \notin f$ and another $g \in F$ such that $y_q \in g$ and $x_p \notin g, \forall p, q \in L \backslash \{0\}$

Remark 4.2.1. We take the definition of L-points $x_\lambda, 0 < \lambda \leqslant 1$ so as to include all crisp singletons. Hence every crisp T_1 topology is a T_1-L topology by identifying it with its characteristic function. If τ is any topology on a finite set, then τ is T_1, if and only if it is discrete. But in a T_1-L topological space every L-point need not be closed.

Example 4.2.1. Let $X = \{a, b, c\}$ and $L = \wp(X)$, power set of X, then $F = \{\underline{0}, \mu_{\{a\}}, \mu_{\{b\}}, \mu_{\{c\}}, \mu_{\{a,b\}}, \mu_{\{a,c\}}, \mu_{\{b,c\}}, \underline{1}\}$ is a T_1-L topology. Let $a_\lambda, b_\lambda, c_\lambda, 0 \neq \lambda \in L$ are L-points. The complements of $a_\lambda, b_\lambda, c_\lambda$ are not open in F so that $a_\lambda, b_\lambda, c_\lambda$ are not closed.

Definition 4.2.3. [22] An element $p \in L$ is called prime if $p \neq 1$ and whenever $a, b \in L$ with $a \wedge b \leqslant p$, then $a \leqslant p$ or $b \leqslant p$. The set of all prime elements of L will be denoted by $P_r(L)$.

Definition 4.2.4. [73] The scott topology on L is the topology generated by the sets of the form $\{t \in L : t \nleq p\}$, where $p \in P_r(L)$. Let

(X, τ) be a topological space and $f : (X, \tau) \to L$ be a function where L has its scott topology, we say that f is scott continuous if for every $p \in P_r(L), f^{-1}\{t \in L : t \nleq p\} \in \tau$.

Remark 4.2.2. When $L = [0, 1]$, the scott topology coincides with the topology of topologically generated spaces of Lowen[35]. Every Scott continuous function need not be lower semi continuous.

Remark 4.2.3. The set $\omega_L(\tau) = \{f \in L^X; f : (X, \tau) \to L$ is scott continuous$\}$ is an L-topology. It is the largest element in W_τ. If τ is a T_1 topology $\omega_L(\tau)$ is a T_1-L topology, we can denote it by $\omega_{1L}(\tau)$. An L-topology F on X is called an induced T_1-L topology if there exists a T_1 topology τ on X such that $F = \omega_{1L}(\tau)$.

Definition 4.2.5. [24] Let (X, τ) be a topological space and $\alpha \in X$. A function $f : (X, \tau) \to L$, where L has its scott topology, is said to be completely scott continuous at $\alpha \in X$ if for every $p \in P_r(L)$ with $f(\alpha) \nleq p$, there is a regular open neighbourhood U of α in (X, τ) such that $f(x) \nleq p$ for every $x \in U$. That is $U \subset f^{-1}\{t \in L : t \nleq p\}$ and f is called completely scott continuous on X, if f is completely scott continuous at every point of X.

Note 1.

Let F be a T_1-L topology on the set X, let F_c denote the $0 - 1$ valued members of F, that is, F_c is the set of all characteristic mappings in F. Then F_c is a T_1-L topology on X. Define $F_c^* = \{A \subset X : \mu_A \in F_c$, where μ_A is the characteristic function of $A\}$. The T_1-L topological space (X, F_c) is same as the T_1 topological spaces(X, F_c^*)

Definition 4.2.6. A T_1-L topological space (X, F) is said to be a weakly induced T_1-L topological space, if for each $f \in F, f$ is a scott continuous function from (X, F_c^*) to L.

Definition 4.2.7. If F is the collection of all scott continuous functions from (X, F_c^*) to L, then F is an induced space and $F = \omega_{1L}(F_c^*)$.

Definition 4.2.8. [34] An element of a lattice L is called an atom if it is the minimal element of $L\backslash\{0\}$.

Definition 4.2.9. [34] An element of a lattice L is called a dual atom if it is the maximal element of $L\backslash\{1\}$.

Definition 4.2.10. [34] A bounded lattice is said to be complemented if for all x in L there exists y in L such that $x \vee y = 1$ and $x \wedge y = 0$.

4.3 Lattice of weakly induced $T_1 - L$ topologies

For a given T_1-topology τ on X, the family $W_{1\tau}$ of all weakly induced T_1-L topologies defined by families of scott continuous functions from (X, τ) to L forms a lattice under the natural order of set inclusion. The least upper bound of a collection of weakly induced T_1-L topologies belonging to $W_{1\tau}$ is the weakly induced T_1-L topology which is generated by their union and their greatest lower bound is their intersection. The smallest element is the crisp cofinite topology denoted by 0 and the largest element is $\omega_{1L}(\tau)$. Also for a T_1 topology τ on X, the family $CW_{1\tau}$ of all weakly induced T_1-L topologies defined by families of completely scott continuous functions from (X, τ) to L forms a lattice under the natural order of set inclusion.

Since every completely scott continuous function is scott continuous, it follows that $CW_1\tau$ is a sublattice of $W_{1\tau}$. We note that $W_{1\tau}$ and $CW_{1\tau}$ coincide when each open set in τ is regular open. When $\tau = D$, the discrete topology on X, these lattices coincide with lattice of weakly induced T_1-L topologies on X.

Theorem 4.3.1. *[18] The Ultra spaces on a set E are exactly the topologies of the form $\mathfrak{S}(x, \mathscr{U}) = \wp(E - x) \cup \mathscr{U}$, where $x \in E, \mathscr{U}$ is an ultrafilter on E not containing $\{x\}$.*

Theorem 4.3.2. *[62] Let (X, F) and (X, G) be two fuzzy topological spaces on X. Then G covers F if and only if $G = F(g)$ for every $g \in G - F$, where $F(g)$ is the simple extension of F by g.*

Theorem 4.3.3. *The lattice $W_{1\tau}$ is complete.*

Proof. Let S be a subset of $W_{1\tau}$ and let $G = \bigcap_{F \in S} F$. Clearly G is a T_1-L topology. Let $g \in G$. Since each $F \in S$ is a weakly induced T_1-L topology, g is a scott continuous mapping from (X, F_c^*) to L. That is $g^{-1}(\{t \in L : t \nleq p$, where $p \in P_r(L)\}) \in F_c^*$ for each $F \in S$. Therefore $g^{-1}(\{t \in L : t \nleq p$ where $p \in P_r(L)\}) \in \bigcap_{F \in S} F_c^*$. Hence g is a scott continuous function from (X, G_c^*) to L, where $(X, G_c^*) = (X, \bigcap_{F \in S} F_c^*)$. That is $G \in W_{1\tau}$ and G is the greatest lower bound of S. Let K be the set of upper bounds of S. Then K is non empty since $1 = \omega_{1L}(\tau) \in K$. Using the above argument K has a greatest lower bound say H. Then this H is a least upper bound of S. Thus every subset S of $W_{1\tau}$ has greatest lower bound and least upper bound. Hence $W_{1\tau}$ is complete. $\square$

Note 2.
Let CFT denote the crisp cofinite topology, where $CFT = \{\mu_A : A$ is a subset of X whose complement is finite$\}$ together with $\underline{0}$, μ_A is the charcteristic function of A.

Theorem 4.3.4. *$W_{1\tau}$ is not atomic in general.*

Example 4.3.1. Take $\tau =$ Cocountable topology on the set R of real numbers and $L = [0, 1]$. The smallest element in $W_{1\tau}$ is the crisp cofinite topology CFT denoted by 0 and largest element is $\omega_{1L}(\tau) = \{f | f : (X, \tau) \to L$ is a scott continuous function$\}$. Then the T_1-L topologies of the form $CFT \cup \underline{\alpha}$, $CFT \cup \mu_A$ where $X - A$ is countably infinite are weakly induced. But T_1-L topology of the form $CFT \cup h$ where h is a scottcontinuous function which is neither constant function nor a characteristic function is not weakly induced. Hence weakly induced T_1-L topologies of the form $CFT \cup f$ are atoms in $W_{1\tau}$ and hence $\omega_{1L}(\tau)$ cannot be expressed as join of atoms. Thus $W_{1\tau}$ is not atomic.

Theorem 4.3.5. *[4] $\Lambda(X)$ is not modular and hence not distributive.*

Theorem 4.3.6. *$W_{1\tau}$ is not distributive in general.*

Proof. Since every distributive lattice is necessarily modular, we prove that $W_{1\tau}$ is not modular. This can be illustrated with an example. Take X as any infinite set and $\tau = D$, discrete topology on X. Then $W_{1\tau}$ becoms lattice of all weakly induced T_1-L topology on X and $\Lambda(X)$, the lattice of T_1 topologies on X(identifying by its characteristic functions) is a sublattice of W_{1D}. We know that by theorem 4.3.5 $\Lambda(X)$ is not modular and hence not distributive. Thus $W_{1\tau}$ is not distributive in general. $\square$

Theorem 4.3.7. *If L has dual atoms, then $W_{1\tau}$ has dual atoms.*

Proof. **Case 1.**
Let X be a non empty set and L be a finite pseudo complemented chain.

Let τ be a dual atom in the lattice of T_1 topologies on X. Then by theorem 4.3.1, τ must be of the form $\mathfrak{S}(a, \mathscr{U}) = \wp(X - a) \cup \mathscr{U}$, where $a \in X, \mathscr{U}$ is non principal ultrafilter not containing $\{a\}$. Then $\omega_{1L}(\tau) = \{f | f : (X, \tau) \to L$ is a scott continuous function$\}$. Then $a_\lambda \notin \omega_{1L}(\tau), 0 \neq \lambda \in L$. Let β be the dual atom in L and $F = \omega_{1L}(\tau) \vee a_\beta$ Then F is the ultra L-topology $\mathfrak{S}(a_\beta)$ in $\Omega(X)$ since the simple extension of F by a_1 is the discrete L-topology. Let $F_c =$ the $0 - 1$ valued functions in F and $F_c^* = \{A \subset X | \mu_A \in F_c\}$. Then the weakly induced T_1-L topology defined by Scott continuous funcions from $(X, (F_c^*)$ to L is a dual atom in $W_1\tau$.

Case 2.
Let X be a non empty set and L is not a finite pseudo complemented chain.

Let τ be a dual atom in the lattice of T_1 topologies on X. Then by theorem 4.3.1, τ must be of the form $\mathfrak{S}(a, \mathscr{U}) = \wp(X - a) \cup \mathscr{U}$, where $a \in X, \mathscr{U}$ is non principal ultrafilter not containing $\{a\}$. Then $\omega_{1L}(\tau) = \{f | f : (X, \tau) \to L$ is a scott continuous function$\}$. Then $a_\lambda \notin \omega_{1L}(\tau), 0 \neq \lambda \in L$. Let $\beta_1, \beta_2, ..., \beta_m$ are dual atoms in L and $F(a_{\beta 1}) = \omega_{1L}(\tau) \vee a_{\beta 1}, F(a_{\beta 2}) = \omega_{1L}(\tau) \vee a_{\beta 2}, ..., F(a_{\beta m}) = \omega_{1L}(\tau) \vee a_{\beta m}$. Let $F_{\beta j}$ is the L-topology generated by $(m-1)$ $F(a_{\beta i})$ from m $F(a_{\beta i}), i = 1, 2, ...m, j = 1, 2, ...m, i \neq j$. Then as in case 1. $F_{\beta j}$ is the ultra L-topology $\mathfrak{S}_{\beta j}$ in $\Omega(X)$ since the simple extension of $F_{\beta j}$ by $a_{\beta j}$ is the discrete L-topology. Let $G = F_{\beta j}$, $G_c =$ The $0 - 1$ valued functions in G

and $G_c^* = \{A \subset X | \mu_A \in G_C\}$. Then the weakly induced T_1-Ltopology defined by Scott continuous functions from (X, G_c^*) to L is a dual atom in $W_{1\tau}$.
In both cases since L has dual atoms, $W_{1\tau}$ has dual atoms. Hence the theorem. □

Theorem 4.3.8. *If L has no dual atoms, then $W_{1\tau}$ has no dual atoms.*

Proof. Let F be any weakly induced T_1-L topology other than $1 = \omega_{1L}(\tau)$. Then we claim that there exists atleast one weakly induced T_1-L topology finer than F. Since F is a weakly induced T_1-L topology different from $\omega_{1L}(\tau)$, F cannnot contain at the same time all characteristic functions of opensets in τ and all constant L-subsets. Since L has no dual atoms, the collection S of L subsets not belonging to F is infinite. Since F is a T_1-L topology, $g \in S$, we have $F(g)$, the simple extension of F by g is also a T_1-L topology. Let $G = F(g)$, G_c denote the $0-1$ valued members of G and $G_c^* = \{A \subset X | \mu_A \in G_c\}$, where μ_A is the characteristic function of A. Then there exists a weakly induced T_1-L topology K defined by Scott continuous functions from (X, G_c^*) to L. Thus for any weakly induced T_1-L topology F there exists a weakly induced T_1-L topology K such that $F \subset K \neq 1$. Hence the proof of the theorem is completed. □

Combining Theorem 4.3.7 and Theorem 4.3.8 we have

Theorem 4.3.9. *The lattice of weakly induced T_1-L topologies $W_{1\tau}$ has dual atoms if and only if L has dual atoms.*

Theorem 4.3.10. *$W_{1\tau}$ is not dually atomic in general.*

Proof. This follows from Theorem 4.3.8. □

4.4 Complementation in the lattice of weakly induced T_1-L topologies

Theorem 4.4.1. *$W_{1\tau}$ is not complemented in general.*

Example 4.4.1. Let $X = R$, set of real numbers and $\tau =$ Cocountable topology on R. Take $L = [0, 1]$. The weakly induced T_1-L topology of the form $F = CFT \cup \underline{\alpha}$ where $\alpha \in (0, 1)$ has no complement. For, clearly 1 is not a complement of F, since $F \wedge 1 \neq 0$. Let G be any weakly induced T_1-L topology in $W_{1\tau}$ other than 1. If $F \subset G$, then G is not a complement of F. Hence suppose that F is not contained in G. Since $G \neq 1$, G cannot contain simultaneously all constant L-subsets and all characteristic functions of open sets in τ. Then $F \vee G = H \neq 1$ and so G is not a complement of F.

Remark 4.4.1. When $\tau = D$, the discrete topology on $X, W_{1D} = W_1(X)$, the collection of all weakly induced L-topologies on X. The family of all weakly induced T_1-L topologies is defined by scott continuous functions where each scott continuous function is a characteristic function, is a sublattice of $W_1(X)$ and is a lattice isomorphic to the lattice of all T_1 topologies on X. The elements of this lattice are called crisp T_1 topologies.

Theorem 4.4.2. *The lattice of weakly induced T_1-L topologies $W_1(X)$ is not complemented.*

Proof. This follows from theorem 4.4.1. □

Note 3.
Several types of T_1 topologies have complements in $\Lambda(X)$ [57].

Theorem 4.4.3. *An induced T_1-L topology has complement if the corresponding T_1 topology has complement.*

Proof. Let F be an induced T_1-L topology. Since F is induced, F is the collection of all Scott continuous functions from (X, F_c^*) to L. Let $F_c^* = \tau$. If τ has complement, there exists τ' such that $\tau \wedge \tau'$ equal to the cofinite topology and $\tau \vee \tau'$ equal to the discrete topology on X. Then $F \wedge \tau'$ is CFT and $F \vee \tau'$ is the discrete L-topology. □

Chapter 5

Lattice of Stratified T_1-L topologies

5.1 Introduction

This chapter aims at investigating the lattice structure of the set of all stratified T_1-L topologies on a given set X. In [11] Birkhoff proved that the set of all T_1 topologies, $\Lambda(X)$ is a complete sublattice of $\Sigma(X)$, the lattice of all topologies. $\Lambda(X)$ possess atoms, but is not atomic [67]. Also $\Lambda(X)$ ia not a complemented lattice [57]. $\Lambda(X)$ is not modular and hence not distributive [4]. In [64], Liu determined dual atoms in $\Lambda(X)$ and Frolich [18] proved this lattice is dually atomic. In [32], Jose and Johnson studied the lattice structure of the set $L(X)$ of all stratified L-topologies on a given set X. The lattice of all stratified T_1-L topologies on a given set X is denoted by $S_1(X)$. In this chapter we prove that $S_1(X)$ has atoms and dual atoms if and only if the membership lattice L has atoms and dual atoms respectively and it is neither atomic nor dually atomic.

5.2 Preliminaries

Let X be a nonempty ordinary set and $L = L(\leqslant, \vee, \wedge, ')$ be a F-lattice, i.e, a completely distributive lattice with a smallest element 0 and a largest element $1((0 \neq 1)$ and with an order-reversing involution $a \to a'(a \in L)$ [34]. Assume L has more than two elements. An L-fuzzy subset on X is a mapping $f : X \to L$. The family of all L-fuzzy subsets on X is denoted L^X. We denote the constant function in L^X taking the value $\alpha \in L$ by $\underline{\alpha}$. Here we call L-fuzzy subsets as L-subsets and $F \subseteq L^X$ is called an L-topology in the sense of Chang [13] and Goguen [23] as in [34], if

(i) $\underline{0}, \underline{1} \in F$,

(ii) $f, g \in F \Rightarrow f \wedge g \in F$,

(iii) $f_i \in F$ for each $i \in I \Rightarrow \bigvee_{i \in I} f_i \in F$.

A subset F of L^X is called a stratified L-topology, if

(i) $\underline{\alpha} \in F$ for all $\alpha \in L$,

(ii) $f, g \in F \Rightarrow f \wedge g \in F$,

(iii) $f_i \in F$ for each $i \in I \Rightarrow \bigvee_{i \in I} f_i \in F$.

(The idea goes up to Lowen [35], while the term "stratified" has appeared for the first time in [42]).

Example 5.2.1. For a T_1 topological space (X, τ) and a F-lattice L, $F = \{\mu_A : A \in \tau, \mu_A$ is the characteristic function of$A\}$ is a T_1-L topology on X. If we add all constant functions $\underline{\alpha}, \alpha \in L$ in to F and consider it as a subbase, then we get a new L-topology different from the original one, which is a stratified T_1-L topology.

Example 5.2.2. Let $X = R$. Then $F = \{f \in L^X : f(x) > 0$ for all but finite number of points of $X\}$ together with $\underline{0}$. Then F is a stratified T_1-L topology.

Definition 5.2.1. [44] A fuzzy point x_λ in a set X is a fuzzy set in X defined by

$$x_\lambda(y) = \begin{cases} \lambda & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases} \quad \text{where } 0 < \lambda \leqslant 1$$

In an L-topological space we call x_λ an L-point.

Definition 5.2.2. [44] An L-topological space (X, F) is said to be a T_1-L topological space if for every two distinct fuzzy points x_p and y_q, with distinct support, there exists an $f \in F$ such that $x_p \in f$ and $y_q \notin f$ and another $g \in F$ such that $y_q \in g$ and $x_p \notin g, \forall p, q \in L \backslash \{0\}$.

Remark 5.2.1. We take the definition of L points $x_\lambda, 0 < \lambda \leqslant 1$ so as to include all crisp singletons. Hence every crisp T_1 topology is a T_1-L topology by identifying it with its characteristic function. If τ is any topology on a finite set, then τ is T_1, if and only if it is discrete. However, the same is not true in L-topology.

Example 5.2.3. Let X be a nonempty set, $A \subset X$ and $L = \{0, \alpha, \beta, 1\}$, the diamond lattice. Then $F = \{\mu_A | X - A \text{ is finite}\} \cup \{\underline{\alpha} | \alpha \in L\}$ is a stratified T_1-L topology. Let $x \in X$, then the complement of the L-point x_α is not open in F so that x_α is not closed.

Definition 5.2.3. [22] An element $p \in L$ is called prime if $p \neq 1$ and whenever $a, b \in L$ with $a \wedge b \leqslant p$, then $a \leqslant p$ or $b \leqslant p$. The set of all prime elements of L will be denoted by $P_r(L)$.

Definition 5.2.4. [73] Scott topology on L is the topology generated by the sets of the form $\{t \in L : t \nleq p\}$, where $p \in P_r(L)$. Let (X, τ) be a topological space and $f : (X, \tau) \to L$ be a function where L has its Scott topology, we say that f is Scott continuous if for every $p \in P_r(L)$, $f^{-1}\{t \in L : t \nleq p\} \in \tau$.

Remark 5.2.2. When $L = [0, 1]$, the Scott topology coincides with the topology of topologically generated spaces of Lowen [35]. Every Scott continuous function need not be lower semi continuous. The set $\omega_L(\tau) = \{f \in L^X ; f : (X, \tau) \to L \text{ is Scott continuous}\}$ is an L-topology. If τ is a T_1 topology $\omega_L(\tau)$ is a stratified T_1-L topology, we can denote it by $\omega_{1L}(\tau)$. An L-topology F on X is called an induced T_1-L topology if there exists a T_1 topology τ on X such that $F = \omega_{1L}(\tau)$.

Definition 5.2.5. [34] An element of a lattice L is called an atom if it is the minimal element of $L \backslash \{0\}$.

Definition 5.2.6. [34] An element of a lattice L is called a dual atom if it is the maximal element of $L \backslash \{1\}$.

5.3 Lattice of Stratified $T_1 - L$ topologies

Let $S_1(X) = \{F|F$ is a stratified T_1-L topology on X $\}$ and $\Lambda(X) = \{\tau|\tau$ is a T_1 topology on $X\}$. The family $S_1(X)$ of all stratified T_1-L topologies form a lattice under the natural order of set inclusion. The smallest stratified T_1-L topology is CFT $\cup\{\underline{\lambda}|\lambda \in L\}$, denoted by 0 and the largest stratified T_1-L topology is the discrete L-topology which consists of all L-subsets denoted by 1, where the crisp cofinite topology $CFT = \{\mu_A : A$ is a subset of X whose complement is finite$\}$ together with $\underline{0}$, μ_A is the charcteristic function of A.

Definition 5.3.1. [10] A lattice L is said to be join-complemented provided that for every x in L, there exists y in L such that $x \vee y = 1$.

Definition 5.3.2. [10] A lattice L is said to be meet-complemented provided that for every x in L, there exists y in L such that $x \wedge y = 0$.

Definition 5.3.3. [10] A lattice L is said to be complemented provided that for every x in L, there exists y in L such that $x \wedge y = 0$ and $x \vee y = 1$.

Definition 5.3.4. [10] A lattice L is said to be semi-complemented if it is either join complemented or meet-complemented.

Let $\Lambda(X)$ be the set of all T_1 topologies on X and let $\mathscr{G} = \{G|G \subset X, X \sim G$ is finite$\} \cup \{\phi, X\}$.

Theorem 5.3.1. *[11] $\Lambda(X)$ is a complete sublattice of $\Sigma(X)$. The least element in $\Lambda(X)$ is the cofinite topology $\mathscr{G}$, and the greatest element is the discrete topology.*

Theorem 5.3.2. *[67] $\Lambda(X)$ possesses atoms, but is not atomic. The atoms are precisely the topologies of the form $\mathscr{G} \cup \{x\}$.*

Theorem 5.3.3. *[64] τ is an anti atom in $\Lambda(X)$ iff $\tau = \{G | x \notin G \text{ or} G \in \mathscr{U}\}$,where $x \in X$ and$\mathscr{U}$ is a non principal ultrafilter on X.*

Theorem 5.3.4. *[18] The Ultra spaces on a set E are exactly the topologies of the form $\mathfrak{S}(x, \mathscr{U}) = \wp(E - x) \cup \mathscr{U}$, where $x \in E, \mathscr{U}$ is an ultrafilter on E not containing $\{x\}$.*

Theorem 5.3.5. *[18] $\Lambda(X)$ is anti-atomic.*

Theorem 5.3.6. *[57] $\Lambda(X)$ is not a complemented lattice.*

Theorem 5.3.7. *[4] $\Lambda(X)$ is not modular and hence not distributive.*

Theorem 5.3.8. *The lattice of stratified $T_1 - L$ topologies $S_1(X)$ on a set X is complete.*

Proof. Let S be any subset of $S_1(X)$. Then S has greatest lower bound and least upper bound, since arbitrary intersection of stratified T_1-L topologies is a stratified T_1-L topology and $S_1(X)$ has greatest element 1. □

Theorem 5.3.9. *The collection $S_1'(X)$ of all induced stratified T_1-L topologies on any set X is a complete sublattice of the complete lattice $S_1(X)$.*

Proof. Clearly $S_1'(X)$ is a subset of $S_1(X)$. Let $F, G \in S_1'(X)$. Then there exists topologies τ and τ' in $\Lambda(X)$ such that $F = \omega_{1L}(\tau)$ and $G = \omega_{1L}(\tau')$. Then $F \vee G = \omega_{1L}(\tau \vee \tau')$ and $F \wedge G = \omega_{1L}(\tau \wedge \tau')$. Hence $F \vee G$

and $F \wedge G$ are in $S_1'(X)$ sothat $S_1'(X)$ is a sublattice of $S_1(X)$.

Let H be any subset of $S_1'(X)$. Then H has the greatest lower bound since arbitrary intersection of T_1 topologies are T_1 topology so that arbitrary intersection of induced stratified T_1-L topologies are induced stratified T_1-L topology.

Let K be the set of upper bounds of H. Then K is nonempty, since $1 \in K$. Using the above argument, K has the greatest lower bound, say M. Then this M is the least upper bound of H. Thus every subset H of $S_1'(X)$ has the greatest lower bound and least upperbound. Hence $S_1'(X)$ is a complete sublattice of $S_1(X)$ $\square$

Remark 5.3.1. Let $F = CFT \cup \{\underline{\lambda} | \lambda \in L\} \cup \mu_A^\alpha$, where $X - A$ is countably infinite, α is an atom and μ_A^α is defined by $\mu_A^\alpha(x) = \begin{cases} \alpha & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$
Then$F$ is a stratified T_1-L topology but it is not induced, since we cannot find a T_1 topology τ such that $\omega_{1L}(\tau) = F$.

Proposition 5.3.1[72]
For a stratified L-topology $(X, \omega_L(\tau))$, the family $\beta = \{f_A^\alpha | A \in \tau, \alpha \in L\}$ where $f_A^\alpha(x) = \begin{cases} \alpha & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$ is a base for $\omega_L(\tau)$.

Proposition 5.3.2 [72]
For a stratified L-topology $(X, \omega_L(\tau))$, the family $S = \{\mu_A | \mu_A$ is the characteristic function of the open set A in $\tau\} \cup \{\underline{\alpha} | \alpha \in L\}$, is a subbase for $\omega_L(\tau)$.

Theorem 5.3.10. *The collection $S_1'(X)$ of all induced stratified T_1-*

L-topologies on any set X forms a lattice isomorphic to $\Lambda(X)$.

Proof. Let X be a non empty set and L be a F-lattice with its Scott topology. Define $\theta : \Lambda(X) \to S_1'(X)$ by $\theta(\tau) = \omega_{1L}(\tau)$, where τ is a T_1 topology on X.
Let τ_1 and τ_2 are two T_1 topologies on X. $\tau_1 \vee \tau_2 = T_1$ topology generated by τ_1 and τ_2.

$$\begin{aligned}\omega_{1L}(\tau_1) &= \{f|f \text{ is a Scott continuous function from } (X, \tau_1) \to L\} \\ &= L - \text{topology generated by } \{\mu_A|A \in \tau_1\} \cup \{\underline{\alpha}|\alpha \in L\}\end{aligned}$$

$$\begin{aligned}\omega_{1L}(\tau_2) &= \{f|f \text{ is a Scott continuous function from } (X, \tau_2) \to L\} \\ &= L - \text{topology generated by } \{\mu_A|A \in \tau_2\} \cup \{\underline{\alpha}|\alpha \in L\}\end{aligned}$$

$$\begin{aligned}\omega_{1L}(\tau_1 \vee \tau_2) &= \{f|f \text{is a Scott continuous function from } (X, (\tau_1 \vee \tau_2) \to L\} \\ &= L - \text{topology generated by } \{\mu_A|A \in (\tau_1 \vee \tau_2)\} \cup \{\underline{\alpha}|\alpha \in L\},\end{aligned}$$

$$\begin{aligned}\omega_{1L}(\tau_1) \vee \omega_{1L}(\tau_2) &= \text{stratified } T_1 - L \text{ topology generated by } \omega_{1L}(\tau_1) \text{ and } \omega_{1L}(\tau_2) \\ &= L - \text{topology generated by } \{\mu_A|A \in \tau_1\} \cup \{\mu_A|A \in \tau_2\} \cup \\ &\quad \{\underline{\alpha}|\alpha \in L\} \\ &= L - \text{topology generated by } \{\mu_A|A \in (\tau_1 \vee \tau_2)\} \cup \{\underline{\alpha}|\alpha \in L\} \\ &= \{f|f \text{ is a Scott continuous function from } (X, (\tau_1 \vee \tau_2) \to L\} \\ &= \omega_{PL}(\tau_1 \vee \tau_2).\end{aligned}$$

Hence $\theta(\tau_1 \vee \tau_2) = \theta(\tau_1) \vee \theta(\tau_2)$.

Similarly,

$$\begin{aligned}
\omega_{1L}(\tau_1 \wedge \tau_2) &= \{f | f \text{ is a Scott continuous function from } (X, \tau_1 \wedge \tau_2) \to L\} \\
&= L - \text{topology generated by } \{\mu_A | A \in \tau_1 \wedge \tau_2\} \cup \{\underline{\alpha} | \alpha \in L\} \\
&= L - \text{topology generated by } \{\mu_A | A \in \tau_1\} \cup \{\underline{\alpha} | \alpha \in L\} \wedge \\
&\quad\ L - \text{topology generated by } \{\mu_A | A \in \tau_2\} \cup \{\underline{\alpha} | \alpha \in L\} \\
&= \{f | f \text{ is a Scott continuous function from } (X, \tau_1) \to L\} \wedge \\
&\quad\ \{f | f \text{ is a Scott continuous function from } (X, \tau_2) \to L\} \\
&= \omega_{1L}(\tau_1) \wedge \omega_{1L}(\tau_2).
\end{aligned}$$

Hence $\theta(\tau_1 \wedge \tau_2) = \theta(\tau_1) \wedge \theta(\tau_2)$.

$$\begin{aligned}
\tau_1 \neq \tau_2 &\Rightarrow \{f | f : (X, \tau_1) \to (L, S) \text{ is Scott continuous}\} \neq \{f | f : (X, \tau_2) \to (L, S) \\
&\qquad \text{is Scott continuous}\} \\
&\Rightarrow \omega_{1L}(\tau_1) \neq \omega_{1L}(\tau_2) \\
&\Rightarrow \theta(\tau_1) \neq \theta(\tau_2).
\end{aligned}$$

Hence θ is one-one. Corresponding to an induced stratified T_1-L topology $\omega_{1L}(\tau)$ in $S_1'(X)$, there is a topology τ in $\Lambda(X)$ such that $\theta(\tau) = \omega_{1L}(\tau)$. Hence θ is on to. So θ is an isomorphism. □

Remark 5.3.2. Since $S_1'(X)$ is isomorphic to $\Lambda(X)$, $S_1'(X)$ possesses all the properties of $\Lambda(X)$. That is $S_1'(X)$ is complete, not atomic, dually atomic, not complemented and not modular since Λ has these properties. But $S_1(X)$ has no atoms and dual atoms when $L = [0, 1]$.

Theorem 5.3.11. *The lattice of stratified T_1-L topologies $S_1(X)$ on a set X is not modular.*

Proof. Lattice of T_1-L topologies $\Lambda(X)$ is isomorphic to $S_1'(X)$ and $\Lambda(X)$ is not modular by theorem 5.3.7. So $S_1'(X)$ is not modular. Since $S_1'(X)$ is a complete sublattice of $S_1(X)$, $S_1(X)$ is not modular. □

Theorem 5.3.12. *If L has atoms, then the lattice of stratified T_1-L topologies $S_1(X)$ on a set X has atoms.*

Proof. Let α be an atom in L and let A be a proper subset of X. The stratified T_1-L topology of the form F_A^α, where F_A^α is generated by $0 \cup f_A^\alpha$, where $f_A^\alpha(x) = \begin{cases} \alpha & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$ for each atom α in L, is an atom in $S_1(X)$. □

Theorem 5.3.13. *[62] Let (X, F) and (X, G) be two fuzzy topological spaces on X. Then G covers F if and only if $G = F(g)$ for every $g \in G - F$, where $F(g)$ is the simple extension of F by g.*

Theorem 5.3.14. *If the lattice of stratified T_1-L topologies $S_1(X)$ on a set X has atoms, then L has atoms.*

Proof. Let F be an atom in $S_1(X)$. Since F is an atom, F is a cover of 0(zero element of $S_1(X)$). So by theorem 5.3.13 there exists an element g in $F - 0$ such that $F = 0(g)$, the simple extension of 0 by g. i.e $0(g) = \{h \vee (k \wedge g) | h, k \in 0, g \notin 0\}$. This g must be of the form f_A^α, where $A \subset X, \alpha$ is an atom in L. Otherwise we can find a stratified T_1-L topology G smaller than F and greater than 0, which is a contradiction to the hypothesis. □

By combining theorem 5.3.12 and theorem 5.3.14, we get

Theorem 5.3.15. *Lattice of stratified T_1-L topologies $S_1(X)$ on a set X has atoms if and only if L has atoms.*

Remark 5.3.3. Atoms in $S_1(X) = 0(f_A^\alpha)$, simple extension of 0 by f_A^α, where $0 = CFT \cup \{\underline{\lambda} | \lambda \in L\}$. Atoms in $S_1'(X) = \omega_{1L}(\tau)$, where τ is an atom in $\Lambda(X)$, lattice of T_1 topologies on X. Atoms in $S_1'(X)$ and $S_1(X)$ are different. Atoms in $S_1'(X)$ is independent of atoms in L. But $S_1(X)$ has atoms if and only if L has atoms.

Theorem 5.3.16. *Lattice of stratified T_1-L topologies $S_1(X)$ on a set X is not atomic even if L has atoms.*

Example 5.3.1. Let X be an infinite set and L be a F-lattice with atoms. Let $P = \{f | f \in L^X$ and $f(x) > 0$ for all but finite number of points of $X\}$ together with $\underline{0}$. Then P is a stratified T_1-L topology on X and it cannot be expressed as join of atoms.

Theorem 5.3.17. *If the lattice of stratified T_1-L topologies on a set X has dual atoms, then L has dual atoms.*

Proof. **Case 1.**
Let X be a nonempty set and L be a finite pseudo complemented chain.

Suppose that F is a dual atom in $S_1(X)$. Then F is of the form $\mathfrak{S}(a_\beta)$ and β must be the dual atom in L. Otherwise there exists an element G greater than F and less than 1. Which is a contradiction to the hypothesis.

Case 2.
Let X be a non empty set and L is not a finite pseudo complemented chain.

Suppose that F is a dual atom in $S_1(X)$. Then F is of the form $\mathfrak{S}_{\beta j}$ and $\beta_1, \beta_2, ...$ must be dual atoms in L. Otherwise there exists an element G greater than F and less than 1. Which is a contradiction to the hypothesis.
So in either case if $S_1(X)$ has dual atoms, then L has dual atoms. Hence the proof of the theorem is completed □

Theorem 5.3.18. *If L has dual atoms, then the lattice of stratified T_1-Ltopologies $S_1(X)$ on a set X has dual atoms.*

Proof. **Case 1.**
Let X be a non empty set and L be a finite pseudo complemented chain.

Let τ be a dual atom in the lattice of T_1 topologies on X. Then by theorem 5.3.4, τ must be of the form $\mathfrak{S}(a, \mathscr{U}) = \wp(X - a) \cup \mathscr{U}$, where $a \in X, \mathscr{U}$ is non principal ultrafilter not containing $\{a\}$. Then $\omega_{1L}(\tau) = \{f | f : (X, \tau) \to L$ is a scott continuous function$\}$. Then $a_\lambda \notin \omega_{1L}(\tau), \lambda \in L$. Let β be the dual atom in L and $F = \omega_{1L}(\tau) \vee a_\beta$. Then F is the ultra L-topology(dual atom) $\mathfrak{S}(a_\beta)$ in $S_1(X)$, since the simple extension of F by a_1 is the discrete L-topology.

Case 2.
Let X be a non empty set and L is not a finite pseudo complemented chain.

Let τ be a dual atom in the lattice of T_1 topologies on X. Then by theorem 5.3.4, τ must be of the form $\mathfrak{S}(a, \mathscr{U}) = \wp(X-a) \cup \mathscr{U}$, where $a \in X, \mathscr{U}$ is non principal ultrafilter not containing $\{a\}$. Then $\omega_{1L}(\tau) = \{f | f : (X, \tau) \to L$ is a scott continuous function$\}$. Then $a_\lambda \notin \omega_{1L}(\tau), \lambda \in L$.

Let $\beta_1, \beta_2, \ldots, \beta_m$ are dual atoms in L and $F(a_{\beta 1}) = \omega_{1L}(\tau) \vee a_{\beta 1}, F(a_{\beta 2}) = \omega_1(\tau) \vee a_{\beta 2}, ..., F(a_{\beta m}) = \omega_{1l}(\tau) \vee a_{\beta m}$. Let $F_{\beta j}$ is the L-topology generated by $(m-1)$ $F(a_{\beta i})$ from m $F(a_{\beta i}), i = 1, 2, ...m, j = 1, 2, \ldots, m, i \neq j$. Then as in case 1. $F_{\beta j}$ is the ultra L-topology $\mathfrak{S}_{\beta j}$ in $S_1(X)$ since the simple extension of $F_{\beta j}$ by $a_{\beta j}$ is the discrete L-topology.
In both cases since L has dual atoms, $S_1(X)$ has dual atoms. Hence the theorem. □

Combining theorem 5.3.17 and theorem 5.3.18, we have

Theorem 5.3.19. *The lattice of stratified T_1-L topologies $S_1(X)$ has dual atoms if and only if L has dual atoms.*

Theorem 5.3.20. *$S_1(X)$ is not dually atomic in general*

Proof. This follows from theorem 5.3.19. □

Remark 5.3.4. If τ is a dual atom in $\Lambda(X)$, then $\omega_{1L}(\tau)$ need not be a dual atom in $S_1(X)$.

For example, take X any non empty set and $L = \{0, \alpha, 1\}$, where $0 < \alpha < 1$. Since τ is a dual atom in the lattice of T_1-topologies, the only topology greater than τ is the discrete topology on X. Suppose A is a subset of X such that $A \notin \tau$. Then $\omega_{1L}(\tau)$ consists of all Scott continuous functions from(X, τ) to L, and the characteristic function of A does not belong to $\omega_{1L}(\tau)$. Thus $\omega_{1L}(\tau) < \omega_{1L}(\tau) \vee \mu_A^\alpha < 1$ where

$$\mu_A^\alpha(x) = \begin{cases} \alpha & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$$

Thus $\omega_{1L}(\tau)$ is not a dual atom in $S_1(X)$.

Theorem 5.3.21. *$S_1(X)$ is not complemented.*

Proof. Let F be the stratified T_1-L topology generated by 0 in $S_1(X)$ and $\{x_\lambda\}$. Then 1 is not a complement of F, since $F \wedge 1 \neq 0$. Let H be any stratified T_1-L topology other than 1, the discrete L topology. If $F \subset H$, then H cannot be the complement of F. If $F \nsubseteq H$ and let $K = \{k : h$ is a Scott continuous function from (X, τ) to L and $k \notin H\}$. Then K is non empty. Also $\{f \wedge h | f \in F, h \in H\}$ is a subbase for $G = F \vee H$. Then at least one subset of K is not contained in G and so $G \neq 1$. Hence H is not a complement of F. $\square$

Chapter 6

Lattice of Principal L-topologies

6.1 Introduction

The concept of fuzzy topology was introduced by Chang [13] in 1968, and later in a different way by Lowen [35] and Hutton [27]. Mean while Goguen [23] introduced the concept of L fuzzy sets and consequently the Chang's definition of a fuzzy topology has been extended to L-topology [34]. In this chapter we investigate the lattice structure of the set $\beta(X)$ of all principal L-topologies on a given set X. In [11], Birkhoff proved that the set $\Sigma(X)$ of all topologies on a fixed set X, forms a complete lattice

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* Some results of this chapter are included in the following paper.
Raji George and T.P. Johnson : On the Lattice of Principal L-topologies. Far East Journal of Mathematical Sciences, Volume 58, Number 1, 2011

with natural order of set inclusion. Vaidyanathaswamy [66] showed that this lattice is not distributive in general. Steiner [58] proved that the lattice of topologies on a set with more than two elements is not even modular. Vaidyanathaswamy [66] determined atoms in this lattice and proved that it is an atomic lattice. Frolich [18] determined dual atoms of this lattice and proved that it is also dually atomic. Van Rooji [68] and Steiner [58] independently proved that the lattice of topologies is complemented. In [30], Johnson studied the lattice structure of the set of all L-topologies on a given set X. It is quite natural to find sublattices in the lattice of L-topologies and study their properties. The collection $\beta(X)$ of all principal L-topologies on a given set X forms one of the sublattice of the lattice of L-topologies on X. Lattice of principal L-topologies is a complete sublattice of lattice of L-topologies. Also $\beta(X)$ is neither modular nor complemented. The concept of principal topologies in the crisp case was studied by Steiner [58]. The lattice of principal topologies is a complete lattice whose least element is the indiscrete topology and greatest element is the discrete topology. This lattice is both atomic and dually atomic. It's atoms coincide with those of $\Sigma(X)$. However we prove that the lattice $\beta(X)$ has dual atoms if the membership lattice L is a finite pseudocomplemented chain or a Boolean lattice and if the lattice $\beta(X)$ has dual atoms, then L has atoms and dual atoms. It is not dually atomic in general. Also it is proved that induced principal L-topologies have complements.

6.2 Preliminaries

Let X be a non empty ordinary set and L be a F-lattice. . We denote the constant function in L^X taking the value $\alpha \in L$ by $\underline{\alpha}$. The fundamental definitions of L fuzzy set theory and L-topology are assumed to be familiar to the reader. A topological space is called principal if it is discrete or if it can be written as the meet of principal ultra topologies. Steiner [58] proved that this is equivalent to requiring that arbitrary intersection of open sets is open. Analogously we define principal L-topology.

Definition 6.2.1. An L-topology is called principal L-topology if arbitrary intersection of open L-subsets is an open L-subset.

Example 6.2.1. Let $F = \{f | x_p \leqslant f\}$ together with $\underline{0}$. Then F is a principal L-topology.

Example 6.2.2. Let $X = \{a, b, c\}, L = [0, 1]$ and $F = \{\underline{0}, \underline{1}, \mu_{\{a\}}, \mu_{\{a,b\}},$

$$\left.\begin{array}{llllll} f : a \to .6 & g : a \to 1 & h : a \to 1 & i : a \to 1 & j : a \to .6 & k : a \to .6 \\ \quad b \to .5 & \quad b \to 1 & \quad b \to .5 & \quad b \to .5 & \quad b \to .5 & \quad b \to 0 \\ \quad c \to .4, & \quad c \to .4, & \quad c \to .4, & \quad c \to 0, & \quad c \to 0, & \quad c \to 0 \end{array}\right\}$$

Then F is a principal L-topology.

Example 6.2.3. Let $X = \{x, y, z\}, L = \{0, a, b, 1\}$, diamond lattice. Let $F = \{\underline{0}, \underline{1}, x_a, y_b, x_a \vee y_b\}$. Then F is a principal L topology.

Example 6.2.4. Let $X = \{x, y, z\}, L = \{0, a, 1\}$, a pseudocomplemented chain. Then $F = \{\underline{0}, \underline{1}, x_a\}$ is a principal L topology.

Example 6.2.5. Let $X =$ Set of all real numbers, $L = [0, 1]$ and $F =$

$\{f|f(x) > 0$ for all but finite number of points$\}$ together with $\underline{0}$. Then F is not a principal L topology.

Definition 6.2.2. [34] A F-lattice is a complete and completely distributive lattice with an order reversing involution.

Definition 6.2.3. [34] An element of a lattice L is called an atom, if it is the minimal element of $L\backslash\{0\}$.

Definition 6.2.4. [34] An element of a lattice L is called a dual atom, if it is the maximal element of $L\backslash\{1\}$.

Definition 6.2.5. [15] A lattice is said to be bounded if it posses 0 and 1.

Definition 6.2.6. [34] A bounded lattice L is said to be join complemented if for all x in L, there exist $y \in L$ such that $x \vee y = 1$.((Refer figure 6.1)

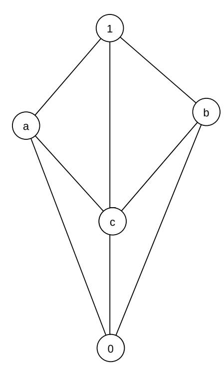

Figure 6.1:

Definition 6.2.7. [34] A bounded lattice L is said to be meet complemented if for all $x \in L$, there exist $y \in L$ such that $x \wedge y = 0$.(Refer figure 6.2)

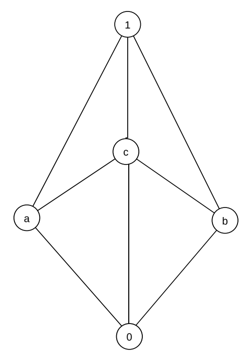

Figure 6.2:

Definition 6.2.8. [34] A bounded lattice is said to be complemented if it is both join complemented and meet complemented.(Refer figure 6.3)

Definition 6.2.9. [34] A bounded lattice L is said to be semi complemented if it is either join complemented or meet complemented.

Definition 6.2.10. [22] An element $p \in L$ is called prime if $p \neq 1$ and whenever $a, b \in L$ with $a \wedge b \leqslant p$, then $a \leqslant p$ or $b \leqslant p$. The set of all prime elements of L will be denoted by Pr (L).

Definition 6.2.11. [73] The Scott topology on L is the topology S, generated by the sets of the form $\{t \in L : t \nleqslant p\}$, where $p \in \Pr(L)$. Let (X, τ) be a topological space and L be a fuzzy lattice. $f : (X, \tau) \to L$ is

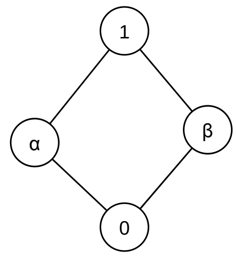

Figure 6.3:

said to be Scott continuous if $f : (X, \tau) \rightarrow (L, S)$ is continuous. i.e. if for every $p \in$ pr (L), $f^{-1}\{t \in L : t \nleq p\} \in \tau$.

Remark 6.2.1. When $L = [0, 1]$, the scott topology coincides with a topology of topologically generated spaces of Lowen [35]. Every Scott continuous function need not be lower semi continuous. The set $\omega_L(\tau) = \{f \in L^X | f : (X, \tau) \rightarrow L$ is scott continuous $\}$ is an L-topology. An L-topology F on X is called an induced L-topology, if there exists a topology τ on X such that $F = \omega_L(\tau)$. If τ is a principal L-topology, then $\omega_L(\tau)$ is a principal L-topology and is denoted by $\omega_{PL}(\tau)$.

Note.
If a lattice L is modular if and only if it has no sublattice isomorphic to N_5 where N_5 is the standard non modular lattice [20].

6.3 Lattice of principal L-topologies

The family $\beta(X)$ of all principal L-topologies on a given set X forms a lattice under the natural order of set inclusion. The least upper bound of a collection of principal L-topologies belonging to $\beta(X)$ is the principal L-topology, which is generated by their union and the greatest lower bound is their intersection. The smallest principal L-topology is the indiscrete L-topology denoted by 0 and the largest principal L-topology is the discrete L-topology denoted by 1.

Theorem 6.3.1. *[18] The Ultra spaces on a set E are exactly the topologies of the form $\mathfrak{S}(x, \mathscr{U}) = \wp(E - x) \cup \mathscr{U}$, where $x \in E, \mathscr{U}$ is an ultrafilter on E not containing $\{x\}$.*

Theorem 6.3.2. *[62] Let (X, F) and (X, G) be two fuzzy topological spaces on X. Then G covers F if and only if $G = F(g)$ for every $g \in G - F$, where $F(g)$ is the simple extension of F by g.*

Theorem 6.3.3. *The lattice $\beta(X)$ is complete.*

Proof. Let S be a subset of $\beta(X)$ and let $G = \bigcap_{H_\alpha \in S} H_\alpha$, let $f_\alpha \in G$ then $f_\alpha \in H_\alpha$ for each α. Since H_α is a principal L-topology, $\wedge f_\alpha \in H_\alpha$ for each α. Therefore $\wedge f_\alpha \in \cap H_\alpha$ and so $\wedge f_\alpha \in G$. Thus G is closed under arbitrary intersection. That is $G \in \beta(X)$ and G is the greatest lower bound of S. Let K be the set of upper bounds of S, then K is non empty, since $1 \in K$. Using the above argument K has a greatest lower bound, say H. Then this H is the least upper bound of S. Thus every subset S of $\beta(X)$ has greatest lower bound and least upper bound. Hence $\beta(X)$ is complete. □

Theorem 6.3.4. *$\beta(X)$ is atomic.*

Proof. Atoms in $\beta(X)$ are of the form $\mathcal{F}(g) = \{\underline{0}, \underline{1}, g\}$, where g is an L-subset of X. Let P be any element of $\beta(X)$, then $P = \bigvee_{g \in P} \mathcal{F}(g)$. Hence $\beta(X)$ is atomic. $\square$

Theorem 6.3.5. *$\beta(X)$ is not modular.*

For example let $X = \{a, b, c\}$ and $\tau = \{\phi, X, \{a\}, \{b\}, \{a, b\}\}$. Suppose $F_1 = \{\underline{0}, \underline{1}\}$, $F_2 = \{\underline{0}, \underline{1}, \mu_{\{a\}}\}$, $F_3 = \{\underline{0}, \underline{1}, \mu_{\{b\}}\}$, $F_4 = \{\underline{0}, \underline{1}, \mu_{\{a\}}, \mu_{\{a,b\}}\}$, $F_5 = \{\underline{0}, \underline{1}, \mu_{\{a\}}, \mu_{\{b\}}, \mu_{\{a,b\}}\}$, where $\mu_{\{a\}}$, $\mu_{\{b\}}$ and $\mu_{\{a,b\}}$ are the characteristic functions of open subsets $\{a\}$, $\{b\}$ and $\{a, b\}$ of (X, τ) respectively. Then each element in the collection $S = \{F_1, F_2, F_3, F_4, F_5\}$ belong to $\beta(X)$ and S is a sublattice of $\beta(X)$, isomorphic to N_5. Therefore $\beta(X)$ is not modular.

Theorem 6.3.6. *If the lattice of principal L-topologies $\beta(X)$ on a set X has dual atoms, then L has dual atoms and atoms.*

Proof. **Case 1.**
Let X be a non empty set and L be a finite pseudo complemented chain.

If $\mathfrak{S} = \mathfrak{S}(a, \mathscr{U}(b_\lambda)) = \{f | f(a) = 0\} \cup \{f | f \geqslant b_\lambda\}$, then the principal ultra L-topology $= \mathfrak{S}(a, \mathscr{U}(b_\lambda), a_\beta) = \mathfrak{S}(a_\beta)$ is the simple extension of $\mathfrak{S}$ by a_β, i.e., $\mathfrak{S}(a_\beta) = \{f \vee (g \wedge a_\beta), f, g \in \mathfrak{S}, a_\beta \notin \mathfrak{S}\}$, where $a, b \in X, \lambda$ and β are the atom and dual atom in L respectively (from chapter 2). So $\mathfrak{S}(a_\beta)$ is a dual atom in the lattice of principal L topologies. Since the simple extension of $\mathfrak{S}(a_\beta)$ by the L point a_1 is 1(discrete L-topology), by theorem 6.3.2, 1 is a cover of $\mathfrak{S}(a_\beta)$.

Suppose that F is a dual atom in $\beta(X)$. Then F is of the form $\mathfrak{S}(a_\beta)$ $= \mathfrak{S}(a, \mathscr{U}(b_\lambda), a_\beta)$ and β must be the dual atom and λ must be the atom in L. Otherwise there exists an element G greater than F and less than 1. Which is a contradiction to the hypothesis.

Case 2.
Let X be a non empty set and L be a finite Boolean lattice.

If $\mathfrak{S} = \mathfrak{S}(a, \mathscr{U}(b_\lambda)) = \{f|f(a) = 0\} \cup \{f|f \geqslant b_\lambda\}$, where $a, b \in X, \lambda$ is an atom, then a principal ultra L-topology $\mathfrak{S}_{\beta j}(a, \mathscr{U}(b_\lambda)) = \mathfrak{S}_{\beta j}$ $=L$-topology generated by any $(m-1)$ $\mathfrak{S}(a_{\beta i})$ among m $\mathfrak{S}(a_{\beta i}), i = 1, 2, ...m, j = 1, 2, ...m, i \neq j$ if there are m dual atoms $\beta_1, \beta_2, ..., \beta_m$, where $\mathfrak{S}(a_{\beta i})$ is the simple extension of $\mathfrak{S}$ by $(a_{\beta i})$, i.e, $\mathfrak{S}(a_{\beta i}) = \{f \vee (g \wedge a_{\beta i}), f, g \in \mathfrak{S}, a_{\beta i} \notin \mathfrak{S}\}$. So $\mathfrak{S}(\beta_j)$ is a dual atom in the lattice of principal L topologies. Since the simple extension of $\mathfrak{S}(\beta_j)$ by the L point $a_{\beta j}$ is 1(discrete L-topology), by theorem 6.3.2, 1 is a cover of $\mathfrak{S}_{\beta j}$.

Suppose that F is a dual atom in $\beta(X)$. Then F is of the form $\mathfrak{S}_{\beta j} = \mathfrak{S}_{\beta j}(a, \mathscr{U}(b_\lambda))$ and $\beta_1, \beta_2, ...$ must be dual atoms and λ must be atom in L. Otherwise there exists an element G greater than F and less than 1. Which is a contradiction to the assumption that F is a dual atom in $\beta(X)$. So in either case if $\beta(X)$ has dual atoms, then L has dual atoms and atoms . Hence the proof of the theorem is completed □

Theorem 6.3.7. *If L is a finite pseudo complemented chain or a Boolean lattice, then $\beta(X)$ has dual atoms.*

Proof. **Case 1.**
Let X be a non empty set and L be a finite pseudo complemented chain.

Since L is a finite pseudo complemented chain, it has atom and dual atom. Let τ be a dual atom in the lattice of principal topologies on X. Then by theorem 6.3.1, τ must be of the form $\mathfrak{S}(a, \mathscr{U}) = \wp(X - a) \cup \mathscr{U}$, where $a \in X, \mathscr{U}$ is an ultrafilter not containing $\{a\}$. Since τ is a principal topology, $\mathscr{U}$ is a principal ultra filter so that $\tau = \mathfrak{S}(a, \mathscr{U}(b)) = \wp(X-a) \cup \mathscr{U}(b)$. Then $\omega_{PL}(\tau) = \{f \geqslant b_\lambda | f : (X, \tau) \to L \text{ is a scott continuous function}\}, b \in X$ and λ is an atom in L. Then $a_\alpha \notin \omega_{PL}(\tau)$ for $0 \neq \alpha \in L$. Let β be the dual atom in L and $F = \omega_{PL}(\tau) \vee a_\beta$. Then F is the ultra L-topology $\mathfrak{S}(a_\beta)$ in $\beta(X)$ since the simple extension of F by a_1 is the discrete L-topology.

Case 2.

Let X be a non empty set and L be a finite Boolean lattice.

Since L is a Boolean lattice, it has atoms and dual atoms. Let τ be a dual atom in the lattice of principal topologies on X. Then by theorem 6.3.1, τ must be of the form $\mathfrak{S}(a, \mathscr{U}) = \wp(X - a) \cup \mathscr{U}$, where $a \in X, \mathscr{U}$ is an ultrafilter not containing $\{a\}$. Since τ is a principal topology, $\mathscr{U}$ is a principal ultra filter so that $\tau = \mathfrak{S}(a, \mathscr{U}(b)) = \wp(X - a) \cup \mathscr{U}(b)$. Then $\omega_{PL}(\tau) = \{f \geqslant b_\lambda | f : (X, \tau) \to L \text{ is a scott continuous function}\}, b \in X$ and λ is an atom. Then $a_\alpha \notin \omega_{PL}(\tau)$ for $0 \neq \alpha \in L$. Let $\beta_1, \beta_2, ..., \beta_m$ are dual atoms in L and $F(a_{\beta 1}) = \omega_{PL}(\tau) \vee a_{\beta 1}, F(a_{\beta 2}) = \omega_{PL}(\tau) \vee a_{\beta 2}, ..., F(a_{\beta m}) = \omega_{PL}(\tau) \vee a_{\beta m}$. Let $F_{\beta j}$ is the L-topology generated by $(m-1)$ $F(a_{\beta i})$ from m $F(a_{\beta i}), i = 1, 2, ...m, j = 1, 2, ...m, i \neq j$. Then as in Case 1. $F_{\beta j}$ is the ultra L-topology $\mathfrak{S}_{\beta j}$ in $\beta(X)$ since the simple extension of $F_{\beta j}$ by $a_{\beta j}$ is the discrete L-topology.

In both cases since L has dual atoms and atoms, $\beta(X)$ has dual atoms(Ultra L-topology). Hence the theorem. □

Theorem 6.3.8. *Lattice of principal L-topologies $\beta(X)$, on a set X*

is not dually atomic in general.

Proof. This follows from Theorem 6.3.7. $\square$

Proposition 6.3.1.
If τ is a dual atom in Π, then $\omega_{PL}(\tau)$ is not a dual atom in $\beta(X)$, in general.

Proof. For example, take X as any set and $L = \{0, \alpha, 1\}$, where $0 < \alpha < 1$. Since τ is a dual atom in the lattice of principal topologies, the only principal topology greater than τ is the discrete topology on X. Suppose A is a subset of X such that $A \notin \tau$. Then $\omega_{PL}(\tau)$ consists of all Scott continuous functions from (X, τ) to L and the characteristic function of A does not belong to $\omega_{PL}(\tau)$. Thus $\omega_{PL}(\tau) < \omega_{PL}(\tau) \vee \{\underline{0}, \underline{1}, \chi_A^\alpha\} < 1$(discrete topology), where $\chi_A^\alpha(x) = \begin{cases} \alpha & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$
Thus $\omega_{PL}(\tau)$ is not a dual atom in $\beta(X)$. $\square$

Proposition 6.3.2.
If τ is a dual atom in Π, then $\omega_{PL}(\tau)$ is a dual atom in $\beta(X)$ if and only if $L = \{0, 1\}$.

Proof. Can be easily proved. $\square$

6.4 Complementation problem in the lattice of principal L- topologies

Theorem 6.4.1. *Let X be a non empty set and L be a F-lattice. Then the lattice of principal L-topologies $\beta(X)$ is not complemented in general.*

Proof. Let $F = \{\underline{0}, \underline{1}, h\}$, where h is not a characteristic function of an open set in X be a principal L topology. We claim that F has no complement. Here 1 is not a complement of F. Let G be any principal L-topology in $\beta(X)$ other than 1. If $h \in G$, then G is not a complement of F, since $F \wedge G \neq 0$. Suppose that $h \notin G$. Clearly G cannot contain simultaneously all characteristic functions of subsets of X and all constant L-subsets. The set $K = \{k | k$ is a function from X to L and $k \notin G\}$ is non empty and two cases arise:
(i) K contains constant L subsets (ii) K contains at least one characteristic function corresponding to a subset of X. In either case $H = \{f \wedge g | f \in F, g \in G\}$ is a base for $P = F \vee G$. Then at least one subset of K is not contained in P. Hence $P \neq 1$,i.e., G is not a complement of F. □

Remark 6.4.1. Let X be a non empty set and $L = \{0, \alpha, 1\}$ ordered by $0 < \alpha < 1$. Then $F = \{\underline{0}, \underline{1}, \underline{\alpha}\}$ is an atom in $\beta(X)$ and $\underline{\alpha}$ is not a characteristic function. Let $G = \{\underline{0}, \underline{1}\} \cup \{\chi_A : A \subset X\}, \chi_A$ is the characteristic function of A. Clearly G is a principal L-topology. Then $F \wedge G = \{\underline{0}, \underline{1}\}$ and $F \vee G = 1$. Hence G is a complement of F.

Theorem 6.4.2. *If F is any principal L-topology on X such that the topology corresponding to the characteristic functions in F is neither* *discrete nor indiscrete, then F has atleast one join complement.*

Proof. Let τ be the principal topology corresponding to the characteristic function in F. Since the lattice of principal topologies on X is complemented [58], we can find a principal topology τ' such that $\tau \wedge \tau' = 0$ and $\tau \vee \tau' = 1$ in the lattice of principal topologies. Then the principal L-topology generated by $F \cup \omega_{PL}(\tau') = 1$ in $\beta(X)$ where $\omega_{PL}(\tau') = \{f \in L^X | f : (X, \tau') \to L$ is scott continuous $\}$. Hence the proof. $\square$

Theorem 6.4.3. *If F is any induced principal L-topology on X, then F has atleast one complement in $\beta(X)$.*

Proof. Since F is induced there exists a principal topology τ in the lattice of principal topologies such that $\omega_{PL}(\tau) = F$. Since the lattice of principal topologies is complemented [58], there exist atleast one topology τ' in the lattice of principal L-topologies such that $\tau \wedge \tau' = 0$ and $\tau \vee \tau' = 1$ in the lattice of principal L-topologies. Then $F \vee \omega_{PL}(\tau') = 1$ and $F \wedge \omega_{PL}(\tau') = 0$ in $\beta(X)$. $\square$

Remark 6.4.2. For a given principal topology τ, the family F_τ of all principal L-topologies defined by families of Scott continuous functions from (X, τ) to L, form a lattice under the natural order of set inclusion. From this lattice, we can deduce the properties of $\beta(X)$.

Chapter 7

Lattice of Weakly Induced Principal L-topologies

7.1 Introduction

The concept of induced fuzzy topological space was introduced by Weiss [75]. Lowen called these spaces a topologically generated spaces. Martin [38] introduced a generalized concept, weakly induced spaces, which was called semi induced space by Mashhour et al. [40]. The notion of lower semi continuous functions plays an important tool in defining the above concepts. In ([24],[5]), Aygun et al. introduced a new class of functions from a topological space (X, τ) to a fuzzy lattice(F-lattice) L with its

* Some results of this chapter are included in the following paper.
Raji George and T.P. Johnson : The lattice structure of weakly induced principal L-topologies. Annals of Fuzzy Mathematics and Informatics, Volume 4, Number 2, 2012

scott topology called (completely) scott continuous functions, as a generalization of (completely) lower semi continuous functions from (X, τ) to $[0, 1]$.

It is known that [30] lattice of L-topologies is complete, atomic and not complemented. In [31], Jose and Johnson generalized weakly induced spaces introduced by Martin [38] using the tool (completely) scott continuous functions and studied the lattice structure of the set $W(X)$ of all weakly induced L-topologies on a given set X. A related problem is to find subfamilies in $W(X)$ having certain properties. The collection of all weakly induced principal L topologies $W_P(X)$ form a lattice with the natural order of set inclusion. The concept of principal topologies in the crisp case was studied by Steiner [58]. The lattice of principal topologies is both atomic and dually atomic. Analogously we study the lattice structure of the set of all weakly induced principal L-topologies on a given set X. Here we study properties of the lattice $W_{P\tau}$ of all weakly induced principal L topologies defined by families of (completely) scott continuous functions with reference to τ on X. From the lattice $W_{P\tau}$ we deduce the lattice $W_P(X)$ of all weakly induced principal L-topologies on X. It is join complemented. Also we prove that if L is a finite pseudocomplemented chain or a complemented F-lattice, then $W_P(X)$ has dual atoms and if L has neither dual atoms nor atoms, then $W_P(X)$ has no dual atoms.

7.2 Preliminaries

Let X be a nonempty ordinary set and $L = (\leqslant, \vee, \wedge, ')$ be a completely distributive lattice with smallest element 0 and largest element 1, $0 \neq 1$, and with an order reversing invalution $a \to a'$ $(a \in L)$ called a F-lattice. We identify the constant function from X to L with value α by $\underline{\alpha}$. The fundamental definition of L-fuzzy set theory and L-topology are assumed to be familiar to the reader in the sense of Chang [13].

A topological space is called principal if it is discrete or if it can be written as the meet of principal ultra topologies. Steiner [58] proved that this is equivalent to requiring that the arbitrary intersection of open sets is open. Analogously we define principal L-topology as

Definition 7.2.1. An L-topology is called principal L-topology if arbitrary intersection of open L subsets is an open L subset.

Definition 7.2.2. [34] An element of a lattice L is called an atom if it is the minimal element of $L \backslash \{0\}$.

Definition 7.2.3. [34] An element of a lattice L is called a dual atom if it is the maximal element of $L \backslash \{1\}$.

Definition 7.2.4. [15] A lattice is said to be bounded if it possess smallest element 0 and largest element 1.

Definition 7.2.5. [34] A bounded lattice L is said to be join complemented if for all x in L, there exists y in L such that $x \vee y = 1$.

Definition 7.2.6. [34] A bounded lattice L is said to be meet complemented if for all x in L, there exist y in L such that $x \wedge y = 0$.

Definition 7.2.7. [34] A bounded lattice is said to be complemented if it is both join complemented and meet complemented.

Definition 7.2.8. [34] A bounded lattice L is said to be semi-complemented if it is either join complemented or meet complemented.

Definition 7.2.9. [22] An element p of L is called prime if $p \neq 1$ and whenever $a, b \in L$ with $a \wedge b \leqslant p$, then $a \leqslant p$ or $b \leqslant p$. The set of all prime elements of L will be denoted by Pr (L).

Definition 7.2.10. [73] The scott topology on L is the topology S, generated by the sets of the form $\{t \in L : t \nleqslant p\}$ where $p \in$ pr (L). Let (X, τ) be a topological space and $f : (X, \tau) \to L$ be a function, where L has its scott topology. We say that f is scott continuous if for every $p \in \mathrm{Pr}\,(L), f^{-1}\{t \in L : t \nleqslant p\} \in \tau$.

Remark 7.2.1. When $L = [0, 1]$, the scott topology coincides with the topology of topologically generated spaces of Lowen [35].
The set $\omega_L(\tau) = \{f \in L^X; f : (X, \tau) \to L$ is scott continuous $\}$ is an L-topology. It is the largest element in W_τ, where W_τ is the lattice of weakly induced L-topologies defined by families of scott continuous functions with reference to τ on X. If τ is a principal topology $\omega_L(\tau)$ is a principal L-topology, we can denote it by $\omega_{PL}(\tau)$. An L-topology F on X is called an induced principal L-topology if there exist a principal topology τ on X such that $F = \omega_{PL}(\tau)$.

Definition 7.2.11. ([24], [5]) Let (X, τ) be a topological space and $a \in X$. A function $f : (X, \tau) \to L$, where L has its scott topology, is said to be completely scott continuous at $a \in X$ if for every $p \in$ Pr (L) with $f(a) \nleqslant p$, there is a regular open neighbourhood U of a in (X, τ) such that $f(x) \nleqslant p$ for every $x \in U$. That is $U \subset f^{-1}(\{t \in L : t \nleqslant p\})$ and f is called completely scott continuous on X, if f is completely scott continuous at every point of X.

Note.

Let F be a principal L-topology on the set X, let F_c denote the 0-1 valued members of F, that is, F_c is the set of all characteristic mappings in F. Then F_c is a principal L-topology on X. Define $F_c^* = \{A \subset X : \mu_A \in F_c$ where μ_A is the characteristic function of $A\}$. The principal L-topological space (X, F_c) is same as the principal topological space (X, F_c^*).

Definition 7.2.12. A principal L-topological space (X, F) is said to be a weakly induced principal L topological space, if for each $f \in F$, f is a scott continuous function from (X, F_c^*) to L.

Definition 7.2.13. If F is the collection of all scott continuous functions from (X, F_c^*) to L, then F is an induced space and $F = \omega_{PL}(F_c^*)$.

7.3 Lattice of weakly induced principal L-topologies

For a given principal topology τ on X, the family W_{P_τ} of all weakly induced principal L-topologies defined by families of scott continuous functions from (X, τ) to L forms a lattice under the natural order of set inclusion. The least upperbound of a collection of weakly induced principal L-topologies belonging to W_{P_τ} is the weakly induced principal L-topology which is generated by their union and the greatest lowerbound is their intersection. The smallest element is the indiscrete L-topology, denoted by 0 and the largest element is denoted by $1 = \omega_{PL}(\tau)$.

Also for a principal topology τ on X, the family CW_{P_τ} of all weakly induced principal L topologies defined by families of completely scott continuous function from (X, τ) to L forms a lattice under the natural order of set inclusion. Since every completely scott continuous function is scott continuous, it follows that $CW_{P\tau}$ is a sublattice of $W_{P\tau}$. We note that $W_{P\tau}$ and $CW_{P\tau}$ coincide when each open set in τ is regular open.

When $\tau = D$, the discrete topology on X, these lattices coincide with lattice of weakly induced principal L-topologies on X.

Theorem 7.3.1. *[18] The Ultra spaces on a set E are exactly the topologies of the form $\mathfrak{S}(x, \mathscr{U}) = \wp(E - x) \cup \mathscr{U}$, where $x \in E, \mathscr{U}$ is an ultrafilter on E not containing $\{x\}$.*

Theorem 7.3.2. *The lattice $W_{P\tau}$ is complete.*

Proof. Let S be a subset of $W_{P\tau}$ and let $G = \bigcap_{F \in S} F$. Clearly G is a principal L-topology. Let $g \in G$. Since each $F \in S$ is a weakly induced principal L topology, g is a scott continuous mapping from $(X, (F_c^*)$ to L, that is $g^{-1}\{t \in L : t \nleq p$ where $p \in Pr(L)\} \in F_c^*$ for each $F \in S$. Therefore $g^{-1}\{t \in L : t \nleq p$ where $p \in \Pr L\} \in \bigcap_{F \in S}(F_c^*$. Hence g is a scott continuous function from (X, G_c^*) to L, where $(X, G_c^*) = (X, \bigcap_{F \in S} F_c^*)$. That is $G \in W_{P\tau}$ and G is the greatest lower bound of S. Let K be the set of upperbounds of S. Then K is non empty, since $1 = \omega_{PL}(\tau) \in K$.

Using the above argument K has a greatest lowerbound, say H, then this H is a least upper bound of S. Thus every subset S of $W_{P\tau}$ has greatest lowerbound and least upperbound. Hence $W_{P\tau}$ is complete. $\square$

Theorem 7.3.3. *$W_{P\tau}$ is not atomic in general.*

Proof. Atoms in $W_{P\tau}$ are either of the form $\{\underline{0}, \underline{1}, \underline{\alpha}\}$ or $\{\underline{0}, \underline{1}, \mu_A\}$, where μ_A is the characteristic function of open subsets A of (X, τ) and $\alpha \in (0, 1)$. Let $X = \{a, b, c\}$, $\tau = \{\phi, X, \{a\}, \{a, b\}\}$ and $F = \{\underline{0}, \underline{1}, \mu_{\{a\}}, \mu_{\{a,b\}},$

$$\left.\begin{array}{rrrrrr} f : a \to .6 & g : a \to 1 & h : a \to 1 & i : a \to 1 & j : a \to .6 & k : a \to .6 \\ b \to .5 & b \to 1 & b \to .5 & b \to .5 & b \to .5 & b \to 0 \\ c \to .4, & c \to .4, & c \to .4, & c \to 0, & c \to 0, & c \to 0 \end{array}\right\}$$

$F_c = \{\underline{0}, \underline{1}, \mu_{\{a\}}, \mu_{\{a,b\}}\}$. $F_c^* = \{\phi, X, \{a\}, \{a, b\}\} = \tau$ and $F \in W_{P\tau}$. But this F cannot be expressed as join of atoms. Hence $W_{p\tau}$ is not atomic. □

Theorem 7.3.4. *$W_{P\tau}$ is not distributive.*

Proof. Since every distributive lattice is necessarily modular, we prove that $W_{P\tau}$ is not modular. This can be illustrated with an example.
Let X be an infinite set and τ be the discrete topology D on X. Then $W_{P\tau}$ becomes lattice of all weakly induced principal L-topologies on X and $\Pi(X)$, the lattice of principal topologies on X(identifying its charecteristic functions) is a sublattice of W_{PD}. We know that $\Pi(X)$ is not modular and hence not distributive. Thus $W_{P\tau}$ is not distributive in general. □

Theorem 7.3.5. *If L is a finite pseudo complemented chain or a complemented F-lattice, then $W_P(X))$ has dual atoms.*

Proof. **case 1.**
Let X be a non empty set and L be a finite pseudo complemented chain.

Since L is a finite pseudo complemented chain, it has atom and dual atom. Let τ be a dual atom in the lattice of principal topologies on X.

Then by theorem 7.3.1, τ must be of the form $\mathfrak{S}(a, \mathscr{U}) = \wp(X - a) \cup \mathscr{U}$, where $a \in X, \mathscr{U}$ is an ultrafilter not containing $\{a\}$. Since τ is a principal topology, $\mathscr{U}$ is a principal ultra filter so that $\tau = \mathfrak{S}(a, \mathscr{U}(b)) = \wp(X - a) \cup \mathscr{U}(b)$. Then $\omega_{PL}(\tau) = \{f \geqslant b_\lambda | f : (X, \tau) \to L$ is a scott continuous function from (X, τ) to $L\}, b \in X$ and λ is an atom in L . Then $a_\alpha \notin \omega_{PL}(\tau)$ for $0 \neq \alpha \in L$. Let β be the dual atom in L and $F(a_\beta) = \omega_{PL}(\tau) \vee a_\beta$. Then $F(a_\beta)$ is the ultra L-topology $\mathfrak{S}(a_\beta)$ in $\beta(X)$ since the simple extension of $F(a_\beta)$ by a_1 is the discrete L-topology. Let $G = F(a_\beta)$, $G_c =$ the $0 - 1$ valued functions in G and $G_c^* = \{A \subset X | \mu_A \in G_c\}$. Then the weakly induced principal L-topology defined by Scott continuous funcions from $(X, (G_c^*)$ to L is a dual atom in $W_P(X)$.

Case 2.

Let X be a non empty set and L is a finite complemented F-lattice.

Since L is a complemented F-lattice, it has atoms and dual atoms. Let τ be a dual atom in the lattice of principal topologies on X. Then by theorem 7.3.1, τ must be of the form $\mathfrak{S}(a, \mathscr{U}) = \wp(X - a) \cup \mathscr{U}$, where $a \in X, \mathscr{U}$ is an ultrafilter not containing $\{a\}$. Since τ is a principal topology, $\mathscr{U}$ is a principal ultra filter so that $\tau = \mathfrak{S}(a, \mathscr{U}(b)) = \wp(X - a) \cup \mathscr{U}(b)$. Then $\omega_{PL}(\tau) = \{f \geqslant b_\lambda | f : (X, \tau) \to L$ is a scott continuous function$\}, b \in X$ and λ is an atom. Then $a_\alpha \notin \omega_{PL}(\tau)$ for $0 \neq \alpha \in L$. Let $\beta_1, \beta_2, ..., \beta_m$ are dual atoms in L and $F(a_{\beta 1}) = \omega_{PL}(\tau) \vee a_{\beta 1}, F(a_{\beta 2}) = \omega_{PL}(\tau) \vee a_{\beta 2}, ..., F(a_{\beta m}) = \omega_{PL}(\tau) \vee a_{\beta m}$. Let $F_{\beta j}$ is the principal L-topology generated by $(m-1)$ $F(a_{\beta i})$ from m $F(a_{\beta i}), i = 1, 2, ...m, j = 1, 2, ...m, i \neq j$. Then $F_{\beta j}$ is the ultra L-topology $\mathfrak{S}_{\beta j}$ in $\beta(X)$ since the simple extension of $F_{\beta j}$ by $a_{\beta j}$ or a_1 is the discrete L-topology. Take $G = F_{\beta j}$ and let $G_c =$ the $0 - 1$ valued functions in G and $G_c^* = \{A \subset X | \mu_A \in G_c\}$. Then the weakly induced principal L-topology defined by Scott continuous functions from

(X, G_c^*) to L is a dual atom in $W_P(X)$
In both cases, $W_P(X)$ has dual atoms. Hence the theorem. □

Theorem 7.3.6. *If L has neither dual atoms nor atoms, then $W_P(X)$ has no dual atoms.*

Proof. Let F be any weakly induced principal L-topology other than 1. Then we claim that there exists at least one weakly induced principal L-topology finer than F. Since F is a weakly induced principal L-topology different from 1, F cannot contain all characteristic functions of subsets of X. Since L has neither dual atoms nor atoms, the collection S of Scott continuous functions not belonging to F is infinite. If $g \in S$, then $F(g)$, the simple extension of F by g is a principal L-topology. Take $G = F(g)$. Let G_c denote the $0-1$ valued members of G and $G_c^* = \{A \subset X | \mu_A \in G_c\}$, where μ_A is the characteristic function of A. Then there exists a weakly induced principal L-topology H defined by Scott continuous functions from (X, G_c^*) to L. Thus for any weakly induced principall L-topology F there exists a weakly induced principal L-topology H such that $F \subset H \neq 1$. Hence the proof of the theorem is completed. □

Theorem 7.3.7. *The lattice $W_P(X)$ of all weakly induced principal L-topologies on any set X is not dually atomic in general.*

Proof. This follows from theorem 7.3.6. □

7.4 Complementation problem in the lattice of weakly induced principal L-topologies

Proposition 7.4.1
If L has no dual atoms, then atoms in $W_{P\tau}$ of the form $\{\underline{0}, \underline{1}, \underline{\alpha}\}$ have no complements in $W_{P\tau}$.

Proof. Let $F = \{\underline{0}, \underline{1}, \underline{\alpha}\}$ be atom in $W_{P\tau}$. We claim that F has no complement. 1 is not a complement of F since $1 \wedge F \neq 0$. Let P be a weakly induced principal L-topology in $W_{P\tau}$ other than 1. If $F \subset P$, then P cannot be the complement of F, since $F \wedge P \neq 0$. If $F \not\subseteq P$, let $F \vee P = G$ and G has the subbase $\{f \wedge p | f \in F, p \in P\}$. Then G cannot be equal to 1. Hence P is not a complement of F. □

Remark 7.4.1. The above proposition is not true for an arbitrary lattice L. For example, take $L = \{0, \alpha, 1\}$ ordered by $0 < \alpha < 1$. If (X, τ) is a principal L topological space and $K = \{\underline{0}, \underline{1}, \underline{\alpha}\}$, then clearly K is an atom in $W_{P\tau}$, when $\underline{\alpha}$ is not a characteristic function. Let $H = \{\underline{0}, \underline{1}\} \cup \{\mu_A : A \in \tau\}$. Then H is an element of $W_{P\tau}$ and $K \wedge H = 0$ and $K \vee H = 1$. Hence H is a complement of K.

Theorem 7.4.1. *$W_{P\tau}$ is not complemented.*

Proof. This follows from the Proposition 7.4.1. □

Remark 7.4.2. When $\tau = D$, the discrete topology on X then $W_{PD} = W_P(X)$, the collection of all weakly induced principal L-topologies on X. Let Δ denote the family of all weakly induced principal L-topologies defined by scott continuous functions where each scott continuous function is a characteristic function. Then Δ is a sublattice of $W_P(X)$ and is a lattice isomorphic to the lattice of all principal topologies on X. The elements of Δ are called crisp principal topologies.

Theorem 7.4.2. *The lattice of weakly induced principal L-topologies $W_P(X)$ is not complemented.*

Proof. This follows from theorem 7.4.1. □

Theorem 7.4.3. *Every atom in $W_P(X)$ of the form $\{\underline{0}, \underline{1}, \mu_A\}$ has complement.*

Proof. Let $F = \{\underline{0}, \underline{1}, \mu_A\}$. Then F is an element of Π, lattice of principal topologies on X. Since Π is complemented there exists τ in Π such that $\tau \vee F$ equal to the discrete principal topology and $\tau \wedge F$ equal to the indiscrete principal topology on X. Then $F \vee \omega_{PL}(\tau) = 1 = \omega_{PL}(D)$ and $F \wedge \omega_{PL}(\tau) = 0$. □

Theorem 7.4.4. *The lattice $W_P(X)$ of all weakly induced principal L-topologies on any set X is semi complemented.*

Proof. Let $F \in W_P(X)$. Since F is weakly induced principal L-opology, there is a principal topology $\tau = F_c^*$ on X such that each element $f \in F$ is a scott continuous function from (X, F_c^*) to L. Since the lattice of principal topologies is complemented, we can find a principal topology τ' such that $F \vee \omega_{PL}(\tau') = 1 = \omega_{PL}(D)$ where D is a discrete topology and $F \wedge \omega_{PL}(\tau')$ need not be equal to 0, the indiscrete principal L-topology on X. Thus every F in $W_P(X)$ has a join complement. Hence $W_P(X)$ is semi complemented. $\square$

Chapter 8

Lattice of Stratified Principal L-topologies

8.1 Introduction

In this chapter we investigate the lattice structure of the set of all stratified principal L-topologies on a given set X. In [11], Birkhoff described a technique of comparison of topologies and noted that the set of all topologies on a fixed set forms a complete lattice with the natural order of set inclusion. In [24], Ayg$\ddot{u}$n, Warner and Kundri introduced a new class of functions from a topological space (X, τ) to a fuzzy lattice L with its Scott topology called Scott continuous functions as a generalization of lower semi continuous functions from (X, τ) to $[0, 1]$. It is known [30] that

* Some results of this chapter are included in a paper accepted for publication in International journal of Fuzzy Information and Engineering, Springer.

the lattice of L-topologies on a given set X is complete and atomic. In [32], Jose and Johnson studied the lattice structure of the set $L(X)$ of all stratified L-topologies on a given set X. A related problem is to find subfamilies in $L(X)$ having certain properties. The collection of all stratified principal L-topologies $S_P(X)$ forms a lattice with the natural order of set inclusion. The concept of principal topology in the crisp case was studied by Steiner [58]. The lattice of principal topologies is both atomic and dually atomic. Analogously, we study the lattice structure of the set of all stratified principal L-topologies $S_P(X)$ on a given set X. This lattice has atoms if and only if the membership lattice L has atoms. If the lattice $S_P(X)$ has dual atoms, then L has dual atoms and atoms. Also if L is a finite pseudocomplemented chain or a Boolean lattice, then $S_P(X)$ has dual atoms. It is also complete and join-complemented.

8.2 Preliminaries

Let X be a non empty set and L be a completely distributive lattice with an order reversing involution called F-lattice [34]. We denote the constant function in L^X taking the value $\alpha \in L$ by $\underline{\alpha}$. Here we call L-fuzzy subsets as L-subsets and a subset F of L^X is called an L-topology in the sense of Chang [13] and Goguen [23] as in [34], if

i. $\underline{0}, \underline{1} \in F$
ii. $f, g \in F \Rightarrow f \wedge g \in F$
iii. $f_i \in F$ for each $i \in I \Rightarrow \vee_{i \in I} f_i \in F$.

A subset F of L^X is called a stratified L-topology, if

i. $\underline{\alpha} \in F$ for all $\alpha \in L$

ii. $f, g \in F \Rightarrow f \wedge g \in F$

iii. $f_i \in F$ for each $i \in I \Rightarrow \vee_{i \in I} f_i \in F$.

(The idea goes up to Lowen [35], while the term "stratified" has appeared for the first time in [42]). Steiner [58] proved that a topology τ is a principal topology if and only if arbitrary intersections of open sets are open(such kind of spaces are also called Alexandroff spaces [1]). Analogously, we define principal L-topology.

Definition 8.2.1. An L-topology is called principal L-topology provided that arbitrary intersections of open L-subsets are open L-subsets.

Example 8.2.1. Let X be an infinite set. Then $F = \{f \in L^X : \underline{\alpha} \leqslant f\}$ together with $\underline{0}$, where $x \in X$ and α is an atom in L, is a stratified principal L-topology.

Example 8.2.2. Let $X = R$ and $F = \{f \in L^X : f(x) > 0$ for all but finite number of points of $X\}$ together with $\underline{0}$. Then F is a stratified L-topology, which is not a principal L-topology.

Definition 8.2.2. A principal L-topology is called stratified principal L-topology provided that it contains every constant L-subset .

Definition 8.2.3. An element p of L is called prime if $p \neq 1$ and whenever $a, b \in L$ with $a \wedge b \leqslant p$, then $a \leqslant p$ or $b \leqslant p$. The set of all prime elements of L will be denoted $Pr(L)$.

Definition 8.2.4. [73] The Scott topology on L is the topology S, generated by the sets of the form $\{t \in L : t \nleq p$ where $p \in Pr(L)\}$. Let

(X, τ) be a topological space and let L be a fuzzy lattice. $f : (X, \tau) \to L$ is said to be *Scott continuous* if $f : (X, \tau) \to (L, S)$ is continuous, i.e., if for every $p \in Pr(L)$, $f^{-1}\{t \in L : t \nleq p\} \in \tau$.

Remark 8.2.1. When $L = [0, 1]$, the Scott topology coincides with the topology of topologically generated spaces of Lowen [35]. The set $\omega_L(\tau) = \{f \in L^X | f : (X, \tau) \to L \text{ is Scott continuous}\}$ is a stratified L-topology. If τ is a principal topology, then $\omega_L(\tau)$ is a stratified principal L-topology, which is denoted $\omega_{PL}(\tau)$. A stratified principal L-topology F on X is called induced provided that there exists a principal topology τ on X such that $F = \omega_{PL}(\tau)$.

8.3 Lattice of Stratified Principal L-topologies

Let $S_P(X) = \{F | F \text{ is a stratified principal } L\text{-topology on } X\}$ and Π is the lattice of principal topologies on X. The family $S_P(X)$ of all stratified principal L-topologies forms a lattice under the natural order of set inclusion. The smallest stratified L-topology is the indiscrete L-topology, with all constant L-subsets, is denoted 0 and the largest stratified principal L-topology is the discrete L-topology, consisting of all L-subsets and is denoted 1.

Definition 8.3.1. [10] A lattice L is said to be join-complemented provided that for every x in L, there exists y in L such that $x \vee y = 1$.

Definition 8.3.2. [10] A lattice L is said to be meet-complemented provided that for every x in L, there exists y in L such that $x \wedge y = 0$.

Definition 8.3.3. [10] A lattice L is said to be complemented provided that for every x in L, there exists y in L such that $x \wedge y = 0$ and $x \vee y = 1$.

Definition 8.3.4. [10] A lattice L is said to be semi-complemented provided that it is either join-complemented or meet-complemented.

Theorem 8.3.1. *[18] The Ultra spaces on a set E are exactly the topologies of the form $\mathfrak{S}(x, \mathscr{U}) = \wp(E - x) \cup \mathscr{U}$, where $x \in E, \mathscr{U}$ is an ultrafilter on E not containing $\{x\}$.*

Theorem 8.3.2. *[58] The lattice of topologies Σ on a set E is distributive if E has fewer than three elements. If E has three or more elements, Σ is not even modular.*

Theorem 8.3.3. *[58] The lattice Π of principal topologies is a complemented lattice.*

Theorem 8.3.4. *The lattice of stratified principal L-topologies $S_P(X)$ on a set X is complete.*

Proof. Let K be any subset of $S_P(X)$. Then K has the greatest lower bound and the least upper bound, since arbitrary intersections of stratified principal L-topologies are stratified principal L-topologies and $S_P(X)$ has the greatest element 1. $\square$

Theorem 8.3.5. *The collection $S'_P(X)$ of all induced stratified principal L-topologies on any set X is a complete sublattice of the complete lattice $S_P(X)$.*

Proof. Clearly $S'_P(X)$ is a subset of $S_P(X)$. Let $F, G \in S'_P(X)$. Then there exists topologies τ and $\tau^{'}$ in Π such that $F = \omega_{PL}(\tau)$ and $G = \omega_{PL}(\tau^{'})$. Then $F \vee G = \omega_{PL}(\tau \vee \tau^{'})$ and $F \wedge G = \omega_{PL}(\tau \wedge \tau^{'})$. Hence $F \vee G$ and $F \wedge G$ are in $S_P^{'}(X)$ so that $S_P^{'}(X)$ is a sublattice of $S_P(X)$.

Let H be any subset of $S_P^{'}(X)$. Then H has the greatest lower bound since arbitrary intersections of principal topologies are principal topologies so that arbitrary intersections of induced stratified principal L-topologies are induced stratified principal L-topologies.

Let K be the set of upper bounds of H. Then K is nonempty, since $1 \in K$. Using the above argument, K has the greatest lower bound, say M. Then this M is the least upper bound of H. Thus every subset H of $SP^{'}(X)$ has the greatest lower bound and least upper bound. Hence $S_P^{'}(X)$ is a complete sublattice of $S_P(X)$. □

Proposition 8.3.1[72]
For a stratified L-topology $(X, \omega_L(\tau))$, the family $\beta = \{f_A^\alpha | A \in \tau, \alpha \in L\}$ where $f_A^\alpha(x) = \begin{cases} \alpha & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$
is a base for $\omega_L(\tau)$.
Proposition 8.3.2. [72]
For a stratified L-topology $(X, \omega_L(\tau))$, the family $S = \{\mu_A | \mu_A$ is the characteristic function of the open set A in $\tau\} \cup \{\underline{\alpha} | \alpha \in L\}$ is a subbase for $\omega_L(\tau)$

Theorem 8.3.6. *The collection $S_P^{'}(X)$ of all induced stratified principal L-topologies on any set X forms a lattice isomorphic to* Π.

Proof. Let X be a nonempty set and L be an F-lattice with its Scott

topology. Define $\theta : \Pi \to S'_P(X)$ by $\theta(\tau) = \omega_{PL}(\tau)$, where τ is a principal topology on X.
Let τ_1 and τ_2 are two principal topologies on X
$\tau_1 \vee \tau_2 =$ principal topology generated by τ_1 and τ_2

$$\begin{aligned}\omega_{PL}(\tau_1)) &= \{f|f \text{ is a Scott continuous function from } (X, \tau_1) \to L\}\\ &= L - \text{topology generated by } \{\mu_A|A \in \tau_1\} \cup \{\underline{\alpha}|\alpha \in L\}\end{aligned}$$

$$\begin{aligned}\omega_{PL}(\tau_2) &= \{f|f \text{ is a Scott continuous function from } (X, \tau_2) \to L\}\\ &= L - \text{topology generated by } \{\mu_A|A \in \tau_2\} \cup \{\underline{\alpha}|\alpha \in L\}\end{aligned}$$

$$\begin{aligned}\omega_{PL}(\tau_1 \vee \tau_2) &= \{f|f\text{is a Scott continuous function from } (X, (\tau_1 \vee \tau_2) \to L\}\\ &= L - \text{topology generated by } \{\mu_A|A \in (\tau_1 \vee \tau_2)\} \cup \{\underline{\alpha}|\alpha \in L\}\end{aligned}$$

$$\begin{aligned}\omega_{PL}(\tau_1) \vee \omega_{PL}(\tau_2) &= \text{stratified principal } L - \text{topology generated by}\\ &\quad\ \omega_{PL}(\tau_1) \text{ and } \omega_{PL}(\tau_2)\\ &= L - \text{topology generated by } \{\mu_A|A \in \tau_1\} \cup\\ &\quad\ \{\mu_A|A \in \tau_2\} \cup \{\underline{\alpha}|\alpha \in L\}\\ &= L - \text{topology generated by } \{\mu_A|A \in (\tau_1 \vee \tau_2)\} \cup \{\underline{\alpha}|\alpha \in L\}\\ &= \{f|f \text{ is a Scott continuous function from } (X, (\tau_1 \vee \tau_2) \to L\}\\ &= \omega_{PL}(\tau_1 \vee \tau_2)\end{aligned}$$

Hence $\theta(\tau_1 \vee \tau_2) = \theta(\tau_1) \vee \theta(\tau_2)$

Similarly

$$\begin{aligned}
\omega_{PL}(\tau_1 \wedge \tau_2) &= \{f|f \text{ is a Scott continuous function from } (X, \tau_1 \wedge \tau_2) \rightarrow L\} \\
&= L - \text{topology generated by } \{\mu_A | A \in \tau_1 \wedge \tau_2\} \cup \{\underline{\alpha} | \alpha \in L\} \\
&= L - \text{topology generated by } \{\mu_A | A \in \tau_1\} \cup \{\underline{\alpha} | \alpha \in L\} \wedge \\
&\quad\; L - \text{topology generated by } \{\mu_A | A \in \tau_2\} \cup \{\underline{\alpha} | \alpha \in L\} \\
&= \{f|f \text{ is a Scott continuous function from } (X, \tau_1) \rightarrow L\} \wedge \\
&\quad\; \{f|f \text{ is a Scott continuous function from } (X, \tau_2) \rightarrow L\} \\
&= \omega_{PL}(\tau_1) \wedge \omega_{PL}(\tau_2)
\end{aligned}$$

Hence $\theta(\tau_1 \wedge \tau_2) = \theta(\tau_1) \wedge \theta(\tau_2)$

$$\begin{aligned}
\tau_1 \neq \tau_2 &\Rightarrow \{f|f : (X, \tau_1) \rightarrow (L, S) \text{ is Scott continuous}\} \neq \{f|f : (X, \tau_2) \rightarrow (L, S) \\
&\qquad \text{is Scott continuous}\} \\
&\Rightarrow \omega_{PL}(\tau_1) \neq \omega_{PL}(\tau_2) \\
&\Rightarrow \theta(\tau_1) \neq \theta(\tau_2)
\end{aligned}$$

Hence θ is one-one. Corresponding to an induced stratified principal L-topology $\omega_{PL}(\tau)$ in $S'_P(X)$, there is a topology τ in Π such that $\theta(\tau) = \omega_{PL}(\tau)$. Hence θ is on to. So θ is an isomorphism. □

Remark 8.3.1. Since $S'_P(X)$ is isomorphic to Π, $S'_P(X)$ possesses all the properties of Π. That is $S'_P(X)$ is complete, atomic, dually atomic, complemented and not modular since Π has these properties [58].

Theorem 8.3.7. *The lattice of stratified principal L-topologies $S_P(X)$ on a set X is not modular.*

Proof. Lattice of principal L-topologies Π is isomorphic to $S'_P(X)$ and Π is not modular [58]. So $S'_P(X)$ is not modular. Since $S'_P(X)$ is a complete sublattice of $S_P(X)$, $S_P(X)$ is not modular. $\square$

Theorem 8.3.8. *If L has atoms, then the lattice of stratified principal L-topologies $S_P(X)$ on a set X has atoms.*

Proof. Let α be an atom in L and let A be a proper subset of X. The stratified principal L-topology of the form F_A^α, where F_A^α is generated by $0\cup f_A^\alpha$, where 0 is the zero element of $S_P(X)$ and $f_A^\alpha(x) = \begin{cases} \alpha & \text{if } x \in A \\ 0 & \text{otherwise} \end{cases}$ for each atom α in L, is an atom in $S_P(X)$. $\square$

Theorem 8.3.9. *[62] Let (X, F) and (X, G) be two fuzzy topological spaces on X. Then G covers F if and only if $G = F(g)$ for every $g \in G-F$, where $F(g)$ is the simple extension of F by g.*

Theorem 8.3.10. *If the lattice of stratified principal L-topologies $S_P(X)$ on a set X has atoms, then L has atoms.*

Proof. Assume L has more than two elements. Let F be an atom in $S_P(X)$. Since F is an atom, F is a cover of 0(zero element of $S_P(X)$). So by theorem 8.3.9 there exists an element g in $F - 0$ such that $F = 0(g)$, the simple extension of 0 by g. i.e $0(g) = \{h\vee(k\wedge g)|h, k \in 0, g \notin 0\}$. This g must be of the form f_A^α, where $A \subset X, \alpha$ is an atom in L. Otherwise we can find a stratified principal L-topology G smaller than F and greater than 0, which is a contradiction to the hypothesis. $\square$

Combining theorem 8.3.8 and theorem 8.3.10, we get the following

Theorem.

Theorem 8.3.11. *The lattice of stratified principal L-topologies $S_P(X)$ on a set X has atoms if and only if L has atoms.*

Remark 8.3.2. Atoms in $S_P(X) = 0(f_A^\alpha)$, where $0 = \{\underline{\lambda} | \lambda \in L\}$. Atoms in $S_P^{'}(X) = \omega_{PL}(\tau)$, where τ is an atom in Π, lattice of principal topologies. Atoms in $S_P^{'}(X)$ and $S_P(X)$ are different. Atoms in $S_P^{'}(X)$ is independent of atoms in L. But $S_P(X)$ has atoms if and only if L has atoms.

Theorem 8.3.12. *The lattice of stratified principal L-topologies $S_P(X)$ on a set X is not atomic in general.*

Proof. Follows from theorem 8.3.11. □

Theorem 8.3.13. *If the lattice of principal L-topologies $S_P(X)$ on a set X has dual atoms, then L has dual atoms and atoms.*

Proof. **Case 1.**
Let X be a non empty set and L be a finite pseudo complemented chain.

If $\mathfrak{S} = \mathfrak{S}(a, \mathscr{U}(b_\lambda)) = \{f | f(a) = 0\} \cup \{f | f \geqslant b_\lambda\}$, then the principal ultra L-topology $= \mathfrak{S}(a, \mathscr{U}(b_\lambda), a_\beta) = \mathfrak{S}(a_\beta)$ is the simple extension of $\mathfrak{S}$ by a_β, i.e., $\mathfrak{S}(a_\beta) = \{f \vee (g \wedge a_\beta), f, g \in \mathfrak{S}, a_\beta \notin \mathfrak{S}\}$, where $a, b \in X, \lambda$ and β are the atom and dual atom in L respectively (from chapter 2). So $\mathfrak{S}(a_\beta)$ is a dual atom in the lattice of principal L topologies. Since the simple extension of $\mathfrak{S}(a_\beta)$ by the L point a_1 is 1(discrete L-topology), by theorem 8.3.9, 1 is a cover of $\mathfrak{S}(a_\beta)$.

Suppose that F is a dual atom in $S_P(X)$. Then F is of the form $\mathfrak{S}(a_\beta)$ $= \mathfrak{S}(a, \mathscr{U}(b_\lambda), a_\beta)$ and β must be the dual atom and λ must be the atom in L. Otherwise there exists an element G greater than F and less than 1. Which is a contradiction to the hypothesis.

Case 2.

Let X be a non empty set and L be a finite Boolean lattice.

If $\mathfrak{S} = \mathfrak{S}(a, \mathscr{U}(b_\lambda)) = \{f|f(a) = 0\} \cup \{f|f \geqslant b_\lambda\}$, where $a, b \in X, \lambda$ is an atom, then a principal ultra L-topology $\mathfrak{S}_{\beta j}(a, \mathscr{U} b_\lambda)) = \mathfrak{S}_{\beta j}$ $=L$-topology generated by any $(m-1)$ $\mathfrak{S}(a_{\beta i})$ among m $\mathfrak{S}(a_{\beta i}), i = 1, 2, ...m, j = 1, 2, ...m, i \neq j$ if there are m dual atoms $\beta_1, \beta_2, ..., \beta_m$, where $\mathfrak{S}(a_{\beta i})$ is the simple extension of $\mathfrak{S}$ by $(a_{\beta i})$, i.e, $\mathfrak{S}(a_{\beta i}) = \{f \vee (g \wedge a_{\beta i}), f, g \in \mathfrak{S}, a_{\beta i} \notin \mathfrak{S}\}$. m can be assumed infinite value (from chapter 2). So $\mathfrak{S}(\beta_j)$ is a dual atom in the lattice of principal L topologies. Since the simple extension of $\mathfrak{S}(\beta_j)$ by the L point $a_{\beta j}$ is 1(discrete L-topology), by theorem 8.3.9, 1 is a cover of $\mathfrak{S}_{\beta j}$.

Suppose that F is a dual atom in $\beta(X)$. Then F is of the form $\mathfrak{S}_{\beta j} =$ $\mathfrak{S}_{\beta j}(a, \mathscr{U} b_\lambda))$ and $\beta_1, \beta_2, ...$ must be dual atoms and λ must be atom in L. Otherwise there exists an element G greater than F and less than 1. Which is a contradiction to the assumption that F is a dual atom in $S_P(X)$.

So in either case if $S_P(X)$ has dual atoms, then L has dual atoms and atoms . Hence the proof of the theorem is completed. □

Theorem 8.3.14. *If L is a finite pseudo complemented chain or a Boolean lattice, then $S_P(X)$ has dual atoms.*

Proof. **Case 1.**

Let X be a non empty set and L be a finite pseudo complemented chain.

Since L is a finite pseudo complemented chain, it has atom and dual atom. Let τ be a dual atom in the lattice of principal topologies on X. Then by theorem 8.3.1, τ must be of the form $\mathfrak{S}(a, \mathscr{U}) = \wp(X - a) \cup \mathscr{U}$, where $a \in X, \mathscr{U}$ is an ultrafilter not containing $\{a\}$. Since τ is a principal topology, $\mathscr{U}$ is a principal ultra filter so that $\tau = \mathfrak{S}(a, \mathscr{U}(b)) = \wp(X-a) \cup \mathscr{U}(b)$. Then $\omega_{PL}(\tau) = \{f \geqslant b_\lambda | f : (X, \tau) \to L$ is a scott continuous function from(X, τ) to $L\}, b \in X$ and λ is an atom in L . Then $\omega_{PL}(\tau)$ is a stratified principal L-topology and $a_\alpha \notin \omega_{PL}(\tau)$ for $0 \neq \alpha \in L$. Let β be the dual atom in L and $F = \omega_{PL}(\tau) \vee a_\beta$. Then F is the ultra L-topology $\mathfrak{S}(a_\beta)$ in $S_P(X)$ since the simple extension of F by a_1 is the discrete L-topology.

Case 2.

Let X be a non empty set and L be a finite Boolean lattice.

Since L is a Boolean lattice, it has atoms and dual atoms. Let τ be a dual atom in the lattice of principal topologies on X. Then by theorem 8.3.1, τ must be of the form $\mathfrak{S}(a, \mathscr{U}) = \wp(X-a) \cup \mathscr{U}$, where $a \in X, \mathscr{U}$ is an ultrafilter not containing $\{a\}$. Since τ is a principal topology, $\mathscr{U}$ is a principal ultra filter so that $\tau = \mathfrak{S}(a, \mathscr{U}(b)) = \wp(X-a) \cup \mathscr{U}(b)$. Then $\omega_{PL}(\tau) = \{f \geqslant b_\lambda | f : (X, \tau) \to L$ is a scott continuous function from(X, τ) to $L\}, b \in X$ and λ is an atom. Then $a_\alpha \notin \omega_{PL}(\tau)$ for $0 \neq \alpha \in L$. Let $\beta_1, \beta_2, ..., \beta_m$ are dual atoms in L and $F(a_{\beta 1}) = \omega_{PL}(\tau) \vee a_{\beta 1}, F(a_{\beta 2}) = \omega_{PL}(\tau) \vee a_{\beta 2}, ..., F(a_{\beta m}) = \omega_{PL}(\tau) \vee a_{\beta m}$. Let $F_{\beta j}$ is the L-topology generated by $(m-1)$ $F(a_{\beta i})$ from m $F(a_{\beta i}), i = 1, 2, ...m, j = 1, 2, ...m, i \neq j$. Then as in case 1, $F_{\beta j}$ is the ultra L-topology $\mathfrak{S}_{\beta j}$ in $\beta(X)$ since the simple extension of $F_{\beta j}$ by $a_{\beta j}$ is the discrete L-topology.

In both cases, $S_P(X)$ has dual atoms. Hence the theorem. $\square$

Theorem 8.3.15. *The lattice of stratified principal L-topologies $S_P(X)$ on a set X is not dually atomic in general.*

Proof. Follows from theorem 8.3.14. □

8.4 Complementation problem in the lattice of stratified principal L-topologies

Theorem 8.4.1. *If F is any stratified principal L-topology on X such that the topology corresponding to the characteristic functions in F is neither discrete nor indiscrete, then F has at least one join-complement.*

Proof. Let τ be the principal topology corresponding to the characteristic functions in F. Since the lattice Π is complemented [58], we can find $\tau^{'}$ in Π such that $\tau \wedge \tau^{'} = 0$ and $\tau \vee \tau^{'} = 1$ in Π. Then $F \vee \omega_{PL}(\tau^{'}) = 1$ and $F \wedge \omega_{PL}(\tau^{'}) \neq 0$ in $S_P(X)$. □

Theorem 8.4.2. *The lattice of stratified principal L-topologies $S_P(X)$ on a set X is semi-complemented.*

Proof. Let F be any stratified principal L-topology on X and τ be the topology corresponding to the characteristic functions in F. Let $\tau^{'}$ be a complement of τ in Π. Then $F \vee \omega_{PL}(\tau^{'}) = 1$ in $S_P(X)$. □

Theorem 8.4.3. *If F is an induced stratified principal L-topology $S_P(X)$ on X, then F has at least one complement in $S_P(X)$.*

Proof. Since F is induced, there exists a topology τ in Π such that $\omega_{PL}(\tau) = F$. Since Π is complemented, there exists at least one topology $\tau^{'}$ on Π such that $\tau \wedge \tau^{'} = 0$ and $\tau \vee \tau^{'} = 1$ in Π. Then $F \vee \omega_{PL}(\tau^{'}) = 1$ and $F \wedge \omega_{PL}(\tau^{'}) = 0$ in $S_P(X)$. □

Remark 8.4.1. We have analyzed the lattice structure of the set of all stratified principal L topologies on an arbitrary set X and have obtained characterization for certain properties of it. This study reveals more about the interplay between L-topology and lattice theory. Also for a given principal topology τ on X, the family $F_{p\tau}$ of all stratified principal L-topologies defined by families of Scott continuous functions from (X, τ) to L, forms a lattice under the natural order of set inclusion. From this lattice, we can deduce properties of $S_P(X)$ and $S_P^{'}(X)$.

Chapter 9

Lattice of L-closure operators

9.1 Introduction

In 1965 Zadeh [77] introduced fuzzy sets as a generalization of ordinary sets. After that Chang [13] introduced fuzzy topology and that led to the discussion of various aspects of L-topology by many authors. The Čech closure spaces introduced by Čech. [12] is a generalization of the topological spaces. The theory of fuzzy closure spaces has been established by Mashhour and Ghanim [39] and Srivastava et. al ([45],[54]). The definition of Mashhour and Ghanim is an analogue of Čech closure spaces and Srivastava et. al. have introduced it as an analogue of the definition of closure space given by Dikranjan et. al.[16]. Based on [54], Rekha Srivas-

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* Some results of this chapter are included in the following paper.
Raji George and T.P. Johnson : On the Lattice of L-closure operators, International Journal of Science and Research, Volume 2, Number 3, 2013

tava and Manjari Srivastava studied the subspace of a fuzzy closure space. The notion of T_0-fuzzy closure spaces and T_1 fuzzy closure spaces were also introduced in [45]. In [43] Ramachandran studied some properties of lattice of closure operators. In [28] Johnson studied some properties of the lattice $L(X)$ of all fuzzy closure operators on a fixed set X. In [76] Wu-Neng Zhou introduced the concept of L-closure spaces and the convergence in L-closure spaces. In this chapter we study properties of the lattice $LC(X)$ of L-closure operators and L-closure spaces which is a generalization of the concept of fuzzy closure spaces. Here we proved that the complete lattice $LC(X)$ is not modular. Also we identify the infra L-closure operator and ultra L-closure operator and establish the relation between ultra L-topology and ultra L-closure operator. We proved that an L-closure operator is an ultra L-closure operator if and only if it is the L-closure operator associated with an ultra L-topology. Also proved that infra L-closure operators are less than or equal to any non principal ultra L-closure operator and no non principal ultra L-closure operator has a complement so that the lattice of L-closure operators is not complemented in general.

9.2 Preliminaries

A completely distributive lattice L is called a F-lattice, if there is an order reversing involution from L to L. Let X be any nonempty set and L is a F-lattice. The fundamental definition of L-fuzzy set theory and L-fuzzy topology are assumed to be familiar to the reader as in [34]. Here we call L-fuzzy subsets as L subsets and L-fuzzy topology as L-topology.

Definition 9.2.1. [39] A Čech fuzzy closure operator on a set X is a function $\chi : I^X \to I^X$, satisfying the following three axioms

(i). $\chi(0) = 0$,

(ii). $f \leqslant \chi(f)$ for every f in I^X,

(iii). $\chi(f \vee g) = \chi(f) \vee \chi(g)$ where $I = [0, 1]$.

For convenience it is called fuzzy closure operator on X and (X, χ) is called fuzzy closure space. In [76] Wu-Neng Zhou defined L-closure operator as follows.

Definition 9.2.2. A mapping $C : L^X \to L^X$ is called an L-closure operator or an L-closure, if it satisfies the following conditions for any $A, B \in L^X$:

(i). $C(0_X) = 0_X$,

(ii). $A \leqslant C(A)$,

(iii). $A \leqslant B$ implies $C(A) \leqslant C(B)$,

(iv). $C(C(A)) = C(A)$.

But in this chapter we take the definition of L-closure operator as a generalization of fuzzy closure operator in [39]

Definition 9.2.3. Let X be a non empty set and L be a F lattice. An L-closure operator on L^X is a mapping $\psi : L^X \to L^X$ satisfying the following conditions:

(i) $\psi(\underline{0}) = \underline{0}$,

(ii) $f \leqslant \psi(f)$,

(iii) $\psi(f \vee g) = \psi(f) \vee \psi(g)$ for every $f, g \in L^X$.

The pair (X, ψ) is called an L-closure space. An L-subset f of X is said to be an L-closed set in (X, ψ) if $\psi(f) = f$. An L-subset f of X is open if its complement is closed in (X, ψ). The set of all open L-subsets of (X, ψ) form an L-topology on X called the L-topology associated with the L-closure operator ψ.

Let F be an L-topology on a set X. Then a function $\psi : L^X \to L^X$ defined by $\psi(f) = \bar{f}$ for all $f \in L^X$, where $\bar{f}$ denotes the closure of f with respect to F is called the L-closure operator associated with the L-topology F.

An L-closure operator on a set X is called L-topological if it is the L-closure operator associated with an L-topology on X. That is $\psi(\psi(f)) = \psi(f)$ for all $f \in L^X$. Note that different L-closure operators can have the same associated L-topology. But different L-topologies cannot have the same associated L-closure operator.

Example 9.2.1. Let $X = \{a, b, c\}, L = \{0, \alpha, \beta, 1\}$. Let $\psi_1 : L^X \to$

L^X defined by

$$\psi_1(f) = \begin{cases} \underline{0} & \text{if } f = \underline{0} \\ \underline{\beta} & \text{if } f(x) < \beta, \forall x \in X \\ \underline{1} & \text{otherwise} \end{cases}$$

Then ψ_1 is a fuzzy closure operator.
$\psi_2 : L^X \to L^X$ defined by

$$\psi_2(f) = \begin{cases} \underline{0} & \text{if } f = 0 \\ \underline{1} & \text{otherwise} \end{cases}$$

Then ψ_2 is a fuzzy closure operator.
Associated fuzzy topologies of ψ_1 and ψ_2 are same, which is the indiscrete fuzzy topology.

9.3 Lattice of L-closure operators

Let ψ_1 and ψ_2 be L-closure operators on X. Then $\psi_1 \leqslant \psi_2$ if and only if $\psi_2(f) \leqslant \psi_1(f)$ for every f in L^X. The relation $\leqslant$ defined above is a partial order on the set of all L-closure operators on L^X. We denote the poset by $LC(X)$. Then $LC(X)$ is a lattice. The L-closure operator D on X defined by $D(f) = f$ for every f in L^X is called the discrete L-closure operator. The L-closure operator I on X defined by $I(f) = \begin{cases} \underline{0} & \text{if } f = \underline{0} \\ \underline{1} & \text{otherwise} \end{cases}$ is called the indiscrete L-closure operator.

Remark 9.3.1. D and I are the L-closure operators associated with the discrete and indiscrete L-topologies on X respectively. Moreover D is the unique L-closure operator whose associated L-topology is discrete. Also I and D are the smallest and the largest elements of $LC(X)$ respectively.

Theorem 9.3.1. *$LC(X)$ is a complete lattice.*

Proof. It is enough to show that every subset of $LC(X)$ has greatest lower bound in $LC(X)$. Let $S = \{\chi_j | j \in J\}$ be a subset of $LC(X)$. Then $\sup\limits_{j\in J}\{\chi_j(f)\} = \inf\limits_{j\in J}\{\chi_j\}$ is an L-closure operator and is the greatest lower bound of S in $LC(X)$. □

Definition 9.3.1. [20] A lattice L is called modular if it satisfies the condition $x \geqslant z$ implies that $(x \wedge y) \vee z = x \wedge (y \vee z), \forall x, y, z \in L$.
Lattice of L closure operators $LC(X)$ is modular if and only if $\chi \geqslant \eta \Rightarrow \chi \wedge (\psi \vee \eta) = (\chi \wedge \psi) \vee \eta, \forall \chi, \psi, \eta \in LC(X)$.

Theorem 9.3.2. *$LC(X)$ is not modular.*

Proof. Let X be any set and $x \in X$. Define ψ_x, χ_x, η_x from $L^X \to L^X$ by

$\psi_x(\underline{0}) = \underline{0}$

$$\psi_x(f)(y) = \begin{cases} f(y) & \text{if } y \neq x \\ 1 & \text{if } y = x \end{cases}$$

$\chi_x(\underline{0}) = \underline{0}$

$$\chi_x(f)(y) = \begin{cases} 1 & \text{if } y \neq x \\ f(y) & \text{if } y = x \end{cases}$$

$\eta_x(\underline{0}) = \underline{0}$

$$\eta_x(f)(y) = \begin{cases} 1 & \text{if } y \neq x \\ \beta & \text{if } y = x \end{cases} \quad \text{and } \beta \geqslant f(y)$$

Then $\chi_x(f)(y) \leqslant \eta_x(f)(y), \forall y$. Hence $\chi_x \geqslant \eta_x$.

$$\begin{aligned} \chi_x \wedge \psi_x &= \inf(\chi_x, \psi_x) \\ &= \sup(\chi_x(f)(y), \psi_x(f)(y)) \\ &= 1 \end{aligned}$$

$$\begin{aligned} (\chi_x \wedge \psi_x) \vee \eta_x &= \inf(1, \eta_x(f)(y)) \\ &= f(y) \end{aligned}$$

$$\begin{aligned} \psi_x \vee \eta_x &= \sup(\psi_x, \eta_x) \\ &= \inf(\psi_x(f)(y), \eta_x(f)(y)) \\ &= f(y) \end{aligned}$$

$$\begin{aligned} \chi_x \wedge (\psi_x \vee \eta_x) &= \sup(\chi_x(f)(y), f(y)) \\ &= 1 \end{aligned}$$

There fore $\chi_x \wedge (\psi_x \vee \eta_x \neq (\chi_x \wedge \psi_x) \vee \eta_x$

So $LC(X)$ is not modular. □

Definition 9.3.2. An L-closure operator on X is called an infra L-closure operator if the only L-closure operator on X strictly smaller than it is I.

Let X be any set and $a, b \in X$ such that $a \neq b$. Define $\psi_{a,b} : L^X \longrightarrow L^X$ by $\psi_{a,b}(f) = \begin{cases} f & \text{if } f = \underline{0} \\ g_{\alpha,b} & \text{if } f = a_\alpha \\ \underline{1} & \text{otherwise} \end{cases}$,

α is a dual atom in L and $g_{\alpha,b}$ is defined by $g_{\alpha,b}(a) = \begin{cases} 1 & \text{if } a \neq b \\ \alpha & \text{if } a = b \end{cases}$

In the topological context Ramachandran [43] proved that a closure operator on X is an infra closure operator if and only if it is of the form $V_{a,b}$ for some a, b in $X, a \neq b$, where $V_{a,b}$ is defined by

$$V_{a,b}(A) = \begin{cases} \phi & \text{if } A = \phi \\ X - \{b\} & \text{if } A = \{a\} \\ X & \text{otherwise} \end{cases}$$

Analogously in the L-topological context we prove the following theorem.

Theorem 9.3.3. *An L-closure operator is an infra L-closure operator if and only if it is of the form $\psi_{a,b}$ for some $a, b \in X, a \neq b$.*

Proof. Let ψ be an L-closure operator on X strictly smaller than $\psi_{a,b}$, then $\psi(a_\alpha)$ will be strictly greater than $\psi_{a,b}(a_\alpha) = g_{\alpha,b}$ and hence equal to $\underline{1}$ so that $\psi(f) = \underline{1}, \forall f \in L^X$ other than $\underline{0}$. Hence $\psi = I$. Thus all L-closure operators of the form $\psi_{a,b}$ are infra L-closure operators.

Conversely let ψ be any L closure operator other than I. Then we can find a non zero L-subset f such that $\psi(f) \neq I(f) = \underline{1}$ (ie $\psi(f) \neq \underline{1}$) and elements a_α, b_β such that $a_\alpha \leqslant f$ and b_β not in $\psi(f)$. Then b_β is not an element of $\psi(a_\alpha)$. That is $b_\beta \nleq \psi(a_\alpha) \Rightarrow g_{\alpha,b} \nleq \psi(a_\alpha)$. That is $\psi_{a,b}(a_\alpha) \nleq \psi(a_\alpha)$. Also $\psi_{a,b}(k) = \underline{1}$ for every nonzero L-subset k other than a_α. So $\psi_{a,b}(f) \geqslant \psi(f), \forall f$. That is $\psi_{a,b} \leqslant \psi$. Thus all infra L-closure operators are of the form $\psi_{a,b}$ for $a, b \in X$ such that $a \neq b$. $\square$

Remark 9.3.2. When $L = I$ there is no infra L-closure operator.

Definition 9.3.3. An L-topology F on X is an ultra L-topology if the only L-topology on X strictly finer than F is the discrete L-topology.

Let X be a non empty set and L is a finite pseudo complemented chain. If $\mathfrak{S} = \mathfrak{S}(a, \mathscr{U}(b_\lambda)) = \{f|f(a) = 0\} \cup \{f|f \geqslant b_\lambda\}$, then a principal ultra L-topology $= \mathfrak{S}(a, \mathscr{U}(b_\lambda), a_\beta) = \mathfrak{S}(a_\beta)$, which is the simple extension of $\mathfrak{S}$ by a_β, i.e,$\mathfrak{S}(a_\beta) = \{f \vee (g \wedge a_\beta), f, g \in \mathfrak{S}, a_\beta \notin \mathfrak{S}\}$, where $a, b \in X, \lambda$ and β are the atom and dual atom in L respectively.

Let X be a nonempty set and L is a finite Boolean lattice. If $\mathfrak{S} = \mathfrak{S}(a, \mathscr{U}(b_\lambda)) = \{f|f(a) = 0\} \cup \{f|f \geqslant b_\lambda\}$, where $a, b \in X, \lambda$ is an atom,then a principal ultra L-topology denoted by $\mathfrak{S}_{\beta j} = L$-topology generated by any $(m-1)$ $\mathfrak{S}(a_{\beta i})$ among m $\mathfrak{S}(a_{\beta i}), i = 1, 2, , ...m, j = 1, 2, ...m, i \neq j$ if there are m dual atoms $\beta_1, \beta_2, , ..., \beta_m$, where $\mathfrak{S}(a_{\beta i}) =$ simple extension of $\mathfrak{S}$ by $a_{\beta i}$

Let X be an infinite set and L is a finite pseudo complemented chain. If $\mathfrak{S} = \mathfrak{S}(a, \mathscr{U}) = \{f|f(a) = 0\} \cup \mathscr{U}$ where $\mathscr{U}$ is a non principal ultra L-filter not containing $a_\lambda, 0 \neq \lambda \in L$. Then the non principal ultra L-topology $= \mathfrak{S}(a, \mathscr{U}, a_\beta) = \mathfrak{S}(a_\beta)$, is the simple extension of $\mathfrak{S}$ by a_β, where $a \in X, \beta$ is the dual atom in L.

Let X be an infinite set and L is a finite Boolean lattice. If $\mathfrak{S} = \mathfrak{S}(a, \mathscr{U}), a \in X$, then a non principal ultra L-topology $\mathfrak{S}_{\beta j} = L$-topology generated by any $(m-1)$ $\mathfrak{S}(a_{\beta i})$ among m $\mathfrak{S}(a_{\beta i}), i = 1, 2, ..., m, j = 1, 2, ..., m, i \neq j$, if there are m dual atoms $\beta_1, \beta_2, ..., \beta_m$, where $\mathfrak{S}(a_{\beta i}) =$ simple extension of $\mathfrak{S}$ by $a_{\beta i}$.

If X is a non empty set and L is a diamond lattice $\{0, \alpha, \beta, 1\}$ then the

L-closure operator ψ associated with an ultra L-topology $\mathfrak{S}(a, \mathscr{U}, a_\beta), a \in X$, is given by

$$\psi(f) = \begin{cases} f & \text{if } f = \underline{0} \text{ or } a_\alpha \leqslant f \text{ or } cf \in \mathscr{U} \\ f \vee a_\alpha & \text{otherwise} \end{cases}$$

In topological context it is known that a closure operator on X is an ultra closure operator if and only if it is the closure operator associated with some ultra topology on X and in L-topological context we prove the following theorem.

Theorem 9.3.4. *Let X is a non empty set and L is a diamond lattice $\{0, \alpha, \beta, 1\}$. Then an L-closure operator on X is an ultra L-closure operator if and only if it is the L-closure operator associated with some ultra L-topology on X .*

Proof. Let $\mathfrak{S}(a, \mathscr{U}, a_\beta)$ be an ultra L-topology on X and ψ be the associated L-closure operator. Let ψ' be an L-closure operator on X strictly larger than ψ. Then there exists an L subset f of X such that $\psi'(f) < \psi(f)$. But $\psi'(f) \neq \psi(f)$. Then $\psi(f) = f \vee a_\alpha$ and$\psi'(f) = f$, which means that complement of f is open in (X, ψ') and not open in (X, ψ). Also every open set in (X, ψ) is open in (X, ψ') . Thus the associated L- topology of ψ' is strictly larger than the ultra L-topology and hece is discrete. Thus $\psi' = D$. Hence the L-closure operator associated with an ultra L-topology is an ultra L-closure operator.

Next to prove that every ultra L-closure operator is the L-closure operator associated with an ultra L-topology.

Let ψ be an L-closure operator on X other than D. It suffices to prove that there exists an L-closure operator associated with an ultra L-topology larger than ψ. Since $\psi \neq D$ there exists an element a of X such that a_α is not open in (X, ψ). Now consider $\mathfrak{S} = \{f | f(a) = 0\} \cup \mathscr{U}$ where $\mathscr{U}$ is an ultra L-filter not containing $a_\lambda, 0 \neq \lambda \in L$. Then a_α is not an element of $\mathfrak{S}$. Now consider the ultra L-topology $\mathfrak{S}(a, \mathscr{U}, a_\alpha)$ = simple extension of $\mathfrak{S}$ bya_α. Let ψ' be the L-closure operator associated with it . Then $\psi \leqslant \psi'$. Otherwice if $\psi' \leqslant \psi$, then every open set in ψ' is open in ψ. But a_α is open in ψ' . So it must be open in ψ, which is a contradiction. □

Remark 9.3.3. In a similar way we can prove the above theorem when L is a finite pseudo complemented chain or other Boolean lattices.

Definition 9.3.4. Let $x \in X, \lambda \in L$. An L point x_λ is defined by
$$x_\lambda(y) = \begin{cases} \lambda & \text{if } y = x \\ 0 & \text{if } y \neq x \end{cases} \quad \text{where } 0 < \lambda \leqslant 1$$

Definition 9.3.5. An L-closure operator ψ on X is T_1 if every L point is closed. That is $\psi(x_\lambda) = x_\lambda, \forall x \in X, \lambda \in L$.

Definition 9.3.6. [62] Let $\psi_1 = \{f | \psi(f)) = f\}$. A fuzzy closure space (X, ψ) is called quasi-separated if and only if for any two fuzzy points x_λ and y_γ with $x_\lambda \in C(y_\gamma)$, there exist $f, g \in \psi_1$ such that $x_\lambda \in f \leqslant C(y_\gamma)$ and $y_\gamma \in g \leqslant C(x_\lambda)$.

Theorem 9.3.5. *[62]*

A fuzzy closure space is quasi-separated if and only if every fuzzy point in X is Čech-fuzzy closed.

Proposition 9.3.1.
Let $\psi_1 = \{f \in L^X | \psi(f)) = f\}$. An L-closure space (X, ψ) is said to be T_1

if for every pair of distinct L points x_λ and y_γ, there exist $f, g \in \psi_1$ such that $x_\lambda \in f \leqslant C(y_\gamma)$ and $y_\gamma \in g \leqslant C(x_\lambda)$.

Proof. Necessary part
Suppose that the L-closure operator ψ is T_1. Then by definition $\psi(x_\lambda) = x_\lambda$ Then by theorem 9.3.5 the L-closure space(X, ψ) is quasi separated. Hence for every pair of distinct L points x_λ and y_γ, there exist $f, g \in \psi_1$ such that $x_\lambda \in f \leqslant C(y_\gamma)$ and $y_\gamma \in g \leqslant C(x_\lambda)$.
Sufficient part
Suppose that for every pair of distinct L points x_λ and y_γ, there exist $f, g \in \psi_1$ such that $x_\lambda \in f \leqslant C(y_\gamma)$ and $y_\gamma \in g \leqslant C(x_\lambda)$. Then by definition (X, ψ) is quasi separated. Then by theorem 9.3.5, (X, ψ) is a $T_1 L$-closure space. □

Proposition 9.3.2. [62]
An L-closure space (X, ψ) is T_1 if and only if the associated L topological space (X, F) is T_1

Theorem 9.3.6. *Infra L-closure operators are less than or equal to any non principal ultra L-closure operator.*

Proof. Let $\psi_{a,b}$ be an infra L-closure operator and ψ be a non principal ultra L-closure operator. Since $\psi_{a,b}(f) = \underline{1}$ for all f in L^X other than $\underline{0}$ and a_α, it is enough to show that $\psi(a_\alpha) < \psi_{a,b}(a_\alpha) = g_{\alpha,b}$. Since all non principal ultra L-topologies are T_1, the corresponding L-closure operators are T_1 by the above proposition. Hence by the definition $\psi(a_\alpha) = a_\alpha$. That is $a_\alpha < g_{\alpha,b} \Rightarrow \psi(a_\alpha) < \psi_{a,b}(a_\alpha)$
That is $\psi(f) \leqslant \psi_{a,b}(f) \forall f \Rightarrow \psi_{a,b} \leqslant \psi$. □

Theorem 9.3.7. *No non principal ultra L-closure operator has a complement.*

Proof. Assume the contrary. Let ψ be a non principal ultra L-closure operator with a complement ψ' in the lattice $LC(X)$. Since ψ' is not indiscrete there exists an infra L-closure operator $\psi_{a,b} \leqslant \psi'$ by the proof of the theorem 9.3.3. But $\psi_{a,b} \leqslant \psi$ by theorem 9.3.6. This contradicts the fact that ψ and ψ' are complements in the lattice $LC(X)$ and hence the proof of the theorem. □

Remark 9.3.4. The lattice of L-closure operators is not complemented in general.

If L is a diamond lattice, the principal ultra L-closure operator associated with the principal ultra L-topology $\mathfrak{S}(a, \mathscr{U}(b_\beta), a_\beta)$ is given by

$$\phi_{a,b}(f) = \begin{cases} f & \text{if } f = \underline{0} \text{ or } a_\alpha \leqslant f \text{ or } cf \in \mathscr{U}(b_\beta) \\ f \vee a_\alpha & \text{otherwise} \end{cases}$$

Theorem 9.3.8. *An infra L-closure operator $\psi_{a,b}$ and $\phi_{b,a}$ are in comparable if L is a diamond lattice.*

Proof. We have $\psi_{a,b}(a_\alpha) = g_{\alpha,b}$ and
$\phi_{b,a}(a_\alpha) = a_\alpha \vee b_\beta$
Since α and β are not comparable, $\psi_{a,b}$ and $\phi_{b,a}$ are not comparable. □

Remark 9.3.5. In a similar way, we can discuss the above theorem if L is a finie pseudo complemented chain or other Boolean lattices.

Definition 9.3.7. An L-closure space (X, ψ) is said to be T_0 if for all $x, y \in X, x \neq y, \exists$ a closed L-subset f such that $f(x) \neq f(y)$.

Example 9.3.1. Let $X = \{a, b, c\}$ and $L = \{0, \alpha, \beta, 1\}$, a diamond lattice. Consider the L-topology

$$\phi = \left\{\underline{0}, a_\alpha, b_\beta, \mu_{\{a\}}, \mu_{\{b\}}, \mu_{\{a,b\}}, \underline{1}, \begin{array}{r} f : a \to \alpha \\ b \to \beta \\ c \to 0, \end{array} \quad \begin{array}{r} g : a \to \alpha \\ b \to 1 \\ c \to 0, \end{array} \quad \begin{array}{r} h : a \to 1 \\ b \to \beta \\ c \to 0, \end{array} \right\}$$

Define $c : L^X \to L^X$ by $c(f) = \wedge\{g \in \phi : g \geqslant f\}$, for all $f \in L^X$. Then (X, c) is a T_0 L-closure space.

Definition 9.3.8. An L-closure space (X, ψ) is said to be T_1, if every L point x_λ is closed.

Example 9.3.2. Let $X = \{a, b, c\}$ and $L = \{0, \alpha, \beta, 1\}$. Consider the discrete L-topology L^X. Define $c : L^X \to L^X$ by $c(f) = \wedge\{g \in L^X : g \geqslant f\}$ for all $f \in L^X$. Then (X, c) is a T_1 L-closure space.

Remark 9.3.6. Every T_1 L-closure space is T_0. But the converse need not be true.

Example 9.3.3. Let $X = \{a, b\}$ and $L = \{0, \alpha, \beta, 1\}$, a diamond lattice. Consider the L-topology
$\phi = \{\underline{0}, a_\alpha, \mu_{\{a\}}, \underline{1}\}$. Define $c : L^X \to L^X$ by $c(f) = \wedge\{g \in \phi : g \geqslant f\}$, for all $f \in L^X$. Then (X, c) is a T_0 L-closure space. But it is not a T_1 L-closure space, since b_β is not closed in (X, c).

Concluding remarks and suggestions for further study

We have identified the principal and non principal ultra L-topologies and determined the number of ultra L-topologies on an arbitrary set. Also we have analyzed the lattice structure of some sublattices of Lattice of L-topologies. However it is not yet analyzed in detail that under what condition on the F-lattice L, the above lattices are dually atomic.

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- *Raji George, T. P. Johnson*: On the Lattice of L-closure operators, International Journal of Science and Research.
- *Raji George, T. P. Johnson*: On the Lattice of Stratified Principal L-topologies, International Journal of Fuzzy Information and Engineering, Springer. (Accepted for publication)

"This is the LORD's doing; it is marvelous in our eyes."

Psalm 118 : 23

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