SOME BIVARIATE LIFE TIME MODELS IN DISCRETE TIME

THESIS SUBMITTED TO THE
COCHIN UNIVERSITY OF SCIENCE AND TECHNOLOGY
FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY
UNDER THE FACULTY OF SCIENCE

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MAY 1995

CERTIFICATE

Certified that the thesis entitled **"Some bivariate life time models in discrete time"** *is a bonafide record of work done by* **Smt. Asha Gopalakrishnan** *under my guidance in the Division of Statistics, School of Mathematical Sciences, Cochin University of Science and Technology and that no part of it has been included any where previously for the award of any degree or title.*

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DECLARATION

This thesis contains no material which has been accepted for the award of any other Degree or Diploma in any University and to the best of my knowledge and belief, it contains no material previously published by any other person, except where due references are made in the text of the thesis.

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ACKNOWLEDGEMENTS

I owe my inspiration and sincere gratitude to Professor N. Unnikrishnan Nair for his guidance and persistent encouragement throughout the development and writing of this thesis.

The co-operation rendered by all my teachers and my colleagues is also thankfully acknowledged.

I also wish to express great happiness at the encouragement rendered by my family.

Financial assistance extended by the The National Board of Higher Mathematics is duly acknowledged.

I thank Datasystems for their speedy and accurate typing.

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May 15, 1995 *ASHA GOPALAKRISHNAN*

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Chapter I

INTRODUCTION

1.1. RELIABILITY MODELLING.

The term reliability of an equipment or device is often meant to indicate the probability that it carries out the functions expected of it adequately or without failure and within specified performance limits at a given age for a desired mission time when put to use under the designated application and operating environmental stress. A broad classification of the approaches employed in relation to reliability studies can be made as probabilistic and deterministic, where the main interest in the former is to device tools and methods to identify the random mechanism governing the failure process through a proper statistical frame work, while the latter addresses the question of finding the causes of failure and steps to reduce individual failures thereby enhancing reliability. In the probabilistic attitude to which the present study subscribes to, the concept of life distribution, a mathematical idealisation that describes the failure times, is fundamental and a basic question a reliability analyst has to settle is the form of the life distribution. It is for no other reason that a major share of the literature on the mathematical theory of reliability is focussed on methods of arriving at reasonable models of failure times and in showing the failure patterns that induce such models. The application of the methodology of life time distributions is not confined to the assesment of endurance of equipments and systems only, but ranges over a wide variety of scientific investigations where the word life time may not refer to the length of life in the literal sense, but can be concieved in its most general form as a non-negative random variable. Thus the tools developed in connection with modelling life time data have found applications in other areas of research such as actuarial science, engineering, biomedical sciences, economics, extreme value theory etc.

The probabilistic behaviour of a device or system is generally expressed in terms of the failure density

$f(x;\theta)$ defined on the positive real axis specifying the instantaneous probability of failure at a given time x. A failure model, then consists of specification of the functional forms of f and the values of its parameters. This choice of a functional form of f can be accomplished in several ways. When theoretical investigations is of interest one can arbitrarily choose a well known classical density and derive results that are of relevance to the analysis. Alternatively, a density can be derived from known physical properties of the components or system by appealing to the well known practices and methods in mathematical modelling. Most of the distributions thus derived often turn out to have classical forms as seen from for eg. from Mann, Schafer and Singpurwalla (1974), Barlow and Proschan (1975), Bain(1978) and Martz and Waller (1982). Another alternative is to fit a density to the existing data, a method that can work even when knowledge about the physical condition under which the system is designed and operated is not substantial. However, the fact that a particular data passes a test of fit with respect to a model, by itself cannot be accepted as evidence to argue that the underlying failure mechanism is dictated by such a law. This is due to the fact that for data generated from life tests, the observations at the tails are sparse so that in the case of moderately skewed data more than one of the competing models like gamma, inverse guassian, Weibull or lognormal may produce a satisfactory fit. Even when there is a good fitting distribution it is essential to search among the possible generating mechanisms for the most appropriate explanation of the failure process. In the present study the view held is that one has to identify notions that can adequately explain the failure pattern in a practical situation and thereby characterize the probability distribution exhibiting the hypothesised behaviour. The emphasis on characterization theorems made here is prompted by the fact that, they are the only methods by which an exact identification of a distribution is possible.

1.2 RELIABILITY ANALYSIS IN DISCRETE TIME

With the advancement of technology, the increase of high cost sophisticated equipments demands method of analysis that could ensure some sort of assurance regarding the period of their failure - free operation over a pre-designated mission time. The conventional analysis developed over the last three decades pertaining to reliability, treats the random variable representing life time as continuous. The consideration along these lines requires equally sophisticated measuring devices to generate accurate observations. Xekalaki (1983) points out this problem and advocates the use of discrete distributions when measurements lack the accuracy it needs to undergo analysis in continuous time. Gupta and Gupta (1983), Lawless (1982) Shaked et.al (1995) cite situations where discrete random variables arise naturally in association with reliability studies. Further the existence of very good approximations for continuous distributions via discrete models renders the evaluation of reliability from the discrete set up to the continuous one, if needed. The overall gains resulting from man power requirements and the use of less advanced measuring devices, (for example, if the number of failures are observed only in completed units of time such as at the stroke of each hour or at a fixed time every day) on many ocassions, outweigh the marginal increase in accuracy gained otherwise from continuous measurements. Thus there is a strong case for looking at reliability aspects in the discrete time domain and therefore, a development of concepts and methods when length of life is treated as discrete random variable appears to be in the right place.

1.3 REVIEW OF RESULTS.

As already pointed out, with the exception of a few results most of the research work in life time model, make use of continuous distributions. We briefly look into the state of the art in the discrete time domain. Among widely circulated publications Cox (1972), Kalbfleisch and Prentice (1980) and Lawless (1982) have provided the basic formulations to the study of discrete life distributions. If $f(x)$ is the probability mass function of a discrete non-negative random variable, X and

$$R(x) = P(X \geq x), \quad x = 0,1,2... \tag{1.1}$$

is the survival function of X, then the failure rate is given by

$$h(x) = P(X=x \mid X \geq x) = f(x)/R(x). \tag{1.2}$$

The failure rate h(x) determines the life distribution uniquely through the formula

$$R(x) = \prod_{y=0}^{x-1} [1-h(y)]. \tag{1.3}$$

Xekalaki (1983) has shown that if X is a random variable taking values in the set { 0,1,2,...,m}, $m \in [0,1,...] \cup [+\infty]$ then $h(x) = (a+bx)^{-1}$, iff X has geometric distribution for b=0, Waring distribution for b>0 and negative hyporgeometric distribution for b<0. With the help of the slope to mean ordinate ratio, he has derived the beta and Pareto type II models as continuous approximations to the negative hypergeometric and Waring models. The mean residual life of X,

$$r(x) = E\,[X-x \mid X>x] \tag{1.4}$$

$$= [R(x+1)]^{-1} \sum_{y=x+1}^{\infty} R(y)$$

is another concept that can explain the pattern of failure, which can be used either independently or in combination with the failure rate. In terms of the mean residual life

$$R(x) = \prod_{u=1}^{x-1} \frac{r(u-1)-1}{r(u)} [1-f(0)] \tag{1.5}$$

where f(0) is determined such that $\sum f(x) = 1$. (Nair and Hitha, 1989). Further,

$$1-h(x+1) = \frac{r(x) - 1}{r(x+1)} \quad , \quad x=0,1,2,... \tag{1.6}$$

Nair (1983) has used the function r(x) to define the notion of memory of life distributions and also to clarify them as possessing no memory, negative memory and positive memory. Salvia and Bollinger (1982) besides obtaining (1.3) proved that for increasing failure rate distributions

$$E\,(x) \le (1-h_0)/h_0 \quad \text{and} \quad R(x) \le (1-h_0)^k \doteq e^{-h_0 x}$$

where $h_0 = h_{(0)}$

and f(x) defines a proper probability mass function if and only if $\sum_{I=0}^{\infty} h(j)$ diverges to $+\infty$. They also demonstrate some limiting behaviour through the equations

$$r = (R-1)^{-1}, \quad f = R^{-1} \text{ and } h = (r+1)$$

where

$$r = \lim r(x), \quad h = \lim h(x), \quad f = \lim(f(x)/f(x+1))$$

and $R = \lim [\, R(x)/R(x+1) \,]$. The classification of discrete life distribution through virtual age, virtual hazard rate, mean remaining life etc were studied by Abouammoh (1990) and Roy and Gupta (1992). Studies in the same direction are available in Ebrahami (1986) who discusses the class of discrete decreasing and increasing mean residual life functions.

The increasing or decreasing nature of failure rates can be subsumed into the concept of bathtub-shaped rates in which the failure rate at first increases (decreases) then remain constant and there upon starts decreasing (increasing). The papers by Guess and Park (1988) and Mi (1993) discuss the properties of the class of discrete life distributions with bathtub-shaped failure rates. Characterization of geometric distribution and discrete IFR(DFR)distributions using order statistics are estalbished in Neweichi and Govindarajulu (1979), while the relation

$$m(x) = \mu + (a_0 + a_1 x + a_2 x^2)\ h(x+1)$$

a_0, a_1, a_2 being real constants, $\mu = E(x)$ and $m(x) = E(X|X>x)$, is shown in Nair and Sankaran (1991) to characterize the Ord family

$$f(x+1) - f(x) = -(x+d)\ f(x)/(b_0 + b_1 x + b_2 x^2)$$

where

$$d=(a_1 - a_2 - \mu)/(2a_2+1)$$

and

$$b_i = a_i\ /(2a_2+1),\ i=0,1,2.$$

The last result is a generalisation of a characterization of the negative binomial distribution proved in Osaki and Li (1988).

The probability mass function of a random variable Y with

$$g(y) = \frac{P(x>y)}{\mu},\ y=0,1,2...$$

is called the distribution based on partial sums of X or the renewal distribution corresponding to X. Gupta (1979) has shown that the failure rate of Y is the reciprocal of the mean residual life function of X and further obtained characterization of the geometric destrubution. Continuing the work along these lines Nair and Hitha (1989) obtained mutual characterizations of the distributions of X and Y also that linear mean residual life (reciprocal linear failure rate) characterizes the geometric (Waring, negative hypergeometric) distribution when the slope is zero (greater than zero, less than zero). Some other ageing concepts used in replacement problems such as NBU, NBUE etc are discussed in Hitha (1991), Klefsjo (1981), Sankaran (1992) which are not included here as they are not relevant to the present study. An interesting feature of the extension of the failure rate concept into higher dimensions is that there is no unique way by which this can be done.

Nair and Nair (1988) considered the two dimensional case by defining the mean residual life function of the random Vector (X_1, X_2) in the support of $I_2^+ = \{(x_1, x_2) | x_1, x_2 = 0, 1, 2 \ldots\}$ as

$$\underline{r}(x_1, x_2) = (r_1(x_{1,}x_2)r_2(x_{1,}x_2))$$

where,

$$r_i(x_{1,}x_2) = E\ [X_i - x_i \mid X_1 \geq x_1,\ X_2 \geq x_2] \tag{1.7}$$

and proved that $r(x_1, x_2) = (C_1, C_2)$, with C_i as constants independent of X_1 and X_2 if and only if X_1 and X_2 are independent and geometrically distributed. Such models being of no practical utility in dealing with two-component system in which the component life times have association, they looked at models with the property

$$\underline{r}(x_1, x_2) = (a_1(x_2),\ a_2(x_1)) \tag{1.8}$$

(i.e, mean residual lives of the components are locally constant and characterized the bivariate geometric distributin with survival function

$$P\ (X_1 \geq x_1, X_2 \geq x_2) = P_1^{x_1}\ P_2^{x_2}\ \theta^{x_1 x_2} \tag{1.9}$$

$$0 < P_1, P_2 < 1\ ,\ 0 \leq \theta \leq 1\ ,\ 1-\theta \leq (1-p_1\theta)(1-P_2\theta).$$

The distribution (1.9) turns out to be the discrete analogue of the bivariate exponential of Gumbel (1960). An identity that permits the conversion of the exponential probabilities into geometric (1.9) and vice-versa was also derived in Nair (1993). In an earlier paper Nair and Nair (1990) introduced a formal definition of a vector valued bivariate failure rate as

$$\underline{b}(x_1,x_2) = (b_1(x_1,x_2),\ b_2(x_1,x_2))$$

where

$$b_i(x_1,x_2) = \frac{P(X_i=x_i\ ,\ X_j\geq x_j)}{P(X_1\geq x_1\ ,\ X_2\geq x_2)}\ ,\ i,j = 1,2;\ i\neq j, \qquad (1.10)$$

and used the identity

$$b_1(t_1,t_2) = 1- \frac{r_1(t_1,t_2)}{1+r\ (t_1+1,t_2)}$$

to prove that the (1.9) is the only bivariate geometric distribution with failure rate (1.10), whichhas the form $(c_1(x_2),c_2(x_1))$. Along with this the following are also established in Nair (1990).

(i) A necessary and sufficient condition for the local lack of memory property

$$P\ [X_i>x_i+s\ |\ X_1\geq x_1,X_2\geq x_2] = P[X_i>s\ |\ X_j\geq x_j] \qquad (1.11)$$

to hold for all non - negative integers x_1, x_2 and s is that (X_1, X_2) has distribution (1.9).

(ii) $E[(X_i - x_i)^{(k)} \mid X_1 \ge x_1, X_2 \ge x_2] = E[(X_i)^{(k)} \mid X_j \ge x_j]$

where, $t^{(k)} = t(t-1)....(t-k+1)$, holds if and only if the distribution is (1.9).

(iii) The vector, $b(x_1, x_2)$ of (1.10) determines the distribution of (X_1, X_2) in the support of I_2^+ uniquely through the formula

$$P[X_1 \ge x_1, X_2 \ge x_2] = \prod_{r=1}^{x_1} [1 - b_1(x_1 - r, x_2)] \prod_{r=1}^{x_2} [1 - b_2(0, x_2 - r)]$$

The last result is very general as it is the key to substantiate characterization of any bivariate distribution, given the functional form of the failure rate. All these results have been re-obtained by Roy (1993) also. We notice that the definition (1.10) expressed in vector form treats the two components separately. It is sometimes desirable to have a single quantity (scalar) that represents the failure rate, as provided for by Puri and Rubin (1974) (see also Puri (1973)). They define a multi variate failure rate as

$$a(x_1,x_2\ldots x_n) = \frac{P(X_1=x_1,\ X_2=x_2,\ \ldots X_n=x_n)}{P(X_1>x_1, X_2>x_2,\ \ldots X_n>x_n)} \tag{1.12}$$

and demonstrate that the only distributions of $(X_1,X_2,..X_n)$ where each X_i is non-negative integer valued, for which $a(x_1,x_2,\ \ldots x_n) = \beta$. a constant are mixtures of geometric distribution given by

$$P\ (X_i = x_i \ldots X_n = x_n) = \int_0^1 \ldots \int_0^1 \prod_{i=1}^{n} p_i^{n_i}\ H\ (dp_1,\ldots dp_n),$$

where the probability measure H is concentrated on the set $B=\left\{\Pi(\frac{p_i}{1-p_i})=\beta,\ 0<p_i<1,\ i-1,2,\ldots n\right\}$. Other than providing this characterization, the paper does not throw any light on the properties of $a(x_1,x_2,\ \ldots x_n)$ or its role in reliability modelling. It appears that the denominator of (1.12) should be modified to $P\ (X_1 \geq x_1,\ldots,\ X_n \geq x_n)$ to give a proper physical meaning to (1.12) as a rate and such a modified version of (1.12) is discussed in the following Chapter.

A third alternative definition of a failure rate is suggested in Kotz and Johnson (1991), who view it again as a vector

$$\underline{C}(x_1,x_2) = (C_1(x_1,x_2)\ ,\ C_2(x_1,x_2))$$

where

$$C_i(x_1,x_2) = \frac{P(X_i=x_i,\ X_j=x_j)}{P(X_i\geq x_i,\ X_j=x_j)}\ ,\ i,j=1,2;\ i\neq j. \qquad (1.13)$$

No formal discussion of this rate vis-a-vis its ability to model discrete life times is made in their paper.

A comprehensive treatment of failure rates in the multivariate setting is attempted in a recent work by Shaked et. al (1995). If (X_1,X_2) is a random vector in N_+^2 where $N_+=\{1,2,...\}$, they define the bivariate conditional hazard rate functions of (X_1,X_2) as

$$\lambda_1(x) = P(X_1=x,\ X_2\geq x|\ X_1\geq x,\ X_2\geq x),\ x \in N_+$$

$$\lambda_2(x) = P(X_2=x,\ X_1\geq x|\ X_1\geq x,\ X_2\geq x)$$

$$\lambda_{12}(x) = P(X_1=x,\ X_2=x|\ X_1\geq x,\ X_2\geq x)$$

$$\lambda_1(x|x_2) = P(X_1=x,|\ X_1\geq x,\ X_2=x_2),\ x > x_2,\ (x,x_2) \in N_+^2$$

$$\lambda_1(x|x_1) = P(X_1=x,|\ X_1=x,\ X_2\geq x),\ x > x_1,\ (x_1,x\ ) \in N_+^{\ 2}$$

provided, the condition in the above conditional probabilities have positive probabilities. These rates determine the distribution of (X_1, X_2) uniquely. Their paper providesnecessary and sufficient conditions for the λ's to be the hazard rate of some discrete random vector (X_1, X_2), along with applications of the rates to imperfect repair modelling. The system of definitions generated through the above five functions have all the desiderata for evolving a good model as they characterize the model through reasonable physical interpretation of the component life times. However, one should not fail to notice that the assessment of the reliability of the system is facilitated through the behavior of five quantities which is a little too much demanding. The definitions already given in terms of vectors have lesser number of components and they could be advantageous from the application point of view, once their desirable properties are investigated and shown to have requisites comparable to those of Shaked et.al (1995).

The review and discussion of the work available in literature on modelling multivariate life time data in discrete time, reveal that the work in this direction is still in the formative stages. Though various definitions have been proposed in different contexts, a systematic study of them have been lacking and therefore, for reasons mentioned at the beginning of this section an attempt along these lines is called for. The motivation towards undertaking a discussion of the discrete case also arrives from the wide acceptance received for the research in the continuous case from scientists in different fields of activity. Accordingly in the present thesis, our objective is to introduce and examine the basic concepts and provide the necessary theorems that enable a reasonable preliminary theoretical setting for modelling multivariate discrete life time data. The problems considered in this connection with a summary of the main results are presented in the next section.

1.4. ORGANISATION OF THE PRESENT STUDY.

The present work is organised into six chapters. In chapter II, the bivariate scalar failure rate (1.12) with the modification suggested in the discussion that followed the definition, is studied. It is shown that this rate, in general, does not determine the life time distribution uniquely. However the knowledge of the scalar rate together with one of the marginal failure rates (or equivalently one of the marginal distributions) is a necessary and sufficient condition to arrive at a unique survival function. Through examples we indicate the forms of scalar rates that characterize bivariate geometric, Waring and negative hypergeometric distributions. Characterization of bivariate models that have independent geometric marginals, bivariate geometric law and the relationship the scalar rate has with the mean residual life and bivariate lack of memory property are also proved.

The vector failure rates of Nair and Nair (1990) and Kotz and Johnson (1991) form the subject matter in Chapter III. The identities connecting the former with the scalar rate and mean residual life are considered. Among the various properties studied in this chapter, include the result that there exist no bivariate model with dependency among the constituent variables, which has a constant failure rate vector. Conditions on the rate and the mean residual life that guarantee equivalence with the lack of memory property are explored. Since constant failure rate must imply independent geometric marginals, to generate a meaningful law that ensures dependent life times for the components, this condition is relaxed to one possessing piece-wise constant rates. The geometric distribution characterized by this form is identified. The second vector rate discussed here is that in (1.13), referred to in the current study as the conditional failure rate. A new mean residual life that is in conformity with this rate is introduced and their inter-relationship is derived. One fundamental question that is examined with reference to any definition of failure rate, is whether it can pave way for the unique evaluation of the corresponding distribution. We examine the conditions under which the answer to this question is in the affirmative. This is followed by three characterization theorems and a discussion of the inter-relationship between the three types of rates already under discussion.

A standard problem in reliability modelling is the demarcation of distributions based on the monotone behaviour of failure rates, thereby giving rise to what are generally called increasing or decreasing failure rate distributions. In Chapter IV the vector failure rate and conditional failure rates that guarantee one-one correspondence between them and the corresponding life time distributions, are considered for this classification. Unlike in the univariate case, several modes of criteria exist that can lead to an increasing (decreasing) failure rate. Accordingly various criteria are introduced and the boundary classes in each case is identified along with the chain of implications existing between the different classes.

It is observed that geometric distributions that exhibit constant, piece-wise constant and locally constant failure rates have the property that they are both increasing and decreasing. A bivariate geometric distribution that serves as the discrete analogue of the Marshall - Olkin (1967) distribution is studied in Chapter V. After discussing the standard properties of the model, we establish some characterizations, and methods of estimating its parameters.

In view of the simplicity of analysis bivariate notions have been the objects of study in chapter II to IV. The thesis is concluded in Chapter VI by pointing out definitions and theorems in the general multivariate setup.

Chapter II

SCALAR FAILURE RATE

2.1 INTRODUCTION

The discussions initiated in section 1.3 pointed out that there is no unique generalization of the univariate concept of failure rate into higher dimensions and as such an extension can be accomplished on the basis of different considerations on the manner in which the failure mechanism is perceived. In the present chapter we look at the joint behaviour of the component life times as our starting point and define bivariate failure rate as a scalar quantity, which is analogous to Basu's (1971) definition in the continuous case.

Let $\underline{X} = (X_1, X_2)$ be a discrete random vector representing failure times of a two-component system in the support of $I_2^+ = [(x_1, x_2) | \ x_1, x_2 = 0, 1, 2, \ldots]$ with joint survival function

$$R(\underline{x}) = P(\underline{X} \geq \underline{x})$$

and probability mass function

$$f(\underline{x}) = P(\underline{X}=\underline{x})$$

where $\underline{x}=(x_1,x_2)$ and $\underline{X}\geq\underline{x}$ means $X_j \geq x_j$, j=1,2.

Associated with X, we define the marginal survival function of X_j with support $I_1^+ = \{x|x=0,1,2,...\}$

$$R_j(x_j) = R(x_j,0) = P[X_j \geq x_j], \quad j=1,2$$

and the corresponding mass functions by

$$f_j(x_j) = P[X_j=x_j] = R_j(x_j) - R_j(x_j+1).$$

Then from (1.2), the failure marginal rate of X_j is

$$h_j(x_j) = f_j(x_j)/R_j(x_j), \ j=1,2.$$

2.2. SCALAR FAILURE RATE

Definition 2.1

The scalar failure rate is defined at those points for which R(x)>0, by

$$a(\underline{x})=f(\underline{x})/R(\underline{x}) \tag{2.1}$$

$$= \frac{R(x_1,x_2)-R(x_1+1,x_2)-R(x_1,x_2+1)+R(x_1+1,x_2+1)}{R(x_1,x_2)}.$$

Rewriting (2.1) as

$$a(\underline{x}) = P[X_1=x_1,\ X_2=x_2 | X_1 \geq x_1, X_2 \geq x_2],$$

it is apparent that $0<a(\underline{x})\leq 1$ at all points in the support of $\underline{X}$ with the equality $a(\underline{x}) = 1$ holding good only if (X_1,X_2) has a finite support. Further the last representation says that for a two component system which has survived the time point (x_1,x_2), the scalar failure rate provides the probability that the system fails at the time point (x_1,x_2).

One of the important objectives in introducing the notion of failure rate is to examine the possibility of identifying the failure time distribution or equivalently to explore means for assessing the reliability of the system, once a realistic judgement of the behaviour of the failure rate function is made out of physical characteristics governing the system. In other words, the first question to be settled is, given the functional form of the failure rate would it be possible to determine the distribution of $\underline{X}$ uniquely? The following example shows that the answer is not in the affirmative.

Example 2.1.

Consider the survival functions

$$R(\underline{x}) = p^{x_1}(1-p)^{x_2}; \quad 0<p<1, \ \underline{x} \in I_2^+$$

and

$$G(\underline{x}) = (1/2)\left[p^{x_1}(1-p)^{x_2} + (1-p)^{x_1}(p)^{x_2} \right]; \ 0<p<1, \ \underline{x}\in I_2^+$$

with the corresponding probability mass functions

$$f(\underline{x}) = p^{x_1+1} (1-p)^{x_2+1} ; \qquad 0<p<1, \ \underline{x} \in I_2^+$$

and

$$g(\underline{x}) = (p/2)(1-p) \left[p^{x_1}(1-p)^{x_2} + (1-p)^{x_1} p^{x_2} \right];$$

$$0<p<1, \ \underline{x} \in I_2^+$$

respectively. From (2.1) it is seen that both have the same scalar failure rate, $a(\underline{x}) = p(1-p)$. This naturally leads one to probe into the additional requirements that are needed to ensure the unique determination of $R(\underline{x})$. This is settled in the following Theorem.

Theorem 2.1

One of the marginal failure rates,

$$h_j(x_j) = P(X_j=x_j | X_j \geq x_j), \ j=1,2$$

(or equivalently a marginal distribution) along with $a(\underline{x})$ determine $R(\underline{x})$ uniquely.

Proof:

We first derive a recurrence relation connecting the failure rates and the survival function. Using (1.3) first we have,

$$R(x_1,0) = \prod_{y=0}^{x_1-1} [(1-h_1(y)], \ R(0,x_2) = \prod_{y=0}^{x_2-1} [(1-h_2(y)] \cdot$$

From definition 2.1,

$$f(\underline{x}) = a(\underline{x})R(\underline{x}); \qquad \underline{x} \in I_2^+.$$

When $x_2 = 0$,

$$f(x_1,0) = a(x_1,0)R(x_1,0),$$

$$= a(x_1,0) \prod_{y=0}^{x_1-1} [(1-h_1(y)].$$

Thus $f(x_1,0)$ is determined.

Further, when $x_2 = 1$

$$R(x_1,1) = R(x_1,0) - P[X_1 \geq x_1,\ X_2=0],$$

$$= R(x_1,0) - \sum_{t_1=x_1}^{\infty} f(t_1,0),$$

$$= R(x_1,0) - \sum_{t_1=x_1}^{\infty} R(t_1,0)a(t_1,0).$$

Making use of this specified value of $R(x_1,1)$ and proceeding along the same lines for $x_2=2,3,4...,$ we arrive at the recurrence relation

$$R(\underline{x}) = R(x_1,x_2-1) - \sum_{t_1=x_1}^{\infty} R(t_1,x_2-1)a(t_1,x_2-1). \qquad (2.2)$$

When the iteration is carried over x_1, another equivalent form is obtained as,

$$R(\underline{x}) = R(x_1-1,x_2) - \sum_{t_2=x_2}^{\infty} R(x_1-1,t_2)a(x_1-1,t_2). \qquad (2.3)$$

The equation in (2.2), when used iteratively starting with $x_2=1$ along with (1.3) establishes our assertion. If $h_2(x_2)$ is given then the result is established using equation (2.3) and iterating on x_1 starting with $x_1=1$. □

Example 2.2.

If a discrete random vector $\underline{X}$ has scalar faiure rate of form

$$a(\underline{x}) = [(1-p_1\theta^{x_2+1})(1-p_2\theta^{x_1+1})+\theta-1]\theta^{-1}$$

and the marginal failure rate of X_1 is $(1-p_1)$, $0<p_1<1$ then,

$$f(x_1,0) = [(1-p_1\theta)(1-p_2\theta^{x_1+1})+\theta-1]\theta^{-1}\, p_1^{x_1}$$

From (2.2)

$$R(x_1,1) = p_1^{x_1} - \sum_{t=x_1}^{\infty} [(1-p_1\theta)(1-p_2\theta^{t+1})+\theta-1]\theta^{-1}p_1^{t},$$

$$= p_1^{x_1} - (1-p_1\theta)\theta^{-1}\left[\sum_{t=x_1}^{\infty} p_1^{t} - \sum_{t=x_1}^{\infty} p_2\theta(\theta p_1)^{t}\right] -(1-\frac{1}{\theta})\sum_{t=x_1}^{\infty} p_1^{t}.$$

$$= p_1^{x_1} - [(1-p_1\theta)/\theta]\left[\frac{p_1^{x_1}}{1-p_1} - \frac{p_2\theta(\theta p_1)^{x_1}}{1-p_1\theta}\right] - (1-\frac{1}{\theta})\frac{p_1^{x_1}}{1-p_1}$$

$$= p_1^{x_1}\, p_2\theta^{x_1}.$$

Thus if,

$$R(x_1,x_2-1) = p_1^{x_1} p_2^{x_2-1} \theta^{x_1(x_2-1)} ,$$

we must have

$$R(x_1,x_2) = p_1^{x_1} p_2^{x_2-1} \theta^{x_1(x_2-1)} - \frac{1}{\theta} \sum_{t=x_1}^{\infty} p_1^{t} p_2^{x_2-1} \theta^{t(x_2-1)}$$

$$[(1-p_1\theta^{x_2})(1-p_2\theta^{t+1})-(1-\theta)] ,$$

$$= p_1^{x_1} p_2^{x_2-1} \theta^{x_1(x_2-1)} - [(1-p_1\theta^{x_2})\frac{1}{\theta} p_2^{x_2-1} \sum_{t=x_1}^{\infty} p_1^{t} \theta^{t(x_2-1)}]$$

$$+ (1-p_1\theta^{x_2})p_2^{x_2} \sum_{t=x_1}^{\infty} \theta^{t} p_1^{t} \theta^{t(x_2-1)}$$

$$+(1-\theta)\frac{1}{\theta} p_2^{x_2} \sum_{t=x_1}^{\infty} p_1^{t} \theta^{t(x_2-1)}$$

$$= p_1^{x_1} p_2^{x_2-1} \theta^{x_1(x_2-1)} - p_2^{x_2-1} \left(p_1\theta^{(x_2-1)}\right)^{x_1} + p_1^{x_1} p_2^{x_2} \theta^{x_1 x_2}$$

$$= p_1^{x_1} p_2^{x_2} \theta^{x_1 x_2}$$

Then by induction on x_2 the form of the survival function of

$\underline{X}$ is as in equation (1.9). Since $R(0,x_2)=p_2^{x_2}$; it follows that $0<p_2<1$ and from $0\le R(\underline{x})\le 1$ one must have $0\le\theta\le 1$. Also appealing to the probability mass function where $f(0,0)\ge 0$,

$$[(1-p_1\theta^{x_2+1})(1-p_2\theta^{x_1+1})+\theta-1]\ge 0, \text{ for all } \underline{x}\in I_2^+,$$

it follows that

$$(1-p_1\theta)(1-p_2\theta)+\theta-1]\ge 0$$

or

$$(1-p_1\theta)(1-p_2\theta)\geq 1-\theta.$$

This is the bivariate geometric distribution of Nair and Nair (1988) and we have thus established a characterization of that model in terms of $(a(\underline{x}),h_1(x_1))$

Example 2.3.

Let $\underline{X}$ have a scalar failure rate of the form

$$a(\underline{x}) = \frac{n(n+1)}{(m+n+x_1+x_2+1)(m+n+x_1+x_2)}, \quad m,n>0, \ \underline{x}\in I_2^+$$

and a marginal failure rate

$$h_1(x_1) = \frac{n}{(m+n+x_1)}.$$

Then for $x_2=0$ equation (2.2) can be written as

$$R(x_1,1)=\prod_{y=0}^{x_1-1}\left[1-\frac{n}{m+n+y}\right]-\sum_{t=x_1}^{\infty}\left\{\prod_{y=0}^{t-1}\left[1-\frac{n}{m+n+y}\right]\frac{n(n+1)}{(m+n+t+1)(m+n+t)}\right\},$$

$$=\frac{(m)_{x_1}}{(m+n)_{x_1}}-\sum_{t=x_1}^{\infty}\frac{(m)_t}{(m+n)_t}\frac{n(n+1)}{(m+n+t+1)(m+n+t)},$$

$$=\frac{(m)_{x_1}}{(m+n)_{x_1}}-\sum_{t=x_1}^{\infty}\frac{(m)_t}{(m+n)_t}\left[1-\frac{2(m+t)}{(m+n+t)}+\frac{(m+t)(m+t+1)}{(m+n+t)(m+n+t+1)}\right],$$

$$=\frac{(m)_{x_1}}{(m+n)_{x_1}}-\sum_{t=x_1}^{\infty}\frac{(m)_t}{(m+n)_t}+2\sum_{t=x_1}^{\infty}\frac{(m)_{t+1}}{(m+n)_{t+1}}-\sum_{t=x_1}^{\infty}\frac{(m)_{t+2}}{(m+n)_{t+2}},$$

$$=\frac{(m)_{x_1+1}}{(m+n)_{x_1+1}}\ ;\ m,n>0,$$

where $(a)_x = a\,(a+1)\ldots.(a+x-1)$. Now if we can write

$$R(x_1,x_2-1)=\frac{(m)_{x_1+x_2-1}}{(m+n)_{x_1+x_2-1}},$$

then from (2.2)

$$R(x_1,x_2)= \frac{(m)_{x_1+x_2-1}}{(m+n)_{x_1+x_2-1}} - \sum_{t=x_1}^{\infty} \frac{(m)_{t+x_2-1}}{(m+n)_{t+x_2-1}} \left[1- \frac{2(m+t+x_2-1)}{(m+n+t+x_2-1)} + \frac{(m+x_2+t-1)(m+x_2+t)}{(m+n+x_2+t-1)(m+n+x_2+t)}\right]$$

$$= \frac{(m)_{x_1+x_2-1}}{(m+n)_{x_1+x_2-1}} - \sum_{t=x_1}^{\infty} \frac{(m)_{t+x_2-1}}{(m+n)_{t+x_2-1}} + 2 \sum_{t=x_1}^{\infty} \frac{(m)_{t+x_2}}{(m+n)_{t+x_2}} - \sum_{t=x_1}^{\infty} \frac{(m)_{t+x_2+1}}{(m+n)_{t+x_2+1}}$$

$$= \frac{(m)_{x_1+x_2}}{(m+n)_{x_1+x_2}} \; ; \; m,n>0, \; \underline{x}\in I_2^+.$$

Then by induction the survival function of $\underline{X}$ has the expression given in the last equation. The marginal distributions are Waring and therefore, $\underline{X}$ has a bivariate Waring distribution.

Example 2.4

Suppose that $\underline{X}$ has scalar failure rate

$$a(\underline{x}) = \frac{k(k-1)}{(k+n-x_1-x_2)(m+n-x_1-x_2-1)} \; ; \quad k,n>0, \; x_1+x_2 \le n,$$

$$x_1, x_2 = 0,1,\ldots.,n$$

and a marginal failure rate

$$h_1(x_1) = \frac{k}{k+n-x_1}; \quad k,n>0, \; x_1=0,1,2,\ldots.,n.$$

The marginal survival function of X_1 is

$$R_1(x_1) = \prod_{y=0}^{x_1-1} \left[1 - \frac{k}{k+n-y}\right] ,$$

$$= \binom{k+n-x_1}{n-x_1} \Big/ \binom{k+n}{n}, \; k,n>0, \; x_1=0,1,2,\ldots,n.$$

Then from (2.2),

$$R(x_1,1) = \left[\binom{k+n-x_1}{n-x_1} - \sum_{t=x_1}^{\infty} \binom{k+n-t}{n-t}\right.$$

$$\left. + \; 2\sum_{t=x_1}^{\infty} \binom{k+n-t-1}{n-t-1} \sum_{t=x_1}^{\infty} \binom{k+n-t-2}{n-t-2}\right] \Big/ \binom{k+n}{n},$$

$$= \binom{k+n-x_1-1}{n-x_1} \Big/ \binom{k+n}{n},$$

Arguing as in the last two examples, with

$$R(x_1, x_2-1) = \binom{k+n-x_1-x_2-1}{n-x_1-x_2-1} \Big/ \binom{k+n}{n}$$

and equation (2.2)

$$R(\underline{x}) = \left[\binom{k+n-x_1-x_2-1}{n-x_1-x_2-1} - \sum_{t=x_1}^{\infty} \left[\binom{k+n-t-x_2-1}{n-t-x_2-1} + 2\binom{k+n-t-x_2-2}{n-t-x_2-2} - \binom{k+n-t-x_2-3}{n-t-x_2-3}\right]\right] \Big/ \binom{k+n}{n},$$

$$= \binom{k+n-x_1-x_2}{n-x_1-x_2} \Big/ \binom{k+n}{n}; \quad k,n>0, x_1+x_2 \le n, \quad x_1, x_2 = 0,1,2,\ldots n,$$

which is by induction the survival function of X. The model specified by this survival function has a bivariate negative hypergeometric form.

2.3. BIVARIATE MEAN RESIDUAL LIFE FUNCTION.

As an alternative to the failure rate, the bivariate mean residual life function defined as

$$\underline{r}(\underline{x}) = (r_1(\underline{x}), r_2(\underline{x}))$$

where

$$r_j(\underline{x}) = E[X_j - x_j | \underline{X} > \underline{x}], \quad j=1,2. \qquad (2.4)$$

can also be used for modelling failure time data and understanding the failure mechanism. Like the failure rate function the bivariate mean residual life also determines the distribution of failure times uniquely. The essential differences between the two are

a) The former accounts only for the immediate future in assessing the failure of the component whereas the latter incorporates the failure pattern for the entire duration after time t.

b) If the components progressively wear out the mean residual life will be a decreasing function of time while failure rate will be increasing in time. However, decreasing mean residual life does not imply increasing failure rate.

Although the analysis of mean residual life can be done independently without access to the failure rate, there exist an identity connecting these two which permits the study of the former once the behaviour of the failure rate is known without having to calculate the mean residual life separately. The relevant result is given in the following Theorem.

Theorem 2.2.

For all $\underline{x} \in I_2^+$,

$$1-a(x_1+1,x_2+1) = \frac{r_1(\underline{x})-1}{r_1(x_1+1,x_2)} + \frac{r_2(\underline{x})-1}{r_2(x_1,x_2+1)} + \frac{[r_2(x_1+1,x_2)-1][r_1(\underline{x})-1]}{r_2(x_1+1,x_2+1)\ r_1(x_1+1,x_2)}. \qquad (2.5)$$

Proof:

From (2.4) ,

$$R(x_1+1,x_2+1)r_1(\underline{x}) = \sum_{x_1+1}^{\infty} \sum_{x_2+1}^{\infty} (t-x_1)P[X_1=t_1,X_2=t_2],$$

$$= \sum_{t=x_1+1}^{\infty} (t-x_1)P[X_1=t,X_2>x_2],$$

$$= \sum_{t=x_1+1}^{\infty} R(t,x_2+1) \qquad (2.6)$$

and hence

$$\frac{r_1(\underline{x})-1}{r_1(x_1+1,x_2)} = \frac{R(x_1+2,x_2+1)}{R(x_1+1,x_2+1)} \quad . \tag{2.7}$$

Similarly,

$$\frac{r_2(\underline{x})-1}{r_2(x_1,x_2+1)} = \frac{R(x_1+1,x_2+2)}{R(x_1+1,x_2+1)} \tag{2.8}$$

so that

$$\frac{r_2(x_1+1,x_2)-1}{r_2(x_1+1,x_2+1)} = \frac{R(x_1+2,x_2+2)}{R(x_1+2,x_2+1)} \ . \tag{2.9}$$

Equation (2.7) and (2.9) leads to

$$\frac{R(x_1+2,x_2+2)}{R(x_1+1,x_2+1)} = \frac{[r_2(x_1+1,x_2)-1][r_1(\underline{x})-1]}{r_2(x_1+1,x_2+1)\ r_1(x_1+1,x_2)} \ . \tag{2.10}$$

Now,

$$1-a(x_1+1,x_2+1) = \frac{[R(x_1+2,x_2+1)+R(x_1+1,x_2+2)-R(x_1+2,x_2+2)]}{R(x_1+1,x_2+1)} \tag{2.11}$$

Using (2.7),(2.8),(2.9) and (2.10), to substitute for the ratio of R(.,.)'s we get (2.5). □

2.4. CHARACTERIZATIONS USING SCALAR FAILURE RATE. (Asha and Nair, 1994a)

In partial modification of the earlier assertion in Theorem 2.1 giving the condition for the unique determination of the survival function, in this section an alternate set of requirements that enable characterizations of models using the scalar failure rate are explored.

A reasonable requirement for a bivariate definition is that when the component variables are independent, there should be a meaningful interpretation of the bivariate concepts in terms of the established univariate concepts. We now show that this is indeed the case with the definition of scalar failure rate.

Theorem 2.3

The random variables X_1 and X_2 are independent if and only if

$$a(\underline{x}) = h_1(x_1)h_2(x_2)$$

Proof:

If X_1 and X_2 are independent, then from definition (2.1)

$$a(\underline{x}) = f_1(x_1)f_2(x_2)/R_1(x_1)R_2(x_2)$$

$$= h_1(x_1)\ h_2(x_2).$$

To prove the converse, we use the method of induction, by first noting that from (2.2) we can see that the result holds for $x_2=1$. Let the result be true for x_2.

$$R(x_1,x_2+1) = R(\underline{x}) - \sum_{t_1=x_1}^{\infty} R(t_1,x_2)a(t_1,x_2),$$

$$= R(\underline{x}) - \sum_{t_1=x_1}^{\infty} R(t_1,0)h_1(t_1)R(0,x_2)h_2(t_2)$$

$$= R(x_1,0)R(0,x_2) - f_2(x_2)\sum_{t_1=x_1}^{\infty} f_1(t_1)\ .$$

$$= R(x_1,0)[R(0,x_2) - f_2(x_2)]$$

$$= R(x_1,0)R(0,x_2+1).$$

The last equation along with the induction argument establishes that X_1 and X_2 are independent.

As has been pointed out already, assessment of the functional form of scalar failure rate based on failure time data or physical conditions of the system is the starting point in reliability modelling. However as the practice in most mathematical modelling problems, certain standard forms of scalar failure rate are worth consideration for preservation and use in situations where the actual behaviour coincides with the hypothesised models. Further such investigations can also lead to forms of scalar failure rate that characterize some well known bivariate distributions whose properties are readily available. When the simplicity of the final model is looked upon as desirable criterion, it becomes apparent that the scalar failure rate has to be assigned simple mathematical structures that render ease in computation of the survival function and other reliability characteristics. Accordingly in the next few theorems we discuss the situations where the scalar failure rate meets such requirements. The choice of such forms are also motivated from the desire to develop bivariate models that serve as extensions to the univariate laws derived under similar conditions.

Theorem 2.4

The functional forms $a(\underline{x})=c$, and $h_1(x_1)=h_1$, $0<c<h_1<1$, where c and h_1 are constants for all $\underline{x}\in I_2^+$, if and only if X_1 and X_2 are independent geometric variables with parameters h_1 and ch_1^{-1} respectively.

Proof:

When X_1 and X_2 are independent geometric random variables with the given parameters

$$R(\underline{x})=(1-h_1)^{x_1}\ (1-ch_1^{-1})^{x_2};\ \underline{x}\in I_2^+$$

with

$$R_1(x_1) = (1-h_1)^{x_1},\quad x_1\in I_1^+$$

and

$$f(\underline{x})= h_1(1-h_1)^{x_1}\ ch_1^{-1}\ (1-ch_1^{-1})^{x_2}.$$

Thus $a(\underline{x})=c$ and $h_1(x_1)=h_1$. Conversely from (2.2),

$$R(\underline{x})=R(x_1,x_2-1) - c \sum_{t_1=x_1}^{\infty} R(t_1,x_2-1).$$

When $x_2=1$,

$$R(x_1,1) = R(x_1,0) - c \sum_{t_1=x_1}^{\infty} R(t_1,0),$$

$$= \prod_{t=0}^{x_1-1} [1-h_1(t)] - c \sum_{t_1=x_1}^{\infty} \prod_{t=0}^{t_1-1} [1-h_1(t)],$$

$$= [1-h_1]^{x_1} - c \sum_{t_1=x_1}^{\infty} [1-h_1]^{t_1},$$

$$= [1-h_1]^{x_1} - ch_1^{-1}[1-h_1]^{x_1},$$

$$= [1-h_1]^{x_1} [1-ch_1^{-1}].$$

Thus if for some integer x_2-1,

$$R(x_1,x_2-1) = [1-h_1]^{x_1} [1-ch_1^{-1}]^{x_2-1},$$

then

$$R(\underline{x}) = [1-h_1]^{x_1} [1-ch_1]^{x_2-1} - c \sum_{t=x_1}^{\infty} [1-h_1]^{t} [1-ch_1^{-1}]^{x_2-1},$$

$$= [1-ch_1^{-1}]^{x_2-1} \{(1-h_1)^{x_1} - [c(1-h_1)^{x_1}/h_1]\},$$

$$= [1-ch_1^{-1}]^{x_2-1} (1-h_1)^{x_1}[1-ch_1^{-1}],$$

$$= [1-ch_1^{-1}]^{x_2}(1-h_1)^{x_1}, \quad \underline{x}\in I_2^{+}.$$

The conclusion follows by induction. □

Note.

That $a(\underline{x})=c$ alone does not guarantee neither independence nor geometric marginals is seen from the second survival function in Example 2.1. Since the above model does not work with systems in which the component life times are dependent, we relax the condition on scalar failure rate by requiring it to be piece-wise constant in the partitions of the sample space defined by $x_1<x_2$, $x_1>x_2$ and $x_1=x_2$.

Theorem 2.5.

The scalar failure rate is of the form

$$a(\underline{x}) = \begin{cases} c_1 & x_1>x_2 \\ c_2 & x_1<x_2 \\ c_3 & x_1=x_2, \end{cases} \tag{2.12}$$

where c_i's are constants such that $0<c_i<1$, $i=1,2,3$ and that marginal failure rate $h_j(x_j)$ is $(1-p_j)$ $0<p_j<1$, $j=1,2$ if and only if the random vector $\underline{X}$ has the bivariate geometric law with survival function.

$$R(\underline{x}) = \begin{cases} p^{x_2} p_1^{x_1-x_2} & x_1 \geq x_2 \\ p^{x_1} p_2^{x_2-x_1} & x_1 \leq x_2 \end{cases} \qquad (2.13)$$

$$1+p \geq p_1+p_2, \quad 0<p<\leq p_j<1, \quad j=1,2,$$

where $p=p_1+p_2-c_3-1$.

Proof:

When the distribution of $\underline{X}$ is specified by (2.13), the probability mass function of $\underline{X}$ is obtained as

$$f(\underline{x}) = R(\underline{x})-R(x_1+1,x_2)-R(x_1,x_2+1)-R(x_1+1,x_2+1)$$

$$= \begin{cases} (p_1-p)(1-p_1)p_1^{-1} \; p^{x_2} \, p_1^{x_1-x_2} & , \quad x_1>x_2 \\ (p_2-p)(1-p_2)p_2^{-1} \; p^{x_1} \, p_2^{x_2-x_1} & , \quad x_1<x_2 \\ (1+p-p_1-p_2)p^{x_1} & , \quad x_1=x_2, \end{cases}$$

$$1+p \geq p_1+p_2, \; 0<p\leq p_j<1, \quad j=1,2.$$

Thus

$$a(\underline{x}) = \frac{f(\underline{x})}{R(\underline{x})} = \begin{cases} (p_1-p)(1-p_1)p_1^{-1} & , \quad x_1>x_2 \\ (p_2-p)(1-p_2)p_2^{-1} & , \quad x_1<x_2 \\ (1+p-p_1-p_2) & , \quad x_1=x_2. \end{cases} \qquad (2.14)$$

Also the marginal distributions of (2.13) are obtained as

$$R(x_1,0) = p_1^{x_1}, \qquad 0<p_1<1, \quad x_1 \in I_1^+,$$

and

$$R(0,x_2) = p_2^{x_2}, \qquad 0<p_2<1, \quad x_2 \in I_1^+$$

so that $h_j(x_j) = 1-p_j$, $j = 1,2$.

Conversely, suppose that (2.12) holds. Then from (2.1) for $i=1,2,3$

$$c_i=[R(\underline{x})-R(x_1,x_2+1)-R(x_1+1,x_2)+R(x_1+1,x_2+1)][R(\underline{x})]^{-1} \quad (2.15)$$

according as $x_1>x_2, x_1<x_2$ and $x_1=x_2$ respectively.

Consider the equation in the first region $x_1>x_2$. When $x_2=0$,

$$c_1R(x_1,0) = [R(x_1,0)-R(x_1,1)-R(x_1+1,0)+R(x_1+1,1)].$$

By the assumption of constancy of the marginal failure rates the marginal distributions are geometric so that

$$R(x_1,0)=p_1^{x_1} \quad \text{and} \quad R(0,x_2)=p_2^{x_2}.$$

Thus,

$$R(x_1,1)-R(x_1+1,1) = p_1^{x_1} [1-p_1-c_1].$$

Summing over x_1 provides,

$$R(x_1,1) = p_1^{x_1} [1-p_1-c_1][1-p_1]^{-1}, \quad x_1=1,2,\ldots . \qquad (2.16)$$

For $x_2=1$

$$c_1 R(x_1,1) = [R(x_1,1)-R(x_1,2)-R(x_1+1,1)+R(x_1+1,2)]$$

$$x_1=2,3,\ldots$$

Substituting for $R(x_1,1)$ from equation (2.16),

$$R(x_1,2)-R(x_1+1,2) = p_1^{x_1}(1-p_1-c_1)^2(1-p_1)^{-1}$$

and summation over x_1 provides

$$R(x_1,2) = p_1^{x_1}(1-p_1-c_1)^2(1-p_1)^{-2}.$$

Iterating for $x_2=3,4,\ldots,x_1-1$ we obtain

$$R(\underline{x}) = p_1^{x_1}(1-p_1-c_1)^{x_2} (1-p_1)^{-x_2}. \qquad (2.17)$$

Next observe that by putting $x_1=0$ in (2.15)

$$c_2 R(0,x_2)=[R(0,x_2)-R(0,x_2+1)-R(1,x_2)+R(1,x_2+1)], \quad x_2=1,2,\ldots \qquad (2.18)$$

Proceeding as above, by the assumption of geometric marginals we get

$$R(1,x_2)-R(1,x_2+1) = p_2^{x_2}(1-p_2-c_2).$$

Now summation over x_2 provides,

$$R(1,x_2) = p_2^{x_2}(1-p_2-c_2)(1-p_2)^{-1}, \quad x_2=1,2,... \tag{2.19}$$

and subsequently as continuation of the same process of computation,

$$R(\underline{x}) = p_2^{x_2}(1-p_2-c_2)^{x_1}(1-p_2)^{-x_1}, \quad x_1=0,1,2,...,x_2-1. \tag{2.20}$$

Whenever $x_1=x_2$, we observe that for $x_1=x_2=0$

$$R(1,1) = p_1+p_2+c_3-1 = p.$$

Specialising for $x_1=x_2=x=1,2,3...$ it is seen that

$$R(x,x) = p^x, \qquad x\in I_1^+.$$

Also (2.16) and (2.19) reveal that

$$p = R(1,1) = [p_1+p_2+c_3-1] = p_1(1-p_1-c_1)/(1-p_1),$$

$$= p_2(1-p_2-c_2)/(1-p_2). \tag{2.21}$$

Using (2.21), the form of the bivariate geometric law (2.13) is recovered and the proof of the theorem is completed.

The lack of memory property of distributions, the constancy of the failure rate and the constancy of the mean residual life are well known characteristic properties of the geometric distribution, among the class of distributions in the support of the set of non-negative integers. Though the results concerning each was established independently, Galambos and Kotz (1978) points out the equivalence of these properties in the continuous univariate case. In view of the possibility of alternative definitions of failure rates in higher dimensions it is of interest to examine the implications of definition of failure rate to a meaningful definition of bivariate lack of memory property. Therefore, we propose one form of definition of bivariate lack of memory property and examine its relationship with the constancy of the failure rate being currently discussed.

2.5. BIVARIATE LACK OF MEMORY PROPERTY

Definition 2.2

A random vector X is said to possess the bivariate lack of memory property if it satisfies the equation

$R(\underline{x}+\underline{t}) = R(\underline{x})R(\underline{t})$, for all $\underline{x},\underline{t}=(t,t);\underline{x}\in I_2^+, t\in I_1^+$.

or

$$P[X_1\geq x_1+t, X_2\geq x_2+t \mid X\geq x] = P(X_1\geq t, X_2\geq t) \quad \text{for all}$$

$\underline{x},\underline{t}=(t,t);\underline{x}\in I_2^+, t\in I_1^+$.

Theorem 2.6

If $\underline{X}$ is a discrete random vector in the support of I_2^+, with geometric marginals, then $\underline{X}$ possesses the bivariate lack of memory property if and only if $\underline{X}$ is distributed as the bivariate geometric law (2.13).

Proof:

When bivariate lack of memory property is satisfied the equation

$$R(x_1+t, x_2+t) = R(\underline{x})R(\underline{t}) \tag{2.22}$$

holds for all $\underline{x}\in I_2^+$ and $\underline{t}=(t,t), t\in I_1^+$.

The unique solution to (2.22) subject to conditions of geometric marginals for $\underline{X}$ are obtained first by setting $x_1=x_2=x$ and $R(x,x)=G(x)$ for all x in I_1^+ in (2.22).

Then (2.22) becomes the Cauchy functional equation

$$G(x+t) = G(x)G(t), \qquad x,t \in I_1^+.$$

whose only solution that satisfies the conditions of a survival function is

$$G(x) = p^x, \text{ for some } 0<p<1.$$

Now, taking $x_2=0$ in (2.22),

$$R(x_1+t,t) = R(x_1,0)p^t,$$

$$= p^t p_1^{x_1}, \quad 0<p_1<1.$$

Thus,

$$R(\underline{x}) = p^{x_2} p_1^{x_1-x_2} \quad \text{for } x_1 \geq x_2$$

Similarly,

$$R(\underline{x}) = p^{x_1} p_2^{x_1-x_2} \text{ for } x_1 \leq x_2 \; .$$

Conversely if X is distributed as (2.13) then,

$$R(x_1+t,x_2+t) = \begin{cases} p^{x_2} p_1^{x_1-x_2} p^t & x_1+t \geq x_2+t \\ p^{x_1} p_2^{x_2-x_1} p^t & x_1+t \leq x_2+t \end{cases}$$

$$= R(\underline{x})\, R(\underline{t}) \qquad \underline{x} \in I_2^+ \text{ and } \underline{t}=(t,t), t \in I_1^+.$$

which establishes the theorem.

The following theorem is a direct consequence of the last two theorems.

Theorem 2.7

If $\underline{X}$ has geometric marginals then the piece-wise constancy of the scalar failure rate is equivalent to the bivariate lack of memory property.

The deliberations carried out in the present chapter to define a failure rate in the two-dimensional discrete time domain, viewed it as a scalar quantity, providing a single over all index that depicts the pattern of failure. However, in many cases as exemplified above, we require additional information on atleast one of the failure rates of the components to determine the distribution of life lengths uniquely. Thus further development of the analysis of failure time distributions based on scalar failure rate appears to be complicated and alternative formulations which are more productive seem to be in order. These will be discussed in the following chapter.

Chapter III

VECTOR FAILURE RATES

3.1 INTRODUCTION

Following the observations made in the concluding part of the previous chapter, in the succeeding sections alternative formulations of the concept of bivariate failure rate are taken up and their properties are studied. Instead of providing a single cumulative index, the proposed definitions build up the concepts through a mechanism that evaluates the spot changes in the joint survival times in terms of each of the component life times. Such an approach has been successfully employed in the continuous case by Marshall(1975a) and Johnson and Kotz (1975) to define vector failure rates that are amenable to algebra and produce reasonably good physical interpretation. Following this lead, analogous definition in the discrete domain introduced in Section 1.3 of Chapter I are further analysed and several new results in this connection are established in the present chapter.

Recall from Section 1.3 that the vector failure rate of a bivariate random vector $\underset{\sim}{X}$ can be defined, (Nair and Nair,(1990)) as

$$\underset{\sim}{b}(\underset{\sim}{x}) = (b_1(\underset{\sim}{x}), b_2(\underset{\sim}{x}))$$

where

$$b_j(\underset{\sim}{x}) = P(X_j = x_j \mid \underset{\sim}{X} \geq \underset{\sim}{x}), \; j=1,2. \tag{3.1}$$

It follows that

$$\left.\begin{aligned} 1-b_1(\underset{\sim}{x}) &= R(x_1+1, x_2)/R(\underset{\sim}{x}) \\ \text{and} \quad 1-b_2(\underset{\sim}{x}) &= R(x_1, x_2+1)/R(\underset{\sim}{x}) \end{aligned}\right. \tag{3.2}$$

Other than defining this rate with the object of characterizing certain distributions, Nair and Nair, (1990) do not explicitly take up a discussion of the properties of these rates. Accordingly we address to this problem in the following section.

3.2 PROPERTIES OF THE VECTOR FAILURE RATE

Theorem 3.1

The bivariate mean residual life in Definition 2.4 is related to vector failure rate as

$$1-b_1(x_1+1,x_2+1) = [r_1(\underline{x})-1]/r_1(x_1+1,x_2)$$

and (3.3)

$$1-b_2(x_1+1,x_2+1) = [r_2(x)-1]/r_2(x_1,x_2+1).$$

Proof:

By definition,

$$r_1(x) = E\,[(X_1-x_1)|\ X_1>x_1,\ X_2>x_2],$$

$$= [R(x_1+1,x_2+1)]^{-1} \sum_{y=x_1+1}^{\infty} (y-x_1)\ P[X_1=y,X_2>x_2],$$

$$= [R(x_1+1,x_2+1)]^{-1} \sum_{n=1}^{\infty} n\ P[X_1=x_1+n,\ X_2>x_2],$$

$$= [R(x_1+1,x_2+1)]^{-1} \sum_{n=1}^{\infty} R(x_1+n,\ x_2+1),$$

or

$$r_1(x)R(x_1+1,x_2+1)= \sum_{n=1}^{\infty} R(x_1+n,\ x_2+1). \qquad (3.4)$$

Incrementing x_1 to x_1+1,

$$r_1(x_1+1,x_2)R(x_1+2,x_2+1)= \sum_{n=1}^{\infty} R(x_1+n+1,\ x_2+1). \qquad (3.5)$$

Subtracting (3.5) from (3.4),

$$r_1(\underline{x})R(x_1+1,x_2+1) - r_1(x_1+1,x_2)R(x_1+2,x_2+1) = R(x_1+1,x_2+1),$$

from which it is seen that

$$1-b_1(x_1+1,x_2+1) = [r_1(\underline{x})-1]/r_1(x_1+1,x_2).$$

Similarly,

$$1-b_2(x_1+1,x_2+1) = [r_2(\underline{x})-1]/r_2(x_1,x_2+1).$$

Theorem 3.2.

The scalar failure rate $a(\underline{x})$ is related to the vector failure rate $\underline{b}(\underline{x})$ as either

$$a(x) = b_1(x)-b_1(x_1,x_2+1) + b_1(x_1,x_2+1)b_2(\underline{x})$$

or

$$a(\underline{x}) = b_2(\underline{x})-b_2(x_1,x_2+1) + b_1(x)b_2(x_1+1,x_2).$$

Proof:

The scalar failure rate

$$a(\underline{x}) = \frac{R(\underline{x})-R(x_1+1,x_2)-R(x_1,x_2+1)+R(x_1+1,x_2+1)}{R(x)}$$

From (3.2), the assertion follows. □

Note: One cannot choose the components $b_1(x)$ and $b_2(\underline{x})$ arbitrarily as they should satisfy the consistency condition

$$[1-b_1(x_1,x_2+1)][1-b_2(x)] = [1-b_2(x_1+1,x_2)][1-b_1(x)].$$

3.3. CHARACTERIZATIONS USING VECTOR FAILURE RATE. (Asha and Nair, 1994a).

Our objectives in developing characterization theorems of distributions in terms of failure rates have been explained in connection with the discussion of similar problem involving the scalar failure rate. The following theorems relating to vector failure rate help spotting the suitable models and also to describe the random mechanism that generates such models.

Theorem 3.3.

The following statements are equivalent.

(i) X_1 and X_2 are independent .

(ii) $b_j(\underline{x}) = h_j(x_j)$; j=1 or 2, $\underline{x} \in I_2^+$.

Proof:

If (i) holds then by equation (3.1), (ii) is verified. Conversely if (ii) holds for every $\underline{x} \in I_2^+$ we have from (1.2) and (3.1)

$$P(X_j = x_j \mid \underline{X} \geq \underline{x}) = P(X_j = x_j \mid X_j \geq x_j),\ j=1 \text{ or } 2,\ \underline{x} \in I_2^+$$

In particular for j=1, this is equivalent to

$$R(\underline{x}) - R(x_1+1,x_2) = R(x)\ h_1(x_1)$$

or,

$$[1-h_1(x_1)]\ R(\underline{x}) = R(x_1+1,x_2).$$

Thus

$$R(\underline{x}) = R(0,x_2) \prod_{r=0}^{x_1-1} [1-h_1(r)].$$

The second expression on the right is $R(x_1,0)$ by equation (1.3). This means that

$$R(\underline{x}) = R(x_1,0)\ R(0,x_2)$$

for every $\underline{x}\in I_2^+$. Thus X_1 and X_2 are independent □

Theorem 3.4

The following statements are equivalent.

(i) X_1 and X_2 are independent geometric variables.

(ii) $\underline{b}(\underline{x}) = (b_1,b_2)$ where b_j's are constants for all $\underline{x}\in I_2^+$.

(iii) $R(\underline{x}+\underline{s}) = R(\underline{x})R(\underline{s})$ for every $\underline{x}$ and $\underline{s}=(s_1,s_2) \in I_2^+$.

Proof:

If (i) holds then (ii) follows from (3.1) and Theorem 3.3. Conversely if (ii) holds, then from (1.3), we obtain

$$R(x) = [1-b_1]^{x_1} [1-b_2]^{x_2}.$$

and therefrom (i) is established.

The equation (iii) follows from (i) directly. Now, if (iii) holds then on setting $x_1, x_2 = x$ and $s_1, s_2 = s$ in (iii), it reduces to the Cauchy functional equation.

$$G(x+s) = G(x)\, G(s)$$

where $G(x)=R(x,x)$, whose only solution that satisfies the conditions of a survival function is

$$G(x) = p^x, \quad \text{for some } 0<p<1,\ x \in I_1^+$$

Further when $x_j = s_j = 0$, (iii) once again reduces to the Cauchy Functional equation giving a solution

$$R(x_1, 0) = p_1^{x_1}, \qquad 0<p_1<1,\ x \in I_1^+$$

and

$$R(0, x_2) = p_2^{x_2}, \qquad 0<p_2<1,\ x \in I_1^+.$$

Also $s_1=x_2=0$ implies, $R(x_1, s_2) = R(x_1, 0)R(0, s_2)$ for all $x_1, s_2 \in I_1^+$, thus establishing (i). □

It becomes apparent from the above deliberations that distributions with independent marginals demand for constancy of the vector failure rate. Thus there exist no bivariate model in which the variables are dependent whose vector failure rate exhibits constancy. The investigation of the model in which the vector failure rate is locally constant resulted in the bivariate model (1.9). The results in this direction have already been reviewed. We further proceed to realise another non-trivial bivariate distribution toying with the idea of keeping the constancy of the vector failure rate in some manner. The following paves way for such a property.

Theorem 3.5

The bivariate lack of memory property is equivalent to

(i) $b_j(\underline{x}+\underline{t})=b_j(\underline{x})$, j=1,2 for all $\underline{x}\in I_2^+$ and $\underline{t}=(t,t), t\in I_1^+$.

(ii) $\underline{r}(\underline{x}+\underline{t}) = \underline{r}(\underline{x})$, for all $\underline{x}\in I_2^+$ and $\underline{t}=(t,t), t\in I_1^+$.

where $\underline{r}(.,.)$ is the mean residual life function defined in equation (2.4).

Proof:

When the bivariate lack of memory property holds (i) is seen to be true from (3.2). Conversely note that (i) implies

$$\frac{R(x_1+t+1,x_2+t)}{R(x_1+t,x_2+t)} = \frac{R(x_1+1,x_2)}{R(\underset{\sim}{x})} \tag{3.6}$$

and

$$\frac{R(x_1+t,x_2+t+1)}{R(x_1+t,x_2+t)} = \frac{R(x_1,x_2+1)}{R(\underset{\sim}{x})} \tag{3.7}$$

From the last two equations when $x_1=0$, $x_2=0$,

$$R(t+1,t)=R(1,0)R(t),$$

and

$$R(t,t+1)=R(0,1)R(t), \text{ for all } t\in I_1^+.$$

Thus the bivariate lack of memory property holds for the points (0,1) and (1,0) from which it follows

$$R(t+1,t+1) = R(t,t+1)[R(1,1)/R(0,1)]$$

$$= R(1,1)R(t,t),$$

, proving that it holds for the point (1,1). The generality of the assertion is proved by the method of induction by assuming that the result holds for (i+1,j) and (i,j+1). Thus from (3.6) and (3.7) it follows that

$$R(i+t+2,j+t) = R(i+t+1,j+t)[R(i+2,j)/R(i+1,j)]$$

$$= R(t,t)R(i+2,j), \text{ for all } t\in I_1^+$$

and similarly

$$R(i+t,j+t+2) = R(t,t)R(i,j+2), \text{ for all } t\in I_1^+,$$

which proves the bivariate lack of memory property.

Now, if (ii) holds, then from (3.3), it seen that (3.6) and (3.7) are true which implies the bivariate lack of memory property.

Conversely if the bivariate lack of memory property holds then,

$$r_1(x_1+t,x_2+t) = [R(x_1+t+1,x_2+t+1)]^{-1} \sum_{y=x_1+1}^{\infty} (y-x_1)\, P[X_1=y+t, X_2>x_2+t],$$

$$= [R(x_1+1,x_2+1)R(t,t)]^{-1} \sum_{y=x_1+1}^{\infty} (y-x_1)\, P[X_1=y+t, X_2>x_2+t],$$

$$= [R(x_1+1,x_2+1)R(t,t)]^{-1} \sum_{y=x_1+1}^{\infty} (y-x_1)[R(y+t,x_2+t+1)-R(y+t+1,x_2+t+1)],$$

$$= [R(x_1+1,x_2+1)]^{-1} \sum_{y=x_1+1}^{\infty} (y-x_1)[R(y,x_2+1) - R(y+1,x_2+1)],$$

$$= [R(x_1+1,x_2+1)]^{-1} \sum_{y=x_1+1}^{\infty} (y-x_1)P[X_1=y,X_1>x_2],$$

$$= r_1(\underline{x}).$$

In a similar manner it can be proved that the bivariate lack of memory property implies

$$r_2(x_1+t,x_2+t) = r_2(\underline{x}).$$

Thus

$$\underline{r}(\underline{x}+\underline{t}) = \underline{r}(\underline{x}); \text{ for all } \underline{x}\in I_2^+ \text{ and } \underline{t}=(t,t), t\in I_1^+. \quad \square$$

Theorem 3.6

A random vector $\underline{X}$ with support I_2^+ has distribution (2.13) if and only if its vector failure rate is of the form

$$\underline{b}(\underline{x}) = \begin{cases} (a_1,b_1), & x_1>x_2 \\ (a_2,b_2), & x_1<x_2 \\ (a_1,b_2), & x_1=x_2 \end{cases} \qquad (3.8)$$

for all $\underline{x}$ in I_2^+, $0< a_j,b_j<1$, $j=1,2$.

Proof:

If $\underline{X}$ is distributed as (2.13) then by (3.1)

$$b_1(\underline{x}) = 1 - \frac{R(x_1+1,x_2)}{R(\underline{x})},$$

$$
= \begin{cases}
1 - \dfrac{p^{x_2} p_1^{x_1 - x_2 + 1}}{p^{x_2} p_1^{x_1 - x_2}} \; ; & x_1 > x_2 \\[2ex]
1 - \dfrac{p^{x_1 + 1} p_2^{x_2 - x_1 - 1}}{p^{x_1} p_2^{x_2 - x_1}} \; ; & x_1 < x_2 \\[2ex]
1 - \dfrac{p^{x_1} p_1}{p^{x_1}} \; ; & x_1 = x_2
\end{cases}
$$

and

$$
b_2(\underset{\sim}{x}) = 1 - \frac{R(x_1, x_2 + 1)}{R(x)}
$$

$$
= \begin{cases}
1 - \dfrac{p^{x_2 + 1} p_1^{x_1 - x_2 - 1}}{p^{x_2} p_1^{x_1 - x_2}} \; ; & x_1 > x_2 \\[2ex]
1 - \dfrac{p^{x_1} p_2^{x_2 - x_1 + 1}}{p^{x_1} p_2^{x_2 - x_1}} \; ; & x_1 < x_2 \\[2ex]
1 - \dfrac{p^{x_2} p_2}{p^{x_2}} \; ; & x_1 = x_2
\end{cases}
$$

giving

$$\underline{b}(\underline{x}) = \begin{cases} (1-p_1, 1-pp_1^{-1}) & , \quad x_1>x_2 \\ (1-pp_2^{-1}, 1-p_2) & , \quad x_1<x_2 \\ (1-p_1, 1-p_2) & , \quad x_1=x_2. \end{cases}$$

Now, if (3.8) holds, then from the recurrence relation

$$[1-b_1(\underline{x})]\ R(\underline{x}) = R(x_1+1, x_2), \tag{3.9}$$

we get on iteration,

$$R(\underline{x}) = \begin{cases} (1-a_1)^{x_1-x_2-1}\ R(x_2+1, x_2), & x_1 \geq x_2 \\ (1-a_2)^{x_1}\ R(0, x_2), & x_1<x_2\ . \end{cases} \tag{3.10}$$

Similarly,

$$[1-b_2(x)]\ R(x) = R(x_1, x_2+1) \tag{3.11}$$

and

$$R(\underline{x}) = \begin{cases} (1-b_2)^{x_2-x_1-1}\ R(x_1, x_1+1), & x_1 \leq x_2 \\ (1-b_1)^{x_2}\ R(x_1, 0), & x_1>x_2. \end{cases} \tag{3.12}$$

Combining (3.11) and (3.12)

$$R(\underline{x}) = \begin{cases} (1-b_2)^{x_2-1}(1-a_2)^{x_1} R(0,1), & x_1<x_2 \\ (1-b_1)^{x_2}(1-a_1)^{x_1-1} R(1,0), & x_1>x_2 \end{cases}$$

and

$$R(x,x)= (1-a_1)^{x-1}(1-b_1)^{x} R(1,0) = (1-a_2)^{x}(1-b_2)^{x-1} R(0,1),$$

so that

$$R(\underline{x}) = \begin{cases} (1-b_2)^{x_2-1}(1-a_2)^{x_1} R(0,1), & x_1\leq x_2 \\ (1-b_1)^{x_2}(1-a_1)^{x_1-1} R(1,0), & x_1\geq x_2. \end{cases}$$

Also,

$$R(0,0)=1=(1-a_1)^{-1} R(1,0) =(1-b_2)^{-1} R(0,1)$$

giving

$$R(1,0) = (1-a_1) \text{ and } R(0,1) = 1-b_2.$$

Thus for all $\underline{x}$ in I_2^+ we must have,

$$R(\underline{x}) = \begin{cases} (1-b_2)^{x_2}(1-a_2)^{x_1} & x_1\leq x_2 \\ (1-b_1)^{x_2}(1-a_1)^{x_1}) & x_1\geq x_2. \end{cases}$$

Specialising for $x_1=x_2$,

$$(1-a_1)(1-b_1) = (1-a_2)(1-b_2).$$

Writing $1-a_1=p_1$, $1-b_2=p_2$ and $(1-a_1)(1-b_1)=p$ we get the form (3.8) and $0<p\le p_1,p_2<1$, $1+p\ge p_1+p_2$. □

From the above two results and Theorem 2.6, the following result can be proved.

Theorem 3.7

The following statements are equivalent

1. $b_j(\underline{x}+\underline{t}) = b_j(\underline{x})$, $h_j(x_j+t) = h_j(x_j)$.

   for all $\underline{x}\in I_2^+$, $\underline{t}=(t,t)$; $x_j,t\in I_1^+$, $j=1,2$.

2. $R(\underline{x}+\underline{t})=R(\underline{x})R(\underline{t})$, $R_j(x_j+t)=R_j(x_j)R_j(t)$ for all $\underline{x}\in I_2^+$, $\underline{t}=(t,t)$, $t,x_j\in I_1^+$.

   where $R_j(x_j) = P(X_j > x_j)$.

3. X has bivariate geometric distribution specified by (2.13).

Proof:

That the statements (i) and (ii) are equivalent follows from Theorem 3.5 and the analogous result in the univariate case (See Section 1.3). The equivalence of (ii) and (iii) follows from Theorem 2.6. □

3.4 CONDITIONAL FAILURE RATE

There are practical situations (see Castillo and Galambos,(1989)) cited in literature where the access to the conditional distributions are more likely than to the joint distribution, bringing to therefore the problem of determining the latter through the former. In life testing situations the behaviour of each of the components hypothesised by the survival time of the other component, taken together, formed the basis of all definitions so far considered. Instead of conditioning on the survival times, in a new definition proposed by Kotz and Johnson, (1991), the conditional event is taken to be the survival time of the given component and failure time of the other component. This definition too assumes a vector form given by,

$$\underline{c}(\underline{x})=(c_1(\underline{x}),c_2(\underline{x}))$$

where

$$c_j(\underline{x})=P(X_j=x_j|X_k=x_k,X_j\geq x_j),\quad j,k=1,2 \text{ and } j\neq k. \qquad (3.13)$$

The failure rate so defined will be for obvious reasons called the conditional failure rate in the sequel. Notice that $c_1(\underline{x})$ gives the probability of the failure of the first component at time x_1, given that its life time is at least x_1 units of time and that the second component has survived only upto time x_2. An analogous interpretation is given for $c_2(\underline{x})$.

From equation (3.13), we see that

$$c_1(\underline{x}) = P(X_1=x_1 | X_1 \geq x_1, X_2=x_2),$$

$$= \frac{f(\underline{x})}{R(x)-R(x_1,x_2+1)},$$

$$= \frac{R(\underline{x})-R(x_1+1,x_2)-R(x_1,x_2+1)+R(x_1+1,x_2+1)}{R(\underline{x})-R(x_1,x_2+1)}.$$

Similarly,

$$c_2(\underline{x}) = \frac{R(\underline{x})-R(x_1+1,x_2)-R(x_1,x_2+1)+R(x_1+1,x_2+1)}{R(\underline{x})-R(x_1+1,x_2)}.$$

Therefore, we have

$$1-c_1(\underline{x}) = \frac{R(x_1+1,x_2)-R(x_1+1,x_2+1)}{R(x)-R(x_1,x_2+1)}$$

and (3.14)

$$1-c_2(\underline{x}) = \frac{R(x_1,x_2+1)-R(x_1+1,x_2+1)}{R(x)-R(x_1,x_2+1)}$$

The bivariate mean residual life defined in equation (2.4) is related to the conditional failure rate by

$$1-c_1(x_1+1,x_2+1)=\frac{[r_1(\underline{x})-1]\left[1-\dfrac{(r_2(x_1+1,x_2)-1)}{r_2(x_1+1,x_2+1)}\right]}{r_1(x_1+1,x_2)\left[1-\dfrac{(r_2(\underline{x})-1)}{r_2(x_1,x_2+1)}\right]}$$

and

$$1-c_2(x_1+1,x_2+1)=\frac{[r_2(\underline{x})-1]\left[1-\dfrac{(r_1(x_1,x_2+1)-1)}{r_2(x_1+1,x_2+1)}\right]}{r_2(x_1,x_2+1)\left[1-\dfrac{(r_1(\underline{x})-1)}{r_1(x_1+1,x_2)}\right]}.$$

In this connection it is worthwhile to observe that the conditioning event in respect of the conditional failure rate and bivariate mean residual life are different and therefore while interpreting one in terms of the other we are obviously speaking of different kind of events to make comparision between the two rather inconvenient. But if the requirement is the mean residual life function or failure rate in question, then the relationship is useful. However, if the purpose is the comparative behaviour of the two concepts, it is more meaningful to have a new definition for mean residual life function in which the conditioning events are identical. Accordingly we propose the following new definition of the mean residual life function.

Definition 3.1

The mean residual life of $\underline{X}$ in the support of I_2^+ is defined as the vector

$$\underline{m}(\underline{x}) = (m_1(\underline{x}), m_2(\underline{x}))$$

where

$$m_1(\underline{x}) = E((X_1 - x_1) | X_1 \geq x_1, X_2 = x_2)$$

and (3.15)

$$m_2(\underline{x}) = E((X_2 - x_2) | X_1 = x_1, X_2 \geq x_2).$$

Then

$$m_1(\underline{x})[R(\underline{x}) - R(x_1, x_2+1)] = \sum_{y=x_1+1}^{\infty} [R(y, x_2) - R(y, x_2+1)] \quad (3.16)$$

and

$$m_1(x_1+1, x_2)\ [R(x_1+1, x_2) - R(x_1+1, x_2+1)]$$
$$= \sum_{y=x_1+2}^{\infty} [R(y, x_2) - R(y, x_2+1)]. \quad (3.17)$$

Subtracting (3.17) from (3.16),

$$m_1(\underline{x})[R(x)-R(x_1,x_2+1)]-m_1(x_1+1,x_2)[R(x_1+1,x_2)-R(x_1+1,x_2+1)]$$

$$= R(x_1+1,x_2) - R(x_1+1,x_2+1).$$

Therefore,

$$m_1(\underline{x})\left[\frac{R(\underline{x})-R(x_1,x_2+1)}{R(x_1+1,x_2)-R(x_1+1,x_2+1)}\right] = 1+m_1(x_1+1,x_2)$$

From (3.14)

$$c_1(\underline{x}) = 1 - \frac{m_1(\underline{x})}{1+m_1(x_1+1,x_2)} .$$

Similarly

$$c_2(\underline{x}) = 1 - \frac{m_2(\underline{x})}{1+m_2(x_1,x_2+1)}$$

On the question of unique determination of the survival function in terms of conditional failure rate we show in the following theorem that the answer is not in the affirmative.

Theorem 3.8

In general, the failure rate $\underline{c}(\underline{x})$ need not determine the distribution of $\underline{X}$ uniquely, but if $f(\underline{x})>0$ for all $\underline{x}\in I_2^+$ then $f(\underline{x})$ is determined uniquely by $\underline{c}(\underline{x})$.

Proof:

The first part follows from the consideration of the two distributions

(i)

$f(\underline{x})=$

X_2 \ X_1	1	2	3
1	$\frac{1}{4}$	$\frac{1}{8}$	0
2	0	$\frac{1}{8}$	0
3	0	0	$\frac{1}{2}$

(ii)

$f(\underline{x})=$

X_2 \ X_1	1	2	3
1	$\frac{1}{3}$	$\frac{1}{6}$	0
2	0	$\frac{1}{6}$	0
3	0	0	$\frac{1}{3}$

which are different but have the same conditional failure rate viz

$c(\underline{x})=$

X_2 \ X_1	1	2	3
1	(2/3,1)	(1,1/2)	(0,0)
2	(0,0)	(1,1)	(0,0)
3	(0,0)	(0,0)	(1,1)

To prove the second part, note that

$$1-c_1(\underline{x}) = \frac{H(x_1+1, x_2)}{H(\underline{x})} \tag{3.18}$$

where

$$H(x) = R(\underline{x}) - R(x_1, x_2+1),$$

$$= P[X_1 \geq x_1, X_2 = x_2].$$

Solving (3.18),

$$H(\underline{x}) = \prod_{r=0}^{x_1-1} [1-c_1(r, x_2)]H(0, x_2),$$

$$= \prod_{r=0}^{x_1-1} [1-c_1(r, x_2)]\ P[X_2 = x_2].$$

Thus

$$P[X_1 \geq x_1 | X_2 = x_2] = \prod_{r=0}^{x_1-1} [1-c_1(r, x_2)],$$

or

$$P[X_1 = x_1 | X_2 = x_2] = \prod_{r=0}^{x_1-1} [1-c_1(r, x_2)]c_1(\underline{x}). \tag{3.19}$$

Similarly

$$P[X_2 = x_2 | X_1 = x_1] = \prod_{r=0}^{x_2-1} [1-c_2(x_1, r)]c_2(\underline{x}). \tag{3.20}$$

Thus $c(\underline{x})$ determines the conditional distribution of X_j given $X_k = x_k$ uniquely for $j,k=1,2$ and $j \neq k$. Since $f(\underline{x})>0$ for all $\underline{x}$, it follows from Gourieroux and Monfort, (1979) that the conditional distributions uniquely determine $f(\underline{x})$. An alternative set of conditions that ensures the determination of $f(\underline{x})$ through the conditionals is provided in the following theorem.

Theorem 3.9

A necessary and sufficient condition for the conditional densities $f(x_1|x_2)$ and $f(x_2|x_1)$ in (3.19) and (3.20) to determine the joint probability mass function is that

$$f(x_1|x_2)/f(x_2|x_1) = A_1(x_1)/A_2(x_2),$$

where $A_j(x_j)$ are positive with finite sums over the set of non-negative integers satisfying

$$\sum_{x_1=0}^{\infty} A_1(x_1) = \sum_{x_2=0}^{\infty} A_2(x_2).$$

Proof:

The proof of this result is on lines similar to that in the continuous case given in Abrahams and Thomas, (1984).

Let $f(\underset{\sim}{x})$ be a joint probability mass function with conditionals $f(x_1|x_2)$ and $f(x_2|x_1)$. Then,

$$f(x_1|x_2)f_2(x_2) = f(x_2|x_1)f_1(x_1),$$

or

$$f(x_1|x_2)/f(x_2|x_1) = f(x_1)/f(x_2) = A_1(x_1)/A_2(x_2),$$

where A_1 and A_2 are obtained after cancelling common factors if any in f_1, f_2. Conversely, writing

$$\frac{f(x_1|x_2)}{f(x_2|x_1)} = \frac{A_1(x_1)/\Sigma A_1(x_1)}{A_2(x_2)/\Sigma A_2(x_2)}$$

and defining

$$f_j(x_j) = A_j(x_j)/\Sigma A_j(x_j), \quad j=1,2,$$

f_1 and f_2 are probability mass functions and the joint density is obtained either as $f(x_1|x_2)f_2(x_2)$ or as $f(x_2|x_1)f_1(x_1)$. □

3.5 CHARACTERIZATIONS USING CONDITIONAL FAILURE RATE. (Asha and Nair, 1994a)

Theorem 3.10

For a random vector $\underline{X}$ in the support of I_2^+ the following statements are equivalent

(i) X_1 and X_2 are independent

(ii) $c_j(\underline{x}) = h_j(x_j)$ for j=1 or 2 and $x \in I_2^+$

Proof:

If X_1 and X_2 are independent then by (3.13) it follows that

$$\underline{c}(\underline{x})=(h_1(x_1),h_2(x_2)). \tag{3.21}$$

Now consider

$$c_1(x)=h_1(x_1), \text{ for all } x \in I_2^+.$$

Then from (3.14),

$$\frac{R(x_1+1,x_2)-R(x_1+1,x_2+1)}{R(\underline{x})-R(x_1,x_2+1)} = 1-h_1(x_1).$$

That is

$$H(x_1+1,x_2)=[1-h_1(x_1)]H(\underline{x})$$

where

$$H(\underline{x}) = [R(\underline{x})-R(x_1,x_2+1)]$$

which gives

$$H(\underline{x}) = \prod_{r=0}^{x_1-1} [1-h_1(r)]H(0,x_2).$$

Therefore

$$P[X_1 \geq x_1 | X_2 = x_2] = \prod_{r=0}^{x_1-1} [1-h_1(r)] \text{ for all } \underline{x} \in I_2^+ \qquad (3.22)$$

$$= P[X_1 \geq x_1],$$

from Section 1.3. Thus X_1 and X_2 are independent. □

Corollary 3.1

X_1 and X_2 are independent geometric variables if and only if

$$\underline{c}(\underline{x}) = [(h_1(x_1), h_2(x_2)]$$

$$= (c_1, c_2) \qquad (3.23)$$

where $0 < c_j < 1$, $j=1,2$ are constants.

Proof:

If X_1 and X_2 are independent geometric variables (3.23) follows from (3.13). The converse is obtained by substituting for $h_1(x_1)$ in (3.22) and analogous result for $c_2(\underline{x})$. □

Theorem 3.11

For all $x \in I_2^+$ and t_j, $s_j=0,1,2,\ldots$, $j=1,2$ the following statements are equivalent.

(i) $\underline{c}(\underline{x}) = [c_1(x_2), c_2(x_1)]$

(ii) $P[X_j \geq t_j+s_j | X_j \geq s_j, X_k=x_k] = P[X_j \geq t_j | X_k=x_k]$, $j=k=1,2$, $j \neq k$ (3.24)

(conditional lack of memory property)

(iii) $\underline{X}$ has a bivariate distribution specified by probability mass function

$$f(\underline{x}) = \alpha\, p_1^{x_1} p_2^{x_2}\, \theta^{x_1 x_2},\quad 0<p_j<1, j=1,2; 0<\theta\leq 1, \qquad (3.25)$$

where

$$\alpha^{-1} = \sum_{r=0}^{\infty} p_1^r/(1-p_2\theta^r) = \sum_{s=0}^{\infty} p_2^s\,/(1-p_1\theta^s).$$

Proof:

The equivalence of (ii) and (iii) are established in Nair and Nair (1991). If $\underline{X}$ is distributed as (3.24)

$$\underline{c}(\underline{x}) = [1-p_1\theta^{x_2}, 1-p_2\theta^{x_1}]$$

which has the form (i). Conversely if

$$\underline{c}(\underline{x}) = [c_1(x_2), c_2(x_1)]$$

then from (3.19),

$$P(X_1 \geq x_1 | X_2 = x_2) = [1-c_1(x_2)]^{x_1} \quad (3.26)$$

from which

$$P(X_1 \geq x_1 | X_2 = x_2) = [1-c_1(x_2)]^{x_1}$$

and similarly

$$P(X_2 \geq x_2 | X_1 = x_1) = [1-c_2(x_1)]^{x_2} \; c_2(x_1). \quad (3.27)$$

From the equations (3.26) and (3.27), it follows that

$$P(X_1 \geq x_1 + t | X_2 = x_2) = [1-c_1(x_2)]^{x_1+t},$$

$$= [1-c_1(x_2)]^{x_1}[1-c_1(x_2)]^{t},$$

$$= P(X_1 \geq x_1 | X_2 = x_2) P(X_1 \geq t | X_2 = x_2),$$

for all $\underline{x} \in I_2^+$ and $t=0,1,2,...$ and

$$P(X_2 \geq x_2 + t | X_1 = x_1) = [1-c_2(x_1)]^{x_2+t}$$

$$= P(X_2 \geq x_2 | X_1 = x_1) P(X_2 \geq t | X_1 = x_1)$$

for all $\underline{x} \in I_2^+$ and $t=0,1,2,...$

which is the conditional lack of memory property and therefore, from Nair and Nair (1991), the distribution of $\underline{X}$ should be of the form (3.25). □

Theorem 3.12

A random vector $\underline{X}$ defined on I_2^+ has the bivariate geometric distribution (2.13) if and only if its conditional failure rate is of the form

$$\underline{c}(\underline{x}) = \begin{cases} (c_{11}, c_{11}), & x_1 > x_2 \\ (c_{21}, c_{22}), & x_1 < x_2 \\ (c_{31}, c_{32}), & x_1 = x_2 \end{cases} \tag{3.28}$$

where $0 < c_{ij} < 1, i=1,2,3$ and $j=1,2$.

Proof:

If $\underline{c}(\underline{x})$ is of the form (3.28), then from (3.14) for $x_1 > x_2$,

$$\begin{aligned} P[X_1 \geq x_1+1, X_2 = x_2] &= (1-c_{11}) P[X_1 \geq x_1, X_2 = x_2], \\ &= (1-c_{11})^{x_1 - x_2} P[X_1 \geq x_2+1, X_2 = x_2], \end{aligned} \tag{3.29}$$

From (3.28) we have

$$P[X_1 \geq x_2+1, X_2 = x_2] = (1-c_{31}) P[X_1 \geq x_2, X_2 = x_2],$$

which on substitution in (3.29) gives

$$P[X_1 \geq x_1+1, X_2=x_2] = (1-c_{11})^{x_1-x_2}(1-c_{31})P[X_1 \geq x_2, X_2=x_2]$$

or equivalently

$$P[X_1 \geq x_1, X_2=x_2] = (1-c_{11})^{x_1-x_2-1}(1-c_{31})P[X_1 \geq x_2, X_2=x_2],$$

$$x_1 \geq x_2. \qquad (3.30)$$

Also,

$$P[X_1=x_1, X_2 \geq x_2+1] = (1-c_{12})P[X_1=x_1, X_2 \geq x_2],\ x_1 \geq x_2+1.$$

Summation over x_1 reduces the above to

$$R(x_1, x_2+1) = (1-c_{12})R(\underline{x}), \qquad x_1 \geq x_2+1$$

$$= (1-c_{12})^{x_2+1} R(x_1, 0), \qquad x_1 \geq x_2+1$$

or equivalently

$$R(x) = (1-c_{12})^{x_2} R(x_1, 0), \qquad x_1 \geq x_2. \qquad (3.31)$$

Setting $x_2=0$ in (3.30)

$$P[X_1 \geq x_1, X_2=0] = (1-c_{11})^{x_1-1}(1-c_{31})P[X_2=0]$$

or

$$P[X_1 \geq x_1, X_2 \geq 0] - P[X_1 \geq x_1, X_2 \geq 1] = (1-c_{11})^{x_1 - 1}(1-c_{31})P[X_2=0]$$

(3.32)

Substituting from (3.31), the terms in the left hand side of the equation (3.32) yields

$$R(x_1,0)c_{12} = (1-c_{11})^{x_1 - 1}(1-c_{31})P[X_2=0] \quad (3.33)$$

Since $R(0,0) = 1$,

$$1-c_{11} = c_{12}^{-1}(1-c_{13})\ P(X_2=0)$$

Thus from (3.33)

$$R(x_1,0) = (1-c_{11})^{x_1}, \qquad x_1 \in I_1^+.$$

Substituting in (3.31)

$$R(\underline{x}) = (1-c_{12})^{x_2}(1-c_{11})^{x_1}, \qquad x_1 \geq x_2. \quad (3.34)$$

Proceeding in the same manner with $c_2(x)$ in the region $x_1 < x_2$,

$$R(x) = (1-c_{22})^{x_2}(1-c_{21})^{x_1}, \qquad x_1 \leq x_2 \quad (3.35)$$

From (3.34) and (3.35)

$$(1-c_{22})(1-c_{21}) = (1-c_{12})(1-c_{11}) = R(1,1)$$

and so

$$R(x,x) = [R(1,1)]^x$$

Writing

$$p_1 = (1-c_{11}), \quad p_2 = (1-c_{22})$$

and

$$p = (1-c_{12})(1-c_{11}) = (1-c_{22})(1-c_{21})$$

we obtain the form (2.13). The conditions on the parameters follow directly since $0<c_{ij}<1$, $i=1,2,3$ and $j=1,2$. Conversely by direct calculations from the survival function (2.13)

$$\underset{\sim}{c}(x) = \begin{cases} (1-p_1, 1-pp_1^{-1}), & x_1>x_2 \\ (1-pp_2^{-1}, 1-p_2), & x_1<x_2 \\ \left(\dfrac{1+p-p_1-p_2}{1-p_2}, \dfrac{1+p-p_1-p_2}{1-p_1} \right), & x_1=x_2 \end{cases}$$

This completes the proof. □

3.6 INTER-RELATIONSHIPS AND THEIR IMPLICATIONS.

From the definitions of $a(\underset{\sim}{x})$, $b(\underset{\sim}{x})$ and $c(\underset{\sim}{x})$ it can be noted that

$$c_1(\underline{x})b_2(\underline{x}) = c_2(\underline{x})b_1(\underline{x}) = a(\underline{x})$$

The vector failure rate and the conditional failure rate reduce to the vector of univariate failure rates of the individual components in the case of independence while scalar failure rate is the product of the marginal rates in the same situation. Thus $a(\underline{x})$ becomes the products of the components of the vectors when X_1 and X_2 are independent. We now prove that the converse is also true.

Theorem 3.13

For all $\underline{x} \in I_2^+$, the relationship

$$a(\underline{x}) = b_1(\underline{x})b_2(\underline{x}) = c_1(\underline{x})c_2(\underline{x}) \tag{3.36}$$

holds only if X_1 and X_2 are independent.

Proof:

The equation (3.36) is equivalent to

$$\frac{f(\underline{x})}{R(\underline{x})} = \left[1 - \frac{R(x_1+1,x_2)}{R(\underline{x})}\right]\left[1 - \frac{R(x_1,x_2+1)}{R(\underline{x})}\right]$$

and

$$\frac{f(\underline{x})}{R(\underline{x})} = \left[\frac{f(\underline{x})}{R(\underline{x}) - R(x_1,x_2+1)}\right]\left[\frac{f(\underline{x})}{R(\underline{x}) - R(x_1+1,x_2)}\right].$$

Simplification of these equations lead to the functional equation,

$$R(x_1+1,x_2+1)R(x) = R(x_1+1,x_2)R(x_1,x_2+1) \tag{3.37}$$

for all $x \in I_2^+$.

But (3.37) implies,

$$\frac{R(x_1+1,x_2)}{R(x)} = \frac{R(x_1+1,0)}{R(x_1,0)}$$

$$\frac{R(\underline{x})}{R(x_1-1,x_2)} = \frac{R(x_1)}{R(x_1-1,0)}$$

so that

$$R(\underline{x}) = R(x_1,0)\,\frac{R(x_1-1,x_2)}{R(x_1-1,0)} ,$$

$$= R(x_1,0)\,\frac{R(x_1-2,x_2)}{R(x_1-2,0)} ,$$

$$\cdots$$

$$= R(x_1,0)R(0,x_2) .$$

Thus X_1 and X_2 are independent. □

Corollary 3.2

For all $\underline{x} \in I_2^+$, the relationship

$$a(\underline{x}) = b_1(\underline{x})b_2(\underline{x}) = c_1(\underline{x})c_2(\underline{x}) = c$$

where $0<c<1$, holds if and only if X_1 and X_2 are independent and geometrically distributed.

The proof follows directly from Theorems 2.4, 3.4, 3.13 and Corollary 3.1. □

Theorem 3.14

If $\underline{X}$ in support of I_2^+ has geometric marginals with parameter p_i and if the additional assumption $R(x,x)=p^x$ for some $0<p\leq p_j<1$, $j=1,2$, then the relationship

$$a(\underline{x}) = b_1(x)b_2(x) = c_1(x)c_2(\underline{x}) \tag{3.38}$$

holds for all $\underline{x}$ in $I' = \{\underline{x}=(x_1,x_2)\,|\,x_1\neq x_2,\ x_j=0,1,2\ldots,\ j=1,2\}$ if and only if $\underline{X}$ has bivariate geometric distribution (2.13)

Proof:

Under the condition (3.38), it is readily seen that for $x_1\neq x_2$, equation (3.37) continuous to hold. Solving under the assumption of geometric marginals, for $x_1=0$

$$R(1,x_2+1) = R(1,x_2)\ p_2$$

$$= R(1,1)\ p_2^{x_2} = pp_2^{x_2},\quad x_2 = 1,2,\ldots$$

On putting $x_1=1$ in (3.38),

$$R(2,x_2) = R(2,2)\, p_2^{x_2-2}, \quad x_2=2,3,\ldots$$

In general

$$R(x) = R(x_1,x_1)\, p_2^{x_2-x_1}, \quad x_2 \geq x_1$$

$$= p^{x_1}\, p_2^{x_2-x_1}, \quad x_2 \geq x_1. \qquad (3.39)$$

Interchanging the argument x_j in the above equations in the region $x_1 > x_2$ we get

$$R(\underline{x}) = p^{x_2}\, p_1^{x_1-x_2}, \quad x_1 \geq x_2 \qquad (3.40)$$

From (3.39) and (3.40), the form (2.13) is recovered. The conditions on the parameters follow directly from the assumptions.

The theorem provides an important and interesting feature when compared with Theorem 3.13 and its corollary. The fact of independent distributions for the component life times can be modified to impart dependence to them once the sample point that falls on the line $x_1 = x_2$ is singled out and are given arbitrary geometric probabilities.

Chapter lV

CLASSES OF BIVARIATE LIFE TIME MODELS WITH MONOTONE FAILURE RATES

4.1 INTRODUCTION:

In the choice of life distributions in modelling equipment behaviour various notions of aging play fundamental role. The different concepts that describe wear out or aging are conveniently expressed in terms of failure rates or mean residual life. The classes of life distributions based on the monotonicity of the failure rate function, when life lengths are continuous, have been treated in Barlow and Proschan (1975), Brindley and Thompson (1977), Harris (1970), Marshall (1975), Buchanan and Singpurwalla (1977). Consistent with the criteria and notions discussed in these investigations we discuss the corresponding results in discrete time. A treatment of such classes in the discrete case described here have not appeared in the literature so far and this motivates the discussion of the topic in the subsequent sections, although the concepts developed in the present study have close resemblance to the basic approach made in the continuous case. Since we have already distinguished three types of failure rates, naturally there exist separate classifications of distributions based on each definition. However in the light of our finding that $a(\underline{x})$ cannot by itself determine the corresponding distribution uniquely the classification based on this type of failure rate is not extensively studied in our subsequent investigations.

In the possible classes that are considered in the sequel, it is assumed that monotonicity of the failure rate reflects the extent of aging or wear out of the components of the system. The different notions envisaged here take into consideration two aspects of the system, viz. the present age which is equivalent to the intensity of aging and the time point at which the wearing out is reckoned. This leads to the following three different cases.

(i) The age of each component are different and they wear out with different intensities. This is accommodated by comparing the failure rates at times (x_1+t_1, x_2+t_2) and $\underline{x}$.

ii) The age of components are different and wearing out of only one of them is considered at a time. In this case the failure rate at (x_i+t, x_j) are compared with that at (x_i, x_j), $i,j=1,2$, $i \neq j$.

(iii) The components have different ages initially but they wear out with the same intensity. Here the objects of comparison are the rates at (x_1, x_2) and (x_1+t, x_2+t).

4.2 CLASSES WITH MONOTONE VECTOR FAILURE RATE (Nair and Asha 1994a)

4.2.1 b-IFR_1 (b-DFR_1) CLASS

Definition 4.1

The distribution of a random vector $\underline{X}$ in the support of I_2^+ is defined as a b-IFR_1 (b-DFR_1) distribution if for all $\underline{x}$, $\underline{s} = (s_1, s_2)$ in I_2^+,

$$b_j(\underline{x}+\underline{s}) \geq (\leq)\ b_j(\underline{x}), \quad j=1,2 \tag{4.1}$$

The physical interpretation of (4.1) is the aging behaviour indicated as (i) above. From (4.1) it is clear that $\underline{X}$ is b-IFR$_1$ (b-DFR$_1$) if the ratios

$$\frac{R(x_1+s_1+1, x_2+s_2)}{R(x_1+1, x_2)}$$

and

$$\frac{R(x_1+s_1, x_2+s_2+1)}{R(x_1, x_2+1)}$$

are non-increasing (non-decreasing) in each x_j, j = 1,2. A matter of primary interest in any criteria of classification, is to investigate whether there exist a boundary class separating the b-IFR$_1$-b-DFR$_1$ classes or one that is both b-IFR$_1$ and b-DFR$_1$. In the continuous univariate theory, the definitions of notions of aging are so fashioned that the exponential distribution exhibiting a constant failure rate becomes the only distribution that belongs to the boundary class. In the discrete case analogously one would expect a bivariate geometric law to be the only model that is both, b-IFR$_1$ and b-DFR$_1$. Thus corresponding to each definition we seek the bivariate geometric distribution characterized by that property. The answer corresponding to Definition 4.1 is covered in the following theorem.

Theorem 4.1

The only class of distributions in I_2^+ which is both b-IFR$_1$ and b-DFR$_1$ is the bivariate geometric distribution with independent marginals.

Proof:

Let $\underline{X}$ be such that

$$R(\underline{x}) = p_1^{x_1} \; p_2^{x_2} \quad , \quad 0<p_j<1, \; j=1,2.$$

Then $\underline{b}(\underline{x}) = (1-p_1, 1-p_2)$ implies

$\underline{b}(\underline{x} + \underline{s}) = \underline{b}(\underline{x})$ for all $\underline{x}, \underline{s} \in I_2^+$.

Conversely if $\underline{b}(\underline{x}+\underline{s})=\underline{b}(\underline{x})$, then $\underline{b}(\underline{x})=\underline{b}(\underline{0})$, where $\underline{0}=(0,0)$. Then by Theorem 3.4, the joint distribution of $\underline{X}$ is given by

$$R(\underline{x}) = p_1^{x_1} \; p_2^{x_2} \quad , \quad 0<p_j<1, \; j=1,2.$$

From a practical point of view, the implication of Theorem 4.1 is that the Definition 4.1 highly restrictive in the sense that it represents a system in which the components are independent of one another. A more realistic situation would be that in which the life of one component depends on the life of the other. Thus we are led to the following modified definitions.

4.2.2. b-IFR$_2$ (b-DFR$_2$) CLASS

Definition 4.2

The distribution of $\underline{X}$ is said to have a b-IFR$_2$ (b-DFR$_2$) distribution if for all $\underline{x} \in I_2^+$ and $t \in I_1^+$,

$$\begin{aligned} & b_1(x_1+t, x_2) \geq (\leq)\ b_1(\underline{x}) \\ \text{and} & \\ & b_2(x_1, x_2+t) \geq (\leq)\ b_2(\underline{x}). \end{aligned} \tag{4.2}$$

According to this definition a system functioning at age $\underline{x}$ has a higher failure rate whenever any one of its components is replaced by an older one. The bivariate geometric distribution, that serves as the boundary of the b-IFR$_2$ and b-DFR$_2$ classes is the one which satisfies

$$b_1(x_1+t, x_2) = b_1(\underline{x}) \quad \text{and} \quad b_2(x_1, x_2+t) = b_2(\underline{x}),$$

for all $\underline{x} \in I_2^+$ and $t \in I_1^+$.

From section 1.3 it is easily seen that this distribution is that specified by equation (1.9).

4.2.3 b-IFR_3 (b-DFR_3) CLASS

An alternative way of relaxing the Definition 4.1 is realised by assuming that the failure rate of the system with two components of different ages are observed after each component has worked through the same time. Thus we have

Definition 4.3

The distribution of the random vector $\underline{X}$ in the support of I_2^+ is defined as a b-IFR_3 (b-DFR_3) distribution if and only if

$$h_j(x_j+t) \geq (\leq) \; h_j(x_j)$$

and

$$b_j(\underline{x}+\underline{t}) \geq (\leq) \; b_j(\underline{x}), \; j=1,2$$

for all $\underline{x} \in I_2^+$ and $\underline{t}=(t,t)$, $t \in I_1^+$. (4.3)

In the following theorem we identify the bivariate geometric distribution which forms a boundary class for the $b\text{-}IFR_3$ class and $b\text{-}DFR_3$ class.

Theorem 4.2

A bivariate distribution with support I_2^+ is both $b\text{-}IFR_3$ and $b\text{-}DFR_3$ if and only if it is a bivariate geometric with survival function (2.13).

Proof:

By definition, a bivariate distribution in support of I_2^+ is both $b\text{-}IFR_3$ and $b\text{-}DFR_3$ if and only if

$$h_j(x_j+t) = h_j(x_j) \qquad \text{for all } x_j, j=1,2,\ t \in I_1^+$$

and

$$b_j(\underline{x}+\underline{t}) = b_j(\underline{x}), \quad \text{for all } \underline{x} \in I_2^+ \text{ and } \underline{t}=(t,t),\ t\in I_1^+$$

The assertion follows from Theorem 3.7.

It is possible to have a stronger class than that provided by equation (4.3) by defining it as $\underline{X}$ belongs to $b\text{-}IFR_4$ ($b\text{-}DFR_4$) class if and only if

$$h_j(x_j+t) \geq (\leq)\ h_j(x_j) \quad \text{for all } x_j,\ t \in I_1^+,\ \ j=1,2$$

and

$b_j(\underline{x}+\underline{t}) \geq (\leq)\ b_j(\underline{x})$, where $\underline{x} = (x,x)$ and $\underline{t}=(t,t)$ for all $x,t \in I_1^+$.

But there is no characterization of the corresponding boundary class.

4.3 INTER-RELATIONSHIPS

From the definitions given above it is clear that

(i) $\quad$ b-IFR$_2$ $\Leftarrow$ b-IFR$_1$ $\Rightarrow$ b-IFR$_3$ $\Rightarrow$ b-IFR$_4$

and

(ii) $\quad$ b-DFR$_2$ $\Leftarrow$ b-DFR$_1$ $\Rightarrow$ b-DFR$_3$ $\Rightarrow$ b-DFR$_4$

Further the following counter examples illustrate that there exist no other implications between the different classes.

Example 4.1

Consider the bivariate geometric law specified by equation (1.9) with survival function

$$R(\underline{x}) = p_1^{x_1}\ p_2^{x_2}\ \theta^{x_1 x_2},$$

$$0<p_j<1,\ 0\leq\theta\leq 1,\ 1+p_1p_2\theta \geq p_1+p_2,\ \underline{x} \in I_2^+,$$

whose vector failure rate is

$$\underline{b}(\underline{x}) = [1-p_1\theta^{x_2}, 1-p_2\theta^{x_1}], \quad \underline{x} \in I_2^+.$$

Since for all $\underline{x}$ in I_2^+ and t in I_1^+,

$$b_1(x_1+t,x_2) = 1-p_1\,\theta^{x_2} \geq b_1(\underline{x}).$$

and

$$b_2(x_1,x_2+t) = 1-p_2\,\theta^{x_1} \geq b_2(\underline{x}).$$

$R(\underline{x})$ exhibits b-IFR$_2$ property. Also

$$b(x_1+s_1,x_2+s_2) = [1-p_1\theta^{x_2+s_2}, 1-p_2\theta^{x_1+s_1}],$$

$$\geq [1-p_1\theta^{x_2}, 1-p_2\theta^{x_1}] = \underline{b}(\underline{x}),$$

which proves that $R(\underline{x})$ is b-IFR$_1$. But

$$\underline{b}(x+t,x+t) = [1-p_1 r\theta^{x+t}, 1-p_2\theta^{x+t}],$$

$$\not< [1-p_1\theta^{x}, 1-p_2\theta^{x}],$$

$$= b(x,x).$$

Thus $R(\underline{x})$ is not b-DFR$_4$. Hence we conclude that

$$\text{b-IFR}_2 \not\Rightarrow \text{b-DFR}_4 \quad \text{and} \quad \text{b-IFR}_1 \not\Rightarrow \text{b-DFR}_4$$

Example 4.2

Consider a random vector $\underline{X}$ distributed in the bivariate Waring form specified by

$$R(\underline{x}) = \frac{(m)_{x_1+x_2}}{(m+n)_{x_1+x_2}} \quad ; \; m,n > 0 \;, \underline{x} \in I_2^+.$$

where $(m)_x = m(m+1)\dots(m+x-1)$.

From (3.2), the vector failure rate is obtained as

$$b_1(\underline{x}) = 1 - \frac{R(x_1+1, x_2)}{R(\underline{x})}$$

$$= 1 - \frac{(m)_{x_1+x_2+1}}{(m+n)_{x_1+x_2}}$$

$$= 1 - \frac{R(x_1, x_2+1)}{R(\underline{x})}$$

$$= b_2(\underline{x})$$

$$= \frac{n}{m+n+x_1+x_2}$$

Note that for every $\underline{x}$, $\underline{s}$ in I_2^+ and t in I_1^+,

$$b_j(x_1+s_1, x_2+s_2) \leq b_j(\underline{x}); \quad j=1,2$$

and

$$b_j(x+t, x+t) \leq b_j(\underline{x}); \quad j=1,2.$$

Also, $b_1(x_1+t, x_2) \not\gtrless b_1(\underline{x})$ and $b_1(x_1, x_2+t) \not\gtrless b_2(\underline{x})$

Thus, $R(\underline{x})$ is both b-DFR_1 and b-DFR_2, giving the implications

$$\text{b-}DFR_2 \not\Rightarrow \text{b-}IFR_2 \quad \text{and} \quad \text{b-}DFR_2 \not\Rightarrow \text{b-}IFR_4.$$

Example 4.3

Consider the bivariate geometric distribution discussed in Theorem 3.6 specified by the survival function

$$R(\underline{x}) = \begin{cases} p^{x_2} p_1^{x_1-x_2}, & x_1 \geq x_2 \\ p^{x_1} p_2^{x_2-x_1}, & x_1 \leq x_2, \end{cases}$$

$$1+p \geq p_1+p_2,\ 0<p\leq p_j<1,\ j=1,2,\ x\in I_2^+,$$

whose vector failure rate is

$$\underline{b}(\underline{x}) = \begin{cases} 1-p_1\ ,\ 1-pp_1^{-1} & x_1>x_2 \\ 1-pp_2^{-1},\ 1-p_2 & x_1<x_2 \\ 1-p_1\ ,\ 1-p_2 & x_1=x_2 . \end{cases}$$

That $R(\underline{x})$ is both b-IFR_3 and b-DFR_3 has been proved in Theorem 3.6. It can also be noted that $b_1(x_1+t,x_2)$ and $b_2(x_1,x_2+t)$ need not exhibit monotone behaviour. From which we infer that

$$\text{b-}IFR_3 \not\Rightarrow \text{b-}IFR_2 \text{ and b-}DFR_3 \not\Rightarrow \text{b-}DFR_2.$$

Example 4.4

Consider the negative hyper geometric distribution specified by

$$R(\underline{x}) = \frac{\binom{k+n-x_1-x_2}{n-x_1-x_2}}{\binom{k+n}{n}}, \quad x_1, x_2 = 0,1,2,\ldots n; \; x_1+x_2 \le n, k, n > 0.$$

The vector failure rate is obtained as

$$b_1(\underline{x}) = 1 - \frac{R(x_1+1, x_2)}{R(\underline{x})},$$

$$= 1 - \left[\frac{\binom{k+n-x_1-x_2-1}{n-x_1-x_2-1}}{\binom{k+n-x_1-x_2}{n-x_1-x_2}} \right],$$

$$= 1 - \frac{(k+n-x_1-x_2-1)\,!}{k\,!\,(n-x_1-x_2-1)\,!} \cdot \frac{k\,!\,(n-x_1-x_2)\,!}{(k+n-x_1-x_2)\,!},$$

$$= \frac{k}{k+n-x_1-x_2}.$$

Similarly $b_2(\underline{x}) = k\,(k+n-x_1-x_2)^{-1}$

Here

$$b_j(x_1+s_1, x_2+s_2) \ge b_j(\underline{x}); \text{ for every } \underline{x}, \underline{s} = (s_1, s_2) \in I_2^+,$$

so that $R(\underline{x})$ is b-IFR_1. But

$$b_1(x_1+t, x_2) = \frac{k}{k+n-x_1-x_2-t},$$

$$\nleq \frac{k}{k+n-x_1-x_2},$$

$$= b_1(\underline{x})$$

and

$$b_2(x_1, x_2+t) = \frac{k}{k+n-x_1-x_2-t}$$

$$\nleq \frac{k}{k+n-x_1-x_2}$$

$$= b_2(\underline{x}).$$

Also

$$\underline{b}(x+t, x+t) = \left[\frac{k}{k+n-2x-2t}, \frac{k}{k+n-2x-2t}\right]$$

$$\nleq \left[\frac{k}{k+n-2x}, \frac{k}{k+n-2x}\right]$$

$$= \underline{b}(x, x)$$

which proves that $R(\underline{x})$ is neither b-DFR_2 nor b-DFR_4. Hence

$$\text{b-IFR}_1 \not\rightarrow \text{b-DFR}_2 \text{ and b-IFR}_1 \not\rightarrow \text{b-DFR}_4$$

Example 4.5:

Consider the random vector $\underline{X}$ with bivariate mixture geometric distribution specified by survival function

$$R(\underline{x})=\begin{cases} \alpha\, p^{x_1+x_2} + (1-\alpha)\, p^{x_1}, & x_1\geq x_2 \\ \alpha\, p^{x_1+x_2} + (1-\alpha)\, p^{x_2}, & x_1\leq x_2, 0<p,\alpha<1,\ \underline{x}\in I_2^+. \end{cases}$$

The vector failure rate $\underline{b}(\underline{x})$ is obtained as

$$b_1(\underline{x})=1 - \frac{R(x_1+1,x_2)}{R(\underline{x})}$$

$$= \begin{cases} 1-p, & x_1\geq x_2+1 \\ 1-\left[[\alpha p^{x_1+1}+(1-\alpha)]/[\alpha p^{x_1}+(1-\alpha)]\right], & x_1+1\leq x_2 \\ 1-p, & x_1=x_2 \end{cases}$$

and

$$b_2(\underline{x}) = 1 - \frac{R(x_1,x_2+1)}{R(\underline{x})}$$

$$= \begin{cases} 1-\left[[\alpha p^{x_2+1}+(1-\alpha)]/[\alpha p^{x_2}+(1-\alpha)]\right], & x_1>x_2 \\ 1-p, & x_2>x_1 \\ 1-p, & x_1=x_2 \end{cases}$$

so that

$$\underline{b}(\underline{x})=\begin{cases} 1-p, \quad 1-\left[[\alpha p^{x_2+1}+(1-\alpha)] \,/\, [\alpha p^{x_2}+(1-\alpha)]\right], & x_1>x_2 \\ 1-\left[[\alpha p^{x_1+1}+(1-\alpha)] \,/\, [\alpha p^{x_1}+(1-\alpha)]\right], \quad 1-p, & x_1<x_2 \\ 1-p, & x_1=x_2. \end{cases}$$

Observe that

$$\underline{b}(x+t,x+t) = (1-p,1-p)$$
$$= \underline{b}(x,x)$$

implying that $R(\underline{x})$ is both b-IFR_4 and b-DFR_4. But $\underline{b}(\underline{x})$ does not exhibit a monotone behaviour as $\underline{x}$ varies in the sub regions $x_1<x_2$, $x_1>x_2$ and $x_1=x_2$. Thus we conclude that

$$\text{b-}IFR_4 \not\Rightarrow \text{b-}IFR_3 \quad \text{and} \quad \text{b-}DFR_4 \not\Rightarrow \text{b-}DFR_3$$

Example 4.6

Consider a random vector $\underline{X}$ with a bivariate geometric distribution whose survival function is specified by

$$R(\underline{x}) = p_1^{x_1} \quad , \; p_1^{x_1} \le p_2^{x_2}$$

$$= p_2^{x_2} \quad , \; p_1^{x_1} \ge p_2^{x_2} \; , \quad 0<p_j<1, j=1,2, \; \underline{x} \in I_2^+.$$

Then the vector failure rate is

$$\underline{b}(\underline{x}) = \begin{cases} 1-p_1, 0, & p_1^{x_1} \le p_2^{x_2+1} \\ 0,\ 1-p_2, & p_1^{x_1+1} \ge p_2^{x_2} \\ 1-p_1,\ 1-\dfrac{p_2^{x_2+1}}{p_1^{x_1}}, & p_1^{x_1} \le p_2^{x_2},\ p_1^{x_1} \ge p_2^{x_2+1} \\ 1-\dfrac{p_1^{x_1+1}}{p_2^{x_2}},\ 1-p_2, & p_1^{x_1} \ge p_2^{x_2},\ p_1^{x_1+1} \le p_2^{x_2} \end{cases}$$

Observe that

$$b_1(x_1+t, x_2) = \begin{cases} 1-p_1, & p_1^{x_1+t} \le p_2^{x_2+1} \\ 0, & p_1^{x_1+t+1} \ge p_2^{x_2} \\ 1-p_1, & p_1^{x_1+t} \le p_2^{x_2},\ p_1^{x_1+t} \ge p_2^{x_2+1} \\ 1-\dfrac{p_1^{x_1+t+1}}{p_2^{x_2}}, & p_1^{x_1+t} \ge p_2^{x_2},\ p_1^{x_1+t+1} \le p_2^{x_2} \end{cases}$$

$$\ge b_1(\underline{x}).$$

Also,

$$b_2(x_1, x_2+t) = \begin{cases} 0, & p_1^{x_1} \le p_2^{x_2+t+1} \\ 1-p_2, & p_1^{x_1+1} \ge p_2^{x_2+t} \\ 1-\dfrac{p_2^{x_2+t+1}}{p_1^{x_1}}, & p_1^{x_1} \le p_2^{x_2+t},\ p_1^{x_1} \ge p_2^{x_2+t+1} \\ 1-p_2, & p_1^{x_1} \ge p_2^{x_2+t},\ p_1^{x_1+1} \le p_2^{x_2+t} \end{cases}$$

$$\ge b_2(\underline{x}),$$

so that $R(\underline{x})$ is b-IFR_2. But

$$\underline{b}(x+t,x+t) = \begin{cases} 1-p_1, 0, & p_1^{x+t} \le p_2^{x+t+1} \\ 0,\ 1-p_2, & p_1^{x+t+1} \ge p_2^{x+t} \\ 1-p_1,\ 1-\dfrac{p_2^{x+t+1}}{p_1^{x+t}}, & p_1 \le p_2,\ p_1^{x+t} \ge p_2^{x+t+1} \\ 1-\dfrac{p_1^{x+t+1}}{p_2^{x+t}},\ 1-p_2, & p_1^{x+t} \ge p_2^{x+t},\ p_1^{x+t+1} \le p_2^{x+t} \end{cases}$$

which does not exhibit a monotone behaviour on varying x or t. Thus $R(\underline{x})$ is not b-IFR_4, so that

$$\text{b-}IFR_2 \not\Rightarrow \text{b-}IFR_4.$$

The various implications of the examples are summarised in Table I from which it is clear that no other implications exist other than that made in the assertion.

4.4 CLASSES WITH MONOTONE CONDITIONAL FAILURE RATE (Asha and Nair, 1994b)

4.4.1 C-IFR_1 (C-DFR_1) CLASS

Definition 4.4

A random vector $\underline{X}$ in the support of I_2^+ is defined to have c-IFR_1 (c-DFR_1) distribution if

$$h_j(x_j+t) \geq (\leq)\ h_j(x_j)\ ,\ \text{for all } x_j, j=1,2,\ t\in I_1^+ \qquad (4.4)$$

and

$$\underline{c}(\underline{x}+\underline{t}) \geq (\leq)\ \underline{c}(\underline{x})\ ,\ \text{for all } \underline{x}\ ,\ \underline{t}=(t_1,t_2) \in I_2^+.$$

The physical interpretation of this type of classification is same as in 4.2.1, but with obvious difference in the meaning of the conditional failure rate. We next identity the boundary class of the c-IFR_1 and c-DFR_1 class.

Theorem 4.3

The only class of distribution in I_2^+ which is both c-IFR_1 and (c-DFR_1) is the bivariate geometric with independent marginals.

Proof:

Let $\underline{X}$ be a bivariate geometric distribution with independent geometric (p_j) ,j=1,2 marginals. Then

$$\underline{c}(\underline{x}) = [(1-p_1),(1-p_2)] \text{ for all } \underline{x} \in I_2^+ ,$$

from which it is clear that $\underline{X}$ is both c-IFR_1 and c-DFR_1.

Conversely if $\underline{X}$ is both c-IFR$_1$ and (c-DFR$_1$), then,

$$h_j(x_j+t) = h_j(x_j) \text{ for all } x_j, t \in I_1^+, \; j=1,2 \qquad (4.5)$$

and

$$\underline{c}(\underline{x}+\underline{t}) = \underline{c}(\underline{x}) \;, \text{ for all } \underline{x}, \underline{t} = (t_1,t_2) \in I_2^+ \qquad (4.6)$$

From (4.5) the univariate failure rate must be a constant and hence each X_j has a geometric distribution, say with parameter p_j, j=1,2. Equation (4.6) implies

$$P\,[X_1=x_1 \mid X_1 \geq x_1, X_2=x_2] = P\,[X_1=0 \mid X_1 \geq 0, X_2=0]$$

$$= c_1 \text{ (a constant)}$$

and

$$P\,[X_2=x_2 \mid X_1=x_1, X_2 \geq x_2] = P\,[X_2=0 \mid X_1=0, X_2 \geq 0]$$

$$= c_2 \text{ (a constant)}$$

From Corollary 3.9, the constancy of the conditional failure rate implies the statement of the Theorem.

4.4.2 C-IFR$_2$ (C-DFR$_2$) CLASS

Definition 4.5

The distribution of $\underline{X}$ belongs to c-IFR$_2$ (c-DFR$_2$)

class if for all $\underline{x} \in I_2^+$ and $t \in I_1^+$

$$h_j(x_j+t) \geq (\leq)\ h_j(x_j)\ ,\ j = 1,2$$

and

$$c_1(x_1+t,x_2) \geq (\leq)\ c_1(\underline{x})$$

$$c_2(x_1,x_2+t) \geq (\leq)\ c_2(\underline{x})$$

The boundary class of distributions of the $c\text{-}IFR_2\text{-}c\text{-}DFR_2$ class is the one which satisfies

$$h_j(x_j+t) = h_j(x_j)\ ,\ j = 1,2 \tag{4.7}$$

and

$$c_1(x_1+t,x_2) = c_1(\underline{x})$$

$$c_2(x_1,x_2+t) = c_2(\underline{x}). \tag{4.8}$$

From (4.7) it is obvious that the boundary class should be a bivariate distribution with geometric marginals. Also from (4.8),

$$c_1(\underline{x}) = c_1(0,x_2)$$

$$c_2(\underline{x}) = c_2(x_1,0),$$

which has already been proved to be a characteristic property of 3.25 in Theorem 3.10. For this distribution to have geometric marginals it is necessary and sufficient that $\theta = 1$. More precisely the bivariate distributions that are both c-IFR_1 and c-IFR_2 are those with independent geometric marginals.

Having come to the conclusion that the boundary classes of c-IFR_1 and c-IFR_2 are the same, it remains to prove that the two classes are not identical. For this refer to Example (4.8) given below.

4.4.3. c-IFR_3(c-DFR_3) CLASS

Definition 4.6

The distribution of $\underline{X}$ belongs to the c-IFR_3(c-DFR_3) class if for all $\underline{x}, \in I_2^+$ $\underline{t}=(t,t)$, $t \in I_1^+$

$$h_j(x_j+t) \geq (\leq)\ h_j(x_j), \qquad j = 1,2$$

and

$$\underline{c}(\underline{x}+\underline{t}) \geq (\leq)\ \underline{c}(\underline{x}). \tag{4.9}$$

The unique form of the bivariate law which emerges as the boundary class defined by (4.9) appears to be lacking. However, from Theorem 3.7 it is seen that the bivariate geometric (2.13) belongs to this class.

4.5 INTER-RELATIONSHIPS BETWEEN THE CLASSES

From the definitions given above it is clear that

(i) $c\text{-IFR}_2 \leftarrow c\text{-IFR}_1 \rightarrow c\text{-IFR}_3$

and

(ii) $c\text{-DFR}_2 \leftarrow c\text{-DFR}_1 \rightarrow c\text{-DFR}_3$

The following examples illustrates that there exist no other implications.

Example 4.7

Consider the bivariate geometric distribution in Example 4.3 Since the marginals are geometric (p_j), $j=1,2$, the marginal failure rate $h_j(x_j) = (1-p_j)$, $0<p_j<1$, $j=1,2$ and the conditional failure rate

$$\underline{c}(\underline{x}) = \begin{cases} (1-p_1),\ (1-pp_1^{-1}), & x_1>x_2 \\ (1-pp_2^{-1}),\ (1-p_2), & x_1\le x_2 \\ (1+p-p_1-p_2)(1-p_2)^{-1},\ (1+p-p_1p_2)(1-p_1)^{-1}, & x_1=x_2 \end{cases}$$

$= \underline{c}(\underline{x}+\underline{t})$, for all $\underline{x} \in I_2^+$, $\underline{t} = (t,t)$, $t \in I_1^+$.

Thus by definition (3.4.3), $R(\underline{x})$ is both c-IFR_3 and c-DFR_3. Also notice that since $c_1(x_1+t,x_2)$ and $c_2(x_1,x_2+t)$, for all $t \in I_1^+$ and all $\underline{x} \in I_2^+$ need not exhibit a monotone behaviour, c-IFR_3 and c-DFR_3 need not imply neither c-IFR_2 nor c-DFR_2.

Example 4.8

Consider the bivariate geometric distribution (Pathak and Sreehari, 1981) specified by the probability mass function

$$f(\underline{x}) = \binom{x_1+x_2}{x_1} p_1^{x_1} p_2^{x_2} (1-p_1-p_2); \quad \underline{x} \in I_2^+, \ 0<p_1,p_2<1, p_1+p_2<1.$$

since the marginals are geometric (p_j),

$$h_j(x_j) = 1-p_j, \ j=1,2$$

and the conditional failure rate is given by

$$c_1(\underline{x}) = \frac{\binom{x_1+x_2}{x_1} p_1^{x_1}}{\sum\limits_{y=x_1}^{\infty} \binom{y+x_2}{y} p_1^y};$$

$$= \left[1 + \left(1 + \frac{x_2}{x_1+1}\right)p_1 + \frac{1}{x_1+1}\left(1 + \frac{x_2}{x_1+1}\right)p_1^2 + \ldots\right]^{-1}$$

Also

$$c_1(x_1+t,x_2)=\left[1+\left(1+\frac{x_2}{x_1+t+1}\right)p_1+\frac{1}{x_1+t+1}\left(1+\frac{x_2}{x_1+t+1}\right)p_1^2+\dots\right]^{-1}$$

$$\geq \left[1+\left(1+\frac{x_2}{x_1+1}\right)p_1+\frac{1}{x_1+1}\left(1+\frac{x_2}{x_1+1}\right)p_1^2+\dots\right]^{-1}$$

$$= c_1(\underline{x}); \text{ for all } \underline{x}\in I_2^+,\quad t\in I_1^+.$$

Similarly,

$$c_2(x_1,x_2+t)=\left[1+\left(1+\frac{x_1}{x_2+t+1}\right)p_1+\frac{1}{x_2+t+1}\left(1+\frac{x_1}{x_2+t+1}\right)p_1^2+\dots\right]^{-1}$$

$$\geq \left[1+\left(1+\frac{x_1}{x_2+1}\right)p_1+\frac{1}{x_2+1}\left(1+\frac{x_1}{x_2+1}\right)p_1^2+\dots\right]^{-1}$$

$$= c_2(\underline{x}); \text{ for all } \underline{x}\in I_2^+,\quad t\in I_1^+.$$

so that $\underline{c}(\underline{x})$ is $\in$ IFR_2. But $\underline{c}(\underline{x}+\underline{t})$ does not exhibit any monotone behaviour. Thus

$$\text{c-IFR}_2 \not\Rightarrow \text{c-DFR}_3$$

$$\text{c-IFR}_2 \not\Rightarrow \text{c-IFR}_3$$

Example 4.9

Consider a random vector $\underline{X}$ specified by the survival function

$$R(\underline{x}) = 2(x_1+x_2)^{-1}, \; x_j = 1,2,3,\ldots,j=1,2$$

The marginal survival function is specified by

$$R_j(x_j) = 2(x_j+1)^{-1}, \; x_j = 1,2,\ldots,j=1,2.$$

and the marginal failure rate

$$h_j(x_j) = \frac{2(x_j+1)^{-1} - 2(x_j+2)^{-1}}{2(x_j+1)^{-1}}$$

$$= (x_j+2)^{-1}$$

for which $h_j(x_j+t) \le h_j(x_j)$ for all x_j ,t = 1,2,...,j=1,2.
From (3.14) the conditional failure rate is obtained as

$$\underline{c}(\underline{x}) = \left[\, 1-(1+(2/x_1+x_2))^{-1}, \; 1-(1+(2/x_1+x_2))^{-1} \right].$$

Also

$$\underline{c}(\underline{x}+\underline{s}) \le \underline{c}(\underline{x}) \; ; \text{ for all } \underline{x},\underline{s}=(s_1,s_2)\in I_2^+$$

and

$$c_1(x_1+t,x_2) \le c_1(\underline{x}),$$

$$c_2(x_1,x_2+t) \le c_2(\underline{x}), \text{ for all } \underline{x} \in I_2^+ \text{ and } t\in I_1^+.$$

Thus $R(\underline{x})$ is c-DFR_1 and c-DFR_2 but not c-IFR_2 and c-IFR_3.

Example 4.10

Consider a random vector $\underline{X}$ with the bivariate negative binomial distribution as in Example(4.4).

$$R(\underline{x}) = \frac{\binom{k+n-x_1-x_2}{n-x_1-x_2}}{\binom{k+n}{n}};$$

$$x_1+x_2 \le n, \ k>0$$

$$x_j = 0,1,2,...n.$$

Then the marginal distribution is

$$R_j(x_j) = \frac{\binom{k+n-x_j}{n-x_j}}{\binom{k+n}{n}};$$

$$x_j = 1,2,...n; \ j = 1,2; \ k > 0$$

and the marginal failure rate

$$h_j(x_j) = k \ / \ (k+n-x_j) \ ; \ x_j = 0,1,2,...n; \ j = 1,2.$$

for which

$$h_j(x_j+t) \ge h_j(x_j), \quad x_j,t = 0,1,2,...n; \ j = 1,2.$$

From (3.14) the conditional failure rate is obtained as

$$\underline{c}(\underline{x}) = \left[(k-1)/(k+n-x_1-x_2-1) \ , \ (k-1)/(k+n-x_1-x_2-1) \right]$$

for which

$$\underline{c}(\underline{x}+\underline{s}) \geq \underline{C}(\underline{x}), \text{ for all } \underline{x}, \underline{s}=(s_1,s_2) \in I_2^+.$$

Thus $R(\underline{x})$ is c-IFR_1. But since $c_1(x_1+t,x_2) \leq c_1(\underline{x})$ and $c_2(x_1,x_2+t) \leq c_2(\underline{x})$ and $\underline{c}(\underline{x}+\underline{t}) \leq \underline{c}(\underline{x})$, $R(\underline{x})$ is not c-DFR_2 or c-DFR_3. The above implications are presented in Table II.

In the discussions so far made in this chapter, we have made only a brief study of the application of various definitions of failure rates in describing alternative criteria for aging of equipments or devices. The literature in this area when time is treated as continuous is abundant with different criteria as well as classification of life distributions based on them. Analogous treatment in the discrete time domain is an unresolved problem that requires detailed investigation, which will be presented else where.

Table I

Bivariate Distribution $R(\underline{x})$	$(h_1(\underline{x}), h_2(\underline{x}))$	Example for
i) Bivariate Geometric $p_1^{x_1} p_2^{x_2} \theta^{x_1 x_2}$ $0<p_i<1, 1+p_1p_2\theta \ge p_1+p_2.$ $x_i = 0,1,\ldots, \; i=1,2$	$(1-p_1\theta^{x_2}, 1-p_2\theta^{x_1})$	$b\text{-}IFR_2 \not\Rightarrow b\text{-}DFR_4$ $b\text{-}IFR_1 \not\Rightarrow b\text{-}DFR_4$
ii) Bivariate Waring $\dfrac{(m)_{x_1+x_2}}{(m+n)_{x_1+x_2}}$ $m,n>0$; $x_i=0,1,\ldots$ $i=1,2$	$\left(\dfrac{n}{m+n+x_1+x_2}, \dfrac{n}{m+n+x_1+x_2}\right)$	$b\text{-}DFR_2 \not\Rightarrow b\text{-}IFR_2$ $b\text{-}DFR_2 \not\Rightarrow b\text{-}IFR_4$
iii) Bivariate Geometric $p^{x_2} p_1^{x_1-x_2}, \; x_1 \ge x_2$ $p^{x_1} p_2^{x_2-x_1}, \; x_1 \le x_2$ $0<p\le p_i<1, \; x_i=0,1,2,\ldots$ $1+p\ge p_1+p_2, \; i=1,2$	$(1-p_1, 1-p/p_1)$ $x_1>x_2$ $(1-p/p_2, 1-p_2)$ $x_1<x_2$ $(1-p_1, 1-p_2)$ $x_1=x_2$	When $p\ge p_1p_2$ $b\text{-}DFR_3 \not\Rightarrow b\text{-}DFR_2$ When $p\le p_1p_2$ $b\text{-}IFR_3 \not\Rightarrow b\text{-}IFR_2$
iv) Bivariate Negative Hypergoemetric $\binom{k+n-x_1-x_2}{n-x_1-x_2} \Big/ \binom{k+n}{n}$ $x_1+x_2\le n. \; x_i=0,1,2,\ldots,n$	$\left(\dfrac{k}{k+n-x_1-x_2}, \dfrac{k}{k+n-x_1-x_2}\right)$	$b\text{-}IFR_1 \not\Rightarrow b\text{-}DFR_2$ $b\text{-}IFR_1 \not\Rightarrow b\text{-}DFR_4$
v) Bivariate geometric mixture $\alpha p^{x_1+x_2}+(1-\alpha)p^{x_1}, \; x_1\ge x_2$ $\alpha p^{x_1+x_2}+(1-\alpha)p^{x_2}, \; x_1\le x_2$ $0<p,\alpha<1, x_i=0,1,2\ldots$ $i=1,2$	$\left(1-p, 1-\dfrac{\alpha p^{x_2+1}+(1-\alpha)}{\alpha p^{x_2}+(1-\alpha)}\right)$ $x_1\ge x_2+1$ $\left(1-\dfrac{\alpha p^{x_1+1}+(1-\alpha)}{\alpha p^{x_1}+(1-\alpha)}, 1-p\right)$ $x_1+1\le x_2$ $(1-p, 1-p), \; x_1=x_2$	$b\text{-}IFR_4 \not\Rightarrow b\text{-}IFR_3$ $b\text{-}DFR_4 \not\Rightarrow b\text{-}DFR_3$

Bivariate Distribution	$(h_1(\underline{x}), h_2(\underline{x}))$	Example for
vi) Bivariate Geometric $p_1^{x_1}$, $p_1^{x_1} \le p_2^{x_2}$ $p_2^{x_2}$, $p_1^{x_1} \ge p_2^{x_2}$ $0<p_i<1, x_i=0,1,2...$ $i=1,2$	$(1-p_1, 0)$, $p_1^{x_1} \le p_2^{x_2+1}$ $(0, 1-p_2)$, $p_1^{x_1+1} \ge p_2^{x_2}$ $\left(1-p_1,\ 1-\frac{p_2^{x_2+1}}{p_1^{x_1}}\right)$ $p_1^{x_1} \le p_2^{x_2}, p_1^{x_1} \ge p_2^{x_2+1}$ $\left(1-\frac{p_1^{x_1+1}}{p_2^{x_2}},\ 1-p_2\right)$ $p_1^{x_1} \ge p_2^{x_2}, p_1^{x_1+1} \le p_2^{x_2}$	$b\text{-}IFR_2 \not\Rightarrow b\text{-}IFR_4$

Table II

Bivariate Distribution	Marginal and Conditional Failure rates	Example For
i) Bivariate Geometric $R(\underline{x})=\begin{cases} p^{x_2}\, p_1^{x_1-x_2}, & x_1\geq x_2 \\ p^{x_1}\, p_1^{x_2-x_1}, & x_1\leq x_2 \end{cases}$ $1+p\geq p_1+p_2,\ 0<p\leq p_j<1$ $\underline{x}\in I_2^+,\ j=1,2.$	$h_j(x_j)=1-p_j$ $\underline{c}(\underline{x})=\begin{cases} 1-p_1,\ 1-pp_1^{-1}, & x_1>x_2 \\ 1-p_2,\ 1-pp_2^{-1}, & x_1<x_2 \\ \frac{1+p-p_1-p_2}{1-p_2},\ \frac{1+p-p_1-p_2}{1-p_1}, & x_1=x_2 \end{cases}$	$c\text{-}IFR_3 \not\leftrightarrow c\text{-}IFR_2$ $c\text{-}IFR_3 \not\leftrightarrow c\text{-}DFR_2$ $c\text{-}DFR_3 \not\leftrightarrow c\text{-}IFR_2$ $c\text{-}DFR_3 \not\leftrightarrow c\text{-}DFR_2$
ii) Bivariate Geometric $f(\underline{x})=\binom{x_1+x_2}{x_1} p_1^{x_1} p_2^{x_2} (1-p_1-p_2)$ $\underline{x}\in I_2^+, 0<p_1<p_2<1, p_1+p_2<1.$	$h_j(x_j)=1-p_j$ $\underline{c}(\underline{x})=([1+(1+\frac{x_2}{x_1+1})p_1+\ldots]^{-1},$ $[1+(1+\frac{x_2}{x_2+1})p_2+\ldots]^{-1})$	$c\text{-}IFR_2 \not\leftrightarrow c\text{-}DFR_3$ $c\text{-}IFR_2 \not\leftrightarrow c\text{-}IFR_3$
iii) $R(\underline{x}) = 2(x_1+x_2)^{-1}$ $x_j=1,2,\ldots,\ j=1,2.$	$h_j(x_j)=(x_j+2)^{-1}$ $\underline{c}(\underline{x})=\left[1-\left(1+\frac{2}{x_1+x_2}\right)^{-1},\ 1-\left(1+\frac{2}{x_1+x_2}\right)^{-1}\right]$	$c\text{-}DFR_2 \not\leftrightarrow c\text{-}IFR_3$
iv) Negative Binomial $R(\underline{x})=\frac{\binom{k+n-x_1-x_2}{n-x_1-x_2}}{\binom{k+n}{n}}$ $x_1+x_2\leq n,\ k>0,\ x_j=0,1,2,\ldots,n.$	$h_j(x_j)=k/(k+n-x_j),\ x_j=0,1,\ldots,n$ $\underline{c}(\underline{x})=\left[\frac{k-1}{k+n-x_1-x_2-1},\ \frac{k-1}{k+n-x_1-x_2-1}\right]$	$c\text{-}IFR_1 \not\leftrightarrow c\text{-}DFR_2$ $c\text{-}IFR_1 \not\leftrightarrow c\text{-}DFR_3$

Chapter V

PROPERTIES OF A BIVARIATE GEOMETRIC DISTRIBUTION

5.1 INTRODUCTION

Arising from the discussions in the three preceeding chapters we have been able to identify different forms of bivariate geometric distributions that are characterized by interesting functional forms of failure rates and form the boundary class when these failure rates exhibit monotone behaviour. Of these the bivariate form specified by the survival function

$$R(x) = \begin{cases} p^{x_2} p_1^{x_1 - x_2}, & x_1 \geq x_2 \\ p^{x_1} p_2^{x_2 - x_1}, & x_1 \leq x_2 \end{cases} \tag{5.1}$$

$0 < p \leq p_j < 1,\ 1+p \geq p_1+p_2,\ j=1,2,$

though have repeatedly appeared in literature in different contexts, has not been subjected to a detailed investigation viz-a-viz its distributional properties and characterizations. The present chapter is an attempt in this direction.

Hawkes (1972) in his attempt to construct a bivariate exponential distribution by generalizing to two - dimensions a simple random shock model supposed that two-components behave in such a way that the number of shocks needed to lead to a failure follows a bivariate geometric model which he derived as follows.

If A_1 and A_2 are two events with joint probability distribution given by

	A_1	$\bar{A}_1$	
A_2	p_{11}	p_{01}	P_2
$\bar{A}_2$	p_{10}	p_{00}	Q_2
	P_1	Q_1	1

Define X_j as the number of trails completed upto the first occurence of the event A_j, then the probability generating function is obtained as

$$\phi(s_1,s_2) = s_1 s_2 \left\{ p_{11} + \frac{p_{10} P_2 s_2}{1-Q_2 s_2} + \frac{p_{10} P_1 s_1}{1-Q_2 s_1} + p_{00}\, \phi(s_1,s_2) \right\}$$

and hence

$$R(\underline{x}) = \begin{cases} p_{00}^{x_1} (p_{10}+p_{00})^{x_2-x_1} & x_1 \le x_2 \\ p_{00}^{x_2} (p_{01}+p_{00})^{x_1-x_2} & x_1 \ge x_2 \end{cases}$$

where $p_{00}+p_{01}+p_{10}+p_{11}=1$, $p_{10}+p_{11}<1$ and $p_{01}+p_{11}<1$.

For the relationship of the model 5.1 to that of Paulson and Uppulari (1972) and other details we refer to Block (1977) who also used this same model to generate bivariate geometric and bivariate exponential distributions introduced by various authors, through geometric compounding. The bivariate geometric model (5.1) is a slightly reparameterized form of the bivariate geometric distributions considered above.

An alternative way of interpreting the model is by the concept of non-aging. Assuming that (i), the probability of survival of a two component system with components of age x_1 and x_2 respectively is the same as that of a new system or

$$P[\underline{X}\ge\underline{x}+\underline{t} \mid \underline{X}\ge\underline{t}] = P[\underline{X}\ge\underline{x}]; \quad \underline{x}\in I_2^+, \; \underline{t}=(t,t), t\in I_1^+ \qquad (5.2)$$

and (ii) each of the components taken individually does not wearout over time or

$$P[X_j \geq x_j + t \mid X_j \geq t] = P[X_j \geq x_j], j=1,2. \qquad (5.3)$$

The first condition simplifies to the bivariate lack of memory property and the second is equivalent to assuming that X_j is distributed geometrically. The unique solution of (5.2) that satisfies (5.3) is the bivariate geometric distribution (5.1). It may be recalled that this fact has been proved in Chapter II.

The method of trivariate reduction can also be employed to construct the bivariate geometric model by making use of three independent geometric variables Y_0, Y_1 and Y_2 with parameters γ_0, γ_1 and γ_2 respectively and defining $X_1 = \mathrm{Min}(Y_0, Y_1)$ and $X_2 = \mathrm{Min}(Y_0, Y_2)$. The distribution of $\underline{X}$ can be observed as

$$R(\underline{x}) = \begin{cases} P[Y_0 \geq x_1, Y_1 \geq x_1, Y \geq x_2], & x_1 \geq x_2 \\ P[Y_0 \geq x_2, Y_1 \geq x_1, Y_2 \geq x_2], & x_1 \leq x_2 \end{cases}$$

$$= \begin{cases} \gamma_0^{x_1} \gamma_1^{x_1} \gamma_2^{x_2} & x_1 \geq x_2 \\ \gamma_0^{x_2} \gamma_1^{x_1} \gamma_2^{x_2} & x_1 \leq x_2 \end{cases}$$

Writing $\gamma_0\gamma_1\gamma_2=p$, $\gamma_0\gamma_1=p_1$ and $\gamma_0\gamma_2=p_2$, the form of the bivariate geometric is recovered. it should be noted that this particular method always generates a bivariate geometric for which $p \leq p_1 p_2$.

5.2 DISTRIBUTION THEORY

In the present section, we investigate the distributional properties of the bivariate geometric distribution.

1. The joint probability mass function of $\underline{X}$ is

$$f(\underline{x})=R(\underline{x})-R(x_1+1,x_2)-R(x_1,x_2+1)+R(x_1+1,x_2+1)$$

This is calculated as seen in Chapter two as

$$f(\underline{x}) = \begin{cases} (p_1-p)(1-p_1)\, p^{x_2}\, p_1^{x_1-x_2-1}, & x_1>x_2 \\ (p_2-p)(1-p_2)\, p^{x_1}\, p_2^{x_2-x_1-1}, & x_1<x_2 \\ (1+p-p_1-p_2)p^{x}, & x_1=x_2=x \end{cases} \tag{5.4}$$

From the condition of rectangular positively it is evident that the parameter must satisfy the constraints $1+p \geq p_1+p_2$ apart from the condition $0<p\leq p_j<1$, $j=1,2$.

2. As already mentioned, the marginal distribution of X_j is geometric with parameter p_j, $j=1,2$, therefore $R_j(x_j)=p_i^{x_j}$ and the conditional probability mass function of X_i given $X_j=x_j$, $j\neq i$ is

$$f(x_i|x_j) = \begin{cases} p_i^{x_i-1}\left(\frac{p}{p_1p_2}\right)^{x_j} \frac{(1-p_i)(p_i-p)}{(1-p_j)} & ,x_i>x_j \\ p^{x_i}\, p_j^{-(x_i+1)}\,(p_j-p), & x_i<x_j \\ \frac{(1+p-p_1-p_2)}{1-p_j}\left(\frac{p}{p_j}\right)^{x_j} & x_i=x_j \end{cases} \quad (5.5)$$

The regression functions are accordingly

$$E\,[X_i|X_j=x_j] = \frac{(1-p)(p_1p_2-p)}{(1-p_1)(1-p_2)(p_j-p)}\left(\frac{p}{p_j}\right)^{x_j} + \left(\frac{p}{p_j-p}\right)$$

3. The correlation structure of $\underline{X}$ derives from

$$E\,(X_1,X_2) = \frac{1-p_1p_2}{(1-p_1)(1-p_2)(1-p)} \quad (5.6)$$

and

$$\text{Cov}\,(X_1,X_2) = \frac{p-p_1p_2}{(1-p_1)(1-p_2)(1-p)} \tag{5.7}$$

The coefficient of corelation γ has the expression,

$$\gamma = \frac{p-p_1p_2}{(p_1p_2)^{1/2}(1-p)} \tag{5.8}$$

Thus when $p=p_1p_2$ the variables are uncorelated in which case X_1 and X_2 are independent as well. Notice that γ is an increasing function of p and attains its maximum value unity, when p is the geometric mean between p_1 and p_2. This happens when $p=p_1=p_2$ in which case the two components survive or fail together. Further, γ is minimum when p is $\text{Max}(0,p_1+p_2-1)$ in which case the corresponding values of γ are

$$-(p_1p_2)^{1/2} \quad \text{and} \quad \frac{(1-p_1)(1-p_2)}{(p_1p_2)^{1/2}(p_1+p_2-2)}$$

4. The probability generating function of the bivariate geometric distribution (5.1) is derived as

$$s(t_1,t_2) = \frac{(1-p_1)(p_1-p)t_1}{(1-t_1p_1)(1-t_1t_2p)} + \frac{(1-p_2)(p_2-p)t_2}{(1-t_2p_2)(1-t_1t_2p)}$$
$$+ \frac{(1+p-p_1-p_2)}{(1-t_1t_2p)}$$

5. A comparision of the relative life length of the components in the system is offered by the following probabilities

$$P[X_1>X_2] = \frac{p_1-p}{1-p}$$

$$P[X_1<X_2] = \frac{p_2-p}{1-p}$$

$$P[X_1=X_2] = \frac{1+p-p_1-p_2}{1-p}$$

6. If $Z=\text{Min}(X_1,X_2)$, the distribution of Z is geometric with parameter p. On the other hand $W=\max(X_1,X_2)$ has a survival function of the form

$$\begin{aligned} P[W\geq w] &= 1-P[W<w] \\ &= 1-P[\text{Max}(X_1,X_2)<w] \\ &= 1-[1-P(X_1\geq w)-P(X_2\geq w)+P(X_1\geq w,X_2\geq w)] \\ &= p_1^w+p_2^w-p^w. \end{aligned}$$

7. The event $Z=z$ is independent of the events $X_1<X_2$, $X_1>X_2$ and $X_1=X_2$. To prove this assertion we note that the conditional probability mass function of Z given $X_1<X_2$,

$$f(z|X_1<X_2) = P[\min(X_1,X_2) = z|X_1<X_2]$$

$$= P[X_1=z,\ X_2>z|X_1<X_2]$$

$$= \frac{1-p}{p_2-p} P[X_1=z,\ X_2\geq z+1]$$

$$= \frac{1-p}{p_2-p} [R(z,z+1) - R(z+1,z+1)]$$

$$= (1-p)\ p^z$$

$$= P[Z=z]$$

Thus $Z=z$ is independent of $X_1<X_2$. The independence of the other two events $X_1>X_2$ and $X_1=X_2$ with $Z=z$ follows from similar arguments. Our result implies that which of the two components that fail first does not depend on the relative maginitude of the life lengths of the components.

8. If $\underline{X}$ and $\underline{Y}$ are two independent random vectors having distribution (5.1) with $p_1p_2>p$ then the component wise minimum of $\underline{X}$ and $\underline{Y}$ also exhibit the distribution (5.1). This can be proved as follows.

$$P[\mathrm{Min}(X_1,Y_1)\geq z_1\ ,\ \mathrm{Min}(X_2,Y_2)\geq z_2]$$

$$= P[X_1 \geq z_1, X_2 \geq z_2, Y_1 \geq z_1, Y_2 \geq z_2]$$

$$= P[\underline{X} \geq \underline{z}]\, p[\underline{Y} \geq \underline{z}]$$

$$= \begin{cases} (p^2)^{z_2}(p_1^2)^{z_1 - z_2} & , \quad z_1 \geq z_2 \\ (p^2)^{z_1}(p_2^2)^{z_2 - z_1} & , \quad z_1 \leq z_2 \end{cases}$$

Also it follows that $0<p^2 \leq p_j^2<1$ and since $p_1 p_2>p$, $1+p^2 \geq p_1^2+p_2^2$ is implied, thus satisfying the conditions in the parameters.

5.3 CHARACTERIZATIONS BY RELIABILITY CONCEPTS

In this section we establish some characteristic properties of the bivariate geometric law 5.1. Characterizations of the model by the form of its scalar failure rate, vector failure rate and conditional failure rates have already been proved in the previous chapters. Another concept of direct relevance to reliability analysis is that of residual life distribution (Nair and Asha, 1994).

When modelling equipment behaviour in reliability studies, concepts such as the mean residual life, variance residual life etc... are extensively used in literature. Such concepts and their functional forms provide characterizations of various life distributions. Since these functional forms arise as summary measure of the distribution of residual life, the concept of residual life distribution is more basic and revealing and provides deeper insight into the failure phenomenon. It will be seen shortly that the basic properties inherent in the residual life distribution is responsible for the physical characteristics governing the life length of the system.

Definition 5.1

Let $\underline{X}$ be a discrete random vector in the support of $I_2^+ = \{x=(x_1,x_2);\ x_j = 0,1,2,...;\ j=1,2\}$ with survival function

$$R(\underline{x})=P[\underline{X}\geq\underline{x}],\ \underline{x}\in I_2^+ \ . \tag{5.9}$$

The random vector

$$\underline{Y}(\underline{x}) = (Y_1(\underline{x}),Y_2(\underline{x})),$$

where

$$Y_j(\underline{x}) = (X_j-x_j|\underline{X}\geq\underline{x}),\ j=1,2$$

is defined as the residual life function corresponding to $\underline{X}$. The distribution of $\underline{Y}(\underline{x})$ is specified by the survival function

$$S(y_1,y_2;x_1,x_2) = \frac{R(x_1+y_1,x_2+y_2)}{R(\underline{x})} \; ; \quad y_1,y_2=0,1,\ldots \qquad (5.10)$$

at all points $\underline{x}$ for which $R(\underline{x})>0$. The distribution of $\underline{Y}(\underline{x})$ with survival function (5.10) is defined as the residual life distribution corresponding to the random vector $\underline{X}$.

A most convenient property of the residual life distribution that imparts reduction in effort to make inferences is that the residual life distribution has the same form as that of the orginal life distribution except for a change in parameters. In other word the life distribution is closed with respect to the formation of residual distribution. It is easy to see from (5.10) that the parent distribution and the residual life distribution are identical in all respects if and only if X_1 and X_2 are independent geometric variables, which does not provide a meaningful bivariate model. Hence one has to relax the condition of identical distribution for $\underline{X}$ and $\underline{Y}(\underline{x})$ in some meaningful manner so as to end up with bivariate distributions having a dependence structure. In the following theorem it is shown that the identical nature of the survival functions of $\underline{X}$ and $\underline{Y}(\underline{x})$ in a subset of the sample space is enough to characterize the bivariate geometric model (5.1).

Theorem 5.1 (Asha and Nair, 1994 c)

If $\underline{X}$ is a bivariate random vector with a survival function $R(\underline{x})$ in the support of I_2^+, then

$$S(y_1,y_2;x_1,x_2) = R(\underline{y}) \tag{5.11}$$

for ever $\underline{x}$, $\underline{y}$ in Q_j , where

$$Q_j = \left\{ \underline{x},\underline{y} \,\middle|\, x_1 \le x_{3-j},\ y_3 \le y_{3-j}\ ;\ x_j, y_j = 0,1,2\ldots \right\};\ j=1,2.$$

if and only if $\underline{X}$ follows the bivariate geometric model (5.1)

Proof:

By (5.10)

$$S(y_1,y_2\ ;\ x_1,x_2) = \frac{R(x_1+y_1,\ x_2+y_2)}{R(\underline{x})}$$

in Q_j

$$S(y_1,y_2\ ;\ x_1,x_2) = \frac{p^{x_{3-j}+y_{3-j}}\ p_j^{x_j+y_j-x_{3-j}-y_{3-j}}}{p^{x_{3-j}}\ p_j^{x_j-x_{3-j}}}$$

$$= p^{y_{3-j}}\ p_j^{y_j-y_{3-j}}\ ,\ j=1,2$$

$$= R(\underline{y})$$

Conversely, let (5.11) be satisfied.

Then by definition of residual life distribution,

$$R(x_1+y_1,x_2+y_2) = R(\underline{x})\ R(\underline{y}) \qquad (5.12)$$

for every $\underline{x}$, $\underline{y}$ in Q_j , j=1,2.

In particular, where $x_1=x_2=x$ and $y_1=y_2=y$

$$R(x+y,x+y) = R(x,x)\ R(y,y), \qquad (5.13)$$

implying

$$R(x,x) = p^x \text{ for some } 0<p<1 \text{ and all } x \in I_1^+ \qquad (5.14)$$

Also when $x_1=y_1=0$

$$R(0,x_2+y_2) = R(0,x_2)\ R(0,y_2)$$

or

$$R(0,x_2) = p_2^{x_2} \text{ for some } 0<p_2<1 \text{ and all } x_2 \in I_1^+ \qquad (5.15)$$

similarly when $x_2=y_2=0$

$$R(x_1,0) = p_1^{x_1} \text{ for some } 0<p_1<1 \text{ and all } x_1 \in I_1^+ \qquad (5.16)$$

Now, the equation (5.10) is equivalent to

$$R(x_1+y_1,x_2+y_2) = R(x_1,x_2)\ R(x_1-x_2,0)\ R(y_1,y_2)\ R(y_1-y_2,0)$$

$$\text{for all } x_1\ge x_2 \text{ and } y_1\ge y_2 \qquad (5.17)$$

and

$$R(x_1+y_1,x_2+y_2) = R(x_1,x_1)\ R(0,x_2-x_1)\ R(y_1,y_1)\ R(0,y_2-y_1)$$

$$\text{for all } x_1\le x_2 \text{ and } y_1\le y_2 \qquad (5.18)$$

Using (5.14), (5.15) and (5.16) in (5.17) and (5.18) and replacing x_1+y_1 by x_1 and x_2+y_2 by x_2 , the form of the bivariate geometric model (5.1) is recovered.

The next theorem establishes that stationarity of the r^{th} factorial moment of $\underline{Y}(\underline{x})$ is equivalent to the bivariate lack of memory property.

Theorem 5.2:

If $\underline{X}$ is a discrete bivariate random vector in the support of I_2^+ with survival function $R(\underline{x})$ then

$$E\ \underline{Y}^{(r)}(x_1+t,x_2+t) = E\ \underline{Y}^{(r)}(\underline{x}) \tag{5.19}$$

where $Y^{(r)} = Y(Y-1)\ldots(Y-r+1)$, for any two consecutive values of r and all $\underline{x},\underline{y} \in I_2^+$, $t \in I_1^+$, if and only if $\underline{X}$ satisfies the bivariate lack of memory property.

Proof:

Consider

$$E\ \underline{Y}^{(r)}(\underline{x}) = \left(E\ Y_1^{(r)}(\underline{x}), E\ Y_2^{(r)}(\underline{x}) \right)$$

where,

$$EY_1^{(r)}(\underline{x}) = E((X_1-x_1)^{(r)} | \underline{X} \geq \underline{x}),$$

$$=[R(\underline{x})]^{-1} \sum_{y_1=0}^{\infty} y_1^{(r)}[R(x_1+y_1,x_2+y_2)-R(x_1+y_1+1,x_2+y_2)].$$

When the bivariate lack of memory property is satisfied,

$$E\ Y_1^{(r)}(x_1+t,x_2+t) = [R(x_1+t,x_2+t)]^{-1}$$

$$\sum_{y_1=0}^{\infty} y_1^{(r)}\ [R(x_1+y_1+t,x_2+y_2+t) - R(x_1+y_1+t+1,x_2+y_2+t)],$$

$$=[R(\underline{x})R(t,t)]^{-1}$$

$$\sum_{y_1=0}^{\infty} y_1^{(r)} R(t,t)[R(x_1+y_1,x_2+y_2)-R(x_1+y_1+1,x_2+y_2)],$$

$$= E\ Y_1^{(r)}(\underline{x}).$$

Similarly,

$$E\ Y_2^{(r)}(x_1+t, x_2+t) = E\ Y_2^{(r)}(\underline{x})$$

Thus condition (5.19) holds for any r=1,2... and $\underline{x}$, $\underline{y} \in I_2^+$, $t \in I_1^+$. To prove the converse, note that

$$E\ Y_1^{(r)}(\underline{x}) = [R(\underline{x})]^{-1} \sum_{y_1=0}^{\infty} y_1^{(r)}\ [R(x_1+y_1, x_2+y_2)$$

$$- R(x_1+y_1+1, x_2+y_2)],$$

$$= [1^{(r)}-0^{(r)}]\ R(x_1+1, x_2+1)$$

$$+ [2^{(r)} - 1^{(r)}]\ R(x_1+2, x_2+2)\ + \ldots ,$$

$$= \sum_{y_1=0}^{\infty} [\ y_1^{(r)} - (y_1-1)^{(r)} [R(x_1+y_1, x_2+y_2),$$

$$= r \sum_{y_1=0}^{\infty} [(y_1-1)\ldots(y_1-r-1)[R(x_1+y_1, x_2+y_2),$$

$$= r \sum_{y_1=0}^{\infty} (y_1-1)^{(r)} [R(x_1+y_1, x_2+y_2). \qquad (5.20)$$

Changing x_1 to x_1+1,

$$R\ (x_1+1, x_2)\ EY_1^{(r)}(x_1+1, x_2) = \sum_{y_1=0}^{\infty} (y_1-1)^{(r-1)} [R(x_1+y_1+1, x_2+y_2)$$

Subtracting the above from (5.20) we get

$$[R(\underline{x})]E\ Y_1^{(r)}(\underline{x}) - R\ (x_1+1,x_2)\ E\ Y_1^{(r)}(x_1+1,x_2)$$

$$= \sum_{y_1=0}^{\infty} (y_1-1)^{(r-1)}[R(x_1+y_1,x_2+y_2) - [R(x_1+y_1+1,x_2+y_2)],$$

$$= R\ (x_1+1,x_2)\ E\ Y_1^{(r)}(x_1+1,x_2),$$

which implies

$$\frac{R(x_1+1,x_2)}{R(x_1+t,x_2+t)} = \frac{E\ Y_1^{(r)}(\underline{x})}{E\ Y_1^{(r)}(x_1+1,x_2) + E\ Y_1^{(r-1)}(x_1+1,x_2)}.$$

When (5.19) holds

$$\frac{R(x_1+t+1,x_2+t)}{(x_1+t,x_2+t)} = \frac{E\ Y_1^{(r)}\ (x_1+t,x_2+t)}{E\ Y_1^{(r)}(x_1+t+1) + E\ Y_1^{(r-1)}(x_1+t+1,x_2+t)},$$

$$= \frac{E\ Y_1^{(r)}(\underline{x})}{E\ Y_1^{(r)}(x_1+1,x_2) + E\ Y_1^{(r-1)}(x_1+1,x_2)},$$

$$= \frac{R(x_1+1,x_2)}{R(\underline{x})} \text{ for all } \underline{x} \in I_2^+ ,t \in I_1^+ . \qquad (5.21)$$

Similarly it can be proved that

$$\frac{R(x_1+t,x_2+t+1)}{R(x_1+t,x_2+t)} = \frac{R(x_1,x_2+1)}{R(\underline{x})} \text{ for all } \underline{x} \in I_2^+ , \ t \in I_1^+ \quad (5.22)$$

That the equation (5.21) and (5.22) imply the bivariate lack of memory property has already been proved in Theorem (3.5) in Chapter III.

Corollary 5.1

If $\underline{X}$ has geometric marginals then (5.19) is satisfied if and only if $\underline{X}$ has the bivariate geometric distribution (5.1).

5.4 CHARACTERIZATIONS BY DISTRIBUTIONAL PROPERTIES

In the present section, we characterize the bivariate geometric law using its distributional properties. First we prove a characterization on the marginal and conditional distributions of the same component, which incidentally also provides a characterization of the univariate geometric distribution in terms of the bivariate distribution.

Theorem 5.3.

If $\underline{X}$ is a random vector with support I_2^+ such that the conditional distribution of X_1 given X_2 is of the form (5.5), then X_1 is geometric with parameter p_1 if and only if X_2 is geometric with parameter p_2.

Proof:

If X_2 is geometric, then its probability mass function is

$$f_2(x_2) = p_2^{x_2} (1 - p_2), \qquad x_2 \in I_1^+ \tag{5.23}$$

Using the expression for $f(x_1|x_2)$ from (5.5) and that of $f_2(x_2)$, the joint probability mass function of $\underline{X}$ turns out to be

$$f(x_1,x_2) = f\ (x_1|x_2)f_2(x_2)$$

$$= \begin{cases} p_1^{x_1-1} \left(\frac{p}{p_1}\right)^{x_2} (1-p_1)(p_1-p), & x_1>x_2 \\ p_1^{x_1} (p_2-p)(1-p_2)p_2^{x_2-x_1-1} & x_1<x_2 \\ (1+p-p_1-p_2)p^x & x_1=x_2=x. \end{cases} \tag{5.24}$$

Summation of (5.24) over the support of X_2 yields

$$f_1(x_1) = p_1^{x_1} (1-p_1). \qquad x_1=0,1,2,...$$

Conversely, assuming X_1 to be geometric and $f(x_1|x_2)$ to be as in (5.5),

$$(1-p_1)p_1^{x_1} = \sum_{x_2=0}^{x_1-1} p_1^{x_1-1} \left(\frac{p}{p_1p_2}\right)^{x_2} \frac{(1-p_1)(p_1-p)}{(1-p_2)} f_2(x_2)$$

$$+ \left(\frac{1+p-p_1-p_2}{1-p_2}\right) \left(\frac{p}{p_2}\right)^{x_1} f_2(x_1)$$

$$+ \sum_{x_2=x_1+1}^{\infty} p_1^{x_1} p_2^{-(x_1+1)} (p_2-p)f_2(x_2). \quad (5.25)$$

When $x_1 = 0$ we have from (5.25),

$$(1-p_1) = \left(\frac{1+p-p_1-p_2}{1-p_2}\right) f_2(0) + \left(\frac{p_2-p}{p_2}\right) \sum_{x_2=1}^{\infty} f_2(x_2)$$

Which is equivalent to writing

$$(1-p_1) = \left(\frac{1+p-p_1-p_2}{1-p_2}\right) [1-R_2(1)] + \left(\frac{p_2-p}{p_2}\right) R_2(1)$$

On simplification

$R_2(1)=p_2$, so that

$f_2(0) = 1-R_2(1) = 1-p_2.$

Similarly when $x_1 = 1$, we have from (5.25),

$$p_1(1-p_1) = (1-p_1)(p_1-p)+p(1+p-p_1-p_2)+p\left(\frac{p_2-p}{p_2^2}\right) R_2(2).$$

So that

$R_2(2) = p_2^2.$

In general let the result be true for $x_2=x_1$. Then from (5.25),

$$(1-p_1)p_1^{x_1} = p_1^{x_1-1}(1-p_1)(p_1-p)\sum_{x_2=0}^{x_1-1}\left(\frac{p}{p_1}\right)^{x_2} + (1+p-p_1-p_2)p^{x_1}$$

$$+\left(\frac{p_2-p}{p_2}\right)\left(\frac{p}{p_2}\right)^{x_1} R_2(x_1+1)$$

Thus

$$R_2(x_1+1) = p_2^{x_1+1},$$

which proves our assertion.

Corollary 5.3

The conditional distribution of $X_1(X_2)$ given $X_2=x_2$ $(X_1=x_1)$ is of the form (5.5) and $X_1(X_2)$ is geometric is a necessary and sufficient condition for $\underline{X}$ to have a bivariate geometric law.

The proof follows directly from Theorem 5.3 and equation (5.5).

Theorem 5.4

The vector $\underline{X}$ has the bivariate geometric law 5.1 with $p_1p_2<p$ if and only if there exist independent geometric variables Y_1, Y_2 and Y_3 such that $X_1 = \mathrm{Min}(Y_1,Y_3)$ and $X_2=\mathrm{Min}(Y_2,Y_3)$.

Proof:

If Y_1, Y_2 and Y_3 are geometric with parameters γ_1, γ_2 and γ_3, $X_1 = \mathrm{Min}(Y_1,Y_3)$ and $X_2=\mathrm{Min}(Y_2,Y_3)$ then,

$$R(\underline{x}) = P[Y_1\geq x_1, Y_2\geq x_2, Y_3\geq \mathrm{Max}(x_1,x_2)] = \begin{cases} \gamma_1^{x_1}\gamma_2^{x_2}\gamma_3^{x_1}, & x_1\geq x_2 \\ \gamma_1^{x_1}\gamma_2^{x_2}\gamma_3^{x_3}, & x_2\geq x_1 \end{cases}$$

Taking $p=\gamma_1\gamma_2\gamma_3$, $p_1=\gamma_1\gamma_3$ and $p_2=\gamma_2\gamma_3$ the form of the bivariate geometric in (5.1) is recovered. Conversely, if $\underline{X}$ has the bivariate geometric law (5.1) with $p_1p_2<p$, there exist number $0<\gamma_j<1$, $j=1,2,3$ such that

$$p=\gamma_1\gamma_2\gamma_3,\quad p_1=\gamma_1\gamma_3,\quad p_2=\gamma_2\gamma_3 \tag{5.26}$$

that satisfy the conditions on the parameters. To see this note that

$$(1-\gamma_3)+(1-\gamma_1)(1-\gamma_2)\gamma_3 \Leftrightarrow 1+\gamma_3\{(1-\gamma_1)(1-\gamma_2)-1\}\geq 0$$
$$\Leftrightarrow 1+\gamma_3\{\gamma_1\gamma_2-\gamma_1-\gamma_2\}\geq 0$$
$$\Leftrightarrow 1+\gamma_1\gamma_2\gamma_3-\gamma_1\gamma_3-\gamma_2\gamma_3\geq 0$$
$$\Leftrightarrow 1+\gamma_1\gamma_2\gamma_3\geq\gamma_1\gamma_3+\gamma_2\gamma_3\geq 0$$
$$\Leftrightarrow 1+p\geq p_1+p_2.$$

Also obviously $p<p_j$, $j=1,2$ and $p_1p_2<p$. We can solve for γ_1, γ_2 and γ_3 as

$$\gamma_1 =(p/p_2),\ \gamma_2=(p/p_1),\ \gamma_3= (p_1p_2/p) \qquad (5.27)$$

Now choosing γ_1, γ_2 and γ_3 to be the parameters of independent geometric random variables Y_1, Y_2 and Y_3 respectively. We can write (5.1) with $p_1p_2<p$ as

$$R(\underline{x}) = \begin{cases} \gamma_1^{x_1}\ \gamma_2^{x_2}\ \gamma_3^{x_1}, & x_1\geq x_2 \\ \gamma_1^{x_1}\ \gamma_2^{x_2}\ \gamma_3^{x_3}, & x_2\geq x_1 \end{cases}$$

$$= \begin{cases} P[Y_1\geq x_1, Y_2\geq x_2, Y_3\geq x_1], & x_1\geq x_2 \\ P[Y_1\geq x_1, Y_2\geq x_2, Y_3\geq x_2], & x_2\geq x_1 \end{cases}$$

$$= \begin{cases} P[\mathrm{Min}(Y_1,Y_3)\geq x_1, Y_2\geq x_2], & x_1\geq x_2 \\ P[Y_1\geq x_1, \mathrm{Min}(Y_2,Y_3)\geq x_2], & x_2\geq x_1 \end{cases}$$

$= P[\mathrm{Min}(Y_1,Y_3)\geq x_1, \mathrm{Min}(Y_2,Y_3)\geq x_2]$ for all $\underline{x}\in I_2^+$. Hence our assertion.

Note:

Let us consider the bivariate geometric distribution $p=1/8$, $p_1=3/8$ and $p_2=5/8$. Evidently $o<p<p_1, p_2<1$ and $1+p=9/8> p_1+p_2$. But $p_1p_2=15/64 \nless 1/8$. In this case there does not exist a representation of p, p_j, $j=1,2$ in terms of $0<\gamma_i<1$, $i=1,2,3$ as in (5.27). Since $\gamma_1=(p/p_2)=1/5$, $\gamma_2=(p/p_1)=1/3$ and $\gamma_3=(p_1p_2/p)=15/8>1$. Therefore it is necessary that $p_1p_2<p$ for $0<\gamma_3<1$.

Theorem 5.5

The bivariate random vector $\underline{X}=(X_1,X_2)$ in the support of I_2^+ with geometric marginals has the bivariate geometric (5.1) if and only if

(i) $Z=\min(X_1,X_2)$ is geometric.

(ii) Z is independent of $V = (X_1-X_2)$.

Proof:

Assume that $\underline{X}$ has bivariate geometric (5.1). Then the probability mass function of $\underline{X}$ is

$$f(\underline{x}) = \begin{cases} (p_1-p)(1-p_1)\, p^{x_2}\, p_1^{x_1-x_2-1}, & x_1>x_2 \\ (p_2-p)(1-p_2)\, p^{x_1}\, p_2^{x_2-x_1-1} & x_1<x_2 \\ (1+p-p_1-p_2)p^{x} & x_1=x_2 \end{cases}$$

Now the joint probability mass function of $Z=\mathrm{Min}(X_1,X_2)$ and $V=(X_1-X_2)$,

$$f(z,v)= \begin{cases} p^{z}\; p_1^{v-1}(1-p_1)(p_1-p) & v\geq 1 \\ p^{z}\; p_2^{-(v+1)}(1-p_2)(p_2-p) & v\leq -1 \\ p^{z}\; (1+p-p_1-p_2) & v=0 \end{cases} \qquad (5.28)$$

and $z\in I_2^{+}$.

The marginal probability mass function of Z and V are respectively

$$f_1(v)=\begin{cases} p_1^{v-1}\dfrac{(1-p_1)(p_1-p)}{(1-p)} & v\geq 1 \\ p_2^{-(v+1)}\dfrac{(1-p_2)(p_2-p)}{(1-p)} & v\leq -1 \\ \dfrac{1+p-p_1-p_2}{1-p} & v=0 \end{cases}$$

and

$$f_2(z)=p^z(1-p), \qquad z\in I_2^+.$$

Thus Z is geometric and further $f(z,v)=f_1(z)f_2(v)$, which implies that Z and V are independent. Conversely, if (i) and (ii) holds, then for $x_2\geq x_1$

$$P[X_1\geq x_1, X_2\geq x_2] = P[X_1\geq x_2, X_2\geq x_2]+P[x_1\leq X_1<x_2, X_2\geq x_2],$$

$$= P[Z\geq x_2]+P[x_1\leq X_1<x_2, -X_2\leq -x_2],$$

$$= P[Z\geq x_2]+P[x_1\leq X_1<x_2, X_1-X_2\leq X_1-x_2],$$

$$= P[Z\geq x_2]+\sum_{z=x_1}^{x_2-1} P[Z=z, V\leq Z-x_2],$$

$$= P[Z\geq x_2]+\sum_{z=x_1}^{x_2-1} P[Z=z]P[V\leq z-x_2]. \qquad (5.29)$$

Now,

$$P[X_1=x_1,X_2\geq x_2] = P[X_1\geq x_1,X_2\geq x_2]-P[X_1\geq x_1+1,X_2\geq x_2],$$

$$= P[V\leq x_1-x_2]P[Z=x_1],$$

$$= P[V\leq x_1-x_2]p^{x_1}(1-p).$$

Now consider

$$R(x_1+t,x_2+t)= P[Z\geq x_2+t]+ \sum_{z=x_1+t}^{x_2+t-1} P[Z=z]P[V\leq Z-(x_1+t)]$$

Also,

$$P[X_1=t,X_2\geq x_2+t] = P[V\leq -x_2]p^{t}(1-p).$$

$$P[X_1=t+1,X_2\geq x_2+t] = P[V\leq 1-x_2]p^{t+1}(1-p).$$

...

...

...

$$P[X_1=t+i,X_2\geq x_2+t] = P[V\leq i-x_2]p^{t+i}(1-p).$$

$$= p^{t}\ P[V\leq i-x_2]p^{i}(1-p).$$

$$= p^{t}P[X_1=i,X_2\geq x_2].$$

Therefore

$$\sum_{i=x_1}^{\infty} P[X_1=t+i,X_2\geq x_2+t] = p^{t} \sum_{i=x_1}^{\infty} P[X_1=i,X_2\geq x_2].$$

which implies

$$R(x_1+t,x_2+t)=R(t,t)R(x_1,x_2) \text{ for } x_1 \le x_2.$$

Proceeding in a similar manner for $x_1 \ge x_2$ we obtain

$$R(x_1+t,x_2+t)=R(x_1,x_2)\ R(t,t) \text{ for } \underline{x} \in I_2^+.$$

That this functional equation which is the bivariate lack of memory property along with geometric marginals gives the unique solution for R(.,.) of the bivariate geometric law has already been proved in Chapter II.

Theorem 5.6

Let $\underline{X}$ and $\underline{Y}$ be two independent and identically distributed random vectors with support I_2^+. Then $Z=(Z_1,Z_2)$ where $Z_i=\text{Min}(X_i,Y_i)$ i=1,2 has bivariate geometric law if and only if $\underline{X}$ and $\underline{Y}$ are distributed as the bivariate geometric law (5.1) with $p_1p_2>p$,

$$P[Z\ge z] = P[Z\ge z_1, Z_2\ge z_2],$$

$$= P[\text{Min}(X_1,Y_1)\ge z_1, \text{Min}(X_2,Y_2)\ge z_2],$$

$$= P[X_1\ge z_1, Y_1\ge z_1, X_2\ge z_2, Y_2\ge z_2],$$

since $\underline{X}$ and $\underline{Y}$ are independent.

$P[Z \geq z] = P[X_1 \geq z_1, X_2 \geq z_2] P[Y_1 \geq z_1, Y_2 \geq z_2]$

$$= \begin{cases} p^{z_2} p_1^{z_1 - z_2} \; p^{z_2} p_1^{z_1 - z_2}, & z_1 \geq z_2 \\ p^{z_1} p_2^{z_2 - z_1} \; p^{z_1} p_2^{z_2 - z_1}, & z_1 \leq z_2 \end{cases}$$

$$= \begin{cases} (p^2)^{z_2} (p_1^2)^{z_1 - z_2}, & z_1 \geq z_2 \\ (p^2)^{z_1} (p_2^2)^{z_2 - z_1}, & z_1 \leq z_2 \end{cases}$$

That the parameters satisfy the conditions $o<p^2 \leq p_j^2 < 1$, $j=1,2$ and $1+p^2 \geq p_1^2 + p_2^2$ follow directly. Hence Z has the bivariate geometric law with parameters p_1^2, p_2^2 and p^2. The converse is obtained by retracing the steps.

Theorem 5.7

If X_1 and X_2 are geometric variables and Z in the support of non-negative integers is independent of X_1 and X_2 such that

$$P[X_1 - z \geq x_1 + t, X_2 - z \geq x_2 + t] = P[X_1 - z \geq t, X_2 - z \geq t] P[X_1 \geq x_1, X_2 \geq x_2] \quad (5.30)$$

for every $x_1, x_2, t = 0,1,2,\ldots$ if and only if $\underline{X}$ has bivariate geometric law (5.1).

Proof:

If $S(x_1,x_2) = P[X_1-z\geq x_1, X_2-z\geq x_2]$, condition (5.30) reduces to the functional equation

$$S(x_1+t,x_2+t)= S(t,t)R(x_1,x_2);\ \underline{x}\in I_2^+,\ t\in I_1^+ \qquad (5.31)$$

Now since

$$S(x_1,x_2) = S(0,0)R(t,t)R(x_1,x_2),\ \underline{x}\in I_2^+,\ t\in I_1^+$$

equation (5.31) becomes

$$R(x_1+t,x_2+t)S(0,0) = S(0,0)R(t,t)R(x_1,x_2),\quad \underline{x}\in I_2^+,\ t\in I_1^+$$

which is the bivariate lack of memory property. The Theorem is now evident.

The next theorem is a modification of the above result in which we remove the, restriction on Z to be independent of $\underline{X}$.

Theorem 5.8

Let $\underline{X}$ be a discrete random vector in the support of I_2^+ with geometric marginals. Let $Z = \mathrm{Min}(X_1,X_2)$. Then the random vector (X_1-Z,X_2-Z) is independent of Z if and only if $\underline{X}$ has bivariate geometric law 5.1.

Proof:

The conditional distribution of (X_1-Z, X_2-Z) given $Z=z$ is

$$P[X_1-Z=u_1, X_2-Z=u_2 \mid Z=z] = \frac{P[X_1=u_1+z, X_2=u_2+z]}{P[Z=z]} . \qquad (5.32)$$

When $\underline{X}$ is distributed as the bivariate geometric law by direct calculation of the quantities involved in (5.32)

$$P[X_1-z=u_1, X_2-z=u_2 \mid Z=z] = \begin{cases} p^{u_1} p_2^{u_2-u_1-1} (1-p_2)(p_2-p), & u_1<u_2 \\ p^{u_2} p_1^{u_1-u_2-1} (1-p_1)(p_1-p) & u_1>u_2 \\ p^{u} & u_1=u_2=u \end{cases}$$

which is independent of z. Conversely let (X_1-Z, X_2-Z) be independent of Z. Then $G(u_1,u_2) = P[X_1-z=u_1, X_2-z=u_2 \mid Z=z]$ is a function independent of z for all $\underline{u} \in I_2^+$, $z \in I_1^+$.

$P[X_1=u_1+z, X_2=u_2+z] = G(u_1,u_2)P[Z\geq z]$, for all $\underline{u} \in I_2^+$, $z \in I_1^+$.

Which is equivalent to

$$R[u_1+z, u_2+z) = G(u_1,u_2)R(z,z) \text{ for all } \underline{u} \in I_2^+,\ z \in I_1^+. \qquad (5.33)$$

In particular for $z=0$, from (5.33) we obtain

$$R(u_1,u_2)=G(u_1,u_2) \text{ for all } \underline{u}\in I_2^+.$$

Thus (5.33) can be rewritten as

$$R[u_1+z,u_2+z) = R(u_1,u_2)R(z,z) \text{ for all } \underline{u}\in I_2^+,\ z\in I_1^+. \qquad (5.34)$$

which is the bivariate lack of memory property. As seen in Chapter II the bivariate lack of memory property when solved along with the assumption of geometric marginals yields the bivariate geometric model (5.1).

Let (X,Y) be a continious non-negative random vector admitting a survival function $S(x,y) = P[X>x,Y>y]$. Further assume that for some r, $s>0$ (X_r,Y_s) be a discrete random vector in the support of I_2^+ with

$$R_{r,s}(u,v) = P[X_r\geq u,Y_s\geq v]$$

$$= P[X\geq ur,Y\geq vs] = S(ur,vs) \qquad (5.35)$$

at points u,v for which $0<S(u,v)<1$.

Theorem 5.9. (Asha and Nair 1995)

The distribution of (X,Y) is Marshall-Olkin bivariate exponential with survival function

$$S(x,y) = e^{-\lambda_1 x_1 - \lambda_2 y - \lambda_{12} \max(x,y)}, \quad \lambda_1, \lambda_2, \lambda_{12} > 0, \; x,y > 0$$

if and only if for every r,s >0

$$R_{r,s}(u,v) = S(ur,vs) \tag{5.36}$$

where

$$R_{r,s}(u,v) = \begin{cases} p^v \, p_1^{u-v}, & u \geq v \\ p^u \, p_2^{v-u}, & u \leq v, \; 1+p \geq p_1+p_2, 0<p \leq p_j<1, j=1,2, p_1 p_2<p. \end{cases}$$

Proof: When condition (5.36) is satisfied

$$S(ur,vs) = \begin{cases} p^v \, p_1^{u-v}, & u \geq v \\ p^u \, p_2^{v-u}, & u \leq v, \; 1+p \geq p_1+p_2, \; 0<p \leq p_j<1, \; j=1,2, \end{cases} \tag{5.37}$$

so that

$$p_1 = S(r,0) \; ; \; p_2 = S(0,s) \text{ and } p = S(r,s)$$

Thus

$$S(ur,vs) = \begin{cases} [S(r,s)]^v \, [S(r,0)]^{u-v} & u \geq v \\ [S(r,s)]^u \, [S(0,s)]^{v-u} & u \leq v. \end{cases} \tag{5.38}$$

Setting v=0

$$S(ur,0) = [S(r,0)]^u$$

writ ing g(r) = [S(r,0)] we get

$$g(ur) = [g(r)]^u \ , \ r>0 \qquad (5.39)$$

When r = 1/u ,

$$g(1) = [g(1/u)]^u$$

$$g(1/u) = [\ g(1)\ ]^{1/u}$$

and for r = 1/m

$$g(u/m) = [g(1/m)]^u$$

$$= [g(1)]^{u/m} \text{ for } m \in I_1^+.$$

Define

$$a = -\log g(1) = -\log S(1.0)$$

Thus

$$g(y) = [g(1)]^y$$

$$= e^{-ay} \ , \ y>0,\ a>0,$$

where y is any positive rational number. Now there exist sequences y_u' and y_u'' satisfying $y_u' \le x \le y_u''$ and $\lim y_u' = \lim y_u'' = x$, for irrational x. Since g(.) is non-increasing this would mean that $g(r) = e^{-ar}$ for all r>0 and some a>0. Similarly, by setting u=0.

$$S(0,s) = e^{-bs}\ ,\ b>0,\ s>0$$

where $b = -\log S(0,1)$

Equation (5.38) can now be written as

$$S(ur,vs) = \begin{cases} [S(r,s)]^{v}\ e^{-ar(u-v)} & u \geq v \\ [S(r,s)]^{u}\ e^{-br(v-u)} & u \leq v. \end{cases} \qquad (5.40)$$

Now, $v=u$, $r=s=x$

$$S(xu,xu) = [S(x,x)]^{u} \qquad (5.41)$$

with

$G(x) = S(x,x)$, equation (5.41) becomes

$G(xu) = [G(x)]^{u}$

By similar arguments from (5.39)

$$G(x) = [G(1)]^{x}$$

$$= e^{-cx}\ ,\ x>0,\ c>0$$

where $c = -\log G(1) = -\log S(1,1)$ and x is some positive real. Thus in (5.40) setting $r=s=x$

$$S(xu,xv) = \begin{cases} [S(x,x)]^{v}\ e^{-ax(u-v)} & ,\ u \geq v \\ [S(x,x)]^{u}\ e^{-bx(v-u)} & ,\ u \leq v \end{cases}$$

$$= \begin{cases} e^{-cxv}\, e^{-bx(u-v)} & , u \geq v \\ e^{-cxu}\, e^{-ax(v-u)} & , u \leq v. \end{cases}$$

or

$$S(x_1, y_1) = \begin{cases} e^{-cy_1}\, e^{-a(x_1 - y_1)} & , x_1 \geq y \\ e^{-cx_1}\, e^{-b(y_1 - x_1)} & , x_1 \leq y. \end{cases} \quad (5.42)$$

Where $xu = x_1$ and $xv = y_1$. Since coordinate wise convergence of rationals to reals is enough to guarantee such convergence in $R_2^+ = \{(x,y) \mid x, y \geq 0\}$, it follows that (5.42) holds for all reals.

Since $S(x, y)$ is survival functions $S(x,y)$ decreases monotonically which implies

$$S(x,x) \leq S(x,0) \leftrightarrow e^{-cx} \leq e^{-ax} \leftrightarrow c \geq a$$

$$S(x,x) \leq S(0,x) \leftrightarrow e^{-cx} \leq e^{-bx} \leftrightarrow c \geq b.$$

Let λ_2 = c-a and λ_1 = c-b and λ_{12} = a+b-c.

Since S(.,.) is a survival function,

$$G(x) = 1 - S(x, 0) - S(0, x) + S(x,x) \geq 0, \text{ for every } x$$

$$= 1 - e^{-ax} - e^{-bx} + e^{-cx} \geq 0$$

$$G'(x) = a\, e^{-ax} + b\, e^{-bx} - c\, e^{-cx} \geq 0$$

$$\lim_{x\to 0} G'(x) = a + b - c \geq 0$$

Thus $\lambda_{12} \geq 0$. Since $\lambda_{12}+\lambda_1+\lambda_2 = c$, $\lambda_2+\lambda_{12} = b$ and $\lambda_1+\lambda_{12} =$ a, the survival function (5.42) is

$$S(x,y) = \begin{cases} e^{-(\lambda_1+\lambda_{12})(x-y)} & x\geq y \\ e^{-(\lambda_2+\lambda_{12})(y-x)} & x\leq y \end{cases}$$

$$= \begin{cases} e^{-(\lambda_1+\lambda_2+\lambda_{12})y-(\lambda_1+\lambda_{12})(x-y)} & x\geq y \\ e^{-(\lambda_1+\lambda_2+\lambda_{12})x-(\lambda_2+\lambda_{12})(y-x)} & x\leq y \end{cases}$$

$$= e^{-\lambda_1 x-\lambda_2 y-\lambda_{12}\max(x,y)} \quad ,\lambda_1,\lambda_2,\lambda_{12}>0,\ x,y>0.$$

Conversely if for any r,s>0

$$R_{rs}(u,v) = e^{-\lambda_1 ur-\lambda_2 sv-\lambda_{12}\mathrm{mar}(ur,sv)}$$

Setting ur = xu' and vs = xv' for some x>0 and u', v' $\in I_2^+$

$$R_{rs}(u,v) = e^{-\lambda_1 xu'-\lambda_2 xv'-\lambda_{12} x \max(u',v')}$$

$$= \begin{cases} r_1^u\ r_2^v\ r_3^u & u\geq v \\ r_1^u\ r_2^v\ r_3^v & u\leq v \end{cases}$$

where $r_1 = e^{-\lambda_1 r}$, $r_2 = e^{-\lambda_2 s}$ $r_3 = e^{-\lambda_{12} r}$ and $r_4 = e^{-\lambda_{12} s}$.
Since the above two form of R(u,v) must agree at the boundary for which u=v, one must have $\gamma_3=\gamma_4$.

$$= \begin{cases} p^{v'} p_1^{u'-v'}, & u' \geq v' \\ p^{u'} p_1^{v'-u'}, & u' \leq v'. \end{cases}$$

Where $p = r_1 r_2 r_3$, $p_1 = r_1 r_3$, $p_2 = r_2 r_3$. Thus the proof is complete.

5.5 ESTIMATION OF PARAMETERS.

When the process of model selection as the bivariate geometric distribution (5.1) is completed either on the basis of the physical characteristics of the problem at hand or on the basis of the similarity to the properties the model possess, the second stage is to examine its reasonableness in the light of the data. While this can be done exactly through an appropriate characterization theorem, one often has to attend to the question of fitting the model to the given data. This requires some reasonable estimates of the parameters and therefore problem of estimation of the parameters of the bivariate geometric distribution (5.1) is undertaken in this section for the sake of completion of the modelling procedure. The discussion aims only at providing some rough and ready estimates rather than looking comprehensively into the inferential procedures and assessing their relative merits.

5.5.1. METHOD OF MOMENTS

For estimating the parameters p, p_1, p_2 of the bivariate geometric distribution (5.1), by the method of moments we make use of the fact that the marginals are geometric with parameter p_1, p_2 and $Z=Min(X_1,X_2)$ is also geometric with parameter p. Therefore

$$E(X_j) = \frac{p_j}{1-p_j}, \qquad j=1,2$$

and

$$E(Z)= \frac{p}{1-p}.$$

Let (X_{1i},X_{2i}); $i=1,\ldots,n$ denote a random sample from the bivariate geometric (5.1) and $Z_i=\text{Min}(X_{1i},X_{2i})$. Defining

$$\bar{x}_j = \frac{1}{n}\sum_{i=1}^{n} x_{ji}; \qquad j=1,2$$

and

$$\bar{z} = \frac{1}{n}\sum_{i=1}^{n} z_i;$$

the estimates $\tilde{p}_1$, $\tilde{p}_2$, and $\tilde{p}$ of p_1, p_2 and p are obtained by solving the equations

$$\frac{p_j}{1-p_j} = \bar{x}_j; \quad j=1,2$$

$$\frac{p}{1-p} = \bar{z}.$$

Thus

$$\tilde{p}_j = \frac{\bar{x}_j}{1+\bar{x}_j}; \quad j=1,2$$

and

$$\tilde{p} = \frac{\bar{z}}{1+\bar{z}} .$$

The sampling behaviour of these estimates will now be investigated. From the earlier discussions, it follows that X_{1i}, X_{2i} and Z_i follow geometric distribution with parameter p_1, p_2 and p respectively. Consequently $\sum_{i=1}^{n} X_{1i}$, $\sum_{i=1}^{n} X_{2i}$ and $\sum_{i=1}^{n} Z_i$ have negative binomial distributions with parameters (p_1,n), (p_2,n) and (p,n) respectively. Hence

$$E(\tilde{p}_j) = E\left(\frac{\bar{x}_j}{1+\bar{x}_j}\right)$$

$$= E\left(\frac{t_1}{n+t_1}\right) \quad \text{where } t_1 = \sum_{i=1}^{n} x_{1i}$$

$$= \sum_{t_1=0}^{\infty} \left(\frac{t_1}{n+t_1}\right)\binom{n+t_1-1}{t_1} p_1^{t_1} (1-p_1)^n.$$

$$= (1-p_1)^n\left[\sum_{t_1=1}^{\infty}\binom{n+t_1-2}{n-1} p_1^{t_1} - \sum_{t_1=1}^{\infty} \frac{1}{n+t_1}\binom{n+t_1-2}{n-1} p_1^{t_1}\right]$$

$$= p_1 - (1-p_1)^n \sum_{t_1=1}^{\infty} \frac{1}{n+t_1}\binom{n+t_1-2}{n-1} p_1^{t_1}.$$

Thus the bias in the estimator $\tilde{p}_1$ is given by

$$\text{Bias}(\tilde{p}_1) = E(\tilde{p}_1) - p_1$$

$$= \sum_{t_1=1}^{\infty} \frac{1}{n+t_1}\binom{n+t_1-2}{t_1} p_1^{t_1}. \tag{5.43}$$

The bias of $\tilde{p}_2$ and $\tilde{p}$ follow in the same manner.

5.5.2 METHOD OF MAXIMUM LIKELIHOOD

a. If a two component system is connected in series, the system fails when any one of the component fails and accordingly in such a system the interest is in $Z=\mathrm{Min}(X_1,X_2)$. When the life times follow the bivariate geometric (5.1), in such a system, instead of the usual procedure it is more advantageous to have a sampling scheme in which $\mathrm{Min}(X_{1i},X_{2i})$ is observed, which will be called a series sampling.

In a series sample of size n, with $n_1 = \sum_{i=1}^{n} \delta_{1i}$, $n_2 = \sum_{i=1}^{n} \delta_{2i}$ and $n_3 = \sum_{i=1}^{n} \delta_{3i}$ where

$$\delta_{ji} = \begin{cases} 1 & \text{if } X_{ji} > X_{(3-j)i} \\ 0 & \text{otherwise} \end{cases}, \quad j=1,2$$

and $\delta_{3i} = 1-\delta_{1i}-\delta_{2i}$, the likelihood becomes

$$L(p,p_1,p_2|n_1,n_2,n,z) = p^{\Sigma z_i} (1-p)^n \left(\frac{p_1-p}{1-p}\right)^{n_1} \left(\frac{p_2-p}{1-p}\right)^{n_2} \left(\frac{1+p-p_1-p_2}{1-p}\right)^{n-n_1-n_2}$$

and the likelihood equations are

$$\frac{\partial \ln L}{\partial p} = \frac{\Sigma z_i}{p} - \frac{n_1}{p_1 - p} - \frac{n_2}{p_2 - p} + \frac{n_3}{1+p-p_1-p_2} = 0$$

$$\frac{\partial \ln L}{\partial p_1} = \frac{n_1}{p_1 - p} - \frac{n_3}{1+p-p_1-p_2} = 0$$

$$\frac{\partial \ln L}{\partial p_2} = \frac{n_2}{p_2 - p} - \frac{n_3}{1+p-p_1-p_2} = 0$$

which when solved gives

$$\hat{p} = \frac{\Sigma z_i}{n+\Sigma z_i} \quad \hat{p}_1 = \frac{n_1+\Sigma z_i}{n+\Sigma z_i} \quad \text{and } \hat{p}_2 = \frac{n_2+\Sigma z_i}{n+\Sigma z_i}$$

Notice that $\hat{p}$ is same as $\tilde{p}$ the moment estimator of p.

Since $\sum_{i=1}^{n} z_i = z$ has negative binomial with parameter (p,n) and $\delta = (\delta_1, \delta_2, \delta_3)$ a multinomial $\left(1, \frac{p_1-p}{1-p}, \frac{p_2-p}{1-p}\right)$

$$E(\hat{p}_1) = E\left[\frac{n_1+z}{n+z}\right]$$

$$= E_z\left[E_n\left(\frac{n_1+z}{n+z}\middle| z\right)\right]$$

$$= E_z\left[\frac{n\theta_1+z}{n+z}\right]$$

where

$$\theta_1 = \frac{p_1 - p}{1-p}$$

$$= \sum_{z=0}^{\infty} \frac{z}{n+z} \binom{n+z-1}{n-1} p_1^z (1-p_1)^n + n\theta_1 \sum_{z=0}^{\infty} \frac{1}{n+z} \binom{n+z-1}{n-1} p_1^z (1-p_1)^n$$

$$= p_1 - (1-p_1)^n \left[\sum_{z=0}^{\infty} \left(\frac{1}{n+z-1} - \frac{n\theta_1}{n+z} \right) \binom{n+z-1}{n-1} p_1^z \right]$$

$$\text{Bias}(\hat{p}_1) = E(\hat{p}_1) - p_1$$

$$= (1-p_1)^n \sum_{z=0}^{\infty} \left(\frac{1}{n+z-1} - \frac{n\theta_1}{n+z} \right) \binom{n+z-1}{n-1} p_1^z .$$

The bias of $\hat{p}_2$, can be derived in the same manner.

b. When the components are connected in parallel apart from $Z = \text{Min}(x_1, X_2)$ and $\delta = (\delta_1, \delta_2, \delta_3)$ we need observe $\text{Max}(X_1, X_2)$ also. If $W_i = \text{Max}(X_{1i}, X_{2i}) - \text{Min}(X_{1i}, X_{2i})$ then the joint probability mass function of (Z_i, W_i, δ) is given by

$$f(z, w, \delta, \delta_1, \delta_2) = f(z) f(\delta_1, \delta_2) f(w | \delta_1, \delta_2)$$

$$= p^z (1-p) \left(\frac{p_1 - p}{1-p} \right)^{\delta_1} \left(\frac{p_2 - p}{1-p} \right)^{\delta_2} \left(\frac{1+p-p_1-p_2}{1-p} \right)^{\delta_3}$$

$$p_1^{w\delta_1} (1-p_1)^{\delta_1} (1-p_2)^{\delta_2} p_2^{w\delta_2} [I(w=0)]^{\delta_3}.$$

Thus the likelihood is

$$L = p^{\Sigma z_i} (p_1-p)^{\Sigma\delta_{1i}} (p_2-p)^{\Sigma\delta_{2i}} (1+p-p_1-p_2)^{\Sigma\delta_{3i}} p_1^{\Sigma w_i\delta_{1i}}$$

$$(1-p_1)^{\Sigma\delta_{1i}} (1-p_2)^{\Sigma\delta_{2i}} p_2^{\Sigma w\delta_{2i}} .$$

The equations to be solved for the parameter values are

$$\frac{\Sigma z_i}{p} - \frac{n_1}{p_1-p} - \frac{n_2}{p_2-p} + \frac{n_3}{1+p-p_1-p_2} = 0$$

$$\frac{n_1}{p_1-p} - \frac{n_3}{1+p-p_1-p_2} + \frac{\Sigma w_i\delta_{1i}}{p_1} - \frac{n_1}{1-p_1} = 0$$

$$\frac{n_2}{p_2-p} - \frac{n_3}{1+p-p_1-p_2} + \frac{\Sigma w_i\delta_{2i}}{p_2} - \frac{n_2}{1-p_2} = 0.$$

Let $P[X_1>X_2]=\theta_1$, $P[X_1<X_2]=\theta_2$ and $P[X_1=X_2]=\theta_3=1-\theta_1-\theta_2$. in a sample of size n, the probability that there are n_1 observations for which $X_{1i}>X_{2i}$, n_2 observations for which $X_{1i}< X_{2i}$, is

$$L(\theta_1,\theta_2,\theta_3 \mid n) = \frac{n!}{n_1!n_2!n_3!} \theta_1^{n_1} \theta_2^{n_2} \theta_3^{n_3}$$

where $n_3 = n-n_1-n_2$, the multinomial distribution, Thus

$\theta_1=n_1/n$, $\theta_2=n_2/n$, $\theta_3=n_3/n$ inherit all the properties of the maximum likelihood estimates of the parameter of a multinomial distribution.

Chapter VI

SOME MULTIVARIATE EXTENSIONS.

6.1 INTRODUCTION

In connection with the study of life times in discrete time, the main bivariate models encountered were

i) bivariate geometric with independent marginals specified by the survival function

$$R(\underline{x}) = p_1^{x_1} p_2^{x_2}; \qquad 0<p_j<1, j=1,2, \ \underline{x}\in I_2^+ \tag{6.1}$$

ii) bivariate geometric law (Nair and Nair(1988)),

$$R(\underline{x}) = p_1^{x_1} p_2^{x_2} \theta^{x_1 x_2}; \quad 0<\theta\leq 1, 0<p_j<1, j=1,2,$$

$$1+p\theta\geq p_1+p_2, \ \underline{x}\in I_2^+ \tag{6.2}$$

iii) bivariate distribution with geometric conditionals (Nair and Nair (1991)) specified by probability mass function

$$R(\underline{x}) = \alpha p_1^{x_1} p_2^{x_2} \theta^{x_1 x_2}; \quad 0<\theta\leq 1, 0<p_j<1, j=1,2, \tag{6.3}$$

where

$$\alpha^{-1} = \sum_{r=0}^{\infty} \frac{p_1^r}{(1-p_2\theta^r)} = \sum_{s=0}^{\infty} \frac{p_2^s}{(1-p_1\theta^s)}$$

and the

iv) bivariate geometric law specified by the survival function

$$R(\underline{x}) = \begin{cases} p^{x_2} p_1^{x_1-x_2} ; & x_1 \geq x_2; \quad 0<p\leq p_j<1 \\ p^{x_1} p_2^{x_2-x_1} ; & x_1 \leq x_2; \quad 1+p\geq p_1+p_2, j=1,2 \end{cases} \tag{6.4}$$

These distributions arise as boundary classes of monotone failure rates or are characterized by simple functional forms of their failure rates.

It is of natural interest to explore the extensions of the bivariate concepts so far discussed and of the corresponding distributions in the general multivariate cases. Though most of these multivariate generalizations can be obtained as straight forward extensions, in some cases the conditions attached to them become more restrictive and demanding. In this chapter we briefly sketch the multivariate forms of the definitions and models since the ideas were already conveyed in the bivariate case itself. The proofs and explanations in the more general cases are only touched up on in the following discussions.

Let $\underline{X}=(X_1,\ldots X_n)$ be a discrete random vector representing failure times of a n-component system in the support of $I_n^+=\{(x_1,\ldots,x_n)|x_k=0,1,\ldots,k=1,2,\ldots,n\}$ with joint survival function

$$R_n(\underline{x}) = P[\underline{X}\geq\underline{x}]$$

and probability mass function

$$f_n(\underline{x}) = P[X\geq\underline{x}]$$

where $\underline{x}=(x_1,x_2,\ldots,x_n)$ and $\underline{X}\geq\underline{x}$ implies $X_k\geq x_k$ $k=1,2,\ldots,n$.

6.2 SCALAR FAILURE RATE

The scalar failure rate in I_n^+ discussed in section 1.1 has the following properties.

1. If $a_r(\underline{x}) = \dfrac{f_r(x_1,x_2,\ldots,x_n)}{R_r(x_1,x_2,\ldots,x_n)}$; then $a_r(\underline{x});r=1,2,\ldots n$ determines the corresponding $R_n(x_1,\ x_2,\ \ldots,\ x_n)$ uniquely.

Proof:

An relationship analogous to (2.2) and (2.3) can be obtained as

$$R_n(\underline{x}) = R_n(x_1,\dots,x_{k-1}, x_k-1, x_{k+1}, \dots, x_n)$$

$$- \sum_{t_1=x_1}^{\infty} \dots \sum_{t_{k-1}=x_{k-1}}^{\infty} \sum_{t_{k+1}=x_{k+1}}^{\infty} \dots$$

$$\sum_{t_n=x_n}^{\infty} R_n(t_1,\dots,t_{k-1},x_k-1,t_{k+1},t_n)$$

$$a(t_1,\dots,t_{k-1},x_k-1,t_{k+1},t_n)$$

$$\text{for } k=1,2,3,\dots,n;\ \underline{x}\in I_n^+. \qquad (6.5)$$

The proof follows in the same manner as the bivariate case after noticing that $R_{n-1}(x_1,\dots x_{n-1}) = R_n(x_1,\dots,x_{n-1},0)$.

2. The random variable $X_1,X_2,\dots X_n$ are independent if and only if

$$a_n(\underline{x}) = \prod_{r=1}^{n} h(x_r); \quad \underline{x}\in I_n^+$$

3. The scalar failure rate

$$a_r(\underline{x}) = C \quad \text{for } r = 1, 2, \dots n$$

if and only if $X_1, X_2, \ldots, X_n$ are independent geometric variables.

4. A random vector $\underline{X}$ in the support of I_n^+ has scalar failure rate of the form.

$$a_r(\underline{x}) = C_{r.i_1,i_2,\ldots,i_r} ;$$

$$x_{i_1} > x_{i_2} > \ldots > x_{i_k} = x_{i_{k+1}} = \ldots x_{i_{k+j}} > \ldots > x_{i_r}, k=1,2,\ldots,r,$$

$$j=0,1,2,\ldots,r-k, \; r=1,2,\ldots,n \qquad (6.6)$$

for each $(i_1,i_2,\ldots i_r)$ is the permutation of $(1,2,\ldots,r)$, if and only if $R_n(\underline{x})$ has a survival function of the form specified by

$$R_n(\underline{x}) = p_{i_1}^{x_{i_1}} \left(\frac{p_{i_1 i_2}}{p_{i_1}}\right)^{x_{i_2}} \left(\frac{p_{i_1 i_2 i_3}}{p_{i_1 i_2}}\right)^{x_{i_3}} \cdots \left(\frac{p_{i_1 i_2 i_3 \ldots i_n}}{p_{i_1 i_2 \ldots i_{n-1}}}\right)^{x_{i_n}}$$

$$x_{i_1} \geq x_{i_2} \geq \ldots \geq x_{i_n} . \qquad (6.7)$$

The parameters are such that

$$0 < p_{i_1}, \ldots, p_{i_n} \leq \ldots \leq p_{i_1, i_2} \leq p_1, p_2, \ldots, p_n < 1$$

$$p_{i_1,i_2,\ldots,i_k} = p_{123\ldots k} \text{ for } k=2,\ldots,n$$

and

$$1-\sum_{k=1}^{n} p_k - \sum_{k<l} p_{kl} + \ldots + (-1)^n p_{123\ldots n} \geq 0.$$

Proof:

The proof can be arrived at by the principle of mathematical induction. That the result is true for n=1 and n=2 has been proved in Theorem 2.5. Assuming that the result holds for r=n-1, by repeatedly using 6.5, the result can be arrived at as in the bivariate set up. To illustrate, let $x_1 > x_2 > \ldots > x_n$. Then if $a_r(\underline{x}) = C_{n.12\ldots n}$; $x_1 > x_2 > \ldots > x_n$ (6.5) can be written as

$$R_n(\underline{x}) = R_n(x_1,\ldots,x_{n-1})$$

$$- C_{n.12\ldots n} \sum_{t_1=x_1}^{\infty} \cdots \sum_{t_{n-1}=x_{n-1}}^{\infty} R_n(t_1,\ldots,t_{n-1},x_n-1)$$

For $x_n=1$

$$R_n(x_1,\ldots,x_{n-1},1) = R_{n-1}(x_1,\ldots,x_{n-1})$$

$$- C_{n.12\ldots n} \sum_{t_1=x_1}^{\infty} \cdots \sum_{t_{n-1}=x_{n-1}}^{\infty} R_{n-1}(t_1,\ldots,t_{n-1})$$

From (6.7)

$$R_n(x_1,\ldots,x_{n-1},1) = p_1^{x_1}\left(\frac{p_{12}}{p_1}\right)^{x_2}\cdots\left(\frac{p_{12\ldots n-1}}{p_{12\ldots n-2}}\right)^{x_{n-1}}$$

$$\left[1-\frac{c_{n.12\ldots n}\, p_1 p_{12}\cdots p_{12\ldots n-2}}{(1-p_1)(p_1-p_{12})\cdots(p_{12\ldots n-1}-p_{12\ldots n-1})}\right]$$

Repeating the above procedure for $x_n=2,3,\ldots, x_{n-1}$, we obtain

$$R_n(\underline{x}) = p_1^{x_1}\left(\frac{p_{12}}{p_1}\right)^{x_2}\cdots\left(\frac{p_{12\ldots n-1}}{p_{12\ldots n-2}}\right)^{x_{n-1}}$$

$$\left[1-\frac{c_{n.12\ldots n}\, p_1 p_{12}\cdots p_{12\ldots n-2}}{(1-p_1)(p_1-p_{12})\cdots(p_{12\ldots n-1}-p_{12\ldots n-1})}\right]^{x_n}$$

for $x_1>x_2>\ldots>x_n$.

Proceeding on similar lines for other regions, $R_n(\underline{x})$ as in (6.8) is recovered along with the conditions on the parameters.

6.3 VECTOR FAILURE RATE.

An direct extension of the vector failure rate of Nair and Nair (1990) to the multivariate case has the following definition.

Definition 6.1.

The vector failure rate of a n dimensional random vector $\underline{X}$ in the support of I_n^+ is defined as

$$\underline{b}(\underline{x}) = (b_1(\underline{x}),\ldots,b_n(\underline{x}))$$

where

$$b_i(\underline{x}) = P[X_i=x_i \,|\, \underline{X}\geq\underline{x}]$$

$$= 1-[R(x_1,\ldots,x_{i-1},x_i+1,x_{i+1},\ldots,x_n)/R(\underline{x})] \qquad (6.8)$$

1. The failure rate $\underline{b}(\underline{x})$ determines the distribution of $\underline{X}$ uniquely through the formula.

$$R_n(\underline{x}) = \prod_{r=0}^{x_1-1} [1-b_1(x_1-r-1,x_2,\ldots,x_n)]$$

$$\prod_{r=0}^{x_2-1} [1-b_2(0,x_2-r-1,\ldots,x_n)]$$

$$\ldots$$

$$\prod_{r=0}^{x_n-1} [1-b_n(0,0,\ldots,0,x_n-r-1)] \qquad (6.9)$$

as seen from (6.8).

The results concerning the behaviour of $\underline{b}(\underline{x})$ under independence and the model characterized by constancy of

$\underline{b}(\underline{x})$ can be obtained using the same logic as in the bivariate case. However, multivariate model characterized by the local constancy of $\underline{b}(\underline{x})$ as an extension of the equation 1.9 needs some elaboration in view of the conditions needed for its validity.

2. A random vector $\underline{X}$ has a multivariate geometric distribution specified by the survival function

$$R_n(\underline{x}) = \left(\prod_{i=1}^{n} p_i^{x_i}\right)\left(\prod_{i<j} p_{ij}^{x_i x_j}\right) \cdots \left(p_{12\ldots n}^{x_1 x_2 \ldots x_n}\right) \tag{6.10}$$

where

$$0<p_i<1,\ 0<p_{ij}, p_{ijk}, \ldots, p_{12\ldots n} <1$$

and

$$1-\sum_i p_i - \sum_{i<j}\sum p_i p_j p_{ij} + \ldots + (-1)^n p_{12\ldots n} \geq 0.$$

if and only if its vector failure rate is of the form

$$\underline{b}(\underline{x}) = (b_1(\underline{x}_1^*), \ldots, b_n(\underline{x}_n^*)) \text{ for all } \underline{x} \tag{6.11}$$

where

$$\underline{x}_k^* = (x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)$$

Proof:

When the distribution is as stated in (6.10)

$$1-b_i(\underline{x})=p_i\left(\prod_{i<j} p_{ij}^{x_i}\right)\left(\prod_{i<j<k} p_{ijk}^{x_i x_j}\right)\cdots\left(p_{12\ldots n}^{x_1\ldots,x_{i-1},x_{i+1},\ldots x_n}\right)$$

(6.12)

which is independent of x_i for all i = 1,2,...n. Thus the conditions is necessary. To establish the sufficiency part, we use the method of induction. In the bivariate case the failure rate having the functional form

$$\underline{b}(\underline{x})=(b_1(x_1^*),b_2(x_2^*)) \text{ for all } \underline{x}\in I_2^+$$

characterizes the distribution (6.10) for n=2. Assume that the conditions holds for every subset if m variables in $\underline{X}$. Then, let

$$1-b_i(x_1,\ldots,x_{m+1}) = B_i(x_i^*).$$

Thus,

$$R(x_1,x_2,\ldots,x_{m+1}) = [B_i(x_i^*)]^{x_i} R(x_1,\ldots,x_{i-1},0,x_{i+1},\ldots,x_{m+1})$$

$$= [B_i(x_i^*)]^{x_i}\left(\prod p_j^{x_j}\right)$$

$$\left(\prod_{j<k} p_{jk}^{x_j x_k}\right)\cdots\left(p_{12\ldots m+1}^{x_1 x_2,\ldots,x_{i-1},x_{i+1},\ldots,x_{m+1}}\right)$$

$$j=1,2,\ldots,m+1, j\neq i. \qquad (6.13)$$

For i=1,2,...,n we can write equivalent expressions for $R(x_1,x_2,\ldots,x_{m+1})$. Dividing each of them by

$$\left(\prod_{i=1}^{m+1} p_i^{x_i}\right)\left(\prod_{i<j} p_{ij}^{x_i x_j}\right)\cdots\left(p_{12\ldots m}^{x_1 x_2 \ldots x_m}\right)$$

and taking $(x_1, x_2, \ldots, x_{m+1})^{th}$ root we find

$$\left[B_1(x_1^*)\left[p_1^{x_1}\left(\prod_{j\neq 1} p_{1j}^{x_1 x_j}\right)\cdots\left(p_{13\ldots m+1}^{x_1 x_3 \ldots x_{m+1}}\right)\right]^{-1}\right]^{(x_1 x_2 \ldots x_{m+1})^{-1}}$$

$$=\left[B_2(x_2^*)\left[p_2^{x_2}\left(\prod_{j\neq 2} p_{2j}^{x_2 x_j}\right)\cdots\left(p_{23\ldots m+1}^{x_2 x_3 \ldots x_{m+1}}\right)\right]^{-1}\right]^{(x_1 x_2 \ldots x_{m+1})^{-1}}$$

$$= \ldots$$

$$=\left[B_{m+1}(x_{m+1}^*)\left[p_{m+1}^{x_{m+1}}\left(\prod_{j\neq m+1} p_{m+1,j}^{x_{m+1} x_j}\right)\cdots\left(p_{1\ldots m}^{x_1 \ldots x_m}\right)\right]^{-1}\right]^{(x_1 \ldots x_{m+1})^{-1}}$$

This however means that

$$\left[B_i(x_i^*)\left[p_i^{x_i}\left(\prod_{j\neq i} p_{ij}^{x_i x_j}\right)\cdots\left(p_{1\ldots i\ldots m}^{x_1 \ldots x_i \ldots x_m}\right)\right]^{-1}\right]^{(x_1 x_2 \ldots x_{m+1})^{-1}} = \text{a constant}$$

independent of $x_1, \ldots, x_{m+1}$, say, $p_{12\ldots m+1}$. Then

$$B_i(x_i^*) = p_i^{x_i}\left(\prod_{j\neq i} p_{ij}^{x_j}\right)\cdots\left(p_{12\ldots m+1}^{x_1 \ldots x_{i-1} x_{i+1} \ldots x_{m+1}}\right) \qquad (6.14)$$

Subtitituting (6.14) into (6.13) we recover the survival function (6.10) for n=m+1. The conditions on the parameters are obtained from the relationships

$R(x_1,\ldots,x_{i-1}) \geq R(x_1,\ldots,x_i)$, $i=1,2,\ldots,n$ and $f(0,0,\ldots,0) \geq 0$.

Multivariate lack of memory property.

Definition 6.2

Let $\underline{X}$ be a n - dimensional random vector with survival function $R_n(\underline{x})$ with support I_n^+. Then $\underline{X}$ is said to possess the multivariate lack of memory property if and only if

$$R_n(x_1+t,\ldots,x_n+t) = R_n(\underline{x})R_n(\underline{t})$$

for all $\underline{x} \in I_n^+$ and $\underline{t}=(t,\ldots,t)$ where $t \in I_1^+$.

3. For all $x \in I_n^+$ and $t=(t,\ldots,t)$; $t \in I_1^+$ the following statments are equivalent

(i) $b_j(x_1+t,\ldots.x_r+t)=b_j(x_1,\ldots.x_r)$; $j=1,2$, $r=1,2,\ldots,n$

(ii) $R_r(\underline{x}+\underline{t}) = R_r(x)R_r(\underline{t})$, $r=1,2,\ldots,n$

(iii) $\underline{X}$ has a multivariate geometric distribution specified by survival function

$$R_n(\underline{x}) = p_{i_1}^{x_{1i}} \left(p_{i_1 i_2}/p_{i_1}\right)^{x_{i_2}}$$

$$\cdot \left(p_{i_1 i_2 i_3}/p_{i_1 i_2}\right)^{x_{i_3}} \cdots \left(p_{i_1 \ldots i_2}/p_{i_1 \ldots i_{n-1}}\right)^{x_{i_n}}$$

$$x_{i_1} \geq x_{i_2} \geq \ldots \geq x_{i_n}, \quad (6.15)$$

for each permutation $(i_1, i_2, \ldots, i_n)$ of the integers from 1 to m. The parameters are such that

$$0 < p_{i_1, i_2, \ldots, i_m} \leq \ldots \leq p_{i_1, i_2} \leq p_1, p_2, \ldots, p_n < 1,$$

$$p_{i_1, i_2, \ldots, i_j} = p_{123\ldots j} \text{ for } j=2,\ldots,n$$

and

$$1 - \sum_{j=1}^{n} p_j - \sum_{j<k}\sum p_{jk} + \ldots. + (-1)^{n-1} p_{123\ldots n} \geq 0.$$

Proof:

We first prove (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iii)

Let,

$$\underline{b}(x_1+t, \ldots, x_r+t) = \underline{b}(x_1, \ldots, x_r) \quad \text{for all } r \leq n. \quad (6.16)$$

First we prove that (6.16) is equivalent to the multivariate lack of memory property (MLMP)

$$R(x_1+t, \ldots, x_r+t) = R(x_1, \ldots, x_r) R(t, \ldots, t) \text{ for all } r \leq n. \quad (6.17)$$

When (6.16) holds, from (6.8) for i=1,

$$\frac{R(x_1+t+1,x_2+t,\ldots,x_r+t)}{R(x_1+1,x_2,\ldots,x_r)} = \frac{R(x_1+t,x_2+t,\ldots,x_r+t)}{R(x_1,\ldots,x_r)}$$

$$= \frac{R(t,x_2+t,\ldots,x_r+t)}{R(0,x_2,\ldots,x_r)} . \quad (6.18)$$

Similarly using the defintions of $b_2(x_1,\ldots,x_r)$

$$\frac{R(x_1+t,x_2+t,\ldots,x_r+t)}{R(x_1,x_2,\ldots,x_r)} = \frac{R(x_1+t,t,\ldots,x_r+t)}{R(x_1,0,x_3,\ldots,x_r)} . \quad (6.19)$$

Setting $x_2=0$ in (6.18) and substituting in (6.19)

$$\frac{R(x_1+t,x_2+t,\ldots,x_r+t)}{R(x_1,x_2,\ldots,x_r)} = \frac{R(t,t,x_3+t,\ldots,x_r+t)}{R(0,0,x_3,\ldots,x_r)} .$$

Successively using $b_3,\ldots,b_r$ and noting $R(0,0,\ldots,0)=1$ we obtain (6.17). The converse is obtained by using (6.17) in the expression for $\underline{b}(\underline{x}+\underline{t})$. To complete the proof it remains to establish that the only solution of (6.17) is (6.15).

For r=1, the only solution is $R(x_j) = p_1^{x_j}$ for some $0<p_1<1$ and for r=2,

$$R(x_1+t,x_2+t)=R(x_1,x_2)R(t,t).$$

Setting $x_2=0$,

$$R(x_1+t,t)=p_1^{x_1}\ R(t,t) \tag{6.20}$$

Further

$$R(y+t,y+t) = R(y,y)R(t,t)$$

gives $R(t,t) = p_{12}^t$ for some $0<p_{12}<1$. Thus for $x_1 \geq x_2$, from (6.20)

$$R(x_1,x_2) = p_1^{x_1}(p_{12}/p_1)^{x_2}. \tag{6.21}$$

Assuming the solution (6.15) to hold for any r variables in $\underline{X}$,

$$R(x_1+t,\ldots,x_r+t,t) = R(x_1,x_2,\ldots,x_{r+1})R(t,\ldots,t)$$

specializes to

$$R(x_1+t,x_2+t,\ldots,x_r+t,t)=R(x_1,x_2,\ldots,x_r,0)R(t,\ldots,t)$$

$$= p_1^{x_1}(p_{12}/p_1)^{x_2}\ldots\ (p_{1\ldots r}/P_{12\ldots r-1})^{x_r}R(t,\ldots,t)$$

$$x_1 \geq x_2 \geq\ \ldots\ \geq x_r. \tag{6.22}$$

Also,

$$R(y+t,\ldots,y+t) = R(y,\ldots,y)R(t,\ldots,t)$$

$$R(t,\ldots,t) = p_{12\ldots r+1}^t, \quad 0<p_{12\ldots r+1}<1.$$

$$R_{\gamma}(\underline{x})=p_1^{x_1}(p_{12}/p_1)^{x_2}\ldots(p_{1\ldots r+1}/P_{12\ldots r})^{x_{r+1}}\ \ x_1 \geq x_2 \geq \ldots \geq x_{r+1}.$$

By induction we have derived the result for $x_1 \geq x_2 \geq \ldots \geq x_{r+1}$. The expression for $R_r(\underline{x})$ in other regions of the sample space are similarly obtained as the bivariate case. The converse is straight forward.

6.4 CONDITIONAL FAILURE RATE

We define the conditional failure rate of $\underline{X}$ as the vector

$$\underline{c}(\underline{x}) = (c_1(\underline{x}), \ldots, c_n(\underline{x})) \tag{6.23}$$

where

$$c_i(x) = P[X_i = x_i \mid X_1 \geq x_1, \ldots, X_i = x_i, \ldots, X_n = x_n] \quad i=1,2,\ldots,n.$$

As in the bivariate case it can be showed that $\underline{c}(\underline{x})$ determines the univariate conditional distributions through the relationship

$$P[X_i = x_i \mid X_1 = x_1, \ldots, X_{i-1} = x_{i-1}, X_{i+1} = x_{i+1} \ldots, X_n = x_n]$$

$$= \prod_{r=0}^{x_i - 1} [1 - c_i(x_1, \ldots, x_{i-1}, r, x_{i+1}, \ldots, x_n)] c_i(\underline{x}); i=1,2,\ldots,n.$$

The above relationship is arrived at by noting that

$$1 - c_i(\underline{x}) = \frac{H_i(x_1, \ldots, x_i + 1, \ldots, x_n)}{H_i(\underline{x})}$$

where

$$H_i(\underline{x}) = P[X_1=x_1,\ldots,X_i \geq x_i, X_{i+1}=x_{i+1}\ldots,X_n=x_n]$$

so that

$$H_i(\underline{x}) = [1-c_i(\underline{x})]H_i(x_1,\ldots,x_i-1,\ldots,x_n)$$

$$= \prod_{r=0}^{x_i-1} [1-c_i(x_1,\ldots,x_{i-1},r,x_{i+1},\ldots,x_n)]$$

$$H_i(x_1,\ldots,0,\ldots,x_n)$$

$$H_i(\underline{x}) = \prod_{r=0}^{x_i-1} [1-c_i(x_1,\ldots,x_{i-1},r,x_{i+1},\ldots,x_n)]$$

$$P[X_i=x_i | X_1=x_1,\ldots,X_{i-1}=x_{i-1},X_{i+1}=x_{i+1}\ldots,X_n=x_n]$$

Thus

$$P[X_i \geq x_i | X_1=x_1,\ldots,X_{i-1}=x_{i-1},X_{i+1}=x_{i+1}\ldots,X_n=x_n]$$

$$= \prod_{r=0}^{x_i-1} [1-c_i(x_1,\ldots,x_{i-1},r,x_{i+1},\ldots,x_n)]$$

or

$$P[X_i=x_i | X_1=x_1,\ldots,X_{i-1}=x_{i-1},X_{i+1}=x_{i+1}\ldots,X_n=x_n]$$

$$= \prod_{r=0}^{x_i-1} [1-c_i(x_1,\ldots,x_{i-1},r,x_{i+1},\ldots,x_n)]c_i(\underline{x}); \qquad (6.24)$$

Thus $\underline{c}(\underline{x})$ determines the conditional distributions of $\underline{X}$ uniquely. Appealing to the result by Gourieroux and Monfort (1979) we conclude that $\underline{c}(\underline{x})$ determines the corresponding distribution uniquely if and only if $f(\underline{x})>0$ for all $\underline{x}\in I_n^+$.

The behaviour of the conditional failure rate under independence of the components and the model characterized by global constancy of the conditional failure rate are direct extensions of the bivariate case and is therefore, not pursued here.

In the next theorem we derive the multivariate model which is characterized by the local constancy of $\underline{c}(\underline{x})$.

Theorem.

A random vector $\underline{X}$ the support on I_n^+ has the conditional failure rate (6.19) of the form

$$\underline{c}(\underline{x})=(c_1(\underline{x}_1^*),\ldots,c_n(\underline{x}_n^*)) \qquad (6.25)$$

if and only if probability mass function of $\underline{X}$

$$f(\underline{x})=\alpha\left(\prod_{i=1}^{n} p_i^{x_i}\right)\left(\prod_{i<j}\theta_{ij}^{x_i x_j}\right)\ldots\left(\theta_{12\ldots n}^{x_1 x_2 \ldots x_n}\right) \qquad (6.26)$$

where

$$0<p_i<1,\ 0<\theta_{ij},\theta_{ijk},\ldots,\theta_{12\ldots n}\le 1$$

with

$$\alpha^{-1}=\sum_{\underline{x}_i}{}_{*}\ \prod_{j\neq i}^{n} p_j^{x_j}\ \prod_{\substack{i<j\\ i\neq j}}\theta_{jk}^{x_jx_k}\ \ldots\theta_{12\ldots i-1,i+1,\ldots,n}^{x_k\ldots x_{i-1}x_{i+1}\ldots x_n}\ .$$

$$i=1,2,\ldots,n.$$

Proof:

When (6.21) holds

$$P[X_i=t_i+s_i|X_1=x_1,\ldots,X_i\ge s_i,\ldots,X_n=x_n]$$

$$= P[X_i=t_i|X_1=x_1,\ldots,X_n=x_n] \qquad (6.27)$$

or

$$p_i(t_i+s_i,\underline{x}_i^*) = p_i(t_i,\underline{x}_i^*)p_i(s_i,\underline{x}_i^*)$$

$$p_i(t_i,x_i^*) = [p_i(\underline{x}_i^*)]^{t_i}$$

where

$$p_i(t_i,\underline{x}_i^*)= P(X_i\ge t|X_i^*=x_i^*)$$

Thus

$$f(x_i|\underline{x}_i^*) =q_i(x_i^*)\ [p_i(x_i^*)]^{x_i} \qquad (6.28)$$

where

$$q_i(.) =1-p_i(.).$$

For notational simplicity we shall present the proof for n=3. From (6.28) we have

$$
\begin{aligned}
f(\underline{x}) &= q_1(x_1^*)\,[p_1(\underline{x}_1^*)]^{x_1} f_1(\underline{x}_1^*) \\
&= q_2(x_2^*)\,[p_2(x_2^*)]^{x_2} f_2(x_2^*) \\
&= q_3(x_3^*)\,[p_3(\underline{x}_3^*)]^{x_3} f_3(\underline{x}_3^*)
\end{aligned} \tag{6.29}
$$

Changing x_3 to x_3+1 and dividing the resulting equation by (6.29)

$$
\begin{aligned}
\frac{f(x_1,x_2,x_3+1)}{f(\underline{x})} &= \frac{q_1(x_2,x_3+1)}{q_1(x_2,x_3)}\left[\frac{p_1(x_2,x_3+1)}{p_1(x_2,x_3)}\right]^{x_1}\frac{f_1(x_2,x_3+1)}{f_1(x_2,x_3)} \\
&= \frac{q_2(x_1,x_3+1)}{q_2(x_2,x_3)}\left[\frac{p_2(x_1,x_3+1)}{p_2(x_1,x_3)}\right]^{x_2}\frac{f_2(x_1,x_3+1)}{f_2(x_1,x_3)} \\
&= p_3(x_1,x_2)
\end{aligned}
$$

so that

$$
\frac{p_3(x_1+1,x_2)}{p_3(x_1,x_2)} = \frac{p_1(x_2,x_3+1)}{p_1(x_2,x_3)} = \theta_{13}(x_2) \tag{6.30}
$$

and

$$
\frac{p_3(x_1,x_2+1)}{p_3(x_1,x_2)} = \frac{p_2(x_1,x_3+1)}{p_2(x_1,x_3)} = \theta_{23}(x_1) \tag{6.31}
$$

The equations (6.30) and (6.31) together imply

$$
\begin{aligned}
p_1(x_2,x_3) &= p_1(x_2,0)[\theta_{13}(x_2)]^{x_3} \\
p_2(x_1,x_3) &= p_2(x_1,0)[\theta_{23}(x_1)]^{x_3}
\end{aligned} \tag{6.32}
$$

and

$$p_3(x_1,x_2) = p_1(x_2,0)[\theta_{13}(x_2)]^{x_3} = p_2(x_1,0)[\theta_{23}(x_1)]^{x_3}$$

Adopting the same procedure as above the last equation gives

$$\frac{\theta_{13}(x_2+1)}{\theta_{13}(x_2)} = \frac{\theta_{23}(x_1+1)}{\theta_{23}(x_1)} = \theta_{123}, \text{ a constant}$$

so that

$$\theta_{13}(x_2) = [\theta_{123}]^{x_2}\theta_{13}$$

and (6.33)

$$\theta_{23}(x_1) = [\theta_{123}]^{x_1}\theta_{23}$$

which on substitution in (6.32) gives

$$p_1(x_2,x_3) = p_1(x_2,0)\theta_{123}^{x_2x_3}\ \theta_{13}$$

$$p_2(x_1,x_3) = p_2(x_1,0)\theta_{123}^{x_1x_3}\ \theta_{23} \qquad (6.34)$$

$$p_3(x_1,x_2) = p_1(x_2,0)\theta_{123}^{x_2x_3}\ \theta_{13}$$

$$= p_2(x_1,0)\theta_{123}^{x_1x_3}\ \theta_{23}$$

Proceeding on the same lines as above by incrementing x_2 and x_1 to x_2+1 and x_1+1 respectively in (6.29) gives the following

$$p_1(x_2,x_3) = p_1(0,x_3)\theta_{123}^{x_2x_3}\ \theta_{12}^{x_2}$$

$$p_2(x_1,x_3) = p_2(0,x_3)\theta_{123}^{x_1x_3}\ \theta_{12}^{x_1} \qquad (6.35)$$

The equations (6.34) along with (6.35) gives

$$p_1(x_2,x_3) = p_1\ \theta_{13}^{x_3}\ \theta_{12}^{x_2}\ \theta_{123}^{x_3x_2}$$

$$p_2(x_1,x_3) = p_2\ \theta_{23}^{x_3}\ \theta_{12}^{x_1}\ \theta_{123}^{x_3x_2}$$

$$p_3(x_1,x_2) = p_3\ \theta_{13}^{x_1}\ \theta_{23}^{x_2}\ \theta_{123}^{x_1x_2}$$

which substitution in (6.29) yields

$$f(\underline{x}) = \left[1-\ p_1\ \theta_{13}^{x_3}\ \theta_{12}^{x_2}\ \theta_{123}^{x_3x_2}\right]\left[p_1\ \theta_{13}^{x_3}\ \theta_{12}^{x_2}\ \theta_{123}^{x_3x_2}\right] f_1(\underline{x}_1^*)$$

$$= \left[1-\ p_2\ \theta_{23}^{x_3}\ \theta_{12}^{x_1}\ \theta_{123}^{x_3x_2}\right]\left[p_2\ \theta_{23}^{x_3}\ \theta_{12}^{x_1}\ \theta_{123}^{x_3x_2}\right] f_2(\underline{x}_2^*)$$

$$= \left[1-\ p_3\ \theta_{13}^{x_1}\ \theta_{23}^{x_2}\ \theta_{123}^{x_1x_2}\right]\left[p_3\ \theta_{13}^{x_1}\ \theta_{23}^{x_2}\ \theta_{123}^{x_1x_2}\right] f_3(\underline{x}_3^*)$$

from which $f_i(\underline{x}_i^*)$ is solved as

$$f_1(\underline{x}_1^*) = \alpha p_3^{x_3}\ p_2^{x_2}\ \theta_{23}^{x_2x_3}\left[1-\ p_1\ \theta_{13}^{x_3}\ \theta_{12}^{x_2}\ \theta_{123}^{x_3x_2}\right]^{-1}$$

$$f_2(\underline{x}_2^*) = \alpha p_3^{x_3} p_1^{x_1} \theta_{13}^{x_1 x_3} \left[1 - p_2 \theta_{23}^{x_3} \theta_{12}^{x_1} \theta_{123}^{x_3 x_2}\right]^{-1}$$

$$f_3(\underline{x}_3^*) = \alpha p_2^{x_2} p_1^{x_1} \theta_{12}^{x_1 x_2} \left[1 - p_3 \theta_{13}^{x_1} \theta_{23}^{x_2} \theta_{123}^{x_1 x_2}\right]^{-1}$$

and so

$$f(\underline{x}) = \alpha\, p_1^{x_1} p_2^{x_2} p_3^{x_3} \theta_{12}^{x_1 x_2} \theta_{13}^{x_1 x_3} \theta_{23}^{x_2 x_3} \theta_{1}^{x_1 x_2 x_3}$$

where α is a constant independent of $\underline{x}$ such that

$$\alpha^{-1} = \sum_{x_2, x_3} p_3^{x_3} p_2^{x_2} \theta_{23}^{x_2 x_3} \left[1 - p_1 \theta_{13}^{x_3} \theta_{12}^{x_2} \theta_{123}^{x_3 x_2}\right]^{-1}$$

$$= \sum_{x_1, x_3} p_3^{x_3} p_1^{x_1} \theta_{13}^{x_1 x_3} \left[1 - p_2 \theta_{23}^{x_3} \theta_{12}^{x_1} \theta_{123}^{x_3 x_2}\right]^{-1}$$

$$= \sum_{x_1, x_2} p_2^{x_2} p_1^{x_1} \theta_{12}^{x_1 x_2} \left[1 - p_3 \theta_{13}^{x_1} \theta_{23}^{x_2} \theta_{123}^{x_1 x_2}\right]^{-1}$$

The case for n>3 follows by repeating the above logic.

6.5. PLAN FOR FUTURE WORK

Compared to the vast literature available in the continuous case pertaining to reliability concepts and modelling, the attempt in the present work was confined to explore the concept of failure rate in the bivariate discrete set up and to offer discrete counter parts of what are known as lack of memory models through some characterization. Other concepts like mean residual life, vitality function, variance and percentile residual life, residual life distributions, memory and models based on them need critical examination and detailed analysis in the discrete time domain. Models based on their monotone behaviour are also of interest. Concepts that are of importance in replacement policy like NBU, HNBUE etc are to be defined meaningfully and their properties investigated. A third area that is of primary importance is the development of discrete life time models which show marked departure from the geometric laws already introduced in the present study. Hopefully, the result in these and in other related areas will be presented in a future work.

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