Some Aspects of History of Indian Mathematics in 18th and early 19th Century

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C E R T I F I C A T E

This is to certify that this thesis is a bona fide record of work by Syed Aftab Husain Rizvi, carried out in the Department of Mathematics and Statistics, University of Cochin, Cochin 682 022 under my supervision and guidance and that no part thereof has been submitted for a degree in any other University.

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C O N T E N T S

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S Y N O P S I S

SYNOPSIS

This thesis is an attempt to throw light on the works of some Indian Mathematicians who wrote in Arabic or Persian. In the Introductory Chapter on outline of general history of Mathematics during the eighteenth and nineteenth century has been sketched. During that period there were two streams of Mathematical activity. On one side many eminent scholars, who wrote in Sanskrit, held the field as before without being much influenced by other sources. On the other side there were scholars whose writings were based on Arabic and Persian texts but who occasionally drew upon other sources also.

In Chapter II importance of Geometry and transmission of Euclidean Geometry in India has been discussed and contribution of Indian Mathematicians in this area outlined.

The Third Chapter is devoted to the life and work of an important Indian Mathematician, G͟hulām Husain Jaunpūrī (b. 1790).

One of his works, JĀMĒ-I-BAHĀDURK͟HĀNĪ, has been called on "Encyclopaedia of Mathematical Sciences". A portion of this work dealing with Euclidean Geometry has been translated into English.

Chapter IV is based on Section I of JĀME-I-BAHĀDUR KHĀNĪ which deals with Definitions, Objects and Principles of Geometry. A detailed critical study of the concepts and comparison with the similar literature has been made. In this connection the works of contemporaries like Mohammad Hasan b. Dildār Alī, Mīr Mohammad Hāshim, Sheikh Barkat Allahābādī, Jegannātha Samrāta, Diwān Kanhjī Mal Kāyastha, and some modern writers have been specially mentioned.

Chapter V is concerned with the Theorems and Problems contained in Section Two dealing with the Properties of straight lines, Angles and Rectilinear Surfaces. It consists of 49 Propositions in which 38 Propositions are taken from the Book I, and 11 from Book II of TAHRĪR-I-UQLĪDES by Nasīr al-Dīn al-Tūsī, 1 from Apollonius' work and in 9 Propositions proofs are given by Ghulam Husain.

In Chapter VI Properties of Circles and Arcs, Lines and Angles, which are produced by the comparison of Circles are established. This is Section Three of JĀME-I-BAHĀDUR KHĀNĪ. It consists of 35 Propositions in which 28 are from Book III, 6 from Book IV, 1 from Blunt's book. A comparative study of these Propositions have been done.

The first 21 Propositions of the Chapter VII bring out the logic behind the Theory of Ratios and the Rules of Simple, Compound and Derivative Proportion. This theory of Proportion is applied in proving 18 Propositions of Book VI concerning similar triangles, parallelograms and polygons. Three Propositions are taken from Nasīr al-Dīn al Tūsī's version of Ptolemy's MEGALE SYNTAX one from Conic Sections of Archimedes, two from TĀLIMAT of Ibn Haitham. Fifteen Propositions are claimed by Ghulām Husain to be his own discovery.

In the last Chapter some suggestions for future study have been given. A Bibliography is also given as an Appendix.

S T A T E M E N T

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This thesis contains no material which has been accepted for the award of any other degree or diploma in any University and, to the best of my knowledge and belief, it contains no material previously published by any other person, except where due reference is made in the text of the thesis.

SYED AFTAB HUSAIN RIZVI

A C K N O W L E D G M E N T S

A C K N O W L E D G M E N T S

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C H A P T E R O N E

CHAPTER ONE

MATHEMATICAL ACTIVITY DURING 18th and 19th CENTURY.

(A brief Historical Survey)

During the medieval period there were two streams of mathematical activity some used Sanskrit and the others Arabic and Persian. It may be noted here that most of their works are commentaries or translation of classic but of course some contain original results also. A few compendiums are also composed which are based on important classical works. Unlike in Europe where different new mathematical theories were discovered.

1.1 Astronomy:

In Mathematics emphasis was laid on astronomy. A number of treatises have been written during this period, for the purpose of making calendar, calculation of time and exact position of the planets and the celestial bodies for fixing dates of festivals and ritual purposes.

Sūrya-siddhānta of Āryabhaṭa (b. 476 A.D.) was the main astronomical text on which number of commentaries were written. Among them are: Siddhānta - tattvaviveka by Kamalākara (c. 1616 A.D.), in which he refuted opinion of Āryabhaṭa regarding the motion of the earth, equatorial region, longitude of different places, diameter and circumference of the earth; Kiraṇāvati by

Dādābhai (1720 A.D.); and Naksatra - Vivaran by Kodandarama (1854 A.D.) etc. are more important. Apart from these Siddhantaratnavali of Rajanarayana (1789 A.D.), Golananda of Cintamani (1791A:D.) and Karanabhusana of Cunnirama (1825 A.D.) are some of the astronomical works, Laksmipati (1740 A.D.) wrote three books: Dhruva - bhramana on the stability orinstability of the polar star, Samarāt - yantra dealing with astronomical instrument and last one which has special feature Sodasa - yoga - vyākhyāna, which is based on some Arabic text. It describes the 16 yogas viz. Ikkavala etc. An earlier book of the same category was Jagannāth Samrāth's Samrāthsiddhānta, a Sanskrit translation of Tahrīr al-Majistī, an improved recension of Ptolemy's Megale Syntaxis composed in 1247 A.D. by Nasīr al-Dīn Tūsī.

Later with a view to reform the old and antiquated state of astronomical knowledge and to make it up to date as far as possible, Venkatesaketakara (1898 A.D.) composed Jyotirganita. The book embodies the astronomical discoveries and researches made during last four centuries. For preparing this book, the author has taken help of the works of the eminent astronomers such as Leverrier, Hansen and Newcomb (85; 241).

For the other stream of mathematicians, astronomy was also very popular but mainly they were interested in Greeco astronomical literature who with their efforts enriched their contents. It was Khair Allāh "Muhandis" b. Lutf Allāh, who composed a commentary on Tahrir al-Majisti entitled Taqrib al-Tahrīr in

1774 A.D. in the third decade of the reign of Muhammad Shāh. M. Zaman b. M. Sādiq b. Abi Yazid Anbalaji Dihlawi, who completed in 1718 - 19 a commentary of al-Mulakkhas fi'l-hai'at of Mahmud b. M. b. Umar Chaghmini, an Arabic work completed in 618/1221. Moreover Wajid al-Dīn Gujrati, Maulana Imam al-Dīn Riazi (1656 - 1733), Maulāna Abdul Hai Firangimahali etc. also wrote commentaries on the same work.

Another favourite book for Indians was Bahā' al-Dīn M.b. Husain al-Amilī's Tashrīh al-Aflāk, an Arabic manual of astronomy, which was prescribed for higher studies in Madresah. Numerous commentaries and Made Easy booklets were composed by different authors. But the chapters dealing with the so-called structure of the even sphere was translated into Persian as Bab-i-Tashrih al-Aflāk by Ismat Allāh b. Āzam Saharanpūrī. Later on a Made Easy of Bāb-i-Tashrih al-Aflak was prepared by Imām al-Din b. Lutf Allāh Muhandis in 1103/1691 and called Tasrīh-i-Tashrih al-Aflāk.

Apart from such commentaries, some original texts were also composed during this period, those are Dil-Pasand by Pandit Raj Nemd'har in 1204/1789-90 which was dedicated to Nawāb Amir al-Daulāh of Tonk, and Hada'iq al-Nujum by Rājah Ratan Singh "Zakhmī" (1197/1782 - 1851), who composed this astronomical work in 1253/1837.

As regards the Zījs preparation, Indians were not behind in comparison to Central Asia. The most important Zīj named

"Zij-i-Shāhjahani" was prepared by Mulla Farid in 17th Century, and was dedicated to Shāh Jahan. On account of its great utility and usefulness, this Zij became more popular than Ulugh Beg's Zij. Later on a translation was made for the scholars of other stream whose language was not Persian. This Zij was studied by Rājah Jai Singh before he undertook the preparation of his "Zij-i-Mohammad Shāhi" (54; 208). Khair Allāh Khān wrote a commentary on this Zij, in which he asserted that he has found, not only that the orbit of the sun is excentric, but that the orbits of all the signs are of an Elliptical form (43; 576).

However, along the activity of writing books and making Zijs Indian astronomers set up many astronomical observatories. In the 18th century Jai Singh built monumental masonary observatories in several cities like Jaipūr, Delhi, Ujjain and Banaras. The speciality of these observatories was that the instruments which were used in it were of the most improved form, which were prepared under the direct supervision of Jai Singh.

1.2 Arithmetic.

It may be noted here that Varāha-Mihira's Brihat-Samhita which was translated into Persian as Tarjamah-i-Bārāhi by Abd al-Aziz during the reign of Sultan Firoz Shāh Tughlaq, which was the only work of this kind, but no other Sanskrit astronomical work is known to have been translated from Sanskrit into Arabic or Persian in India. But in arithmetic the situation was different.

Lilavati of Bhaskaracharya (1150 A.D.) was the most important text on Arithmetic and Geometry. Numerous commentaries on it were composed by different authors in Sanskrit, e.g. Sridhara Mahapatra wrote Sarvabodhini-Vyakhya in 1717 A.D. , Mahidasa prepared as Lilavativivarana in 1722 A.D. and Nilambara Jha composed Lilavatyudaharana in 1825 A.D.

It is interesting to note that about 437 years later at the instance of Akbar, Lilavati was translated into Persian by Faidi Nagori as Nuskhah-i-Lilavati in 1587 A.D. Again in 1663 A.D. during the reign of Aurangzeb, Medni Mal b. Dharam Narayan b. Kalyan Mal Kayasth composed his Bada'-i-Funun based on the Lilavati and divided it into 9 chapters.

There was another important book which was prescribed for higher students was Khulasat al-Hisab of Baha al-Din M. b. Husain al-'Amili (d. 1622 A.D.) which is an Arabic compendium of Arithmetic Storey (76; 11-12) has given a list of 14 commentaries and translation of this important book. Some of them which were written by Indians are: Muntakhab by Lutf Allah Muhandis in 1092/1681, Kifait al-Hisab fi Sharh Khulasat al-Hisab by Gulshan Ali Jaunpuriprepared in 1227/1812, Faid al-Wahab fi Sharh Khulasat al-Hisab by Nizam al-Din Ahmad b. M. 'Abd Allah al-Shahid, Ghayah-i-Juhdal-Hussab by M. Zaman Faiyad b. M. Sadiq Anbalaji Dihlawi in 1130/1718, Hashiyah Khulasat al-Hisab by Mohammad Barkat b. Abdur Rehman Allahabadi, Lawami al-Lawami al-Lubab fi Sharh Khulāsat al-Hisab by Amin al-Dīn al-Lahorī in 1770 A.D However, Abdul Hai (4; 376-378) gives a list of 36 books on Arithmetic which were written by Indians in Arabic and Persian during this period. Some important books are: Dastūr al-Mahasbīn by Rafi al-Dīn composed in 1164 A.H., Dastūr al-Hisab by Siraj al-Dīn Hasan in 1205 A.H. etc.

Apart from this M. Irtada Alī Khān (d. 1835-36) compiled a book entitled Naqud al-Hisab in 1235/1819. Bālmiki Tiwarī during the reign of Aurangzeb wrote a book in Sanskrit based on Lilāvat and translated it into Persian, but retained the title Vikat Chintamani.

1.3 Algebra.

On other part of Siddhantasiromani, Vija-Ganita, a well-known algebrical work, many commentaries were written in Sanskrit. Not many seem to have been written during this period other then Krparama Misra composed Bālabodhinī in 1792 A.D. However, an edition with expository notes was prepared by Sudhakara Dvivedin (c. 1892) under the title Bijaganitamavyaktaganitam vā.

It was also translated in Persian by 'Ata Allah "Rushdi" b. Ahmad-i-Nadir during the reign of Shāh Jahān in 1634-35. It is interesting to note that Europe came to know about this great classic first through the translation from Rushdī's version into English by E. Strachey with notes by Devis, published in London in 1813 as Bija Ganita or the Algebra of the Hindus. It was only latter that H.T. Colebroke was able to publish an English translation from the original Sanskrit in London in 1817.

Moreover, following books were also composed by Indians as reported by Abdul Hai (4; 378) : Jabra-wa-Mūqablah by Tafaddul Husain Khān Lakhnawī, Resalah Jabra-wa-Muqablāh by Raushan Alī Jaunpūrī, Al-Sitta al-Jabriyā Manzūm by Najm al-Dīn Alī Khān Kakorawī, Al-Makhrūtāt al-Jabrīya by Muftī Alī Kabīr Jaunpūrī.

Another type of work that was done in early 19th century was writing of compendiums. Nilambara Jhā (1825 A.D.) composed Golaprakasa in Sanskrit in which he has incorporated three types of knowledge, viz. (1) For several demonstrations, he has taken help of English Books, (ii) applied his own knowledge on the subject which is different from the ancient, and (iii) ancient Siddhanta theories. The work consists of five chapters dealing with (i) canon of Science (ii) trigonometry (iii) spherical geometry, (iv) spherical trigonometry, and (v) questions on plane and spherical trigonometry. In Persian also such work as Khizānat al-ʿIlm was written by Diwān Kanhji Kayasth 'Azimābādī (Now Patna) was composed in 1229/1814 which was dedicated to Francis Hakas. This work consists of ten chapters mainly dealing with Arithmatic, mensuration and algebra.

'Abd al-Rahim Corakhpūri commonly called 'Abd al-Rahim Dahriyah, who composed an Encyclopaedic work entitled Shigarf Bayan in the first quarter of 19th Century which deals with (1) a history of the genesis and evolution of the human race (2) a brief autobiography (3) an Arabic treatise on Astronomy (4) five important reasons for translating into Arabic and Persian the standard European works on astronomy, geography and mathematics (5) reasons for preferring the work of Simpson to that of Nasīr al-Dīn Tūsī on Euclid's Elements.

Another important compendium is Ghulām Husain Jaunpūrī's JĀME-I-BAHADUR KHANI which we are considering in detail which is the main subject matter in this thesis.

1.4 Geometry.

Geometry in ancient India was basically "constructional". It arose and developed with the needs of rituals and cosmographic speculations. The earliest manual of geometry Sulbasutras which may be called "manuals for geometrical construction" of Vadic Period laid down formulas for construction of Vedis (altars) of various shapes, depending on the particular ritual. Thus this involved geometrical problems like (i) construction of square on a given straight line (ii) making a circle equal in area to square and vice versa (iii) (a) doubling a circle (b) Geometrical method to find the value of $\sqrt{2}$ (iv) (a) A theorem on the square of the diagonal which runs as : "the diagonal of a square produces an area double of the original square". (b) A general

statement on the square of the diagonal of a rectangle now known as Pythagoras theorem runs thus : "the diagonal of a rectangle produces both (areas) which its length and breadth produce separately" (v) construction of a square equal to the sum of two unequal squares. (vi) Transformation of a rectangle into a square and vice versa (18; 120). Post Vadic geometry is mainly concerned with mensuration formulae and solution of certain rectinear figures such as triangles and quadrilaterals of different varieties and some geometers have showed considerable proficiency and indeed they obtained some remarkable results. (33; 125). For example : The first Indian mathematician after the Sulbasutra is Āryabhata who wrote Aryabhatya (c. 499 A.D.) which consists of four sections of which, the section of Ganitapada (or mathematics) deals with geometry (18; 140). Its position in India is somewhat akin to that of the Elements of Euclid in the Greek some eight centuries before. Both are summaries of earlier developments. Āryabhata gives mainly some mensuration formula and solution of certain rectinear figures such as triangle, circle and trapezium and the volume of a sphere and pyramid. He also gives the property of right angled triangles as the square of the bhuja (perpendicular side) plus the square of the Koti (side), being equal to square of the Karna (hypotenuse); that is $x^2 + y^2 = z.^2$ General solution for such rational triangles
8 are given by Brahmagupta, Mahāvira, Bhāskara and other writers in the form $m^2 - n^2$, $2mn$ and $m^2 + n^2$ for the bhuja, Koti (71;184).

Area of triangle as the product of the perpendicular and half of the base is given by Aryabhata (15; 38) which was followed by other authors. For area of a circle he states, "half of the circumference, multiplied by the semi-diameter". Regarding relation between chord and circumference, he enunciates that "the chord of one-sixth of the circumference is equal to half the diameter".

Another eminent mathematician of India was Brahmagupta (628 A.D who deals with geometry in Chapter 12 of Brahmsphuta-Siddhānta. His principal interest is in arithmetic and algebra, but some of his geometry attracted attention. He makes a clear distinctio between exact results and approximation. The geometrical figure which interested Brahmagupta particularly is the cyclic quadrilateral. He gave general rules in two stanzas which are most important contributions to the geometry of the cyclic quadrilateral are as follows:

"The gross area of a triangle or a quardrilateral is the product of half the sums of the opposite sides; the exact area is the square root of the product of four sets of half the sum of the sides (respectively) diminished by the sides". (68; 88) that is, if a, b, c, d be the four sides of a quadrilateral taken in order, we have

$$\text{Area} = \sqrt{(s - a)\ (s - b)\ (s - c)}$$

where $s = 1/2\ (a + b + c + d)$.

Another stanza runs as under:

"The sums of the products of the sides about both diagonals being divided by each other, multiply the quotients by the sum of the products of opposite sides; the square roots of the results are the diagonals in a trapezium".

The meaning of this is that, if a quadrilateral having sides a, b, c, d, in order and diagonals, x and y, then

$$x^2 = \frac{ad + bc}{ab + cd} (ac + bd)$$

$$y^2 = \frac{ab + cd}{ad + bc} (ac + bd)$$

This formula was also given by Mahāvira (c. 850 A.D.) and Bhaskara II (c.1150 A.D.). The same formulae were rediscovered in Europe in 1619 A.D. by W. Snell.

To find the radius of the circle describe round a quadrilateral, Brahmagupta gives the following rule (33; 141).

"The diagonal of an isosceles trapezium being multiplied by its flank side and divided by twice its altitude gives its heart line; in case of a quadrilateral of unequal sides, it is half the square root of the sum of the squares of the opposite sides".

Bhaskara (c. 1114 - 1200) is also known as Bhāskara II or Bhāskaracārya, who was the leading mathematician of the twelth century. It was he who filled some of the gaps in Brahmagupta's work. He was much interested in the problem of finding the length of a side of a regular inscribed polygon. He discards the old crude methods of finding the chord in terms of the arc and vice versa and gives much better approximations. He reproduces Brahamagupta's theorem for the area of a quadrilateral, adding the area so obtained will be gross.

Many commentaries were written on the texts of these authors. The style of geometry that was determined by these texts was by and large retained in 18th and 19th century also. However, some Euclidean geometry had started entering in Sanskrit mathematical literature. In a way Jagannatha was pioneer of this trend. In 1718 A.D. he translated Euclid's Elements from Arabic version Tahrīr-i-Uqlīdes by Nasīr al-Dīn Tūsī into Sanskrit, as Rekhaganita. Nayansukhopādhyāya translated the Arabic version of a treatise on a spherical geometry into Sanskrit in 1730 A.D. as Ukārākhya-grantha. Yogadhyāna Misra translated Hutton's Euclidean Geometry into Sanskrit in 1873 A.D entitled Ksitra - tattva - pradipikā.

There was much more activity in this field amongst Arabic and Persian scholars which will be discussed in the next chapter.

C H A P T E R T W O

CHAPTER TWO

EUCLID'S ELEMENTS, ITS IMPORTANCE AND TRANSMISSION INTO INDIA

The Greeks developed a theoretical geometry from an empirical science, that is, they deduced all possible geometrical relationships by the laws of logic from a limited number of postulates or axioms which were laid down from the empirical facts.

For the development of such deductive geometry Thales (sixth Century B.C.), the earliest Greek mathematician who "showed the road leading to the principles to his successors"(13; 31) Pythagoras (572 B.C.), Hippocrate (5th Century B.C.) and Eudoxus (c. 400 B.C.) invented many geometrical problems including substantially advanced number theory by demonstrating the reality of irrational quantities and theory of proportion etc.

Euclid (b. 365 B.C.) compiled his Elements (c. 300 B.C.) from a number of works of earlier men. Among these are Hippocrates of Chios. The latest compiler before Euclid was Theudius, whose text book was used in the Academy and was probably that was used by Aristotle. However,the whole design of the Elements are due to Euclid (80; 1020).

2.1. CONTENTS OF THE ELEMENTS.

Euclid's Elements are not only devoted to geometry but contain a good deal of number theory and elementary geometric algebra. The chief merit of the work lies in the skillful selection of propositions and their arrangement into a logical sequence. The work comprises thirteen books with a total of 465 propositions and definitions, postulates and axioms. The manner of presentation of a proposition is that there is first of all the enunciation of the proposition; secondly, the statement of the precise data; thirdly, the statement of what we are required to do with reference to the precise data mentioned; fourthly, the constructions, the addition when necessary of more lines to the figure; fifthly, the proof; and sixthly, the conclusion, stating what has actually been done, which generally follows the wording of the original enunciation and finally a stereotyped closing sentence : quod erat demonstrandum. (12; 22).

Euclid's Book I is in three sections. The first 26 propositions deal mainly with properties of triangles and include the three congruence of triangles. Propositions 27 to 32 are based on the theory of parallels and a very controvertial proposition that the sum of the angles of a triangle is equal to two right angles. The remaining propositions deal with parallelograms, triangles and squares with special reference to area.

Book II deals with the results on the dissection of rectangles that would now be expressed as algebraical formulas e.g. $(x - y)^2 = x^2 + 2\ xy + y^2$.

Book III deals with circles including their intersections and touching.

Book IV consists of problems about circles, particularly the inscribing or circumscribing of rectineal figures.

In Book V Eudoxus general theory of proportion have been dealt with Borrow observes, "there is nothing in the whole body of the Elements of a more subtile invention, nothing more solidly established and more accurately handled, than the doctrine of proportions". In like spirit Cayley says, "There is hardly anything in mathematics more beautiful than this wondrous fifth book"(81; 419).

Book VI is based on the Eudoxian theory of proportions to plane geometry. Along others a generalization of the Pythagorean theorem in which instead of squares, three similar figures are described on the three sides of a right angled triangle.

Books VII, VIII and IX deal with questions in the theory of numbers. In Book IX it is elegantly proved in the 20th Proposition that there are an infinite number of prime numbers. In Proposition 35, there is a beautiful geometric derivation of what is practically equivalent to our algebraic formula for the sum of the first n terms of a geometric progression and in the 36th, it is proved that if $S_n = 2^n - 1$ is prime, then $P = 2^{n-1} S_n$ is a perfect number, that is, a number which is equal to the sum of its divisors smaller than itself.

Book XI, XII and XIII deal with theory of three dimensions. In propositions 16 and 17 of the last book, all the details are carried out for the actual construction of an icosahedron and of a dedecahedron, inscribed in a sphere.

2.2. ELEMENTS IN ARABIC

During reign of Hārūn al-Rashīd (786 - 809) began the translation of classical scientific works of Greek into Arabic. His son al-Mamūn (813 - 833) who set up 'House of Wisdom' (Beit al-Hokmah) at Baghdad to further the works started by his father. He procured among others, a manuscript of Euclid's Elements from Byzantine Emperor. Al-Hajjāj ibn Yūsuf ibn Matar was the first to translate the Elements into Arabic. He twice translated the Elements, first under Harūn al-Rashīd then again under al-Mamūn, therefore, the first translation was known as the "al-Hārūnī" version, the second bore the name "al-Mamūnī" version. It was next translated by Ishāq ibn Husain (d. 900) and this translation was improved by Thabīt ibn Qurra (826 - 901) (24; 1).

The third form which is the most important work in this line is Tahrīr Kitab Usul al-Handasah wa'l-Hisâb al-Mansūb ila

Uqlīdes (popularly known as Tahrīr Uqlīdes) by Nasīr al-Dīn Muhammad bin Muhammad al-Tūsī (1201 - 1274.). This edition appeared in two forms, a large and a smaller one. This work is however, not a translation of Euclid's text but a re-written Euclid based on the older Arabic translations (47,i; 79). In the introduction of Tarīr, al-Tūsī writes :

"I thought it right to compose the book which contains the principles of number and calculation attributed to Euclid's or "Sur" with a brevity not injurious; also to inquire into its object and design with thorough investigation, and to add to it what I found worthy from amongst those (principle) which I had gathered from the writings of experts in this science or had discovered by my ingenuity; also to improve upon the texts of the two editions of Hajjāj and Thabīt by giving hints either about additional points or about the differences between their descriptions of Propositions or their Proofs (9; 1

No doubt al-Tūsī's version was the main source for introducing Euclidean Geometry in Europe as well as India and scholars whose language was Arabic or Persian treated it as the main text.

The result of these translations was that many new results, alternative proofs and concepts started developing.

2.3 IMPORTANCE OF EUCLID'S ELEMENTS.

The importance of Euclid's Elements being its deductive nature is manifold. It introduced into mathematical reasoning new standards of rigor and marked a decisive step in the geometrization of mathematics (81; 124).

al-Beruni (in the book "Kital al-Tafhim li-awa'il Sina' at al-Tanjim". (Book on the teaching the Elements of Science about Stars) (70; 58) says that : "It converts the science in numbers from practical to general and takes astronomy out of the area of the guess work and hypothesis to the sphere of truth".

It is an important fact that proof of the propositions of this geometry are based on Aristotelian logic which, of course, according to Fritz (13; 29) "had developed from the art of disputation between two participants in a dialogue". So it provides apparently for thinking.

Moreover Ghulām Husain Jaunpūri gives four profound importance in the following words : (43; 152).

Firstly, this science provides the pleasure of conviction to the soul; and this art leads from mixture of conjectures to the obvious certainty.

Secondly, it provides such matters which are best remedy of the extremely complicated ignorance which is the most fatal sickness of soul.

Thirdly, because man is of civic nature and for management of house and civic politics, he is always in great need of various types of implements of production and the instruments of creation

So there must be some superior law, so that with its help, every possible objective may be obtained.

Fourthly, by means of this cscience, knowledge of astronomy and celestial and terrestrial bodies get so much clear that it leads the knower to the stage of "Lau Koshefa'l Ghita-o-la mazdatt-o-Yaqina" (When the covers are removed, certainty is increased, from the path).

Another fact of importance relative to the learner point of view is that the geometry is a means of perfecting the learner's understanding with reference to the whole of geometry. For we shall be able to acquire knowledge of the other parts of this science as well, while without them, it is impossible for us to get a grasp of so complex a subject, and knowledge of the rest is unattainable (47; i; 116). Tseng, a Chinese scholar-general presented his opinion in this context and compared with traditional mathematics and says "according to our traditional mathematics each section derive its name from a specific (practical) function The students all follow rules in solving their problems. All their lives they use mathematics knowing only how to do it and not why it is done. Therefore, they consider mathematics as a very difficult subject simply because they are confused, knowing the method, but not the principle..... Euclid's Geometry (on the other hand) deals not with method, but with principles (77; 172-173).

Further, it is, according to Smogorzhevsky (75; 20), adequate for solving most practically important problems with a very high degree of accuracy; and since it is at the same time, characterized by great simplicity, its wide application is always permanently guaranteed.

2.4 TRANSMISSION OF EUCLID'S ELEMENTS INTO INDIA

It is difficult to affirm when and how Euclid's Elements was introduced in India. But according to Datta and Singh (33; 124), the earliest attempt, as far as known, to introduce Euclid's Elements into India, in the garb of Sanskrit verses, was made by the eminent philosopher, mathematician and astronmer, Abu Raihan Muhammad ibn Ahmad al-Berūnī (973 - 1048). Al-Berūnī himself noted in his "Kitab al-Hīnd" (8; 137):

"Most of their books are composed in Sloka, in which I am now exercising myself, being occupied in composing for the Hindus a translation of the books of Euclid and of the Almagest, and dictating to them a treatise on the construction of the astrolabe being simply guided herein by the desire of spreading science".

But no such book is mentioned either by Mūjtabaī (58; 15-17), who gives a list of 27 titles of al-Beruni's works or in the Bibliography of Sanskrit works on Astronomy and Mathematics (85).

With the establishment of the Mughal power in India, the Samarqand school of mathematicians and astronomers began to play a prominent part in the study of mathematics and astronomy among the

Muhammedans in India (71; 205).

Akbar (1575) included the Euclid's Elements into the course of study (3, i; 288). Abul Fadl (1590) referred some propositions from Euclid's Element (3,ii; 418). This shows that he had good knowledge of this work. However, it was confined in a small circle of Indian Scholars. Only in 1658 A.D. Kamalāka the Court astronomer of the Emperor Jahāngīr (18; 171), wrote Siddhānttattva-Viveka in which he utilised some definitions and propositions from some version of Euclid's Elements. For example, Chapter 3 Śloke 22 - 27 states (51; 105):

"The line is that which has always the length, but not the breadth and which is very fine. That it should be know by the intelligent as of two kinds, curved and non-curved. Of them, the line which is non-curved and whose other extremity is not visible to the keenest eye placed on one extremity of it is called a straight line. The curved line is like a circle or a bow. That which has neither length nor breadth and which always lies within the circle at a distance of the radius from the circumference is called centre. The line around and which is always at a distance of the radius from the centre is called the circumference. The straight line which passes through the centre of the circle from circumference to circumference is called the diameter. Other straight lines are called full chords or chords of complete arcs".

Besides these, some other definitions and propositions are mentioned which correspond to Euclid's Proposition I.15,19,21; VI. 8 etc. However, no proofs have been given. It will be reasonable to think that Kamalākara had good knowledge of Elements.

In fact, al-Berūnī's desire of introducing Euclid's Elements was fulfilled when Jagannātha translated in 1718 A.D. from Nasīr al-Dīn Tūsī's Tahrīr Uqlīdes, by the order of his patron Rajā Jayasīmha of Jaipur into Sanskrit under the title Rekhāganita.

A Persian translation of Tahrīr Uqlīdes was composed by Khair Allāh Khān b. Lūtf Allah Khān (1731-32), another colleague of Jayasīmha.

But it seems that Rekhāganita could not influence the scholars whose language was Sanskrit. Although it waw much popularised among the other stream and a number of commentaries on Tahrīr Uqlīdes appeared.

2.5 TASHRIH AND HASHIA (COMMENTARIES AND GLOSSES) BY INDIANS.

Most of the commentaries are written in Arabic and only a few in Persian.

Mir Mohammad Hāshim b. Qasim Alī Husein (d. 1061-1652) (5;406), who was well versed in Mathematics as well as medicine, wrote a commentary on al-Tūsī's Tahrīr Uqlīdes in Arabic upto nine

Muqala, in the reign of Shah Jahan. In the introduction of his Sharh Tahrir-i-Uqlides (46; 1) he remarks:

"I have planned to make some useful additions which I have drawn from the works of experts in this field of science".

In writing this commentary he utilised Kitab-i-Hall-i-Shükük, Masadirat of Abu Ali Hasan bin Haitham (965 - 1039 A.D.), a commentary written by Abu'l Fath Umar al-Khayyam (c,1080-1123 A.D.), Al-Islah of Athir al-Din Abhari, Al-Shifa of Abu Ali Sina, Sharh of Abu'l Abbas Fadl bin Qasim and commentary by Abbas b. Saeed al- Jauhari and a commentary by Abu'l Qasim Ali b. Ahmad (46; 3).

Another commentary was written by Mohammad Barkat b. Abdul Rahman Allahabadi (cir. 1225/1810). His commentary is based only on first two Meqala.

Syed Hasan b. Dildar Ali Nasirabadi (1787 - 1842 A.D.) composed Hawashih Tahrir-i-Uqlides (11; 405).

A commentary called "Mukhtalif Sharh Tahrir-i-Uqlides" also was written, which contains commentaries from Mohammad Abid, Taqi al-Din Mohammad, Syed Hasan, Mirza Mohammed etc. Neither the name of the author nor the date of writing is known but a copy of Manuscript in the Nasiriya Library of Lucknow gives the date of purchase (1217/1799). There is conjecture that Syed Hasan is the author because some commentaries ascribed Syed.

Besides these, there are some other important commentators like Fakhr al-Dīn Lakhnawī, Mūftī Moḥammad Abbās b. Syed Alī Abbas (1112/1763 - 1241/1889) (11; 33).

CHAPTER THREE

CHAPTER THIRD

LIFE, WORKS AND SIGNIFICANCE OF GHULAM HUSAIN OF JAUNPUR

This Chapter is devoted to the life and works of an important Indian Mathematician GHULĀM HUSAIN JAUNPŪRĪ, (b. 1790). A brief account of his work JAME-I-BAHĀDUR KHANI which is called "Encyclopaedia of Mathematical Sciences" and reason of writing such a book in Persian and its contents are discussed.

3.1. LIFE

Abū'l Qāsim alias Ghulām Ḥusain was born in Jaunpur in the year 1205/1790 (5; 359). His birth place has been an important seat of learning since the days of Sharqu Kings and was righly called the 'Shiraz of India'. Shāhjahān proddly used to say "East is my Shīrāz", i.e. a centre of intellectuals (6; 44). Ghulam Husain came from a learned family. His father Fatech Moḥammad Karbalāi graduated from an institute at Karbalā - one of the main centres of Oriental education. Ghulām Ḥusain received his early education from his father. According to John Tytler: (78; 254).

> He had devoted himself to the study of mathematics and astronomy, not only as far as they zare contained in the Arabic and Brahminical writings but also acquired the knowledge of many valuable particulars which are not to be found in the writings of the Greeks by the opportunity of associating with Europeans". From his writings it appears that he had mastered in science of Geometry and Astronomy.

He served the court of various Rajahs. In fact those were the only places where scholarship was encouraged and patronised. For along time he was associated with the Rajah of Tikari. He served the court of Raja of

Banaras as Sadr al - Sadūr (Chief Justice). Last years of his life he served at Murshidabad. While he was returning from Murshidābād to his native place, he breathed his last at a place called Dawoodpur in 1279/ 1862 and was buried there (57; 191-193).

3.2 GHULĀM HUSAIN'S WORKS

He wrote a large number of comprehensive and small treatises dealing with mainly astronomy and mathematics. As noted in Nuzhat al-Khawatir, "Ghulam Husain was one of the greatest scholar of arithmetic, astronomy, geometry and other branches of Mathematics" (5; 359). His remark is not an exaggeration as is evident from his master piece JĀME-I-BAHĀDUR KHĀNĪ. Most of his writings are in Persian and a few in Arabic. The following are his works which are noted by Story (78; 19) and Abdul Hai (4; 373,381,385).

3.2.1 GEOMETRY

(1) Sharh Tahrīr-i-Uqlīdes

(2) Raiḍ al-Nufūs Sharh-i-Ukar-i-Thaudhusiyus

Sharh Tahrīr-i-Uqlīdes is a comperehnsive commentary on an Arabic recension Tahrīr Kitab Usul al-Handasah wa-'l-ḥisab al-mansub ila Uqlides (Commonly called Tahrīr-i-Uqlides by Nasīr al-Dīn Mohammad b. Mohammad al-Tūsī, an Arabic version of Euclid's Elements, composed in 646/1248). Shabbir Aḥmad Ghorī says that this commentary is not extant any where. But the book is available in Kāshi Naresh Granthalaya, Ramnagar, Varanasi. It covers 1478 pages composed in 1275/1857. Perhaps this in his last work in Arabic.

Raid al-Nufus is the only known commentary by an Indian, of Tahrie Ukar Thaudhusiyus (651/1253) by Nasir al-Din Tusi of Theodecius Spherics.

3.2.2 ASTRONOMY

1. Zīj-i-Bahādūr Khānī
2. Istilahat al-Taqwim
3. Sharh Tahrīn al-Majisti
4. Tenbīh al-Munkarin

Zij-i-Bahadur Khani was composed in 1241/1825-26. This is a compilation of Astronomical Tables in Persian and was dedicated to Bahādur Khān.

Istilāhāt al-Taqwīm was written in early nineteenth century and is a small treatise in Persian on the compilation of al-manaces in fourteen short chapters.

Sharh Tehrīm al-Majisti is a commentary on al-Majisti by Nasir-al-Dīn Tusi, which is an improved recension of Megale Syntexis of Claudius Ptolemaeus, who is known as Ptolemy in English and in Islamic language Batlimus.

3.2.3 MATHEMATICS/ASTRONOMY

1. Jāmi-i-Bahādur Khānī.
2. Anis al-Ahabab fi Bayan-i-Masail-i-Usturlab. Jāme-i-Bahādur Khānī was completed in 1249/1833 and the whole book is divided into a Muqaddamah and six Khezinahs (Treasuries) dealing with Geometry, Optics, Arithmetic, Trigonometry, Astronomy and Astronomical Tables and Almaces.

Anīs al-Ahbēb fi Bayān-i-Masa'il-i-Usturlāb was composéd in 1233/1818-19. This is a commentary on Beha' al-Din al-'Āmilī's treatise As-Sufaihātul-Imkān on the construction and use of astrolab; a discourse on fundamentals of geometry, physics and astronomy. Some diagrams illustrate the motion and position of plantes and stars from specific latitudes and longitudes.

A comprehensive study of the Mathematics and Astronomy of Ghulam Husain undoubtedly is an important task. No serious study has been made as yet. We shall here analyse the principles of Euclidean Geometry and Propositions which are spread over a long chapter of this great compendium JAME-I- BAHADUR KHANI in coming chapters of this thesis.

3.3. JĀMI-I-BAHĀDUR KHĀNĪ

This compendium which has been called the Encyclopaedia of Mathematical Sciences in Persian, occupies an important places in the Indian Literature of Mathematics. The author commenced writing it on 15th Safar 1248/1832 and completed on 15th Jamāda I 1249/1833 i.e. within a period of one year two months and seventeen days. It was lithographed at Calcutta in 1835. This was written on the instructions of Mirza Khan Bahādur son of Rajah Mitrajit Singh of Tikari, a princlly state in Bihar.

A Tytler notes: (78;254)

" It was a work of very considerable merit and information, compared with the author's opportunities".

Shabbir Ahmad Khān Ghori remarks: (54;238):

" No other book on astronomy on the pattern of this book is known except al'-Berūnī's Kitab al-Tafhim of which the first two parts are devoted to Astronomy, Astrolab and Astrology, but their treatment of mathematical sciences is not so thorough and exhaustive as in JĀME-I-BHĀDUR KHANĪ".

3.3.1 REASONS OF WRITING JĀME-I-BAHĀDUR KHĀNĪ.

Most of the Central and West Asian Scientific literature was mainly in Arabic and Indian scholars who were associated with that school of science also depended on Arabic. But by 17th and 18th century Persion became more popular among the scholars who belonged to this cschool of science but there was very little literature available in that language. Ghulam Husain writes in his JĀME-I-BAHĀDUR KHANĪ:

" Once Rajah Khan Bahādur Khān, illustrious son of Maha Rajah Mitrajit Singh of Tikarī, addressed me saying that during these days mathematical sciences and arts, inspite of elegance of problems, firmness of arguments and pleasure due to confirmation and deriving benefits have been so much out of fashion and forsaken that not even one among the elite and the commoner in our regime has any inclination towards it. This is not free from a few reasons: Firstly, they do not know its importance and objective, so they could not turn towards it. Secondly, from the reading and study of the books on this art, it is clear that the acceptance of propositions to the extant of belief is dependant on the existence upon a large number of books and treatises such that some depend on another; besides they are not free from much repetitions and difficulty of words and obscurity of meaning. Hence non-availability of these scientific books and also the huge volume and the toughness of expression becomes forbading for most of the people. Thirdly, although rational sciences have no particular language but most of the books on these sciences through which maximum benefit can be obtained are in Arabic. Therefore, majority of the Persian knowing scholars who have not spent much of their time in learning Arabic remain deprived and it is also known that by rational contemplation, the problems, become finer and better day by day, rest which occur between different voices; therefore, this arrangement is called Music (Ilm-i-Musuqi).

Fourthly, knowledge of form and magnitudes of the heavenly bodies and the differences between the positions among themselves and the relation to its sub-lunary bodies; this is known as astronomy (Ilm-i-Haiyat).

Later on he classifies Mathematics (Ilm-i-Rayāzi) into subsidiary sciences as Optics (Ilm-i-Manāzir), Trigonometry (Ilm-i-Īnekās), Algebra (Ilm-i-al-Jabr wa al- Muqābalah), and sculpture (Ilm-i-Ijad). Chapter One on Geometry deals with the definition, objectives and principles of geometry; principles of straight lines, angles, rectlineal surfaces, general quantities and the rule of simple and compound and derivative proportions and principle of cylinder, pyramid, cones, spheres and the circles, arcs, angles which exist on the surface of a sphere and ellipse. This chapter comprises of 274 propositions which are extracted from the various writings through Arabic sources. For example, from the writings of Nasīr al-Dīn al-Tūsī, Abū Alī al-Hasan bin al-Hasan al-Haitham (about 965-1039) known by the name Ibn al-Haitham, Archmides, Ptolemy, Banu Moosa Yahyā bin Abi-Shukr, Tafaddul Husain Khān etc. and some original discoveries of the author which is a valuable contribution of Medieval Indian Mathematics.

Chapter two, which is based on Optics, deals with the principles of vision, the science of optics and reflection. It consists of 59 problems in which 7 problems are taken from Rasāla al-Māhānī, 17 from Manazir-i-Uqlīdes by Nasīr al-Din al-Tūsī, 7 from Zia of Abū al-Mansoor, 4 from Lam' at of Abu Raihān al-Berūnī, 9 from Talimāt al Manāzir by Nasīr el-Dīn Tūsī and 4 from Sharh Manāzir of Abu Jāfar-e-Khāzīn al-Makkī, 3 from Vajiz of Banu Moosa al-Baghdādī and 8 problems are the original discovery of the author.

Third Chapter deals with definitions of Arithmetic, objective, operations, of arithemetic of integers, fractions, decimal fractions and the rules of logarithms and an exhaustive table of logarithm. Operation of arithmetic by sexagesimal figures and finding of unknown quantities by means of the regula falsi are also discussed.

Chapter Four is on trigonometry and comprises of one introduction and seven sub-chapters dealing with the magnitude of chords and sines, tangent and their exhaustive tables in sexagesimal system accurately upto the fourth subdivisions, division of circles, magnitudes of the site and angles of plane and spherical triangles, and computation of elevations, breadth that are both subjectively and objectively independant of matter, while inferior science (Ilm-i-Tabaii or Ilm-i-Adnā) deals with problems dependant on matters both subjectively and objectively which includes Physics, Medicine, Geology, Zoology, Botany etc. but medium science (Ilm-i-Ausat) can be made subjectively independant of matter. These problems constitute the area of mathematics (Ilm-i-Riazī). Thus mathematics occupied a position in between the two. Ghulam Husain gives further classification of the second category of science i.e. Ilm-i-Riazī on the basis of foundations which are four:

Firstly, knowledge of continuous statistics, magnitudes and their ramification by lines, surfaces and bodies, angles and relation existing between magnitudes, this is known as geometry (Ilm-i-Handasa).

Secondly, knowledge of the properties and rules of discrete magnitudes which are numbers, this is called arithmetic (Ilm-i-Hisāb).

Thirdly, knowledge of the arranged and the resultant of combination and merger of homogenious magnitudes is called the science of combination, as it is used for separating voices, depending upon the ratio of the intensity, shortness and magnitude of interval, motion and

And since the time of Abd-al Alī al-Barjandī (cir. 1500 A.D.) till now i.e. approximately three hundred lunar years have passed, no comprehensive book has been compiled with clear expression which gives the essence of Majisti and commentary on TAZKIRAH etc. in Persian language dealing with principles of Astronomy and detailed account and rules of astronomical observations. Due to this demand since long, we have wished that if a book covering on the fundamental and subsidiaries of Geometry, Trigonometry, Arithmetic and Astronomy is compiled, definitely de novo, these sciences will become popular and the interest which has become dim due to lapse of time, will once again be lighted and in this Destructible World its memory will remain for a long time." (43;2-3).

3.3.3 CONTENTS OF JĀME-I-BAHĀDUR KHĀNĪ

After the usual prayer to God,(Hamd)Prophet Mohammad and his descedents, Ghulām Husain gives classification of the Hikmat-i-Nazarī (Theoretical Sciences), first on the basis of foundation and then subsidiary in the Muqadmah (Introduction) of this work. According to him (43;4) superior science (Ilm-i-Ilahi or Ilam ma-baad el-Tabaiyāt or Ilam-i-ma qābl al-Tabaiyāt) in modern terminology Metaphysics is concerned with the problems of rivers and the depth of wells etc. In the sub-section on division of surface, the author gives a very interesting method of determining the ratio between the diameter of the circle and its circumference. For which Tytler remarks : (78; 258)

> "All the other Mohammedan mathematicians whom I had oever seen, contended themselves with the coarse approximation of 7 to 22, but it will here be seen that the author carries it on to seven places of decimals".

In fact he finds this ratio in sexagesimal as

$$3^{o}\ 8^{\prime}\ 29^{\prime\prime}\ 43^{\prime\prime\prime}\ 30^{iv}\ 12^{v}$$

that is,in decimal system

$$= 3 + 8/60 + 29/60^2 + 43/60^3 + 30/60^4 + 12/60^5$$

$$= 3.1415903$$

Which is correct upto five decimal places.

In chapter five, the author gives firstly the origin of the science of astronomy and then the system of a·tronomy according to the Hindu, Ptolemy and Copernicus and an account of astronomical instrument, nature and motion of terestial bodies, form of the earth and the particularities of its elevation etc.

Ghulām Ḥusain gives the method for determining the true position of a planet, which is undoubtedly remarkable.

He rejected the Platonic idea of circular planetary orbits and explained the ellipticity of the orbits. In his words: (43; 576)

> "The majority of ancient and most modern observers have determined the orbit to be an excentric circle, and have calculated the partial equation on this supposition, but Mīrza Khair al-Lah, the Engineer in his Commentary on Zīj-i-Mohammad Shāhī, has asserted that he has found not only that the orbit of the sun is excentric, but that the orbits of all the signs are of an elliptical form; by this proof, that if we reckon the place of the Sun and planets, according to the equation of a circle, we shall not find them agreeable to observation; contrary to what takes place in the equation which is produced in the case of the ellipse and if we determine the place from that latter calculation, determination will be more agreeable to observation. Hence the rule of conversion proves that the orbits are elliptical."

In the last chapter of this book, the method of the construction of astronomical tables and explanation of the technical terms of an almanac and formulas for conversion of calendars from one system to another are also discussed in detail.

It may be added here that some very interesting problems on geometry were raised by Ghulām Husain on his visit to Calcutta before the students of the Madrasah, or Mohammedan College there, as reported by Tytler, (78; 268) which is as follows:

1. Produce a finite straight line, so that its square shall be equal to the rectangle between the whole line so produced, and the part produced.
2. Within a circle describe another circle touching the first, and cutting out of it a given part.
3. Describe a circle equal to a given number of other circles.
4. Determine the length of the perpendicular from the apex of a given scalene triangle to the base.
5. Prove that the area of an equilateral triangle is equal to the square root of thrice the square of the furth part of the square of the side.
6. If a line be drawn from one extremity of the diameter of a circle, to a tangent raised from the other extremity the rectangle contained by the whole line thus drawn, and the part of it within the circle, is equal to the square of the diameter.
7. Is the proposition whose translation is given in this extract.
8. From the cube of a given number to find the cube

of the next number above and below it in the natural series

9. Having given the value of $X^2 Z$ and $X Z^2$ to find X and Z.

C H A P T E R   F O U R

CHAPTER FOUR

DEFINITIONS, POSTULATES AND AXIOMS

To prove a theorem in a deductive system is to show that the theorem is a necessary logical consequence of some previously proved propositions; these, in turn, must themselves be proved, and so on. The process of mathematical proof would, therefore, be the impossible task of an infinite regression unless, in going back, one is permitted to stop at some point. Hence there must be a number of statements called definitions, postulates or axioms, which are accepted as true and for which proof is not required. From these we may attempt to deduce all other propositions by purely logical argument.

This Chapter is concern with a critical comparative study on definitions, objects and principles of geometry which are based on Section One of Jāme-i-Bahādur Khānī.

4.1. DEFINITIONS:

1. A point is that which is sensible and can be divided.*

The Pythagoreans defined a point as "a monad having position" (unity in position) and this definition was adopted by Aristotle (c. 340 B.C.) (69; 274). But this is a statement of a philosophical theory rather than a definition. Plato objected to calling a point a "geometrical fiction". He defined a point as "the beginning of a line" (25: 26). Euclid gave the definition "a point is that which has no part". The statement that "the point has no parts", according to Szabo (13; 125-126), is strikingly reminiscent of Zeno of Elea, who spoke of "size without parts". The "size without parts" in Zeno's terminology denoted with referrnce to space that for which he used the word "now" with reference to time. It is by no means incidental that later even Proclus compared the concept of the geometrical "point" with the concept of the temporal "now".
Nasīr al-Dīn Tūsī defined (9; 2) the point is without parts, that is among those which have positions (auza plural of waẓ‘). Jagannāth translates the definition as "athing which is visible and cannot be divded, is called point". (50; 3). Ghulam Husain's definition is almost similar to it.

Regarding the condition of "position", Mohammad Hāshīm remarks (46; 6) that it has been added here, so that, 'moment' (‘aan) is excluded; otherwise it will also be included in the definition of the point. Because "point has position, but 'moment' has no position". Similar to this reasoning, the author of Mukhtalif Sharh Tahrir-i-Uqlides also states that "this condition is essential otherwise the definition of the point will fail when defining simple body (Mujarrad) and 'Unit' (wahdat)". (55; 1).

Atma Ram says that "the geometrical point is the imagination of a smallest place which we cannot conceive an idea of extension" (17; 1).

Thus from the above definitions, it appears that the geometrical point is not a body and has no weight and the point which is used in Euclidean geometry is indivisible and have position.

2. A LINE is that which is divisible only in one dimdnsion, which is length.*

The Platonists defined a line as length without breadth and Euclid did the same. The latter Greek writer defined it as a magnitude "divisible in one way only", Proclus suggested defining a line as the "flux of a point", an idea also going back to Aristotle, who remarked that "a line by its motion produces a surface, and a point by its motion a line". This, According to Smith, occasionally appears as "a line is the path of a moving point". (69; 274-275).

Nasīr al-Dīn Tūsī (9; 23) defines as "A line is length without breadth". Mohammad Hāshīm remarks that "here the negation of depth is inevitable employed; because without depth, breadth is not possible. When there will be no breadth, how the depth is possible; because it is the third dimension, and it cannot be found out without the second dimension". He further says, "in the definition of a line, the meaning of length (ar■) is the 'first dimension', and it should be noted here that all the three (length, breadth and depth) are associated with magnitude and they are different from (46; 6). Ghulām Husain also takes length as 'one dimension'. Jagannāth (50; 3) defined "A thing which is divisible in length but breadthless", whereas Gūlshan Alī al-Hūsainī states "A line is a magnitude which is one dimensional i.e. it could be divided only in length" (44; 155).

CLASSIFICATION OF LINES:

Euclid mentions only one kind of line that is straight line, although another kind of line appears in the definition of a circle in Book III. Mohammad Hāshīm classifies line first into two kinds: (1) straight (Mustaqim) and (2) circular (Mūstadir). He further divides the first one into (a)apprently straight (Mūstaqim al-mūstawī) (b) supposedly straight line (Mūstaqim al-Mūtafarris). But Ghūlām Husain classifies a line into two: (1) those which are 'straight' and (2) those which are not (or curved), He omitted other kinds of lines; perhaps they are unnecessary for his purpose, Gulshan Ali al-Husainī (44; 156-158) divides non-straight lines into (a) circular and (b) non-circular. He gives ten names that are straight line viz. Zila (Side), Sāq (arm), Masqat al-Hajar, Amūd (Perpendicular), Qāida (Base), Jānibiyah (side), Qūtra (Diameter), Watra (Chord), saham (axis), Irtafa (elevation). He further adds, a straight line does not enclose the surface with another straight line, because the meaning of enclosed space is this that all the parts of the periphery surround all the parts of the enclosure and this thing can not be obtained by two straight lines.

3. A STRAIGHT LINE is that on which all points are opposite (Mūtaqabīl) to each other.

Euclid defined straight line as "A line which lies evenly with the points on itself". Proclus explains that Euclid (69; 275) "shows by means of this that the straight line alone (of all lines) occupies a distance equal to that between the points on it". Archimedes stated this idea more tersely by saying that "Of all lines which have the same extremities the straight line is the least". (14; 404). Perhaps this is the source of definition given by Gūlshan Alī Al-Husainī as "A straight line is the shortest line among those which join two points". (44; 155). But often it is found in text books "a straight line is the shortest distance between two points" (Legendre used this). Although 'line and distance' (also 'between') are two radically different concepts. The shortest path between two points is an expression that is less objectionable, but it mearly shifts the difficulty.

Mohammad Hāshīm (46; 7) says "a line is said to be straight if whatever points be supposed on it, each one of them will be opposite (Mūtaqābīl - face to face) to the remaining points Ghulām Husain follows the same definition but Jagannath defines (50; 3) with slight different versions of ites "the points taken on it get covered look like, a single point". The meaning of Mūtaqābīl as defined in Sharh Tadkīrah (a commentary on Nasīr al-Dīn Tūsī's al-Tadhīrāh fi'l-hai'at, cir. 1255) that any point should not be up and down in respect of others, but in one direction (55; 1).

Thus we have so many notions of straight line, but indeed, as Pfleiderer says (47i; 168) "the notion of a straight line, owing to its simplicity, cannot be explained by any regular definite words already containing in themselves, by implication, the notion to be defined (such e.g. are direction, evenness of position), and as thought if were impossible, if a person does not already know what the term straight here means, to teach it to him unless by putting before him in some way a picture or a drawing of it". Perhaps because of keeping such ideas in his mind, Jagannâth defined straight line not under the heading of definition, but "characteristic of a straight line".

4. A non-straight line is the contrary of a straight line.

5. If the formation of non-straight line is of the type that on its concave side a point can be obtained such that all straight lines drawn from that point towards the non-straight line, are equal then line is called a farjārī.*

6. A Line ends in a point.*

Aristotle defined a point as the "extremity of a line". But Euclid and latter mathematicians defined a point differently. But those who difined a line as the collection of points for them "beginning" or "end" is a point and intersection of two lines is also a point. But there are certain type of lines e.g. circle, ellipse i.e. a closed figure having no "extremities". Heath rightly remarks that there is "for fetched distincion between two aspects of a circle or ellipse as a line and a closed figure (thus which we are describing a circle, we have two extremities at any moment but they disappear when it is finished) is an unnecessarily elaborate attempt to establish the literal universality of the "definition , which is really no more than an explanation that, if a line has extremities, those extremities are point". (47,i; 165).

7. PARALLEL LINES are those lines whose formations neither converge nor diverge and if they are extended in two directions without end, essentially they will not meet.

The word "parallel" means "along side". Euclid defined parallel straight lines as "the straight lines which, being in the same plane and being produced indefinitely in both directions, do not meet one another in either direction".

Another definition which is rather less satisfactory is given by Posidonius (100 B.C.) that "parallel lines are those, which in one plane, neither converge nor diverge, but have all the perpendiculars equal which are drawn from the points of one line to the other". (69; 279). Simplicius (about 500 A.D.) also followed the same. But in this definition the there is an objection that it should be proved that the distance between two parallel lines is the perpendicular to them (47,i; 190).

Now question arises whether it is necessary that straight line should be in the same plane and lines should be straight. Some say that since the distance between the lines is every where the same, and one does not incline at all towards the other, they must for that reason, be in thesame plane. For the same question, Mohammad Hāshim replies that "for the parallelism, it is not necessary that all the lines should be in one plan, and those which are in two planes are not parallel". He further says that it has been imagined that the property of parallelism exists only in straight lines, perhaps there is no real parallelism in circular lines, but it never happens. Because parallelism exists in concentric circles also. (46; 11) Jagannātha Samrātha while giving definition omitted the words "parallel lines should be in the same plane". Ghulām Ḥusain goes one step further and omitted the words "straight lines" and "situated in the same plane" from the definition which are in fact, unnecessary conditions imposed in the definition by Euclid.

8. A SURFACE is a magnitude which has length and breadth and is divisible only in these two dimensions.*

The notion of surface is ancient and might have come when we measure areas and mark their boundaries.
The definition of a surface corresponds to definition of a line. Because a line is one dimensional magnitude whereas a surface is of two and is divisible in these two dimensions. Gulshan Ali remarks that these two dimensions should intersect at right angle and there should not be third dimension (44; 159).

Classification of Surfaces: Mir Mohammad Hāshim (46; 7) as well as Ghulam Husain divide the surfaces into two classes, i.e. (i) plane surface and (ii) non-plane surface.

9. PLANE SURFACE is that on which appearance of straight lines in all directions are possible.

Euclid stated that " a plane surface is a surface which lies evenly with the straight line on itself". Archimedes used a modification of this definition, "a surface such that a straight line fits on all points of it", or "such that the straight line fits on all ways". But in fact there was no material improvement in these statements until the 18th century, when Simson (1758) suggested the definition that "a plane superficies is that in which any two points being taken, the straight line between them lies wholly in that superficies, a statement which Gauss (c. 1800) characterised as redundant. (69; 276). But this definition is a modified form of the definition given by Mohammad Hāshim (46; 7).

According to him, "all the points taken on a surface should be opposite to each other", the meaning of "opposite" is same as stated earlier. Gulshān Alī also followed the same. In his words " a plane surface is that surface on which if any line be drawn in any direction it lies in that surface and does not go outside". He further says that this condition exclude the surfaces of sphere, cone and circular cylinder, because when lines are drawn on them,some lie inside and some go outside. Here meaning of line is straight line (44; 159).*

10. NON-PLANE SURFACE is contrary of the Plane Surface.*

11; A surface terminates in a line. *

12. A surface is also called superficies.

Mohammad Háshim remarks that sometimes it happens that a surface instead of terminating in line, ends at a point, for example, the cone which terminates at one end in a point and other in a circle. (46; 7).

13. AN ANGLE is a corner of a surface which is situated between two lines which get joined at a point in a way that they do not get united.*

Euclid's definition of an angle is as follows:
"A plane angle is the inclination to one another of two lines in a plane which meet one another and do not lie in straight line".

This definition of angle excludes "straight angle". Ghulam Husain also held the same view but he called an angle as "the corner of the surface" which lies between two lines. According to Mohammad Hashim, the condition of "corner" is applied here so that the surface which is obtained by intersection of two radii of a circle in a major segment may be excluded from this definition. Because if an angle is defined without using the word Mûnhadib (corner) but it is taken to be a surface which lies between two lines" only then the definition will include that angle also which is obtained in major segment of the circle. Although that is not an angle. (46; 7). Thus Munfaḍela angle is not an angle. Syed Hasan also supports the same view and gives a reason that when two arcs of a circle unite on the same line, then this definition holds good but it cannot be called an angle. He further says that the discussion does not end by calling it "a pair of intersecting lines", because here the surface exists but comes under the lines, not between them (55; 2).

There was conceptual controversy among the philosophers and geometers as to the particular category in which an angle should be placed, is it, a magnitude, (Māqoola-e-Kam) quality (Māqoola-e-Kaif) or state (Māqoola-e-Wad'a). Same who put it in the category of magnitudeargue that an angle can be partitioned in two dimensions similar to a surface (46; 7). Thus in their view an angle is a surface, and therefore, magnitude. Hence Ghulam Husain regarded angles as magnitude or quantity. Those who place an angle in the category of quantity give four reasons in support of their opinion (16; 6).

1. It is not a line, because it has breadth, nor a body, because it does not necessarily have depth, nor a surface, because it can not be divided up in a way that a surface can be divided, for no angle is devided breadthwise, but only lengthwise.

2. Every quantity remains a quantity when it is doubled, but a certain angle does not remain an angle when it is doubled, does not remain a quantity. Therefore, an angle, by its very

nature, does not seem to be a quantity.

3. No quantity is an accident of another quantity. But an angle is an accident of another quantity; therefore, it is not a quantity. For it is an accident of a surface or a body to be angular.

4. Nothing that is in itself a kind of quantity it also a quantity. But an angle is a kind of quantity. Therefore, it is not a quantity. For an angle, by its very nature, cuts a figure up, and this ability to divide is a kind of quantity.

We also find four reasons given in favour of the opposite 1 point of view.

1. Whatever can be increased and decreased is a quantity, but an angle can be increased and decreased. Therefore, it is a quantity. For an obtuse angle is greater than a right angle, and an acute angle is less.

2. The subject of an attribute belongs to the same category as that of which the attribute is descriptive. Acuteness and obtuseness are conditions of quantity and conditions that describe an angle. Therefore, an angle is a quantity.

3. That which divisibility necessarily suits is a quantity. But that divisibility is suitable for an angle. Therefore, an angle is a quantity, for an angle is divisible lengthwise.

4. Everything having dimension or dimensions is a quantity. But an angle has dimensions, length, and breadth. Therefore, t is a quantity.

It seems then, that we must say that an angle is a quantity. The same condition holds in the solid angle.

Classification of "Angles" on the basis of lines:

Mohammad Hāshīm gives an elaborate classification of angles which are given below: (46; 8).

1. Rectineal angle made by straight lines.
2. Non-rectineal angle made by non-straight lines.
   (a) Convex-Convex.
   (b) Concave-Concave.
   (c) Convex-Concave.
   (d) Line-Convex.
   (e) Line-Concave.

Here the term "Convex-Convex" means an angle bounded by two circular arcs with convexity outwards; line-convex means an angle bounded by a straight line and an arc with convexity outwards, and so in every case.

14. RIGHT ANGLE is one of the two angles which are formed by setting a straight line vertically on another straight line. In this case the vertical straight line is called Perpendicular.

15. AN ACUTE ANGLE is that which is less than a right angle.

16. AN OBTUSE ANGLE which is greater than a, right angle. Be it known that for a right angle the condition is that the bounding sides must be straight lines and for the acute and obtuse angles the conditions are more general as bounding sides may both be straight or non-straight or unlike. So when we say "angle" only, the meaning thereby will be a plane angle.

17. SOLID ANGLE means a corner of a body which is bounded by three plane angles or more than this, provided that the sum of the angles be less than four right angles.*

18. A FIGURE is that which is enclosed by one boundary or many boundaries.*

Some geometers include line also in figure. Mohammad Hāshim has the same view. (46; 9).

19. A CIRCLE is a plane surface which is bounded by one line, in which a point can be found such that all straight lines drawn from it terminating on that line, are mutually equal. That point is called the CENTRE of the circle. The line is called the CIRCUMFERENCE, and metaphorically a "circle" also means the "circumference". *

The definition, as it stands, has no genetic character. It says nothing as to the existence or non-existence of the thing defined or as to the method of constructing it. It simply explains what is meant by the word "circle" and is a provisional definition which cannot be used until the existence of circles is proved or assumed. (47;i; 184). A genetic definition as defined by Gūlshan Alī that a circle is figure described when one fixed point is supposed and another moves about that point until it returns to its first position. He further says that the word "circle" in fact is the property of the point that by its movement, circular line (Khat-e-Mūstadīr) is obtained. (44; 160).

20. A DIAMETER of a circle is a straight line which passes through the centre of the circle and ends in both sides on the circumference and bisects the surface of the circle.

And that the diameter bisects the circle, this thing, in view of Syed Hasan (33; 2) and Mohammad Hāshīm (46; 10) could be placed in postulates and putting it in definitions is not suitable place. *

21. A CHORD is that which does not pass through the centre and divides the circle into two unequal segments; and each two segments of the surface of the circle is called minor or major segments; and the two parts of the circumference of the circle are called ARCH; and the chord corresponding to two parts are called the BASES.*

22. A TRIANGLE is a plane surface such that three straight lines become its boundary. If all the three sides are equal, the triangle is called EQUILATERAL. And if only two sides

are equal, the triangle will be ISOSCELES, otherwise SCALENE.

If in the triangle there is a right angle, it is called RIGHT ANGLED TRIANGLE.

And if an obtuse angle occurs, the triangle is called OBTUSE ANGLED; and otherwise ACUTE ANGLED.

And because existence of right and obtuse angles in an equilateral triangle is impossible. Therefore, by combining the two properties of sides and angles, seven kinds of triangles are obtained and when we say triangle, we mean a rectilineal triangle.

Thus G̱hulām Husain classified triangles into seven, according to the sides and angles, but details are given by Mohammad Hashim.

They are as follows (46; 9):

The triangle which has:

(1) equal sides and acute angles.

(2) two equal arms (Sāq) and a right angle.

(3) two equal arms and an obtuse angle. Here the right and obtuse angles are possible only when the base (Qāida) be longer than the arms.

(4) two equal arms and acute angles.

(5) different sides and a right angle.

(6) different sides and an obtuse angle.

(7) different sides and acute angles.

In Hindu Geometry, triangles are classified according to the sides but not the angle, viz Sama-tribhuja (equilateral triangle), dvisama-tribhuja (isosceles triangle) and Visma-tribhuja (scalene triangle). Only the right-angled triangle is called by the name of Jatya-tribhuja by Brahmagupta and others. Here the Sanskrit word "Jatya" means "noble", "well-born", "genuine". Shukla observes (33; 129) that the name Jatya-tribhuja for the right angled triangle seems to imply that all other triangles are derived from it.

23. A SQUARE is a right angular surface such that four equal lines bound it.*

24. A RECTANGLE is a right angular surface which two opposite sides are longer than the two other opposite sides.*

25. A RHOMBUS is a surface whose four lines willbe equal such that none of its angle will be right angle.*

26. A RHOMBOID is that none of the sides are equal and none of the angles is right, but every one of the opposite side are equal.*

27. A PARALLELOGRAM surface means those four surfaces mentioned above and the quadrilateral other than these four figures is called Munharif.*

28. A, rectilineal surface whose number of sides are more than four, is called POLYGON. If the sides and angles are equal,

they are called REGULAR PENTANGON, HEXAGON upto DECAGON; and if the sides and angles are unequal, they are called Pentagon and Hexagon.

29. ELLIPTIC figure is a surface, which has a single line as the boundary and it has two dimeters - one is major and other minor; and they intersect at right angles; and on two sides of intersection on the major diameter, two points can be found at equal distances wsuch that from these two points two straight lines are drawn, meet at a point on the boundary and the sum of these two lines is always equal to the major diameter; and the boundary of the elliptic surface is called ELLIPTICAL LINE.*

30. ELLIPTICAL SURFACE (Ahlaljī) is the one such that two arcs of two equal circles, when each one is less than a semi-circle by opposite convex directions bound it; and the line which joins between the two angles of the figure is called major diameter of the Elliptical surface; and the line which bisects the major diameter at right angles is called minor diameter of the Elliptical surface.*

31. TURNIP LIKE SURFACE (Shaljamī) is the one such that two arcs of different convexity of two equal circles which are greater than semi-circle, enclose it; and the line which joins the meeting point of each twoarcs is the minor diameter of the turnip; and other line which bisects it at right angles is the major diameter.

32. CRESCENT LIKE SURFACE is the one such that two arcs enclose it, one from the concave direction and the other convex, provided that none of the two arcs are greater than semicircle.*

33. HORSE SHOE SURFACE is the one such that two arcs enclose it like the crescent but each one is greater than semi-circle.*

34. A SECTOR is a surface such that an arc of the circumference of a circle and two radii are the boundaries and if the arc is smaller than semi circumference, the sector is minor and if greater, major. *

35. A BODY is that which is divisible in three dimensions which are length, breadth and height; and it ends in surface.*

36. A CUBIC SOLID is that for which six squares form the boundary.*

37. A RECTANGULAR SOLID is that which has six rectangular surfaces as its boundary.*

38. A NON RECTANGULAR SOLID is that which has four rectangular surfaces and two non rectangular parallellogramic surfaces as its boundary.*

39. SIMILAR SOLIDS are those whose bounding surfaces are same in number and each corresponding surfaces are similar i.e. each corresponding side will be proportional and corresponding angles equal.*

40. A SPHERE is a solid figure which is bounded by a single surface and in its middle there is a point such that all the straight lines drawn from that point to the boundary, are equal and that point is the centre of the sphere.*

41. PARALLEL SPHERICAL SURFACES are those whose centres are common.*

42. CIRCLES AT EQUAL DIMENSIONS from the centre of the sphere are that whose straight lines joining the centre of the sphere and their centres are equal. *

43. A PERPENDICULAR ON A SURFACE is the one such that every straight line in the surface passing through the spot of its fall encloses a right angle, and if it does not make right angle the line is oblique.*

44. A SURFACE PERPENDICULAR TO A SURFACE is that when from the inter-section a perpendicular is set upon one of the surfaces, it falls on the other surface; and in case that perpendicular does not fall on the other surface, that surface is enclined to the other surface at acute angle.

Equi-inclination surfaces are those whose angles of inclination are equal. *

45. PARALLEL SURFACES are those that when they are extended in their own directions indefinitely, they not meet at all.*

46. A CIRCULAR CONE is a solid which has for its boundary a circle and a fire like surface emanating from the circumference

of the same circle and gradually becoming narrower ending at a point. The said circle is called the BASE of the cone and the point which is the extremity of the fire like surface is named the VERTEX of the cone; and the line joining the vertex of the cone and the centre of the base, is the AXIS of the cone. Hence if the axis is perpendicular to the surface of the base, the cone is right-angled. Otherwise it is called OBLIQUE; Also if the axis of the cone is equal to the radius of the base, the cone is RIGHT ANGLED and is greater, ACUTE ANGLED; and if smaller, OBTUSE ANGLED.*

47. A PYRAMID is that which has for its boundary a base whose sides and angles are equal and also some triangles whose bases are equal to the sides of the base and their number is equal to the number of sides of the base, in a way that the angles of the verticies of the triangles cover the angle of the vertex of the pyramid; its axis is a line which joins the vertex and centre of the base; as in the case of cone, the pyramid is vertical and inclined.*

48. SIMILAR CONES (or Pyramids) are those whose axes have the same ratio as their diameters of the bases. If a plane surface cuts the cone and is parallel to its base, the section of the cone which is connected with the base, is called the frustum of the cone (pyramid).*

49. A CYLINDER is a solid which has for its boundary two parallel circular surfaces and a circular surface which is connected between the two circumferences of the two circles; and the two afore-mentioned circles are the BASE of the cylinder and the line joining the two centres is the AXIS of the cylinder. If the axis is perpendicular to the surface of the base, the cylinder is RIGHT and otherwise OBLIQUE; and if the base of the cylinder is lateral the cylinder will also be lateral.*

50. A PRISM is a solid for which the boundaries are the two triangles and three parallellograms.

Solids of equal altitude are those whose perpendiculars falling from the vertex to the surface of their bases are equal.*

4.2 POSTULATES (USUL-I-MAUZU')

First it should be inferred from Divine Science that the entites i.e. points, lines, surfaces, and circles exists in essence (Nafeu'l-Amr), not that their existence is like the canine teeth of a ghost.

Regarding entities, that is, point, line, surface, and circle, Mohammed Hāshīm agrees with Ghulām Husain's view and says "they exist in reality", and advances the argument that because state (Waz'a) is essential point, line, surface and circle also exist because all the rules of geometry are applicable. Hence all these exist even if in imagination only, just as the rules of Arithmetic are real, although number is a matter of belief. (45; 11).*

1. It is possible to fix a point or points on every line or every surface.*

2. We can fix a straight line on every surface which passes through the point taken on it. *

3. Each point and straight line and plane surface corresponds to its like.*

4. A common intersection of the two lines is a point, and of every two surfaces it is a straight line.*

Mohammed Hāshīm argues that common section (Fasl-i-Mūshtarak) has its "state" between two magnitudes, such that it is the extremity of one magnitude and beginning of the other and the converse, or it is beginning of both the magnitudes or in the same way the extremity.

The common section is that which if merged in any one, there should not be addition and if it is removed, then there should not be any shortage (46; 12). In fact this notion of common section gives the idea of intersection of sets.

5. It is possible to for a straight line to join two points.* Here the object of the Postulate is rather to draw an imaginary straight line between two points than that it can actually be realised in practice. Mohammad Hāshim holds the same opinion and says: "The straight line can be drawn between two points in imagination and is impossible to draw actual straight line between two poles of the universe and the method to draw the imaginary straight line is that we fix two points and suppose a set of points between them in the same direction and suppose one more point corresponding to one of the two (first) points and if we suppose this point moves along all the points, then the path of this point will be a straight line between two given points" (46; 12).

6. Every finite straight line can be extended in its own direction.*

The above Postulate 5 asserts the possibility of drawing a "unique" straight line from one point to another. In the same way this Postulate maintains the possibility of producing a finite straight line only in "one way" in either direction.

To establish this Postulate, Mohammad Hāshīm gives a method which is attributed to Athir al-Dīn Abhari who mentioned in his book Istilah Kitab al-Uclides that "we fix a point in the direction in which the line is to be produced and then join this point with the ray of the line. Thus, if by joining, an angle is not formed, then the line will be straight, otherwise, we suppose motion in the line such that the angle becomes gradually greater and greater until it vanishes. Then, it will be a straight line. This is what is required. (46; 12). This proof is indeed for the claim to produce a finite straight line.

7. Making every point a centre, we can draw a circle for every distance that we like.*

It may be observed here that there is no restriction imposed on the radius of the circle. So according to Frankland (47; i; 199) it may (1) be indefinitely small, and this implies that space is "continuous", not discrete, with an irreducible minimum distance between contiguous points in it. (2) The circle may be indefinitely large, which implies the fundamental hypothesis of "infinitude" of space.

8. Right angles are all mutually equal.*

According to Mohammed Hāshim (46; 13), this Postulate should not be regarded as a Postulate but as an Axiom. Because it is obvious that without addition, equal things remain equal. Its analysis is as follows:

We suppose the angles A B C, A B D, E G H, E G T are right angles and each two exist on either side of the given straight line. They are equal by definition. Now we say that the angle A B C, A B D are equal to e.g. E G H, E G T. Because when twe superimpose the point B on G and line C D on H T, then inevitably A B will be superimposed on E G. Hence the result is proved. If it be not accepted, then A B is like K G and it is also well-known that if nothing is added in equal things, they will be equal. Therefore, if angle K G T is equal to A B D, then K G H will be equal to A B C, and angle A B D in already equal to A B C by supposition. It is also known that when things are equal to a thing, then they will mutually be equal. Hence angle K G H will be equal to A B D; although the angle K G T was equal to A B D. Therefore, K G T will be equal to K G H. But it has been supposed that the angle E G T is equal to E G H. Therefore, angle E G T will be greater than the angle K G T which was equal to K G H which is greater than K G T.

Hence the angle E G H which is equal to E G T will be greater than the angle K G H, which is its part. Hence part will be greater than its whole, which is false. In other words it may be said that B is superimposed on G and C D on H T. Hence the first two angles will be superimposed on the latter. Therefore, their total will be equal. Hence their half will also be equal. Some others argue that if a right angle will be smaller than a right angle, then one will be acute and other obtuse, othersise, that will be neither smaller nor greater than the right angle. For this argument Mohammad Hāshīm remarked that before proving that all right angles are equal, it cannot be accepted that the acute angle is absolutely smaller than every right angle.

9. Any angle which is equal to a right angle is a right anlge.*

This Postulate is not included in the Elements and it is not of a general character. Because any angle equal to any other angle will be equal to that angle. Mohammad Hashim observes that here equality is in the sense of magnitude which appears according to the object of the Book (Tahrīr-i-Uqlīdas) and a equality of ratios is excluded from this statement, because it is defined in Proposition 5 of Book V. (46; 16).*

10. It is impossible that two straight lines enclose a surface.*

This Postulate is an interpolation and no doubt, it was taken from the inference of Proposition 1 of 2, that "if the base B C does not coincide with the base E G, then they enclose a space; which is impossible".

11. A straight line does not coincide with more than one adjecent undirected straight lines*

12. For two different magnitudes which are of the same genus, the multiples of smaller one will always become greater than the bigger.

PARALLEL POSTULATE:

This Postulate is referred as Fifth Postulate of Elements, in which Euclid showed great strength of his genius. The statement is as follows:

"I f a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which are the angles less than the two right angles".

There was difference of opinion since antiquity about whether this so-called Postulate should be treated as a Postulate or Axiom or Proposition. But majority of great geometers thought that this Postulate was capable of proof and attempts to prove it were made by Ptolemy and Proclus among others. Even in modern times geometers like Naṣīr al-Dīn Ṭūsī considered this proposition as a problem. Mohammad Hashim followed the same line and stated that this Postulate was not self-evident like the others, which may be proved by paying a little attention towards it (46; 14).

Halsted (45; 72) writes that in a manuscript copy of Euclid in Arabic but in a Persian hand, bought at Ahmedabad, the editor, in the Introduction of the Parallel Postulate says: "I maintain that the last proposition is not among the universally acknowledged truths nor anything that is demonstrated in any other part of geometry.

The best view thereoore, would be that it should be put among questions instead of the principles; and I shall demonstrate it in a suitable place".

Thus between Euclid's time and our own, hundreds of people, finding it complicated and artificial, have tried to deduce it as a proposition. But they only succeeded in replacing it by various equivalent assumptions, such as the following five: (29; 2).

(1) Two parallel lines are equidistant.

(2) If a line intersects one of two parallels, it also intersects the other.

(3) Given a triangle, we can construct a similar triangle of any size whatever.

(4) The sum of the angles of a triangle is equal to two right angles.

(5) Three non-collinear points always lie on a circle.

Thus in the words of Heath: "When we consider the countless successive attempts made through more than twenty centuries to prove the Postulate, many of them by geometers of ability, we cannot admire the genius of the man who concluded that such a hypothesis which he found necessary to the validity of the whole system of geometry, was really indemonstrable" (47,i; 202).

Gauss was the first to have a clear view of a geometry of the Fifth Postulate, but this remain for quite fifty years concealed in the mind. But it was only revealed after the works of Lobachevsky (1829-30) and J. Bolyai (1832), appeared (23; 64). But public recognition they had not gained and in all likelihood, the number of mathematicians acquinted with their work was extremely small. But their works got full recognition after 1868. Thus they were the first to state publicly and to establish rigorously that a consistant system of Geometry can be built upon the assumptions explicit and implicit, of Euclid, when his Parallel Postulate is omitted and another incompatible with it, put in its place (26;11).

Ghulām Husain does not include this Parallel Postulate among Postulates, perhaps he had doubts in its validity. But of course, he does not give its substitute. He omitted Proposition 29 in which Euclid used this postulate for the first time, but he gives his own proof of Proposition 25 in which Proposition 29 is used.

4.3. <u>AXIOMS</u> ('ulūm-i-Mūta'ārifa)

1. Things which are equal to the same thing are also equal to one another.*
2. If equal things are added to or subtracted from equal things, the sum or remainder of both will also be equal.*
3. If equal things are added to or subtracted from unequal things, the sum or remainder will also be unequal.*
4. The one which is greater will be greater and one which is smaller, smaller.*
5. Two things to which equal things are added or subtracted from and the sum or remainder comes out to be equal, then the things must have been equal.*
6. If magnitudes whose fixed multiples or fixed parts, are the same, in both cases the magnitudes will themselves be equal.*
7. A magnitude which is equal to a greater (magnitude) is greater and the one equal to the smaller, smaller.*
8. The whole is greater than its part.*

Euclid included only five axioms which are mentioned in axioms 1,2,8 excluding "things which coincide with one another are equal to one another", which is mentioned neither by Mohammed Ḥāshim nor Ghulām Husain, perpaps because of keeping the idea in their mind that this axiom as Tannery remarked "it is incontestably geometrical in chacter, and

should therefore, have been excluced from the Common Notions "
Thus as Heath observed that we have here a definition of geometrical equality more or less sufficient, but not a real axiom. (47, i; 225).

C H A P T E R   F I V E

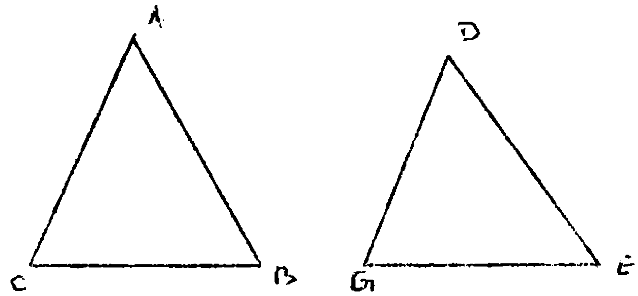
CHAPTER FIVE

PROPERTIES OF STRAIGHT LINES, TRIANGLES AND RECTILINEAR SURFACES.

This Chapter is concerned with the Propositions contained in Section Two of JĀME-I-BAHĀDUR KHĀNĪ. The first 21 Propositions are dealt with properties of straight lines, triangles and comparison of different triangles in respect of their area. Propositions 22 to 29 are based on theory of parallels and 30 to 38 on rectilinear surfaces. In the remaining propositions, geometrical algebra has been discussed.

PROPOSITION 1

If two sides and an angle which is between them of a triangle are equal to their corresponding two sides and an angle between them of another triangle, then the remaining corresponding sides and angles are equal to each other and triangle to triangle.



So, in trinagles A B C, D E G, side A B is equal to D E and A C to D G and angle A equal to D.

In this situation the side B C will necessarily be equal to the side E G and angle B equal to E and angle C to G; and atriangle to triangle;

Because if we suppose super posing the side A B on D E such that B coincides with E and then necesserily the angle A will coincide with angle D; Because of equality of the sides A B, D E and the angles A, D and the sides A C, D G, the point C coincides with G and it is clear that in this case the side B C will also coincide with E G, because the coincidence of the ends has resulted. Otherwise it is necessary that the two lines B C, E G which are straight enclose a surface and this is false.

Hence what was required is proved. *

This proposition is the fourth of the Taḥrīr-i-Uqlīdis. Ghulām Ḥusain placed it as first perhaps because, in the argument neither postulate nor the first three prepositions, except Common Nation (things which coincide with one another are equal to one another), have been used. In this proposition what is usually called the method of super-position, a method depending upon intuitive idea about the coincidebility of equal straight lines and angles and upon the possibility of motion without deformation is implied.

It is true that Common Notion is not quoted in argument but it is simply inferred that "the side B C will coincide with E G, because coincidence of the ends, otherwise it is necessary that the two lines B C, E G which are straight enclose a space and it is impossible. Syed Ḥasan observes that it is possible to coincide the line B C with E G and also possible that it would fall outside or would lie within it. But we are not interested in second as well as the third case because the superposition of the side B C is not possible, otherwise they will be part of the same straight line (55;2). This infers that coincidence of the end points is a must for lines, being coincidence.

Regarding impossibility of the two straight lines enclosing space,

Heath writes: (47,i ; 249)

Heilberg (Paralipomena Zu Euclid in Hermes XXXVIII, 1903, P. 50) pointed out, as a conclusive reason for regarding these words as an early interpolation, that the text of ar-Nairize (Codex Leidensis 399, 1 ed. Betsthorn-Heiberg) does not give the words in this place but after the conclusion Q.E.D., which shows that they constitute a scholium only. They were doubtless added by some commentator who thought it necessary to explain the immediate inference that, since B coincides with E and C with F (G), the straight line B C coincides with E F (EG) and inference that really follows from the definition of a straight line and Postulate 1; and no doubt, the Postulate that "Two straight lines cannot enclose a space (afterwards, placed among the Common Notion) was interpolated at the same time."

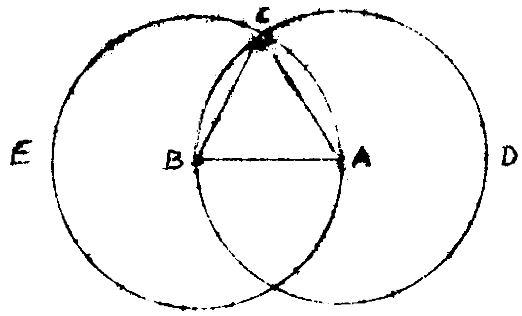
Ghulām Husain does not use any such bracket and two straight lines do not enclose a space takes as Postulate and not as a Common Notion, (43; 12) whereas Atmā Rām placed it among Common Notion (17 ;12).

Proclus claims that Euclid knows or acknowledges the assertion when he takes the first Postulate, which pre-supposes the uniqueness of the straight line connecting to points. Proclus' claim is vague enough to be incontestable.

Euclid undoubtedly does "know" that only one straight line can be drawn through two points, and probably he is relying on this "knowledge" in the proof of I.4. But when Heath (Vol. I page 232) suggests that Euclid could omit Postulate 6 (two straight lines do not enclose a space) because "the fact it states is included in the meaning of Postulate 1 (to draw a straight line from any point to any point)" he is being extremely misleading at best. Postulate 1 implies nothing about uniqueness of the straight line, he surely could have done so - for example, by inserting a definite article before 'straight line' in Postulate 1. The evidence suggests that Euclid never raised the question whether the impossibility of two straight lines enclosing a space was logical consequence of his first principle (60 ; 32).

PROPOSITION 2

We wish to construct a triangle on a finite line whose all the three sides are equal.



Like the line A B;

and taking a point A as centre and the distance A B, draw a circle BCD.

Now by making the point B as centre, with distance A B, draw the circle A C E which interesect the first circle, say on the point C; and we join A C and B C;

and thus the equilateral triangle A B C is formed .

Because the two sides A B, A C which emanate from the centre of the circle B C D and terminate at circumference will be equal and similarly the two sides B A, B C which emanate from the centre of the circle A C E reaches its circumference, will also be equal. Therefore A C, B C which are equal to A B will themselves be equal.*

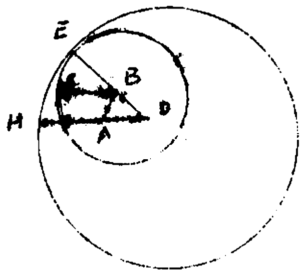
This is first Proposition of Taḥrīr-i-Uqlīdes which got second place in JĀME-I-BAHĀDUR KHĀNĪ.

The general objection of the geometers is that construction of the two circles in this demonstration is certainly justified by Postulate 3, but there is nothing in Euclid's first principles which explicitly guarantee that the two circles shall intersect in a point C and that they will not, somehow or other, slip through each other with no common point. The existence of this point, then must be either postulated or proved........ Only by the introduction of some aadditional assumption can be existence of the point C be established. Therefore, the Proposition does not follow from Euclid's first principles, and the proof of the Propositions is invali (35; 321).

Ghulam Husain, (43; 12) and Speusippus, (73; 266) the successor of Plato at the Academy, on the other hand, held that "eometric objects are eternal things, and hence not brought into being; it is better to say that these objects exist". If we follow them, then the circle will meet and Euclid is vindicated of every fault at least through this Proposition.

PROPOSITION 3

We wish to draw a straight line from a given point which is equal to a given bounded straight line.



Like the point A and the line B C;

We join the point A with one of the two ends of the lines, the line is A B; and construct an equilateral triangle A B D on it; and on point B, with distance B C , the circle C E G; and extend D A straight upto the point E, which is on the circumference of this circle; after that we draw on D with distance D E, the circle H E T and extend the arm D A straight upto the point H, which is on the circumference of this circle.

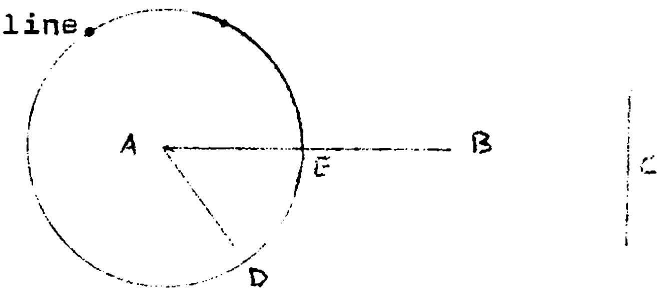
Then the line A H which has been drawn from the point A will be equal to the line B C.

Because the two lines D H, D E which are the radii of the circle E H T, are equal; and as from these we remove two lines AD, BD which are equal, equals AH, BE remain; and B C is equal to B E, being the radii of the circle ECG. Therefore, A H, B C which are equal to B E, are equal. This was required. *

Here we may have a nnmber of different figures by preserving the same method of de-monstration while admitting variations of position of point. The point may be taken either outside the line or on the line, and in the first case, it may be either on the line produced or situated obliquely with regard to it and in the second case, it may be an intermediate point on it. According to Syed Hasan (55 ; 2) the point should be distinct from the line. Mohammed Barkat (19 ; 8) says that the point will neither be the extremity of the line nor situated on it but distinguished from the line.

PROPOSITION 4

We wish to separate from a long straight line equal to a small straight line.

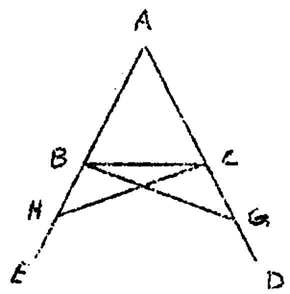


Let A B be the long straight line and C be the smaller one; we draw the line A D from the point A equal to the line C; and construct on the point A, with distance A D, the circle D E. Thus the circumference of this circle separates the line A E from the line A B equal to A D, i.e. equal to C. *

There is an error in the proof as their is nothing to guarantee that a line passing through a point inside a circle, even the centre, will however, for produce, meet the circle.

PROPOSITION 5

The two angles which are above the base of an isesceles triangle are equal and similarly those two angles which are formed below it by producing the arms.



For example, two angles A B C, A C B of the triangle A B C in which the two arms A B, A C are equal and similarly, the two angles C B D B C E which are formed below the base B C after extending the two above mentioned arms towards D E.

For proving the hypothesis, we fix a point G on the line B D and from C E separate C H equal to B G and join the lines B H, C G. I say that in the two triangles C A G, B A H, two sides A C, A G and the angles A are respectively equal to the sides B A, A H and the angle A.

Hence the two sides C G and B H which remain in these two triangles will be equal.

Similarly, the two angles A B H, A C G which mutually correspond and the two angles G, H.

Further we say that in two triangles BGC, BHC, the two sides BG, CG and angle G are respectively equal to two sides GH, BH, and the angle H.

Therefore, the two angles B C G, C D H are equal; and we substract these two equal angles from the angle A C G, A B H, which were equal, the two angles A C B, A B C which are above the base, are left equal. From the above statement the equality of the two angles C B G, B C H which lie below the base, also get established.

The Geometers called this proposition "Mamūnī". ***

According to Proclus (p. 250-20) the discoverer of the fact that in any isosceles triangle, the angles at the base are equal was Thales (47, i; 252) Ghulam Husain says:

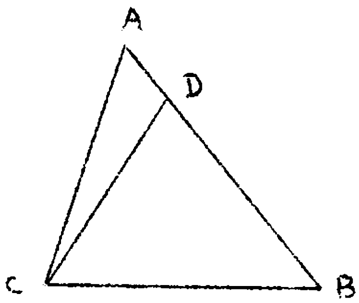
"Geometer call it Mamuni".

Because it is said, according to Ghori, (53; 281)

Mamun al-Reshid had interest specially in Speculative Sciences and Perticularly in the principles of Uqlides. Even he used to have the figure of this Proposition on the sleeves of his Robe, that is why the figure is called "Mamuni".

PROPOSITION 6

If in a triangle two angles are equal, then the two sides respectively facing them will also be equal.



For example, in the triangle A B C, the angles B, C are equal. I say that the two sides A B, A C are also equal and if they are different, then let A B should be longer.

From A B, we separate B D equal to A C;

and join C D;

thus in two triangles A C B, C B D, the two sides A C, B C and the angle A C B are equal to two sides B D, B C and the angle C B D.

Therefore, by proposition 1, both these two triangles are equal. Althouth they are whole and part. This is a contradiction.

Hence the required is proved.

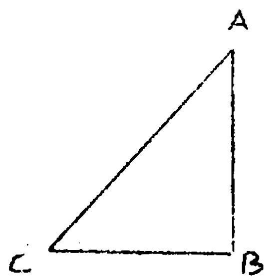
This Proposition is the converse of the preceding Proposition, and also it is the first Proposition in which reductio ad-absurdum is used.

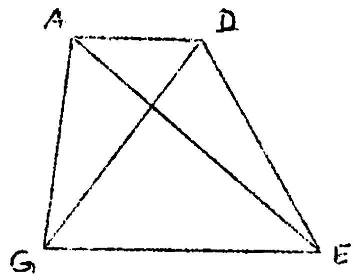
Todhunter points out that I.6, not being wanted till II.4, could be postponed till later and proved by means of I.26 (47,i ; 258). But the author of Mukhtalif Sharh Tahrīr-i-Uqlīdes says that if this Proposition could have been placed after I.18, then it was much easier to prove it, because in I.18, it has been proved that the greater side subtends the greater angle. Then on this basis, when two angles are equal, then their opposite sides will also be equal, other-wise the angles will necessarily be different (55 ; 27).

Ātmā Rām points out that it should be remembered that the smaller side should be cut off from A B from the side of the angle B, otherwise the proof of this theorem with the help of Proposition 4 (= G I. Proposition 1) will be impossible (17 ; 14).

PROPOSITION 7

If three sides of a triangle are respectively equal to three sides of another triangle, then their corresponding angles will also be equal, and triangle to triangle.





For example, in two triangle A B C, D E G, side A B is equal to the side D E and A C to D G and B C to E G.

Therefore, we say that the angle A is equal to the angle D and angle B to E and angle C to G.

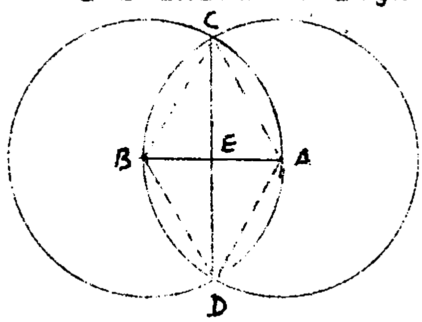
Because if we suppose to superpose the side B C on the side E G and triangle on the triangle. It will not be free from two conditions either point A will coincide with D or not.

In case of coincidence, the result is obvious and in the case of non-coincidence, we join D A; therefore, two triangles E D A, D A E are equal on account of equality of the two sides A E , D E. But the angle G D A is smaller that the angle E D A . Hence it will also be smaller than the angle E A D and the angle E A D is smaller than the angle G A D; therefore the angle G D A will be much smaller than the angle G A D. But it is equal; for two arms A G, D G being equal. This is a contradiction. Hence claim is proved.*

There are certain editions who regard I.7 as being of no use except to prove I.8. But Dodgson points out that "there is a close analogy between I. 7,8 and III.23,24. These analogies give to geometry much of its beauty, and I think that they ought not to be lost sight of". It will, therefore, be appearent how ill advised are those editors who eliminate I. 7 altogether (47,i ; 261). But Ghulām Husain instead of omission, combines these two propositions into one perhaps because of analogy.

PROPOSITION 8

We wish to bisect a bounded straight line.



For example, the line A B.

We draw a circle on A, with length of the line and similarly on B, the circle C A D and join C D which ce certainly bisects A B on point E.

Because when I join faur lines A C, B C, A D, B D, in the two trinagles A C D, B C D, the corresponding equal sides are obtained. Therefore, by preceding proposition, the corresponding angles are also equal;

and because the two angles A CD, B C D correspond will be equal. Thus in two triangles A C E, B C E, the two sides A C, C E and the angle A C E are equal to two sides B C, C E and the angle B C E. Hence by Proposition 1, the two lines A E, B E are equal.

This is what was required. *

This is an alternative method of bisecting a given straight line like that of Proposition 2 by Ghulām Husain. Apollonius also adopted the same method. But the objection to this proof is that, instead of assuming the bisection n of the angle A C B, as already effected by I. 9, Apollonius goes a step further back and embodies a construction for bisecting the angle. That is, he unnecessarily does over gain, what has been done before, which is open to objection. from a theoretical point of view (47,i; 268). But it may be noted here that Ghulam Husain placed this P Proposition 9 after 10, Hence this objection does not apply on this construction.

Euclid bisects a straight line A B as follows:-

Let the equilateral triangle A B C be constructed on it (I. 1) and let the angle A C B be bisected by the straight line C D (I. 9),

then the straight line A B has been bisected at the point D.

For, since A C = C B,

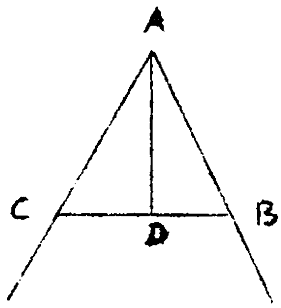
C D is common

So A D = B D (I. 4)

Therefore, A B is bisected at D.

PROPOSITION 9

We wish to bisect a given angle.

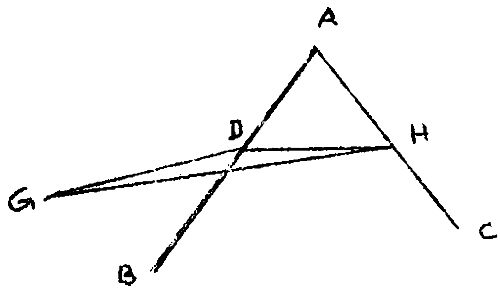


Like the angle A.

So, we fix a point B on one side of it and take A C equal to A B.

We join B C and bisect it at point B with the help of the preceding Proposition and join A D, Hence this line must equally divide into two angles B A D, C A D; because of equality of the corresponding sides of the triangles A D B, A D C and the two angles being corresponding to each other. This is what was required.*

The proof of this Proposition is given by Ghulam Husain. Euclid gives this construction in a different way. First, he takes a random point D on A B and cut off A E from A C equal to A D and joins D E. Further he supposes the equilateral triangle D E F on D E and then joins A F. This straight line A F bisects the angle B A C.
From this construction it can be observed that Euclid does not say in his description of the construction, that the equilateral triangle should be constructed on the side of D E opposite to A and he leaves this to be inferred from his figure. But three cases may be considered according to the vertex of the equilateral triangle falls on A, above A or below it. Again if the equilateral triangle be described on the side of D.E. opposite to A, then its vertex may fall either on one of the line forming the angle or outside it altogether because it is not necessary that A will lie within the line A B and A C. Last two possibilities are answered in the Nasīr al-Dīn Tūsī's Tahrīr-i-Uglīdes (9;8-9) and also in Rekha ganita (50; 15-16) in the following words:-
"The point G must be within the space included by the lines A B and A C. Why? If it is not within this space, it must be on one of the lines or outside the space between the two lines as in the marginal figure.



Then the angles G D H and G H D will be equal, but the angle C H D is equal to the angle B D H. Thus if the point G be on the side P D, the whole angle D H C and its parts D H G are equal. This is absurd.

If the point G be outside the side B D, then the angle G D H shall be greater than the angle B D H. It should also be greater than the angle D H C; because the angles B D H and D H G are equal (I.5). But the greater angle G D H is equal to the angle D H G. Therefore, the part D H G is greater than the whole D H G. This is absurd; because part can not be greater than the whole. Therefore, the point G must be within the space included by the two arms."

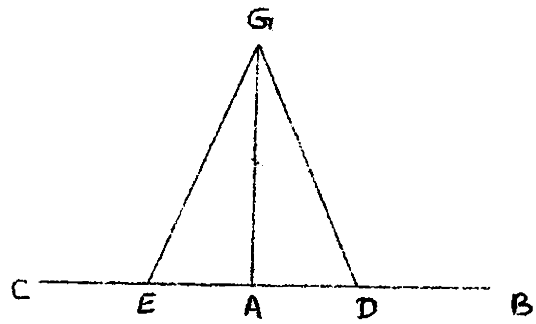
The author of Mukhٰalīf Sharḥ Taḥrīr-i-Uglīdes says (55;3):-

This point should exist in middle of the lines AB, A C, and this statement can be true only when the equilateral triangle be described on D E opposite to the angle D A E. It is impossible that the point will lie on one of the lines neither before nor after extension. It is also impossible that the point will lie on the vertex of the triangle A D E or out of it.

Thus Euclid's method of bisection of an angle creates doubt in the mind of geometers whereas G͟hulām Ḥusain's construction gives unique figure and does not leave any room for doubt.

PROPOSITION 10

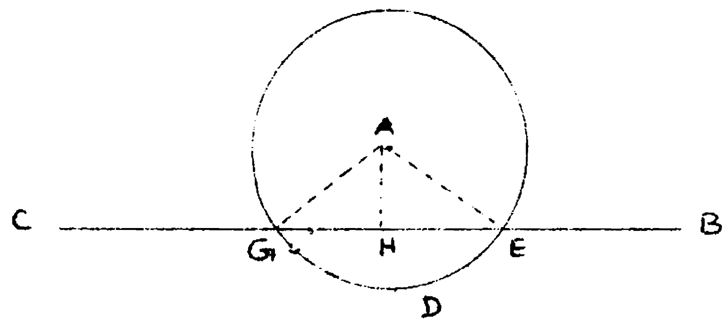
We wish, to draw a perpendicular at a point which is on an unbounded straight line.



Like the point A which is situated on the line B C. From A C, we fix a point D between A, B and separate A E equal to A D and construct on D E on equilaterial triangle D G E and join G A so that this line will be perpendicular to C B; Because in two triangles A G D, A G E, the corresponding sides are equal; hence by proposition 7, the corresponding angles will also be equal; and the two angles G A D, G A E which are corresponding and in two sides of a line A G are obtained will be right angle. Hence the line G A will be perpendicular. *

PROPOSITION 11

We wish to draw a perpendicular to a given unbounded straight line from a point which is not on it.

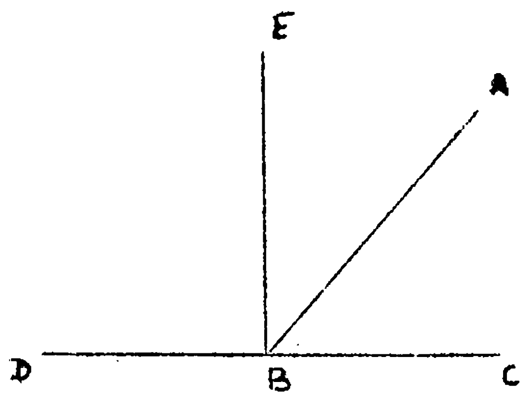


Let A be the point and B C the line.
We fix a point D apposite to A and draw on A, with distance A D, a circle D G E which will inevitably intersect the straight line B C on two points E, G; then we bisect the line E G at point H and join the line A H, which according to the statement of the preceding Proposition will be perpendicular to B C. *

The supposition in this Proposition that, if only D be taken on the side of B C, remote from A, the circle described with A D as radius must necessarily intersect B C in two points. Because, as Ātmā Rām pointed out (17; 16)
We imagine that each one part of the circumsference of the circle will lie in both sides of the line B C and also the circumference is a continuous line, so it manifested that the circumference will cut two times the line B C. Atma Ram further adds that the restriction of unbounded line is imposed because it was possible in certain cases that the circle did not intersect the line at any point or intersected at only one point.
Heath also observes (47, i ; 272) that to satisfy it some postulate of continuity is required e.g. something like that suggested by Killing.
"If a point (here the point describing the circle) moves in a figure which is divided into two parts (by the straight line) and if it belongs at the beginning of the motion, to one part, it must during the motion cut the boundary between the two parts."
and this of course applies to the motion in two directions from D.

PROPOSITION 12

If a straight line be set up on another straight line, it forms two angles on both sides, then they will be either right angles or that one will be acute and the other will be obtuse but together they will be equal to two right angles.

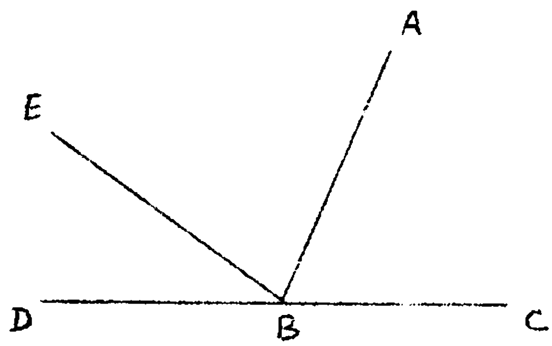


So, the line A B is set up on C D and two angles A B C, A B D are formed;
therefore, if A B is perpendicular to C D, then it is obvious that each two angles formed will be right angles if it is not perpendicular. We draw a perpendicular B E from B; then there will be three angles A B C, A B E, E B D and when we add the second to first, then the sum of the three angles will be two right angles. If we add to the third the same two angles which had earlier been formed, are obtained; hence the sum being equal, establish the two right angles.

Ātmā Rām remarks that in the statement of this Proposition, "the line makes angle with the line", is necessary because if this condition is not imposed here, then there will be another case in which the line may set up at one of the extremities of the other line and in this case only one angle will be formed (17;19).

PROPOSITION 13

If two straight lines meet at a point of a third straight line from its two sides and form with that straight line two angles and their sum equals two right angles, in that case both the lines will meet in the same direction.



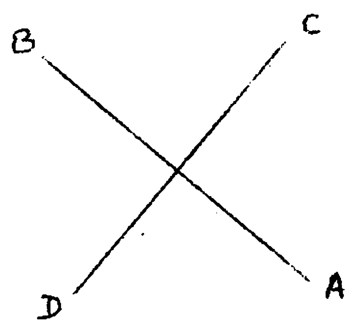
So, the lines C B, D B meet at the point B of the line A B and two angles C B A, D B A equal to two right angles are formed; I say that the line C B D will be single straight line. If it is not so, then the line C B E will be single line and by the preceding Proposition, two angles C B A, E B A, which is equal to two right angles, will be equal to two angles A B C, A B D and when we substract the common angle A B C, the two angles A B D, A B E, the whole and part remain equal. This is a contradiction.
Hence the given statement is proved.*

This is converse of Proposition 12 of 1

It may be observed that the condition of meeting two straight lines B D and C D with the line A B not lying on the same side is necessary in this Proposition. If there is no such condition, then it is possible that the two straight lines which form angles with third straight line would be equal to two right angles, but both the lines would not be in the same straight line as has been shown in the figure, that the straight lines B C and B D make such two angles A B C and A B D with the straight line A B whose sum will be equal to two right angles, but the lines C B and B D will not be in one straight line.

PROPOSITION 14

The two opposite angles which are formed by intersection of two straight lines are equal.



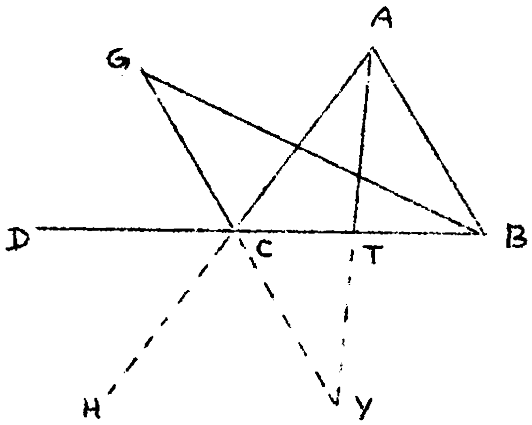
Like the two angles A E D, C E B, that are formed by intersection of the two straight lines A B, C D; and since by Proposition 12, the two angles A E C, C E B are equal to two right angles and similarly the angles A E D, A E C; therefore, by substracting the common angle A E C, the two equal angles A E D, B E C remain and similarly the two angles A E C, B E D are also equal. Hence the claim is proved.

From this statement it is inferred that at a point the angles of any magnitude which are accumulated, will together be four right angles.*

The additional statement which is given in the end of the proof of this Proposition, of course, is not of Euclid but is given in Tahrīr-i-Uqlides as Aqulo (I say). Mohammad Barkat adds that if any number of straight lines intersect at one point, the sum of all the angles formed will be equal to four right angles (19; 3).

PROPOSITION 15

If every triangle from which a side is produced, then the exterior angle formed, will be greater than each of its two interior opposite angles.



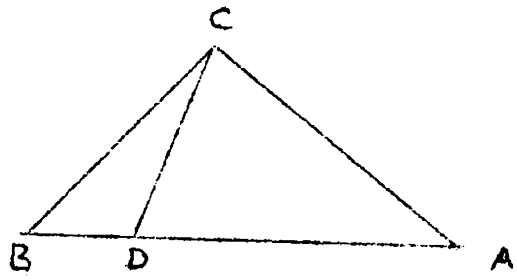
so, in the triangle A B C, the side B C is extended towards D and the exterior angle A C D is formed.
We say that this angle is greater than each of the two interior angles A B C, B A C.
To prove the claim, we bisect the side A C at E and join B E and produce it upto G and we make E G equal to B E and join C G.
Thus in two triangles A E B, C E G, the two sides A E, B E and the opposite angle A E B are equal to two sides C E, E G and the opposite angle C E G.
Hence, by Proposition 1, the remaining corresponding angles of these two triangles will be equal. Therefore the angle B A E will be equal to the angle G C E and the whole angle A C D is greater than the part angle G C E, so it will also be greater than the angle B A C.

Now we extend A C upto H and bisect B C at T and join A T and extend it upto Y and make T Y equal to A T and join Y C.

I say that in the two triangles A T B, Y T C, two sides A T, T B and angle T are equal to two sides Y T, T C and the angle T respectively. Thus the angle A B T will be equal to angle Y C T and the angle H C B, i.e. the angle A C D is greater than the angle Y C T; hence it will also be greater than the angle A B C. This is what was required.*

PROPOSITION 16

Every angle of a triangle whose opposite side is longer will be greater than the angle whose opposite side is shorter.



For example, in the triangle A B C, the side A B is longer than A C. We say that the angle C will be greater than the angle B. We separate A D equal to A C from A B and join C B. It is clear that the whole angle A C B is greater than the angle A C D i.e. the angle A D C and angle A D C is an exterior angle of the triangle B C D, will be much greater than the angle B and hence the angle A C B will be much greater than the angle B. This is what was required.*

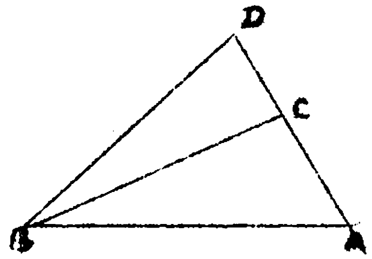
It is a general practice of Euclid to give only one case and leave the others, here also he does the same. Nasir al-Din Tusi too does not prove the Proposition if the side A C be extended. The author of Mukhtalif Sharh Tahrir-i-Uglides (55; 3) and Mohammad Barkat have (19 20) given complete proof of the proposition. Ghulam Husain gives the same proof.

After having proved the Proposition, Nasir al-Din Tusi remarks:-

From this (Proposition) it is also inferred that if two straight lines drawn from the same point intersect a third straight line, then the two angles formed on the same side of the line shall never be equal (9; 11).

Jagannath Samrata says that the direction here is to be taken from the lines drawn from the point and he shows (50;23) "from the point A, draw the lines A B and A C and they meet the line B D in the points B and C. Then the angles A B C and A C B formed in the same direction, shall not be equal. Because the triangle A B C is formed by the meeting of the three lines (AB, AC and DC). The exterior angle A C D is greater than the angle A B C. This is proved.' Therefore, what is stated is proved". This is in fact, another way of putting the Proposition 15.

Mohammad Barkat (19,20) and the author of Mukhtalif Sharh Tahrir-i-Uqlides (55 ; 3) proved this Proposition in a different way as follows:-

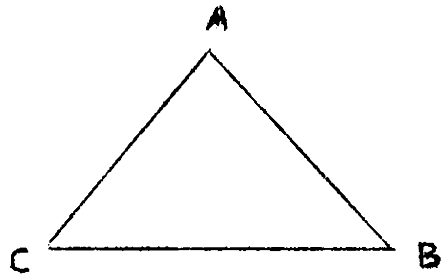


Produce the line A C upto D and make A D equal to A B and join B D;

Since angle A C B is greater than the angle C D B, because the former angle is the exterior angle of the triangle B C D; and the angle A D B is equal to the angle A B D being equal arms A D, A B, by construction; And the angle A B D is greater than the angle A B C. Therefore, the angle A C B will be much greater than the angle A B C. This is what was required.

PROPOSITION 17

In a triangle the opposite sides of any greater angle will be longer than the opposite side of the smaller angle.



So, in the triangle A B C, angle A is greater than the angle B.

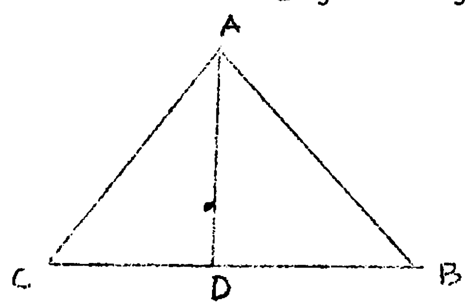
We say that B C is longer than A C.

If it is not longer than it, then it will be equal; and it is inevitable that either angle A will be equal to B by Māmūnī's proposition, or smaller, and it is necessary that the angle A will be smaller than the angle B by preceding Proposition and both contradict each other. Hence the supposed angle A is greater.

So the objective is proved.*

PROPOSITION 18

A sum of two sides of a triangle is greater than the third.



For example, in the triangle A B C the sum of two sides A B, A C is greater than B C.

We bisect the angle B A C by the line A D and say that the angle A D B which is exterior to the triangle A D C, will be greater than the interior angle D A C, that is to say, the angle B A D.

Hence in the triangle A B D, side A B which is the opposite side of the greater angle, will be longer than the side BD, which is the opposite side of the smaller angle and similarly the exterior angle A D C of the triangle A D B is greater than the angle D A B, i.e. the angle D A C. Therefore in the triangle A D C the side A C will be longer than C D. Thus the sum of A B, A C will be greater than the sum of B D, C D which is B C.

The geometers call this Proposition "Hamārī." *

It is said that this Proposition is evident even to an ass and requiring no proof. But Proclus replies truely that a mere perception of the truth of the theorem is a different thing from a scientific proof of it and a knowledge of the reasons why it is true (47,i ; 287).

Some geometers viewed that this Proposition should be placed in axioms but Ātmā Rām commented that the number of axioms should be minimum as far as possible and because of this, this Proposition and some others, although obvious, have not been placed in axioms and are proved with the help of logic (17 ;22).

This is an alternative method of proving the Proposition which is attributed to Nasīr al-Dīn Tūsī. In this proof it may be noted that the Proposition is proved without producing any of its sides.

Another proof which is not free from interest in which the author of Mukhtalif Sharh Tahrīr-i-Uqlīdes used reductio ad absurdum method.

Suppose A C, A B are not greater than B C, but equal. Then extend B A in the direction of B equal to the magnitude of A C and join D C.

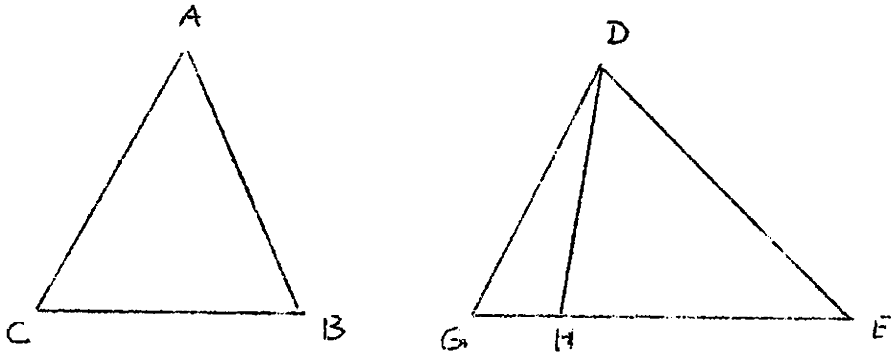
Then B D will necessarily be equal to C D. Hence the angle B C D is equal to the angle B D C. But A C D equals ADC. Therefore the angle B C D will be equal to

A C D, because each of them is equal. But the angle ACB is smaller than BCD, which contradicts the proof. Hence our supposition is false. (55; 3).

But here it may also be noted that he does not show other cases that the sum of A B, A C is less than B C.

PROPOSITION 19

If two angles and one side in a triangle are equal to two angles and one side of another, then the remaining corresponding sides and angle will be equal and triangle to triangle.



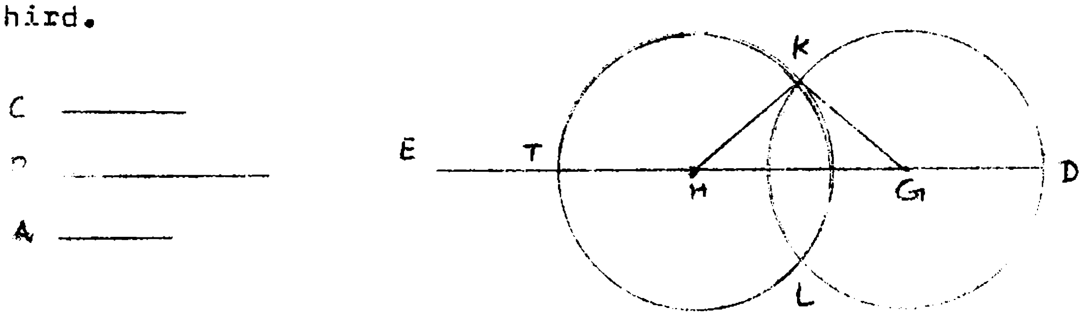
Let in two triangles A B C, D E G, the angle A be equal to the angle B and C to D. Let first two sides BC, GE be equal which are between two given angles.
We suppose the side BC superpose on E G and triangle on the triangle; then the angle A coincides with the angle E, being equal and similarly the angle C with the angle G. Thus A B coincides with D E and A C with D G, and point A with point D.
This is what was required.
If two sides AB, D E are equal, then we suppose superposition of A B on D E and triangle on the triangle; no doubt, the angle B will concide with angle E, being equal; and line B C with the Line E G and hence it follows necessarily the point C also concides with the point G; because

if it falls elsewhere like the point H, then it is necessary that the two angles D H E, D G E which are exterior and interior angles of the triangle D H G, are equal. This is a contradiction. Hence the object is proved. If two sides AC, D G be equal, then similarly the hypothesis is proved.*

This is one of the three "congruence" theorems for triangles in Book I, the other two are 4 and 8 (= Proposition 1 & 7). Euclid proved Proposition 4 and 8 by method of superposition but failed in 26. This constitutes one piece of evidence for a widely held thesis that Euclid found something unsatisfactory in this method of proof (60;22). But other geometers like Naṣīr- al-Dīn Ṭūsī give an alternative proof of this Proposition using superposition method. Here Ghulām Ḥusain also applies the same as an alternative proof.

PROPOSITION 20

We wish to construct a triangle whose three sides will be equal to three given straight lines with the condition that the sum of any two sides should be greater than the third.

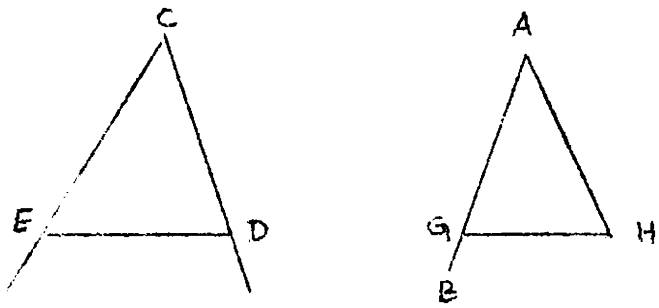


For example, the given lines are A, B, C.
Suppose another straight line D E which is bounded only at D; and from it we cut off D G equal to A and G H equal to B and H T equal to C. We draw on G, with distance D G, the circle D K L and on H with distance H T another circle T K L. These circles intersect at two points K, L. Join one of the two intersecting points, e.g. the point K with the two centres of the circles by the lines G K, H.K.
Hence the required triangle is H. K G. Because the side G K i.e. D G is equal to A and G H equal to B and H. K i.e. H T equal to C.*

It may be observed here that Euclid does not give any reason why the described circles will meet if the condition that any two of the straight lines A, B, C are together greater than the third be fulfilled. But Naṣīr al-Dīn Tūsī (38) (9;13) as also incorporated in Rekhā Ganīta (50;29) answered this question that it has already been demonstrated that "the sum of any two sides of a triangle is greater than the third side (Proposition 18). Therefore, the two circles cut each other. Further, he says that if the sum of A and B be not greater than C, then the line H T shall either be equal to or greater than H D. Then the circle K T L will lie within the circle K D L and will touch when H T is equal to H D and it will not meet in the other case. If the sum of the lines B, C is not greater than A, then the circle K D L will make the circle K T L fall within it, and if the sum of the lines A and C is not greater than B, then the line G H shall be equal to or greater than the sum of the lines H T, G D. Then too, the circle shall not touch. Thus then one circle shall not make another fall within it, but the two circles shall stand and separate if G H is greater than the sum of H T and G D.

PROPOSITION 21

We wish to draw an angle at a given point on straight line equal to a given angle.



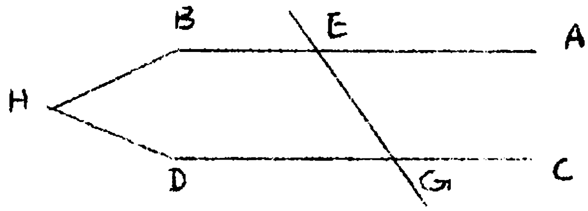
For example, at point A in the straight line A B, equal to the angle C.

We fix two points E, D on the two arms of the angle and join D E and on A B including the point A, we construct a triangle whose sides are equal to the sides of the triangle D C E and that triangle is A G H such that the side A G will be equal to C E and A H to D C and G H to E D. Therefore, by Proposition 7, the angle A constructed will be equal to the angle C.*

In this Proposition this fact has not been noticed that the construction of the triangle is not exactly the same as given in Proposition 20. Heath pointed out that we have here to construct a triangle on a certain finite straight line A G as base, in I. 22 (Proposition 20), we have already only to construct a triangle with sides of given length without any restriction as to how it is to be placed. Thus in I.22 we set out any line whatever and measure successively three lengths along it beginning from the given extremity, and what we must regard as the base is the intermediate length, not the length beginning at the given extremity of the straight line arbitrarily set out. Here the base is a given straight line abutting at a given point.

PROPOSITION 22

If two straight lines on which another straight line be set up and two alternate angles of the interior angles formed, are equal, then these two lines will be parallel.

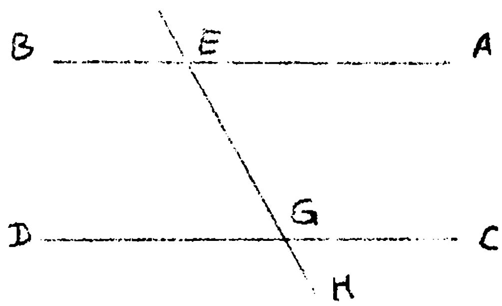


So, on two straight lines A B, C D, a line E G is set up and two alternate angles C G E, B E G are equal.
We say that the lines A B, C D are parallel.
Because if between them parallelity is not established, then in one side they meet after extension, e.g. in the direction B, D, at the point H.
Thus a triangle E G H is formed and one of the two alternate angles which is C G E is the external angle of the triangle, and contrary to this another one which is B E G is interior and these interior and exterior angles are equal; but this implication is impossible according to the Proposition 15.
Hence these two lines A B, C D will never meet; therefore, they are parallel. *

Upto this Proposition Ghulām Husain has dealt mainly with triangles, their construction and their properties in the sense of the relation of their sides and angles and comparison of different triangles in respect of their area, in the particular cases where they are congruent. Now with this proposition begins second section of this chapter which establishes the "Theory of Parallels ".

PROPOSITION 23

If a line be set up on two straight lines and exterior angle is equal to the interior angle or two interior angles in the same side are equal to two right angles, in both cases those two lines will be parallel.



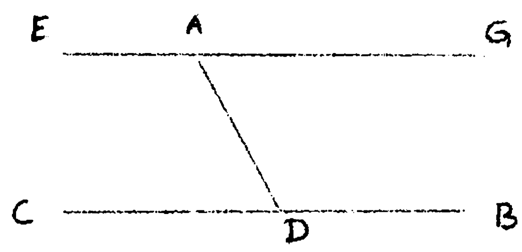
Let the line E G H be set up on the two straight lines A B, CD and the exterior angle HGD, say, is equal to the interior angle B E G and the two interior angles D G E, B E G together are equal to two right angles.

We say that the lines A B, C D are parallel.

Because, exterior angle D G H is equal to the interior angle B E G and the angle C G E which is equal to the angle H G D will also be equal to the angle B E G. Thus the alternate angles are equal. And also if the angles D G E, and B E G together are two right angles, then the two alternate angles will also be equal. Because the angle C G E and also the angle D G E are equal to two right angles. Hence it is inevitable that the said two alternate angles are equal, as the angle C G E together with the angle D G E is also equal to two right angles. Therefore, by preceding Proposition, the two lines AB, C D are parallel.*

PROPOSITION 24

We wish to draw a straight line from a given point which will be parallel to a given straight line.

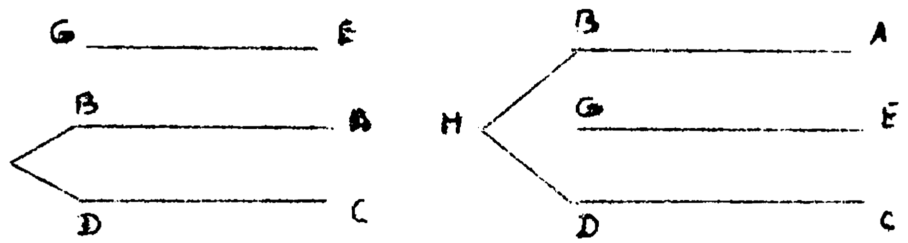


Let A B a given point and B C be the line.

We fix a random point D in the line B. C and join A D and draw an angle D A E at the point A with the line A D equal to the angle A D B and extend the line A E towards G, This line E G will be parallel to B C, by Proposition 22. Because the two angles A D B, D A E are alternate.*

PROPOSITION 25

Two straight lines which are parallel to a straight line, will also be parallel to each other.



For example, two straight lines A B, C D are parallel to the line E G, then they will also be parallel to each other. If E G is situated between these two lines, and if A B, C D are not parallel, then they will meet in one side, say at point H in the direction of B, D. This implies that when E G be extended towards G, definitely it will meet one of the two lines A B, C D; although, they were parallel. This is a contradiction.

Hence the lines A B, C D will not meet in reality and will be parallel.

If E G is situated in one side, then also we say if A B, C D are not parallel, then they will meet at point H. It is clear that these two lines will converage to H, and diverge in the side of A C; and since the line A B H is parallel to E G, so they will neither converge nor diverge. But the line C D H is converging and diverging along with the line A B H. Therefore, it should also be same with the line E G. Although by supposition it is parallel. This is a contradiction. Hence AB, CD do not meet and therefore, they are parallel.*

The present proposition establishes the transitive property of parallel lines.

This logical proof of the proposition is given by Ghulam Husain, in which he used only definition of parallel lines. Euclid while proving it, used Proposition 29 that "a straight line falling on parallel straight lines makes the alternative angles equal to one another; the exterior angle equal to the interior and opposite angle, and the interior angles on the same side equal to two right angles, " and in which he found it necessary to introduce first time the most controvertial Parallel Postulate.

It may be interesting to note here that Carl Friederich Gauss (1777-1855), adopted the same method as Ghulam Husain, but he used his own definition of parallel lines. According to him: "If the coplanar straight lines A M, B N, do not intersect each other, while, on the other hand, every straight line through A between A M and A B cuts B N, then A M is said to be parallel to B H". He proved as follows:

If the line (1) is parallel to the line (2) and to the line (3), then (2) and (3) are parallel to each other.

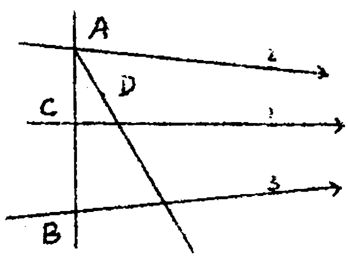
Case (i), Let the line (1) lie between (2) and (3).
Let A and B be two points on (2) and (3), and let A B cut (1) in C. Through A let an arbitrary line A D be drawn

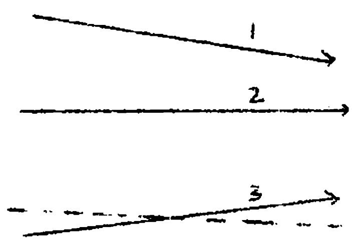
between A B and (2). Then it must cut (1), and on being produced must also cut (3).

Since this holds for every line suchas A D, (2) is parallel to (3). Case (ii), Let the line (1) be outside both (2) and (3), and let (2) lie between (1) and (3).

If (2) is not parallel to (3), through any point chosen at random upon (3), a line different from (3) can be drawn which is parallel to (2).

This, by case (i), is also parallel to (1), which is absurd. (23 ; 67,72)





Labachevsky also proves on the same line as Ghulam Husain did but with the help of his own non-Eucli-dean definition of parallel lines (34 ; 81,95):
"a line a' is said to be parallel to a line a if a' is the boundary line for the set of lines passing through some point of a' and not intersecting a".

For proving the proposition, he assumes that the lines a and b are parallel in the same direction to the line c, and that a and b do not intersect (otherwise through their co-mmon point there would pass lines parallel to c in the same direction, which is impossible).

To prove that a and b are parallel, we consider two cases:

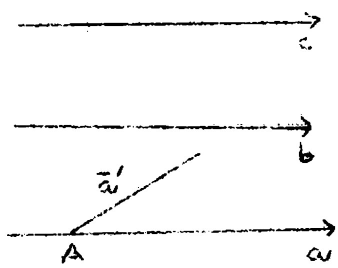
(1) the lines a and b are on the same side of c;

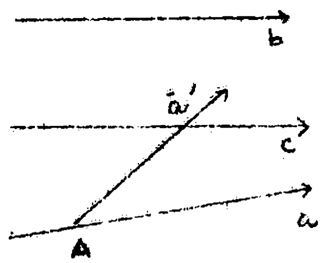
(2) the lines a and b are on the opposite sides of c.

In the first case either a or b lies in the interior zone of the plane determined by the other line together with c. Assume, for instance, that b lies in the interior zone relative to a and c.

Take any point A on a and denote by a the ray on a which originates A and points in the direction of parallelism of lines a and c. We must show that $\bar{a}$ is the boundary of the set of rays originating from A and not intersecting b. Assume the contrary, that is, let there exist a ray $\bar{a}$ which originates from A in the direction of parallelism (that is, which lies on that side of the perpendiculars from A to b and c on which the parallel lines lie) and lies closer to b than the ray $\bar{a}$ but does not meet b. But then, the ray $\bar{a}$, cannot meet either (by a lemma), which contradicts the fact that a and c are parallel, because in this case $\bar{a}$ will not be a boundary for the set of rays originating from A and not intersecting c.

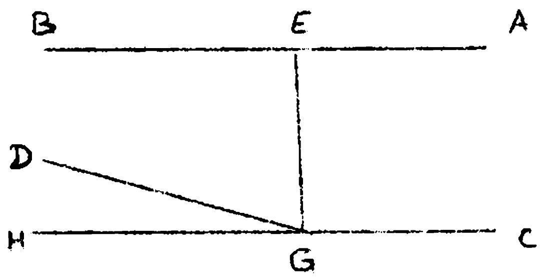
Let us now consider the second case. Assume that a and b lie on the opposite sides of c; then b and c lie on the same side of a. Draw a ray $\bar{a}'$ through any point A on a so that it is closer to b and c than a and passes on that side of the perpendiculars from A to b and c on which the parallel lines lie. Since a and c are parallel, the ray $\bar{a}'$ intersects c, and since c and b are also parallel, this ray intersects b too. Thus $\bar{a}$ is the boundary line for the set of rays passing through A and not intersecting b; accordingly, a and b are parallel (in the same direction in which they are parallel to c).





PROPOSITION 26

If on two straight lines another line be set up and two interior angles which are in the same side, are formed, be less than two right angles, in this case, if those two lines be extended in the same side, then they will certainly meet together.



Let the line E G be set up on the two straight lines A B, C D and two angles B E G, D G E be less than two right angles;

I say that if the straight lines A B, C D be extended towards B, D, they will certainly meet together;

for, if they do not meet they will be parallel and since the angle D G E together with the angle B E G is less than two right angles and the angle A E G together with the angle B E G is equal to two right angles;

therefore, the angle B G E will be less than the angle A E G. We draw an angle E G H at point G with the line E G equal to the angle A E G.

Since these two alternate angles are equal; therefore, the line H G will also be a prallel to the line A B.

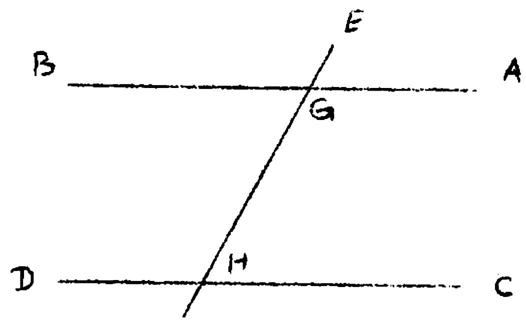
Hence two lines C D, H G which are parallel to A B, by preceding Proposition, are parallel; that they meet at point H G is a contradiction.

Hence C D will necessarily meet the line A B;

and the entire statement is proved.*

PROPOSITION 27

If a straight line be set up on two parallel straight lines, then two internal angles in the same side will be equal to two right angles and alternate angles will also be equal to one another and internal angles will be equal to external angles.



Let two straight lines A B, C D be parallel and a line E G H is set up on them.

We say that the two angles B G H, D H G are equal to two right angles;

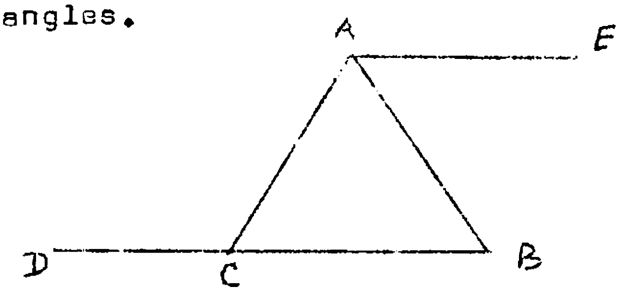
Because if they are less than two right angles, then it is necessary that these two lines will meet in the direction of B, D, be preceding Proposition and if they are greater than two right angles, then certainly the angles A G H, C H G, which are internal angles in second side, will be less than a right angle; because all the four angles being four right angles; therefore, the two lines in the direction of A C, will meet.

This is a contradiction.

Hence the Proposition is proved.*

PROPOSITION 28

In only, triangle if one side of it be produced, the exterior angle so formed will be equal to the sum of the two opposite interior angles and the sum of the two interior angles will be two right angles.



Let the side B C of the triangle A B C be extended upto D.

We say that the exterior angle A C D is equal to the sum of the two interior angles A, B.

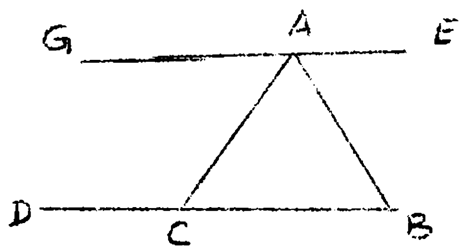
We draw a line A E from the point A parallel to B D.

Thus it is evident that the two alternate angles C A E, A C D which are obtained by setting the line A C on the two parallel lines A E, B D, are equal, by preceding Proposition. Similarly the two alternate angles E A B, A B D are also obtained by setting the line A B, are equal. Therefore, the angle A C D, which is equal to the angle E A C, will be equal to the sum of the two angles E A B, B A C, i.e. the sum of the two angles A B C, B A C, and since it is clear that the angle A C D together with the angle A C B is equal to two right angles. Hence the sum of the two angles A B C, B A C together with the angles A C B will also be equal to two right angles.*

Another alternative proof which is given in a Sharh (55;6) is, in fact, no less elegant than that given by Euclid, and this is indeed an advancement on the above figure. But in the proof later part of the proposition has been supposed.

Let A B C be a triangle.

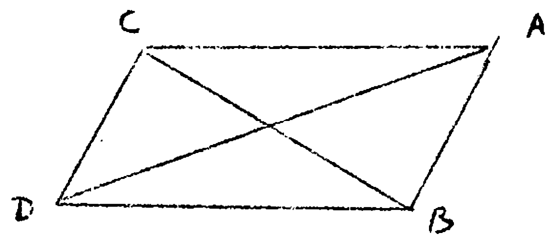
Extend B C upto D and through A draw the line A E parallel to the line B C and then extend it upto G.



Now we say that the angles G A C and A C B are alternate; and both the angles A and C are equal to two right angles; and since the sum of three internal angles of triangle is equal to two right angles. So when we add the angle A C B with the angles B A C and A B C, that will be equal to two right angles and also with the angle A C D. Therefore, external angle is equal to two internal angles, which is required.

PROPOSITION 29

If two straight lines are equal and parallel, then the lines joining both the extremities, which are in the same direction are also equal and parallel.



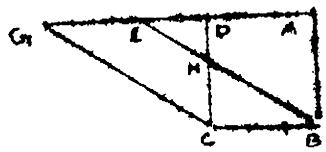
For example, A B, C D that are equal and parallel straight lines and their extremities are joined by the two straight lines A C, B D, then these two straight lines will also be equal and parallel.

Join, B C. So in two triangles A B C, B C D, the sides AB, B C are equal to two sides C D, B C and angles A B C, B C D are alternate angles;
therefore, the remaining sides and angles of these two triangles are equal. Hence A C, B D are equal and being equal alternate angles A C B, C B D, they are also parallel. This is what was required.*

From this description, it is clear that the sides and the alternate angles of a parallelogram are equal and their diagonals are their bisectors.

PROPOSITION 30

If two parallelograms which are on the same base and the same side and between two parallel lines, then they are equal.



For example, the two parallelograms A B C D, E B C G are situated on the same base B C between the two parallel lines B C, A G are equal.

Because the lines A D, E G are parallel to the line B C and so by preceding Proposition they are equal to one another and if we take E D as common, then A E equals D G.

In triangles B A E, C D G, the two sides A B, A E and interior angle B A Eare equal to two sides C D, D G and exterior angle C D G.

Thus these two triangles are equal;

and in these two triangles, the angle H D E is common.

If we subtract it, the two equal trapeziums BADH, CHEG are remained.

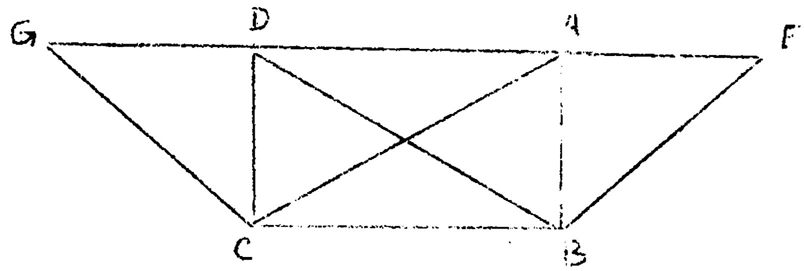
If we add the triangle B H C in both, the trapeziums equal to the aforesaid surface are obtained.

This is what was wanted.*

This Proposition introduced a concept of equality between figures and here equality is in the sense of congruence, as applied to straight lines, angles and triangles.

PROPOSITION 31

If two triangles are on the same base and between the same two parallel lines, then they will be equal.



Let A B C, C B D be two triangles on the base B C, between two parallel lines B C, A D.

For proving equality, we draw a line B E parallel to the line A C and C G parallel to D G till they meet the line A D at two points E, G;

and then two parallograms A C B E, D B C G are formed on the base B C between two parallel lines B C, E G.

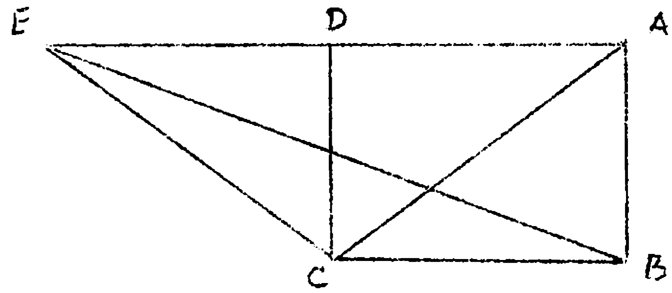
Hence by preceding Proposition, these two sufraces will be equal and by Parism of the Proposition 29 each two triangles which are half of those surfaces, are also equal; therefore, they will mutually be equal,

This is what was required.

It is important to observe that Ghulām Ḥusain like others does not show how the lines B E, C G will meet the line A D. Mohammad Ābīd (85;6) explains that the reason of the lines meeting at points G, E is that the lines E G, B C are parallel and therefore, the angles E A C and A C B will be equal to two right angles and the angle A B E will be equal to the angle C A B; because C A, B E are parallel. Therefore, the angle E B A and A B C together will be equal to the angle E A C and will be less than two right angles. Hence the lines A E, B E will meet at any point.

PROPOSITION 32

If a parallelogram and a triangle which are on the same base in the same side, between two parallel lines, then the surface will be twice of the triangle.



For example, the surface A B C D and the triangle C B E are situated on the base B C between two parallel lines B C, A E.

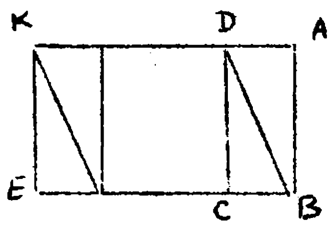
We join A C. In this way triangles A B C, C B E which are on the base B C between two parallel lines B C, A E are equal.

It is evident that the surface A B C D is twice of the triangle A B C; hence it will also be twice of the triangle C B E.

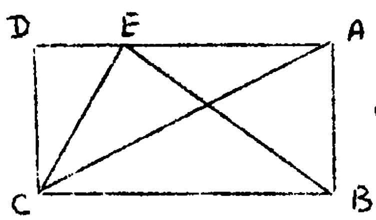
Be it known that in this Proposition and two Propositions earlier that if two surfaces or two triangles on a surface and a triangle are on the two equal bases, by the minor amendments, the same statement will be true.*

This Proposition has two cases, because of the same base, the triangle may have its vertex outside the parallellogram or within. The first case has been proved above if the base of the figures are equal and common, but if the base is equal but not common, then in this case also the Proposition may be proved. Mohammad Barkat has proved it as follows (17;53).

The parallellogram A B C D and the triangle K H E are in the same side of the equal bases B C, H E between the same two parallel lines A K, B E. Join BD. The surface A B C D will be double of the triangle D B C and the triangle D B C is equal to the triangle K H E, by Proposition 37. Then the surface A B C D is double of the triangle K H E.



(i)



(ii)

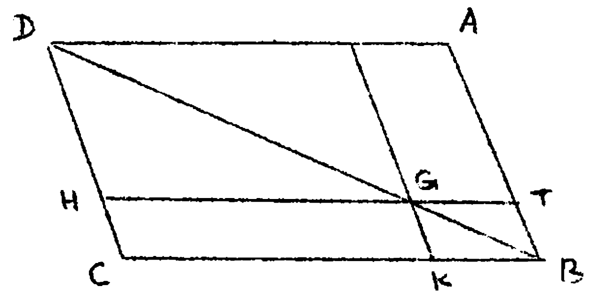
The figure for the second case is as under:-

The parallellogram A is double of the triangle A C D (I.34). But the triangle A C D is equal to the triangle E C D (I.37). Therefore the parallellogram A C is double of the triangle A C D.

Ghulam Husain combines Proposition I.41 and 36 into one and remarks that it is inferred from this and the earlier two Proposition that if two surfaces are two triangles on a surface and a triangle are on two equal bases, between two parallel lines, then by minor amendments the same statement will be true.

PROPOSITION 33

Two compliments will always be equal and the compliments are two parallelograms which are situated in another parallelogram on the two sides of its diagonal, then they meet at a point of the diagonal and have common two angles with the surface.



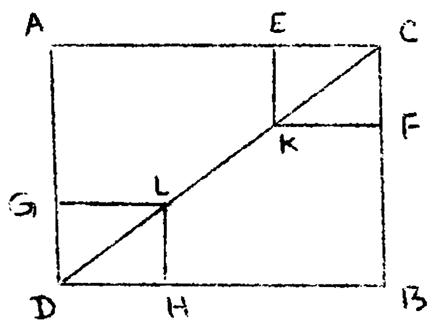
For example, the two parallelograms A T D E, G K C H which are situated in the surface A B C D on either side of the diagonal B D and meet at point G of the diagonal and have two common angles A, C of the surface A B C D;
then we say that both these two surfaces are equal.

Because the diagonal B D bisects the surface A B C D into two triangles A B D, C B D and similarly it also bisects the surfaces E G H D, T B K G.
Because when we subtract from the two triangles E GD, HDG which are equal and two triangles T B G, K G B which are also equal, the two equal compliments remain.

This is what we wanted.*

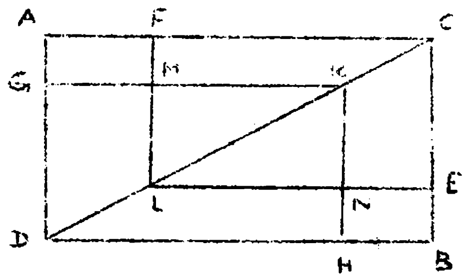
Euclid gives a general form of statement in this Proposition but in proving it, he considers as a particular case. Nasīr al-Dīn Tūsī and Ghulām Husain too modified the statement, as the particular case, as has been stated above and they proved it accordingly.

But, in fact, this Proposition has three cases. The parallellograms about the diameter may either touch one another in a point or are severed from one another by a certain part of the diameter or cut one another. The first case is the one proved above. The second case is represented by the following figure:- (50;44)



In this case the complements A G L K E and B F K L H are not parallellograms. But they can be proved to be equal in the same way as in the first case.

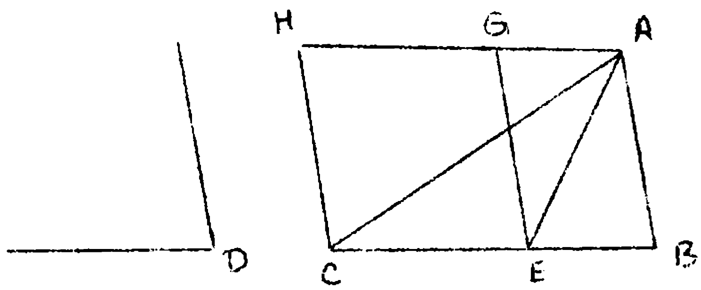
In the third case the parallellogram E F cuts the parallellogram G H. In this case the trapezium G M L D may be proved to be equal to the trapezium L N H D and finally the complements AM to the complement EH.



It may be noted that in each of the three cases the parallellogram about the diameter may not have one angle common with the whole parallellogram; still the demonstration in each of these cases will be the same.

PROPOSITION 34

We wish to construct a parallelogram equal to a given triangle and whose one angle will be equal to a given angle.



For example, the triangle A B C and the angle D.

We bisect E B C at E and join A E.

By Proposition 21 construct at point E with the line C E the angle C E G equal to the angle G and draw a line A H from the point A parallel to C E which undoubtedly meets the line E G at point G;

because of these two lines being drawn from the line making two angles which are less than two right angles.

We draw a line from the point C, the line C H parallel to E G till it meets the line A G after producing upto H.

Thus the parallelogram G E C H forms which along with the triangle A E C is on the base E C, between two parallel straight lines C E, A H;

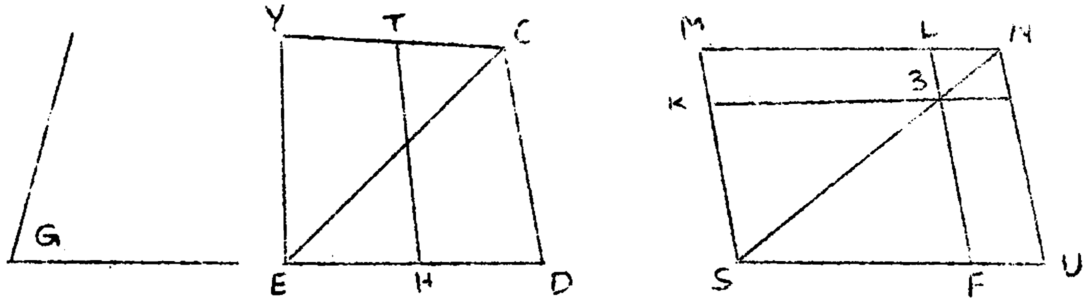
therefore, it will be double of the triangle A E C and the triangle A B C is also the double of the triangle A E C;

because the triangles A B C, A E C are on the two equal bases B E, E C, between two parallel lines B C, A H, will be equal.

Hence the surface G E. C H and the triangle A B C, that both are double of the triangle A E C, will be equal and the angle G E C of the surface is equal to the angle D.*

PROPOSITION 35

We wish to construct on a given line a parallelogram which is equal to a given triangle and one of whose angles will be equal to a given angle.



So the given line is A B, the triangle C D E and the angle G.

Hence first, with the help of the preceding Proposition, we construct a parallelogram H E Y T which will be equal to the given triangle and the angle H will be equal to angle G. Later we produce the line A B straight upto K and make K B equal to H T and draw an angle K B L at point B with the line B K, equal to the angle E H T and make B L equal to D H and draw the lines M K, M L from the points K, L parallel to the lines B K, B L, till they meet at point M and it is obvious that the surface A L will be equal to the surface H E Y T. Later we complete the parallelogram A B L N and join M N and produce the two straight lines N B, M K in the direction of B, K till they meet at point S and from S draw the lines S U parallel to M N and produce N A, L B till they meet the line S U at two points U, F.

Thus the surface B U constructed on the line A B will be equal to the triangle C D E and angle U equal to G.

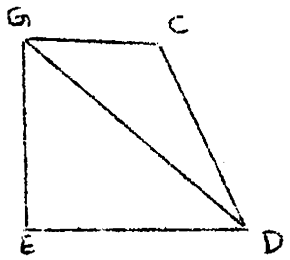
Because two surface B U, B M become complimentary. Therefore the surface B U will be equal to B M, i.e. the surface H Y or equal to the triangle D C E and the angle U i.e. the angle A B F on the angle L B K is equal to the angle G.

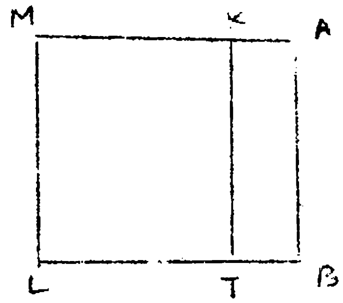
This proposition is Euclid's first example of the "application of areas", one of the most powerful tools of the Greek mathematicians. It is a geometrical equivalent of certain algebraic operations. This method is employed in I,345, II.5, 6 and 11 and VI. 27, 28 and 29 and is "ancient" being discoveries of the must of the Pythagoreans. (81;424) This transformation of areas which can of course be resolved into triangles, by a simple parallellogram, having one side equal to any given straight line and one angle equal to any given rectineal angle. Most important of all such parallellograms is the rectangle, which is one of the simplest forms in which an area can be shown. (47,i;347)

Sulbasutra geometry also evolved purely demonstrative methods for transforming one geometrical figure into another, more especially the square into other equivalent geometrical figures like;: a rectangle into a square, a square into a rectangle, rectangle or square into a trapezium with the shortest parallel side given, an isosceles triangle equal in area to a given square and vice versa etc.

PROPOSITION 36

We wish to construct on a given line a parallelogram equal to a given rectilineal surface whose angle is equal to the given angle.





Let the line be A B and the surface C D E G and the angle H. We dividie the surface into two triangles e.g.C B G, D G E and construct on the line A B the surface A B T K equal to the triangle C D G in a way such that the angle A is equal to the angle H and construct on A T the surface M T equal to the triangle D G E and angle K of it equal to the angle A. In this case the line A K M will be the same straight line; because the angle A K T with the angle A is equal to two right angles.

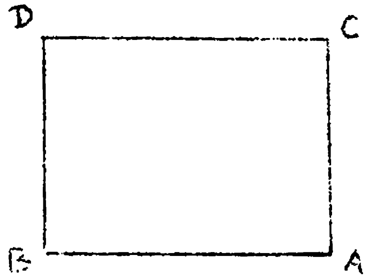
Hence together with the angle T KM will also be equal to two right angles;

and as a result the line B T L will be the same straight line; and the surface A L which is constructed on the line A B will be equal to the surface C E and the angle A is equal to the angle H.

This is what was required.*

PROPOSITION 37

We wish to construct on the given straight line a square.



Let A B be the given line.

We draw a perpendicular A C on the line A B at point A, equal to A D and produce two lines B D, C D, from the points B, C parallel to A B, A C till they meet at point D.

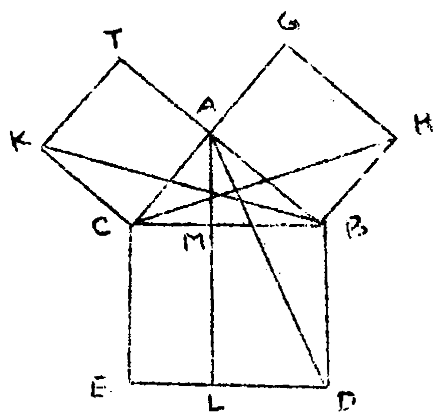
Thus the square A D is obtained;

because of the angle A being a right angle, the angle B will also be right angle. According to the Proposition 27 and 29, the two angles C, D are equal to two angles B, A, then they are also right angles and also the line C D will be equal to A B and B D to A C;

therefore, all the four sides will be equal. *

PROPOSITION 38

In any right angle triangle, the square on the hypotenuse is equal to the sum of the squares on the two sides.



For example, in the triangle A B C, the angle A is right angle. We say that the square of the opposite side B C is equal to the sum of the two squares of A B, A C.

We construct on all the three sides the squares B D E C, B H G A, A T K C and because of the angles G A B, B A C, C A T being right angles, each of the two lines B A T, C A G will be the same straight line, by Proposition 13.

From the point A, we draw the line A M L parallel to the line B D which cuts the line B C, D E at two points M, L;

So the square of the hypotenuse is divided into two parallelograms B L, L C.

We join H C, A D.

We say that in the two triangles D C H, A D B, the two sides B H, B C are equal to the two sides A B, B D and the angle H B C is equal to the angle H B D;

because these two angles are formed by adding the angle A B C in each right angles. Then by Proposition 1, both these triangles are equal. And there is no doubt that the triangle B C H and the square A B H G are situated on the same base B H, between two parallel straight lines H B, G C. Therefore, by Proposition 32, the square H B G C will be double of the triangle H C B; and the triangle A D B and the surface B M L D are situated on the base B D, between two parallel straight lines B D, A L. So the surface B M L D will also be double of the triangle A D B.

Thus the square A B H G will be equal to the surface B M L D which is a part of the square on the hypotenuse.

For equality of their half i.e. the two above mentioned triangles after joining the two lines B K, A E, we similarly prove that the square A T K C will be equal to the surface L E C M which is the second part of the square on the hypotenuse. Now the square on the hypotenuse became equal to the squares on the two sides.

This is what was required. This Proposition is called Bride's Proposition (Shakl-e-Urūs).*

This Proposition is attributed to Pythagoras (540 B.C.) Although Egyptians, Babylonians, Chinese and we our-selves were aware about this proposition, much earlier than Pythagoras himself. The Chineses knew the numerical relation for the particular case $3^2 + 4^2 = 5^2$ probably in the time of Choukong (d. 1105 B.C.) (33; 133). Babylonian records contain a list of Pythagorean number triplets and in Egypt as early as 2000 B.C., a special case of it were known. Now the question arises as to how did then Pythagoras discover the general theorem. It is the conjecture of some writers like Cajori (25; 18) and present days historian mathematician Seidenberg (74; 510) that Pythagoras had probably learned this theorem from the Egyptians, the truth of the theorem in the special case when the sides are 3,4,5 respectively.

According to Burk this theorem was known and proved in all its generality by the Indians long before the date of Pythagoras and the much travelled Pythagoras probably obtained his theory from Indians. But this view is criticised by Canton and he concludes that the theorem of Pythagoras, at least for the (3,4,5) - triangles,....... Pythagoras could have obtained this initial intellectual capital from the Egyptians. Although he confesses that Pythagoras was a pupil of the Indians rather than of the Egyptians. However, who borrowed from whom is still an open question.

However, the Chinese and Babylonians so called Pythagoras theorem have arithmetical aspect. But the full geometrical significance can be found in one of the Sulbasutras (namely Baudhayāna) in the following form (58; 159,160):

> "The diagonal of a rectangle produces what both the longer and the shorter side, each for itself produce".

Apastamba and Katyayana give the above theorem in almost identical terms (33; 133). These three works are supposed to contain rules employed in geometrical constructions used in connection with religious rites, at least as early as 800 B.C. According to Bag, no proof of this theorem is given by Baudhayāna and other Sulba writers, since it is beyond their traditions. (18; 123). Sarasvatiamma also holds the same view, and speculates that very likely they had proofs orally transmitted to the enquiring student. Burk, Hankil, Thibaut and Datta are of the opinion that Baudhayana knew a proof of the theorem (33; 133).

A regorus proof of this theorem, as given above, is due to Euclid himself and not to the Pythagoreans. Proclus remarks: "for my part while I admire those who first observed the truth of this theorem, I marvel more at the writer of the Elements, not only because he made it first by a most lucid demonstration, but because he compelled assent to the still more general theorem by the irre ragable arguments of science in the Six Book (47,i; 350).

Alternative Proofs:

The first proof of this proposition which was given by an Indian mathematician appeared in Bījaganita of Bhāskarācārya. According to him (20; 70):

"Twice the product of the bhuja and koti combined with the square of their difference will be equal to the sum of their squares, just as it is so for two algebraical quantities".

In short, it can be represented as follows:

$$\text{the bigger square} = (\text{bhuja} - \text{koti})^2 + 2\ \text{bhuja}.\ \text{koti}.$$
$$= \text{bhjua}^2 + \text{koti}^2.$$

This is, in fact, a geometrico - algebraical proof.

Another alternative proof which is fully geometrical, is also given by Bhāskara (33; 134)

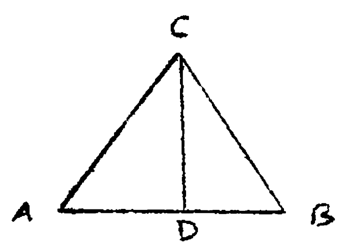
Let A B C be a right-angled triangle of which the angle C is a right angle. From C draw the perpendicular C D on A B. Then the triangle A B C, A C D and C B D are similar.

Therefore, AB: AC : : AC: AD:

or $AC^2 = A\ B.\ AD.$

Similarly, $CB^2 = AB.\ DB.$

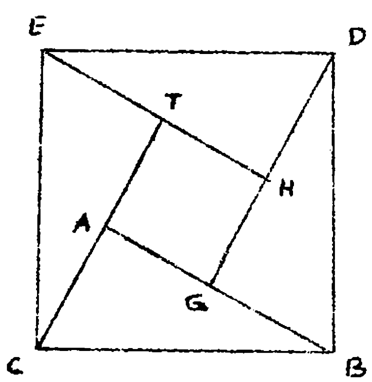
Adding we get $AC^2 + CB^2 = AB^2$.



Later on, this proof was rediscovered in Europe by Wallis in 1963.

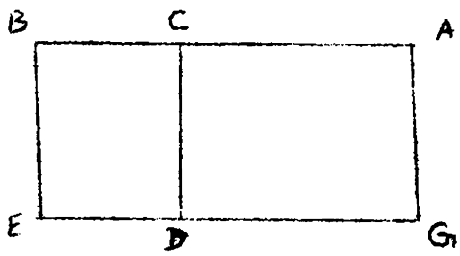
Apart from the usual proof of Euclid, Naṣīr al-Dīn Tūsī incorporates seventeen alternative proofs in Taḥrīr-i-Uqlīdes, among which the last one is quite interesting in the sense that it very much resembles Bhaskara's proof. The proof is as follows:

The squares on the hypotenuse should be described on the triangle. From the point Ddraw the perpendicular D G on A B. From E draw the perpendicular E H on D G. Produce C A to T so as to meet H E at T. In the middle a square is formed which is equal to the difference of the two sides. Thus the four triangles are equal. The sum of any two of these triangles is equal to the rectangle of the two sides. The sum of the four triangles is equal to double of the rectangle of the two sides. This together with the square on the difference of the two sides is equal to the sum of the squares on the two sides. Because, if to this the square A B be added, then the square D C is equal to the sum of the squares on the two sides.



PROPOSITION 39

The surface of a line multiplied by one of its two segments will be equal to the sum of the square on the same segment and the surface of that multiplied by the other segment.



Let line A B be divided at C.

We say that the surface AB. CB is equal to the sum of the square on B C and the surface AC. BC.

We construct a square C B E D on C B and complete the surface A C D G.

Since A G is equal to B E. So it will be equal to C B also. Therefore, the surface A E will be the surface A B. C B and the surface A D will be the surface of both the segments A C, C B;

thus it is seen that the surface A B. C B is equal to the sum of square of C B and the surface A C. C B.*

This is the third proposition of Book II. Ghulam Husain leaves first two, whereas Proposition second and this are the consequence of the first. Although it is a fact that these three propositions are not used by Euclid in Book II. But Heath observes that the proof of all the first ten propositions of Book II are practically independent of each other, though the results are really so interwoven that they can often be deduced from each other in a variety of ways (47; 377). For example, this proposition may be expressed by any of

$$(x + y) = xy + y^2 \text{ .... } (1)$$

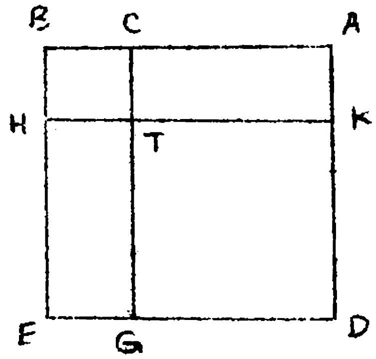
$$xy = (x - y) y + y^2 \text{ .... } (2)$$

$$x (x - y) = y (x - y) + (x - y)^2 \text{ .... } (3)$$

but the fact that different algebraic formulas can be expressed by the same geometric proposition is probably the only stylistic advantage of geometric algebra over its modern counterpart. Unfortunately, this advantage is also an obstacle to the evaluation of the algebraic interpretation. For the algebraic relation of one proposition of geometric algebra to another will depend upon how algebraic variables are assigned to line segments (60; 46).

PROPOSITION 40

The square on a line is equal to the sum of the squares on its segments and twice the surface of one segment into another segment.



Let A B be the line and it be divided at C.

We say that the square on A B is equal to two squares on A C, C B and twice the surface A C. C B.

We construct on A B a square A D E B and draw C G parallel to A D and seperate B H from B E equal to B C and extend the line H T K from H parallel to A B;

thus the square A E is divided into four rectangular surfaces and by Proposition 29, all opposite sides are parallel and it is clear that the surface C H is the square on B C and on account of equality of A C, H E, the the square G K will be equal to the square on A C and the surface A T is the surface A C. C T i.e. B C and the surface E T will also be equal to the surface A T, being A C C T equal to H T, T G respectively.

Hence the sum of the two surfaces AT, ET is twice the surface A C. B C and the square A E consists of two surfaces and the two squares C H, K G which are the squares on the two segments.

This is what was required.

The algebraic representation of this proposition is

$$(x + y)^2 = x^2 + y^2 + 2xy$$

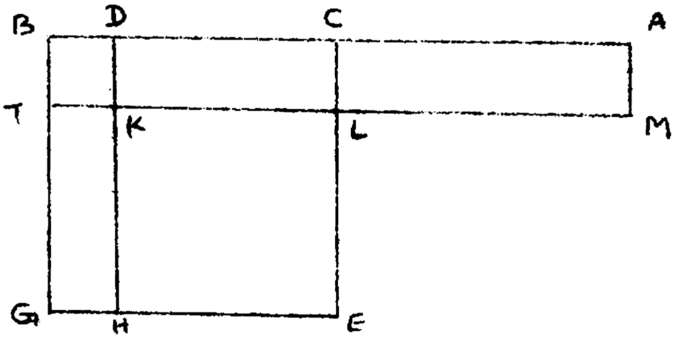
The proof of this proposition is an alternative given by Ghulam Husain.

This proposition is of much use chiefly in surd numbers and square root.

It can also be generalized by dividing a straight line into any number of segments i.e. the square on the whole line is equal to the sum of the squares on all the parts together with twice the rectangles bounded by every pair of the parts

PROPOSITION 41

If a line be bisected and divided into two unequal segments then the sum of their surface of one segment into the other segment and the square of the difference between the half and the segment will be equal to the square of the half.



Let line A B be bisected at C and divided at D.

We say that the surface A D. D B together with the square on C B is equal to the square on B C.

We construct on B C a square B G E C and from D draw the line B H parallel to B G and we separate B T from B G equal to B D and from T draw the line T K L M parallel to A B and from A the line A N M parallel to B G which meets the line T K L M at M.

Because of equality of B T, B D the surface L H will be a square equal to the square on C D;

the surface A K is equal to the surface A D . B D and the surface B D G is equal to the surface A L;

because A C is equal to B C i.e. B G and C L to B D;

Whereas the surface C K together with the surface A L is equal to, the surface A D. B D.
Therefore, the surface D G will also be equal to aforesaid surface;
and when weadd in these two surfaces the square L H which is equal to the square on C D, we get the square C G which is square of the half, is obtained. Hence it is clear that the surface A D . B D together with the square on C D is equal to the square on B C.*

In this proposition and also in 42, the situation becomes more complicated because of the additional line segments involved. This proposition may be treated as speical case of Propositions 43 and 40 respectively which is seen clearly by taking A B as x and D B as y, so these propositions become

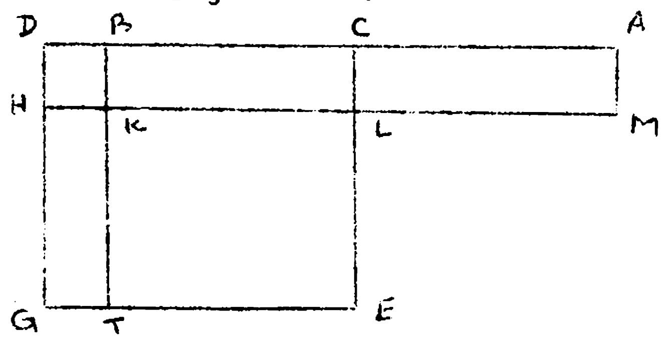
$$(x - y)y + (x/2 - y)^2 = (x/2)^2.$$

and $$(x + y)y + (x/2)^2 = (x/2 + y)^2.$$

Seidenberg states that it was the impression of the Van der Waarden that these propositions, though they are of course differently worded, their diagrams are not the same, were saying much the same thing, and was rightly puzzled for why should we have two propositions for the same thing. To answer this he concluded that these are not propositions, but the solution of the problems (72; 317).

PROPOSITION 42

If a straight line be bisected and another straight line be added in its direction, then the sum of the surface of the line with the addition into the addition and the square of the half is equal to the square of the sum of the half and additional straight line.



Let line A B be bisected at C and B D be added in it.

We say that the sum of the surface A D. B D and the square on B C is equal to the square on C D .

We describe on C D the square C E G D and from G D seperate D H equal to B D and from the point B extend the line B G parallel to D G and from the point H the line H K L M parallel to D A and from A the line A N M parallel to D H till it meets the line H K L M at the point M.

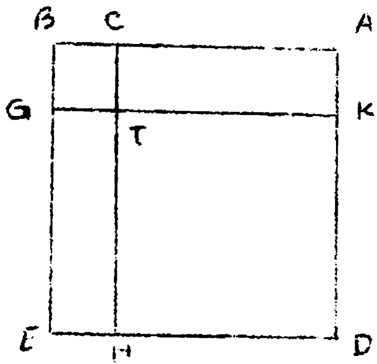
Similar to the description given in the preceding Proposition we say that the surface A H i.e. two surfaces A L, C H in other words two surfaces G K, C H together with the square T L is equal to the square C G.

This is what was required.*

The proof of this proposition is also due to Ghulam Husain.

PROPOSITION 43

The square on a line and the square on one of its segments together is equal to the sum of the twice the surface of the line into that segment plus a square on the other segment.



So, the square on A B together with the square on B C is equal to twice the surface A B . B C and the square on A C. We describe on A B the square A D E B and from B E seperate B G equal to A C and extend the line C H parallel to B D and G T K to A B.

We say that two surfaces C E , A G which are equal to twice the surface A B . B C, are equal to the surfaces A T, E T and twice the square C G;

and when we take square K H, common the sum of the two surfaces A T, T E and twice the square C G and the square K H, that means, the sum of the two squares A E , C G will be equal to twice the surface A B . B C and the square K H. This is what we wanted.*

This Proposition is the geometrical equivalent of the identity

$$(x - y)^2 = x^2 + y^2 - 2xy$$

The Proof of this proposition is an alternative method given by G͟hulām Ḥusain.

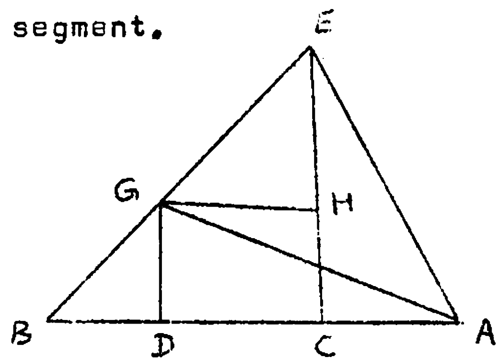
It is not free from interest to note that the addition of the formulas $(x - y)^2 = x^2 + y^2 - 2xy$

and $(x + y)^2 = x^2 + y^2 + 2xy$

gives the algebraic equivalent of the propositions 44 and 45 which really proved the same result.

PROPOSITION 44

If a straight line be bisected and again divided into two unequal segments, then the sum of the two squares of segments will be equal to twice of the sum of the square of the half and the square of the difference between the half and the segment.



Let line A B be bisected at C and again be divided at point D.

We say that the sum of the two squares of A D, B D is equal to twice of the two squares of A C, C D.

In order to prove the claim, we draw a perpendicular CE at point C equal to A C.

Join A E, B E and draw from the point D the line D G parallel to C E and from point G the line A G parallel to C D and join A G.

Hence in two triangles A C E, B C E, two sides A C, B C are equal to the common side C E and two angles A C E, B C E are right angles. Therefore, each of the two angles A E C, B E C will be half of the right angle, by Proposition 5 and

39. Thus the angle A E B which is made up of the combination of these two half right angles will be right angle and also in the triangle B D G became the angle B is semi right angle and the angle B D G is a right angle, so the angle B D G is semi right angle, and hence B D, D G will be equal.

Similar to the aforesaid description the two sides H E, H G will be equal in the triangle H E G.

After this preliminary I say that, due to equality of A C, C E, the square of A E will be double of the square of A C, by Bride's Proposition, and similarly the square of E G will be equal to the double of the square of H G i.e. of CD.

Therefore, the squares of A D, D G i.e. square of A D or the two squares of A D, B D will be double of the squares of AC, C D.

This was our desire. ***

Propositions 44 and 45 prove the same result but have great interest in connection with a problem of indeterminate analysis which received much attention from the ancient Greeks. If we take the straight line A B divided at C and D and if we put C D = x, D B = y, the result obtained by Euclid, namely

$$AD^2 + DB^2 = 2AC^2 + 2CD^2$$

$$\text{or} \quad AD^2 - 2AC^2 = 2CD^2 - DB^2$$

becomes the formula

$$(2x + y)^2 - 2(x + y)^2 = 2x^2 - y^2.$$

If, therefore, x, y be numbers which satisfy one of the two equations

$$2x^2 - y^2 = \pm 1$$

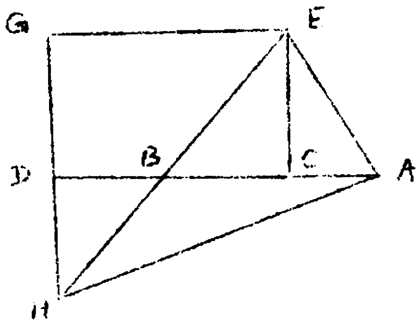
the formula gives us two higher numbers $x + y$ and $2x + y$, which satisfy the other of the equations.

Euclid's propositions thus give a general proof of the very formula used for the formation of the succession of what were called "side" and "diagonal" numbers.

But Fowler commented that Proposition 44 is not only an unnecessary and complicated obfuscation of a simple and elegant geometrical result, but also it leads no-where and indeed Proclus' text reverts, without further comment, at the end of his geometrical proof, to the discussion of side and diameter numbers (41; 205).

PROPOSITION 45

If a straight line be bisected and another straight line be added in the same direction, then the sum of the squares of the line with the extension and the square of the extension will be equal to sum of two times the squares of the half line/the squares of the half together with that extension.
and



Let the line A B be bisected at C and the line B D be extended in the same direction.

We say that the sum of the squares of A B, B D is equal to twice of the sum of the two squares of A C, C D.

From point C, we draw a perpendicular C E equal to A C and join A E, B E and from the point E, the line E G parallel to C D which meets the line D G at G and .extend B E, D G till they meet at point H and join A H.

According to the description of the previous Proposition the angle A E B will be right angle and E G, A D being parallel the angle C, E will also be right angles. After removing the semi right angle C E B, the angle H E G will also be semi right angle and the angle G is right angle;

thus in triangle E G H the angle G H E also remains semi right angle and by Proposition 6 the sides E G, G H will be equal and analogeously we say that in the triangle B D H, the sides B H, D H are also equal.

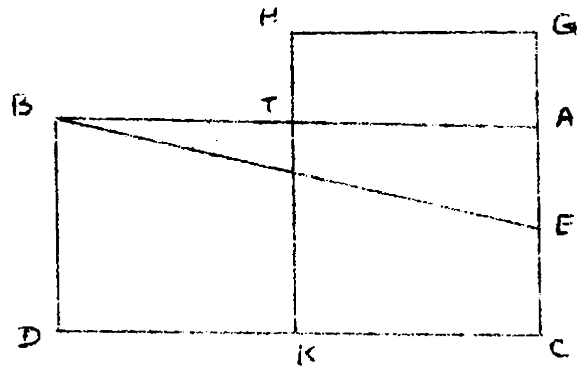
After this preliminary, we say that because A C E C are equal, therefore, the square of A E will be equal to twice of the square A C and due to equality of E G, D G, the square of E H will be equal to twice the square of E G, i.e. twice the square of C D.

Hence the sum of the squares of A E, E H which is equal to the sum of the twice the squares of A C, C D, will be equal to the square of A H i.e. equal to the squares of A B, D H or A D, B D.

This is what was required.*

PROPOSITION 46

We wish to divide a given straight line in two unequal segments in a way that the surface of the original line into a small segment will be equal to the square on the bigger segment.



Let A B be the line.

We construct a square A C D B on it and bisect the side A C at point E and join B E and extend E A to G and make E G equal to B E and construct on A G the square A G H T;

hence the line A B is cut off at point T, but the division is in two unequal segment.

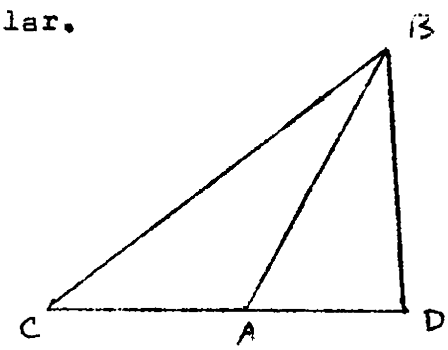
Thus the sum of E A, A B is greater than E B i.e. E G.
We delete the common A E , A B remains which is greater than A G i.e. A T.

Therefore A B is divided at T as needed and extend the line H T upto K and we restrict to the description of the division because the line C A has been bisected at E and extended in its own direction to A G. Therefore, by Proposition 42 the surface C G. A G together with the square on A E is equal to the square on E G i.e.the square on B E or two squares on A E, A B. We delete the common square on E A the remaining surface C G. G A i.e. into G H and that surface is G K equal to the square on A G which is A D and as we delete the common surface A K, the square G T which is the square on the bigger segment remains equal to the surface D T and D T surface is the line B E i.e. A B. B T.

Hence the desire is obtained.*

PROPOSITION 47

In any obtuse angled-triangle, the square on the opposite side of the obtuse angle will be greater than the sum of the squares on two sides by twice the surface of the base i.e. one of the two sides on which a perpendicular falls from one of the two acute angles and the magnitude of the side obtained by extension of that side between the obtuse angle and the feet of the perpendicular.



For example, in the triangle A B C, the angle A is obtuse.

We draw from the point B a perpendicular B D to the extended side A C in the direction of A at point D.

We say that the square on B C exceeds the sum of the two squares on B A, A C by twice the surface A C. A D.

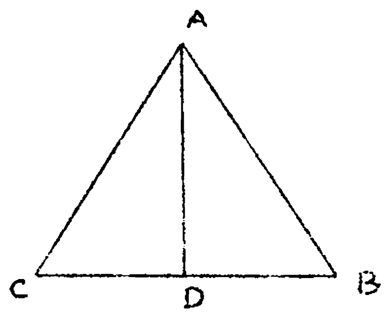
Because C D is divided at A; so, the square on it will be equal to the sum of the two squares on A D, A C together with twice the surface A D. A C, by Proposition 40. When we add the square on B D in each of the two squares, then the sum of the two squares on B D, C D i.e. square on B C will be equal to the sum of the two squares on B D, A D i.e. the square on A B and the square on A C and twice the surface A D into A C.

Hence from this description it is cleared that the square on B C exceeds the sum of the two squares on B A, A C by twice the surface A D. A C.

This is what was required. *

PROPOSITION 48

The square on the opposite side of an acute angle in every triangle will be less than the sum of the squares on the two sides by twice the surface of the base into the magnitude of the base which is obtained from that base between the given angle and the feet of the perpendicular drawn from one of the two remaining angles.



Let in the triangle A B C the angle B be an acute angle and the perpendicular A D be drawn from the angle A to the base which is the side B C.

We say that the square on A C is less than the sum of the two squares on A B, B C by twice the surface B C . B D.

For, B C is divided at A; so the sum of the two squares on B C, B D is equal to the sum of the twice the surface B C. B D and the square on C D, by Proposition 43.

If we take the square on A D as common, then the sum of all the three squares on B C, B D, A D i.e. the two squares on B C, A B will be equal to twice the surface B C . B D together with two squares on C D, A D i.e. the square on A C

Hence from this description it is clear that the square on A C is less than the two squares on B C, A C by twice the surface B C . A D.*

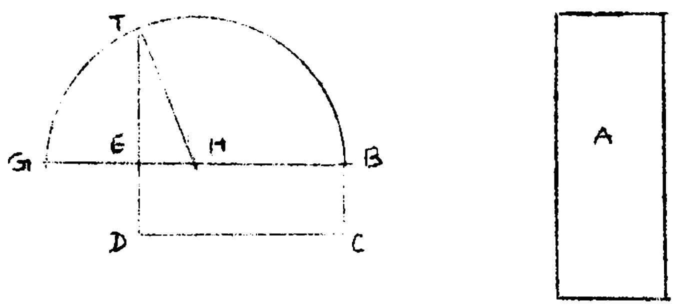
This and the preceding proposition are equivalent to a proof that in any triangle with sides x, y, z and angle A opposite to x.

$$x^2 = y^2 + z^2 - 2yz \cos A.$$

It is probable not withoat significance that this penultimate proposition of Book II is a generalization of Pythagoras' theorem, which was the penultimate proposition of Book I (81; 479).

PROPOSITION 49

We wish to construct a square which will be equal to a rectilineal surface (quadrilateral).



Let A B be the given surface.

First, we constract a rectangle B C D E equal to the given quadrilateral with the help of the Proposition 36. If by construction B C, B E are equal, the purpose is served, other-wise we extend the longer one e.g. B E upto G and make E G equal to E D and bisect B G at point H and describe at point H, with distance HG, the semi-circle B T G and extend the side D E till it meets the circumference at point T.

Hence the line E T will be the side of the required square. Because B G is bisected at H and is divided again at E; hence the surface B E. E G which is exactly the same surface B D together with the square on H E is equal to the square on H G i.e. the square on H T, by Proposition 21 or equal to two squares on H E, E T.

We subtract the common square on E H, the surface B E . E G i.e. surface A remains equal to the square E T.
This is what was required.

This is last proposition which completes the theory of transformation of areas.

The propositions 34, 35, 36 enable us to construct a parallellogram equal to a given triangle or rectineal figure on a given side and angle. By the aid of this proposition we may determine a line such that the square on that line is equal in area to any given rectineal figure or we can square any such figure. As of two squares that is greater which has a greater side, it follows that now one comparison of two areas has been reduced to the comparison of two lines.

In present day the comparison of area is performed in a similar way by reducing all areas to rectangles having a common base. Their altitude give them a measure of their areas (50; 72).

C H A P T E R   S I X

CHAPTER SIX

PROPERTIES OF CIRCLES

In this Chapter properties of circles and arcs, lines and angles which are produced by the comparison of circles are established. This chapter is based on Section Three of JĀME-I-BAHĀDUR KHĀNĪ.

6.1. DESCRIPTIONS:

1. Equal circles are those whose radii are equal. Many editors have held that this should not have been included among definitions. Some call it a postulate and others e.g. Playfair calls it an axiom. But Billingsley, Naṣīr al-Dīn Tūsī and Ghulām Ḥusain admit it as a definition. Heath says that it is a definition in the proper sense of the term. (47, ii; 2). Euclid had no distinct word for "radius", and the diameter is used in relation to the line bisecting a circle and also to mean the diagonal of a square. In India, Aryabhata (c.499) seems to have been the first to use the word "semi-diameter" equivalent to it (15;40), which according to Smith, have passed over to Arabia and thence to Europe. (69; 278). Indian commentators and editors of the Elements in Arabic and Persian preserve the term "semi-diameter" (Nisf Qutra) even in modern times, although radius is used in the same sense.

2. A tangent to a circle is that straight line which meets the circumference of the circle but does not intersect the circle even if it be produced in both directions.*

This definition may be stated in other words as a straight line which has only one point in common with a circle and no where else in the vicinity. In the case of the circle, it is perpendicular to the radius at that point and more over by Proposition 14, we can infer that the tangent meets the circle once only and lies wholly outside and no other line through the point of contact has this property.

Fermat and Descartes defined tangents as the limiting position of a secant when two inter-sections with the curve tend to fall together (28; 45). But such definition is not applicable in Euclidean geometry because movement of a line is not permissible.

3. Touching circles are those whose circumferences meet but there is no inter-section.*

There are different opinions that this definition should include that circles do not cut in the neighbourhood of the point of contact and another opinion is that the definition means that the circles do not cut at all. Todhunter thinks the latter opinion as correct. But Ghulam Husain as Heath also prefer to read the definition as meaning simply that "the circles meet at a point but do not cut at that point". (47,ii;3).

4. Chords equi-distance are those lines whose perpendiculars drop from the centre on the chords are equal and these perpendiculars are distances of the chords from the centre.*

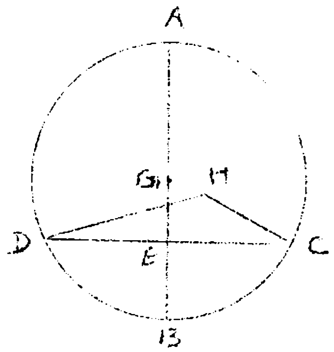
5. Every chords whose perpendicular is greater, its distance would be more.*

6. If two sides of the arc, which is less than a semi-circle be joined with the centre of the circle, the angle so formed is called the CENTRAL ANGLE and the angle of the arc. If they meet at a point on the circumference the angle formed is called ANGLE OF THE CIRCUMFERENCE and because it is situated in the segment, so it is also called ANGLE OF SEGMENT*.

7. Similar segments are those whose angles are equal.*

PROPOSITION 1 :

We wish to locate the centre of a circle.



Let A B be the circle.

We fixed any two points C, D on the circumference and join C D and bisect this line at point E and draw from E a perpendicular line A E B to the line C D which intersects the circumference in both sides at A, B and we bisect A B at G.

Hence G will be on the centre.

And otherwise if any other point (say) H is the centre:

We join H C, H E, H G:

Then the two lines H C, H D will be equal, each being the radius.

In the two triangles C E H, C ED, the corresponding sides are equal; hence the angles C E H, C E D which are formed in either side of the line E H, are right angles and E H will be perpendicular on C D, while A E is already perpendicular; this is a contradiction.

Hence G will be the centre of the circle necessarily.

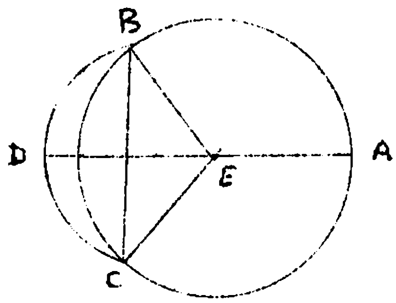
If point H lies on the line A B, another contradiction will arise and it will be the bisection of a line at two points.*

From this Proposition it follows that if two chords intersect each other at right angles, then one of the chords passes through the centre of the circle. It also follows that a perpendicular issuing from the middle point of the chord passes through the centre. The first inference is added by Euclid as Porism, but neither Nasir al-Dīn Tūsī nor Ghulām Husain add any of them as corollaries.

Heath (47,ii;8) observes that the demonstration only shows that the centre of the circle cannot lie on either side of A E, so that it must blie on C E or C D produced. It is, however, taken for granted rather than proved that the centre must be the middle point of A B. The proof of this by reductio ad absurdum is, however, so obvious as to be scarcely worth giving. The same consideration which would prove it may be used to show that a circle cannot have more than one centre, a proposition which, if thought necessary, may be added to this Proposition as a corollary. But it may be noted here that the definition of a circle guarantees the existence and uniqueness of the centre of any given circle. The proof of this proposition itself depends upon this guarantee. Euclid argues indirectly that the centre of a given circle must be the mid-point of the perpendicular bisector of a chord because no other point could be the centre. Obviously, this argument would not work unless it was already known that some point must be the centre. (60; 179)

PROPOSITION 2 :

If a straight line joins two points on the circumference of a circle, then the line falls within the circle.



Let B, C be two points on the circumference of the circle A B C and the line B C join them.

We say that this line will fall within the circle inevitably; and otherwise if it falls aoutside or coincide at the circumference of the circle.

If it falls outside,

Let B D C be the line and E be the centre of the circle.

We join E D which must intersect the circumference at point G. Later on we join E B, E C also.

Since in triangle C D B E, the two sides E B, E C are equal, so two angles E B D, E C D are equal, and the exterior angle C D E will be greater than the interior angle D B E, by Proposition 15 of 2.

Therefore, the angle D C E will also be greater. And by Proposition 17 of 2, the chord C E will be longer than the chord D E, although it is equal to G E, which is the part of D E. This will be a contradiction.

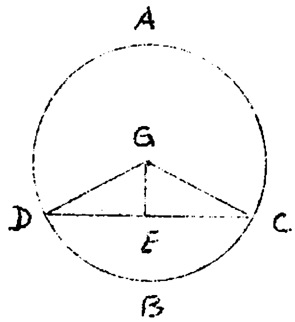
If the line B D C coincides at the circumference, then analogusly it is necessary that C E will be greater than D E, in spite of equality. Hence the joining line B C will fall inevitably within the circle.

This is what was required.*

The proof of this proposition by reductio ad absurdum is not necessary, because the chord cannot be outside or coincide the circle. It may be proved directly. For this, we have only to show that if D be any point on the straight line B C, then D E is less than the radius of the circel.

PROPOSITION 3 :

Every line which is drawn from the centre of a circle towards a chord and if it bisects the chord, then it will inevitably be perpendicular on the chord.



Let line E G be drawn from the centre of the circle A B towards the centre C D and bisecting it at E.

We say that G E will be perpendicular to C D.

For, it we join C D, D G, then in triangles E C G, E D G, the corresponding sides will be equal. Thus equality of the corresponding angles is also proved. And the two angles E which appear on both the sides of the line G E will be equal.

Therefore, they are right angles; and hence G E will be perpendicular.

And also for the converse of the claim, we say that if E G be perpendicular, then it will bisect C D.

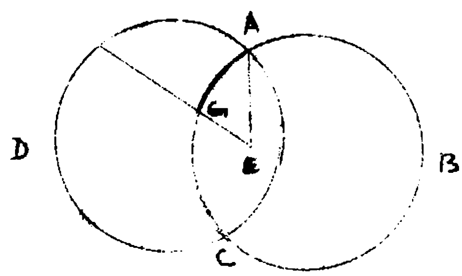
Because in the above two triangles, the two angles E are right and the angles C, D are equal and the sides CG, G D are radii; therefore by Proposition 19 of 2, C E, D E are equal.*

This proposition is the partial converse of the porism of the Proposition 1. So it may be placed just after it. The Porism is a s follows:-

"If in a circle a straight line cut a straight line into two equal parts and at right angles, the centre of the circle is on the cutting straight line". (47,ii;7).

PROPOSITION 4 :

It is not possible that the two circles which intersect at a surface may have the same centre.



Let A B, C D be the two circles, and if possible, let E be the common centre.

We join E A and draw a random line E G D.

Since point E is the centre of both the circles.

Therefore, A E, E G are equal, and similarly E D, E A.

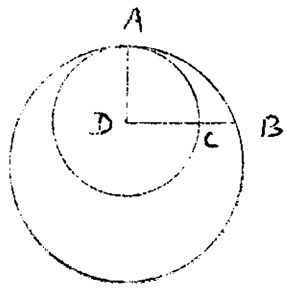
Hence E G , E D both are equal to EA, are equal.

This is impossible.

Hence the claim is proved.*

PROPOSITION 5 :

It is not possible that two circles touching each other will have same centre.



For example, two circles A B, A C touch at point A.

And if it is possible, the point D be their centres.

We join A D and draw a random line D C B.

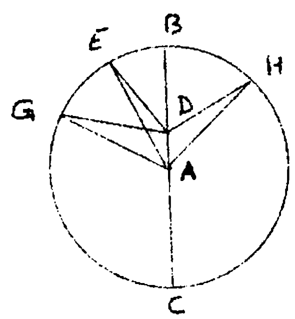
Then as has been discussed in earlier proposition, it is necessary to follow that D B, D C the whole and the part are equal, which is a contradiction.*

This and preceding proposition could be combined in one.

Because it does not make any difference if the circles intersect or meet, so long as they do not coincide together, in either case they cannot have the same centre.

PROPOSITION 6 :

Let any point in a circle other than its centre, be taken and straight lines from it be drawn to the circumference, then the line which passes through the centre will be greater than all of them, and the line which combines this greater line completes the diameter will be the smallest and the line which is near to the said greater line willbe greater than that line which is distant and for every line which is on one of the greatest line in the other side, only one line equal to it can be obtained.



Let A B be the point in the circle B C other than the centre and D the centre.

We draw from A the lines A D B, A E, A G and extend B D A straight upto C; and describe at point D with the line A D an angle A D H equal to the angle A D G, and join A H.

We say that the line A B is greater than all the lines draw from the point A and A C is the smallest of all and A E which is nearer to A B willbe greater than A G which is distant; and A H in the side of H, no other **line except A H** will be equal to A E.

Because the sum of the lines A D, D E i.e. A B is greater than A E, by Proposition Hamari;

Join E A.

We say that the angle A E G is smaller than D E G will be smaller than the angle D G E and angle A G E, is greater than the angle D G E; hence the angle A G E will be much greater than the angle A E G.

Thus chord A E will be greater than the chord A G and in this sequence the rule will be established. And also the sum of D A, A G is greater than D G i.e. D C. When we delete the common line D A from the greater line, the remainder is A C.

Thus A C will be the smallest line.

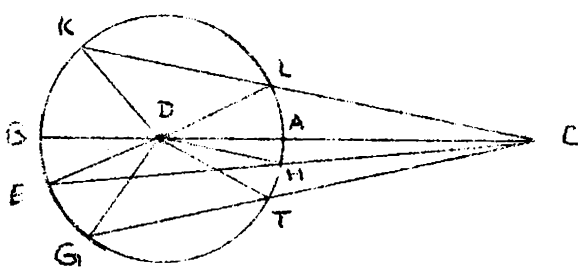
And if in the two triangles A D E, A D H, the sides A D, D E and the angle A D E are equal to the sides A D, D H and the angle A D H;
hence A H will be equal to A E.

It is not possible that from the point A towards A H a line be drawn equal to A E because that line always will be different from the line A H, hence will also be different from A E.

Now all the claims have been established.*

PROPOSITION 7 :

Let any point be taken outside a circle and from that point secant and non-secant lines be drawn to the circle, then a secant which passes through the centre will be greatest and the line which is nearer to the greatest line will be greater than that which is farther and the smallest terminating line is that which will be in the direction of the centre and the terminating line which is nearer to the smallest line will be shorter than the one which is more farther and the secant which lies on one side of the greater secant will have only one line equal to it on the other side and likewise for every terminating line which lies on one side of the shortest other than one terminating line can be found on the other side.



Let C be the point outside the circle A B; and the secants which are drawn from the point C are C B, C E, C G and C B passes through the centre and the terminating lines are C A, C H, C T and C A is directed towards the centre.

I say that C B is greater than C E, C G and C E which is nearer to C B will be greater than C G which is more distant from that; and the terminating line C A which is in the direction of the centre is smallest than C H, C T. Also C H which is nearer to C A is smaller than C T which is more distant than that.

And we join D G and discribe the angle C D K and/D with the line C D equal to the angle C D G; and join C L K; hence in the side of K except the secant C K none will be equal to C G and in the side of L except the terminating line C L, none will be equal to the terminating line C T. And for proof of all the claims, we join the lines D E, G E, D T, D H, H T, D L, D K.

We say that C B i.e. C D, D E is greater than C E and since the two angles D EG, D G E are equal and the angle C E D is smaller than one of them and the angle C D E is greater than other. Therefore, the angle C G E will be much greater than the angle C E G. Thus in the triangle C G E, the chord C E will be greater than the chord C G and also the sum of C H, H D is greater than C D and because when from the two unequal magnitude we delete the equals D A, D H, the remaining A C will be smaller than C H and also if C H is not smaller than C T, then it will either be equal or greater If it be equal then necessarily the sum of the two angles C H T, D H T which is greater than two right angles, will be equal to the sum of the two angles C T H, D T H, which is less than two right angles and this is a contradiction.

If C H be greater, then it necessarily follows that the sum of the former two angles will be smaller than the sum of the latter two angles; this is again a contradiction.

Hence C H will be smaller than C T and since in the two triangles C D H, C D K, the sides CD, C G and the angle C D G are equal to the sides C D, D K and the angle C D K; therefore, the remaining angles and sides in these two triangles are equal.

Thus C K will be equal to C G and in the side of K of the line B C, it is not possible that except the line C K any line will equal to the line C G.

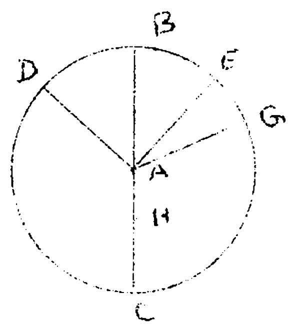
Because if a line be found, then necessarily, it will also be equal to C K and according to the earlier assertion, will be unequal;

hence this will be a contradiction.

We further say that in the two triangles T GD, L KD, the angles K, G are equal and by Mamuni's proposition, the angles T, K are also equal and by proposition 28 of 2, the angle G D T will be equal to the angle K D L. Therefore, G T will be equal to K L and when we subtract the equals G T, K L from the equals G C, K C, the remaining C L, C T are equal. And it is not possible in the side of L of the line A C any other line be equal to C T except the line C L.
Hence all the six claims are established.*

PROPOSITION 8

Let any point be taken in a circle and more than two equal straight lines be drawn from that point to the circumference, then the point taken will be the centre of the circle.



Let B C be the circle and A be the point within it, and let three equal lines A D, A E, A G be drawn towards the circumference.

We say that inevitably the point A is the centre, and otherwise the centre will be H.

We join A H and produce it upto B, C and by propsosition 6 A B is the smallest of the lines and the line A D which is one side of it and on the other side we have two lines A E, A G drawn equal to it.

This is a contradiction.

Hence except point A there will be no other centre. *

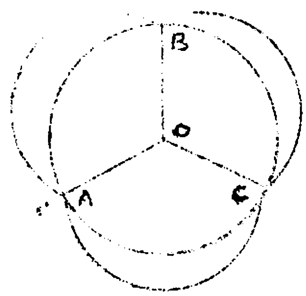
This is an alternative proof proved by the method of reductio ad absurdum by Nasīr al-Dīn Tūsī. (9; 50). Ghulām Husain, as his usual practice, prefers this method.

Euclid gives direct proof in which he bisects A B, B C at E,F and joins E O, F O, and using I.4 (Proposition 1 of 2) to argue that they are the perpendicular bisectors of A B, B C; hence by III.1, the centre lies on each of E O, F O i.e. at O. Mueller (60; 182) remarks that this proof shows essentially that any three noncollinear points A, B, C determine a unique circle with centre at the intersection of the perpendicular bisectors of A B, B C, a fact of fundamental importance in most modern treatments of the circle.

In this proof, it may be noted that the point H might be supposed to fall within the angle D A G. In this case it cannot be shown that A B is smaller than A E and A E than A G. But A B is greater than A E and A E than A G. This establishes the proposition. Some other editors prefer the same proof.

PROPOSITION 9 :

Inter section of two circles will not be on more than two points.



If it be possible, then let them intersect at three points A, B, C; and the centre of one of the two circles is D.

Join the lines D A, D B, D C, and these lines will be equal. The point D is also within the other circlefrom which three equal lines have been drawn towards the circumference;

therefore by preceding proposition, D will also be the centre of the second circle.

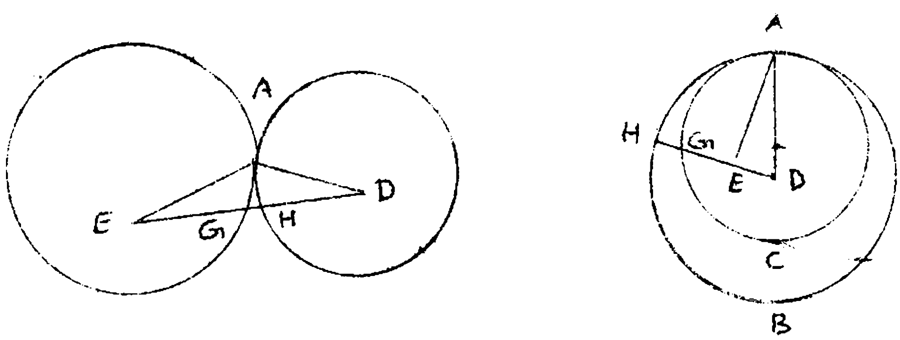
This is a contradiction according to Proposition 5.

Hence inter section will not be except at two points.* Ghulām Husain chose the proof which is given by Naṣīr al-Dīn Tūsī. (9; 51-52). In the proof it is assumed that the point D which is taken as the centre of the circle A B C is within the other circle, but it has not been discussed that the point D is on or outside of the circle. Euclid also proves by reductio ad absurdum method in which he supposes the circle A B C cuts the circle D E F at more than two points B, C, F, H, and join B H, E G and bisects at the points K, L and from these points draws K C, L M at right angles to B H, B G, which pass through the points, A, E. Since in the circle A B C, line A C cuts the line B H at right angles into two equal parts, so the centre of the circle A B C is on A C. Again, since in the same circle A B C, straight line N O cuts a straight line B G into two equal parts and at right angles, the centre of the circle A B C is on N O. But it was also proved to be on A C, and the straight lines A C, N O meet at no point except at P. Therefore, the point P is the centre of the circle A B C.

In this proof Euclid does not prove that the lines bisecting B G, B H at right angles will meet in a point P.

PROPOSITION 1C :

The line which passes through each centre of the two touching circles will also pass through the point of contact.



For example, two circles A B , A C touch at the point A and their centres are D, E.

We say that the line joining D E also passes through the point A, and if possible that it does not pass through A, then necessarily they intersect the circumference of the contacting circles at two points H, G.

Join A D.

Then if they touch internally, we say that the lines D E, A E together are greater than A D, i.e. D H. We delete D E which is common, then the remainder A E will be greater than E H but since E is the centre of the circle A C, so A E will be equal to E G, which is the part of E H;

hence it will be a contradiction.

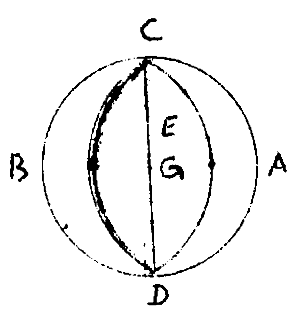
If contact is external, we say that A D, A E are greater than E D, but they are equal to D H, E G, which is part of D E.

This is also a contradiction.

Hence result is established.*

PROPOSITION 11 :

Contact of two circles is not more than one point.



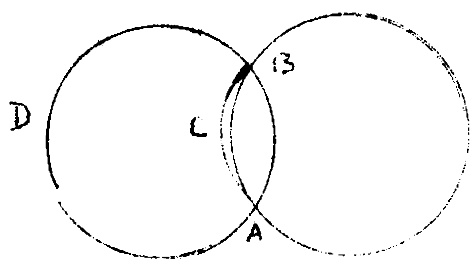
If possible, let the two circles A B, C D first contact at two points C, D internally.

We join their centres that are the two points E, G; and extend E G, then it will pass through two points C, D, according to the previous Proposition; and C E i.e. D E will be less than C G i.e. D G.

This is a contradiction.

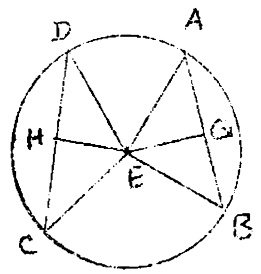
Further let the contact be externally at two points A, B; and the joining chord A B will lie within one of the two circles and outside of the other; this is also a contradiction, by Proposition 2.

Hence the contact will not be except at one point.*



PROPOSITION 12 :

Equal chords that are in a circle, their distance will be equal from the centre and the chords whose distance from the centre are equal will be equal.



For example, in the circle A B C D, the two chords A B, C D are equal and E is the centre of the circle.

From E, we draw the perpendiculars E G, E H to both the chords.

We say that these two perpendiculars are equal.

Because when we join E A, E B, E C, E D, in the two triangles B A E, D CE, the corresponding sides are equal. Therefore, the corresponding angles will also be like that. Thus in the triangles A G E, D H E, the angles A, D are equal and angles G, H are right angles and the sides A E, D E are radii, hence the perpendiculars E G, E H are equal.

Now, if the perpendiculars E G, E H are equal, we say that the chords A B, C D are mutually equal.

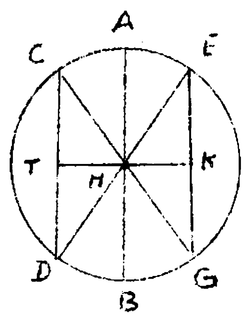
Because the sum of the two squares of A G, E G is equal to the square of AE, i.e. the square of E D, which is the sum of the two squares of D H, H E. And when we delete the two squares of E G, E H which are equal, the two equal squares of A G, D H remain.

Hence A G, D H are equal and their double i.e. the two chords A B, C D also turn out to be equal.

This is what was required. *

PROPOSITION 13 :

The diameter of a circle will be greater than all the chords and the chords which are nearer to the centre will be greater than the chord which is farther.



For example, A B is the diameter of the circle E C D G and H the centre. The chord C D is nearer to the centre in comparison of the chord E G.

From H we draw two perpendiculars H T, H K to the two chords C D, E G and join C H, D H.

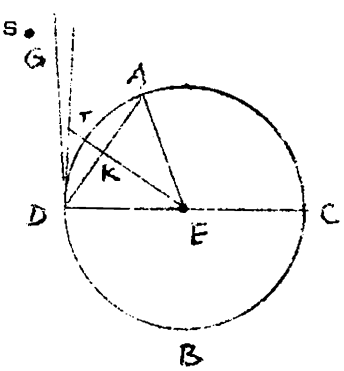
Therefore, the sum of C H, D H i.e. the diameter A B is greater than C D and join H E.

We say that the sum of the squares of H T, C T is equal to the square of C H i.e. the square of H E. In other words the squares of E K, K H and square of H T is smaller than the square of H K; therefore, by deleting the squares of H T, H K, the remaining square of C T is greater than the square of K E;

hence C T will be greater than K S; so twice of C T, which is C D will also be greater than E G, which is twice of E K. This is what was required.*

PROPOSITION 14 :

A perpendicular drawn to the diameter of a circle from its extremity falls outside the surface and it is impossible that between the perpendicular and the circumference of the circle, another straight line may exist, and the angle of the semi circle is greater than any acute rectilineal angle and the angle which is bounded between the perpendicular and the arc of the circle will be less than the remaining all the acute rectilineal angles.



Let A B be the circle about E as centre and C D as diameter. Draw from point D on the above diameter the perpendicular D G.

We say that this perpendicular will fall outside the circle and if possible that it falls within a circle, then it meets the circumference or the circle at point A.

Join E A.

In this case, by equality of the two arcs E A , E D , two angles E A D, E D A are equal and angle E D A is right angle. Hence E A D will also be right angle. This is a contradictior Hence the said perpendicular, in fact, does not fall within the circle but like D G falls outside the circle.

And also if possible that a straight line be existed between perpendicular and the circumference of the circle.
Let H D be the line.

In this case, from the centre E we draw a perpendicular E T to this line; and this perpendicular does not coincide with the line E D, because E D is not perpendicular to D H and also does not exist after producing H D in the direction of B and otherwise in the triangle E D T, the right and abtuse angles will appear together; therefore, this perpendicular does not exist except in the side of H and it is necessary to intercept the circle at point K. Hence in triangle E T D, the side E D which is the opposite side of the right angle will be greater than E T which is the opposite side of the acute angle, by Proposition 17 of 2, i.e.

part E K will be greater than the whole E T. This is a contradiction.

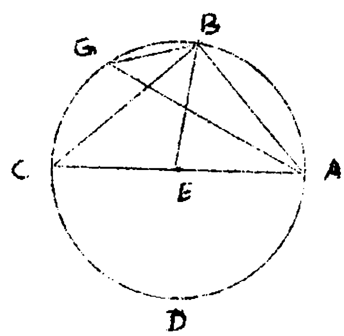
Hence the straight line cannot exist between the perpendicular and circumference. And it is due to this reason the angle E D K A which is the angle of the semi circle is greater than all the acute rectilineal angles and the angle G D K A obtained which is bounded by perpendicular and circumference will be smallest angle.*

This proposition is historically interesting because there was contention in ancient Greeks, both before and after Euclid's time, till the seventeenth century, about the nature of the "angle of a semicircle" and the "remaining angle" between the circumference of the semicircle and the tangent at its extremity. And the latter angle has a recognised name, "horn-like" or "cornicular angle". This angle is a true angle in view of Proclus (47,ii;39-42). But Euclid and Archimedes had tried to avoid the mathematical consideration of horn angles (49; 2). Nasīr al-Dīn Tūsī also followed the same.

Ibn Sīnā (980-1037) wrote a book in which he said that the angle between the circle and the tangent has no magnitude. He assumes that this angle can be so called only in virtue of its quality for it is formed by two tangent lines on the same surface include one towards the other and meeting not along a straight line (66; 8-9). John Wallis (1616-1703) also maintained that the so called angle was not a true angle, and was a quantity. Newton succeeded its mathmatical interpretation in his Calculus and showed that any horn angle is zero (49;3). Ghulām Husain has no room for such an angle.

PROPOSITION 15 :

The angle which exists in the segment of a circle is right, if segment is semi-circle and is acute, if it is major segment, and is obtuse, if minor, and an angle of the major segment will be obtuse and the angle of the minor segment will be acute.



Let the segment A B C be the semi-circle.

We say that the angle A B C which exists in it will be right, the segment being the semi-circle, its centre will be on the diameter A C and that point is E.

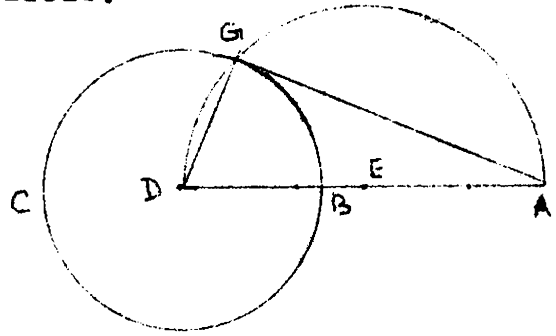
We join A B.

Thus the exterior angle C E B of the triangle A E B will be double of the angle A B E, by proposition 5 and 28 of 2;

immediately deducible from III.20 Proposition 18, if that theorem is extended so as to include the case where the segment is equal to or less than a semi circle, and where consequently the "angle at the centre " is equal to two right angles or greater than two right angles respectively. (47,ii; 63). Ghulam Husain puts III.20 after III.31 (Proposition 15).

PROPOSITION 16:

We wish to draw a straight line from a given point which is a tangent to a given circle.



Let A be the point and B C the circle.

We join the point A with the centre D of the circle, by the line A D, and bisect the line A D at the point E and describe at E with the distance E D the semi-circle A G D which passes through the point A and intersects the circumference of of the circle B C at point G. Join A G.

Hence this line will be the tangent to the circle B C.

Because after joining D G, by preceding proposition angle and on this hypothesis, the exterior angle A E B of the triangle C E B will be double of the angle C B E and because the sum of the angles C E B, A E B is equal to two right angle, half of this sum which is the whole angle A B C will be a right angle.

Next I say further that the segment B A D C which is greater than the semi-circle and the angle B A C which exists in it, is acute although from a right angled triangle, it is not right.

We fix a point G on the arc B C and join G A, G B; hence the segment A B G which is less than the semi-circle, the angle A B G, which exists in it, is greater than the right angle A B C, will be obtuse, and also the angle of the line C B and the arc B A which is greater than the right angle A B C will be obtuse. This is the angle of the major segment and the angle of the line C B and the arc B G which is the angle of the minor segment will be acute because it is less than the angle of the line E B and the arc B G. This angle is the greatest acute, according to the earlier propositions. Hence all the claims have been established.*

In this Proposition the angle of a segment greater or less than a semi-circle seem like the part of Proposition 14. Perhaps because of this reason Ghulām Ḥusain would have placed this proposition just after 14. Although this proposition is

A G D turns out to be right angle. Therefore, A G will be perpendicular to the radius G H and by proposition 14, will be a tangent.

And from the description we also learn that when a line be joined between the centre of the circle and the point of contact, the radius will be perpendicular to the tangent.*

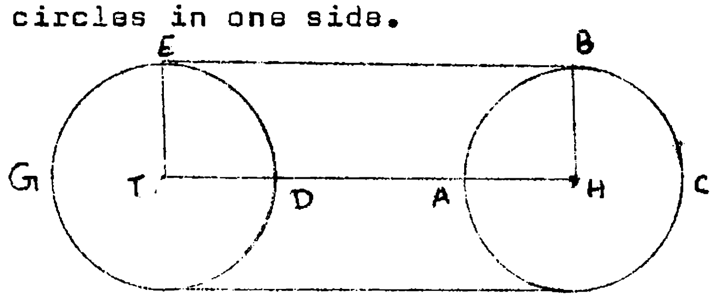
This is an alternative proof given by Ghulām Husain which is an easiest solution, in which we have only to describe a circle on E D as diameter and this circle cuts the given circle at two points of contact. Ghulām Husain leaves out the case where the given point lies on the circumference of the circle, because the construction is so directly indicated by Proposition 14.

To prove this proposition Euclid draws the circle with centre E and radius E A intersecting the given circle at D, draws D F perpendicular to A D and intersecting the drawn circle at F and joins E F intersecting the given circle at B. He uses I. 4 to establish the congruence of triangles A E B, F E D, so that angle A B E is right and by III.16 A B is tangent to the given circle.

From this solution, Ghulām Husain deduced also a useful theorem that "if a line be joined between the centre of the circle and the point of contact, then that line (radius) will be perpendicular to the tangent."

PROPOSITION 17 :

We wish to draw a straight line which will be tangent to two given circles in one side.



Let A B C, D E G be the two circles having centres H, T.

We join H T. If both circles are equal, we draw from H,T, two perpendiculars H B, T E to the line H T, till they terminate at two points B, E on the circumferences of the circles. We join B E which will be tangent to both the circles;

because the perpendiculars H B, T E are equal and parallel. Thus by proposition 29 of 2, B E, H T are also equal and parallel and the angles B, E are right, by proposition 14, the line B E will be tangent to both the circles.

If these circles are un-equal.

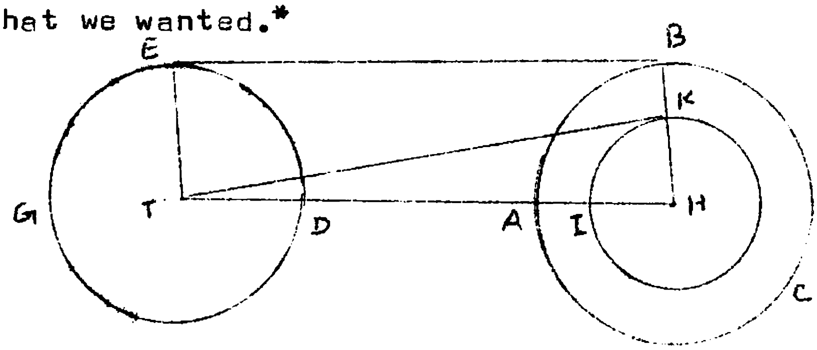
Let A B C be greater.

We separate A I from A H equal to D T and with centre H and distance H I we describe a circle I K and from the point T draw the line K T which will be tangent to the circle I K at point K, by previous proposition.

Join H K and produce it upto B and from the point T draw a perpendicular line T E to the line K T and join B E which will be tangent to both the circles.

Because by previous proposition the angle H K T is right angle. Therefore, the angle B K T will also be right angle and B K i.e. A K or D T is equal to E T. Thus as aforementioned in the case of equal circles, the angles H B E, T E B are right angles and the line B E is tangent to both the circles.

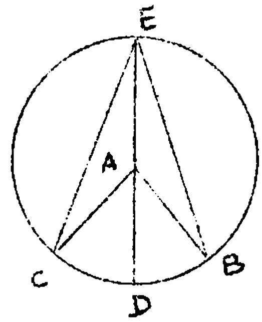
This is what we wanted.*



This proposition is taken from an English book on Geometry by Blunt.

PROPOSITION 18 :

The angle at the centre is twice the angle at the circumference when they are situated on the same arc.



For example, at the arc B D C , the angle B A C which is at the centre is twice the angle B E C at the circumference.

Because when we join A E and extend it upto D, then the exterior angle B A D is equal to double of the interior angle B E A of the isoscales triangle E A B.

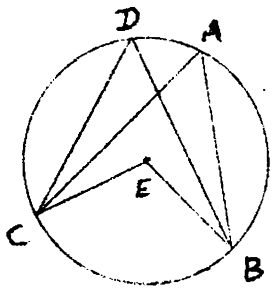
And similarly, the exterior angle C A D is equal to double of the interior angle C E A of the isosceles triangle EA C.

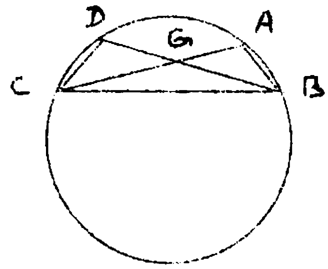
Hence the total angle B A C at the centre which is sum of the two half is double of the total angle B E C on the circumference, which is the sum of the two halves.

In the proof it has not been considered if the angle at the circumference or the angle in the segment is obtuse.

PROPOSITION 19 :

The angles in the same segment are equal.





Let B A C, B D C be the two angles at the same segment B A D C

Let the segment be greater than the semi-circle.

We join the centre E with the points B, C by the lines B E, C E.

In this case, by preceding proposition, each angle which is on the circumference is half of the angle B E C on the centre. Therefore, they are equal.

And if the segment will not be greater than the semi-circle.

We say that the segment A B C D inevitably, will be greater than the semi-circle. Thus two angles A B D, D C A which are in that segment are equal; and in the triangles A B G, D C G, two angles A B G, G C D are equal and likewise the opposite angles G;

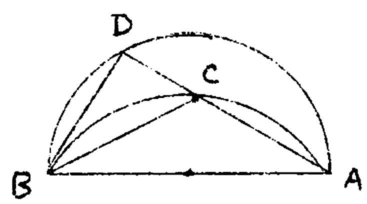
hence each one of the two angles A, D which are supplementary to two right angles, are equal.

This is what was wanted.*

Here also only one case that a segment greater than a semi-circle is considered but the case of a segment less than or equal to a semi-circle has not been discussed.

PROPOSITION 20 :

It is not possible that on the same straight line, in one side, two similar segments may exist and one is greater than the other.



If possible, let on the line A B, two similar and different segments A B C, A D B exist.

We join A C and extend it upto D and join B C, B D.

Hence two interior and exterior angles B D C, B CA, by hypothesis, in triangles B D C, are equal, by the similarity of the two segments. This is a contradiction.

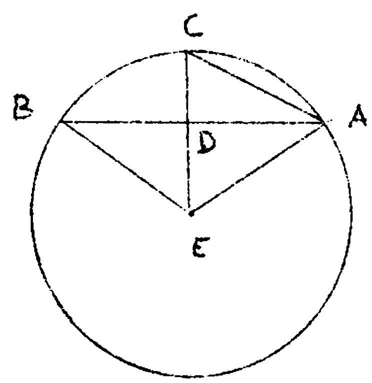
Hence the rule is established;

and also from this statement, it is clear that whenever two similar segments are situated on two equal lines, they will be equal;
because otherwise, after superposition, the same contradiction will arise.

It may be noted here that Ghulām Husain considers in the enunciation, namely if two similar segments are situated on two equal lines, then they will be equal, as corollary of this proposition. But Nasir-al-Din Tūsi like Euclid proves it as a separate proposition.

PROPOSITION 21 :

We wish to construct the complete circle of which a segment is given.



Let A B C be the given segment.

We bisect the chord A B at D and from the point D draw a perpendicular D C to the line A B and join A C and describe at point A, with the line A C, the angle C A E equal to the angle A C D and extend A E, C D till they meet at point E. On account of their extension they make angles less than two right angles with the line A C. Hence the point E will be the centre of the segment of the circle.

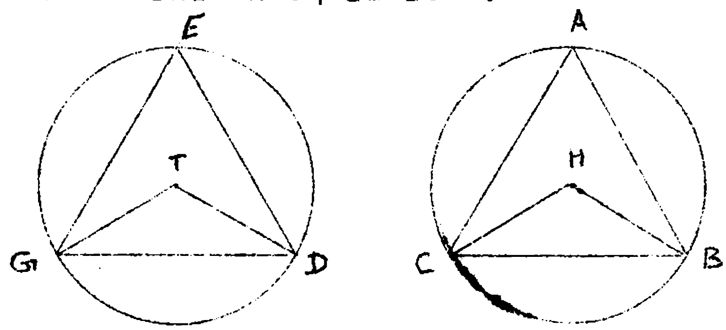
Because the line joining B E will be equal to A E, because in triangles A DE, B DE, two sides A D, D E and right angle

D, are equal to two sides D B, D E and the right angle D; and by equality of the angles A C E, C A E; C E will also be equal to A E. Hence, accordingly from the point E which is inside the circle, three equal lines be drawn towards the circumference.

Thus by proposition 9, the point E will be the centre and when we describe at point E with the distance A E, the arc A G B, complete circle is obtained.*

PROPOSITION 22 :

Equal angles either at the centres or at the circumferences in equal circles stand on equal arcs.



For example, in two equal circles A B C, D E G, two angles A, E at the circumference, are equal, Similarly, the angles H, T at the centres.

We say that the arcs B C, D G are mutually equal.

Because if we join the two chords B C, D G turn out to be equal, due to equality of two sides B H, H C and the angle H of the triangle C B H respectively with the two sides D T, T G and angle T G A of triangle T D G.

Hence the similar segments B A C, D E G which are situated on the two equal lines B C, D G , by preceding proposition, will be equal. If we subtract these two segments from the circumferences, the two arcs B C, D G are also left equal. This is what was wanted.

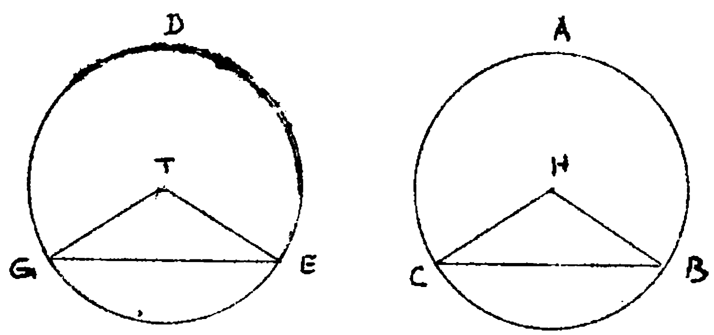
From this assertion the converse of the proposition is also proved, i.e. the angles which stand on equal arcs of two equal circles, are equal whether they stand at centres or on the circumferences.*

In this Proposition Ghulām Ḥusain combines T.U. III.25 and 26 into one, where-as 26 is the converse of 25.

Here also as in Proposition 19 other cases that "angles at the centre or at the circumferences" which are equal to or greater than two right angles has not been considered.

PROPOSITION 23 :

The arcs of equal chords in equal circles are equal, the major with the major and minor with the minor and similarly the chords of equal arcs in equal circles are equal.



Let the chords B C, E G which are in the equal circles A B C, D E G be equal.

I say that the two arcs B A C, E D G and the arcs B C, E G are equal.

Let H, T be the centre of the circles.

We join the lines B H, C H, E T, G T.

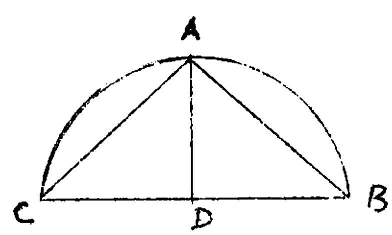
Hence, due to equality of the corresponding sides of the two triangles B H C, E T G, angles H and T are equal; and by previous preposition the twoarcs B A C, D E G and also the two arcs B C, E G will be equal.

Suppose arcs B C, E G are equal, their chords will also be equal, because of equality of these arcs, by converse of the earlier preposition, follows from equivalence of the two angles H, T, and equality of these two angles and equality of the lines B H, C H, E T, G T necessarily imply equality of the two chords B C, E G.

This is what was wanted. *

PROPOSITION 24 :

We wish to bisect an arc.

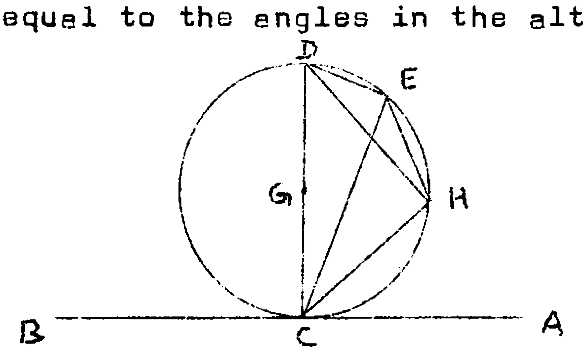


We join the chord B C and bisect it at D, and from D we draw a perpendicular A D on it.

Hence this perpendicular bisects at the arc at point A. Because when we join A B, A C, we get equal lines due to equality of B D and D C; and A D is common to both and the angles D being right. It necessarily follows B A, C A are equal; that their arcs are also equal, by previous proposition. *

PROPOSITION 25:

If a straight line is tangent to a circle and another straight line from the point of contact be drawn which divides the circle into two segments and the two angles will be the tangent, we say that each one of these two angles will be equal to the angles in the alternate segments.



Let the line A B be tangent to the circle E H D at point C and from the point C let line C E be drawn and divide the circle in two segments C H E, E D C and produce with the tangent the two angles A C E , B C E. We say that the angle

E C A will be equal so that angle which lies in the segment C D E and the angle E C B to that which lies in the segment C H E.

To prove the hypothesis, we draw the diameter C G D and join D E.

We say that the two angles D E C, D C A are rights, first by Proposition 15 and second by Proposition 16.

And it is evident that the angles E C A, E D C together with the angle E C D is equal to right angle. Therefore, they are equal to one another.

Hence equality of the angle A C E with the angle C D E which is in the segment C D E is proved.

Further we fix on the arc E C a point H and join E H, C H.

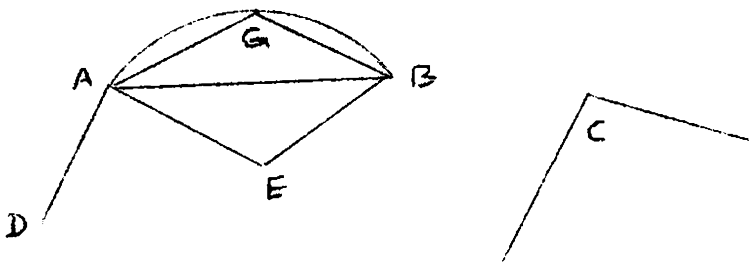
We say that the angle B C E is equal to the angle C H E which lies in the segment C H E.

Because whenever we join H D, the angle C H E is divided into two angles E H D, D H C and the first is equal to the angle E C D, because of both being on the same arc, and the second angle is right, because of its location in the semi segment. Hence whenever the equal angles E H D, E C D be added in two right angles C H D, B C D, two equal angles C H E, B C E are obtained.

This is what was wanted. *

PROPOSITION 26 :

We wish to construct on a given straight line a segment of a circle such that it admits a given angle.



Let A B be the given line and C the given angle.

We describe at point A with the line A B, the angle B A D equal to the angle C and from point A draw a perpendicular A E to the line A D and construct at point B of the line A B, angle A B E equal to the angle B A E and draw A E, BE till they meet at point E. Construct at centre E, with distance A E, the segment A G B, which is required.

Because A D is perpendicular on the radius A E;

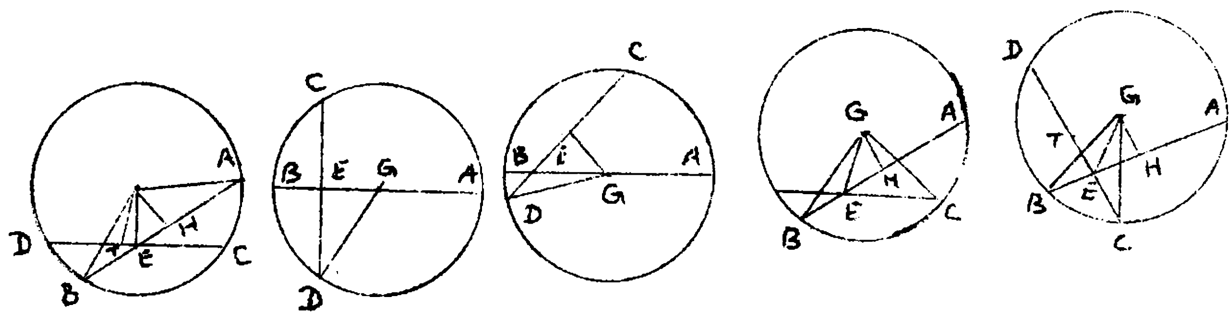
therefore, by proposition 14, will be tangent at point A of the circle of this segment, and by preceding proposition the angle G which lies in the segment A G B will be equal to the angle B A D i.e. the angle H.

This is what was required.*

Here Ghulām Husain does not consider the cases that the given rectineal angle is acute or right or obtuse. Euclid proves all the three cases separately. But Simson remarks on it, that the first and third cases, those namely in which the given angle is acute and obtuse respectively, have exactly the same construction and demonstration, so that there is no advantage in repeating them". (47,ii; 70). Perhaps, because of this reason Ghulām Husain treated with the cases as one. The demonstration of the second case in which the given angle is right angle has already been shown in a round about way.

PROPOSITION 27 :

If two straight lines which intersect in a circle, the surface contained by the two segments of one line will always be equal to the surface contained by the two segments of the other line.



Let two chords A B, C D intersect at point E.

We say that the surface A E . B E is equal to the surface E C . E D.

In this proposition there are many situations, because both the intersecting lines may be diameters or one of them or none is a diameter and secondly, the intersection be at right angle or not right and thirdly, a chord bisects the other or not. Hence in all, there are five cases.

The rule in the first case is obvious;

and in case II, suppoxe the diameter A B intersects the chord C D perpendicularly and G is the centre of the circle. Join G D and it is clear that the line A B is bisected at G and divided at E. On this account the surface A E. B E together with the square on E G is equal to the square B G, by proposition 21 of 2 i.e. the square on G D or the two squaresd G E, E D and by deleting common square on G E, the remaining surface A E . B E is equal to the square on E D, i.e. the surface C E . D E, because C E, D E by proposition 3, are equal.

In case III, from G we draw a perpendicular G H to the chord C D.

We say that the surface A E . B E together with the square on G E i.e. with two squares on G H, H E, is equal to the square on G B, i.e. the square on G D or two squares on GH, H D and when we delete the common square onGH, the remaining surface A E . B E together with the square on E H equals the square on H D, and also the surface C E . E D together with the square on E H is equal to the square H D, and when we delete the common square on H C, the remaining surface A E . E B equals the surface E G . E D.

In case IV, that the chord A B is a bisector of the chord C D.

Join the lines G B, G C, G E and from G draw a perpendicular G H to A B. In this case the surface A E . E B together with the square on E H is equal to the square on B H. If we add the common square on G H, the surface A E . E B together with two squares on E H, H G i.e. square on G E will be equal to two squares on G H, H B i.e, square on G B or the square on G C. i.e. two squares onE G, E C, a nd when we delete the common square on E G, the remaining surface A E . B E equal to the square on E C i.e, equal to the surface E H . D E.

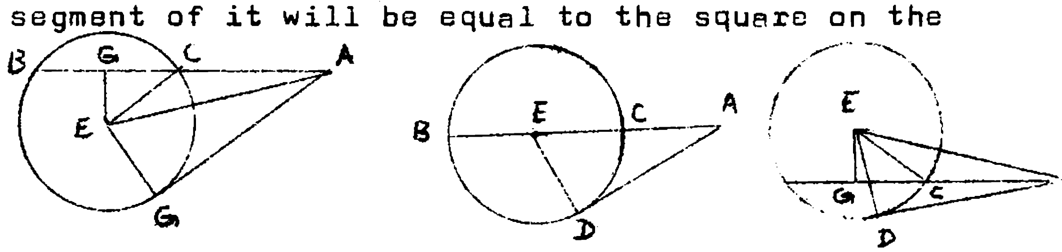
In case V, also, we join the lines G B, G C, G E and from G draw two perpendiculars G H, G T to two chords A B, C D. Accordingly these perpendiculars either they lie in one side of the line G E or in both side of it and for the quantification the surface A E . B E together with the square on H E is equal to the square on H B and we take the common square on H G, we obtain the surface A E . E B together with the two squares on H E, H G, i.e. the square on G E equal to two squares on B H, G H i.e. square on B G and also the surface C E . E B together with the square on T E is equal to the square onT C, and we take the common square on G T, the surface E C . E D together with the two squares T E, T G, i.e. square on G E will be equal to two squares on C T, T G i.e. square on C G or square on B G and when we take common square on E G, the remaining surface A E . B E equal to the surface E C . D E.

This is what was wanted.*

In the proof of this proposition Euclid considers only two cases (1) the two straight lines are diameters and (2) both the lines intersect but do not pass through the centre. Apart from these two cases Ghulam Husain gives three more intermediatry cases namely, (1) that in which one line is a diameter and bisects the other which does not pass through the centre at right angles (2) both the lines intersect at right angle and do not pass through the centre and (3) that in which one chord bisects the other but does not pass through the centre.

PROPOSITION 28:

From a point which is outside a circle and two straight lines are drawn, one of them is secant and the other is tangent, then the surface contained by the entire secant into the external segment of it will be equal to the square on the tangent.



Let A be the point and from it two lines A C B, A D are drawn, towards the circle B C D and the first line intersects the circle and second is tangent to it.

We say that the surface A B . A C is equal to the square on A D.

There are two different cases. Because, either secant passes through the centre or in between the centre and the tangent or does not exist.

Hence if it passes through the centre which is the point E.

Join E D and we say that since the line B C is bisected at E and has been extended straight A C, then by preposition 42 of 2, the surface A B . A C together with the square on E C is equal to the square on A E i.e. two squares on A D, DE or two squares on A D, E C and when we delete the common square on E C, the remaining surface A B . A C equal to the square on A D.

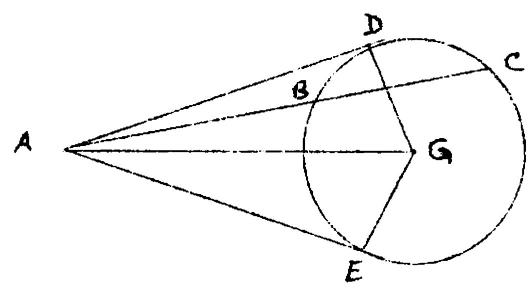
If secant does not pass through the centre,
We join the lines A E, E C and from E draw the perpendicular E G to the secant.

We say that the surface A B . A C together with the square on C G is equal to the square on A G and we take the common square on E G, the surface A B . A C together with two squares on C G, E G i.e. square on C E will be equal to two squares on G E i.e. A E or two squares on E D, A D and when the common squares on E C we delete, the remaining surface A B . A C equals the square on A B.

This is what was required. *

PROPOSITION 29 :

If two straight lines be drawn from a point outside a circle and one of them is secant and other terminates on it, then if surface contained by the whole secant and external segment of it is equal to the square on the terminating line, then terminating line will be tangent to the circle.



For example, from the point A towards the circle B D C, two straight lines in which first is secant and other the terminating lines, are drawn. And the surface A C . A B is equal to the square on A D.

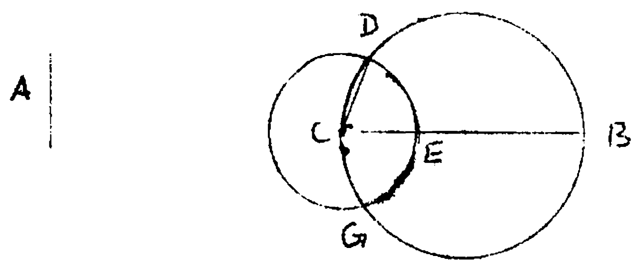
We say that A D will necessarily be tangent.

Join the centre G and the point A; and from A draw the line A E which is tangent to the circle at point E; we state that by preceding preposition square on A E is also equal to the surface A C . A B, for this reason, A D, A E are equal.

In triangles A D G, A E G, the corresponding sides are equal, so the angle A D G will be equal to the angle A E G, and angle A E G, by proposition 16 is right, Hence angle A D G will also be right and by proposition 14 is tangent to the circle.*

PROPOSITION 30 :

We wish to draw a chord in a circle equal to a given line provided that the line is not longer than the diameter of the circle.



Let A be the given line in the circle B C D.

We draw the diameter B C of the circle, and separate C E from the diameter equal to A and describe at point C, distance C E, the circle D E G and join C D, which is the chord of the circle B C D equal to C D, i.e. A.

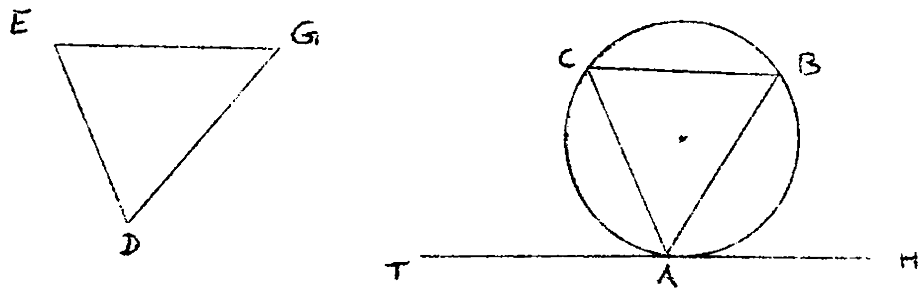
This is what is required.*

The condition that a straight line should be shorter than the diameter of a given circle is because its end points are on the circumference of the circle.

This problem has number of simple solutions.

PROPOSITION 31 :

We wish to construct a triangle in a circle whose angles will be equal to the angles of a given triangle.



Let A B C be the angle and D E G the given triangle. First we draw*the line H T which will be tangent at point A on the circle, and describe at point A with the line A H, the the angle H A B equal to the angle E and angle T A C equal to G, and join B C.

Hence the triangle A B C constructed will be that was required.

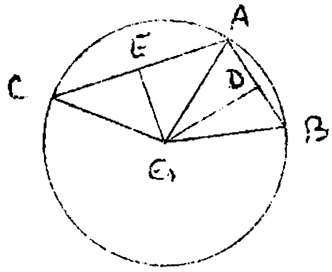
Because by proposition 25, inevitably, the angle C will be equal to the angle H A B i.e. equal to the angle E and angle B equal to the angle T A C i.e. the angle G and due to equality of the angles of two right angle triangles, the remaining angle B A C will be equal to the angle D.

This is what was wanted.*

This proposition may have an infinite number of solutions because infinite number of points may be taken as an angular point of the triangle.

PROPOSITION 32 :

We wish to describe a circle on a given triangle.



For example, on the triangle A B C.

We take two of its sides that from the angle which will not be smaller, like the two sides B A, A C, they are bisected at two points D, E and from the middle point of each of the two perpendiculars D G, G E are drawn so that both the perpendiculars meet at point G and join the lines G A, G B, G C and all these three lines are equal, because in two triangles B D G, A D G, two sides B D, A D are equal and side D G is common and two angles D are right. Therefore A G, B G are equal.

And same conditions hold in the triangles A E G, C E G.

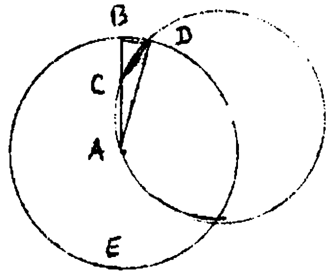
Therefore A G, G C are also equal.

Thus if at point G, with distance of these three lines describe the circle B A C, it passes through all the three angles of the triange.

This is what is required.

PROPOSITION 33:

We wish to construct an isosceles triangle so that every angle at the base will be double of the angle at the vertex.



First, divide a bounded line A B, at point C, with the help of proposition 46 of 2, so that the surface contained by A B . B C is equal to the square on A C. Later at point A, with the distance A B, we draw a circle B D E and from point B draw the chord B C equal to A C. Join A D. The triangle A B D is required.

Join C D and describe the circle A C D on the triangle A C D, in this case, two lines A B, B D which are drawn from B towards the circle A C D. The former is secant and the latter terminating line. The surface B A . B C is equal to the square on A C i.e. B D. Hence line B D will be tangent to the circle A C D and also the line D C which passes through the point of contact which is D.

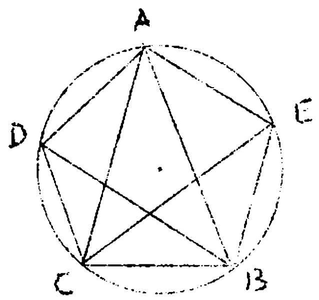
Hence, angle C A D which lies in the segment C A D which will be equal to the angle B D C and taking the angle C D A common, we obtain the angle B D A i.e. angle B equal to the two angles C D A, C A D i.e. exterior angle B C D. Thus in triangle D C B, two sides C D, B D are equal and two lines A C, D C which are equal to D B are equal and angle C A D will be equal to the angle C D A and angle B D C was also equal to the angle C A B. Hence it is established that each one of the angles A D B, A B D is double of the angle. A.

This triangle is called Penta-triangle.*

This construction is the core of the inscription of the regula pentegon, the only very complex problem in Book IV. The other inscription and superscription are straight forward and largely independent of one another. (60; 189). Ghulām Husain does not incorporate such independent problems like IV, 3,4, 6,7,8,9 in this book.

PROPOSITION 34:

We wish to construct a pentagon in a circle.

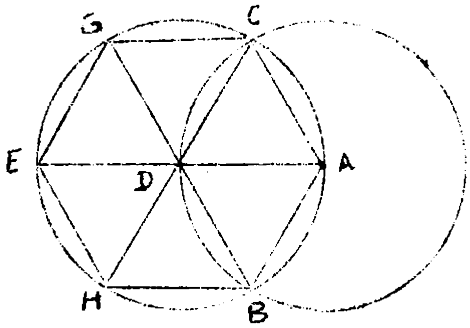


We construct a triangle A B C in the circle similar to the penta triangle, i.e. their angles are equal to the angles of the pente triangle. We bisect the two angles A B C, A C B by the lines B D, C D, and join the lines B E, E A, A D, D C ,till the pentagonal surface A E B C D is obtained.

Because the angles B A C, A B D, D B C, A CE, E C B are equal. Therefore, their arcs and chords are also equal. Thus all the five sides will be equal and since each of the angles stands on three equal arcs, for this reasons, angles are also equal. *

PROPOSITION 35

We wish to construct a hexagon in a circle.



Let A B C be the circle and D the centre.

We describe at point A, which is on the circumference, with distance A D, the circle B D C and join two lines B D, D C, and draw three lines A D, B D, C D from D to the points E, G, H and join the chords A B, B H, H E, E G, G C, C A till a hexagon is completed.

Because it is clear that six equal equilateral triangles are obtained; therefore, each six joining lines are equal and each angle which consists of two angles of the triangle will be equal to the other angles and also from this description it clears that the side of hexagon will be equal to the radius.

Ghulam Husain has shown how to describe regular polygons with 3, 5 and 6 sides. These are sufficient to describe any regular polygon with double the number of sides.

CHAPTER SEVEN

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THEORY OF PROPORTION AND ITS APPLICATION

In the first part of this chapter, the principles of general theory of proportion and the rules of simple, compound and derivative proportion have been discussed. The second part deals with ratios and proportions, related to similar triangles, parallelograms, polygons and bodies. This chapter is based on Section Four of JĀME-I-BAHĀDUR KHĀNI.

7.1 DESCRIPTION:

1. A ratio means the value of a magnitude towards the value of another magnitude when the two are of the same kind i.e. the line be compared with the line and surface with the surface and solid with the solid. Hence in fact ratio is a relative quantity and one of the essential accidents (attributes) is the absolute quantity.

It is evident that as quantification of discrete quantities is not accomplished without attributing certain essential of continuous quantities like assumption of sub dividing without end, similar to quantification of the continuous quantity without attributing certain discrete quantity is not obtained like combination of unit numbers. Besides ratio has some properties of number whose attainment in the continuous quantities is not permitted e.g. the square root towards the original quantity or both towards the cube, a ratio is obtained. Because here is a kind which is absolute that does not admit any change in contrast to a line considered as a side, then it has no ratio to the surface of the square or body of the cube, because of difference in kinds. Whenever between the side and the surface of the square or cubic body, a ratio is obtained, it will not exist, except as attributing number to the side and also some of properties of ratio is like the properties of the continuous quantities e.g. the partition of a line as in Proposition 46 of 2, and the necessary existence of a square between whom the ratio is two or three times, and the existence of incommensurability between the lines which do not have a common factor in contrary to the numbers whose partition is not possible at all. The number which is double or treble of a whole number whose roots are taken, will not be a whole number. Incommensurability of a number is not like the incommensurability of straight lines because in number unity exists as a common factor. Therefore, the quantities such that some have a ratio with some other, are those which on account of separation of some over some, increase.*

The word ratio is a Latin word (From the verb reri, to think or estimate; past-participle, ratus. Hence ratio meant reckoning, calculation, relation, reason) and was commonly used in the arithmetic of the Middle Ages to mean computation.

In the Middle Ages the distinction between ratio and fractions, or ratio and division, become less marked, and in the Renaissance period it almost disappeared except in cases of incommensurability. In one place Leibniz speaks of ratio or fraction (69;480). It is true that there is no explicit definition of ratio, it is a relationship between two homogeneous magnitudes. But a very definite notion of ratio is implied in the definition of proportion.

2. Magnitudes are in the same ratio means the ratio of the first to second will be as the ratio of the third to fourth. If equimultiple whatever be taken without end of the first and the third, and another equimultiple of the second and the fourth, the former multiple will be alike exceed over the latter or less or equal, provided the equimultiples taken are compared in order.*

This definition, in fact, is the principal source of the modern view of irrational numbers, with this theory, which was a major contribution to number theory, mathematics for the first time could take into account incommensurable quantity (e.g. those whose ratio is not the quotient of two integers, such as the diameter and circumference of a circle), as well as such well known commmunsurable as two sides of certain triangles(80;1021). De Morgan put its meaning very clearly:

"For magnitudes, A and B of one kind, and C and D of the same or another kind, are proportional when all the multiples of A can be distributed among the multiples of C among those of D; and if m A lies between n B and (n+1) B, m C lies between m D and (n+1) D. The best testimony to its adequacy is that Weierstrass used it in his definition of equal numbers; (81;419) and according to Heath (47ii;124-126) it exactly corresponds with the modern theory of irrational due to Dedekind.

Saccheri remarks regarding pre-supposed notion of greater, equal and less, that he sees no advantage in this definition. But over all Fine (37;74) remarks that for synthetic geometry no simpler or more elegant definition of proportion than this definition can be desired and the proposition satisfies the criterian for proportionality set out in this definition (40;10-14).

3. And such magnitudes are called Proportional. *

4. If multiples of the two above mentioned types be taken, the first multiple exceeds the second multiple and the third multiple does not exceed the fourth multiple, provided the equality of repetition of the first and the third and the same of the second and the fourth, then the ratio of the first to second will be greater than the ratio of the third to fourth.*

Saccheri remarks (47,ii;130) that the ratio of the first magnitude to the second will also be greater than that of the third to the fourth if, while the multiple of the first is equal to the multiple of the second, the multiple of the third is less than that of the fourth. This case has not been considered by Ghulām Ḥusain in his definition, as Naṣīr al-Dīn Ṭūsī also left.

5. Magnitudes which are proportional the least order is of three terms.*

6. The middle one is repeated and the magnitude which is related named antecedent and to which it is related is termed as consequent.*

7. Inverse ratio means taking the antecedent as consequent and the consequent as the antecedent. *

8. Alternate Ratio means taking the second antecedent as the first consequent and the first consequent as the second antecendent i.e. ratio of the antecedent to the antecedent and the consequent to the consequent. *

9. Composition of a ratio means taking the ratio of the sum of the antecedent and the consequent to consequent.*

10. Separation of a Ratio means taking the ratio of the excess of antecedent over the consequent to consequent.*

11. Converse of a Ratio means taking the ratio of the antecedent to the excess of the antecedent over the consequent.*

12. Equality of ratios are that which occur in the ratio of two kinds of magnitudes whose numbers are some and the ratio of every two magnitudes of one kind will be as the ratio of their corresponding magnitudes of the other kind. The ratio of equality is of two types i.e. ex-equali (Mūntazma) and perturbed (Mūztarabā).*

13. Ex-equali means orderly ratio e.g. the ratio of the antecedent to the consequent is as the ratio of the antecedent of the other to the consequent of the other and the first consequent to the other is as the consequent of the other to their corresponding.*

If x, y, z, ... be one set of the magnitudes, and X, Y, Z, ... another set of magnitudes, such that

x : y = X : Y

y : z = Y : Z

. . . .

. . . .

l : m = L : M

then, x : m = X : M.

14. Perturbed which is not ordered e.g. the ratio of the antecedent to the consequent is as the ratio of the antecedent of the other to the consequent of the other and the consequent of the first to the other is as the antecedent of the other to the consequent of the other. *

If x, y, z be magnitudes and other X, Y, Z, such that

x : y = Y : Z

and y : z = X : Y

then, x : z = X : Z

15. It is not hidden just as a ratio is associated with magnitude, similarly the compasition and division is also associated with the explanation that the original quantity is considered as in itself because it is as a

magnitude and when the original quantity is considered not as itself, then another quantity is associated with it which is the ratio. Thus the ratio of the original quantity in itself when considered is simple.*

16. And if the original quantity is considered in terms of two ratios, however, the ratio obtained, by multiplication of magnitudes of two other ratios, is called Compound Ratio. *

17. If the ratio obtained by division of the magnitude of one of the two ratios by the magnitude of the other, it is the derivative ratio. *

18. If the two simple ratios are similar and those ratios which are compounded by them, it is called double.*

19. If it is compounded twice, the ratio obtained is triple.*

20. Similarly if a ratio is divided by another ratio and the ratio obtained, like partitioned ratio, named first partition. If a ratio is divided two times, the ratio obtained in the end likepartitioned ratio, is called second partition. *

Hence it is known from the previous description that every simple ratio becomes compound and derivative ratio by adjoining the given properties; and every compount and derivative ratio by negating those properties become simple.

21. A straight line cut in mean and extreme ratio means the ratio of the line to the greater segment is as the ratio of the greater segment to the smaller. *

22. Similar Surfaces means their corresponding angles are equal and the bounding sides of equal angles are proportional. *

23. Similar Sided Surfaces are those whose sides are proportional according to order i.e. in every one of those surfaces (equal ratio between) antecedent and consequent exist. *

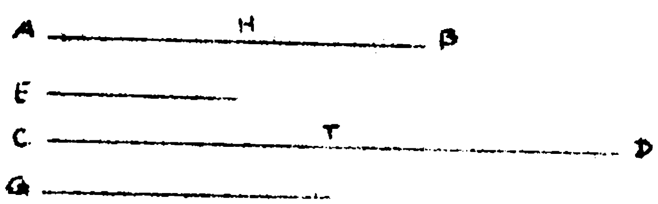
24. The altitude of a figure is a perpendicular which falls from its vertex to the base.

Every small magnitude which is formed by dividing the longer quantity is called the factor and division and the greater one is called the multiple of the smaller one. *

7.2 PROPOSITIONS:

PROPOSITION 1:

If four magnitudes are of the some kind and a first be multiple of a second, a third will be multiple of a fourth, the sum of the first and third will be multiple of the sum of the second and fourth as they were individually. *



For example, the magnitude A B is multiple of E and similar to this C D as G.

We say that the sum of A B, C D is multiple of the sum of E, G as A B is multiple of E alone.

To prove this claim, we divide A B at H equal to the magnitude of E and C D at T equal to G.

Hence the sum of A H, C T is equal to the sum of E, G and the sum of H B, T D, second times, will be equal to the sum of E, G.

Hence the number of multiples in the sum A B, C D of the sum E, G is some as that of the first two multiples, in sense of its parts alone.

This is what was required. *

To find the corresponding formula for the result of this Proposition we suppose x to be the second and y the fourth magnitudes. So according to the proposition, if m x, m y be the first and the third magnitudes which are multiples of x and y, then,

$$m\,x + m\,y = m\,(x+y)$$

The general argument may be represented as follows:

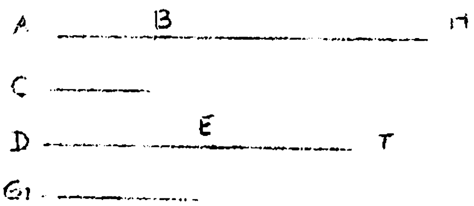
If m x, m y, m z etc. be any multiples of x, y, z, etc. then,

$$m\,x + m\,y + m\,z + \ldots = m\,(x + y + z + \ldots\ldots)$$

The four magnitudes in this proposition must all be of the same kind and Ghulām Husain inserts: "of the same kind" in the enunciation. But neither Euclid nor Nasīr al-Dīn Tūsī have incorporated it in the statement of the proposition.

PROPOSITION 2:

If in the magnitudes of the same kind, a first be multiple of a second as a third of a fourth and a fifth be multiple of the second as a sixth of the fourth, then the sum of the first and fifth will be the multiple of the second as the sum of the third and sixth of the fourth.



For example, A B is multiple of C as D E is of G and B H is multiple of C as E T is of G, then the sum A H will be multiple of C and also sum D T of the G,
Because the number of multiples of C in A B is same as that of G in D E, and similarly the number of multiples in B H is same as in E T; and it is clear that when the equal magnitudes be added in equal, the equal will be obtained. Hence the number of multiples in A H will be equal to the number of multiples in D T.*
In symbolic form the result of this proposition may be expressed as follows:
if x be the second magnitude, and y the fourth and m x is the first, the third is m y and nx be the fifth, ny is the sixth, then,

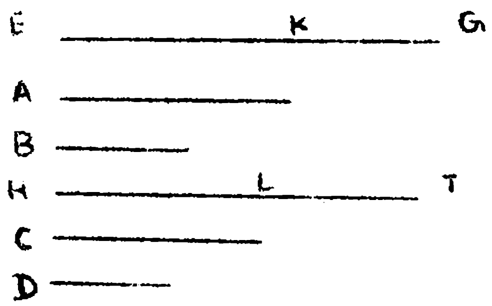
$$m x + n y = (m + n) x$$

and $$m y + n y = (m + n) y$$

where m, n are integers.

PROPOSITION 3:

If a first will be multiple of a second as the third is of a fourth and equi-multiples of the first and third be taken, then the multiple of the first will be multiple of the second as the multiple of the third is the multiple of the fourth.



For example, A is multiple of B as C is the multiple of D and take any equi-multiple of A whatever like and that is E G and another of C and that is H T.

We say that E G will be equi-multiple of B that H T is of D.

Because if we divide E G at K equal to the magnitude of A and H T at L equal to C, then E K i.e. A will be multiple of B, as H L i.e. C is of D, and K G i.e. A is multiple of B as L T i.e. C of D;

hence by preceding Proposition the total E G will be multiple of B as the total H T is of D. *

The enunciation of this porposition may be represented as follows:

If x be the second magnitude and y the fourth and if m x be the first and m y the third and if n. m x, n. my are equi-multiples of m x, m y, then m. n x is the same multiple of x, that n. m y is of y.

Muller remarks (60;124) that insteadof proving

$$m \cdot (n \cdot x) \simeq (m \cdot n) \cdot x$$

Euclid (here Ghulām Husain also) proves the weaker, less explicit

$$y_1 \simeq m \cdot x_1 \text{ and } y_2 \simeq m\, x_2 \text{ and } z_1 \simeq n\, y_1$$

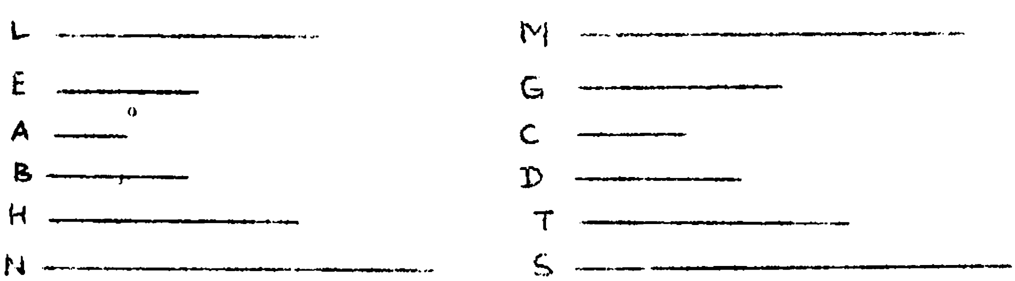
$$\text{and} \quad z_2 \simeq n\, y_2 \longrightarrow z_1 \simeq k \cdot x_1 \text{ and}$$

$$Z_2 \simeq k\, x_2$$

Ghulam Husain proves this proposition for the case n = 2, making it possible to use the preceding proposition.

PROPOSITION 4:

If four magnitudes are propostional and equi-multiples of a first and a third be taken and of a second and a fourth, then the ratio of the first multiple to the third multiple will be as the ratio of the second multiple to the fourth multiple.



For example, A, B, C, D are proportional and take equi-multiples of A, C which are E, G and another equi-multiple of B, D, which are H, T.

We say that the ration of E to H will be as the ratio of G to T.

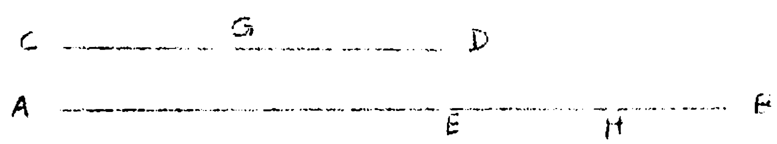
Because of equi-multiples of E, G be taken as L, M and similarly of H, T as N, S, then L, M will be multiple of A, C and N, S of B, D, by previous proposition; and L, M, by preamble which is given in the description, are excess or less or equal respectively of N, S.

Hence in this process if multiples, which have been taken of E, G and H, T first two are either excess over the last two or less or equal. Therefore, by the converse of the preamble, the ratio of E to H will be as the ratio of G to T.

This is what is required. *

PROPOSITION 5 :

If there are two magnitudes, such that one is the multiple of other and subtracted from them are two magnitudes, one being the multiple of other as the whole is of the whole, so whatever remains from the multiple will be a multiple of the other remainder.



For example, A B is the multiple of C D.

We subtract A E, C G from these two and A E is the same multiple of C G.

We say that B E will be same multiple of G D.

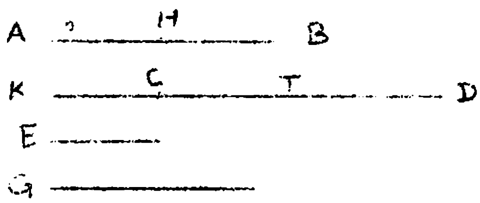
Because if B E is not same multiple of G D, then let the multiple taken of G D be E H and by Proposition 1, the total A H is same multiple of C D and A B will also be so. Hence A H, A B the part and whole which are the same multiple of C D, will be equal. This is a contradiction. Hence result is established.*

This Proposition indeed corresponds to Proposition 1 of 3 with subtraction instead of addition. If x, y are two magnitudes, then according to the statement of the proposition, we have

$$m x - m y = m (x - y).$$

PROPOSITION 6:

If two magnitudes are multiple of the two other magnitudes and from them we subtracted, the two other equi-multiples which are the remainders, are either equal to the same or equi-multiple of them.



For example, A B, C D are equimultiples of E, G and A H which is subtracted from A B is equimultiples of E and C T which is subtracted from C D is same multiple of G. I say that if the remainder H B is equal to E, the remainder T D will also be equal to G and if it is multiple of E, T D will also be same multiple of G; and we take C K equal to G or multiple of G, and similarly B H with respect to E. Then the first A H will be multiple of second E; similarly, the third C T is multiple of the fourth G and the fifth H B of the second E, that the sixth C K is of the fourth G. Therefore, by Proposition 2, the total A B will be multiple of E as that the total K T is the multiple of G and CD being equal to this multiple;

therefore, K T, C D will be equal.

After deleting C T from both, the remainder K C is equal to the remainder T D.

Hence if C K is equal to G, then T D will also be equal to G; and if it is multiple, then D T will also be same multiple.

This is what is required. *

Muller commented (60;124) that instead of proving

$$m \cdot x - n \cdot x \simeq (m - n) \cdot x$$

Euclid (Ghulām Husain also) proves the weaker, less explicit

$Y_1 \simeq m \cdot x_1$ and $y_2 \simeq m \cdot x_2$ and

$z_1 \simeq n \cdot x_1$ and $z_2 \simeq n \cdot x_2 \longrightarrow$

$y_1 - z_1 \simeq kx_1$ and

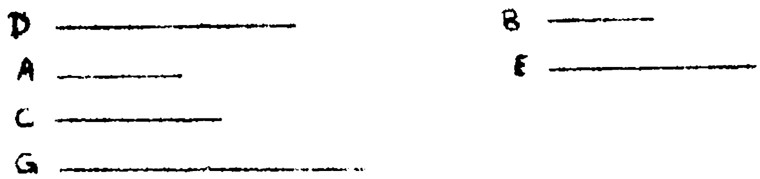
$y_2 - z_2 \simeq k\, x_2$

It is true that Propositions 1 -3 , 5 and 6 of III are independent from anything where as the other propositions of this chapter depend on one or both of the fundamental definitions.

It may be observed here that Ghulam Husain like other editers proves only first case as given in Tahrīr-i-Uglīdīs but leaves without proof the second case that if H B is multiple of E, then E T D will also be same multiple of G.

PROPOSITION 7:

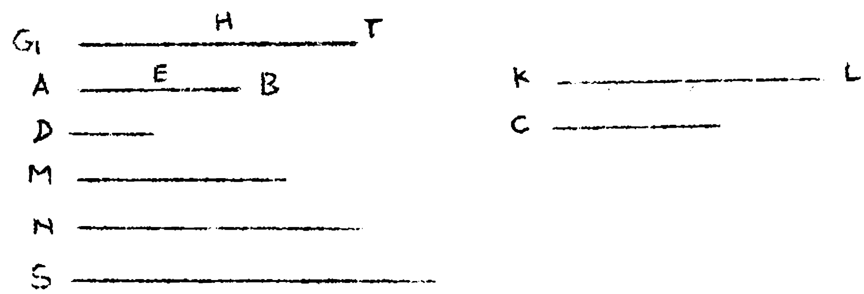
If magnitudes are equal, their ratio to the same magnitudes are equal and similarly, the ratio of the same magnitude to equal magnitudes will also be equal.



Let the magnitudes A, B be equal,
then the ratio of each one to C will be same and the ratio of C to each one of A, B are also same.
Because if equimultiples D, E whatever be possible, of A, B be taken and multiple G of C,
Then , each one of D, E will be either equal or less or greater than G and similarly from other side. Hence by converse of the definition the said ratio will be same.
This is what is required.*
In the consequence of this Proposition, Euclid states a Porism that if magnitudes are proportional, they will also be proportional inversely, But regarding the right place that where it should be put, there is a different opinion. Heath says (47,ii; 149) that "the Porism is no more in place here", whereas Theon put it at the end of V. 4 (Proposition 4 of 3). Ghulām Husain does not put at any place as Nasīr al-Dīn Tūsī also did.

PROPOSITION 8 :

If two magnitudes are unequal then the ratio of the greater among them to a third magnitude will be greater than the ratio of the smaller one to the third and the ratio of the third to greater of them will be smaller than the ratio of that to smaller.



Let A B be greater than C.

We say that the ratio of A B to D is greater than the ratio of C to D and the ratio of D to A B is less than its ratio to C.

We separate B E from A B dqual to C. And one of the two magnitudes A E, E B, which is not greater than its complement, can be multiplied to exceed D. Let A E have this property and we take its multiple such that G H is greater than D. If A E be greater than D without multiplying, we take a random multiple of that and that multiple is G H and of E B another multiple in the same number that is H T and similarly of C which is K L and it necessarily follows that H T, K L are equal and each one of them will be greater than D. We take double of D, that is M, and triple of it which is N. Similarly, we take successive multiples until it reaches its minimal multiples which is greater than K L, and that multiple is S and N which is before it, will not be greater than K L i.e. H T. If we add D in N, the magnitude S is obtained and combining G H with H T, G T is obtained and G H is greater than D. Hence the total G T will be greater than S and the sum G T is multiple of the sum A B. Therefore, K L is multiple of C. Hence equal multiples of A B and C and another multiple of D are obtained. Multiple A B is greater than the multiple D and multiple C does not exceed. Thus by the rule of the previous analysis, the ratio of A B to D will be greater than the ratio of C to it (D), and a multiple of D was obtained which did not exceed a multiple of A B but was greater than a multiple C. Hence the ratio of D to A B will be less than the ratio of it (D) to C.

This is what is wanted.*

This Proposition is in a sense the fundamental Proposition of Book V, It has an unusually complex proof.

If x, y are two magnitudes and $x > y$,

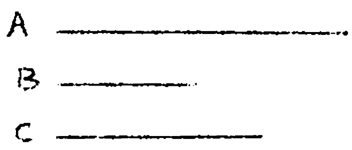
then $x : z > y : z$

and $z : x < z : y$

This Proposition is in a sense the fundamental proposition of Book V.

PROPOSITION 9 :

Magnitudes whose ratio to the same magnitude are equal will be equal and similarly the magnitude whose ratio of the same magnitude to them are same will be equal.



For example, the ratio of A to C is as the ratio of B to C.

I say that A, B are equal, and also the ratio of C to A is as the ratio of C to B, in this case also A, B are equal.

Because, in the case of difference, the ratio will also be different and although the ratios are equal by hypothesis, so a contradiction arises.

Therefore, the said rule is proved.*

If $x : z = y : z$

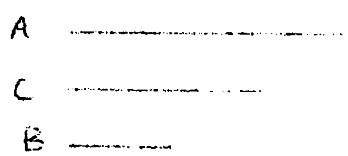
or if $z : x = z : y$

then $x = y$.

Mueller pointed out that this proposition is never explicitly used in Book V, although it probably would play a role in the proof of Case 2 of V. 14 (60;130) to which Euclid dismisses with "Similarly we can prove......". But Ghulam Husain used it in Proposition 10 explicitly.

PROPOSITION 10 :

If there be two magnitudes and ratio of the first to third be greater than the ratio of the second to it, then the first will be greater than the second and any of the two magnitudes, that the ratio of the third to it be greater, will be less.



For example, the ratio A to C is greater than the ratio of B to C.

We say that A will be greater than B.

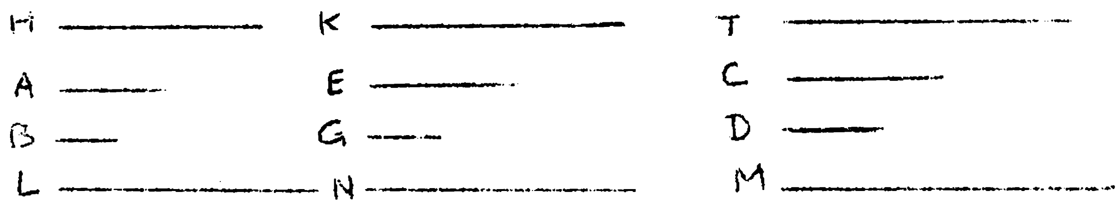
Because, if it be equal or smaller, then by Proposition 8 and 9, necessarily the ratio of A to C will be as the ratio of B to C or less than that. Both these are contradiction.

Hence, necessarily A will be greater than B.

And also the ratio of C to B is greater than its ratio to A. In this case also A will be greater than B, other wise a similar contradiction follows.

PROPOSITION 11

Ratios which are equal to a ratio, are also themselves equal.



For example, ratio of A to B is as the ratio of C to D and ratio of E to G is alsoas the ratio of C to D.

We say that the ratio of E, G isas the ratio of A, B. We take equimultiples whatever be possible, of A, C, E, and those will be H, T, K and we can take for B, D, G, equimultiples, those are L, M, N.

From this it follows that the ratio of A, B isas the ratio of C, D. L, M will be greater, less or equal corresponding to H, T. Because the ratio of C, D isas ratio of E, G; excess or less or equally will also follow correspondingly H, K with M, N. Accordingly, excess, less or equality for H, K with L, N there will be corresponding excess, less or equal.

Thus the ratio of A, B will beas the ratio of E, G. And this statement makes it clear that if one of the two equal ratios be greater than the third ratio, second will also be greater.*

Algebrically if

$$x : y = z : w$$

and $z : w = p : q$

then $x : y = p : q.$

Here Ghulām Husain extract an inference from the description that "if one of the two equal ratios be greater than the third ratio, second will also be greater i.e.

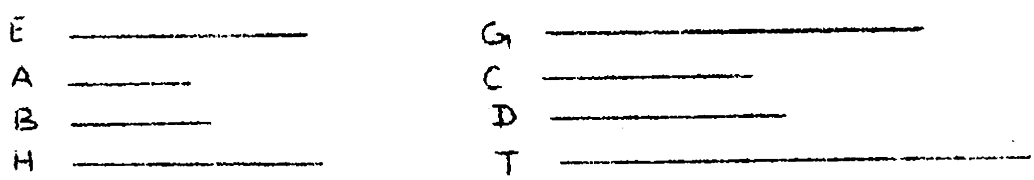
if $x : y = z : w$

and $z : w > p : q$

then $x : y > p : q.$

PROPOSITION 12:

If magnitudes are proportional, then the ratio of an antecedent to its consequent will be equal to the ratio of all the antecedents to all the consequents.



Let ratio of A to B be as the ratic of C to D.

We say that the ratio of A to B will be as the ratio of the sum A, C to B, D.

We take equimultiples of A, C and that are E, G and similarly of B, D and that are H, T.

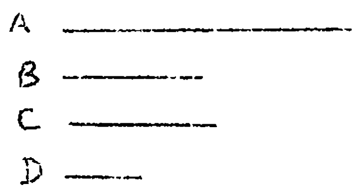
And since the ratio is same, thus if E will be less than H, then G will also be less then T and if equal, equal and if greater, greater.

Therefore, E, G with the sum H, T will be as the relation with E, H;

and it is necessary that the ratio of A to B will be as the ratio of the sum A, C to the sum B, D and on this hypothesis, the rule is established, if antecedents and consequents are taken together.

PROPOSITION 13 :

If four magnitudes are proortional and if the first is greater than the third, the second will also be greater then the fourth and the same holds good in case of being less or equal.



For example, the ratio of A, B is as the ratio of C, D and let the first A be greater than C.

We say that B is also greater than D.

Because, according to the Proposition 10, the ratio of A, which is greater, to B is greater than the ratio of C, lesser to B and the ratio of C to D is as the ratio of A to B. Therefore, the ratio of C to D will be greater than the ratio of C to B.

Hence B will be greater than D.

On the same hypothesis, we can prove, if equal and less.

if $x : y = z : w$

and $x > = < z$

then $y > = < w$.

Like Euclid, Ghulām Husain also omits the proof of the second and third cases of this Proposition.

For second case, if

$x = z$

and $y = w$

then $x : y = z : w$.

for third case, if

$x < z$,

then $y < w$.

for, $z > x$

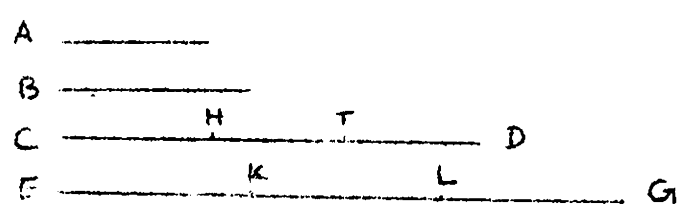
and since $z : w = x : y$

$w > y$ by first case.

Therefore $y < w$.

PROPOSITION 14 :

If two equimultiples be taken of the two magnitudes, then the ratio of the multiples will be as the ratio of the magnitudes.



For example, A, B are two magnitudes.

Let C D be equimultiples of A and E G of B.

We say that the ratio of C D to E G is as the ratio of A to B.

Divide C D at H, T equal to A and E G at K, L equal to B.

It is clear that the ratio of C H to E K is as the ratio of A to B and the ratio of H T to K L and the ratio of T D to L G;

and since C H, H T, T D are the antecedents, so the ratio of their sum to the sum E K, K L, L G which are consequents, is as the ratio of C H to E K, i.e. the ratio of A to B, by Proposition 12.

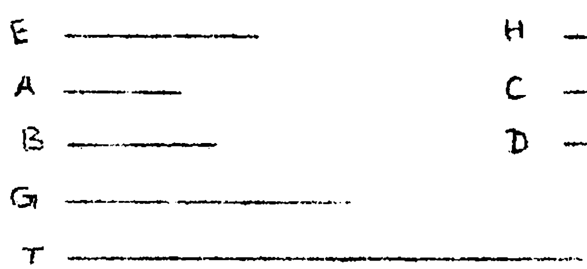
This is what is required.

Algebrically,

$$x : y = mx + my.$$

PROPOSITION 15 :

If four magnitudes of the same kind are proportional, they will also be proportional alternatively.



For example, the ratio of A to B is as the ratio of C to D. We say that the ratio of A to C is as the ratio of B to D.

To prove the claim, we take equimultiples whatever be possible, of A, B and that are E, G and of C, D are H, T.

Hence by previous Proposition, the ratio of A to B is as the ratio of E to G and similarly, the ratio of C to D will be as the ratio of H to T.

Therefore, the ratio of E to G will be as the ratio of H to T, and by Proposition 13, if A is greater than H, then G will also be greater than T, and if it is less, less and if equal, equal. Thus E, G which are the multiples of A, B respectively, are greater than H, T which are the multiple of C, D, or less or equal.

Hence the ratio of A to C will be as the ratio of B to D.

This is what is required. *

Algebrically, if

$$x : y = z : w$$

then $x : z = y : w$.

For (37; 74) it readily follows from definition: "let x,y,z,w be four magnitudes, x of the same kind as y, z of the second kind as w". If for all integral values of m and n, it be the case that accordingly as $mx \gtreqless ny$, so also is $mz \gtreqless nw$, then x is said to be in the same ratio to y as z to w" that

$$mx : my :: nz : nw.$$

By Proposition 13 and 14, accordingly as $mx \gtreqless nz$,

so also is $mx : my \gtreqless nz : mw$,

Therefore, $nz : nw \gtreqless nz : my$,

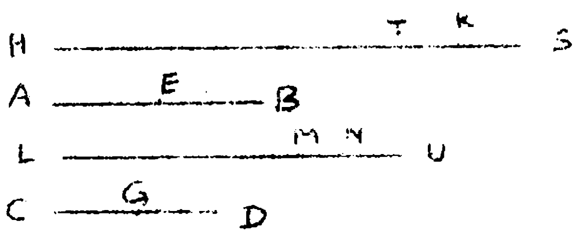
therefore, $my \gtreqless nw$.

Hence $x : z :: y : w$ (by above definition).

Here, the four magnitudes must be of the same kind and Ghulām Husain inserts "of the same kind (Mūtajānis)" in the enunciation.

PROPOSITION 16 :

If magnitudes are proportional componendo, they will also be proportional separando.



For example, the ratio of A B to B E is as the ratio of C D to D G by means of componendo.

We say that the ratio of A E to E B will be as the ratio of C G to G D by means of separando.

Take for each A E, E B, C G, G D every equimultiple possible which are H T, T K, L M, M N and then according to the rule of Proposition 1, the sum H K will be same multiple of the sum A B and similarly L N of C D.

And further we take another equimultiple for E B and G D, and those are K S, H N.

Since K T the first is multiple of E B the second, as N M the third of G D the fourth and K S the fifth of E B the second, similarly N O the sixth is the multiple of G D the fourth. Therefore by the rule of Proposition 2, the sum of T S will be same multiple of E B. Similarly, the sum M O of G D.

It is known that H K, L N are equimultiples of A B, C D and T S , M O are of E B, G D and the ratio of A B to E B is as the ratio of C D to D G.

Therefore, H K, L N are together greater than T S, M O or less or equal. If we delete the common T K, M N the remainder H T, L M are either greater than K S, N O or less or equal and H T, L M are equimultiples of A E, C G and K S, N O of E B, G D. For this reason, by converse of the Preamble which is given in the Description, the ratio of A E to E B will be as the ratio of C G to G D.

This is what is required.*

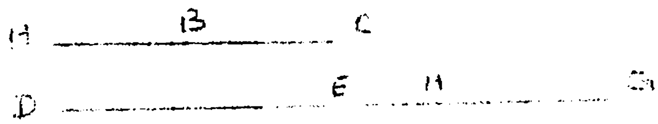
Algebrically we may represent this Proposition as follows:

$$\text{if } x : y = z : w$$
$$\text{then } (x-y):y = (z-w):w$$

It may be noted here that this Proposition could also derive from Proposition 19 ($x:y = z : w \quad x - z : y - w = x : y$) but such a derivation would represent a violation of the policy of making minimal assumptions about homogeneity.

PROPOSITION 17 :

If magnitudes be proportional separando, they will also be proportional componendo.



So, the ratio of A B to B C is as the ratio of D E to E G by means of separando.

We say that the ratio of A C to B C will be as the ratio of D G to E G by means of componendo.

If it is not so, let it be as the ratio of D G to H G which is e.g. less then E G and when according to earlier Proposition we take the separando, the ratio of A B to B C i.e. the ratio of D E to G E or as the ratio of D H to H G and D E is less than D H.

Hence by rule of Proposition 13, E G the total will be less than H G, the part. This is a contradiction.
Hence the result is proved.

To make most clear by using the Algebraic notation the proof implied by Euclid is as follows: (47,ii; 170)

Given, that $x : y = z : w$

Suppose, it is possible that

$(x + y) : y = (z + w) : (w \pm a)$.

Therefore, by Seperando (Proposition 16 of 4

$x : y = (z \mp a) : (w \pm a)$,

whence by Proposition 11 of 4,

$(z \mp a) : (w \pm a) = z : w$.

But $(z - a) < z$,

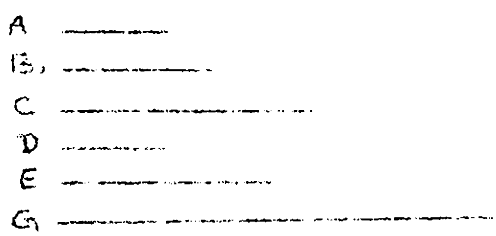
while $(w + a) > w$,

and $(z + a) > z$ while $(w - a) < w$.

which relation respectively contradict Proposition 13 of 4
In fact this proposition is converse of Proposition 16 to 4.

PROPOSITION 18 :

If there are two kinds of magnitudes in equal number and the ratio of every two magnitudes of one kind is as the ratio of the two magnitudes of the other and the ration occurred are in order, then those magnitudes, in the case of equality, will be proportional.



For example, A, B, C are one kind and D, E, G of other, and the ratio of A, B is as the ratio of D, E, and the ratio of B, C is as the ratio of E, G.

We say that the ratio of A, C will be as the ratio of D, G. Because when we take alternando of the magnitudes A, B, D, E; the ratio of A, D will be as the ratio of B, E; and the alternando of the magnitudes B, C, E, G will be the ratio of B, E as the ratio of C, G.

Thus by the rule of Proposition 11, the ratio of A, D will be as the ratio of C, G and by alternando, the ratio of A, C will be as the ratio of B, G.

And this is what is wanted.*

Ghulām Husain omits Proposition 19, 20 of Taḥrīr-i-Uqlīdes Book V and brings Proposition 22 first and then 21. Whereas Propositions 20-23 contain an important part of the theory of Compounded Ratio. Although, as I have already remarked that Euclid has not defined this term.

This Proposition states the fundamental proposition about the ratio ex-aequali in its ordinary form, that if

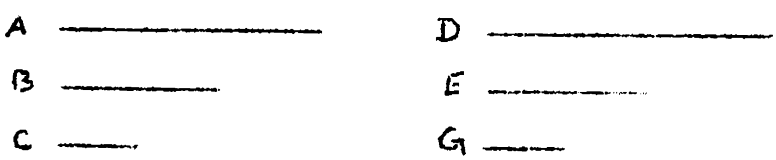
$$x : y = w : u$$

and $y : z = u : v$,

then $x : y = w : v$.

PROPOSITION 19 :

If there are two kinds of magnitudes in equal number and if the ratio of two magnitudes of one kind is the same as the ratio of the two magnitudes of the other and the occurrence of the ratio is perturbed, then, if the first of the first kind is greater than the last one, then the first of the other kind will also be greater than its last, and if it is equal, equal and if less, less.



For example, A, B, C are the magnitudes of the first kind and D, E, G the other; and ratio of A, B is as the ratio of E, G and the ratio of B, C as the ratio of D, E.

We say that if A be greater than C, D will also be greater than G and if equal, equal and if less, less. Let the first be greater. Then the ratio of A to B i.e. the ratio of E to G is greater than the ratio of C to B, by the rule of Proposition 10. Hence D will be greater than G. Similarly, in the case of equality or less, the claim is established. *

Algebraically, if there be three magnitudes x , y, z and others u, v, w and

$$x : y = v : w$$

and $$y : w = u : v,$$

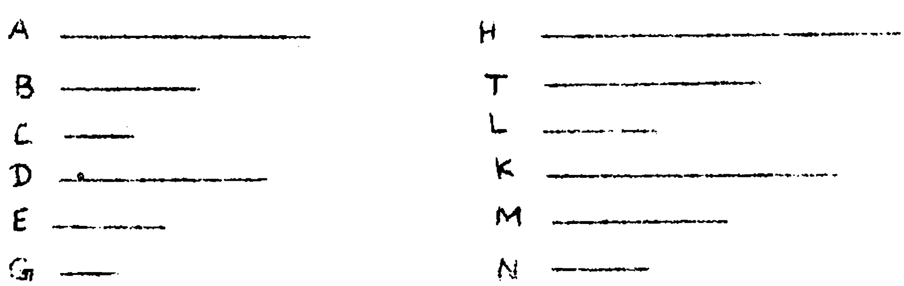
then according to the assertion of the Proposition

$$x > = < z, \quad u > = < w.$$

The enunciation of this Proposition is a generalized form of the statement given by Euclid in V. 21; in which he takes a counter example as the statement. He states "if there be three magnitudes...". Ghulām Husain also like the other editors does not prove the remaining cases, perhaps because of their obviousness.

PROPOSITION 20 :

If the ratio of the magnitudes of the two kinds be the same as in the above Proposition, then in the case of equality, they will be proportional.



For example, A, B, C are of one kind and D, E, G of the other and, the ratio of A, B is as the ratio of E, G and the ratio of B, C as the ratio of D, E.

we say that the ratio of A, C will be as the ratio of D, G.

We take equimultiples whatever be possible of A, B, C and those are H,T,K and similarly for the magnitudes C, E, G and those are L, M, N.

Hence by the rule of Proposition 14, the ratio of H,T will be as the ratio of A, B and the ratio of M, N, as the ratio of E, G. Hence the ratio of H, T will be as the ratio of M,N and similarly, the ratio of T, L will be as the ratio of K,M.

Hence it is clear that the magnitudes H, T, L of one kind together with the magnitudes K, M, N which is of the other, are proportional by means of perturbed like the two origi-onal kinds.

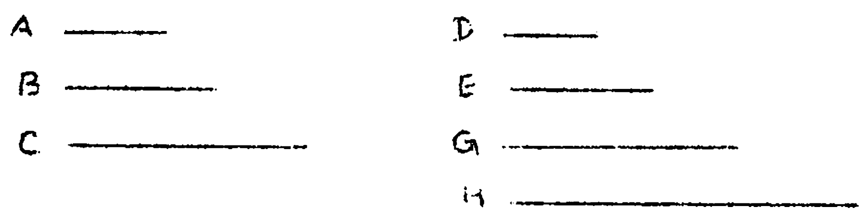
Therefore, by the rule of preceding proposition, H, K will be in excess and less, and equal to L, N respectively.

Therefore the ratio of A, C will be as the ratio of D, G.

This is what is wanted.*

PROPOSITION 21

If there are three magnitudes of the same kind, the ratio of its first to third is the compound ratio of the first to second and the ratio of the second to third.



Let A, B, C be the three magnitudes.

Then the ratio of A, C will be compound ratio of A, B and B, C.

Let D be the unit and take the ratio of D, E as the ratio of A, B and the ratio of E, G as the ratio of B, C. Here by the rule of Proposition 18, it is clear that the ratio of D, G will be equal to the ratio of A, C and since D has been taken in terms of unity. Therefore, E in terms of unity will be the magnitude of the ratio A, B and G, the magnitude of the ratio B, C. Taking the magnitude E to G till H, the magnitude of the ratio formed will be compound; because the multiple of a magnitude over a magnitude means the magnitude whose ratio of unit multiple to that magnitude will be as the ratio of unit to its other multiple. Thus the ratio of E to H will be as the ratio of D, the unit to E i.e. as the ratio of

A to C and the ratio of E, H is compound of the ratio. D, E and E, G. Therefore, the ratio of A, C will also be com-pound of the ratio A, B and B, C.

This is what is required.*

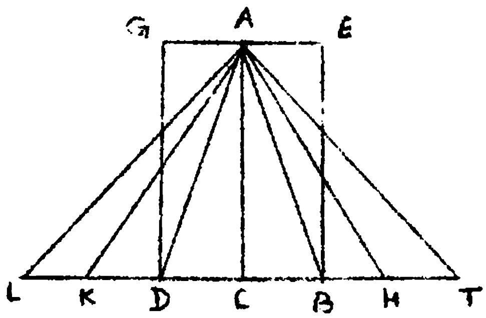
This Proposition is claimed as the original discovery of Ghulam Husain. But Simson states the same statement but in a generalized form as a definition of Compound Ratio derived by the statement of VI. 23, which is as follows:

> "When there are any number of magnitudes of the same kind, the first is said to have to the last of them the ratio compounded of the ratio which the first has to the second, and of the ratio which the second has to the third, and of the ratio which the third has to the fourth, and so on unto the last magnitude".

For example, if A, B, C, D be four magnitudes of the same kind, the first A is said to have to the last D the ratio compounded of the ratio of A to B, and of the ratio of B to C, and of the ratio of C to D. In like manner the same things being supposed, if M has to N the same ratio which A has to D; then, for shortness sake, M is said to have to N the ratio compounded of the ratio of E to F, G to H and K to L.

PROPOSITION 22 :

If altitude of parallellograms or triangles be the same, then the ratio of one another will be as the ratio of their bases.



For example, the parallellograms E C, C G and triangles A B C, A C D have equal altitude.

We say that the ratio of any two parallellograms or two triangles will be as the ratio of B C, C D.

We produce B D in both sides to T, L, and separate for B T equal to B C whatever be possible and those are B H, H T and similarly we separate D K, K L from D L equal to C D and join the lines A H, A T, A K, A L. Hence the triangles A C B, A B H, A H T, on account of the equality C B, B H, H T must be equal, because of the statement given in the end of the Proposition 32 to 2; and similarly, because of the equality of C D, D K , K L, the triangles A CD, A D K, A K L are equal and it is clear that the sum of first three triangles which is the triangle A C T is a multiple of the triangle A C B; and the sum of the three bases C B, B T, H T, i.e. the line C T is same multiple of the base C B; and the sum of last three triangles is multiple of the triangle A C D and the sum of the three bases C D, D K, K L, i.e. the line C L is multiple of the base C D. After this preamble I say that if the total triangle A C T is greater than the total triangle A C L, then the total line C T will also be greater than the total line C L and if equal or less, then C T will also be equal or less.

Thus the ratio of the triangle A B C to the triangle A C D will be equal to the ratio of C B to C D. When the rule is established for two triangles, it will be proved for two surfaces. Because the surface E C is double of the triangle A B C and the surface C G is double of the triangle A C D and the ratio of multiple is also the ratio of their halves.

This is what was required.*

This proposition may be symbolised as follows:-

If T_1, T_2 be triangles and b_1, b_2 their bases, and any multiples of b_1, b_2 be taken, say $m\ b_1$, $n\ b_2$, and the corresponding multiples of T_1, T_2, namely $m\ T_1$, $n\ T_2$, it follows from the simplest geometrical consideration that according as $m\ b_1 \quad n\ b_2$ so also is $m\ T_1 \quad n\ T_2$.

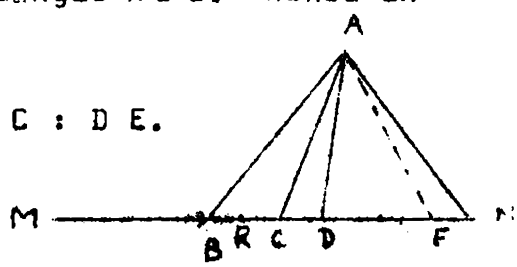
Hence $b_1 : b_2 :: T_1 : T_2$.

This Proposition is an immediate consequence of Propositions 36 and 38 of Z, that triangles are parallelograms which are equal bases and between the same parallel lines, are equal to one another.

Heath remarks "it is of course not necessary that the two given triangles should have a common side, as in the figures. And the object of placing bases in one straight line is to get the triangles and parallelograms within the same parallels." (47, ii; 193)

In modern text books the proof of this proposition is divided into two parts, based on algebraical definitions, The first of which proves the particular theorem where the magnitudes are commensurable but in the second case where they are incommensurable. The commensurable case is as given above. For incommensurable case, let us use simple limit notions, Thus suppose B C and D E are incommensurable. Divide B C into an equal parts, B R being one of the parts (see figure) on D E mark off a succession of segments equal to B R, finally arriving at a point F on D E such that E F, B R. By the commensurable case, triangle A B C : triangle A D F = B C : D F. Now let $n \to \infty$.
Then D $F \to$ D E and triangle A D $F \to$ triangle A D E. Hence in the limit,

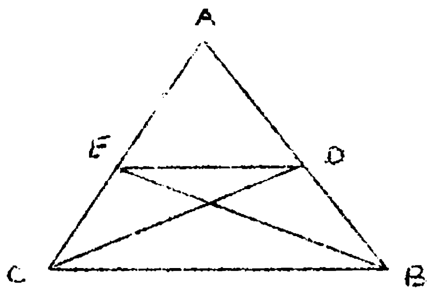
triangle A B C ; triangle A D E = B C : D E.



This approach uses the fact that any irrational number may be regarded as the limit of a sequence of rational numbers, an approach that was rigorously developed in modern times by Georg Cantor (1845 - 1918) (36; 118-119).

PROPOSITION 23 :

If a straight line intersects two sides of a triangle and is parallel to a third side, then it will intersect each of two sides in the same ratio.



So, the line D E intersects two sides A B, A C of the triangle A B C and is parallel to the side BC.

We say that the ratio of A D, D B will be as the ratio of A E, H C.

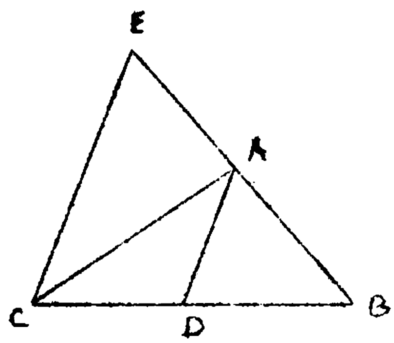
Join B E, C D.

Hence the triangles D BE, E C D which are on the base D E and between two parallel lines D E , B C, are equal and the ratio of the triangle A D E to those two triangles is same, by Proposition 7. But the ratio of those triangles to D B E is as the ratio of A D to D B and to thè triangle E C D is as the ratio of A E to EC. Hence the ratio of A D to D B will beas the ratio of A E to E C. This is what was wanted.*

In the enunciation, this Proposition, it has not been considered the case " if the line parallel to base intersects two remaining sides produced, proportionally."

PROPOSITION 24 :

In any triangle if a straight line be drawn from the angletowards its opposite isde and the line bisects the angle, then the ratio of a segment of the opposite side to other segment will be equal to the ratio of the side which is joined with the first segment to the side of other.



For example, in the triangle A B C, a line A D has been drawn from the angle A and two equal angles B A D, C A D are formed.

We say that the ratio of B D to C D will be as the ratio of A B to A C.

We extend the line A B in the direction of A upto E and from the point C draw the line C E parallel to A D till it meets the extended line A B at point E;

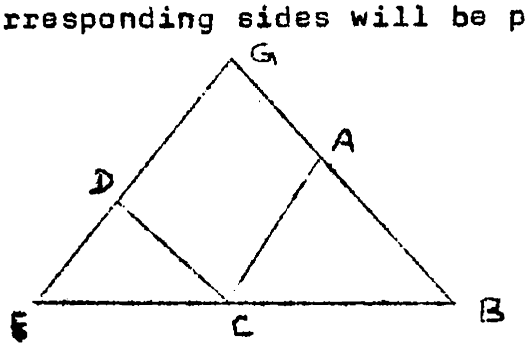
therefore, the interior angle A E C which is equal to the exterior angle B A D will also be equal to the angle D A C i.e. to the angle A CE, which, by hypothesis, is alternative angle B A C. Thus in triangle A E C two sides A E, A C are equal.

Further, we say that the ratio of B D to D C will be equal to the ratio of B A to E A i.e. to A C.

This is what is wanted.*

PROPOSITION 25 :

If in two triangles corresponding angles are equal , then their corresponding sides will be proportional.



For example, in two triangles A B C, D C E, the angle B A C is equal to the angle C D E and the angle B C A to the angle C E D and angle C B A to the angle E C D.

We say that the ratio of A B to C D will be equal to the ratio of A C to D E or the ratio of B C to C E.

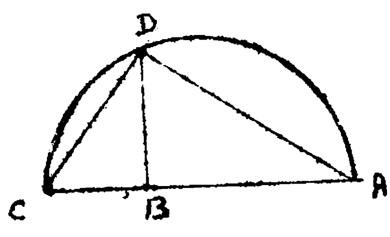
Suppose those two triangles are at the line B C E and we produce B A, E D in the direction of A, D till they meet at G.

On account of equality of interior and exterior angles A B C, D C E, the two lines A B, D C will be parallel and similarly the two lines A C, DE. Therefore, the surface A D will be parallelogram and the ratio of B C to C E will be as the ratio of B A to A G i.e. C D, and also equal to the ratio of D G i.e. A C to D E.

This is what is wanted.*

PROPOSITION 26 :

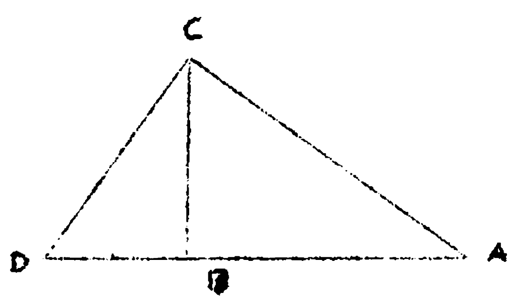
We wish to draw a mean proportional straight line between two given straight lines.



For example, those two given lines are A B, B C, which are joined in the same direction.
We describe a semi-circle A D C on the whole line and from B draw a perpendicular B D upto the circumference.
Hence this perpendicular will be the mean between A B, B C.
Because when we join A D, C D, angle A D C is obtained and by Proposition 15 of 3, since each one of the two angles A D B, D C B together with the angle B D C are equal to the right angle are equal. Hence with the same argument the two angles B A D, B D C which with the angle B D A are equal to right angle, are equal. Therefore, in the two triangles A B D, D B C, the corresponding angles are equal. Thus the corresponding sides are proportional and the ratio of A B to D B will be as the ratio of D B to B C.
This is what was wanted.*

PROPOSITION 27 :

We wish to find a third proportional for two given straight lines.



Suppose two given lines A B, B C bound a right angle A B C.

We join A C and from point C draw a perpendicular C D to the line A C such that the two points B, D be on the same side of C and extend A B in the direction of B till it meets the perpendicular at D. Thus the line B C, a third Proportional, is obtained.

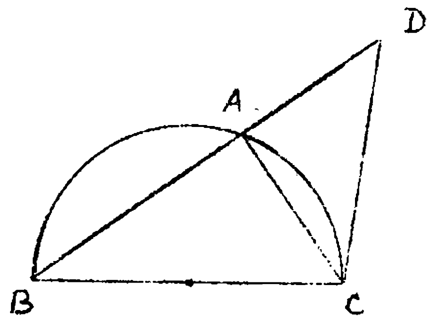
Because according to the analysis of the previous Proposition, the corresponding angles of the two right angled triangles A B C, C B D are equal. Thus the ratio of A B to B C will be as the ratio of B C to B D.

The proof of this Proposition is an alternative proof given by Ghulam Husain.

Euclid's proof in precise runs as follows:-

Let B A, A C be the two given straight lines and let them be placed so as to contain any angle. To find a third proportional to B A, A C. Produce the lines to the point D, E and make B D equal to A C and join B C and from D draw D E parallel to it. Since B C is parallel to D E one of the sides of triangle A D E proportionally, as A B is to B D, so is A C to C E. But B D is equal to A C; therefore, as A B is to A C, so is A C to C E. Therefore, C E, the third proportional of A B, A C, is obtained (47, ii; 214).

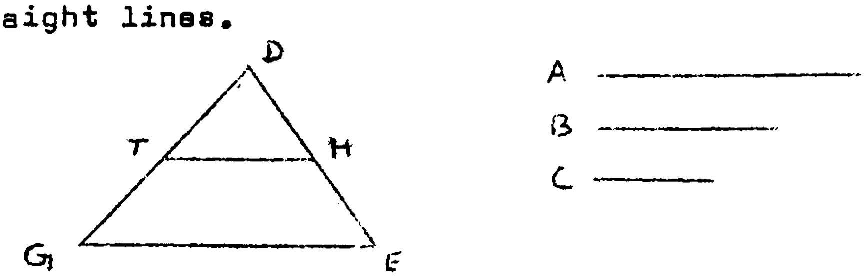
Nasīr al-Dīn Tūsī's proof is also remarkable because he takes a particular angle (right) bounded by two given lines as Ghulam Husain did, whereas Euclid who takes any angle. The proof is as follows:



Let a right angle be formed by the two given lines. Let it be A. Draw the hypotenuse B C. Describe the semi circle A B C. From the point C, draw the perpendicular C D on the line B. C. Produce the line B A so as to meet the line C D in the point D. Then A D shall be the required line. Because the perpendicular C A is drawn from the right angle C on the hypotenuse. Therefore, the ratio of B A to A C will be equal to that of A C to A D. (9; 88)

PROPOSITION 28:

We wish to find a fourth proportional for three given straight lines.



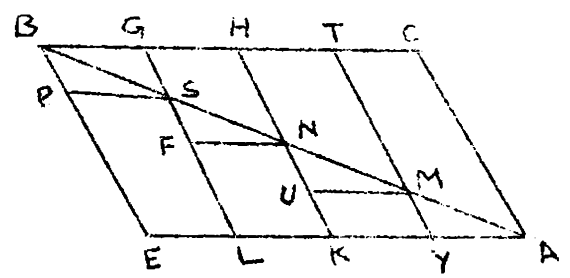
Like the lines A B C.

We draw two lines D E, D G which bound the angle E D G and separate D H from D E equal to A and H E equal to B and D T from D G equal to C and join H T, and from E draw a line E G parallel to H T till it meets D G at G. Thus the line G T, the fourth proportional is obtained. Because, by Proposition 23, the ratio of D H to H E will be as the ratio of D T to T G.

This is what was wanted.*

PROPOSITION 29 :

We wish to divide a given straight line into two equal parts.



Let A B be the line.

We draw a line B C, at point B, making the angle A B C with the line A B. Draw an angle B A E at point A with the line A B equal to the angle A B C upto the line A E parallel to B C is obtained. From B C divide consequitive equal lines B G, G H, H T, T C, in number as desired. Similarly, from A E we cut off A O, O K, K L, L E parallel and equal to the line B G in magnitude and numbers.

Join the lines C A, T O, H K, G L, B E which intersect the line A B at points M, N, S in equal parts and its number will be the number of the segments of the line B C.

Because, if from points M, N, S, we draw the lines M Q, N S, S P parallel to O K, K L, L E, these three parallel lines are obtained. Due to equality of the three lines O K, K L, L E, by Proposition 29 of 2.

Therefore in triangles A K M, M Q N, N F S, S P B, the sides A O, M Q, N F, S P are equal and the angles O A M, Q M N, F N S, P S B, in which some are interior and some exterior. On account of the line A B associated with the parallels A O, M Q, N F, S P, are equal. Similarly the angles A M O, M N Q, N S Q, S B P, associated on the same line on the parallel to T O, H K, G L, B E, are equal.

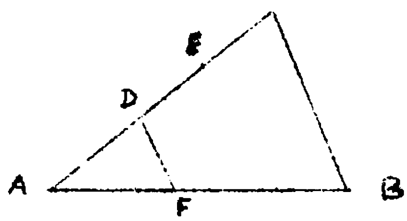
Hence, by Proposition 19 of 2, the remaining corresponding sides of these triangles are equal. Therefore, the lines A M, M N, N S, S B which are the corresponding sides are equal.

This is what is wanted.*

The construction of this Proposition is due to Ghulam Husain.

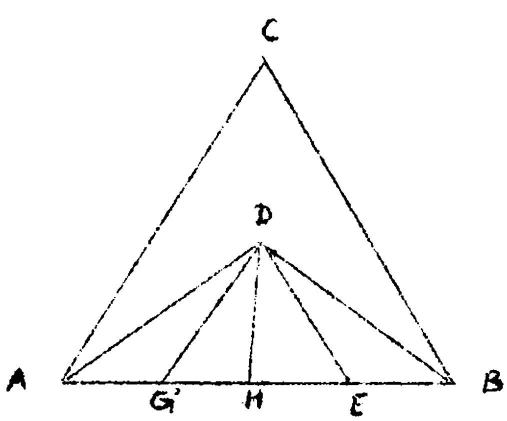
Many authors of modern High School Text books prefer this method, because the demonstration is more general than the Euclid's, in which he divides the third part of a straight line, which is a particular case. This may be seen from the following:

Let A B be the given straight line. It is required to cutt off third part from it. From A draw a straight line A C making any angle with A B. In A C, take any point D, and take D E, E C equal to A D. Join B C and throught D, draw D F parallel to it. Then A F is the third part of the given line A B.



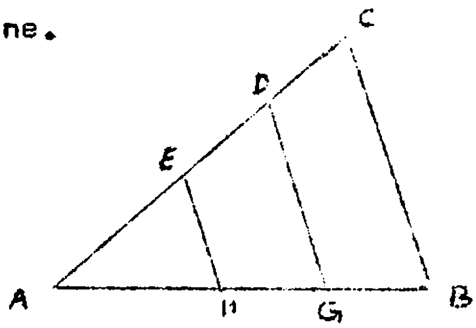
Nasir al-Din Tusi also gives alternative demonstration of trisecting a line (9;90).

Take a line A B and upon it describe the equilateral triangle A B C. Bisect the angle A and B by lines meeting in the point D. Bisect the angle A D B by the line D H and the angles A D H and B D H by lines D G and D E. Then, the line A B is trisected in the points G and E.



PROPOSITION 30

We wish to divide a line in the proportion of the segments of the other line.



Let A B be the line to which we have to divide in the proportion of the segments of the line C A i.e. C D, D E, E A.

We make both the lines the arms for the angle A.

We join C B and from D, E draw two lines D G, E H parallel to C B. Hence two points G, H will divide the line A B in the proportional to the segments of the line C A.

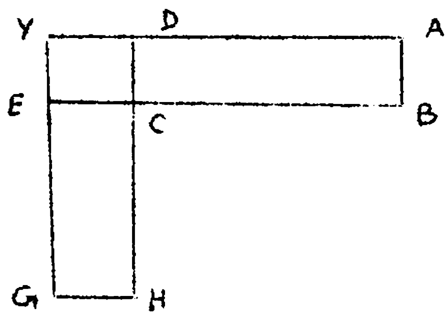
Because, by Proposition 23, the ratio of A E to E D is as the ratio of A H to E G. Similarly, the ratio of A D to D C is as the ratio of A G to G B. Hence the segments of the line A B will be in the same proportion as a segment of the line A C.

This is what is wanted.*

Proposition 27 is the particular case of this proposition. So it should have been placed before it. But it is a perfectly consistant with Euclid's manner to give a particular case first and then its generalization and such an arrangement according to Heath, often has great advantages in that it enables in more difficult parts of a subject to be led upto more easily and gradually (47,ii;218)

PROPOSITION 31 :

If in two parallelograms, two angles are equal and if those two parallelograms are equal, then the sides which bound those surfaces are reciprocal and if sides are reciprocal, the two surfaces are equal to one aother.



For example, two angles C of the parallelograms A C, C G are equal.

First, let both surfaces be equal.

We say that the ratio of B C to C E will be the ratio of H C to C D.

Since two angles C are equal. Therefore, we suppose that two lines B C, C E of these two surfaces are joined and become one. Similarly the lines H C, C D.

Complete the parallelogram C Y.

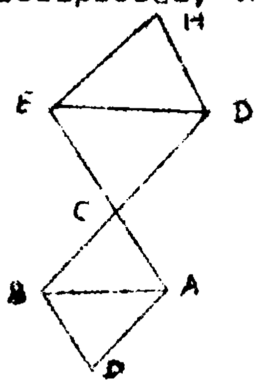
Hence the ratio of each two surfaces to the surface C Y will be the same. But the ratio of first to that surface is as the ratio of B C to C E and the ratio of second is as the ratio of H C to C D. Hence reciprocal sides are proved.

surfaces to the surface C Y are exactly the ratio of the sides and their equal ratio to a thing implies they are equal.

This was what was required.*

PROPOSITION 32 :

If in two triangles, two angles are equal and triangle is equal to triangle, then their sides are reciprocally proportional and if sides are reciprocal, the triangle is equal to triangle.



So, in two triangles A B C, C D E, two angles C are equal. First, those triangles are equal.

We say that the ratio of A C to C E will be as the ratio of D C to C B.

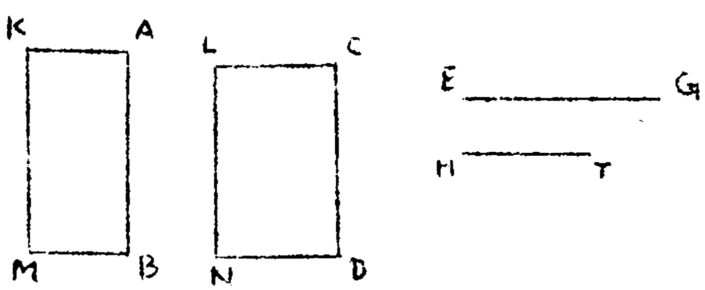
Place both the triangles such that the lines A C , C E are joined in the same direction and similarly, the lines D C, C B. From A, B we draw two lines A G, B G parallel to the lines C B, C A till they meet at G and from two points D, E the lines D H, E H parallel to the lines C E, C D which meet at H. Thus two equal parallelograms are obtained and by preceding proposition, the ratio of the said sides will be reciprocal and also if sides are reciprocal, the triangle is equal to the triangle. Because reciprocal relation between sides is equal to two surfaces and two ecual surfaces imply two equal triangles that each one is the half of the whole surface.

This is exactly what is wanted.*

The Proof of this Proposition is given by Ghulam Husain. It is true that this proposition can be immediately inferred from the preceding proposition, because a triangle is half of a parallelogram with the same base and height. In proving this proposition Ghulam Husain first completes parallellograms and utilised proposition 31. But Euclid does not do so. He gives different method to which Heath favours and remarks: "Euclid's object being to give the student a grasp of methods rather than results, there seems to be no advantage in reducing one proposition form the other instead of using the same method on each".(47,ii;22)

PROPOSITION 33:

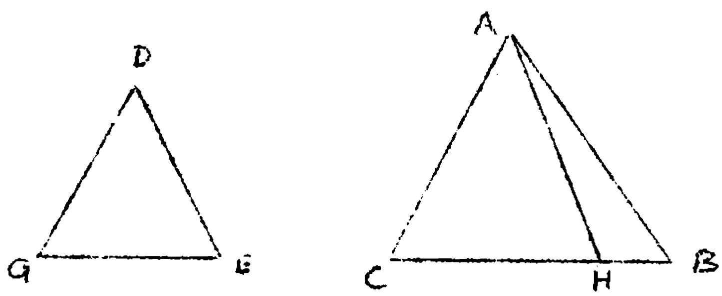
If four lines are proportional, a surface of the first into fourth will be equal to the surface of the second into third.



For example, the lines A B, C D, E G, H T are proportional. We say that the surface A B, H T will be equal to the surface C D. E G and from the points A, C we draw two perpendiculars A K, C L equal to the lines H T, E G and complete the parallelograms A M, C N.
Hence in these two surfaces when two angles A, C e.g. are right angles, are equal and the ratio of A B to C D is as the ratio of C L i.e. E G to A K i.e. H T and this ratio is reciprocal proportional. Therefore, by Proposition 31 the surface AM, C N, necessarily are equal.
This is what was required. *

PROPOSITION 34 :

If two triangles are similar, then the ratio of one to other will be duplicate ratio of the side to its corresponding side.



For example, two triangles A B C, D E G are similar.
We say that the ratio of the triangle A B C to the triangle D E G will be duplicate of the ratio of the side B C to the side E G.
For example, if B C, E G are equal, the proposition is proved. Because, the duplication of the equal ratio is

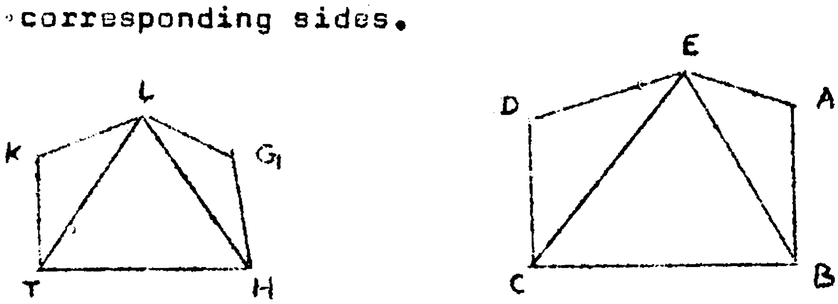
exactly the equal ratio and if they are unequal, let B H be the third proportional of the two sides B C, E G by Proposition 27, in this, case we say that the ratio of B C, B H will be compound of the ratio of B C to E C and the ratio of E G to B H, Proposition 21, Since these two ratios are of the same kind, therefore, the ratio of B C to E H will be duplicate.

Join A H.

Then in two triangles A B H, D E G, the angles B, E are equal and the ratio of the sides of those two angles are reciprocal. Thus by Proposition 32, these two atriangles are equal. And the ratio of the triangle A B C to the triangle A B H i.e. triangle D E G is as the ratio of B C to B H and ratio of B C to B H is duplicate of the ratio B C to E G. Therefore, the ratio of the triangle A B C to the triangle D E G will also be duplicate ratio of B C to E G,

PROPOSITION 35:

If two polygons that are similar are divided into triangles in equal numbers and triangle of the surface is similar to its corresponding triangle of the other surface, and the ratio of the surface to surface is as duplicate ratio of their corresponding sides.



For example, the surface A B C D E, G H T K L are similar. We join the lines B E, E C, H L, L T till both the surfaces are divided into three triangles.

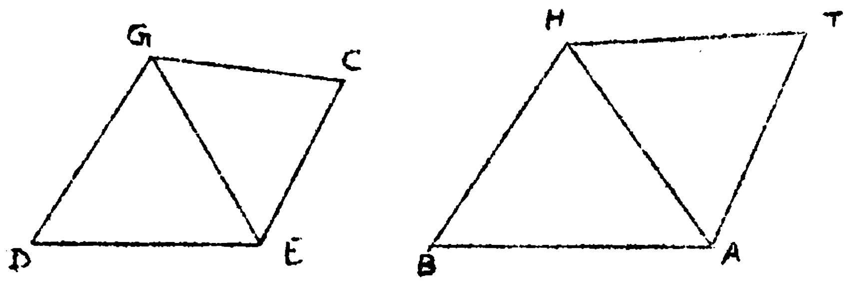
We say that the triangle A B E of first surface is similar to the triangle G H L of the second. Because the angles A, G are equal and the ratio of A B to G H is as the ratio of A E to G L. If we superpose the angle G on the angle A and triangle G H L on the triangle A B E, it is necessary that by converse rule of the Proposition 23, H L will be parallel to B E and hence angle A B E will be equal to the angle G H L and angle A E B to G L H.

Hence by Proposition 25, the triangles A B E, G H L are similar and on this hypothesis the triangle E B C will be similar to the triangle L H T and triangle C E D to T L.K. Because the remaining angles E B C, L H T after excluding two equal angles A B E, G H L from the two equal angles A B C, G H T, are equal. Since the ratios of each triangle to triangle are same. Therefore, by Proposition 12, the ratio of a surface to surface will be as the ratio of the triangle to its corresponding triangle and the ratio of correspon-ding triangles are duplicate of the ratio of their two corresponding sides. Therefore, the ratio of the two surfaces A B C D E, G H T K L will also be duplicate ratio of each two corresponding sides.

This was what was required.

PROPOSITION 36 :

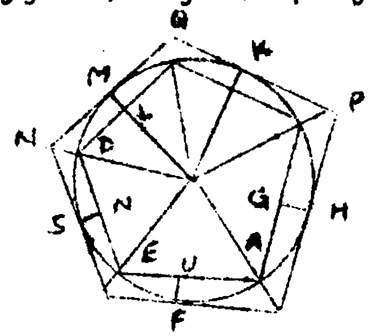
We wish to construct a surface on a given straight line which will be similar to a given surface.



For example, on the line A B similar to the surface C E D G. We divide it by the line E G into two triangles C E G, D E G and construct at point A with the line A B the angle B A H equal to the angle D E G and at B angle A B H equal to the angle E D G and draw two sides AH, B H so as to meet at H and triangle A B H similar to the triangle E D G is obtained.
Further, we construct on the line A H, two angles H A T, A H T equal to the angles G E C, E G C and extend A T, H T till they meet at T and triangle T A H similar to the triangle C E G is obtained. Thus total surface T A B H will be similar to the surface C E D G.
This is what is wanted.*

PROPOSITION 37 :

We wish to construct a figure circumscribing a circle similar to a equilateral and equiangular polygon (regular polygon) which is inscribed in the circle.



Let A B D be the circle and the figure inscribed in the circle is pentagon A B C D E.

From the middle of all the sides draw the perpendiculars G H, T K, L M, N S, U F so as to meet in the circumference of the circle and we draw tangents to the circle at points K, L, M, S, F, H in either sides of the perpendiculars PQ, Q Z, Z I, I J, J P, till each tangent meets its adjacent line at two points.

So the circumscribing surface P Q Z I J similar to the said pentagon is constructed.

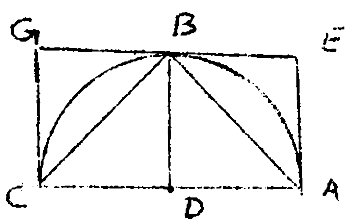
Let O be the centre of the circle, and when we produce the two perpendiculars H G, K T, by Proposition 1 of 3, pass through the centre O, join P B, O B.

I say that the line O P becomes one. Because, when we join B H, K B, in two triangles P B H, P B K, two sides P H, P K which are drawn from one point and are tangent to the circle, are equal and side P B is common, and the chords B H, B K of the two equal arcs are also equal. Hence the angles H B P, K B P which are the corresponding angles of these two triangles are equal. Similarly in the triangles B H O , B K O, their corresponding sides are equal. Hence the angle H B O will be equal to the angle K B O and the sum of the first two angles and these two angles is equal to four right angles. Therefore, the angles P B H, H B O will be equal to two right angles and for this reason, the line O P will be straight. Since the line G H is perpendicular to each one of J P, A B. Therefore, these two lines are parallel and triangles P H O, B G O are similar and likewise the triangles P K O, B T O; and on this hypothesis, when we suppose that the line joining the centre with the angles, and half sides, then each of the subscribingdivide to inscribing pentagon is divided into ten equal and similar triangles. The circumscribing sides each of which is made of two lines from a total of the ten equal lines, are equal, and because of the parallelity of the sides, the circumscribing angles will be equal to the inscribing angles.

This is what is required. *

PROPOSITION 38 :

If a segment is not greater than the semi-circle and with the middle point of the arc two ends of its base are joined by two lines, then the isosceles triangle formed will be greater than the half segment.



For example, in segment A B C, arc A B C has been bisected at point A and join A B, C B , the isosceles triangle is formed.

We say that this triangle will be greater than the half segment.

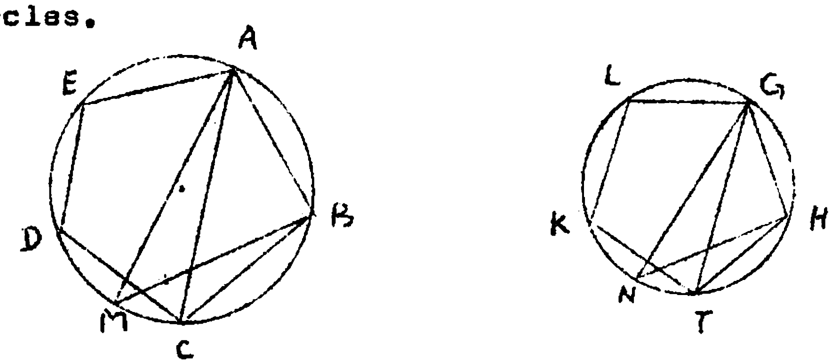
Because, if from the point B we draw a perpendicular B D to the base and from B, the line B E tangent to the segment and from two points A, C draw two lines A E, C G parallel to B D till they meet the tangent in two points E, G. So it is evident that the surface E C is based on four equal triangles A E B, A D B, B D C, C G B and triangle A B C is compound of two out of four triangles and sum of two small segments A H B, C T B, which remain after deducting the triangle A B C from the original segment, is less than the sum of the two triangles. Therefore, the triangle

A B C will necessarily be greater than the half segment and also from this statement it is clear that any rectangle in which a segment is situated, provided the small side is not greater than half of the longer side, that segment will be greater than the half rectangle.*

This Proposition as well as the preceding proposition are the original discovery of Ghulām Husain. These propositions are proved independent to the theory of proportion. So they seem out of place, and should be placed in the end of the preceding chapter.

PROPOSITION 39 :

The ratio of two similar surfaces inscribed in two circles will be as the ratio of the squares on the diameters of those circles.



So, two similar surfaces A B C D E, G H T K L which are inscribed in the two circles A C D, C T K. Let A M, G M be diameters of the circle.

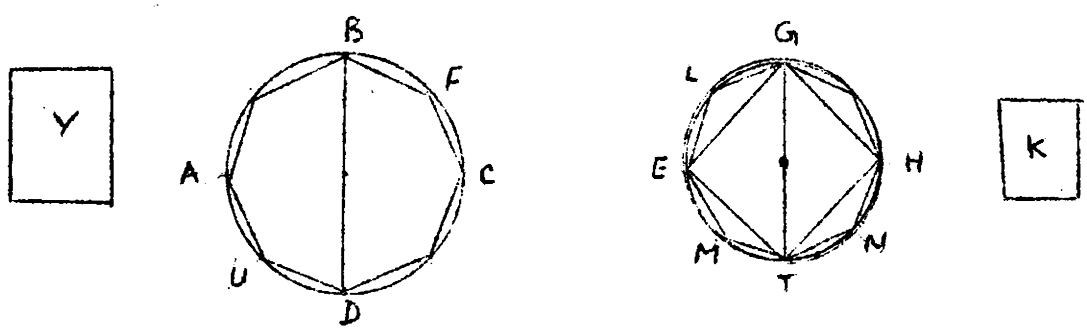
Join A C, B M, G T, H N.

Hence, by Proposition25, the triangles A B C, G H T will be similar, and by Proposition 19 of 3, angle A M B is equal to the angle A C B and angle G N H is equal to the angle G T H. Thus in two triangles A M B, G N H, two angles A M B, G N H are equal and angles A B M, G H N which are in the semi-segment, will be right angle. Therefore, these two triangles will be similar and the ratio of the surface A B C D E to the surface G H T K L will be as the ratio of A B to G H i.e. duplicate ratio of A M to G N and the ratio of the squares A M, G N is also duplicate ratio of AM to GN. Hence the ratio of the surfaces will be equal to the ratio of the squares of the diameters.

This is what is required*.

PROPOSITION 40:

Ratio of two circles will be the ratio of the two squares on their diameters.



Let A B C D, E G H T be circles and B D, G T their diameters.

If the ratio of the sauares on B D to the square on G T is not as the ratio of the circle A B C D to the circle E G H T, then it will be equal to the ratio of the circle A B C D to a surface which is smaller than the circle E G H T or greater.

Let first it be a surface which is smaller than that surface and that surface is Y and let excess of the circle G H T on the surface Y by the surface K.

We bisect the two arcs G ET, G H T at two points, E, H and join G E, E T, T H, H G.

Hence by Proposition 38 the square on G T will be greater than the semi circle. We bisect the four arcs at points L, M, N, S and join their chords so as to form four triangles greater than the half of the each four segments. Similarly by bisecting the arcs till the remaining segments obtained are smaller than the surface K.

Because it is evident that when a greater magnitude is bisected consecutively, necessarily, stagewise smaller and smaller magnitudes are obtained and in this case the partition is of greater order than bisecting. Hence it necessarily follows that partition earlier reaches a limit such that the total of the small segments less than K remain and in such a situation a polygon which here is L N necessarily greater than the surface Y remained.

We construct a polygon in the circle which is similar to L N, and it is U Z.

By preceding Proposition, the ratio of the square on B D to the square G T will be equal to the ratio of the polygon U F to the polygon L N and was equal to the ratio of the circle A B C D to the surface Y but the ratio of the polygon. U F to the surface Y will be greater than its ratio to the polygon M S, by Proposition 8 i.e. than the ratio of the circle A B C D to Y. Therefore, by Proposition 10, the polygon, U F which is the part will be greater than the circle A B C D the whole. This is a contradiction. Hence the ratio of the square B D to the square G T will not be equal to the ratio of the circle A B C D to the surface which will be less than the circle E G H T. And also if the ratio of the square B D to the square G T will be equal to the circle A B C D to a surface which is greater than the circle E G H T.

It we take converse ratio, then the ratio of the square G T to the square B will be as the ratio of the circle E G H T to the surface which will be less than the circle A B C D and the same aforesaid contradiction necessarily reaches, and the aim will be proved.

PORISM: Since by figure 35 it is proved that the ratio of the square to the square is duplicate ratio of the side to side. Hence the ratio of the circle to circle will also be duplicate ratio of the diameter to diameter.*

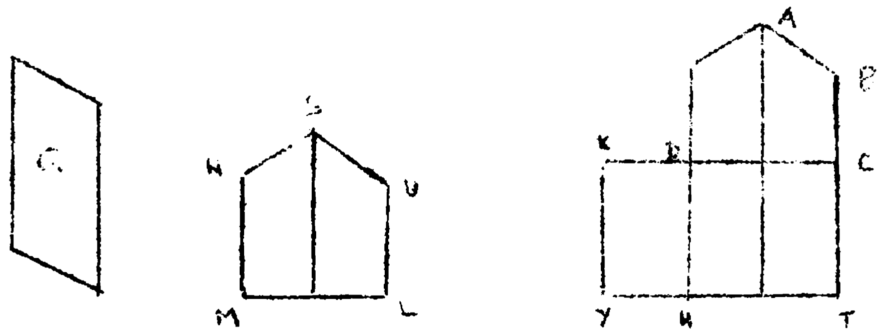
Ghulam Husain infers an interesting result by observing propositions 35 and 40 that "the ratio of two circles will be as the duplicate ratio of their diameters."

This is first proposition to prove it the "method of exhaustion" (or indirect method of limits) is used, particularly approximation method under which, according to Dijksterhuis (56; 219) one inscribes within a curved figure a sequence of polygons and establishes that the difference in content between the curved figure and the polygons becomes less than any preassigned magnitude when the polygons are taken with sufficiently many sides. Usually, each polygon in the sequence is constructed as having double the number of sides of its predecessor and convergence is assured via the principal of successive bisection of the difference. Of course, the same technique will establish the convergence of a sequence of circumscribed polygons downward to the limiting curve.

This method is Eudoxaan-Euclidean method of limits which are implied in throughout Book XII.

PROPOSITION 41 :

We wish to construct a surface similar to a given rectineal surface and equal to another given rectineal surface.



Let the required surface be similar to A B C D E and to be equal to the surface G.

So, we construct rectangular surface on C D equal to the surface A B C D E with the help of the Proposition 36 of 2 and that surface is C D H T. Construct a rectangular figure on D H equal to the surface G whose breadth D K is formed. Now we draw a mean straight line between C D, D K, by Proposition 21 and that line is L M and on L M, we construct a figure which is similar to the figure A B C D E with the help of the Proposition 26 and that surface is S U L M N which is required i.e. similar to the surface A B C D E and equal to the surface G.

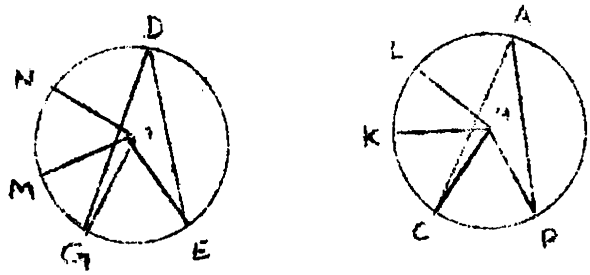
Because the ratio of the line C D to D K i.e. the ratio of the surface C H to the surface D V or the ratio of the surface A B C D E to the surface G is duplicate of the ratio of C D to the line L M by Proposition 21 and the ratio of the figure A B C D E to the figure S U L M N is also duplicate of the ratio of C D, to L M, by Proposition 35. Hence the ratio of the surface A B C D E to the surface S U L M N and to the surface G, the ratio is same.

Therefore, by Proposition 9, the surface S U L M N will be equal to the surface G.

This is what is required.*

PROPOSITION 42:

If in two equal circles, two angles are a t the centres or on the circumferences, the ratio of each two angles will be as the ratio of their two arcs.



Suppose in two equal circles A B C, D E G two angles are
central angles like B H C, E T G.

I say that the ratio of these two angles will be as the ratio of the two arcs B C, E G.

From the circle A B C, we take off the arcs equal to the arc B C whatever be possible and those are C K, K L, and similarly from the circle D EG, we take the arcs equal to the arc E G and these are G M, M N. Join the lines H K, H L, T M, T N.

Hence the sum of the arcs B C, C K, K L, is multiple of the arc B C, and the sum of the angles B H C, C H K, K H L is same multiple of the angle B H C, and similarly the sum of the arcs E G, G M, M N, is multiple of the arc E G and the sum of the angles E T G, G T M, M T N, is multiple of the angle E T G. Hence if the arc B C L is greater than the arc E G N, the sum the angles of the first will also be greater than the sum of the second angles and if equal, equal and if less, less. Thus according to preamble

the ratio of the two angles B H C, E T G, will be as the ratio of the two arcs B C, E G and since the angle on the circumference halves the angle at the centre and in halving the ratio of the multiple is preserved.

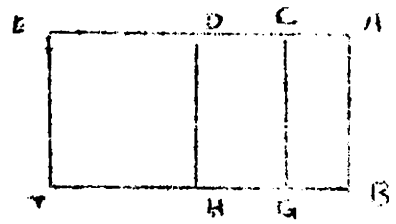
Therefore the ratio of the two angles B A C , E D G on the circumference will be as the ratio of the two arcs B C, E G.

This is what was required.*

This Proposition is in fact a direct deduction from Proposition 33 of 3.

PROPOSITION 43 :

The sum of the surfaces of a line into the lines will be equal to the surface of that line into a line which will be the sum of those lines.



For example, the sum of the surfaces of the line A B into the lines A C, C D, D E is equal to the surface of the line A B into the line A E which is equal to the sum of the afore-mentioned three lines.

Because when we complete the rectangles A G, C H, D T, the sum of these three, inevitably, will be equal to the surface A T which is the surface A B . A G and this is according to the claim.*

The proof of this Proposition is an alternative proof given by Ghulām Ḥusain.

This Proposition is the converse of II.1 that "if there be two straight lines and one of them be cut into any number of segments whatever, the rectangle contained by the two straight lines is equal to the rectangles contained by the uncut straight line and each of the segments".

This Proposition asserts that

A B (A C + CD + DE) = A B. AC + A B . CD + AB . D E

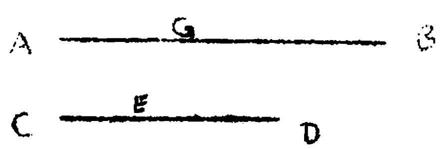
Which is one of the fundamental laws of arithmetic known today as Distributive Law:

$x\,y_1 + xy_2 + \ldots + x\,y_n = x\,(y_1 + y_2 + \ldots\ldots + y_n)$

It may be added here that now-a-days magnitudes are represented by letters that are understood to be numbers (either known or unknown) on which we operate with the algorithmic rules ofalgebra, but in Euclid's days magnitudes were pictured as line segments satisfying axioms and theorems of geometry (21; 121).

PROPOSITION 44 :

The surface of a line into a part of the other line will be equal to the surface of the other line into the same parts of the first line.



For example, the surface of the line A B into one-third of the line C D which is C E is equal to the surface C D. A G, the one-third of A B.

Because it is clear that the ratio of A B to C D is as the ratio of A G to CE. Hence by Proposition 33, the surface of the first A B into the fourth C E will be equal to the surface of the second C D into the third A G.

This is what was required.*

This Proposition is an original discovery of Ghulam Husain.

The theorem can be expressed algebraically,

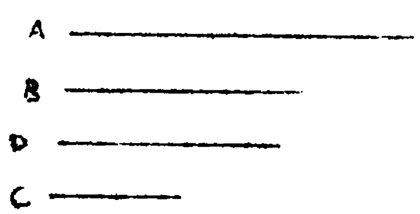
if A B = x, C D = y,

then x (1/3y) = y (1/3x).

Thus in modern terminology this expression is the Commutative Law for multiplication.

PROPOSITION 45 :

In any three lines which are proportionate, the surface of the first into third is equal to the square on the second, and if the surface of the extremities is equal to the square on the middle, the three lines will be proportional.



For example, the three lines A, B, C are proportional and B is the mean of the extremities. Take D equal to it.

Then the proportionals are four and by Proposition 33, the surface A. C will be equal to the surface B . D i.e. square on B, and also if the surface A. C is the square on B, by Proposition 31, it is necessary that the sides of the surface and square are reciprocal i.e. the ratio of A to B will be as the ratio of D i.e. B to C.

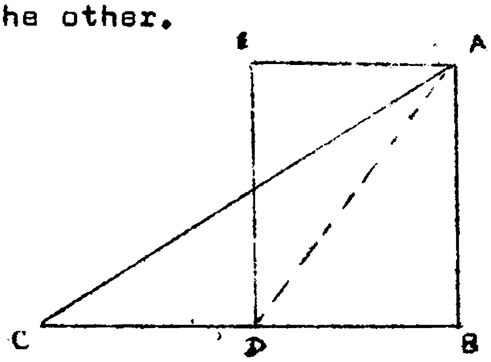
This is what was required.

Remark: The divided line which is given in the Proposition 46 of 2, those divisions are called portion in terms of the mean and extremities because the surface of the whole line with the smaller part is equal to the square of the bigger part. Therefore, according to this Proposition, the ratio of the line to the bigger part is equal to the ratio of Hence here the extremities and the mean is obtained.*

This Proposition is a particular case of the Proposition 33.

PROPOSITION 46 :

The surface of a right-angle triangle will be equal to a surface which is enclosed by the perpendicular side and half of the other.



For example, the trianguler surface A B C whose angle B is right angle is equal to the surface B E which is enclosed by side A B and half of the other side B D.

Because when we join A D, each one of the triangle A B C and the surface B E, the two times of the triangle A B D are obtained.

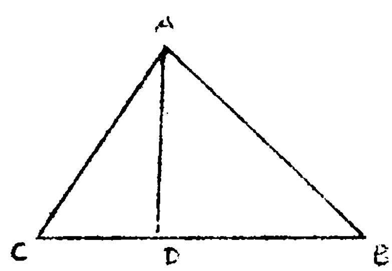
Therefore, what is required will be obtained.*

This Proposition is the discovery of Ghulam Husain.

It may be observed here that the proposition is independent of the proposition of this chapter. So, it should be placed after Proposition 32 of 2.

PROPOSITION 47 :

Any triangle is equal to a surface which is enclosed by the half of the side of that triangle which will not be less than the remaining two sides and from opposite angle of it draw the perpendicular on the same side.



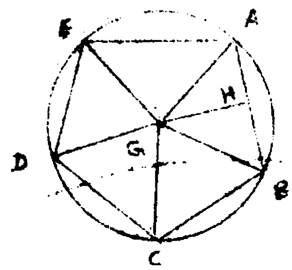
For example, the triangular surface A B C is equal to the surface which will be bounded by that of half of the side B C which is not less than A B and A C and the perpendicular A D which is drawn from the angle A to the side B C.

Because it is manifest that the right angled triangle A D B is equal to the surface of A D into half B D and similarly the triangle A D C is equal to the surface of A D into the half D C, and by Proposition 43, these two surfaces are equal to the surface of A D into the sum of the two halves B D, D C i.e. B C. For this reason, the triangle A B C will be equal to the surface of A D into the half B C and also if the surface of B C into half of the perpendicular A D is formed. Then, this surface will also be equal to the triangle A B C by Proposition 44. *

This Proposition is discovered by Ghulām Husain.

PROPOSITION 48 :

Every equilateral and equiangular figure which is in the circle is equal to a surface which is bounded by a perpendicular drawn from the centre of the circle to a side of that surface and the line which will be equal to the half of the sum of the sides of that surface.



For example, the pentagon A B C D E is equal to the surface whose one side is equal to the perpendicular G H which is drawn from the centre of the circle to the side A B and the other side which will be equal to the half of the sum of A B, B C, C D, D E, E A.

Because when the centre and the angles are joined by the straight lines, the equal triangles are formed as the number of the sides and the perpendicular drawn from the centre to the equal chords, are equal and the surface of the triangle A G B, however, every tiangle will be equal to the surface G H into the half A B, by preceding Proposition,

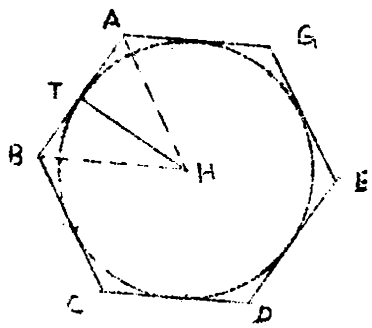
Hence the sum of the surfaces of the triangles i.e. pentagon will be equal to the surface G H into the sum of the half sides.

This is what was wanted.*

This proposition is also due to Ghulām Husain.

PROPOSITION 49 :

Every equilateral and equiangular figure which will be on a circle is equal to a surface which is bounded by the radius of that circle and a line which will be equal to the sum of the semi sides of the figure.



Let the figure circumscribing the circle be e.g. the hexagon A B C D E G and the centre of the circle be the point H and τ the point of contact of the side A B is the point T.

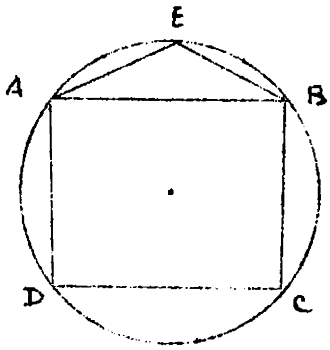
We join the lines H A, H T, H B and the radius H T will be the perpendicular A B and the surface of H T into the half of A B is equal to the triangle A H B and it is evidence that if the lines between the point H and the angles of the figure be joined, triangles in equal numbers of sides of the figures are formed and each of them will be equal to the surface of the radius into half of the side. Thus the surface of the radius into the half of the sum of the sides will be equal to the figure A B C D E G.

This is what we wanted.*

Indeed this proposition is also given by Ghulām Husain.

PROPOSITION 50 :

In every rectilineal surface which is inscribed in a circle, the sum of its sides will be less than the circumference of the circle.



Let the surface be A B C D inscribed in the circle.

I say that the sum of the lines A B, B C, C D, D A is less than the circumference of the circle.

We fixed on the arc A B a point E and join two lines A E, EB and we say that in the triangle A E B, two sides A E, E B are greater than the side A B and the arc A E B is not less than the sum of the two sides A E, E B. Therefore, A B will be less than the arc A E B and on this argument, every chord is less than its arc. For this reason, the sum of the chords will be less than the sum of the arcs which is the circumference of the circle.

This is what was wanted.*

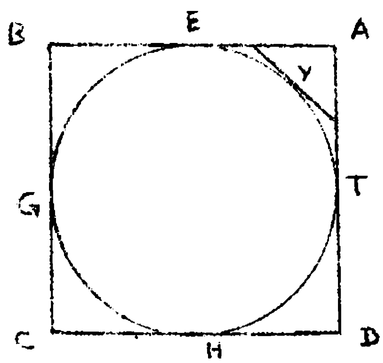
This Proposition indeed is fifth assumption of Archimedes which is given in an English translation of his work to which

Ghulam Husain proved here. The assumption is as follows: (14;40
"These things being premised, if a polygon be inscribed in a circle, it is plain that the perimeter of the inscribed polygon is less than the circumference of the circle; for each of the sides of the polygon is less than that part of the circumference of the circle which is cut off by it".

But Ghulam Husain claims this Proposition as the original discovery of his own. For this it may be conjectured that he might have not recognised it as assumption like Archimedes did or this assumption was not given in the Arabic version which he utilised in this book. But in any case the proof is his own.

PROPOSITION 51 :

In every figure which is circumscribed on a circle, the sum of its sides will be greater than the circumference of the circle.



For example, the figure A B C D which is on the circle E G H T.

We fixed on the arc T E the point Y and from Y drew the tangent K Y L to the circle.

We say that the two lines Y L, L E are not less than the arc Y E and similarly, the two lines Y K, K T are not less than the arc Y T. Thus the sum of the lines T K, K L, L Y will not be smaller than the arc T E and the sum of K A, A L is greater than K L. For this reason, the sum of T A, A E will be greater than the sum of T K, K L, L Y or the arc T E. On this analysis, the sum of E B, B G will be greater than the arc E G and the sum of G C, C H is greater than the arc G H and the sum of H D, D T than the arc H T. For this reason, the sum of the sides will be greater than the sum of the arc which is the circumference of the circle.*

This Proposition is also the discovery of Ghulām Husain.

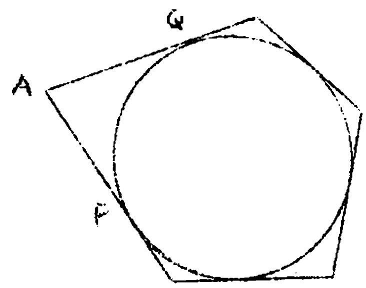
It is not free from interest to note that this Proposition is the Proposition 1 of Book I "on the sphere and cylinder" of Archimedes. The proof is also the same. Ghulām Ḥusain's proof contains an unproved assumption namely "two lines Y L L E are not less than the arc Y E". However, he does not state it any where, such assumption separately as Archimedes put as second assumption.

Archimedes gives the following proof: (14; 405)
Let any two adjacent sides meeting an A, touch the circle at P, Q respectively.

Then (Assumption 2)

• P A + A Q > (arc P Q).

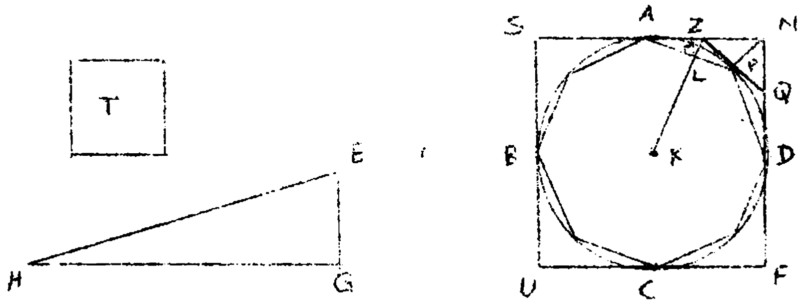
A similar inequality holds for each angle of the polygon; and by addition, the required result follows.



Although there is a similarity in proof; yet it is difficult to infer that Ghulam Husain was aware about Archimedes' assumptions and this Proposition. But Knorr pointed out that "Theon introduces this result into his presentation of isoperimetric figures, he says that "Archimedes assumes this in sphere and cylinder (In Ptolemaeum, ed. A. Rome, page 359) In precisely the same context, Pappus says that the same result "is supposed by Archimedes in sphere and cylinder (Collection V.2, ed. Hultsch, page 312). (56; 218-219). Thus if we conjecture that this Proposition was an axiom of Archimedes and Ghulām Husain treats it as a problem and proved it, will be not be wrong.

PROPOSITION 52 :

Every circle will be equal to a right angled triangle whose one side will be equal to the radius of the circle and other side equal to the circumference of that.



Let A B C D be the circle and E G H the triangle whose side E G is equal to the radius and G H to the circumference.

We say that the circle will be equal to the triangle and otherwise they are different.

Let, first, the circle be greater than the triangle and its excess over the triangle will be the surface T.

We construct the polygon B D in the circle equal to that which has been constructed in the Proposition 40 till the sum of the sum of the small arcs is less than the surface T is obtained and the polygon becomes greater than the triangle E G H.

Later from the centre of the circle which is the point K we draw the perpendicular K L to one of the sides of the given figure, however, extend upto M on the circumference and by Proposition 48, the polygon is equal to the surface

K L into the half of the sum of its sides. Since K M L is less than K M i.e. E G and the sum of the sides is less than the circumference of the circle i.e. G H, and the sum of the sides is less than the circumference of the circle i.e. G H. It is necessary that the polygon will be smaller than the triangle E G H and being greater, this is a contradiction. Hence the circle will not be greater than the triangle.

If the circle will be smaller than the triangle, then take the surface T the excess of the triangle over the circle.

We construct the square N S U F on the circle and bisect the arc A D at P and from P draw the line P Q which will be tangent to the circle at point P and meets the lines D N, A N at Q, Z till the triangle Q N Z greater than the semi-circle D N A whose two sides D N, N A are straight and one side A P D is the arc, is formed. Because whenever, we join N P the hypotenuse N Z will be greater than P Z, the popposite side of the acute angle i.e. Z A and the ratio of the two triangles A P Z, N P Z of equal altitude will be as the ratio of A Z, N Z and N Z is small; therefore, the triangle N P Z will be greater than the triangle A P Z, and the rectilineal triangle A P Z is greater than the rectilineal triangle A M P. Hence the triangle N P Z will be much greater than the triangle A M P Z. On this argument N P Q will be much greater than the triangle D I P Q.

Thus the triangle Q N Z will be much greater than the sum of the two triangles A M P G', D I P Q.

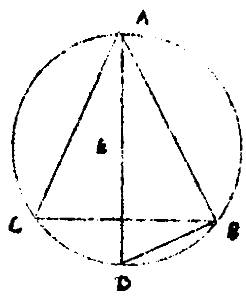
Similarly, tangents bisecting the three arcs A B, B C, C D we draw tangents so that the rectilineal triangle is formed greater than the sum of the two rectilineal and curvilineal triangles are obtained. Likewise, one by one till the last arc obtained is divided and we draw tangents till the sum of the small rectilineal and curvilineal triangles smaller than the surface remains. Therefore, the circumscribed polygon remains less than the tangent E G H but the sum of the sides is less than the circumference of the circle i.e. greater than the line G H. Hence half of the the sum of the sides will also be greater than G H. Thus the surface K M into the half of the sum of the sides, which by Proposition 49, is equal to the polygon will be greater than the triangle E G C; and being smaller, this is a contradicition. Hence, the said rule is established i.e. the magnitude of the surface of the circle is obtained by multiplying its radius with its semi-circumference and also from this statement, it is clear that the magnitude of the sector of the circle is obtained by multiplying the radius with half arc of that sector. *

This Proposition is proved by exhaustion method in its "difference form". In this method, the area or volume of the inscribed or circumscribed figure is regularly increased on decreased until the difference between the desired area or volume and the inscribed or circumscribed figure is less than any preassigned magnitude.

This method was used in Propositions 39 and 47 of 5.

PROPOSITION 53 :

A square on a side of an equilateral triangle which is in a circle (inscribed) will be three times of the square of the radius of the circle.

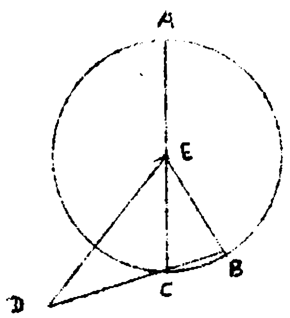


Let A B C be an equilateral triangle inscribed in the circle A D and we bisect the arc B C at D and join A D, which necessarily, passes through the centre E. Because the sum of the two arcs A B, B D is one half of the circumference and the sum of A C , C D, the other half. Since arc B C is one third of the circumference, so the arc B D will be one-sixth and the chord B D, by Proposition 30 of 3, will be equal to the radius and the square A D i.e. four times of the square of E D, the radius will be equal to two squares A B, B D i.e, two squares A B, E D. When we delete the common square E D, the three times of the square E D equal to the square A B remains.

This is what is wanted.*

PROPOSITION 54 :

The side of every hexagon and decagon which are inscribed in a circle, when joined in the same direction, then the sum of the lines divided in mean and extremities ratio and the major segments will be the side of the hexagon.



Let in the circle A C, B C be the side of decagon and C D be joined with it equal to the side of the hexagon andwe draw the diameter C D A and join E B, E D.

Since the arc B C is one fourth of the arc A B and the angle B E C is also one fourth of the angle A E B and angle B E C once together with the angle A E B is equal to two right angles and once together with two angles E B C, E C B is also equal to two right angles. Therefore, the sum of these two angles will be equal to the angle A E B and the angle A E B will be four times of the angle B E C and due to equality of E B, E C, each of the two angles E B C, E C B will be two times of the angle B E C and angle E C B, because of the equality of E C, C D, is also two times of the angle D. For this reason the angle B E C will be equal to the angle D and in two triangles B E C, B D E, these two aforesaid angles are equal and angle B is common. Therefore, both the triangles are similar and the ratio of B D to C E i.e. C D will be as the ratio of C D to B C and by Proposition 3, the surface B D . B C will be equal to the square on C D.

This is what was required. *

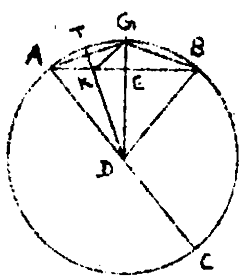
To find the side of the decagon algebraically in terms of radius, if x be the side of decagon, then we have

$D B \cdot B C = E B^2$ or $(r + x)\, x = r^2$, since D C = E B

$$\text{or } x = \frac{r}{2}\left(\sqrt{5} - 1\right)$$

PROPOSITION 55 :

The side of every pentagon which is inscribed in a circle will be greater than the side of the decagon and hexagon inscribed in the same circle i.e. the square on the side of the pentagon is equal to the square on the two sides of the decagon and hexagon.



Let A B be the side of pentagon inscribed in the circle A B C.

We draw the diameter A D C and join the centre D and the point B by the line D B and from D draw the perpendicular D E to A B and join A G, B G and perpendicular D H T to A G, although, it intersesects A B at point K and join G K.

We say that the two angles D A B, K ' B are equal; Because each one is three fifth of the right angle, that is to say, since in the isosceles triangle A D B, the angle A D B is four-fifth of the right angle and the sum of the two angles of the base will be one right angle plus one-fifth of the right angle and every one separately will be three-fifth of the right angle and when the angle A D T the one-fifth of the right angle is deleted from the angle A D B, the fourth-fifth of the right angle, the angle K D B K D B, the three-fifth of the right angle remains also. Thus in two triangles A B D, B D K, the two angles D A B, K D B are equal and the angle D B K, is the common. Thus they are similar. The ratio of A B to B D will be as the ratio of B D to K B and by Proposition 5, the square on B D which is the side of the hexagon will be equal to the surface A B . K B and also in the two triangles A B G, A G K, the two arms G A, G B and the arms K G, KA are equal, and the angle A is common. For this reason, these two triangles are similar. The ratio of A B to A G will be as the ratio of A G to A K and the square on A G which is the side of the decagon will be equal to the surface A B . K A. Thus the sum of the two squares on B D, A G is equal to the sum of the two surfaces of the line A B into its two segments which is K B and K A and according to the Proposition 43, these two surfac[es] are equal to the square on A B which is the side of the Pent[agon]. Thus two squares on B D, A G are together equal to the square on A B.

This is what was required*

Algebraically we can represent this Proposition in terms of the radius r.

We have the result from the preceding Proposition that

$$A\ G = \frac{r}{2}(\ 5-1\ )$$

and by theorem,

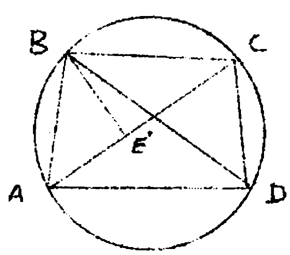
$$(\text{side of Pentagon})^2 = A\ B^2 = B\ D^2 + A\ G^2$$
$$= r^2 + \frac{r^2}{4}\ (6-2\sqrt{5}\ )$$
$$= \frac{r^2}{4}\ (\ 10-2\sqrt{5}\ )$$

so that,

$$\text{Side of Pentagon} = \frac{r}{2}(10-2\sqrt{5}\ )^{1/2}$$

PROPOSITION 56 :

In every four sided figure which is inscribed in a circle, the sum of two surfaces of any side into its opposite side will be equal to the surface enclosed by two diagonals of the figure.



Therefore, the surface A B C D is inscribed in the circle A C and their two diagonals are A C, B D.

We say that the sum of the two surfaces B C. A D and B A. C D is equal to the surface B D. C A.

We make at point B, with the line A B, the angle A B E equal to the angle C B D and take the angle E B D common. Then in two triangles A B D, C B E, the two angles A B D, C B E are equal and similarly the two angles A C B, B D A which stand on the arc A B, are equal and by this reason these two triangles are similar. The ratio B C to C E will be equal to the ratio of B D to D A. Hence the surface B C D A will be equal to the surface B D.• C E and also in the two triangles A B E, B C D, the angles A B E, C B D are equal and similarly, the angles B AE, B D C which stand on the same arc B C; therefore, they are similar. The ratio of A B to A E will be equal to the ratio of B D to C D and the surface A B.• C D will be equal to the surface B D. A E. Then the sum of the two surfaces B C . A D and A B • C D will be equal to the sum of two surfaces B D.CE and B D.• A E i.e. the surface B D into the whole A C. This is what is wanted.

This Proposition is known as "Ptolemy's Theorem". That is

$$A B . C.D + B C . A D = A C . B D.$$

A special case of this general theorem had appeared in Euclid's Data (Proposition 93) (15; 26)

If A B C is a triangle inscribed in a circle, and if B D is a a chord bisecting angle A B C, then

$$(A B + B C) / B D = A C/A D$$

Another and more useful special case of this theorem is that in which one side, say A D, is a diameter of the circle. Then, if A D = 2 r, we have

$$2r \cdot BC + AB \cdot CE = AC \cdot BD.$$

if we let arc $BD = 2\alpha$ and arc $CD = 2\beta$

then,

$$BC = 2r \sin(\alpha - \beta)$$

$$AB = 2r \sin(90 - \alpha)$$

$$BD = 2r \sin\alpha,$$

$$CD = 2r \sin\beta,$$

and $$AC = 2r \sin(90 - \beta)$$

Ptolemy's theorem, therefore, leads sto the result

$$\sin(\alpha - \beta) = \sin\alpha \operatorname{Cos}\beta - \operatorname{Cos}\alpha \sin\beta.$$

Similar reasoning leads to the formula

$$\sin(\alpha + \beta) = \sin\alpha \operatorname{Cos}\beta + \operatorname{Cos}\alpha \sin\beta.$$

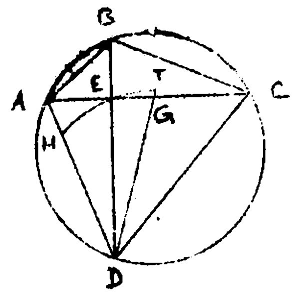
And to the analogous pair

$$\cos(\alpha \pm \beta) = \cos\alpha \cos\beta \pm \sin\alpha \sin\beta.$$

These four sum and the difference formulas consequently are often known as Ptolemy's Formulas.

PROPOSITION 57 :

In every unequal arcs which are not greater than a semi circle the ratio of the chord of the bigger arc to that of the smaller arc will be less than the ratio of the bigger arc to the smaller arc.



Let in the circle A B C the arc C B be greater than the arc B A.

We say that the ratio of the chord C B to the Chord B A will be less than the ratio of their two arcs.

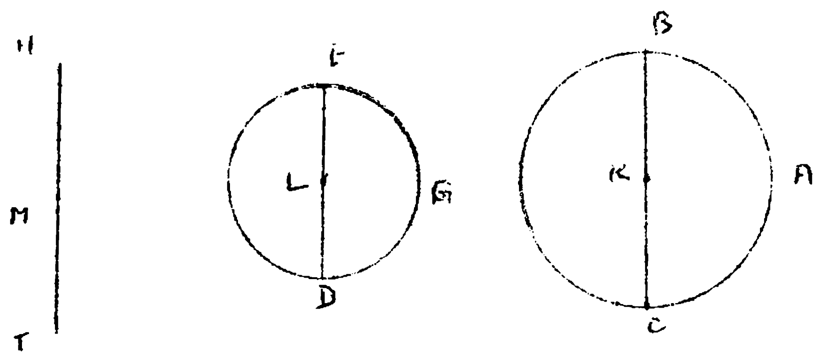
For establishing the objective, we bisect the angle A B C by the line B D and join A C which intersects B D at E and two arcs C D, D A, by Proposition 22 of 3 are equal. Similarly their two chords i.e. the lines CD, D A are also equal, by Proposition 23 of 3. By Proposition 24, the ratio of C E to E A is as the ratio of C B to B A and C B is longer than B A. Therefore, C E will also be longer than E A. From D we draw the perpendicular D G to A C and due to equality of DC, DA, this perpendicular bisects A C at G. Thus G lies between C, E and draw at the centre D, with the distance D E, the arc H E T. We draw D G till it meets this arc at T. Thus the section D E T greater than the triangle D E G is obtained and the section D E H less than the triangle D E A. Hence, the ratio of the triangle to triangle i.e. the ratio of G E to E A will be less than the ratio of the section to section by Proposition 8 i.e. than the ratio of the angle T D E to the angle E D H.

By componendo of the ratio of G A to E A is less than the ratio of the angle G D A to the angle E D A, by Proposition 17 and by duplication of the two antecedents the ratio of A C to A E will be less than the ratio of the angle C D A to the angle E D A and by separation of the ratio C E to E A i.e. the ratio C B to B A will be less than the ratio of the angle C D B to the angle B D A i.e. the ratio of the arc C B to the arc B A.

This is what is required. *

PROPOSITION 58:

The ratio of diameters of circles to the circumferences will be same.

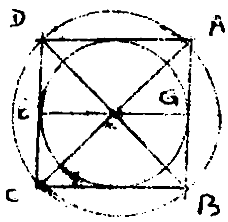


Let two circles A B C, D E G be unequal.

We say that the ratio of the diameter B C to the circumference A B C will be equal to the ratio of the diameter E G to the circumference D E G and otherwise will be equal to the ratio of the diameter E G to the line H T which is different from the circumference D E G and by this supposition, the ratio of the radius K B to the semi-circumference B A C will be equal to the ratio of the radius L E to the half line H T which is H M. The ratio of the rectangle, which is bounded by B K and a line which will be equal to B A C, to a surface which is obtained by enclosing L E., H M is the duplicate ratio of K B to H M, by Proposition 35. By Proposition 52 the circle A B C is equal to the surface K B into the line B A C. Therefore the ratio of this circle to the surface L E . H M will also be duplicate and by porism of Proposition 40, it is proved that the ratio of the circle A B C to the circle D E G is equal to the duplicate ratio of K B, L E. Hence the ratio of the circle A B C to the circle D E G and towards the surface L E . H M will be in one ratio. By Proposition 9, the circle D E G and the surface L E. H M are equal, but here the circumference of the circle D E G is difference from the line H T. The circle D E G by supposition, will also be different from the surface L E H M. This is a contradiction. Hence the claim is proved i.e. the ratio of the diameter B C to the circumference A B C will be as the ratio of the diameter E G to the circumference D E G and also by exchange the ratio of the diameters B C, E G, will be as the ratio of the circumference A B C, D E G.

PROPOSITION 59:

Every circle which circumscribes a square will be double of that circle which is inscribed in the square.

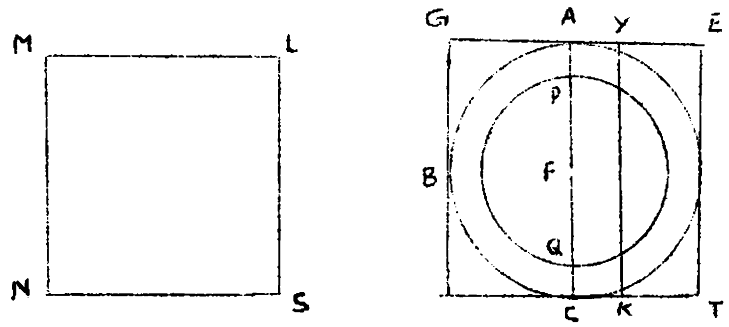


Let A B C D be the square circumscribed in the circle B D and the inscribed the circle E G.

We join D B which will be the diagonal of the square and also the diameter of the circumscribed circle and draw the diameter E G of the inscribed circle parallel to the side of the square and this diameter, of course, is equal to the side of the square and the square on BD is two times of the square on A B i.e. E G, by Bride's proposition and the ratio of the circles will be equal to the ratio of the squares of the diameter. Therefore, the circle B D will also be two times of the circle E G.

PROPOSITION 60:

We wish to separate a ring from a circle which will be a part or parts of that circle.



Let A B C be the circle on the diameter A C and the given part e.g. one-third.

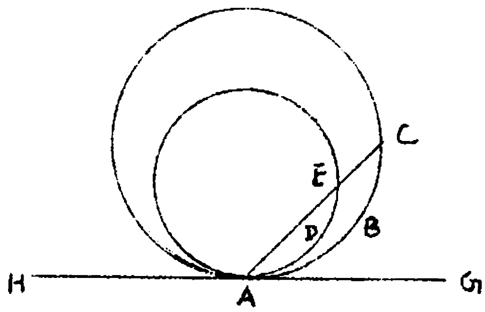
We construct the square E G H T on the circle and separate E Y from E G equal to one-third of that. From the point Y we draw the line Y K parallel to E T. Hence by Proposition 22, the surface E K will be one-third of the square and the surface Y H will be two third. Construct a square which will be equal to the surface Y H, with the help of Proposition 49 of 2 and that tsquare will be L M . N S. Bisect L M at U and let the centre of the circle A B C D be F. We separate F S from F A equal to L U and construct at point F with the distance F P the circle P R. Hence this circle separates the ring from the first circle equal to one-third of it.

Because the ratio of the two circles is equal to the ratio of the two squares L N, E H and the square L M is two-third of the square E H. Hence the circle O P will be two-third of the circle A B C D. Thus one-third ring remains.

This Proposition is the discovery of Ghulam Husain.

PROPOSITION 61 :

In every two circles which touch internally and a straight line is drawn from the point of contact, intersects the circles, then this line separates both circles into two similar sections in one side.



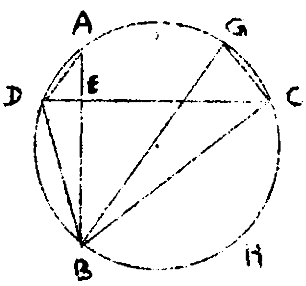
So, two circles A B C, A D E touch at the point A and let the intersecting line which is drawn from the point of contact be A E C.

We say that the two sections A B C, A D E e.g. are similar. Because when from A we draw a line G A H tangent to each of the two circles, then by Proposition 25 of 3, in each two sections the angles which are formed are equal to the angle C A H. Therefore, they are similar and the remaining two sections are also similar.

This is what is wanted.

PROPOSITION 62 :

If two chords of a circle intersect at right angle, then the sum of the square on every four segments of the two chords will be equal to the square on the diameter.



Like two chords A B, C D which intersect at right angle in the point E.

We say that the sum of the squares on A E, E B, C E, E D will be equal to the square on the diameter B G.

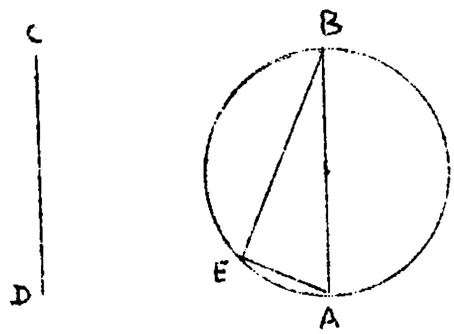
We join the lines G C, C B, B D, D A.

Since in triangle D E B, the angle E is right angle. Hence the sum of the two angles E D B, D B E will be equal to one right angle. Therefore, by Proposition 42, the sum of the two arcs A D and C H B is an arc on which right angle is situated and the right angle does not stand on the circumference but on the semi-circumference. Hence the sum of the two arcs abovementioned will be semi-circumference and B C G is semi-circumference. We delete B H C common, two equal arcs C G, A D, remain and their chords will also be equal. After this introduction, we say that the two squares on A E, E D is equal to the square on A D i.e. square on C G and similarly two squares on C E, E B is equal to the square on B C and two squares on B C, C G i.e. four squares on A E, E D, C E, E B is equal to the square on B G the diameter.

This is what was required.

PROPOSITION 63:

We wish to make a third line in such a way that the greater of the two given lines dominate over the third line and the smaller of the two.



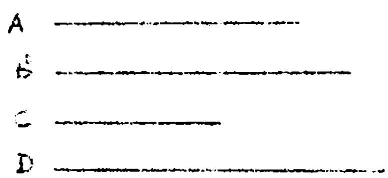
So, two given lines are A B, C D and A B the longer. Then, we describe a circle A E B on the longer line as the diameter and draw the chord B E equal to the line C D with the help of Proposition 30 of 3. We join the line A E which will be the required line.

Because, the angle A E B, by Proposition 35 of 3 is right angle. Thus, by Bride's Proposition, A B is greater than A E, E B.

This is what was wanted.

PROPOSITION 64 :

If four magnitudes are proportional consecutively, then the ratio of the first two fourth will be a cube of ratio of the first to the second i.e. it is obtained by multiplying the simple ratio by itself by two lines.



So, the four magnitudes A, B, C , D are consecutively proportional. I say that the ratio of A to D will be cube of the ratio of A to B.

Because, by Proposition 21 of 4, the ratio of A to C is duplicate ratio of A to B and the ratio of C to D is as the ratio of A to B. Therefore, the ratio of A to D admitting once more repetition becomes a cube. This is what is wanted.

On this analysis if there are five magnitudes, then the ratio of the first to fifth will be 4 times and in case of six, five times and similarly indefinitely.

This interesting property of the theory of proportion is due to Ghulām Ḥusain.

The Proposition may be expressed algebraically as follows:

If x, y, z, w are proportional, then

$$x : w = x : y.$$

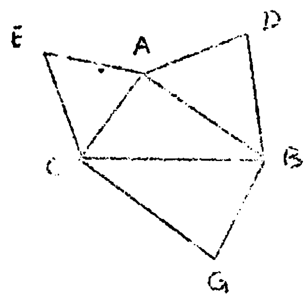
The generalisation of it as stated above is, if

x, y, z, ... l. are proportional,

then, $x : l = (x : y)^{n-1}$, where n is integer.

PROPOSITION 65 :

If similar surfaces be constructed on the sides of a right angled triangle, then the sum of the two surfaces which stand on the two sides of the right angle will be equal to a surface which stands on the hypotenuse.



Let A B C be the triangle and angle A be a right angle and B C the opposite side of the right angle (hypotenuse) and let the similar surfaces stand be e.g. the triangles D A B, E A C, G C B.

We say that the sum of the first two triangles will be equal to the third triangle.

Because, in Proposition 35, generally it is proved that the ratio of each two similar surfaces is duplicate ratio of the corresponding sides and for the squares that stand on these sides, the ratio will also be duplicate. Therefore the ratio of each triangle to the square of its side will be the same. Thus the ratio of the sum of the two triangles D B A, E A C to the triangle G C B will be the ratio of the sum of the two squares on A B, A C to the square on B C, and the sum of the two squares A B, A C is equal to the square on B C, by Bride's Proposition.

Therefore, the sum of the two triangles D B A, E A C will also be equal to the triangle G C B.

This is our objective.

This proposition is the generalization of Proposition 38 of 2. Heath (60; 172) conjectures that this theorem may have been originally established, there seems to be little reason to doubt that Euclid's proof of it as I.47(38 of 2) is derived from the proof of VI.31 (65 of 4). But Euclid chooses to prove it independently of I.47, by using the general theory of proportion. This suggests that he proved I.47 by means of Book I alone, without invoking proportions in order to get it into his first book instead of his sixth. The proof certainly bears the marks of genius (80; 424).

The above proof is an laternative proof which is given by Ghulam Husain. To prove it, he supposes that the triangles on the side of the right angle triangle are similar and uses the Proposition 35 and Bride's Proposition (I.47). But Euclid drops A D perpendicular to B C and invokes VI.8 (Ghulām Husain does not incorporate this proposition in this book), and definition of similarity to get C B : B A = B A : B D, and C B : C A = C A: C D. Hence by Proposition 19 (35 of 4),

(fig. on B C) : (fig. on A B) = C B : B D

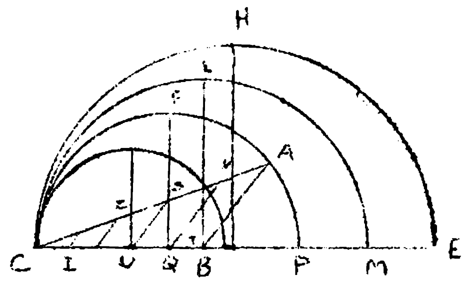
and (fig. on B C) : (fig. on A C) = C B : C D

then (fig. on B C) : (Fig. on AB + fig. on A C)

= C B : B D + C D.

PROPOSITION 66 :

We wish to divide a given triangle into equal parts by lines which will be parallel to one of its sides.



For example, the triangle A B C will be divided into five equal parts by the lines parallel to A B.

So, we separate C B from the side B D equal to one-fifth of B C with the help of the Proposition 29 of 4 and we draw mean proportional line of the two lines C B, C D, with the help of Proposition 26 of 4, such that we extend C B towards E so that B E is equal to C D and bisect E C at G. We draw at a point G with the distance G C the semicircle C H E and from point B draw the perpendicular B H to the line C E which will be mean proportional. We separete from C B, C T equal to B H. It is necessary that the point T lies between G , D and from T we draw the line T Y parallel to B A. Hence this line separates the quadrilateral A B T Y from the triangle A B C equal to one-fifth of that triangle. Because, when we join A D, by Proposition 22 of 4, the triangle A D C will be four-fifth of the triangle A B C and since line C T is mean between the basis of these two triangles and triangle

Y T C is situated on the line C T similar to the triangle A B C. Therefore, by Proposition 40 of 4, triangle Y T C will be equal to the triangle A D C. Therefore, the triangle Y T C will also be four-fifth of the triangle A B C and the quadrilateral A B T Y is one-fifth of the triangle A B C.

Further, we seperate T K from T C equal to one-fourth of it and draw the mean line between CT, C K and draw the line on the semi-circle C L M and that line will be T L. Seperate from C T, C N equal to T L and draw from N the line N S parallel to T Y which, by preceding statement will separate from the triangle Y T C, the quadrilateral Y T N S equal to one-fourth of this triangle i.e. one-fifth of the original triangle. Further, we separate N U from C N equal to one-third and draw the semi-circle C F P and draw N F the mean proportion between two lines C N, C U and separate C F from C N equal to N F and from Q draw the line Q Z parallel to N S till the quadrilateral S N Q Z equal to one-third of the triangle S N C i.e. one-fifth of the triangle A B C is separated. Again from C F, we separate Q J equal to its half and draw the semi-circle C J O. Drew the mean line Q J between C Q, I and from C Q we cut the line C H equal to mean Q J and draw from point H the line H D parallel to Q Z which cuttoff the quadrilateral Z Q H D from the triangle

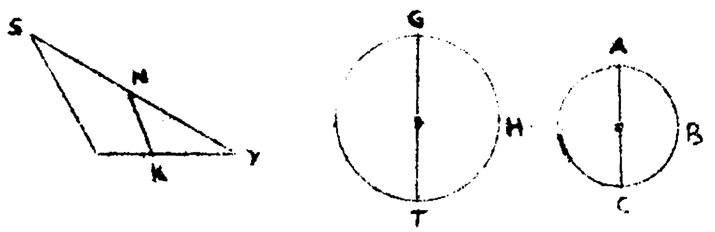
Z Q C equal to its half i.e. equal to one-fifth of the original triangle and the triangle D H C also equal to one-fifth remains.

This was our objective.

PROPOSITION 67 :

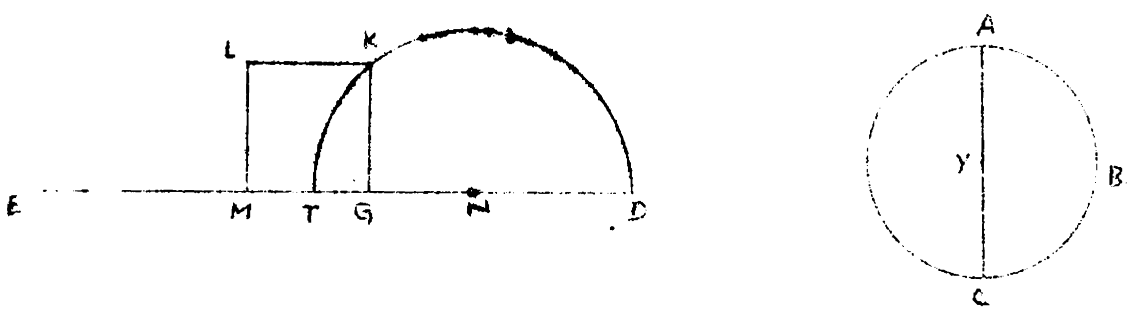
The problem of producing a straight line equal to the circumference of a circle, undoubtedly a section of ancient did not believe in quantifying the circumference of a circle by means of a straight line. Because the straight line and circular line are of different kinds. The straight line in the sense that it is straight, it does not loose its straightness and a circular line being circular never loses its curvature. Straightness is never associated with it and if that is so, that the idea of equality of straight line and circular line will be forbidding. The modern thinkers like Ibn Haitham and Banu Moosa and their followers have treated this statement, as unacceptable, with explanation that it is certain that some of the bodies are capable of joining and twisting. And a straight line which can be spread out on its surface and although it can be twisted in circular form. However, as the twisting increases, its curvature also increases. It is not necessary to rely on straightness and curveness which is dependant upon the nature. When this much is known, we say that there are certain bodies which are long, fine, cylindrical capable of twisting and like metalic wire is wrapped on the circumference of a circle like the inscribed surface of a circular cylinder covering is superpose without gap and because the wire itself is a cylinderical body, it will not be tangential to the outer surface of the curved circular cylinder, but on a straight line which is of its own size from the wire is super imposed without gap on a straight line. Hence the circumference of the circle on the straight line which is equal to the length of the wire will be equal. Otherwise, what is given in the axioms of self-evident truth will be absurd i.e. the things equal to the same thing are not equal and similarly, things super imposed without gap in the sense will be a false conjecture and if it become possible to obtain a straight line equal to the circumference of some circles, according to the laws of geometry for every circle that is taken is a straight line can be obtained equal to its circumference, for example, according to what has been said earlier. From the circles we obtained the circle A B C whose circumference equal to a straight line G H. I say that now the circumference of all the circles can be known. Because by Proposition 58, the ratio of the diameter of every circle to its circumference is as the ratio of A C, D E. Let the given circle be G H T. Hence the ratio of the diameter G T to the circumference G H T will be as the ratio of A C to D E. Now, by rule of the Proposition 28, we obtained fourth line in the ratio for the lines A C, D E, G T and that line will be N S. Therefore, N S will be equal to the circumference of the circle G H T. Because the ratio of G T to the circumference G H T and towards N S is same i.e. the ratio A C to D E. Thus by Proposition 9, N S and the circumference G H T are equal.

This is our object.



PROPOSITION 68 :

We wish to construct a square equal to a given circle.



Let the circle be A B C.

First we produce the line D E which will be equal to the circumference of that circle. Bisect D E at G. Then the surface of the radius into the half of the line D E is equal to the surface of the circle, by Porposition 52. We draw a mean line in the ratio between Y A and D G, with the help of Proposition 16 and that line will be G K. Hence, by Proposition 47, the square of this line i.e.

K M will be equal to the surface Y A. D G i.e. the circle A B C.

This is what is wanted.*

C O N C L U S I O N

C O N C L U S I O N

In this thesis we have analyzed the geometrical concepts and methods used in solving problems, given in a chapter of JAME-I-BAHADUR KHANI. But there are many other works by Indians dealing with the same subject like Sharh Tahrir-i-Uqlidas written by Mir Mohammad Hashim, Sharh Makhrutat-i-Samson by Taffaddul Husain Khan etc. A deeper study of these works is required. Undoubtedly, these commentaries have some original basic geometrical notions which may act as a tool for researchers in developing a new theory of geometry.

N O T A T I O N S

APPENDIX

N O T A T I O N S

*	Represents the closing sentence of JĀME-I-BAHĀDUR KHĀNĪ.
Proposition X of Y	X stands for Proposition Number and Y denotes respective Section of JĀME-I-BAHĀDUR KHĀNĪ.
T.U.X.Y	T.U. is the short form of Tahrīr-i-Uqlīdes of Naṣīr al-Dīn Tūsī; X refers to Muqala Number shown in Roman numeral and Y represents Proposition Number.
X.Y	X denotes Euclid's Book Number in Roman numerals and Y refers to the respective Proposition Number.

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B I B L I O G R A P H Y

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