QUEUEING THEORY AND OTHER APPLICATIONS OF STOCHASTIC PROCESSES

QUEUEING AND INVENTORY MODELS WITH REST PERIODS

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By

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CERTIFICATE

Certified that the work reported in the present thesis is based on the bonafide work done by Sri.Jacob K. Daniel under my guidance in the Department of Mathematics and Statistics, University of Cochin, and has not been included in any other thesis submitted previously for the award of any degree.

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DECLARATION

This thesis contains no material which has been accepted for the award of any other degree or diploma in any University and, to the best of my knowledge and belief, it contains no material previously published by any other person, except where due reference is made in the text of the thesis.

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CHAPTER I

INTRODUCTION

1. QUEUEING THEORY - AN OUTLINE

The development of queueing theory started with the publication of Erlang's paper [19] in 1909 on the M/D/1 queueing system. For this system, which has constant service times and a Poisson arrival process, Erlang explained the concept of statistical equilibrium. This paper touched the essential points of queueing theory, and for a long time research in queueing theory concentrated on questions, first time discussed in theoretical context, by Erlang. The studies of Engset, Molina, Wilkinson, Pollaczek, Vaulot, Crommelin, Kosten and Palm, to mention the most important ones (for details see bibliography given in Syski's book [55]), continued, deepened and enlarged the field of investigation started by Erlang. The mathematical techniques used by Pollaczek were rather diverging from techniques used before 1940. However, they were powerful and made investigation of transient phenomena possible.

In the early fifties, a stormy development of queueing theory took place. Professional mathematicians discovered queueing theory as an interesting field of applications of the results and methods obtained in probability theory. Operations Research turned out to be a new field of applied mathematics and in this field a great need for queueing theory existed, not only for its specific models but also for its theoretical structure. It turned out soon that the basic mathematical models in inventory theory, the theory of counters, dam theory and reliability theory are often identical with or slight modifications of those encountered in queueing theory.

Feller's theory of regenerative events (cf.[22]) inspired Kendall to write his famous paper [30] whose influence on queueing theory has been tremendous. Kendall's exposition created a new technique for analysing certain queueing models which are not Markovian. He showed how to apply the method of Markov chains with discrete state space and discrete time parameter to non Markovian queues. At the same time his approach made the analysis of the transient behaviour of queueing systems much more accessible. As a landmark in the developments of queueing theory, the approach by Takacs [56] is worth mentioning. He stressed the importance of the virtual waiting time process. He analysed, at first, this process for the M/G/1 queue. His ideas and particularly his graphical representation of the realisations of the virtual waiting time process has great influence on further research in queueing theory. This method is successfully applied in many queueing models (see the books by Kleinrock [31], Cohen [14]).

An important variant of service discipline is priority service, which seems to have been treated in literature for the first time in 1954. Cobham [13] published the first paper on non pre-emptive priorities for Poisson input, exponential holding time, single and multiple channels. In 1964, Miller [37], when solving for the waiting time in a priority queueing system under the alternating priority discipline, first introduced and studied the M/G/1 queue with rest periods with first in first out order of service in each priority. In this discipline, customers belong to one among N classes. The server continues serving customers of the same class (say i) until they are depleted and the server starts serving customers of another class and so on, eventually again serving the customers of class i. For class i, the server's rest period is the time elapsing between his departure and return to that class for the first time. Such a model has been partially used by Cooper [15] to analyse a system of queues served in cyclic order.

Consider a class of M/G/1 queueing models where the server is not available over occasional intervals of time. The times, when the server is not available, may initiate some other uninterruptible task (a coffee break, or a tool change), which we shall refer to as a 'rest period'. After completing this task, the server returns and begins serving any back-log that may have accumulated during its absence. We say that such a queueing model has the property of 'exhaustive service' in case each time the server becomes available, he works in a continuous manner until the system becomes empty.

As an example of the exhaustive service system, consider an M/G/1 queueing system with rest periods. This is a queueing model, in which, for each time the system becomes empty and the server goes for rest. If the server returns from rest and finds some units waiting, then he works until the system is empty and begins another rest period. If the server finds the system empty on return from rest, he starts another rest period which is independent of and identically distributed as the previous one. This model (and variations thereof) has been studied by Miller [37], Cooper [15], Levy and Yechiali [33], Heyman [28], Shantikumar [51], Scholl and Kleinrock [50]. They showed that the number of units in the system at a random point in time is distributed as the sum of two or more independent random variables, when the equilibrium solution existed. This property is known as the M/G/1 decomposition property.

Recently, Neuts and Ramalhoto [46], Ali and Neuts [1] studied M/G/1 decomposition property in the case of M/G/1 queueing systems without exhaustive service. Fuhrmann and Cooper [24] gave an intuitive explanation of the reason for the decomposition property for a general class of M/G/1 queueing systems. Doshi [17] examined the intuitive development of GI/G/1 queueing systems with exhaustive service.

2. A SKETCH OF INVENTORY THEORY

The quantitative analysis of the inventory started with the work of Harris [27] in 1915, who formulated and got the optimal solution to a simple inventory situation. Wilson rediscovered the same formula in 1918 and this is referred to as Wilson's formula or Economic lot size formula. Several variations of Wilson's deterministic model have been studied. A stochastic inventory problem was analysed for the first time in 1946 by Màsse [35]. After that several studies were made in this direction (see Arrow, Harris and Marschak [2], Dvoretsky, Kiefer and Wolfowitz [18]).

In [18], the authors obtained the conditions under which optimum inventory levels can be found. The development of the theory upto 1952 have been summarised by Whitin [60].

Berman and Clark [7] developed some specific models which can be applied, for example, to a situation where a central warehouse supplies to a number of field warehouses which in turn supply to distributors. A new method which minimises the variance of the inventory balance under specified conditions using discrete distribution of demand and inventory was developed by Vassian [58].

Consider the situation for setting the overall production levels when there are significant fluctuations in demand. In this case the linear programming technique was effectively applied by Charnes, Cooper and Farr [9]. Through the development of a new technique, Bellman [6] has made an investigation on the feasibility of the approach to dynamic inventory problems. A review of the storage problems was given by Gani [25]. Using the renewal theoretic arguments, a probabilistic treatment of inventory problems was given by Arrow, Karlin and Scarf [3]. The study of random leadtimes in inventory models was provided in the monograph by Ryshikow [49]. Sivazlian [52] investigated an (s,S) inventory system with arbitrary inter-arrival time distribution between unit demands. Recently Srinivasan [53] has made an attempt to study a more general case. In this paper he considered the leadtimes and demands to be arbitrarily distributed.

3. MATRIX GEOMETRIC SOLUTIONS

The classical approach to the analysis of queues is to assume that the variables representing the inter-arrival and service times follow some specific distributions and to define the state of the system by the number of units in the system at a given time. The objective is to find the probability distribution of the number of units in the systems. One way of obtaining the probabilities is to formulate a system of difference-differential equations to represent the behaviour of the queue in time. The solution of this system of equations is known as 'transient solutions'. The computation of the transient solution is a belabored task even for simple models. The computational effort and the interpretation of numerical results of transient solutions depend on the choice of the initial conditions. Under certain conditions, solution to the system of equations, when time tends to infinity, is called steady state solution. For the analysis of steady state solutions we do not require any initial condition.

We can solve the system of difference-differential equations using the methods of generating function, operator etc. The numerical computation of the time dependent probability distributions of interest, by the classical methods based on an application of Rouche's theorem, is in practice quite difficult. Neuts [43] investigated a new approach which has efficient and stable algorithms involving only real arithmetic.

In [41], Neuts shows that a class of infinite, block partitioned, stochastic matrices has an invariant probability vector of a matrix-geometric form. He considers a Markov chain with state space $\{(i,j),\ i \geq o,\ 1 \leq j \leq m\}$ and a transition probability matrix P of the form

$$P = \begin{bmatrix} B_o & A_o & 0 & 0 & \cdots \\ B_1 & A_1 & A_o & 0 & \cdots \\ B_2 & A_2 & A_1 & A_o & \cdots \\ \vdots & \vdots & \vdots & \vdots & \\ B_k & A_k & A_{k-1} & A_{k-2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix} \qquad (1)$$

where the matrices A_k and B_k, $k \geq o$ are m x m non-negative matrices satisfying

$$\sum_{r=o}^{k} A_r e + B_k e = e, \text{ for } k \geqslant o \tag{2}$$

where $e = (1, 1, \ldots, 1)^T$. The substochastic matrix $\sum_{r=o}^{\infty} A_r$ will be denoted by A. Assume that A is stochastic and irreducible. Let $\underline{x}$ be an invariant probability vector such that $\underline{x} \geqslant o$ and

$$\underline{x}\, P = \underline{x} \text{ with } \underline{x}\, e = 1. \tag{3}$$

Partitioning the vector $\underline{x}$ into m-vectors $\underline{x}_o$, $\underline{x}_1$, ..., the equation (3) may be written as

$$\underline{x}_k = \sum_{r=o}^{\infty} \underline{x}_{k+r-1} A_r, \quad \text{for } k \geqslant 1,$$

$$\underline{x}_o = \sum_{r=o}^{\infty} \underline{x}_r B_r, \tag{4}$$

$$\sum_{k=o}^{\infty} \underline{x}_k e = 1.$$

Let X be a non-negative matrix of order m. The spectral radius, sp(X), of X is defined as

$$sp(X) = \min_{\lambda}\left\{|\lambda_i| \leq \lambda \quad , \text{ for } 1 \leq i \leq m\right\},$$

where λ_is' are the eigen values of X. Using the coefficient matrices A_r, $r \geq o$, define the formal series

$$A[X] = \sum_{r=o}^{\infty} X^r A_r. \tag{5}$$

Define the sequence of matrices $\left\{R(n), n \geq o\right\}$ by

$$R(o) = o, \quad R(n+1) = A[R(n)], \text{ for } n \geq o$$

and the vector β by

$$\beta = \sum_{r=1}^{\infty} r A_r e . \tag{6}$$

Note that some components of β may be infinite when A_o is stochastic. This is trivial and will be eliminated from further consideration. In all other cases $\beta \neq o$.

Now examine the existence of a solution to the equations (3) of the form

$$\underline{x}_k = \underline{x}_o R^k, \text{ for } k \geq o \tag{7}$$

where R is non-negative and irreducible. Since $\underline{x}_k \rightarrow o$ as $k \rightarrow \infty$, the matrix R satisfies $sp(R) < 1$. The first equation of (3) yields

$$\underline{x}_o R^{k-1} \left\{ R - A[R] \right\} = \underline{o}, \text{ for } k \geq 1 . \tag{8}$$

If $R = A[R]$, $R \geq \underline{o}$ and $sp(R) < 1$, the above equation is satisfied.

The second equation of (3) yields

$$\underline{x}_o = \underline{x}_o \sum_{r=o}^{\infty} R^r B_r \tag{9}$$

so that $\underline{x}_o$ should be a left eigen vector of the matrix

$$B[R] = \sum_{r=o}^{\infty} R^r B_r . \tag{10}$$

Let $\underline{\pi}$ be the invariant probability vector of A, then we have the following theorem, proof of which is given in [41].

Theorem. If the matrix A is irreducible and $\pi\beta > 1$, let $R \geq 0$, be the minimal solution of $R = A[R]$. If the stochastic matrix $B[R]$ is irreducible, let $\underline{x}_0$ be the left eigenvector, with eigenvalue one of $B[R]$, then $\underline{x}_0 \gg 0$ and $\underline{x}_0$ may be normalized by

$$\underline{x}_0 \, (I-R)^{-1} \, e = 1 \, .$$

The partitioned vector $\underline{x} = (\underline{x}_0, \underline{x}_1, \ldots)$ with

$$\underline{x}_k = \underline{x}_0 \, R^k, \quad k \geq 0$$

is then a strictly positive invariant probability vector of matrix P.

The probabilistic significance of the rate matrix is discussed in Neuts [44]. For discussing the probabilistic significance of the matrix R, first define the taboo probability. Given two states i and j and a set H of states of a Markov chain, the taboo probability ${}_H P_{i,j}^{(n)}$ is the conditional probability that, starting from state i, the system reaches the state j at time n without having visited the set H at any of the times 1,2, ..., n-1. (For details see Chung [10]). For the Markov chain with transition probability matrix P, consider the taboo probability

${}_{i}P^{(n)}_{i,j;i+1,r}$ that starting from the state (i,j), the chain reaches (i+1,r) at time n without visiting the level i in between. This probability is defined for $n \geq 0$, $i \geq 0$, $1 \leq r, j \leq m$ and is clearly equal to zero for $n < 1$. A sample path from (i,j) to (i+1,r) which does not re-enter the level i cannot visit any state (r,j') with $r \leq i$, $1 \leq j' \leq m$. Thus the taboo probability ${}_{i}P^{(n)}_{i,j;i+1,r}$ depends completely on the submatrix of P obtained by deleting all rows and columns with indices (r,j'), $r \leq i$, $1 \leq j' \leq m$. These submatrices are identical for $i \geq 0$. Define

$$R_{j,r} = \sum_{n=1}^{\infty} {}_{i}P^{(n)}_{i,j;i+1,r}, \quad i \geq 0, \; 1 \leq j, \; r \leq m .$$

This is the expected number of visits to the state (i+1,r) before the first return to the level i, given that the chain starts in the state (i,j). The square matrix with elements $R_{j,r}$, $1 \leq j$, $r \leq m$, is denoted by R and is called the rate matrix.

In some cases the matrix A is reducible, the conditions for which is that all the eigenvalues of the matrix R inside the unit disk are different. A detailed account of this type of situation is given in Lucantoni [34].

In some practical situations the structure of the transition probability matrix P is slightly different from (1). Here P has some more complicated structure near the lower boundary. This type of situation is considered by Winsten [61]. This boundary behaviour leads to the consideration of modified matrix geometric invariant vectors. In this case the transition probability matrix is different from (1). The probabilistic significance of the rate matrix is the same, but applies only to non-boundary states. The invariant probability vector $\underline{x}$ in this case is of the form $\underline{x} = [\underline{x}_0, \underline{x}_1, \underline{x}_1 R, \underline{x}_1 R^2, \ldots]$. This type of invariant probability vector is called modified matrix geometric vector.

4. SUMMARY OF RESULTS ESTABLISHED IN THIS THESIS

In this thesis we study the effect of rest periods in queueing systems without exhaustive service and inventory systems with rest to the server. Most of the works in the vacation models deal with exhaustive service. Recently some results have appeared for the systems without exhaustive service.

Chapter II deals with a queueing system in which the inter-arrival times follow a general distribution G(x) and service times are exponentially distributed. We assume that the server takes rest when the system is empty or when the service of every k units are over, whichever occurs first. The rest times are exponentially distributed with parameter α. Using the imbedded Markov chain technique, the stationary behaviour of the process is studied. The waiting time distribution of a unit in the system is given in the stationary case.

In section 4 of this chapter we consider a particular case of the above model. Assume, the arrival process is a Poisson process of rate λ and the service times are exponentially distributed with parameter μ_k, where k is the number of units served consecutively after the previous rest. We obtain the steady state probabilities π_i s' associated with the generator A, of the process as

$$\pi_i = \left(\frac{\alpha}{\mu_i} \right) \left[1 + \sum_{i=0}^{k-1} \frac{\alpha}{\mu_i} \right]^{-1} \text{ for } 0 \leq i \leq k-1$$

$$\pi_k = \left[1 + \sum_{i=0}^{k-1} \frac{\alpha}{\mu_i} \right]^{-1}$$

The condition under which steady state exists is shown to be

$$k\,\alpha > \left[\, 1 + \sum_{i=o}^{k-1} \frac{\alpha}{\mu_i} \,\right].$$

Numerical computation of the rate matrix R is given for various values of α and k. An optimization problem associated with this model is also discussed in the same chapter.

In chapter III, we consider a bulk service queueing model. The size of the batch is (a,b). That is, the minimum and maximum number of units in the batch is 'a' and 'b' respectively. After the service of a group, if the number of customers in the queue is less than 'a', the server takes rest for a random length of time which is exponentially distributed with parameter α. Here a modified matrix geometric solution is investigated and the rate matrix could be given without a numerical computation. The elements of the rate matrix R are obtained as follows:

$$R_{11} = \frac{\lambda}{\lambda+\alpha}$$

$$R_{i,i} = \frac{\lambda}{\lambda+\mu_{a+i-2}}, \quad \text{for } 2 \leq i \leq b-a+1$$

$$R_{b-a+2,b-a+2} = \frac{\lambda}{\lambda+\mu_b} + \frac{\mu_b}{\lambda+\mu_b} R^{b+1}_{b-a+2,b-a+2}$$

$$R_{i,b-a+2} = \left(\frac{\mu_{a+i-2}}{\lambda+\mu_b} \right) \left(\frac{\lambda}{\lambda+\mu_{a+i-2}} \right)^{a+i-1}$$

$$\left[1- \frac{\mu_b}{\lambda+\mu_b} \sum_{j=o}^{b} R^j_{b-a+2,b-a+2} R^{b-j}_{ii} \right]^{-1},$$

for $2 \leq i \leq b-a+1$

and $$R_{1,b-a+2} = \frac{\lambda\alpha}{(\lambda+\alpha)(\lambda+\mu_b)} \left[1- \frac{\mu_b}{\lambda+\mu_b} \sum_{j=o}^{b} R^j_{b-a+2,b-a+2} R^{b-j}_{11}\right]^{-1}$$

This chapter concludes with some discussions on the waiting time distribution of a unit in the system.

Chapter IV deals with two models. One is an Erlangian service model with feedback and rest periods, and the other one is a buffer model. In the Erlangian service model there are k-stages of service and the service time in each stage is exponentially distributed with parameter $k\mu$. After the service of all stages, if a customer feels that the service of some stage is not satisfactory, then he again joins the queue for service with probability θ and leaves the system with probability $1-\theta$. Assume that the server takes rest after serving m consecutive units. We obtain the stationary probabilities and the equilibrium conditions as

$$\pi_{i,j} = \left(1+\frac{m\alpha}{\mu}\right)\left(\frac{\alpha}{k\mu}\right), \text{ for } o \leq i \leq m-1,\ 1 \leq j \leq k$$

$$\pi_m = \left(1+\frac{m\alpha}{\mu}\right)^{-1}$$

and $$\frac{\lambda\left(1+\frac{m\alpha}{\mu}\right)}{(1-\theta)m\alpha} < 1$$ respectively.

In the buffer model, we consider a queueing system with two servers in series and a finite intermediate waiting room between them. The arrival process is a Poisson process of rate λ and service times are exponentially distributed with parameters μ_1 and μ_2 respectively. When the intermediate waiting room is full, the first server takes rest and when the intermediate waiting room is empty, the second server takes rest. These rest times are exponentially distributed with parameters α_1 and α_2 respectively. Using the matrix geometric method, the steady state behaviour is investigated. The conditions for existence of equilibrium solution is shown to be

$$\mu_1 \sum_{i=o}^{k-1} \pi_i \; (w_1R_2 + w_1w_2) > \lambda$$

where w_1R_2 denotes that the first server is working and the second is taking rest, and w_1w_2 denotes that both servers are working. Some numerical examples for computation of π_is are also given for various values of μ_1, μ_2, α_1 and α_2.

The last chapter deals with the inventory models with rest periods to server. We consider an (s,S) inventory system in which the server takes rest whenever the level of inventory falls to zero. The demands are assumed to occur for one unit at a time. The inter-arrival times between successive demands, the lead times and the rest times are assumed to follow general distributions. Using the renewal arguments, we obtain expressions for the transition probabilities and the system size probabilities.

Another inventory model that we study in this chapter is an (s,S) inventory system with two servers S_1 and S_2. When the level of inventory becomes s, an order is given for replenishment and one of the servers is allowed to take rest. But at least one of them must be present always. We conclude the thesis with the expressions for transition time probability density functions and system size probabilities for this model.

CHAPTER-II

A GI/M/1 QUEUE WITH REST PERIODS

In this chapter, we consider a queueing system with rest periods in which the interarrival times follow a general distribution G(.) and service times are exponentially distributed with parameter μ. Several authors, Miller [37], Cooper [15], Scholl and Kleinrock [50] and Fuhrmann [23], studied the role of rest periods in queues when the arrival process is a Poisson process and the service times follow a general distribution G(.). The stationary behaviour of the process, the waiting time distribution of a unit in the system, some numerical examples and an optimization problem associated with this, are discussed through sections 2 to 6.

1. THE MODEL

Consider a GI/M/1 queueing system. We assume that the server takes rest for a random length of time after the service of every consecutive k units or when the system is empty, whichever occurs first. The interarrival times are independent and identically distributed with distribution function G(.), and the service times are exponentially distributed with parameter μ. After the completion of a rest period the server returns to the system and remains idle if there is no unit in the waiting line. The rest times are exponentially distributed with parameter α. Observing the system immediately prior to an arrival point, we obtain an imbedded Markov chain defined on the state space

$$S = (o,o) \cup (o,k) \cup \left\{(i,j) \mid i \geq 1, \ o \leq j \leq k\right\}.$$

The state (o,o) denotes that the system is empty and the server is idle, (o,k) denotes that the system is empty and the server is taking rest. The state (i,j) denotes that the number of units in the system is i and the number of units served after the previous rest is j. We can consider another variation of the above model in which, after the completion of a rest the server returns to the system and if he finds no unit waiting, he extends his rest period. The rest times are independent and identically distributed exponential variates with parameter α. In this case, the state space of the model is more simple than the one considered in this paper. In this case the state space S' is

$$(o,k) \cup \left\{(i,j) \mid i \geq 1, \ \ o \leq j \leq k\right\}$$

where the notations are defined as earlier.

The transition probability matrix P of the system with state space S has a block partitioned structure of the form

$$P = \begin{bmatrix} B_0 & C_0 & 0 & 0 & \cdots \\ B_1 & A_1 & A_0 & 0 & \cdots \\ B_2 & A_2 & A_1 & A_0 & \cdots \\ \vdots & & & & \\ B_n & A_n & A_{n-1} & A_{n-2} & \\ \vdots & & & & \end{bmatrix} \qquad (1)$$

This is different from the canonical form of transition probability matrix of GI/M/1 system. Evans [20] and W.Wallace [59] showed that the block-Jacobi generators of the continuous parameter Markov process of GI/M/1 type, called quasi-birth and death process, have a matrix geometric probability vector. Winsten [61] gave an explanation of the modified geometric stationary density, which yields the solution to the GI/M/c queue with single arrivals.

The matrix C_0 of order 2 x (k+1) is given by

$$C_o = \begin{array}{c} \\ (o,o) \\ (o,k) \end{array} \begin{array}{c} \begin{array}{cccc} (1,o) & (1,1) & \cdots & (1,k) \end{array} \\ \left[\begin{array}{cccc} a_o & o & \cdots & o \\ {}_1a_o & o & \cdots & \hat{a}_o \end{array} \right] \end{array} \tag{2}$$

The element a_o is the probability that during an interarrival time there is no service completion and is given by

$$a_o = \int_o^\infty g(x)\ e^{-\mu x}\ dx\ . \tag{3}$$

The element ${}_1a_o$ is the probability that during an interarrival time, a rest period is over but no service completion and is given by

$${}_1a_o = \int_o^\infty g(x) \int_o^x \alpha e^{-\alpha u}\ e^{-\mu(x-u)} du\ dx \tag{4}$$

and $\hat{a}_o$ is the probability that the server continues his rest in an interarrival time, given by

$$a_o = \int_o^\infty g(x)\ e^{-\alpha x}\ dx\ . \tag{5}$$

The matrix B_o is given by

$$B_o = \begin{array}{c} \\ (o,o) \\ (o,k) \end{array} \begin{array}{cc} (o,o) & (o,k) \\ \left[\begin{array}{c} {}_1a_1 \\ {}_2a_1 \end{array}\right. & \left.\begin{array}{c} \hat{a}_1 \\ {}_1\hat{a}_1 \end{array}\right] \end{array} \tag{6}$$

The entries in B_o are explained along with the entries of B_m. The matrices B_m, for $m \geqslant 1$, are given in a unified way. We write m as nk+r, where k is the maximum number of units served consecutively after a rest and n is the number of rest periods. Here, $n = [m/k]$ and $r = m-[m/k]k$, where $[m/k]$ denotes the integral part of m/k. The matrix B_m, for $m \geqslant 1$, is given by

$$B_m = \begin{array}{c} (m,o) \\ (m,1) \\ \vdots \\ (m,k-i) \\ (m,k-i+1) \\ \vdots \\ (m,k-1) \\ (m,k) \end{array} \begin{array}{c} \begin{array}{cc} (o,o) & (o,k) \end{array} \\ \left[\begin{array}{cc} {}_{n+1}a_{m+1} & {}_n\hat{a}_{m+1} \\ {}_{n+1}a_{m+1} & {}_n\hat{a}_{m+1} \\ \vdots & \vdots \\ {}_{n+2}a_{m+1} & {}_{n+1}\hat{a}_{m+1} \\ {}_{n+2}a_{m+1} & {}_{n+1}\hat{a}_{m+1} \\ \vdots & \vdots \\ {}_{n+2}a_{m+1} & {}_{n+1}\hat{a}_{m+1} \\ {}_{n+2}a_{m+1} & {}_{n+1}\hat{a}_{m+1} \end{array}\right] \end{array} \tag{7}$$

The element ${}_{n+1}a_{m+1}$, which is the probability that m+1 units are served and n+1 rest times are over during an inter-arrival time, is given by

$$ {}_{n+1}a_{m+1} = \int_0^\infty g(x) \int_0^x \gamma_{n+1,\alpha}(u)\, \Gamma_{m+1,\mu}(x-u)\,du\,dx \qquad (8)$$

for n,m = o,1,2, ..., where $\Gamma_{a,b}(\cdot)$ is the gamma distribution with parameters a and b and $\gamma_{a,b}(\cdot)$ the corresponding density. The probability that during an interarrival time m+1 service completion and n rest times are over and the server is in rest, is given by

$$ {}_{n}\hat{a}_{m+1} = \int_0^\infty g(x) \int_0^x (\gamma_{n,\alpha} * \gamma_{m+1,\mu})(u).e^{-\alpha(x-u)}du\,dx \qquad (9)$$

for $n > o$, $m \geq o$, and

$$ \hat{a}_{m+1} = \int_0^\infty g(x) \int_0^x \gamma_{m+1,\mu}(u).e^{-\alpha(x-u)}du\,dx \qquad (10)$$

for $m \geq o$.

Similarly, we can write, for $m,n \geq o$,

$$ {}_{n+2}a_{m+1} = \int_0^\infty g(x) \int_0^x \gamma_{n+2,\alpha}(u)\, \Gamma_{m+1,\mu}(x-u)\,du\,dx, \qquad (11)$$

$${}_{n+1}\hat{a}_{m+1} = \int_0^\infty g(x) \int_0^x (\gamma_{n+1,\alpha} * \gamma_{m+1,\mu})(u) e^{-\alpha(x-u)} du\, dx$$

for $m,n \geq 0$.

For defining the matrices A_m, for $m \geq 0$, we use the same notations as that for B_m. The matrix A_m is given by

$$A_m = \begin{array}{c} \\ (.,0) \\ (.,1) \\ \vdots \\ (.,k-i) \\ (.,k-i+1) \\ \vdots \\ (.,k) \end{array} \begin{array}{c} \begin{array}{ccccc} (.,0) & \ldots & (.,i) & \ldots & (.,k) \end{array} \\ \left[\begin{array}{cccccc} & & {}_n a_m & & & \\ & & & {}_n a_m & & \\ & & & & \ddots & \\ {}_{n+1}a_m & & & & & {}_n\hat{a}_m \\ & {}_{n+1}a_m & & & & \\ & & \ddots & & & \\ & & & {}_{n+1}a_m & & \end{array} \right] \end{array} \quad (12)$$

The element ${}_n a_m$ is the probability that during an interarrival time n rest times and m service completions are over and is given by

$$ {}_{n}a_{m} = \int_{0}^{\infty} g(x) \int_{0}^{x} \gamma_{n,\alpha}(u) . \overline{\Gamma}_{m,\mu}(x-u)\,du\,dx, \text{ for } n,m > o $$

and

$$ a_{m} = \int_{0}^{\infty} g(x)\, \gamma_{m,\mu}(x)\,dx, \quad \text{for } m > o . $$

Similarly

$$ {}_{n}\hat{a}_{m} = \int_{0}^{\infty} g(x) \int_{0}^{x} (\gamma_{n,\alpha} * \gamma_{m,\mu})(u).e^{-\alpha(x-u)}du\,dx, $$

for $n,m > o$, and

$$ \hat{a}_{m} = \int_{0}^{\infty} g(x) \int_{0}^{x} \gamma_{m,\mu}(u).e^{-\alpha(x-u)}du\,dx, \text{ for } m > o, \text{ and} $$

$$ {}_{n+1}a_{m} = \int_{0}^{\infty} g(x) \int_{0}^{x} \gamma_{n+1,\alpha}(u)\, \overline{\Gamma}_{m,\mu}(x-u)\,du\,dx, \text{ for } n \geq o,\ m > o $$

2. STATIONARY DISTRIBUTION

The transition probability matrix (1) can be studied using the method developed by Neuts [41]. The matrices A_i, $i \geq o$ and B_n satisfy the relations

$$\sum_{i=o}^{n} A_i e + B_n e = e, \text{ for } n \geq 1 \tag{13}$$

and $B_o e + C_o e = e$,

where $e = (1,1, \ldots, 1)^T$. Define the matrix A as $A = \sum_{n=o}^{\infty} A_n$.

We can show that A is an irreducible matrix and is stochastic. A is irreducible since there is a path from any state to any other state, either direct or through some other states. There exists an invariant probability vector $\underline{x}$ satisfying $\underline{x} \geq o$ and

$$\underline{x} P = \underline{x} \text{ and } \underline{x} e = 1 . \tag{14}$$

The vector $\underline{x}$ may be partitioned into a 2-vector $\underline{x}_o$ and (k+1) component vectors $\underline{x}_k$, for $k \geq 1$. Then the equation (14) may be written as

$$\begin{aligned} \underline{x}_n &= \sum_{i=o}^{\infty} \underline{x}_{n+i-1} A_i, \text{ for } n \geq 2 \\ \underline{x}_1 &= \underline{x}_o C_o + \sum_{i=1}^{\infty} \underline{x}_i A_i \\ \underline{x}_o &= \sum_{i=o}^{\infty} \underline{x}_i B_i \\ \sum_{i=o}^{\infty} \underline{x}_i e &= 1. \end{aligned} \tag{15}$$

Using the coefficient matrices $A_i, i \geq o$, we define the formal series

$$A[X] = \sum_{n=o}^{\infty} X^n A_n,$$

where X is a non-negative matrix of order k+1. Define the sequence of matrices $\{R(n), n \geq o\}$ by $R(o) = o$, $R(n+1) = A[R(n)]$, for $n \geq o$. Then we have

Theorem. If A is irreducible, the sequence $\{R(n)\}$ converges to a matrix $R > o$, which satisfies

$$sp(R) \leq 1, \; R = A[R], \; \pi R \leq \pi$$

where π is the invariant probability vector of the matrix A.

For proving this theorem see theorem 1 of [41].

Using the theorem given in section 3 of chapter I we can examine the existence of a solution to equations (15) of the form

$$\underline{x}_i = \underline{x}_1 R^{i-1}, \text{ for } i \geq 1. \qquad (16)$$

The matrix R is called the rate matrix. The probabilistic significance of the rate matrix is discussed in the same section. Define B[R] as

$$B[R] = \begin{bmatrix} B_0 & C_0 \\ \sum_{k=1}^{\infty} R^{k-1} B_k & \sum_{k=1}^{\infty} R^{k-1} A_k \end{bmatrix}$$

The second and third equations in (15) yield, $(\underline{x}_0, \underline{x}_1)$ as the left eigen vectors of B[R].

3. WAITING TIME DISTRIBUTION

In this section, we investigate the waiting time distribution of a unit in the system, in the stationary case. If the arriving unit finds the system in (i-1,j), the waiting time of that unit in the system is obtained as follows. We have two cases, (a) $i > k$ and (b) $i \leq k$.

Case (a). $i > k$.

If an arriving unit finds i-1 units in the system and the number of service completion is zero after the previous rest, then the probability that the waiting time of the new arriving unit in the system is less than or equal to t is

$$\sum_{i=k+1}^{\infty} \pi_{i-1,o} \int_0^t \gamma_{i,\mu}(u)\, \Gamma_{[\frac{i-1}{k}],\alpha}(t-u)\,du, \tag{17}$$

where $\pi_{i-1,o}$ is the stationary probability that there are i-1 units in the system and the number of units served after previous rest is zero. If the number of units j, served after the previous rest is greater than zero, then instead of equation (17) we get the required probability as

$$\sum_{i=k+1}^{\infty} \sum_{j=1}^{k-1} \pi_{i-1,j} \int_0^t \gamma_{i,\mu}(u)\, \Gamma_{[\frac{i-k+j-1}{k}]+1,\alpha}(t-u)\,du . \tag{18}$$

Suppose an arriving unit finds, the server in rest and the number of units ahead of him is i-1. Then including the new arriving unit there are i units in the system. Thus the probability that the waiting time of the i^{th} unit in the system is less than or equal to t is

$$\sum_{i=k+1}^{\infty} \pi_{i-1,k} \int_0^t \gamma_{i,\mu}(u)\, \Gamma_{[\frac{i-1}{k}]+1,\alpha}(t-u)\,du \tag{19}$$

Case (b). $i \leq k$.

If an arriving unit finds the number of units in the system to be i-1 and the number of units served after the previous rest completion is zero, then the probability that the waiting time of the new arriving unit is less than or equal to t is

$$\sum_{i=1}^{k} \pi_{i-1,o} \; \Gamma_{i,\mu}(t) . \tag{20}$$

That is, the waiting time is just the service time of i units. If the number of units served after the previous rest is $j > o$, then we get the waiting time distribution of this unit in the system as

$$\sum_{i=1}^{k} \sum_{j=1}^{k-1} \pi_{i-1,j} \int_{o}^{t} \alpha e^{-\alpha u} \; \Gamma_{i,\mu}(t-u)du, \text{ for } i-1+j \geq k \tag{21}$$

and $$\sum_{i=1}^{k} \sum_{j=1}^{k-1} \pi_{i-1,j} \; \Gamma_{i,\mu}(t), \text{ for } i-1+j < k . \tag{22}$$

When the number of units in the system is i-1, which is less than k, and the server is in rest at the time of this arrival, then the probability that the waiting time of the new arriving unit (which is the i^{th} unit in the system on its arrival) in the system is less than or equal to t is

$$\sum_{i=1}^{k} \pi_{i-1,k} \int_0^t \alpha e^{-\alpha u} \Gamma_{i,\mu}(t-u)du \ . \tag{23}$$

Consider the case when the system is empty and the server is idle. The waiting time of the unit which encounters such a system has the distribution

$$\pi_{o,o} \, (1-e^{-\mu t}) \ . \tag{24}$$

When the system is empty and the server is in rest, we have the distribution of the waiting time given by

$$\pi_{o,k} \int_0^t \alpha e^{-\alpha u} \, \mu e^{-\mu(t-u)} du \ . \tag{25}$$

Thus the probability that the waiting time of a unit in the system is less than or equal to t is. by total probability law, the sum of equations (17) to (25).

4. M/M/1 MODEL WITH REST PERIODS

For many models it is advantageous to set up the problem in terms of a Markov process, whenever possible. Here we assume that the arrival process is a Poisson process of rate λ and service rates depend on the number of units served. The server takes rest for a random duration when the system is empty or after the service of every consecutive k units, whichever occurs first. Further we assume that the server returns to the system after a rest period and if he finds the system empty then he remains idle. This model can be studied by a continuous time Markov chain on the state space as defined in section 1.

The infinitesimal generator Q of the Markov chain is

$$Q = \begin{bmatrix} D & E & 0 & 0 & \cdots \\ F & A_0 & A_1 & 0 & \cdots \\ 0 & A_2 & A_0 & A_1 & \ddots \\ \vdots & & \ddots & \ddots & \ddots \end{bmatrix}$$

where

$$D = \begin{bmatrix} -\lambda-\alpha & \alpha \\ 0 & -\lambda \end{bmatrix}$$

$$E = \begin{bmatrix} 0 & 0 & \cdots & 0 & \lambda \\ \lambda & 0 & \cdots & 0 & 0 \end{bmatrix}$$

is a matrix of order 2 x (k+1) and

$$F = \begin{bmatrix} \mu_o & 0 \\ \mu_1 & 0 \\ \vdots & \vdots \\ \mu_{k-1} & 0 \\ 0 & 0 \end{bmatrix}$$

is a (k+1) x 2 matrix. The matrix A_o of order (k+1) is given by

$$A_o = (a_{ij}),\ 1 \leqslant i,j \leqslant k+1,$$

where

$$a_{ii} = -\lambda - \mu, \text{ for } 1 \leqslant i \leqslant k$$

$$a_{k+1,k+1} = -\lambda - \alpha$$

and

$$a_{k+1,1} = \alpha \ .$$

The matrix A_2 of order (k+1) is given by

$$A_2 = (a_{ij}),\ 1 \leqslant i,j \leqslant k+1,$$

where

$$a_{i,i+1} = \begin{cases} \mu_{i-1}, & \text{for } 1 \leqslant i \leqslant k \\ 0 & \text{otherwise} \end{cases}$$

and $A_1 = \lambda I$, I is an identity matrix of order (k+1).

Let $\underline{x}$ be the vector of steady state probabilities associated with Q such that

$$\underline{x}\, Q \;=\; \underline{0} \text{ and } \underline{x}\, e = 1 . \tag{26}$$

Partitioning the vector $\underline{x}$ into vectors $\underline{x}_o$, $\underline{x}_1$, $\underline{x}_2$, ...,
where $\underline{x}_i$ is a (k+1) component row vector, for $i \geq 1$, and $\underline{x}_o = (x_{oo}, x_{ok})$, the equation (26) may be written as

$$\begin{aligned} &\underline{x}_o D + \underline{x}_1 F = \underline{0} \\ &\underline{x}_o E + \underline{x}_1 A_o + \underline{x}_2 A_2 = \underline{0} \\ &\underline{x}_i A_1 + \underline{x}_{i+1} A_o + \underline{x}_{i+2} A_2 = \underline{0} . \end{aligned} \tag{27}$$

Following [45] we examine the existence of a solution of the form

$$\underline{x}_i \;=\; \underline{x}_1\, R^{i-1}, \text{ for } i \geq 1$$

where R is the unique solution to the set of non-negative matrices of order (k+1), which have a spectral radius less than one, of the equation

$$A_1 + RA_o + R^2 A_2 \;=\; 0 . \tag{28}$$

The first equation of (27) may be written as

$$(\underline{x}_o, \underline{x}_1) \begin{bmatrix} D & E \\ F & A_o+RA_2 \end{bmatrix} = 0 .$$

Thus $(\underline{x}_o, \underline{x}_1)$ is the left eigen vector of

$$\begin{bmatrix} D & E \\ F & A_o+RA_2 \end{bmatrix}$$

corresponding to the eigen value zero.

Define $A = A_o+A_1+A_2$. This is a generator of the finite state continuous time Markov chain. Let $\underline{\pi} = (\pi_o, \pi_1, \ldots, \pi_k)$ be the vector of steady state probabilities associated with A such that

$$\underline{\pi}\, A = \underline{0} \text{ and } \underline{\pi}\, e = 1.$$

Putting $\pi_k = c$ in the set of equations obtained from $\underline{\pi}\, A = \underline{0}$, we get

$$\pi_i = \frac{\alpha}{\mu_i} c, \quad o \leq i \leq k-1 .$$

Using the total probability law, $\sum_{i=0}^{k} \pi_i = 1$, we get

$$c = [1 + \sum_{i=0}^{k-1} \frac{\alpha}{\mu_i}]^{-1}.$$

Thus $\pi_i = (\frac{\alpha}{\mu_i})[1 + \sum_{i=0}^{k-1} \frac{\alpha}{\mu_i}]^{-1}$, for $0 \leq i \leq k-1$

and $\pi_k = [1 + \sum_{i=0}^{k-1} \frac{\alpha}{\mu_i}]^{-1}$.

The queue is stable, provided $\underline{\pi} A_2 e > \lambda$. Since $\underline{\pi} A_2 e = (0, \mu_0 \pi_0, \mu_1 \pi_1, \ldots, \mu_{k-1} \pi_{k-1})\, e$, the requirement $\underline{\pi} A_2 e > \lambda$ reduces to $k\alpha c > \lambda$. Thus we get the required steady state condition in terms of the parameters of the process as

$$k\alpha > \lambda [1 + \sum_{i=0}^{k-1} \frac{\alpha}{\mu_i}]. \tag{30}$$

5. AN OPTIMIZATION PROBLEM

We consider an optimization problem associated with the model discussed in the above section. As an example of the model, consider a device which processes a certain type of job. The efficiency of the device is depending on the amount of service given by it. Usually the efficiency decreases as the amount of service given by it increases. So, after certain time the device needs some 'servicing' after which it is as good as new. This 'servicing' time we consider is same as a rest time to the server in queues.

Assume that no queue is allowed. Further, assume that the service rate μ_j decreases as j, the number of units served increases. Let c_1 be the loss due to one customer being lost to the system, c_2 be the service cost per unit time, c_3 the loss per unit time due to the server remaining idle and cost during the rest time be a fixed amount K. Then the question is, what is the optimal value of j? In other words, how many units can be served by a server continuously before proceeding for rest with minimum loss? We also assume that after the rest the server is as good as a fresh one. For convenience, assume $\mu_j = \mu/j$, for $j \geq 1$, $\mu_o = o$. Then the cost of serving n units is

$$c_2 \sum_{i=1}^{n} \frac{i}{\mu} .$$

Total loss by the time n services are completed is

$$c_1 \lambda \sum_{i=1}^{n} \frac{i}{\mu} + c_3 \frac{n}{\lambda} .$$

The average cost of serving one unit is

$$F(n) = [K + c_1 \lambda \sum_{i=1}^{n} \frac{i}{\mu} + c_2 \sum_{i=1}^{n} \frac{i}{\mu} + c_3 \frac{n}{\lambda}]/n . \qquad (31)$$

n^*, the optimum value of n, is obtained by minimizing F(n) with respect to n over the set of positive integers. n^* must satisfy the following conditions

$$F(n^*) - F(n^*+1) \leq 0$$
$$F(n^*) - F(n^*-1) \leq 0 .$$

Thus,

$$[K + c_1 \lambda \sum_{i=1}^{n^*-1} \frac{i}{\mu} + c_2 \sum_{i=1}^{n^*-1} \frac{i}{\mu} + c_3 \frac{(n^*-1)}{\lambda}]/(n^*-1) \geq$$

$$[K + c_1 \lambda \sum_{i=1}^{n^*} \frac{i}{\mu} + c_2 \sum_{i=1}^{n^*} \frac{i}{\mu} + c_3 \frac{n^*}{\lambda}]/n^* \leq$$

$$[K + c_1 \lambda \sum_{i=1}^{n^*+1} \frac{i}{\mu} + c_2 \sum_{i=1}^{n^*+1} \frac{i}{\mu} + c_3 \frac{(n^*+1)}{\lambda}]/(n^*+1) \qquad (32)$$

The left side inequality of the expression (32) gives

$$\frac{2K\mu}{c_1\lambda+c_2} \geqslant n^*(n^*-1) \tag{33}$$

and the right side gives

$$\frac{2K\mu}{c_1\lambda+c_2} \leqslant n^*(n^*+1). \tag{34}$$

From (33) and (34) we can compute n^*, the optimal number of units which can be served by a server continuously before taking rest.

6. NUMERICAL EXAMPLES

For convenience, here we assume that $\mu_i=\mu$ for every i, in the model discussed in section 4. Now we have a queueing system with constant arrival rate λ, constant service rate μ and the rest times are exponentially distributed with parameter α. The steady state condition (30) then reduces to $\mu(1-\frac{\mu}{\mu+k\alpha})>\lambda$. For given values of λ,μ,α and k which satisfy the above condition, we can compute the rate matrix R by successive substitution.

From (28) we have $R = -A_1A_o^{-1} - R^2A_2A_o^{-1}$. We obtain R by successive substitution, starting with R=0, which is continued until the maximum entrywise difference between successive iteratives is less than 10^{-6}.

For $\lambda=1$, $\mu=7$, $\alpha=1$, $k=4$, the rate matrix R is

$$\begin{bmatrix} .125140 & .013784 & .003066 & .000866 & .001124 \\ .000456 & .125354 & .013907 & .003141 & .003645 \\ .001652 & .001228 & .125801 & .014176 & .013215 \\ .007456 & .005139 & .003248 & .127014 & .059651 \\ .066807 & .038699 & .023236 & .014114 & .534457 \end{bmatrix}$$

When α takes values in the order 3, 5 and 7, the entries of R are decreasing and when $\mu = \alpha$ the diagonal entries of R are approximately same upto 4 digits.

When $\lambda = 1$, $\mu = 7$, $\alpha = 1$, $k = 6$, R is

$$\begin{bmatrix} .125015 & .013684 & .002999 & .000823 & .000253 & .000084 & .000117 \\ .000426 & .125034 & .013694 & .003005 & .000826 & .000255 & .000341 \\ .000130 & .000100 & .125006 & .013713 & .003016 & .000833 & .001037 \\ .000423 & .000318 & .000205 & .125126 & .013748 & .003036 & .003380 \\ .001541 & .001120 & .000700 & .000426 & .125256 & .013524 & .012325 \\ .007014 & .004670 & .002850 & .001706 & .001014 & .125602 & .056114 \\ .063729 & .035444 & .020481 & .011987 & .007053 & .004161 & .509832 \end{bmatrix}$$

We also have computed the rate matrix R for k=9 which being a 10 x 10 matrix, is not explicitly given here. From these three cases, k = 4,6,9, we conclude that when k, the number of units served increases, the entries of R decreases.

CHAPTER III

BULK SERVICE QUEUE WITH REST PERIODS

The queueing model with batch service has an extensive literature, dating back to 1954 with the work of Bailey [5]. Fabens [21] and Tackacs [56] considered a batch service with fixed batch size k. Neuts [38] considered a more general case, namely, the batch takes a minimum of 'a' units and a maximum of 'b' units. If immediately after the completion of a service, the server finds less than 'a' units present, he waits until there are 'a' units, whereupon he takes the batch of 'a' units for service. An overview of the effect of group arrivals and/or services on the structure of various classes of queues is given in Neuts [42].

Here we consider a bulk service model with batch size (a,b). Assume that the arrivals are according to a Poisson process of rate λ, and service times of batches depend on the batch size. The service times are exponentially distributed with parameter μ_i, $a \leq i \leq b$. After the service of a batch, if the server finds the number of units waiting is less than 'a', he goes for rest for a random duration which is exponentially distributed with parameter α. Using the matrix geometric method, we investigate the stationary behaviour of the system. The waiting time of a unit in the system is also studied.

1. DESCRIPTION OF THE MODEL

Consider a $M/M^{a,b}/1$ model with unlimited capacity. Assume that the arrivals occur simply in accordance with a Poisson process of rate λ. The units are served in batches according to a bulk service rule such that the service starts only when a minimum number of units, say, 'a' is present in the queue. The maximum size of batch is 'b' and the service time of each batch depends on the size of that batch. The service times are independent exponentially distributed with parameters μ_i, $a \leq i \leq b$. If, immediately after the service of a batch, the server finds less than 'a' units present, he goes for rest for a random duration which is exponentially distributed with parameter α. Further, if the number of units in the queue is less than 'a' upon the server's return, he remains idle.

As an example, consider a taxi car giving service between two cities. The driver starts the trip if he gets at least 'a' customers. If there are more than 'b' customers the driver takes only 'b' customers in the car in a trip. After the trip, if he finds that the number of customers waiting for taxi is less than 'a', then he takes rest for a random duration. After rest, the driver returns and if the number of customers waiting for taxi is still less than 'a' he remains idle until atleast 'a' customers are available.

The state space of the model can be represented as $S = \{(i,1)$, for $i \geq o$, $(i,2)$ for $o \leq i \leq a-1$, (i,j), for $i \geq a$, $a \leq j \leq b\}$. The state $(i,1)$, for $i \geq o$, denotes that the number of units in the system is i and the server is in rest. The state $(i,2)$, for $o \leq i \leq a-1$, denotes that the number of units in the system is i and the server is idle. The state (i,j) for $i \geq a$, $a \leq j \leq b$, denotes the number of units in the system is i, the size of the batch which is being served is j.

The states $\underline{i}$, for $o \leq i \leq a-1$, have the states $(i,1)$ and $(i,2)$, where $(i,1)$ and $(i,2)$ are explained as above. The states $\underline{i}$, for $i \geq a$, have the states $(i,1)$ and (i,j), $a \leq j \leq b$ and j takes the value of $\min(i,j)$. For example $\underline{a}$ denotes $(a,1)$ and (a,a), $\underline{a+1}$ denotes $(a+1,1)$, $(a+1,a)$ and $(a+1,a+1)$ and so on.

2. THE MATRIX GEOMETRIC SOLUTION

The process which has a state space S can be studied by the Matrix Geometric Method. For the sake of convenience in writing the Q matrix, we assume that b-1-a, b-a and b+1-a are less than a. Using the block partitioned method, the infinitesimal generator Q of the Markov chain is

		0	1	2	...	b-1-a	b-a	...	a-1	a	a+1	...	b-1	b	b+1	...
	0	M_1	M_2	0	...											
	1	0	M_1	M_2												
	2	0	0	M_1												
	⋮	⋮	⋮	⋮												
	a-1	0	0	0					M_1	M_2						
	a	M_1^a	0	0					0	M_2^a	M_3^a	...				
	a+1	M_1^{a+1}	M_2^{a+1}	0					0	0	M_3^{a+1}	...				
	⋮	⋮	⋮	⋮												
Q =	b-1	M_1^{b-1}	M_2^{b-1}	M_3^{b-1}		M_{b-a}^{b-1}	0	...					M_{b-a+1}^{b-1}	M_{b-a+2}^{b-1}	0	...
	b	M_1^b	M_2^b	M_3^b			M_{b-a+1}^b	...					0			
	b+1	0	M_1^{b+1}	M_2^{b+1}				...					0			
	b+a	0	0	0					0	M_1^{b+a}	M_2^{b+a}	...	M_{b-a}^{b+a}			
	b+a+1	0	0	0						0	M_1^{b+a+1}	...	M_{b-a-1}^{b+a+1}	C		
	⋮	⋮	⋮	⋮												
	2b-1	0	0	0							0	...	M_1^{2b-1}			
	2b	0	0	0									0			
	⋮	⋮	⋮	⋮									⋮			

(1)

where $\mathcal{C}$ is given by

$$
\mathcal{C} = \begin{array}{c|ccccccccccc|}
 & \underline{b} & \underline{b+1} & \cdots & \underline{b+a} & \underline{b+a+1} & \cdots & \underline{2b-1} & \underline{2b} & \underline{2b+1} & \cdots \\
\hline
\underline{b} & M_{b-a+2} & M_{b-a+3} & & 0 & 0 & \cdots & & & & \\
\underline{b+1} & 0 & M_{b-a+2} & \cdots & & & & & & & \\
\vdots & \vdots & \vdots & & & & & & & & \\
\underline{b+a} & M_{b-a+1} & 0 & \cdots & M_{b-a+2} & M_{b-a+3} & \cdots & & & & \\
\underline{b+a+1} & M_{b-a} & M_{b-a+1} & \cdots & 0 & M_{b-a+2} & \cdots & & & & \\
\vdots & \vdots & \vdots & & & & & & & & \\
\underline{2b-1} & M_2 & M_3 & \cdots & & & \cdots & M_{b-a+2} & M_{b-a+3} & 0 & \cdots \\
\underline{2b} & M_1 & M_2 & \cdots & & & \cdots & 0 & M_{b-a+2} & M_{b-a+3} & \cdots \\
\underline{2b+1} & 0 & M_1 & \cdots & & & \cdots & 0 & 0 & M_{b-a+2} & \cdots \\
\vdots & \vdots & \vdots & & & & & & & & \\
\end{array}
$$

The matrices are

$$M_1 = \begin{bmatrix} -\lambda-\alpha & \alpha \\ 0 & -\lambda \end{bmatrix}$$

$$M_2 = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

$$M_1^a = \begin{bmatrix} 0 & 0 \\ \mu_a & 0 \end{bmatrix}$$

$$M_2^a = \begin{bmatrix} -\lambda-\alpha & \alpha \\ 0 & -\lambda-\mu_a \end{bmatrix}$$

$$M_3^a = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \end{bmatrix}$$

Then the matrices for $\underline{b-1}^{th}$ row are defined as follows.

M_1^{b-1} is a $(b-a+1) \times 2$ matrix $(m_{i,j})$, $1 \leq i \leq b-a+1$, $j=1,2$.

$$M_1^{b-1} = (m_{i,j})$$

where $m_{b-a+1,1} = \mu_{b-1}$

$m_{i,j} = 0$, for all other values of i and j.

$$M_2^{b-1} = (m_{ij}), 1 \leq i \leq b-a+1, \ j=1,2$$

where $m_{b-a,1} = \mu_{b-2}$

$m_{i,j} = 0,$ otherwise.

$$M_3^{b-1} = (m_{i,j}), 1 \leq i \leq b-a+1, \ j=1,2,$$

where $m_{b-a-1,1} = \mu_{b-3}$

$m_{i,j} = 0,$ otherwise.

$$M_{b-a}^{b-1} = (m_{i,j}), \ 1 \leq i \leq b-a+1, \ 1 \leq j \leq b-2a+1$$

for $b-2a+1 \geq a$

where $m_{2,b-2a+1} = \mu_a$

$m_{i,j} = 0,$ otherwise.

For $b-2a+1 < a$

$$M_{b-a}^{b-1} = (m_{i,j}), 1 \leq i \leq b-a+1, \ j=1,2$$

where $m_{2,1} = \mu_a$

$m_{i,j} = 0,$ otherwise.

$$M^{b-1}_{b-a+1} = (m_{i,j}),\ 1 \le i \le b-a+1,\ 1 \le j \le b-a+1$$

where $m_{i,i} = -\lambda - \mu_{a+i-2},\ 2 \le i \le b-a+1$

$m_{1,1} = -\lambda - \alpha$

$m_{1,b-a+1} = \alpha$

$$M^{b-1}_{b-a+2} = (m_{i,j}),\ 1 \le i \le b-a+1,\ 1 \le j \le b-a+2$$

where $m_{i,i} = \lambda\ ,1 \le i \le b-a+1$

$m_{b-a+1,b-a+2} = 0$

$m_{i,j} = 0$, otherwise.

Next we have, for the $\underline{b}^{th}$ row, the matrices as follows:

$$M^{b}_{1} = (m_{i,j}),\ 1 \le i \le b-a+2,\ j=1,2.$$

with $m_{b-a+2,1} = \mu_b$

$m_{i,j} = 0$, otherwise.

$$M^{b}_{2} = (m_{i,j}),\ 1 \le i \le b-a+2,\ j=1,2,$$

with $m_{b-a+1,1} = \mu_{b-1}$

$m_{i,j} = 0$, otherwise.

$$M_3^b = (m_{i,j}), 1 \leq i \leq b-a+2, \quad j=1,2$$

with $m_{b-a,1} = \mu_{b-2}$

$m_{i,j} = 0$, otherwise.

$$M_{b-a+1}^b = (m_{ij}), \quad 1 \leq i \leq b-a+2, \quad 1 \leq j \leq b-2a+2$$

for $b-2a+2 \geq a$.

with $m_{2,b-2a+2} = \mu_a$

$m_{ij} = 0$, otherwise.

For $b-2a+2 < a$

$$M_{b-a+1}^b = (m_{i,j}), 1 \leq i \leq b-a+2, \quad j=1,2$$

with $m_{2,1} = \mu_a$

$m_{i,j} = 0$, otherwise

$$M_{b-a+2} = (m_{i,j}), 1 \leq i, \quad j \leq b-a+2$$

with $m_{1,1} = -\lambda - \alpha$

$m_{i,i} = -\lambda - \mu_{a+i-2}, \quad 2 \leq i \leq b-a+2$

$m_{1,b-a+2} = \alpha$

$M_{b-a+3} = \lambda I$, I is the identity matrix of order $(b-a+2)$

For the b+a[th] row, we have the matrices

$$M_1^{b+a} = (m_{ij}), \quad 1 \leq i \leq b-a+2, \quad j=1,2$$

where $m_{b-a+2,2} = \mu_b$

$m_{i,j} = 0$, otherwise.

$$M_2^{b+a} = (m_{i,j}), \quad 1 \leq i \leq b-a+2, \quad j=1,2,3$$

where $m_{b-a+1,3} = \mu_{b-1}$

$m_{i,j} = 0$, otherwise.

$$M_{b-a}^{b+a} = (m_{i,j}), \quad 1 \leq i \leq b-a+2, \quad 1 \quad j \quad b-a+1$$

where $m_{3,b-a+1} = \mu_{a+1}$

$m_{i,j} = 0$, otherwise.

$$M'_{b-a+1} = (m_{i,j}) \quad 1 \leq i, \; j \lesssim b-a+2$$

where $m_{2,b-a+2} = \mu_a$

$m_{i,j} = 0$, otherwise.

We have the matrices as

$$M_1^{b+a+1} = (m_{i,j}), \quad 1 \leq i \leq b-a+2, \quad j=1,2,3.$$

where $m_{b-a+2,3} = \mu_b$

$m_{ij} = 0$, otherwise.

$$M_{b-a-1}^{b+a+1} = (m_{i,j}) \quad 1 \leq i \leq b-a+2, \quad 1 \leq j \leq b-a+1$$

where $m_{4,b-a+1} = \mu_{a+2}$

$m_{i,j} = 0$, otherwise.

$$M'_{b-a} = (m_{i,j}), \quad 1 \leq i, \; j \leq b-a+2$$

where $m_{3,b-a+2} = \mu_{a+1}$

$m_{i,j} = 0$, otherwise.

$$M_1^{2b-1} = (m_{i,j}), \quad 1 \leq i \leq b-a+2, \quad 1 \leq j \leq b-a+1$$

where $m_{b-a+2,b-a+1} = \mu_b$

$m_{i,j} = 0$, otherwise.

$$M'_1 = (m_{i,j}), \quad 1 \leq i, \; j \leq b-a+2$$

where $m_{b-a+2,b-a+2} = \mu_b$

$m_{i,j} = 0$, otherwise.

$$M'_2 = (m_{i,j}), \quad 1 \leq i, \; j \leq b-a+2$$

where $m_{b-a+1,b-a+2} = \mu_{b-1}$

$m_{i,j} = 0$, otherwise.

$$M_3' = (m_{i,j}),\ 1 \leq i,\ j \leq b-a+2$$

$$m_{b-a,b-a+2} = \mu_{b-2}$$

$$m_{i,j} = 0, \quad \text{otherwise.}$$

Let $\underline{x}$ be the invariant probability vector associated with the infinitesimal generator Q such that

$$\underline{x}\, Q = \underline{0} \text{ and } \underline{x}\ e = 1 \tag{2}$$

where $e = (1,1, \ldots 1)^T$.

Suppose the vector $\underline{x}$ is partitioned as

$$\underline{x} = (\underline{x}_0, \underline{x}_1, \ldots, \underline{x}_{a-1}, \underline{x}_a, \ldots, \underline{x}_b, \ldots) \tag{3}$$

where

$\underline{x}_i$, for $0 \leq i \leq a-1$ are 1×2 vectors

$\underline{x}_i$, for $a \leq i \leq b-1$ are $1 \times (i-a+2)$ vectors

and $\underline{x}_i$, for $i \geq b$ are $1 \times (b-a+2)$ vectors.

Following Neuts [45], we examine the existence of a solution of the following form generated by a non-negative matrix R with spectral radius less than one.

$$\underline{x}_i = \underline{x}_b\, R^{i-b}, \text{ for } i \geq b \tag{4}$$

To get a solution of (4), we find from (2) that

$$\underline{x}_b\left(M_{b-a+3} + R\, M_{b-a+2} + \sum_{i=1}^{b+1-a} R^{a+i}\, M'_{b-a+2-i}\right) = \underset{\sim}{0}. \tag{5}$$

From (5) we get

$$R = -M_{b-a+3}\, M_{b-a+2}^{-1} - \sum_{i=1}^{b+1-a} R^{a+i}\, M'_{b-a+2-i}\, M_{b-a+2}^{-1}\,. \tag{6}$$

Consider $M = M_{b-a+3} + M_{b-a+2} + \sum_{i=1}^{b+1-a} M'_{b-a+2-i}.$

This is reducible. The analysis presented in Neuts [45] is not applicable here. In general (see [41] and [34]), it can be seen that the matrix R is numerically computed using the recurrence relation starting with R=0. Although (6) has a complicated structure, the elements of R can be calculated by solving certain algebraic equations. Comparing the elements on the diagonals of both sides of (6), we obtain

$$R_{11} = \frac{\lambda}{\lambda+\alpha} \tag{7}$$

$$R_{ii} = \frac{\lambda}{\lambda+\mu_{a+i-2}}\,, \text{ for } 2 \le i \le b-a+1 \tag{8}$$

$$\text{and } R_{b-a+2,b-a+2} = \frac{\lambda}{\lambda+\mu_b} + \frac{\mu_b}{\lambda+\mu_b} R^{b+1}_{b-a+2,b-a+2} \tag{9}$$

Since $\lambda, \alpha, \mu_{a+i-2}$, for $2 \leq i \leq b-a+1$, are greater than zero and R_{ii}, for $1 \leq i \leq b-a+1$, are less than one, we can find that the spectral radius of the matrix R is less than one if and only if (9) has a root in (o,1). From (9) we get $R_{b-a+2,b-a+2} < 1$ if and only if

$$\lambda < b\mu_b \,. \tag{10}$$

Comparing the other elements of both sides of (6), we get

$$R_{1,b-a+2} = \frac{\lambda\alpha}{(\lambda+\alpha)(\lambda+\mu_b)} \left[1 - \frac{\mu_b}{\lambda+\mu_b} \sum_{j=o}^{b} R^{j}_{b-a+2,b-a+2} R^{b-j}_{11}\right]^{-1} \tag{11}$$

and

$$R_{i,b-a+2} = \left(\frac{\mu_{a+i-2}}{\lambda+\mu_b}\right) \left(\frac{\lambda}{\lambda+\mu_{a+i-2}}\right)^{a+i-1}$$

$$\left[1 - \frac{\mu_b}{\lambda+\mu_b} \sum_{j=o}^{b} R^{j}_{b-a+2,b-a+2} R^{b-j}_{i\,i}\right]^{-1} \tag{12}$$

for $2 \leq i \leq b-a+1$.

In this model, our task is only to solve (9) so that we can compute all other elements of R. The inequality (10) is the equilibrium condition. The Markov chain is positive recurrent if and only if (10) is satisfied.

From the matrix Q^* and from (2) we find that the vector $\underline{y} = (\underline{x}_o, \underline{x}_1, \ldots, \underline{x}_a, \ldots, \underline{x}_b)$ satisfies $\underline{y}Q^* = \underline{0}$, where Q^* is given by

$$
\overset{*}{Q} = \begin{array}{c} \\ \underline{0} \\ \underline{1} \\ \underline{2} \\ \vdots \\ \underline{a} \\ \underline{a+1} \\ \vdots \\ \underline{b-1} \\ \underline{b} \end{array}
\begin{array}{cccccccccc}
\underline{0} & \underline{1} & \underline{2} & \cdots & \underline{a} & \underline{a+1} & \cdots & \underline{b-1} & & \underline{b} \\
M_1 & M_2 & 0 & & & & & & & \\
0 & M_1 & M_2 & & & & & & & \\
0 & 0 & M_1 & & & & & & & \\
\vdots & \vdots & \vdots & & & & & & & \\
M_1^a & 0 & 0 & & M_2^a & M_3^a & & & & \\
M_1^{a+1} & M_2^{a+1} & 0 & & 0 & M_3^{a+1} & & & & \\
\vdots & \vdots & \vdots & & & & & & & \\
M_2^{b-1} & M_3^{b-1} & & & & & & M_{b-a+1}^{b-1} & & M_{b-a+2}^{b-1} \\
M_1^b & M_2^b+RM_3^{b+1} & M_3^b+RM_2^{b+1}+R^2M_1^{b+2} & & & & & \sum_{i=1}^{b-a+1} R^{b-i} M_i^{2b-i} & & \sum_{i=1}^{b-a+1} M_i' R^{b-i+1} + M_{b-a+2}'
\end{array} \qquad (13)
$$

3. WAITING TIME DISTRIBUTION

The distribution of the waiting time of a unit in the system $M/M^{a,b}/1$ was studied by Medhi [36]. In this section we investigate the waiting time distribution of a unit in the system $M/M^{a,b}/1$ with rest periods. The waiting time of the new arrival depends on the server's state. An arriving unit may find the server in one of the following states

(i) Server is at rest

(ii) Server is working

(iii) Server is idle

(i) Server is at rest

Let $\pi_{i,1}$ be the steady state probability that i units are in the system and the server is at rest. The waiting time of the new arrival depends on the number of units in the system. So we consider three cases which depend on i, the number of units in the queue

(i) $a \leq i \leq b$

(ii) $b+1 \leq i$

(iii) $i < a$

In the first case, namely $a \leq i \leq b$, the probability that the waiting time in the system, of an arriving unit which finds i-1 units ahead of him in the queue, when the server is at rest, is less than or equal to t is

$$\sum_{i \geq a}^{b} \pi_{i,1} \int_{0}^{t} \alpha e^{-\alpha u} [1-e^{-\mu_i(t-u)}]du . \tag{14}$$

When i is greater than or equal to b+1, the waiting time is different for the cases $i-b\,[\frac{i}{b}] \geq a$ and $i-b\,[\frac{i}{b}] < a$. So we have to consider them separately. Thus the probability that the waiting time of a unit, which finds i-1 units ahead of him in the queue, is less than or equal to t is

$$\sum_{\substack{i=b+1 \\ c=i-b[\frac{i}{b}]\geq a}}^{\infty} \pi_{i,1} \int_{0}^{t}\int_{u}^{t} \alpha e^{-\alpha u}\, \gamma_{[\frac{i}{b}],\mu_b}(v)[1-e^{-\mu_c(t-u-v)}]dv\, du +$$

$$\sum_{\substack{i=b+1 \\ c=i-b[\frac{i}{b}]<a}}^{\infty} \pi_{i,1} \int_{0}^{t}\int_{u}^{t} \alpha e^{-\alpha u} \gamma_{[\frac{i}{b}],\mu_b}(v) \left\{ \sum_{m=0}^{b-a} \left(\Gamma_{a-c+m,\lambda}(v) - \right.\right.$$

$$\left. \Gamma_{a-c+m+1,\lambda}(v) \right) (1-e^{-\mu_{a+m}(t-u-v)}) + \sum_{m=b-a+1}^{\infty} \left(\Gamma_{a-c+m,\lambda}(v) - \right.$$

$$\left. \Gamma_{a-c+m+1,\lambda}(v) \right) (1-e^{-\mu_b(t-u-v)})$$

$$+ \sum_{j=o}^{a-c-1} e^{-\lambda v} \frac{(\lambda v)^j}{j!} \left[\int_o^{t-u-v} \gamma_{a-(c+j),\lambda}(w) (1-e^{-\alpha w}) \right.$$

$$(1-e^{-\mu_a (t-u-v-w)}) dw + \int_o^{t-u-v} \sum_{m=o}^{b-a} \left(\Gamma_{a-(c+j)+m,\lambda}(w) - \right.$$

$$\left. \Gamma_{a-(c+j)+m+1,\lambda}(w) \right) \alpha e^{-\alpha w} (1-e^{-\mu_{a+m}(t-u-v-w)}) \, dw$$

$$+ \sum_{m=b-a+1}^{\infty} \left(\Gamma_{a-(c+j)+m,\lambda}(w) - \Gamma_{a-(c+j)+m+1,\lambda}(w) \right)$$

$$\left. \left. \alpha e^{-\alpha w} (1-e^{-\mu_b (t-u-v-w)}) \, dw \right] \right\} dv \, du \qquad (15)$$

where $\gamma_{c,d}(\cdot)$ is the gamma density function and $\Gamma_{c,d}(\cdot)$, the corresponding distribution function with parameters c and d. [i/b] denotes the integer less than or equal to i/b. When i is less than 'a' the probability that the waiting time of a unit, which finds i-1 units ahead of him in the queue is less than or equal to t is

$$\sum_{i<a} \pi_{i,1} \int_0^t \left\{ \alpha e^{-\alpha u} \left[\sum_{m=0}^{b-a} \left(\Gamma_{a-i+m,\lambda}(u) - \Gamma_{a-i+m+1,\lambda}(u) \right) \right. \right.$$

$$(1-e^{-\mu_{a+m}(t-u)}) + \sum_{m=b-a+1}^{\infty} \left(\Gamma_{a-i+m,\lambda}(u) - \Gamma_{a-i+m+1,\lambda}(u) \right)$$

$$\left. \left. (1-e^{-\mu_b(t-u)}) \right] du + \gamma_{a-i,\lambda}(u) \; [1-e^{-\alpha u}](1-e^{-\mu_a(t-u)}) \right\} du \quad (16)$$

(ii) Server is working

Let $\pi_{i,j}$ be the steady state probability that i units are in the system and the server is serving a batch of size j. When $a \leq i \leq b$, the probability that the waiting time of the i^{th} unit in the system is less than or equal to t is

$$\sum_{i \geq a}^{b} \sum_{\substack{j=a \\ c=i-j>a}}^{b} \pi_{i,j} \int_0^t \mu_j e^{-\mu_j u} [1-e^{-\mu_c(t-u)}] \; du$$

$$+ \sum_{i \geq a}^{b} \sum_{\substack{j=a \\ c=0<i-j<a}}^{b} \pi_{i,j} \int_0^t \mu_j e^{-\mu_j u} \left\{ \left[\sum_{m=0}^{b-a} \left(\Gamma_{a-c+m,\lambda}(u) \right. \right. \right.$$

$$- \Gamma_{a-c+m+1,\lambda}(u)\Big) (1-e^{-\mu_{a+m}(t-u)}) + \sum_{m=b-a+1}^{\infty} \Big(\Gamma_{a-c+m,\lambda}(u)$$

$$- \Gamma_{a-c+m+1,\lambda}(u)\Big) \quad (1-e^{-\mu_b(t-u)}) \Big]$$

$$+ \sum_{k=0}^{a-c-1} e^{-\lambda u} \frac{(\lambda u)^k}{k!} \Big[\int_0^{t-u} \gamma_{a-(c+k),\lambda}(v)(1-e^{-\alpha v})(1-e^{-\mu_a(t-u-v)})$$

$$+ \int_0^{t-u} \alpha e^{-\alpha v} \Big[\sum_{m=0}^{b-a} \Big(\Gamma_{a-(c+k)+m,\lambda}(v) - \Gamma_{a-(c+k)+m+1,\lambda}(v) \Big)$$

$$(1-e^{-\mu_{a+m}(t-u-v)}) + \sum_{m=b-a+1}^{\infty} \Big(\Gamma_{a-(c+k)+m,\lambda}(v)$$

$$- \Gamma_{a-(c+k)+m+1,\lambda}(v) \Big) \quad (1-e^{-\mu_b(t-u-v)}) \Big] \, dv \Big] \Big\} \, du. \qquad (17)$$

When $i \geq b+1$, we have to consider the possibilities $i-b[\frac{i}{b}] \geq a$ and $i-b[\frac{i}{b}] < a$ separately. Thus the probability that the waiting time of a unit, which finds $i-1$ units ahead of him, is less than or equal to t is

$$\sum_{i=b+1}^{\infty} \sum_{\substack{j=a \\ c=i-b[\frac{i}{b}]-j \geq a}}^{b} \pi_{i,j} \int_{0}^{t} \mu_j e^{-\mu_j u} \int_{0}^{t-u} \gamma_{[\frac{i-j}{b}],\mu_b}(v)$$

$$(1-e^{-\mu_c(t-u-v)})dv\, du + \sum_{i=b+1}^{\infty} \sum_{\substack{j=a \\ c=i-b[\frac{i-j}{b}]-j<a}}^{b} \pi_{i,j}$$

$$\int_{0}^{t} \mu_j e^{-\mu_j u} \int_{0}^{t-u} \gamma_{[\frac{i-j}{b}],\mu_b}(v) \left\{ \left[\sum_{m=0}^{b-a} \left(\Gamma_{a-c+m,\lambda}(v) - \Gamma_{a-c+m+1,\lambda}(v) \right) \right. \right.$$

$$(1-e^{-\mu_{a+m}(t-u-v)}) + \sum_{m=b-a+1}^{\infty} \left(\Gamma_{a-c+m,\lambda}(v) - \Gamma_{a-c+m+1,\lambda}(v) \right)$$

$$\left. (1-e^{-\mu_b(t-u-v)}) \right] + \sum_{k=0}^{a-c-1} e^{-\lambda v} \frac{(\lambda v)^k}{k!} \left[\int_{0}^{t-u-v} \gamma_{a-(c+k),\lambda}(w) \right.$$

$$(1-e^{-\alpha w})\, (1-e^{-\mu_a(t-u-v-w)})\, dw$$

$$+ \int_{0}^{t-u-v} \sum_{m=0}^{b-a} \left(\Gamma_{a-(c+j)+m,\lambda}(w) - \Gamma_{a-(c+j)+m+1,\lambda}(w) \right)$$

$$\alpha e^{-\alpha w}(1-e^{-\mu_{a+m}(t-u-v-w)})dw + \int_{0}^{t-u-v} \sum_{m=b-a+1}^{\infty} \Big(\Gamma_{a-(c+j)+m,\lambda}(w)$$

$$- \Gamma_{a-(c+j)+m+1,\lambda}(w)\Big) \alpha e^{-\alpha w}(1-e^{-\mu_b(t-u-v-w)})dw\Big]\Big\} \, dv \, du \qquad (18)$$

(iii) Server is idle

After the completion of a rest the server returns to the system and if he finds the number of units in the queue is less than 'a' he remains idle. Let $\pi_{i,2}$, for $i < a$, be the probability that i units are in the system when the server is idle. In this case, the probability that the waiting time of the i^{th} unit is less than or equal to t is

$$\sum_{i<a} \pi_{i,2} \int_{0}^{t} \gamma_{a-i,\lambda}(u) \, (1-e^{-\mu_a(t-u)})du \,. \qquad (19)$$

Thus the probability that the waiting time of a unit in the system, which finds i-1 units ahead of him, is less than or equal to t, is the sum of the quantities given in (14) to (19).

CHAPTER IV

SOME OTHER QUEUEING MODELS WITH REST PERIODS+

In this chapter we study two queueing models with rest periods, one is an Erlangian service with feedback and the other is a buffer model. The utilization of idle time was studied by Levy and Yechiali [33]. Fuhrmann and Cooper [24] studied an M/G/1 system with generalised vacations. Neuts and Ramalhoto [46] and Ali and Neuts [1], showed that the M/G/1 decomposition property holds for the M/G/1 queueing systems without exhaustive service.

Here we consider an $M/E_k/1$ queueing system with feedback and rest after m consecutive services. We assume that after the service of every m consecutive units, the server takes rest for a random length of time. The steady state solutions of the model is investigated. The waiting time distribution of a unit in the system is obtained.

Another model we consider in this chapter is a queueing model with two servers and a finite intermediate waiting room between them. We assume that, when the intermediate

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+ Some results of this chapter was presented as a paper entitled 'Two queues in series with rest periods' in the VI Annual Conference of ISTPA at Trivandrum during December 1984.

waiting room is full the first server takes rest for a random length of time and when the intermediate waiting room is empty the second server takes rest for a random length of time. The rest times are exponentially distributed with parameters γ_1 and γ_2 respectively. The steady state probabilities of the system size are obtained. Some numerical examples are also given at the end of this chapter.

1. AN ERLANGIAN SERVICE SYSTEM WITH FEED BACK

Queueing systems with series stages for service was considered in a general way by Cox [16]. He assumed feedback or feedforward to other stages rather than choosing only between immediate departure and entry into the next stage. We consider a queueing system with arrival process, a Poisson process of rate λ and service time of each stage exponentially distributed with parameter $k\mu$. After the service completion of all k-stages, a customer leaves the system with probability $1-\theta$ and he joins the queue again for service with probability θ. After the service of m units consecutively, the server takes rest for a random length of time, which is exponentially distributed with parameter α. The state space of the model can be written as a triplet (n,i,j), where n denotes the number of units in the system, i the customer in service being in phase i and j the number of units served after the previous rest.

Let the infinitesimal generator Q of the process be denoted by

$$Q = \begin{bmatrix} A_0 & A_1 & 0 & 0 & \cdots \\ A_2 & A_3 & A_4 & 0 & \cdots \\ 0 & A_5 & A_3 & A_4 & \cdots \\ 0 & 0 & A_5 & A_3 & \cdots \\ \vdots & & & & \ddots \end{bmatrix}$$

where A_0 is a square matrix of order (m+1). The matrix A_0 is given by

$$a_{ii} = \begin{cases} -\lambda, & \text{for } 0 \le i \le m-1 \\ -\lambda-\alpha, & \text{for } i=m \end{cases}$$

$$a_{m1} = \alpha$$

and $$a_{ij} = 0, \text{ otherwise.}$$

The matrix A_1 is of order (m+1) x (mk+1) and is equal to $[B_0, B_1, \ldots, B_m]$, where B_i is (m+1) x k matrix for $0 \le i \le m-1$, and B_m is a matrix of order (m+1) x 1. The entries of matrix B_i for $o \le i \le m-1$, is given by

$$b_{i1} = \lambda \text{ , for } o \le i \le m-1$$

$$b_{ij} = 0 \text{ , } 1 < j \le k, \; o \le i \le m$$

and $$B_m = [0, 0, \ldots, 0, \lambda]^T.$$

The matrix A_2 of order (mk+1) x (m+1) is given by

$$A_2 = [C_o, C_1, \ldots, C_{m-1}, \underline{O}]^T$$

where C_is are matrices of order k x (m+1) and $\underline{O}$ is the zero vector. The matrices C_is for $o \le i \le m-1$ are given by

$$c_{k,i+1} = k\mu\,(1-\theta)$$

$$c_{i,j} = 0, \text{ for other values of } i \text{ and } j.$$

The matrix A_3 is given by

$$A_{ii} = \begin{cases} D_1, \text{for } o \le i \le m-1 \\ D_2, \text{for } i=m \end{cases}$$

$$A_{i,i+1} = \begin{cases} D_3, \text{for } o \le i \le m-2 \\ D_4, \text{for } i = m-1 \end{cases}$$

$$A_{m1} = D_1$$

$$A_{i,j} = 0, \text{ otherwise}$$

where D is a square matrix of order k. The matrix D is given by

$$d_{ii} = -(\lambda + k\mu), \text{ for } 1 \leq i \leq k$$

$$d_{i,i+1} = k\mu, \text{ for } 1 \leq i \leq k-1$$

The matrix D_3 of order k is given by

$$d_{k,1} = k\mu\ \theta$$

$$d_{i,j} = 0, \text{ for other values of i and j.}$$

D_4 is a column vector of k elements given by

$$D_4 = [0, 0, \ldots, k\mu\ \theta]^T$$

and

$$D_1 = [\alpha, 0, \ldots, 0]$$

is a row vector of k elements and $D_2 = (\lambda + \alpha)$. The matrix $A_4 = \lambda I$, where I is the identity matrix of order (mk+1). The matrix A_5 is given by

$$A_{i,i+1} = \begin{cases} F, & 0 \leq i \leq m-2 \\ F_1, & \text{for } i = m-1 \\ 0, & \text{otherwise} \end{cases}$$

where the matrix F of order k is given by

$$f_{k,1} = k\mu(1-\theta)$$

$$f_{i,j} = 0, \text{ otherwise}$$

and $$F_1 = [0, 0, \ldots, 0, k\mu(1-\theta)]^T,$$

is a column vector.

1.2. STEADY STATE QUEUE LENGTH

Let $\underline{x}$ be the vector of steady state probabilities associated with the generator Q such that

$$\underline{x}\, Q = \underline{0} \text{ and } \underline{x}\, e = 1 . \qquad (1)$$

We can examine the existence of a solution of the form

$$\underline{x}_i = \underline{x}_1 R^{i-1}, \text{ for } i \geq 1. \qquad (2)$$

The matrix R satisfies the matrix equation

$$A_4 + RA_3 + R^2 A_5 = \underline{0} \cdot$$

Thus $$R = -A_4 A_3^{-1} - R^2 A_5 A_3^{-1} . \qquad (3)$$

We can compute R by successive substitution in the equation (3) starting with R=0. The vector $[\underline{x}_0, \underline{x}_1]$ is obtained by solving the equation

$$[\underline{x}_0, \underline{x}_1] \begin{bmatrix} A_0 & A_1 \\ A_2 & A_3+RA_5 \end{bmatrix} = [\underline{0}, \underline{0}] .$$

Define $A = A_3+A_4+A_5$ as

$$A = \begin{bmatrix} G & H & 0 & 0 & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ 0 & G & H & 0 & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \vdots & & & \ddots & \ddots & & \\ & & & & \ddots & \ddots & \\ & & & & & G & H_1 \\ H_2 & 0 & 0 & & \cdot\cdot & 0 & -\alpha \end{bmatrix}$$

The matrix $G = (g_{ij})$ of order k x k is given by

$$g_{ii} = -k\mu, \text{ for } 1 \leq i \leq k$$

$$g_{i,i+1} = k\mu, \text{ for } 1 \leq i \leq k-1$$

and $$g_{i,j} = 0, \text{ for other values of } i \text{ and } j .$$

The matrix $H = (h_{ij})$ of order k x k is given by

$$h_{k,1} = k\mu$$

and $$h_{i,j} = 0, \text{ otherwise.}$$

H_1 is a column vector of k elements and is given by

$$H_1 = [0, 0, \ldots, k\mu]^T \text{ and}$$

H_2 is a row vector of k elements given by

$$H_2 = [\alpha, 0, \ldots, 0].$$

Thus A is an infinitesimal generator of a finite Markov chain. A is irreducible since there is a path from any state to any other state. Let $\underline{\pi} = (\underline{\pi}_o, \underline{\pi}_1, \ldots, \underline{\pi}_{m-1}, \pi_m)$ be the vector of steady state probabilities associated with A such that $\underline{\pi} A = \underline{0}$ and $\underline{\pi} e = 1$. Since $\underline{\pi} A = \underline{0}$, we get

$$\begin{aligned} \underline{\pi}_o G + \pi_m H_2 &= 0 \\ \underline{\pi}_i H + \underline{\pi}_{i+1} G &= 0, \text{ for } o \le i \le m-2 \\ \underline{\pi}_{m-1} H_1 - \pi_m \alpha &= 0, \end{aligned} \tag{4}$$

using the condition $\sum_{i=o}^{m-1} \sum_{j=1}^{k} \pi_{ij} + \pi_m = 1$.

Solving (4) we get

$$\pi_m = \left(1 + \frac{m\alpha}{\mu}\right)^{-1} \tag{5}$$

and

$$\pi_{ij} = \left(1 + \frac{m\alpha}{\mu}\right)^{-1} \left(\frac{\alpha}{k\mu}\right), \text{ for } o \le i \le m-1, \; 1 \le j \le k. \tag{6}$$

We can write the equilibrium condition in terms of the parameters of the process. The queue is stable if $\underline{\pi}\, A_5\, e \geq \underline{\pi}\, A_4\, e$.

ie. if $\quad \underline{\pi}_o F + \underline{\pi}_1 F + \ldots + \underline{\pi}_{m-1} F \geq \lambda$

Thus we can write the equilibrium condition in terms of the parameters of the process as

$$\sum_{i=o}^{m-1} \pi_{ik} \geq \frac{\lambda}{k\mu(1-\theta)} \,.$$

Substituting the values of $\pi_{i,k}$, for $o \leq i \leq m-1$, we get the equilibrium condition as

$$\frac{\lambda(1+\frac{m\alpha}{\mu})}{(1-\theta)m\alpha} < 1 \tag{7}$$

1.3. NUMERICAL EXAMPLE

For given values of $\lambda, \mu, \alpha, \theta$ and k, which satisifes the condition (7), we can compute the rate matrix R by successive substitution, starting with R=0, in the equation (3). This is continued until the maximum entrywise difference between iterates is less than 10^{-5}.

For $\lambda=1$, $\mu=2$, $\alpha=2$, $k=2$, $m=3$, $\theta=.1$, the rate matrix R is given below.

$$\begin{bmatrix} .221526 & .177221 & .079324 & .063459 & .046345 & .037076 & .053816 \\ .012446 & .209946 & .056344 & .045076 & .028418 & .022735 & .031114 \\ .033239 & .026591 & .229717 & .183774 & .084234 & .067387 & .083076 \\ .020381 & .016305 & .017468 & .213975 & .059355 & .047484 & .050954 \\ .060412 & .048329 & .048127 & .038501 & .238641 & .190913 & .151031 \\ .042569 & .034056 & .030872 & .024698 & .023757 & .219005 & .106424 \\ .171155 & .136924 & .102594 & .082075 & .073411 & .058729 & .427887 \end{bmatrix}$$

2. TWO QUEUES IN SERIES WITH FINITE INTERMEDIATE WAITING ROOM

The queueing model consisting of two servers in series with a finite intermediate waiting room has an extensive literature, dating back to 1956 with the work of G.C.Hunt [29]. The study of blocking in queues with two or more servers in series without intermediate waiting room was initiated by Avi-Itzhak and Yadin [4]. Neuts [39,40] studied models in which there is a finite waiting room between the two servers where the service times with the first server have a general distribution. The queueing models with servers in series is studied by many authors such as Clarke [11,12], Boxma and Konheim [8], Suzuki [54] and Prabhu [47]. Ramanarayanan and Usha [48] investigated two models, one with two waiting rooms in parallel and the other with waiting rooms in series. Latouche and Neuts [32] investigated a model with a group of $r \geq 1$, identical, parallel servers in series with a second group of $c \geq 1$ identical parallel servers.

The system consists of two servers S_1 and S_2 and an intermediate waiting room, say waiting room II, between them with capacity k. The arrival process is a Poisson process with rate λ and the service times by S_1 and S_2 are independent and exponentially distributed with parameters μ_1 and μ_2 respectively. We assume that there is an infinite capacity waiting room, which is waiting room I, in front of the first server S_1. After getting service from S_1, customers take service from S_2. When there are k customers in the waiting room II, the service by S_1 is blocked. We assume that when the service by S_1 is blocked or the waiting room I is empty, S_1 takes rest for a random length of time which is exponentially distributed with parameter α_1. After the rest, S_1 returns to the system and when he finds the number of units in the intermediate waiting room less than k, he starts to serve the units; otherwise he remains idle. When the waiting room II is empty, S_2 takes rest for a random length of time which is exponentially distributed with parameter α_2. After rest S_2 returns to the system and if any unit is waiting in the waiting room II, he starts to serve, otherwise he remains idle.

2.1 THE STRUCTURE OF THE MARKOV PROCESS

Under the assumption of exponential service times and rest times for the servers S_1 and S_2, the queueing model may be described as a continuous parameter Markov chain on the state space

$$\left\{(i,j,R_1,R_2),\ (i,j,W_1,R_2),\ (i,j,R_1,W_2) \text{ and } (i,j,W_1,W_2),\ i \geq 0,\ 0 \leq j \leq k\right\}.$$

The index i denotes the number of units queued up before the server S_1 and j denotes the number of units in the waiting room II. The index R_1,R_2 denote both the servers S_1 and S_2 are in rest. R_1,W_2 denote the server S_1 is in rest and S_2 is serving or is ready to serve. Similarly W_1,R_2 and W_1,W_2 indicate S_1 is serving or ready to serve, S_2 is in rest and both S_1 and S_2 are serving or ready to serve respectively.

The infinitesimal generator Q of the Markov chain has the structure of a block tridiagonal matrix of the form

$$Q = \begin{bmatrix} B_o & A_2 & 0 & 0 & \cdots \\ B_1 & A_1 & A_2 & 0 & \cdots \\ 0 & A_o & A_1 & A_2 & \\ \vdots & & \ddots & \ddots & \ddots \end{bmatrix} \qquad (8)$$

where all the entries are square matrices of order 4(k+1). To describe the above matrices we use a matrix of order (k+1) of the form

$$M = \begin{bmatrix} M_{11} & M_{12} & \cdots & M_{1k+1} \\ M_{21} & M_{22} & \cdots & M_{2k+1} \\ \vdots & & & \\ M_{k+11} & M_{k+12} & \cdots & M_{k+1\ k+1} \end{bmatrix} = \begin{bmatrix} M_{ij} \end{bmatrix} \qquad (9)$$

where M_{ij} are 4 x 4 matrices for $1 \leq j \leq k+1$. The matrix B_o is given by

$$B_o = \begin{bmatrix} M_{ij} \end{bmatrix} \qquad (10)$$

where

$$M_{11} = K_1$$

$$M_{ii} = K, \text{ for } 2 \leq i \leq k+1$$

$$M_{21} = L_1$$

$$M_{i,i-1} = L, \text{ for } 3 \leq i \leq k+1$$

$$M_{ij} = 0, \text{ otherwise.}$$

The matrices K_1 and K are given respectively, by

$$K_1 = \begin{bmatrix} -\lambda-\alpha_1-\alpha_2 & \alpha_1 & \alpha_2 & 0 \\ 0 & -\lambda-\alpha_2 & 0 & \alpha_2 \\ 0 & 0 & -\lambda-\alpha_1 & \alpha_1 \\ 0 & 0 & 0 & -\lambda \end{bmatrix} \tag{11}$$

and

$$K = \begin{bmatrix} -\lambda-\alpha_1-\alpha_2 & \alpha_1 & \alpha_2 & 0 \\ 0 & -\lambda-\alpha_2 & 0 & \alpha_2 \\ 0 & 0 & -\lambda-\alpha_1-\mu_2 & \alpha_1 \\ 0 & 0 & 0 & -\lambda-\mu_2 \end{bmatrix} \tag{12}$$

To describe the matrices L_1 and L, we need a 4 x 4 matrix

$$N = (n_{ij}), \quad 1 \leq i,j \leq 4 . \tag{13}$$

The matrix L_1 is given by

$$L_1 = (n_{ij}) \tag{14}$$

where $n_{31} = n_{42} = \mu_2$

$n_{ij} = 0$, for all other values of i and j.

The matrix L is given by

$$L = (n_{ij}) \tag{15}$$

where $n_{33} = n_{44} = \mu_2$

$n_{ij} = 0$, for all other values of i and j.

The matrix A_1 is given by

$$A_1 = \left[M_{ij}\right] \tag{16}$$

where $M_{11} = D_1$

$M_{ii} = D$, for $2 \leq i \leq k$

$M_{21} = L_1$

$$M_{k+1,k+1} = K$$

$$M_{i,i-1} = L, \text{ for } 3 \leq i \leq k+1$$

$$M_{ij} = 0, \text{ otherwise .}$$

The matrices D_1 and D are square matrices of order 4 and L_1, L and K are the same as described earlier. The matrices D_1 and D are given by

$$D_1 = \begin{bmatrix} -\lambda-\alpha_1-\alpha_2 & \alpha_1 & \alpha_2 & 0 \\ 0 & -\lambda-\alpha_2-\mu_1 & 0 & \alpha_2 \\ 0 & 0 & -\lambda-\alpha_1 & \alpha_1 \\ 0 & 0 & 0 & -\lambda-\mu_1 \end{bmatrix} \tag{17}$$

$$D = \begin{bmatrix} -\lambda-\alpha_1-\alpha_2 & \alpha_1 & \alpha_2 & 0 \\ 0 & -\lambda-\mu_1-\alpha_2 & 0 & \alpha_2 \\ 0 & 0 & -\lambda-\mu_2-\alpha_1 & \alpha_1 \\ 0 & 0 & 0 & -\lambda-\mu_1-\mu_2 \end{bmatrix} \tag{18}$$

The matrix B_1 is given by

$$B_1 = \left[M_{ij}\right] \qquad (19)$$

where

$$M_{12} = F_1$$

$$M_{i,i+1} = F, \text{ for } 2 \leq i \leq k-1$$

$$M_{k,k+1} = F_1$$

$$M_{ij} = 0, \text{ otherwise.}$$

Using (13), the matrices F_1 and F are given by

$$F_1 = (n_{ij}) \qquad (20)$$

where

$$n_{21} = n_{43} = \mu_1$$

$$n_{ij} = 0, \text{ for other values of i and j}$$

and

$$F = (n_{ij}) \qquad (21)$$

where

$$n_{22} = n_{44} = \mu_1$$

$$n_{ij} = 0, \text{ otherwise.}$$

The matrix A_o is given by

$$A_o = \left[M_{ij}\right] \qquad (22)$$

where

$$M_{i,i+1} = F, \text{ for } 1 \leq i \leq k-1$$

$$M_{k,k+1} = F_1$$

$$M_{ij} = 0, \text{ otherwise.}$$

The matrices F_1 and F are same as given in (20) and (21). The matrix $A_2 = \lambda I$, where I is a unit matrix of order 4(k+1). The infinitesimal generator Q is the canonical form of a quasi birth-death process. Thus the system may be studied as a quasi birth-death process. Neuts [45] gives a detailed discussion on quasi-birth-death process. The stationary probability vector $\underline{x} = (\underline{x}_0, \underline{x}_1, \ldots)$ of Q is given by

$$\underline{x}_i = \underline{x}_o R^i, \text{ for } i \geq o \tag{23}$$

where R satisfies the matrix equation

$$R^2 A_o + RA_1 + A_2 = \underline{0} . \tag{24}$$

2.2 STEADY STATE BEHAVIOUR

Let $A = A_o + A_1 + A_2$, which is a finite generator. The block partitioned form of A is given by

$$A = \begin{bmatrix} G_1 & F & 0 & 0 & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ L_1 & G & F & 0 & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ 0 & L & G & F & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ 0 & 0 & L & G & \cdot\cdot & \cdot\cdot & \cdot\cdot \\ \vdots & & & & \ddots & \ddots & \\ 0 & & & & \ddots & \ddots & F \\ 0 & & & & & L & G_2 \end{bmatrix} \tag{25}$$

where G, G_1, G_2, L, L_1 and F are square matrices of order 4. The matrix G_1 is given by

$$G_1 = D_1 + \lambda I \tag{26}$$

where I is the unit diagonal matrix of order 4 and matrix D_1 is given in (17). The matrix G is given by

$$G = D + \lambda I \tag{27}$$

where D is given in (18). The matrix G_2 is given by

$$G_2 = K + \lambda I \tag{28}$$

where K is given in (12) and I is the identity matrix of order 4. The matrices L_1, L and F are the same as given earlier.

Let $\underline{\pi} = (\underline{\pi}_o, \underline{\pi}_1, \ldots, \underline{\pi}_k)$ be the vector of steady state probabilities of A, where $\underline{\pi}_i = (\pi_i R_1 R_2, \pi_i W_1 R_2, \pi_i R_1 W_2, \pi_i W_1 W_2)$ for $o \le i \le k$. It satisfies

$$\underline{\pi} A = \underline{0}, \; \underline{\pi} e = 1 \tag{29}$$

where $e = (1, 1, \ldots, 1)^T$.

From (29) we get

$$\underline{\pi}_o G_1 + \underline{\pi}_1 L_1 = \underline{0}$$
$$\underline{\pi}_o F + \underline{\pi}_1 G + \underline{\pi}_2 L = \underline{0} \quad . \tag{30}$$

From (30) we can write

$$\underline{\pi}_o = - \underline{\pi}_1 L_1 G_1^{-1}$$

and

$$\underline{\pi}_1 = - \underline{\pi}_2 L(G-L_1 G_1^{-1}F)^{-1} \quad .$$

Thus we can write $\underline{\pi}_o$ in terms of $\underline{\pi}_1$ and $\underline{\pi}_1$ in terms of $\underline{\pi}_2$ and so on. Finally we can write $\underline{\pi}_{k-1}$ in terms of $\underline{\pi}_k$. The last equation of (29) is

$$\underline{\pi}_{k-1}F + \underline{\pi}_k G_2 = \underline{0} \; . \tag{31}$$

Since $\underline{\pi}_{k-1}$ can be written in terms of $\underline{\pi}_k$, we can rewrite equation (31) as

$$\underline{\pi}_k (H + G_2) = \underline{0} \; . \tag{32}$$

Thus we find that $\underline{\pi}_k$ is the left eigen vector of $H + G_2$.

As an illustration, consider k=6, the computation of π_i, for $o \leq i \leq 6$ is as follows. For the given matrices L_1, G_1, L, G, F and G_2.

$$\underline{\pi}_o = - \underline{\pi}_1 L_1 G_1^{-1}$$

$$\underline{\pi}_1 = - \underline{\pi}_2 LA^{-1}, \quad \text{where } A = G-L_1G_1^{-1}F$$

$$\underline{\pi}_2 = - \underline{\pi}_3 LB^{-1}, \quad \text{where } B = G-LA^{-1}F$$

$$\underline{\pi}_3 = - \underline{\pi}_4 LC^{-1}, \quad \text{where } C = G-LB^{-1}F$$

$$\underline{\pi}_4 = - \underline{\pi}_5 LD^{-1}, \quad \text{where } D = G-LC^{-1}F$$

$$\underline{\pi}_5 = - \underline{\pi}_6 LE^{-1}, \quad \text{where } E = G-LD^{-1}F$$

$$\underline{\pi}_6(G_2-LE^{-1}F) = \underline{0}$$

That is $\underline{\pi}_6$ is the left eigen vector of $G_2-LE^{-1}F$. Thus first compute $\underline{\pi}_6$, then substitute the value of $\underline{\pi}_6$ in $\underline{\pi}_5$, next substitute the value of $\underline{\pi}_5$ in $\underline{\pi}_4$ and so on.

2.3 THE EQUILIBRIUM CONDITION

The equilibrium condition $\underline{\pi}\ A_0 e > \pi\ A_2 e$ can be explicitly written in terms of the parameters of the model. Since $\underline{\pi} = (\underline{\pi}_o, \underline{\pi}_1, \ldots, \underline{\pi}_k)$ and $\underline{\pi}_i = (\pi_i R_1 R_2, \pi_i W_1 R_2, \pi_i R_1 W_2, \pi_i W_1 W_2)$, for $o \leq i \leq k$, the required condition reduces to

$$\mu_1(\pi_o W_1 R_2 + \pi_o W_1 W_2 + \pi_1 W_1 R_2 + \pi_1 W_1 W_2 + \cdots + \pi_{k-1} W_1 R_2 + \pi_{k-1} W_1 W_2) > \lambda$$

ie. $$\mu_1 \sum_{i=o}^{k-1} \pi_i \, (W_1 R_2 + W_1 W_2) > \lambda , \qquad (33)$$

where μ_1 is the service rate of the first server S_1, λ is the arrival rate to the first server and $\sum_{i=o}^{k-1} \pi_i (W_1 R_2 + W_1 W_2)$ is the steady state probability that the first server is working. Thus the equilibrium condition is , probability that the first server is working is greater than λ/μ_1.

2.4. NUMERICAL EXAMPLES

Assume that the capacity of the intermediate waiting room is 4. For given matrices L_1, G_1, L, G, F and G_2, the stationary probability vectors $\underline{\pi}_o$, $\underline{\pi}_1$, $\underline{\pi}_2$, $\underline{\pi}_3$ and $\underline{\pi}_4$ are computed for various values of μ_1 and μ_2 subject to the condition $\sum_{i=1}^{4} \underline{\pi}_i = 1$. When $\lambda=1$, $\mu_1=5$, $\mu_2=6$, $\gamma_1=\gamma_2=1$

$\underline{\pi}_o$ = (.0930397, .082832, .0930397, .0351743)

$\underline{\pi}_1$ = (0 , .069026, .0310132, .0673253)

$$\underline{\pi}_2 = (\quad 0 \quad , .057522, .036182, .0774446)$$

$$\underline{\pi}_3 = (\quad 0 \quad , .047935, .025323, .0702599)$$

$$\underline{\pi}_4 = (.049248, .049248, .049248, .0492480)$$

When $\lambda=1$, $\mu_1=\mu_2=5$, $\gamma_1=\gamma_2=1$, then π_is are

$$\underline{\pi}_o = (.0849186, .0682233, .0849186, .0306293)$$

$$\underline{\pi}_1 = (\quad 0 \quad , .0568527, .0339693, .0648832)$$

$$\underline{\pi}_2 = (\quad 0 \quad , .0476071, .0412383, .0809727)$$

$$\underline{\pi}_3 = (\quad 0 \quad , .0396725, .0412383, .0809727)$$

$$\underline{\pi}_4 = (.0593833, .0593833, .0593833, .0593833)$$

CHAPTER V

INVENTORY SYSTEM WITH REST PERIODS

A detailed review of the storage systems was given by Gani [25] and the applications of such models to practical situations by Hadley and Whitin [26]. The probabilistic treatment in the study of inventory system using renewal theoretic arguments was given by Arrow, Karlin and Scarf [3]. Srinivasan [53], has discussed an analysis of (s,S) inventory policy with random lead time and unit demand. Recently, Thangaraj and Ramanarayanan [57] studied the (s,S) inventory policy with two ordering levels.

Scholl and Kleinrock [50] analysed an M/G/1 queue with rest periods where they considered the situation in which the server takes rest for a random length of time when the system is empty. Here, we introduce an inventory system with rest to the server. When the inventory level becomes zero, the server takes rest for a random length of time having the distribution function G(.). We consider the following three models, its transition probabilities and system size probabilities.

I. no queue of demands is permitted when the server takes rest.

II. a queue of demands is permitted when the server takes rest, but only after the replenishment.

III. two servers S_1 and S_2 with rest periods alternately to the servers.

1. DESCRIPTION OF THE MODELS I AND II.

Let S be the maximum capacity of a ware house. At time zero the inventory level is S, the stock level goes on decreasing due to incoming demands. The demands are assumed to occur for one unit at a time and the time duration between the arrivals of two consecutive demands are independent and identically distributed (i.i.d) random variables with cumulative distribution function (c.d.f) F(.). As soon as the stock level drops down to s an order is placed for S-s units. The lead time is a random variable with c.d.f G(.). During the lead time there may be a number of demands. If the order of S-s units does not materialise before the level of inventory drops to zero, the server goes for rest for a random length of time.

The time duration for which the level of inventory continuously remains at zero is called a **dry period**. The rest times are i.i.d random variables with c.d.f H(.).

The above distribution functions are assumed to be absolutely continuous with derivatives f(.), g(.) and h(.) respectively. Further, we assume that the random variables such as, time duration between successive demands, the lead times and the rest times, are mutually independent.

In the first model we assume that no queue of demands is permitted when the server is taking rest. In the second model a queue of demands is permitted in the absence of the server, but only after the materialisation of the order. In model II, an agent can work in the absence of the server with the restriction that the maximum number of units sold out by the agent is S-2s-1. So, at the server's arrival epoch the level of inventory is at least s+1. In both the models we assume the renewal epochs to be the demand epochs. (See figs. 1 and 2)

There are a number of applications of the two models described above. It is natural that when the inventory is zero, the firm finds it meaningless to have

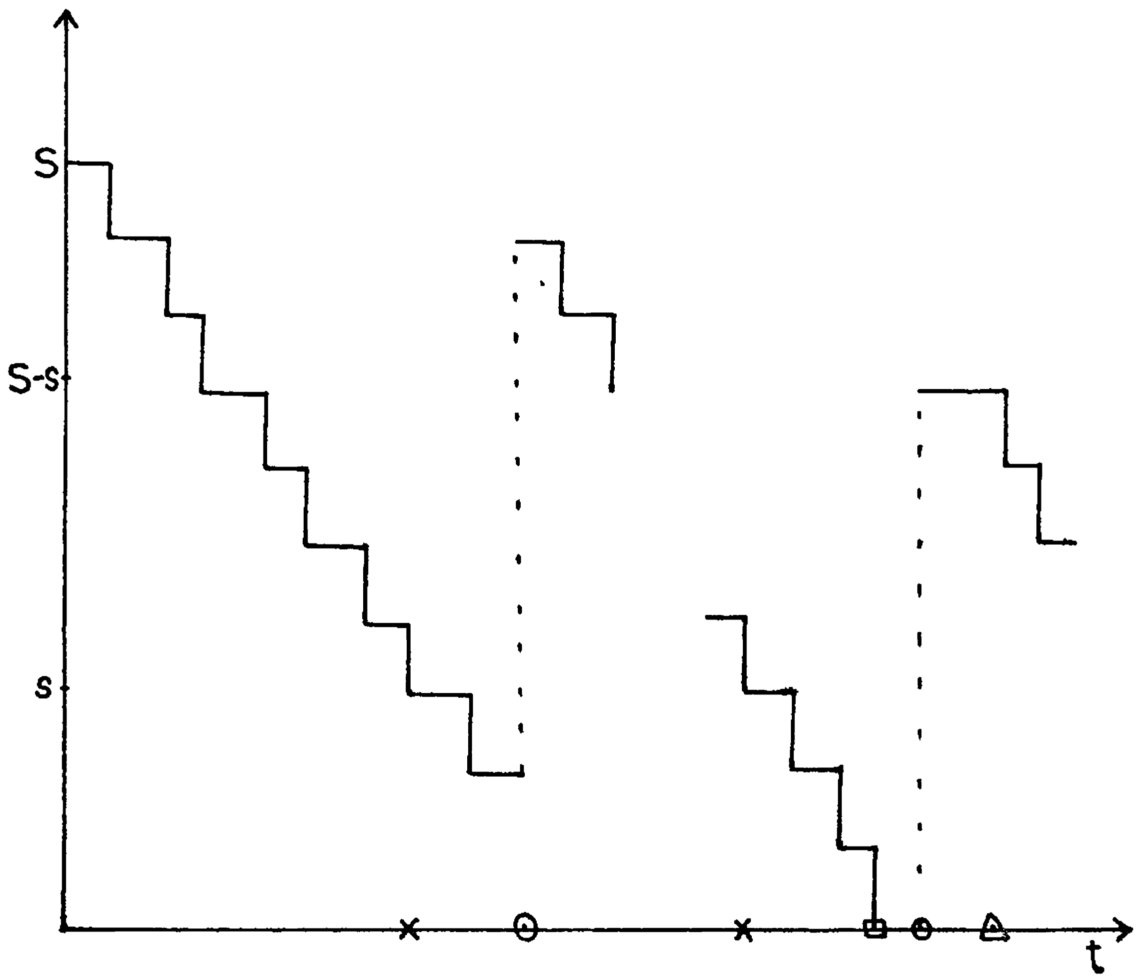

Figure 1.

MODEL I: A typical plot of the stock level versus time

X - order placed

⊙ - order materialised

⊡ - server starts rest

△ - server returns after rest

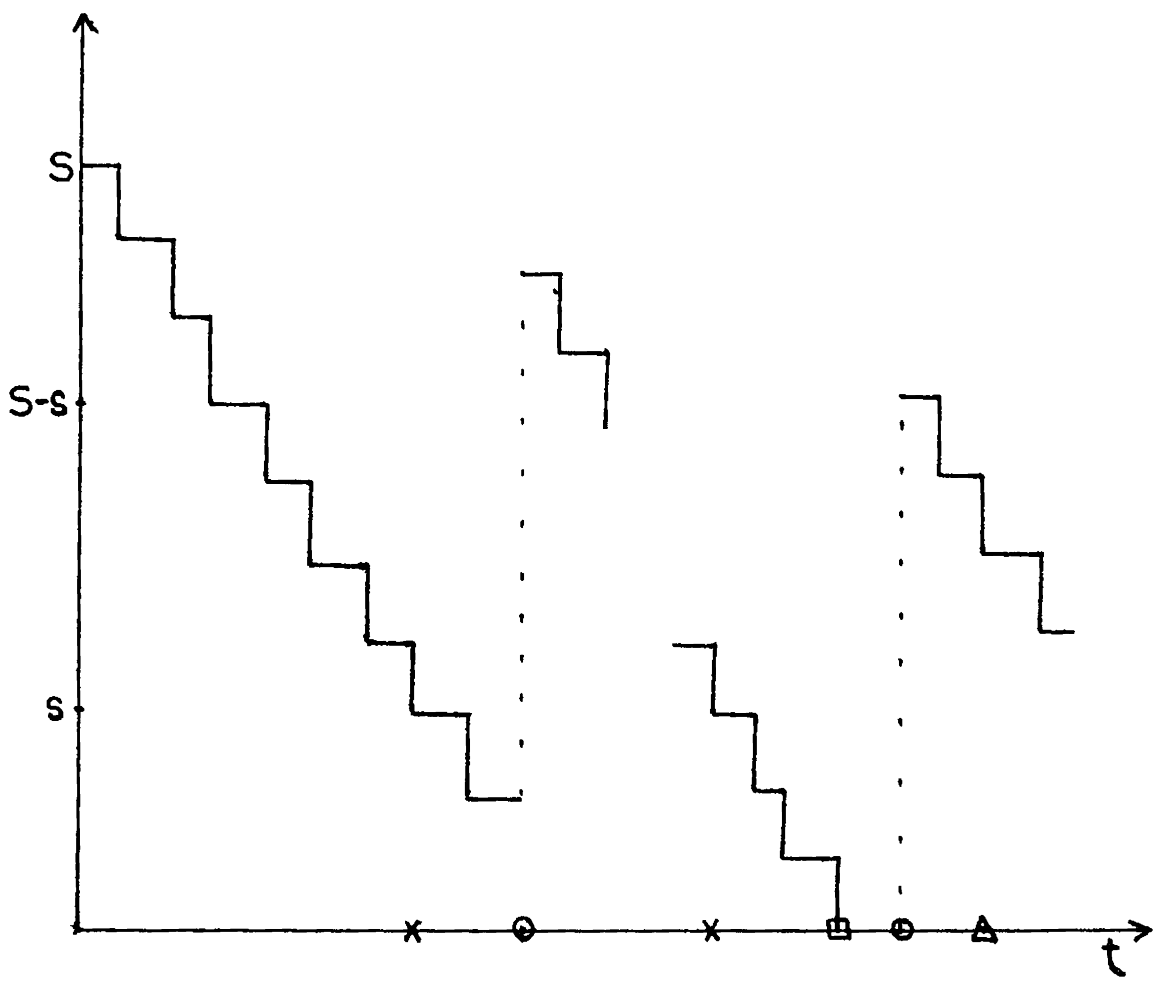

Figure 2.

MODEL II: A typical plot of the stock level versus time

× - order placed
⊙ - order materialised
⊡ - server starts rest
△ - server returns after rest

a server. So he is turned out and as soon as the materialisation of the order takes place, the firm tries to get back the server. But the server, in turn, may not turn up immediately. It can happen that the server returns to the firm even before the materialisation of order and remains there, ready to serve but has to remain idle until replenishment takes place. This situation is represented by model I. As an example of model II, suppose the firm appoints a purely temporary hand to sell the goods in the absence of the server, as soon as the order is materialised. He is permitted to sell atmost S-2s-1 units. If we do not have this restriction, the reordering level may be reached and there is then no need for a permanent server. In this sequel we use the following notations.

Notations:

$p_{x,y}(t)$ - the probability of transition from x to y

$f^{*n}(.)$ - the n-fold convolution of f(.) with itself

$p^{*n}_{s,s}(.)$ - the n-fold convolution of $p_{s,s}(.)$ with itself

K(.) - the renewal density function of f's

$F^{*n}(.)$ - the n-fold convolution of $F(.)$ with itself

$X(t)$ - the inventory level at time t

$\bar{F}(.)$, $\bar{G}(.)$ and $\bar{H}(.)$ are tail distributions corresponding to $F(.)$, $G(.)$ and $H(.)$ respectively.

1.1 TRANSITION PROBABILITIES OF MODEL I

In this model we assume that no queue of demands is permitted when the server takes rest. The probabilities of reaching i from S and S-1 from s in time x are given by

$$p_{S,i}(x) = f^{*(S-i)}(x), \quad s+1 \le i \le S-1 \tag{1}$$

$$p_{s,S-1}(x) = f(x) \; G(x) . \tag{2}$$

Consider $p_{s,S-j}(x)$, $2 \le j \le s$, which is the probability of transition from s to S-j in time x for the first time. When the level of inventory drops to s, an order is placed for S-s units and j-1 units of demands occur upto time u. The order is materialised in (u,x), where x is the first demand epoch after the materialisation of order. Thus

$$p_{s,S-j}(x) = \int_0^x f^{*(j-1)}(u)[G(x)-G(u)]f(x-u)du, \quad 2 \le j \le s . \tag{3}$$

Next we consider the situation when the inventory level drops to zero. As soon as this happens, the server goes for rest. Then we have two different cases.

Case (i): The server returns after rest. By this time the order may or may not be materialised. Even if the order does not materialise the server remains there, ready to serve and starts meeting demands after the order materialises. In this case, we have

$$p_{s,S-s-1}(x) = \int_0^x \int_u^x f^{*s}(u)\ k(v-u)\ [G(x)-G(v)]\ H(x-u)\ f(x-v)dv\ du \quad (4)$$

where k(v-u) represents the renewal function of lost demands during (u,v), and H(x-u) is the probability that the server returns in (u,x].

Case (ii): The server returns after rest. If the order is not materialised, he again goes for rest for a random length of time having the same distribution H(.). This may be repeated several times. Thus we get

$$p_{s,S-s-1}(x) = \int_0^x \int_u^x \int_v^x \int_w^x f^{*s}(u)\ k(v-u)\ m(w-v)$$

$$[G(y)-G(w)]h(y-w)\ f(x-w)\ dy\ dw\ dv\ du \tag{5}$$

where k(.) and m(.) represent renewal functions of lost demands. The probability of first return to s in time x is given by

$$p_{s,s}(x) = \sum_{i=S-s-1}^{S-1} \int_0^x p_{s,i}(u) f^{*(i-s)}(x-u)du \ . \tag{6}$$

1.2. SYSTEM SIZE PROBABILITIES

Define $A_i(t) = \Pr\{X(t)=i \mid X(o) = S\}$ (7)

as the probability of finding i units in the system at time t, given that, at time zero the inventory level is S. We shall now find the probability that the inventory level at time t is a prescribed quantity. Now we have

$$A_S(t)=\bar{F}(t)+\int_0^t f^{*(S-s)}(u) \int_0^{t-u} \sum_{n=o}^{\infty} p^{*n}_{s,s}(v)\bar{F}(t-u-v)$$

$$G(t-u-v)dv\ du \ . \tag{8}$$

The equation (8) can be explained as follows. The first term means that there is no demand in (o,T]. The second term represents the fact that there are S-s demands in (o,u), during (u,u+v) there are n transitions from s to s, n = 1,2, ..., the order materialises in (u+v,t) and no more demand thereafter. Next we have $A_{S-j}(t)$ for $1 \leq j \leq s-1$, given by

$$A_{S-j}(t) = \int_0^t f^{*j}(u)\bar{F}(t-u)du + \int_0^t f^{*(S-s)}(u) \int_0^{t-u} \sum_{n=0}^{\infty} p_{s,s}^{*n}(v) \int_0^{t-u-v} \sum_{i=0}^{j-1} p_{s,S-j+1}(w) \int_0^{t-u-v-w} f^{*i}(y)\ \bar{F}(t-u-v-w-y)dy\ dw\ dv\ du;\ 1 \leq j \leq s-1. \qquad (9)$$

The first term on the right side is obtained by arguing as follows. j demands occur in (o,u] and no more demand in (u,t]. For the second term we have, there are S-s demands in (o,u], during (u,u+v) there are n, (n=1,2,...) transitions from s to s followed by the transition from s to S-j+i in (u+v, u+v+w) and i demand in (u+v+w, u+v+w+y).

We have

$$A_{S-s}(t) = \int_0^t f^{*s}(u)\ \bar{F}(t-u)du + \int_0^t f^{*(S-s)}(u)$$

$$\int_0^{t-u} \sum_{n=0}^{\infty} p_{s,s}^{*n}\ (v) \int_0^{t-u-v} \sum_{j=S-s}^{S-1} p_{s,j}(w)$$

$$\int_0^{t-u-v-w} f^{*(j-S+s)}(y)\ \bar{F}(t-u-v-w-y)dy\ dw\ dv\ du$$

$$+ \int_0^t f^{*(S-s)}(u) \int_0^{t-u} \int_v^{t-u} \int_w^{t-u} \sum_{n=0}^{\infty} p_{s,s}^{*n}(v) f^{*s}(w-v) k(y-w-v)$$

$$\bar{H}(t-u-w-y)\ [G(t-u)-G(v)]dy\ dw\ dv\ du. \qquad (10)$$

For the first two terms of (10) we have similar arguments as given in (9). For the third term, (S-s) units of demands occur in (o,u], in (u,v) there are n transitions from s to $\bar{s}$ and in (v,w) s units of demands occur. Thus the level of inventory drops to zero, the server goes for rest and in (w,y) a number of demands is lost to the system.

The order materialises before t, but the server is not back to the system upto time t. Also

$$A_{S-s-j}(t) = \int_0^t f^{*(s+j)}(u)\, \bar{F}(t-u)du + \int_0^t f^{*(S-s)}(u)$$

$$\int_0^{t-u} \sum_{n=0}^{\infty} p^{*n}_{s,s}(v) \int_0^{t-u-v} p_{s,S-s-1}(w) \int_0^{t-u-v-w} f^{*(j-1)}(y)$$

$$\bar{F}(t-u-v-w-y)dy\, dw\, dv\, du + \int_0^t f^{*(S-s)}(u) \int_0^{t-u} \sum_{n=0}^{\infty} p^{*n}_{s,s}(v)$$

$$\int_0^{t-u-v} \int_w^{t-u-v} \int_y^{t-u-v} \int_z^{t-u-v} f^{*s}(w)\, k(y-w)[G(z)-G(y)]\, h(z-w)$$

$$f^{*j}(\xi -z)\, \bar{F}(t-u-v-\xi)d\xi\, dz\, dy\, dw\, dv\, du, \quad 1 \le j \le S-2s-1. \qquad (11)$$

For the first two terms of (11) we have similar arguments as given in (10). The third term represents the situation, when the level of inventory drops to zero the server goes for rest. The order is materialised before z, where z is the epoch at which the server returns to the system. Now we have

$$A_j(t) = \int_0^t f^{*(S-s)}(u) \int_0^{t-u} \sum_{n=0}^{\infty} p^{*n}_{s,s}(v)\ \bar{G}(t-u-v)$$

$$\int_0^{t-u-v} f^{*(s-j)}(w)\ \bar{F}(t-u-v-w)\ dw\ dv\ du, \text{ for } 1 \leq j \leq s \qquad (12)$$

and

$$A_0(t) = \int_0^t f^{*(S-s)}(u) \int_0^{t-u} \sum_{n=0}^{\infty} p^{*n}_{s,s}(v)\ \bar{G}(t-u-v)$$

$$f^{*s}\ (t-u-v)dv\ du.$$

1.3. TRANSITION PROBABILITIES OF MODEL II

In this model we assume that a queue of demands is allowed after the materialisation of order even when the server has not returned after rest. If the server has not returned after rest, on the materialisation of inventory, an agent is permitted to sell a maximum of

S-2s-1 units. The expressions for transition probabilities $p_{S,i}(x)$, for $s \leq i \leq S-1$; $p_{s,S-1}(x)$ and $p_{s,S-j}(x)$ for $2 \leq j \leq s$ are the same as those given in model I. Also the transition probability from s to S-s-1 in time x has the same expression as in (4). The transition probability from s to S-s-j for $2 \leq j \leq S-2s-1$, in time x is given by

$$p_{s,S-s-j}(x) = \int_0^x \int_u^x \int_u^w \int_u^z f^{*s}(u)\, k(y-u)\, [G(z)-G(y)]$$

$$f(z-y) f^{*(j-2)}(w-z)\, [H(x-u)-H(w-u)]\, f(x-w)\, dy\, dz\, dw\, du \,. \quad (13)$$

This can be explained in the following way. In time u, s demands occur after which the server goes for rest. The order materialises before z but after y, and k(y-u) represents the demands lost in (u,y). The first demand epoch after replenishment is z and an agent sells off j-1 units. The server returns after time w but before x, where x is the first demand epoch after w. Using similar arguments as above, we get

$$p_{s,s}(x) = \int_0^x \int_u^x \int_u^z \int_u^y \int_u^w f^{*s}(u)\ k(v-u)\ [G(w)-G(v)]$$

$$f(w-v) f^{*(S-2s-2)}(y-w)\ k(z-y)\ [H(x-u)-H(z-u)]$$

$$f(x-z)dv\ dw\ dy\ dz\ du \;. \qquad (14)$$

Since the agent cannot sell more than S-2s-1 units, after y there are a number of demands lost to the system and z is the last demand epoch which is lost to the system. Then k(z-y) represents the lost demands in the interval (y,z). The renewal transition probability from s to s is

$$\hat{p}_{s,s}(x) = \sum_{j=s}^{S-1} p_{s,j}(u) f^{*(j-s)}(x-u)du \;. \qquad (15)$$

1.4 SYSTEM SIZE PROBABILITIES

We shall now find the probabilities for various inventory levels at an arbitrary time. Now

$$A_S(t) = \bar{F}(t) + \int_0^t f^{*(S-s)}(u) \int_0^{t-u} \sum_{n=0}^{\infty} \hat{p}^{*n}_{s,s}(w) \bar{F}(t-u-w)$$

$$G(t-u-w)dw\ du \qquad (16)$$

The first term of (16) represents the fact that no demand occurs in (o,T], the second stands for S demands in (o,u], and there are n,s to s renewal transitions in (u,w). No demand occurs in (w,t] but the order materialises in (w,t]. Arguing further we get

$$A_{S-j}(t) = [F^{*j}(t) - F^{*(j+1)}(t)] + \int_0^t f^{*(S-s)}(u)$$

$$\int_0^{t-u} \sum_{n=0}^{\infty} \hat{p}^{*n}_{s,s}(w) \; [F^{*j}(t-u-w) - F^{*(j+1)}(t-u-w)]$$

$$G(t-u-w)dw\, du, \quad 1 \leq j \leq s-1. \qquad (17)$$

For the level to be S-s, at time t, we have

$$A_{S-s}(t) = [F^{*S}(t) - F^{*(s+1)}(t)] + \int_0^t f^{*(S-s)}(u)$$

$$\int_0^{t-u} \sum_{n=0}^{\infty} \hat{p}^{*n}_{s,s}(v) \int_0^{t-u-v} \sum_{j=S-s}^{S-1} p_{s,j}(w)$$

$$[F^{*[j-(S-s)]}(t-u-v-w) - F^{*[j-(S-s)+1]}(t-u-v-w)]dw\,dv\,du$$

$$+ \int_0^t f^{*(S-s)}(u) \int_0^{t-u} \sum_{n=0}^{\infty} \hat{p}^{*n}_{s,s}(v) \int_0^{t-u-v} \int_w^{t-u-v} f^{*s}(w)k(y-w)$$

$$[G(t-u-v) - G(y)] \; \bar{F}(t-u-v-y)\; dy\, dw\, dv\, du\,. \qquad (18)$$

The first two terms have similar arguments as that given for (17) and the third term represents the situtation where the inventory level drops to zero in (v,w), $k(y-w)$ represents the lost demands during (w,y) and thereafter no more demand occurs upto time t, but the order is materialised in $(y, t-u-v)$.

Next we have

$$A_{S-s-j}(t) = [F^{*(s+j)}(t) - F^{*(s+j+1)}(t)] + \int_0^t f^{*(S-s)}(u)$$

$$\int_0^{t-u} \sum_{n=0}^{\infty} \hat{p}^{*n}_{s,s}(v) \int_0^{t-u-v} \sum_{k=S-s-j}^{S-1} p_{s,k}(w)$$

$$[F^{*[k-(S-s-j)]}(t-u-v-w) - F^{*[k-(S-s-j)+1]}(t-u-v-w)]dw\,dv\,du$$

$$+ \int_{0}^{t} f^{*(S-s)}(u) \int_{0}^{t-u} \sum_{n=0}^{\infty} \hat{p}^{*n}_{s,s}(v) \int_{0}^{t-u-v} \int_{w}^{t-u-v} \int_{w}^{\xi} \int_{w}^{z} f^{*s}(w)$$

$$k(y-w)\ [G(z)-G(y)]\ f(z-y) f^{*(j-1)}(\xi - z)\ \bar{H}(\xi - w)$$

$$\bar{F}(t-u-v-\xi) dy\ dz\ d\xi\ dw\ dv\ du,\ 1 \leq j \leq S-2s-2. \qquad (19)$$

Using the arguments that led to (18) we get the first two terms. The third term deals with the following situation. After the s to s transitions the level of inventory drops to zero in (v,w) and the server goes for rest. The order materialises after y but before z and there are j demands occurring in (ξ ,y). The server has not returned after rest upto time ξ .

Now $A_{s+1}(t)$ is given by

$$A_{s+1}(t) = [F^{*(S-s-1)}(t) - F^{*(S-s)}(t)] + \int_{0}^{t} f^{*(S-s)}(u)$$

$$\int_{0}^{t-u} \sum_{n=0}^{\infty} \hat{p}^{*n}_{s,s}(v) \int_{0}^{t-u-v} \sum_{j=s+1}^{S-1} p_{s,j}(w)$$

$$[F^{*(j-s-1)}(t-u-v-w) - F^{*(j-s)}(t-u-v-w) dw\ dv\ du$$

$$+ \int_0^t f^{*(S-s)}(u) \int_0^{t-u} \sum_{n=0}^{\infty} \hat{p}^{*n}_{s,s}(v) \int_0^{t-u-v} \int_w^{t-u-v} \int_w^{\chi} \int_w^{\eta} \int_w^{\xi} \int_w^{z} f^{*s}(w)$$

$$k(y-w)[G(z)-G(y)]f(z-y)f^{*(S-2s-2)}(\xi-z)\ k(\eta-\xi)$$

$$\bar{F}(t-u-v-\eta)\ h(\chi-w)dy\ dz\ d\xi\ d\eta\ d\chi\ dw\ dv\ du$$

$$+ \int_0^t f^{*(S-s)}(u) \int_0^{t-u} \sum_{n=0}^{\infty} \hat{p}^{*n}_{s,s}(v) \int_0^{t-u-v} \int_w^{\xi} \int_w^{z} f^{*s}(w)$$

$$k(y-w)\ [G(z)-G(y)]\ f(z-y).\ [F^{*(S-2s-2)}(t-u-v-z-\xi) -$$

$$F^{*(S-2s-1)}(t-u-v-z-\xi)]\ \bar{H}(t-u-v-w)\ dy\ dz\ dw\ dv\ du \qquad (20)$$

The first two terms can be obtained using arguments similar to the ones which led to (18) and (19). To get the third term, in addition to the arguments that led to the third term of (19), we make use of the fact that in the duration of length $\xi - z$, $S-2s-2$ demands occur and a number of demands is lost thereafter because, the agent cannot sell more than $S-2s-1$ units. The server returns to the system at epoch χ and there is no more demand after the epoch η, where η is the last demand epoch which lost to the system. The fourth term represents the possibility of the server not coming back to the system upto time t. Arguing in a similar manner we get

$$A_j(t) = \int_0^t f^{*(S-s)}(u) \int_0^{t-u} \sum_{n=0}^{\infty} \hat{p}^{*n}_{s,s}(v)\ \bar{G}(t-u-v)$$

$$[F^{*(s-j)}(t-u-v) - F^{*(s-j+1)}(t-u-v)]dv\ du,\ 1 \leq j \leq s$$

and

$$A_0(t) = \int_0^t f^{*(S-s)}(u) \int_0^{t-u} \sum_{n=0}^{\infty} \hat{p}^{*n}_{s,s}(v)\ \bar{G}(t-u-v)$$

$$[F^{*s}(t-u-v) - F^{*(s+1)}(t-u-v)]\ dv\ du\ .$$

2. DESCRIPTION OF THE MODEL III

Let S be the maximum capacity of a ware house. At time zero the inventory level is S, the stock level goes on decreasing due to incoming demands. The demands are assumed to occur for one unit at a time and the time duration between the arrivals of successive demands are i.i.d random variables with c.d.f F(.). We assume that there are two servers S_1 and S_2 in the system. The server S_1 has to order S-s units but S_2 can order only a smaller quantity of 'a' units. Whenever both the servers are present, S_1 will order for a quantity of S-s units. As soon as the stock level becomes s, an order is placed for S-s units by S_1 if he is present. After that S_1 goes for rest for a random length of time. If S_1 is not present S_2 gives an order for 'a' unit. The successive rest times of S_1 are i.i.d with c.d.f $G_1(.)$. The lead time for the quantity of S-s units is a random variable with c.d.f $K_1(.)$. During the lead time there may be a number of demands. Further we assume that at least one of the servers is present in the system always. After completion of rest, S_1 returns to the system and when the level of inventory becomes s, an order is placed by S_1 after which S_2 takes rest for a random length of time. The successive rest times of S_2 are i.i.d random variables with c.d.f $H_1(.)$. The lead time for the small quantity 'a' is a random variable with c.d.f R(.).

The above distribution functions are assumed to be absolutely continuous with derivatives $f(.)$, $g_1(.)$, $k_1(.)$, $h_1(.)$ and $r(.)$ respectively. Further we assume that the random variables such as the lead times and rest times are mutually independent.

2.1. TRANSITION PROBABILITIES

Let $p_{x,y}(t)$ be the transition probability density function from the state x to y in time t. The probabilities of reaching i from S and S-i from s in time x are given by

$$p_{S,i}(x) = f^{*(S-i)}(x), \quad s+1 \leq i \leq S-1 \tag{21}$$

where f^{*n} is the n-fold convolution of f(.) with itself,

$$p_{s,S-i}(x) = \int_0^x f^{*(i-1)}(u)\,[K_1(x) - K_1(u)]\,f(x-u)\,du, \quad \text{for } 1 \leq i \leq s. \tag{22}$$

The argument for (22) is as follows. In time (o,u), i-1 demands occur; after u the order for S-s units materialises and the next demand epoch is at x.

Consider the situation when the level of inventory becomes zero, after which a number of demands are lost to the system. The replenishment was made before x, and x is the first demand epoch after the replenishment. Then

$$p_{s,S-s-1}(x) = \int_0^x \int_u^x f^{*s}(u)\ m(v-u)\,[K_1(x)-K_1(v)]\ f(x-v)\ dv\ du \qquad (23)$$

where m(v-u) represents the renewal function of lost demands during (u,v). The probability of first return to s at time x starting from s due to the order made by the first server S_1 is

$$p_{s,s}(x) = \int_0^x \sum_{j=S-s-1}^{S-1} p_{s,j}(u)\ f^{*(j-s)}(x-u)\ du\ . \qquad (24)$$

Consider another situation when both the servers are present. As soon as the inventory level becomes s an order is placed by the server S_1 after which he takes rest. The order materialises after some time and due to demands that occur, the inventory level decreases and becomes s. By this time the first server might not have returned after rest. Then the second server S_2 will order for a quantity of 'a' units. In this case a transition from s to s+a-i is possible for $1 \leq i \leq s$. This transition probability can be written as

$$\hat{p}_{s,s+a-i}(x) = \int_0^x f^{*(i-1)}(u)[R(x)-R(u)]\, f(x-u)du$$

$$\text{for } 1 \leq i \leq s \qquad (25)$$

where R(.) is the lead time distribution of the quantity 'a'.

When the level of inventory becomes s the order for S-s units is placed by S_1, and then he takes rest. The order materialises after some time and due to demands that occur, the level of inventory decreases and reaches s. If S_1 has not returned upto that time S_2 will order for a quantity of 'a' units. During the lead time s units of demands can occur so that the inventory becomes dry. Further there may be a number of demands lost to the system. Finally the order materialises. Thus

$$\hat{p}_{s,a-1}(x) = \int_0^x \int_u^x f^{*s}(u)\ m(v-u)\ [R(x)-R(v)]$$

$$f(x-v)dv\ du\ . \qquad (26)$$

The first return transition probability from s to s due to the order made by S_2 is

$$\hat{p}_{s,s}(x) = \int_0^x \sum_{j=a-1}^{s+a-1} \hat{p}_{s,j}(u)\ f^{*(j-s)}(x-u)du\ . \qquad (27)$$

Let us denote by $p_{S_1,S_2}(x)$ the transition probability density function of the time between the epoch at which S_1 goes for rest and the epoch at which S_2 goes for rest. Then

$$p_{S_1,S_2}(x) = G_1(x) \int_0^x \sum_{i=S-s-1}^{S-1} p_{s,i}(u)\ f^{*(i-s)}(x-u)du$$

$$+ \int_0^x \sum_{i=S-s-1}^{S-1} \left\{ p_{s,i} * f^{*(i-s)} * \sum_{n=1}^{\infty} \left[\sum_{j=a-1}^{s+a-1} \hat{p}_{s,j} * f^{*(j-s)} \right]^n \right\}(u)$$

$$[\ G_1(x) - G_1(u)]\ \sum_{j=a-1}^{s+a-1} [\ \hat{p}_{s,j} * f^{*(j-s)}]\ (x-u)du\ . \qquad (28)$$

Equation (28) can be explained as follows. The first term represents that at time zero (the time at which the inventory level goes to s) S_1 goes for rest and the order given by S_1 materialises some time later. The server S_1 returns to the system before x and in (x-u) there are (i-s) demands. So the level of inventory becomes s; an order is made by S_1 and S_2 goes for rest. In the second term we consider some more possibilities that the order made by S_1 is materialised. Due to demands that occur, the level of inventory decreases to s. But S_1 has not returned by that time. So S_2 gives an order for 'a' units and due to demands that occur the level of inventory decreases and becomes s. Then S_2 makes an order for 'a' units. This process repeats a number of times upto time u. After u the server S_1 returns to the system and the level of inventory becomes s at x. Then an order is made by S_1 and S_2 takes rest. By a similar argument we can write the transition time probability density function $p_{S_2,S_1}(x)$ of the time between the epoch at which S_2 goes for rest and the epoch at which S_1 goes for rest.

$$p_{S_2,S_1}(x) = H_1(x) \int_0^x \sum_{i=S-s-1}^{S-1} p_{s,i}(u) f^{*(i-s)}(x-u)du$$

$$+ \int_0^x \sum_{i=S-s-1}^{S-1} \left\{ p_{s,i} * f^{*(i-s)} * \sum_{n=1}^{\infty} \left[\sum_{j=S-s-1}^{S-1} p_{s,j} * f^{*(j-s)} \right]^n \right\}(u)$$

$$[H_1(x) - H_1(u)] \sum_{j=S-s-1}^{S-1} [p_{s,j} * f^{*(j-s)}](x-u)du \,. \qquad (29)$$

2.2 SYSTEM SIZE PROBABILITIES

Let $X(t)$ denote the inventory level at time t and define $A_i(t) = \Pr\{X(t) = i \mid X(o) = S\}$, $i = 0,1,2,..S$. That is, $A_i(t)$ is the probability that the inventory level at time t is i given that at time zero the inventory level is S. Now, the probability that the inventory level is S at time t given that at time zero the level is S,

$$A_S(t) = \bar{F}(t) + \int_0^t \left\{ f^{*(S-s)} * \sum_{n=0}^{\infty} [p_{S_1,S_2} * p_{S_2,S_1}]^n \right\}(u) \; \bar{F}(t-u) \; K_1(t-u)du$$

$$+ \int_0^t \left\{ f^{*(S-s)} * \sum_{n=0}^{\infty} [p_{S_1,S_2} * p_{S_2,S_1}]^{n} * p_{S_1,S_2} \right\} (u)$$

$$\bar{F}(t-u)\ K_1(t-u)\ du \ + \int_0^t \left\{ f^{*(S-s)} * \sum_{n=1}^{\infty} [p_{S_1,S_2} * p_{S_2,S_1}]^{n} * \right.$$

$$\left. p_{S_1,S_2} * [\sum_{n=1}^{\infty} p^{n}_{s,s} \bar{H}] \right\} (u) \int_0^{t-u} p_{s,s}(v)\bar{F}(t-u-v)$$

$$K_1(t-u-v)\ dv\ du\ . \qquad (30)$$

The first term of (30) says that no demand occurs upto time t so that the level is S itself at time t. The second term represents that (S-s) demands occur and thereafter there are a number of transitions from S_1 to S_2 and S_2 to S_1 upto time u after which no demand occurs and replenishment was made before t. In the third term we consider one more possibility that after a number of transitions from S_1 to S_2 and S_2 to S_1, another transition takes place from S_1 to S_2 before time u. In the fourth term we consider the situation where after the transition from S_1 to S_2 there are so many s to s cycles upto time u and S_2 is absent during this time. Thereafter a transition from s to s takes place in time v. After this no demand occurs and the replenishment takes place. Next we have

$$A_i(t) = [F^{*(S-i)}(t)-F^{*(S-i+1)}(t)] + \int_0^t \Big\{ f^{*(S-s)} *$$

$$\sum_{n=0}^{\infty} [p_{S_1,S_2} * p_{S_2,S_1}]^n \Big\}(u) \int_0^{t-u} \sum_{j=\max(i,S-s-1)}^{S-1} p_{s,j}(v)$$

$$[F^{*(j-i)}(t-u-v)-F^{*(j-i+1)}(t-u-v)]dv\ du + \int_0^t \Big\{ f^{*(S-s)} *$$

$$\sum_{n=0}^{\infty} [p_{S_1,S_2} * p_{S_2,S_1}]^n * p_{S_1,S_2} \Big\}(u) \int_0^{t-u} \sum_{j=\max(i,S-s-1)}^{S-1} p_{s,j}(v)$$

$$[F^{*(j-i)}(t-u-v) - F^{*(j-i+1)}(t-u-v)]\ dv\ du$$

$$+ \int_0^t \left\{ f^{*(S-s)} * \sum_{n=1}^{\infty} [p_{S_1,S_2} * p_{S_2,S_1}]^n * p_{S_1,S_2} * [\sum_{n=1}^{\infty} p^n_{s,s} \bar{H}_1] \right\}(u)$$

$$\int_0^{t-u} p_{s,s}(v) \int_0^{t-u-v} \sum_{j=\max(i,S-s-1)}^{S-1} p_{s,j}(w) [F^{*(j-i)}(t-u-v-w)$$

$$- F^{*(j-i+1)}(t-u-v-w)] \, dw \, dv \, du, \text{ for } s+a \leq i \leq S-1 . \qquad (31)$$

The second and third terms of (31) are obtained by similar arguments as that given for (30) upto epoch u; after that we consider a transition from s to j and in the fourth term also we include the possibility of this transition after the epoch v.

Further we have

$$A_{s+a}(t) = [F^{*(S-s-a)}(t) - F^{*(S-s-a+1)}(t)]$$

$$+ \int_0^t \left\{ f^{*(S-s)} * \sum_{n=o}^{\infty} [p_{S_1,S_2} * p_{S_2,S_1}]^n \right\}(u) \int_0^{t-u} \sum_{j=S-s-1}^{S-1} p_{s,j}(v)$$

$$[F^{*(j-s-a)}(t-u-v)- F^{*(j-s-a+1)}(t-u-v)]dv\ du + \int_0^t \Big\{ f^{*(S-s)} *$$

$$\sum_{n=o}^{\infty} [p_{S_1,S_2} * p_{S_2,S_1}]^n * p_{S_1,S_2} \Big\} (u) \int_0^{t-u} \sum_{j=S-s-1}^{S-1} p_{s,j}(v)$$

$$[F^{*(j-s-a)}(t-u-v)-F^{*(j-s-a+1)}(t-u-v)]dv\ du + \int_0^t \Big\{ f^{*(S-s)} *$$

$$\sum_{n=1}^{\infty} [p_{S_1,S_2} * p_{S_2,S_1}]^n * p_{S_1,S_2} * [\sum_{n=1}^{\infty} p_{s,s}^n \bar{H}_1] \Big\} (u)$$

$$\int_0^{t-u} p_{s,s}(v) \int_0^{t-u-v} \sum_{j=S-s-1}^{S-1} p_{s,j}(w)\ [F^{*(j-s-a)}(t-u-v-w)$$

$$- F^{*(j-s-a+1)}(t-u-v-w)]\ dw\ dv\ du + \int_0^t \Big\{ f^{*(S-s)} * \sum_{n=o}^{\infty} [p_{S_1,S_2} *$$

$$p_{S_2,S_1}]^n \Big\} (u) \int_0^{t-u} p_{s,s}(v) \int_0^{t-u-v} \sum_{n=o}^{\infty} \hat{p}_{s,s}^{\,n}(w)$$

$$\bar{G}_1(t-u-v-w)\ \bar{F}(t-u-v-w)\ R(t-u-v-w)\ dw\ dv\ du \qquad (32)$$

Using similar arguments that lead to (31) we can explain the first four terms of (32). For the fifth term we consider, at epoch u S_1 takes rest and S_2 is available in the system. The order given by S_1 at time u materialises at v; after that a number of s to s transitions due to the order by S_2 upto epoch w takes place since S_1 has not returned to the system upto w. Thereafter no more demand occurs and the order given by S_2 materialises before time t.

The probability that the inventory level is i at time t, for $s+1 \leq i \leq s+a-1$, is given by

$$A_i(t) = [F^{*(S-i)}(t) - F^{*(S-i+1)}(t)] + \int_0^t \Big\{ f^{*(S-s)} * \sum_{n=o}^{\infty} [p_{S_1,S_2} * p_{S_2,S_1}]^n \Big\}(u) \int_0^{t-u} p_{s,s}(v)$$

$$\int_0^{t-u-v} \sum_{n=o}^{\infty} \hat{p}_{s,s}^{\,n}(w)\ \bar{G}_1(t-u-v-w) \int_0^{t-u-v-w} \sum_{j=max(a-1,i)}^{s+a-1} \hat{p}_{s,j}(\eta)$$

$$[F^{*(j-i)}(t-u-v-w-\eta) - F^{*(j-i+1)}(t-u-v-w-\eta)]\ d\eta\ dw\ dv\ du$$

$$+ \int_0^t \left\{ f^{*(S-s)} * \sum_{n=0}^{\infty} [p_{S_1,S_2} * p_{S_2,S_1}]^n * p_{S_1,S_2} \right\} (u)$$

$$\int_0^{t-u} \sum_{n=0}^{\infty} p^n_{s,s}(v)\, \bar{H}_1(t-u-v) \int_0^{t-u-v} \sum_{j=S-s-1}^{S-1} p_{s,j}(w)$$

$$[F^{*(j-i)}(t-u-v-w) - F^{*(j-i+1)}(t-u-v-w)]\, dw\, dv\, du$$

$$+ \int_0^t \left\{ f^{*(S-s)} * \sum_{n=0}^{\infty} [p_{S_1,S_2} * p_{S_2,S_1}]^n \right\} (u) \int_0^{t-u} \sum_{j=S-s-1}^{S-1} p_{s,j}(v)$$

$$[F^{*(j-i)}(t-u-v) - F^{*(j-i+1)}(t-u-v)]\, dv\, du \qquad (33)$$

The second and third terms are due to the servers S_2 and S_1 respectively. The fourth term is due to the server S_1 but S_1 is absent in the system. Similarly, we can write $A_i(t)$, $1 \leq i \leq s$, as

$$A_i(t) = \int_0^t f^{*(S-s)}(u)[F^{*(s-i)}(t-u) - F^{*(s-i+1)}(t-u)]\bar{K}_1(t-u)du$$

$$+ \int_0^t \left\{ f^{*(S-s)} * \sum_{n=0}^{\infty} [p_{S_1,S_2} * p_{S_2,S_1}]^n \right\}(u) \int_0^{t-u} p_{s,s}(v)$$

$$\int_0^{t-u-v} \sum_{n=0}^{\infty} \hat{p}_{s,s}^{\,n}(w) \bar{G}_1(t-u-v-w)\ \bar{R}(t-u-v-w)\ [F^{*(s-i)}(t-u-v-w)$$

$$- F^{*(s-i+1)}(t-u-v-w)] dw\ dv\ du + \int_0^t \left\{ f^{*(S-s)} * \right.$$

$$\left. \sum_{n=0}^{\infty} [p_{S_1,S_2} * p_{S_2,S_1}]^n * p_{S_1,S_2} \right\}(u) \int_0^{t-u} \sum_{n=0}^{\infty} p_{s,s}(v)$$

$$\bar{H}_1(t-u-v)\ \bar{K}_1(t-u-v)\ [F^{*(s-i)}(t-u-v) - F^{*(s-i+1)}(t-u-v)] dv\ du$$

$$+ \int_0^t \left\{ f^{*(S-s)} * \sum_{n=1}^{\infty} [p_{S_1,S_2} * p_{S_2,S_1}]^n \right\}(u)\ \bar{K}_1(t-u)$$

$$[F^{*(s-i)}(t-u) - F^{*(s-i+1)}(t-u)]\ du \qquad (34)$$

In equation (34) we consider the situation where the replenishment was not made for the last order. Then we get the level of inventory to be less than or at the most equal to s. Otherwise the level can be greater than s.

The probability that the inventory level is zero at time t is given by

$$A_0(t) = \int_0^t f^{*(S-s)}(u) \; \bar{K}_1(t-u) \; [F^{*s}(t-u) - F^{*s+1}(t-u)]$$

$$+ \int_0^t \left\{ f^{*(S-s)} * \sum_{n=0}^{\infty} [p_{S_1,S_2} * p_{S_2,S_1}]^n \right\}(u) \int_0^{t-u} p_{s,s}(v) .$$

$$\int_0^{t-u-v} \sum_{n=0}^{\infty} \hat{p}_{s,s}^{\,n}(w) \; \bar{G}_1(t-u-v-w) \; \bar{R}(t-u-v-w) \; [F^{*s}(t-u-v-w)$$

$$- F^{*(s+1)}(t-u-v-w)] \; dw \; dv \; du + \int_0^t \left\{ f^{*(S-s)} * \right.$$

$$\left. \sum_{n=0}^{\infty} [p_{S_1,S_2} * p_{S_2,S_1}]^n * p_{S_1,S_2} \right\}(u) \int_0^{t-u} \sum_{n=0}^{\infty} p_{s,s}^{\,n}(v)$$

$$\bar{H}_1(t-u-v) \; \bar{K}_1(t-u-v) \; [F^{*s}(t-u-v) - F^{*(s+1)}(t-u-v)] \; dv \; du$$

$$+ \int_0^t \left\{ f^{*(S-s)} * \sum_{n=1}^{\infty} [p_{S_1,S_2} * p_{S_2,S_1}]^n \right\}(u)$$

$$\bar{K}_1(t-u)\ [F^{*s}(t-u) - F^{*(s+1)}\ (t-u)]\ du\ . \qquad (35)$$

3. CONCLUDING REMARKS AND SUGGESTIONS FOR FURTHER STUDY

To sum up we have studied in this thesis, the role of rest periods in queueing systems like GI/M/1, $M/M^{a,b}/1$, erlangian service model with feedback and a buffer model. Also we introduced the concept of rest in inventory models.

We suggest further study in these lines. Assume that the server takes rest after serving a random number of units where this random number depends on the number of units in the system. Investigation of the stationary behaviour, waiting time etc. seems to be hard but quite interesting.

Fuhrmann and Cooper [24] studied the decomposition property in the M/G/1 queueing system with generalised vacations. Doshi [17] investigated the decomposition property in the GI/G/1 queueing systems with exhaustive service. Investigation of the decomposition property in GI/G/1 system without exhaustive service may be interesting and it is more general one than considered in [17].

Consider an inventory system with more than one ordering level and more than one server. Assume that some of the servers takes rest when the inventory level is less than a prescribed quantity. Here again one can investigate the transition probabilities and stock level probabilities.

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