ANALYSIS

DISCRETE FUNCTION THEORY

A STUDY OF DISCRETE PSEUDOANALYTIC FUNCTIONS

BY

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THESIS

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CERTIFICATE

This is to certify that this thesis is a bona fide record of work by Smt.Mercy K.Jacob, carried out in the Department of Mathematics and Statistics, University of Cochin, Cochin 682022 under my supervision and guidance and that no part thereof has been submitted for a degree in any other University.

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CONTENTS

SYNOPSIS

There is a recent trend to describe physical phenomena without the use of infinitesimals or infinites. This has been accomplished replacing differential calculus by the finite difference theory. Discrete function theory was first introduced in 1941. This theory is concerned with a study of functions defined on a discrete set of points in the complex plane. The theory was extensively developed for functions defined on a Gaussian lattice. In 1972 a very suitable lattice H: $\{(\pm q^m x_o, \pm q^n y_o),\ x_o > 0,\ y_o > 0,\ 0 < q < 1,\ m, n \in Z\}$ was found and discrete analytic function theory was developed. Very recently some work has been done in discrete monodiffric function theory for functions defined on H.

The theory of pseudoanalytic functions is a generalisation of the theory of analytic functions. When the generator becomes the identity, ie., (1, i) the theory of pseudoanalytic functions reduces to the theory of analytic functions. Though the theory of pseudoanalytic functions plays an important role in analysis, no discrete theory is available in literature. This thesis is an attempt in that direction. A discrete pseudoanalytic theory is derived for functions defined on H.

In the first chapter an outline of the theory of pseudoanalytic functions in the classical continuous case is given, also emphasising the importance of discretisation. With a historical survey of the discrete function theory, the present developments have been stated. A gist of the results established in the thesis is also given.

The second chapter deals with the definitions of Hölder-type discrete functions and generating vectors. Their properties have been examined. Using q-difference equations modulo-g where g is a generating vector, definitions of discrete g-pseudoanalytic functions of the first and second kind are given and their properties studied. We denote the class of all discrete g-pseudoanalytic functions of the first kind in a discrete domain D by ${}_1P_D(g)$ and that of second kind by ${}_2P_D(g)$. The real and imaginary parts of the elements of ${}_2P_D(g)$ satisfy a linear elliptic system of partial q-difference equations of the second order with Hölder-type coefficients.

Concepts of g and p_g-integration in the discrete system are introduced and their properties examined. It is established in chapter 3 that the g-integral of a discrete function is an element of ${}_1P_D(g)$ and p_g-integral of a discrete function is an element of ${}_2P_D(g)$.

Solutions of partial q-difference equations modulo-g and an analogue of Beltrami's equations are discussed. Properties of solutions thus obtained are established through examples in the fourth chapter.

The discrete g-derivative of an element of ${}_1P_D(g)$ is not in general an element of ${}_1P_D(g)$. However there does exist a generating vector $g^{(1)}$ such that the discrete g-derivative is an element of ${}_1P_D(g^{(1)})$. We call $g^{(1)}$, a successor of g and g, a predecessor of $g^{(1)}$. It is shown that if $g = [g_1\ g_2]$ then $[\frac{g_1}{i}\ \frac{g_2}{i}]$ is a successor of g. Also any generating vector equivalent to $[\frac{g_1}{i}\ \ \frac{g_2}{i}]$ is also a successor of g. We have discussed the concept of a generating sequence and the periodicity of the generating sequence. It is established that if $w \in {}_1P_D(g)$ is not a g-pseudoconstant then g can be embedded in a generating sequence of minimal period one if and only if the first component of the generating vector is equal to the product of the second component and a function of y alone. It is also established that any generating vector g can be embedded in a generating sequence of minimal period 2.

A product of two elements of ${}_1P_D(g)$ is not in general an element of ${}_1P_D(g)$. In the last chapter we have found some sufficient conditions under which w^2, aw + b are elements of ${}_1P_D(g)$, where $w \varepsilon {}_1P_D(g)$, a, b are complex constants. Denoting aw + b by w* and taking the power $(w^*)^2$ we obtain sufficient conditions for a quadratic to be an element of ${}_1P_D(g)$. Also we obtain sufficient conditions for a cubic and in general an n^{th} degree polynomial to be an element of ${}_1P_D(g)$.

In conclusion some applications and further problems of study are suggested.

A bibliography containing 70 references is also listed.

STATEMENT

This thesis contains no material which has been accepted for the award of any other degree or diploma in any University and, to the best of my knowledge and belief, it contains no material previously published by any other person, except where due reference is made in the text of the thesis..

MERCY.K.JACOB.

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CHAPTER I

INTRODUCTION

This thesis is an attempt to formulate a basic theory for discrete pseudoanalytic functions defined on a geometric lattice of the form

$$\left\{(\pm q^m x_o, \pm q^n y_o),\ x_o > 0,\ y_o > 0,\ 0 < q < 1, m, n \in Z\right\}.$$

Accordingly, a brief summary of the work done in the fields of the theory of pseudoanalytic functions, generalised analytic functions, discrete analytic functions, q-difference functions and q-analytic functions is sketched here and a gist of the results obtained is given.

1. Theory of pseudoanalytic functions

In this section we give a brief outline of the theory of pseudoanalytic functions introduced by Bers [1]. The theory of pseudoanalytic functions was developed from the point of view of partial differential equations, much of the motivation being provided by problems in mechanics of continua.

A linear partial differential equation:

$$a_{11}(x,y)\ \alpha_{xx} + 2a_{12}(x,y)\ \alpha_{xy} + a_{22}(x,y)\ \alpha_{yy}$$

$$+ a_1(x,y)\ \alpha_x + a_2(x,y)\ \alpha_y + a(x,y)\alpha = 0 \qquad 1(1)$$

is called elliptic if by a transformation of the independent variables it can be brought into the form:

$$\alpha_{xx} + \alpha_{yy} + b_1(x,y)\ \alpha_x + b_2(x,y)\ \alpha_y + b_3(x,y)\alpha = 0 \qquad 1(2)$$

where b_1, b_2, b_3 are continuous.

The simplest elliptic equation is the Laplace equation

$$\alpha_{xx} + \alpha_{yy} = 0 \qquad 1(3)$$

The theory of this equation is well developed as solutions of 1(3) are the real or imaginary parts of analytic functions. Bers developed the theory of pseudoanalytic functions which bears the same relationship to the general elliptic equation as classical function theory does to Laplace's equation. Vekua [1] established the theory of 'generalised analytic functions' that satisfy a nonhomogeneous elliptic system of linear partial differential equations of the second order.

In classical analysis a function $w(z)$ satisfies a Hölder-condition with constant K and exponent α, $0 < \alpha \leq 1$, if $|w(z) - w(z_o)| \leq K\,|z - z_o|^{\alpha}$ at z_o. The function is said to be Hölder-continuous at z_o if it satisfies a Hölder-condition at z_o.

Suppose that F and G are two Hölder-continuous functions satisfying the condition

$$\mathrm{Im}(\overline{F(z)}G(z)) > 0 \text{ in } D. \qquad 1(4)$$

If F and G are Hölder-continuous in a domain D and 1(4) is satisfied in D then (F,G) is called a generating pair in D.

We know that if $w(z_o)$ is a complex constant then it is of the form $\gamma.1 + \mu.i$ where γ and μ are real constants. The theory of pseudoanalytic functions is based on assigning the part played by 1 and i by two Hölder-continuous functions satisfying the condition 1(4).

It follows from 1(4) that for every z_o in D we can find unique real constants γ_o, μ_o such that

$$w(z_o) = \gamma_o F(z_o) + \mu_o G(z_o)$$

We say that $w(z)$ possesses at z_o the (F,G) derivative $\dot{w}(z_o)$

if the (finite) limit

$$\dot{w}(z_0) = \lim_{z \to z_0} \frac{w(z) - \gamma_0 F(z) - \mu_0 G(z)}{z - z_0} \tag{1(5)}$$

exists.

A function $w(z)$ will be called regular (F,G)-pseudoanalytic of the first kind in a domain D (or simply pseudoanalytic) if $\dot{w}(z)$ exists everywhere in D.

If $\dot{w}(z_0)$ exists then at z_0, w_z and $w_{\bar{z}}$ exist and w at z_0 satisfy the equations

$$w_{\bar{z}} = aw + b\bar{w} \tag{1(6)}$$

$$\dot{w} = w_z - Aw - B\bar{w} \tag{1(7)}$$

where

$$a = \frac{-(\bar{F}G_{\bar{z}} - F_{\bar{z}}\bar{G})}{F\bar{G} - \bar{F}G} \tag{1(8)}$$

$$b = \frac{FG_{\bar{z}} - F_{\bar{z}}G}{F\bar{G} - \bar{F}G} \tag{1(9)}$$

$$A = \frac{-(\bar{F}G_z - F_z\bar{G})}{F\bar{G} - \bar{F}G} \tag{1(10)}$$

and

$$B = \frac{FG_z - F_zG}{F\bar{G} - \bar{F}G} \tag{1(11)}$$

We denote them by $a_{(F,G)}$, $b_{(F,G)}$, $A_{(F,G)}$ and $B_{(F,G)}$ respectively.

Equation 1(6) is equivalent to the real system

$$\left.\begin{aligned} u_x - v_y &= \alpha_{11}u + \alpha_{12}v \\ u_y + v_x &= \alpha_{21}u + \alpha_{22}v \end{aligned}\right\} \qquad 1(12)$$

with real Hölder continuous $\alpha_{ij}(x,y)$. In a certain sense every elliptic equation 1(1) is equivalent to a system of the form 1(12).

If $w = \gamma F + \mu G$ is pseudoanalytic of the first kind in a domain D then $\gamma + i\mu$ is called pseudoanalytic of the second kind in D.

The class of all pseudoanalytic functions satisfies many of the properties of the class of analytic functions, however product of two pseudoanalytic functions and the (F,G) derivative of a (F,G) pseudoanalytic function are not in general (F,G) pseudoanalytic functions.

If the (F,G) derivative of w is not pseudoanalytic then a generating pair (F_1,G_1) can be found so that the derivative is (F_1,G_1)-pseudoanalytic. (F_1,G_1) is called a successor of (F,G) and (F,G) is called a predecessor of (F_1,G_1).

He proved that (F_1,G_1) is a successor of (F,G) if

$$a_{(F_1,G_1)} = a_{(F,G)} \text{ and } b_{(F_1,G_1)} = -B_{(F,G)}$$

where a, b, B are given by 1(8), 1(9) and 1(11) respectively.

A sequence of generating pairs $\{(F_v,G_v)\}$, $v = 0, \pm 1, \pm 2, \ldots$ is called a generating sequence if (F_{v+1},G_{v+1}) is a successor of (F_v,G_v). $\{(F_v,G_v)\}$ is said to have period $\mu > 0$ if $(F_{v+\mu},G_{v+\mu})$ is equivalent to (F_v,G_v) and non-periodic if no such μ exists. Bers did not study the periodicity problem extensively. He considered only some particular cases. But Protter [1] in his paper discussed the problem in detail. He could find the necessary conditions for a generating vector to be embedded in a generating sequence of a prescribed period, in a non-periodic generating sequence etc.

A basic result in the theory of pseudoanalytic functions is the similarity principle proved by Bers [1]. The similarity principle states that with every pseudoanalytic function w can be associated an analytic function f (and vice-versa). Also it is found that mapping by pseudoanalytic functions of the second kind is quasiconformal.

Vekua [1] developed a more general theory, the theory of generalised analytic functions which are solutions of a non-homogeneous elliptic system of partial differential equations of the first order. Vekua showed that pseudo-analytic functions of the second kind satisfies a Beltrami's system of equations. Generalised Cauchy's theorem, Cauchy's formula, power series etc. were obtained. In 1976 Withalm [1] developed the theory of hyperpseudoanalytic functions.

2. Importance of discretization

The differential character of equations of motion implies that a dynamical system is governed by laws operating with a precision beyond the limits of detection by experiment. This is too much of an assumption. It seems logical to introduce the general principle that all physical phenomena can be described without the use of infinitesimals or infinites. It requires use of finite difference calculus in formulating basic physical laws.

For instance, according to Newton's Laws if at a time t we know the position and velocity of a body the equations predict the situation at time t + dt, but in order to verify the prediction one would be obliged to distinguish between two positions x and x + dx by making two measurements separated by an infinitesimal time interval. But as a matter of fact, we have no desire to find the situation after the time dt, we wish to predict the state within limits, after the finite time interval, and this we do by integrating the equations of motion. Each co-ordinate may be expressed as a continuous function of the time and we may calculate the configuration at any time including infinity.

Ruack [1] feels that this may be accomplished replacing differential calculus by the finite difference theory. In the classical finite difference theory, functions which are often defined only on a discrete set of points are usually treated as functions of a continuous variable. Recently methods have also been devised to treat functions defined only at a discrete set of points in the complex plane. Many eminent mathematicians have developed this theory.

Even though the theory of pseudoanalytic functions plays an important role in analysis no discrete theory for pseudoanalytic functions is available in literature. So we have made an attempt in this direction and introduced a theory, which is applicable to geometric difference functions.

3. Discrete analytic function theory

This basic theory is the study of functions defined only at certain lattice points in the complex plane and the lattice of definition is usually taken to be the set of Gaussian integers. This they called 'discrete analytic function theory' does not need the concept of continuity.

In this theory the concept of a monodiffric function plays an important role which was introduced by Isaacs[2] by modifying monogeneity. Instead of derivative he used the difference quotients both along the real and the imaginary axes. In fact, he defined two types of monodiffric functions. Functions satisfying the equation

$$f(z+1) - f(z) = \frac{f(z+i) - f(z)}{i} \qquad 1(13)$$

are called 'monodiffric functions of the first kind' and those satisfying the equation

$$f(z+1) - f(z-1) = \frac{f(z+i) - f(z-i)}{i} \qquad 1(14)$$

'monodiffric functions of the second kind'. In both the cases the lattice taken is the set of gaussian integers of the form m+in, where m, n are integers. Using these definitions he introduced concepts of discrete contour integrals, residues, powers, polynomials and a convolution which served as an analogue for multiplication, provided one of them was a polynomial.

In 1944 Ferrand [1] introduced the so-called 'preholomorphic function' using the diagonal quotient equality

$$\frac{f(z+1+i) - f(z)}{1+i} = \frac{f(z+i) - f(z+1)}{i-1} \qquad 1(15)$$

which is equivalent to the definition of functions of the second kind 1(14) given by Isaacs.

Later Duffin [1] using the definition of Ferrand could establish convolution products, analytic continuation, entire functions and application to practical problems. In 1970 Deeter and Mastin [1] showed that the solution of a minimum problem in the theory of conformal mapping can be approximated by discrete functions. They also showed that the Bergman kernel function can be approximated, for certain regions, by using these discrete functions. In 1977 Zeilberger [1] developed the theory for functions defined on a n-dimensional lattice. He introduced a certain class of binary operations generalising a binary operation defined by Duffin and Rohrer [1], in the set of solutions of partial difference equations. He could find many interesting results in that direction.

Hayabara [1], Deeter and Lord [1] and MacLeod [1] constructed an operational calculus for discrete analytic functions and studied their properties.

Eminent Russian mathematicians like Abdullaev and Babadžanov [1], Meredov [1] and Fuksman [1] have also made a study of the theory of monodiffric functions of the second kind. Berzsenyi [1,2] studied several interesting convolution integrals and algebraic structures for monodiffric functions.

Duffin [2] defined a rhombic lattice and studied the theory. In 1972 Harman [1] defined a geometric lattice of the form $\left\{(\pm q^m x_o, \pm q^n y_o),\ x_o > 0,\ y_o > 0,\ 0 < q < 1,\ m,n \in Z\right\}$ and developed a discrete q-analytic theory.

4. q-Difference functions

The first mention of a q-difference equation appears to have been made by Laplace in 1773, when he considered a functional equation of the form

$$F(x,\mu(x),\ \mu(qx)) = 0$$

Babbage [1] in 1815 studied the properties of the above equation and in particular he considered the equation

$$f(x) = f(qx)$$

Pincherle [1] in 1880 studied the equation $f(x) = f(qx)$ and obtained a solution of the form

$$f(x) = x^{\alpha-\beta} \prod_{n=0}^{\infty} \frac{(1-q^{\alpha+n}x)\ (1-q^{1-\alpha-n}x^{-1})}{(1-q^{\beta+n}x)\ (1-q^{1-\beta-n}x^{-1})} \qquad 1(16)$$

which is called q-periodic function. This function plays the role of an arbitrary constant in q-difference equations.

Jackson [2] studied the theory of q-difference equations extensively. In 1910 he introduced the concept of q-integration which he defined as the inverse of the q-difference operator

$$\theta_x f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad q \neq 1$$

However, it was only in 1949-51 that a real interest in q-integration was revived. Hahn [2] in 1949 and Jackson [1] in 1951 studied the fundamental properties of the inverse operation

$$\theta_x^{-1} f(x) = \frac{1}{(1-q)} \int f(x)\, d(q;x)$$

and showed that in the limiting case ie., when $q \rightarrow 1$, the basic integral is reduced to the ordinary Riemannian integral. The definite q-integrals are defined by

$$\int_0^x \theta_x f(x) d(q;x) = f(x) - f(0)$$

$$\int_x^\infty \theta_x f(x) d(q;x) = f(\infty) - f(x)$$

where

$$\int_a^b = \int_0^b \quad \int_0^a$$

If $\theta F(x) = f(x)$, then

$$F(0) - F(x) = (q-1)x \sum_{j=0}^{\infty} q^{j} f(q^{j}x),$$

$$F(\infty) - F(x) = (q-1)x \sum_{j=1}^{\infty} q^{-j} f(q^{-j}x)$$

A complete bibliography of the Jackson's work is given by Chaundy [1].

In 1960, Abdi [2] developed the theory of q-Laplace transforms which was used in solving certain q-difference and q-integral equations. He also introduced a bibasic functional equation of the form:

$$a(z)\, f(pz) + b(z)\, f(qz) + c(z)\, f(z) = 0$$

It may be noted that q-difference equations occur in the theory of water waves and have been treated by Williams [1] and Peters [1].

5. <u>q-Analytic function theory</u>

In 1972 Harman [1] developed a discrete analytic theory for geometric difference functions.

He defined a lattice with geometric spacing ie, points of the form $H = \{(\pm q^{m} x_0, \pm q^{n} y_0);$ $0 < q < 1,\ x_0, y_0 > 0,\ m, n \in Z\}$. Functions defined on the points of H are called discrete functions.

Functions satisfying

$$\frac{f(x,y) - f(qx,y)}{(1-q)x} = \frac{f(x,y) - f(x,qy)}{(1-q)iy}$$

where $z = (x,y) \varepsilon H$, he called q-analytic functions.

In his work analogues of contour integrals, Cauchy's integral formula etc. established. A discrete analytic continuation operation $\mathcal{C}$ was devised which enables functions defined on the real axis to be continued into the complex plane as q-analytic functions. This process in fact is an analogue of Taylor's theorem. The continuation operator is used to derive q-analogues of multiplication, of the function z^n; n a non-negative integer. Several results were obtained in connection with the representation of q-analytic function as power series. A factorisation theorem analogous to the fundamental theorem of algebra was obtained for the q-polynomials.

He studied q-analytic solutions of linear q-difference equations with both constant and variable coefficients and obtained some results in conformal mapping.

6. Outline of chapters

In this thesis a discrete pseudoanalytic theory for geometric difference functions is introduced. A brief outline of the basic results of the thesis is given.

Functions are defined on the set $\left\{(\pm q^m x_o, \pm q^n y_o), x_o > 0, y_o > 0, 0 < q < 1, m, n \in Z\right\}$ and a class of functions analogous to Hölder-continuous functions is of special importance in this work.

Thus, discrete functions satisfying the inequality $|f(z) - f(z')| \leq k\,\sigma^{\mu}$ where $z' = (x',y') \in D$, a discrete domain, $z \in N(z')$, $\sigma = (q^{-1} - 1)\max(x',y')$, $0 < \mu \leq 1$ have been called discrete Hölder-type at $z' \in D$. If the above inequality holds for all $z \in D$ such that $N(z) \subset D$, then the function is called discrete Hölder-type in D. We denote the class of such functions by $\mathfrak{H}(D)$. If $g_1, g_2 \in \mathfrak{H}(D)$, then the row vector $g = [g_1\ g_2]$ is called a generating vector in D if $\mathrm{Im}(\overline{g_1}\ g_2) > 0$ throughout.

Definitions of discrete pseudoanalytic functions of the first and second kind over a discrete domain are given. The two classes are respectively denoted as ${}_1P_D(g)$ and ${}_2P_D(g)$. Both ${}_1P_D(g)$ and ${}_2P_D(g)$ form vector spaces over the field of real numbers. The real and imaginary parts of elements of ${}_2P_D(g)$ are solutions of linear elliptic system of partial q-difference equations of second order with Hölder-type coefficients.

Concepts of discrete g and p_g-integration analogous to the integrals of Bers [1] are introduced. Properties of the above integrals are studied. It is shown that g-integral of a discrete function is an element of ${}_1P_D(g)$ and p_g-integral of a discrete function is an element of ${}_2P_D(g)$. Bers was able to establish a generalisation of the Cauchy's integral formula for the pseudoanalytic functions but an analogous result is not obtained in the discrete case.

Making use of Jackson's [3] basic integral the solutions of partial q-difference equations modulo-g and an analogue of a Beltrami's system are obtained. Properties of the solutions are examined and some examples discussed.

We can see that the g-derivative of an element of ${}_1P_D(g)$ is not in general an element of ${}_1P_D(g)$. However there does exist a generating vector $g^{(1)}$ such that the g-derivative is an element of ${}_1P_D(g^{(1)})$. We discuss concepts like successors and predecessors of generating vectors, generating sequences, periodicity of the generating sequences. It is shown that if $w \in {}_1P_D(g)$ is not a g-pseudoconstant then g can be embedded in a generating sequence of minimal period one if and only if the first component of the generating vector equal to the second component and a function of y alone. Bers [1] did not discuss the periodicity problem in detail, but in 1956 Protter [1] studied the problem extensively. He has established the conditions when a generating vector can be embedded in a generating sequence with prescribed minimal periods and a non-periodic generating sequence. In our theory we have established that any generating vector can be embedded in a generating sequence of minimal period 2.

Product of two elements of ${}_1P_D(g)$ is not in general an element of ${}_1P_D(g)$. We have found sufficient conditions under which w^2, $\alpha w + \beta$ where $w \in {}_1P_D(g)$, α, β complex constants, are elements of ${}_1P_D(g)$. Denoting $\alpha w + \beta$ by w^* and taking the powers $(w^*)^2$ and $(w^*)^3$ we can find the sufficient conditions for quadratic and cubic polynomials to be elements of ${}_1P_D(g)$. Thus we believe that under certain conditions on f_1, f_2, α, β etc. an n^{th} degree polynomial will be an element of ${}_1P_D(g)$. No similar work is available in the classical continuous case.

CHAPTER 2

DISCRETE PSEUDOANALYTIC FUNCTIONS AND THEIR PROPERTIES

Discrete function theory is a theory of complex valued functions defined at a discrete set of points in the complex plane. Harman [1] used a particular lattice suitable for q-difference functions. Although the theory of pseudo-analytic functions plays an important role in analysis, yet no discrete analogue is available in literature. In this chapter a class of functions analogous to pseudoanalytic functions is defined and its properties studied.

1. The lattice

q-Difference functions of a complex variable are usually defined on a set of points of the form H :

$$\left\{(\pm q^m x_o, \pm q^n y_o),\ x_o > 0,\ y_o > 0,\ 0 < q < 1,\ m, n \in Z\right\} \quad 2(1)$$

For convenience only the first quadrant of the complex plane is considered. Extension to the other three quadrants can be treated as in Appendix 1 of Harman [1].

We define the discrete plane H^1 with respect to some fixed point $z_o = (x_o, y_o)$ in the first quadrant as the set of lattice points:

$$H^1 = \left\{(q^m x_o,\ q^n y_o),\ m, n \in Z\ x_o > 0,\ y_o > 0,\ 0 < q < 1\right\}$$ and z_o will be called the origin of H^1.

2(2)

In the sequel the following notation is used.

Suppose $z = (x,y) \varepsilon H^1$. Then

$$A(z) = \{(qx,y),\ (qx,qy),\ (x,qy),\ (q^{-1}x,\ qy),\ (q^{-1}x,y),\ (q^{-1}x,\ q^{-1}y),\ (x,q^{-1}y),\ (qx,q^{-1}y)\}$$

consists of all points adjacent to z. 2(3)

$$N(z) = \{(qx,y),\ (x,qy),\ (q^{-1}x,y),\ (x,q^{-1}y)\}$$

the set of all points directly adjacent to z. 2(4)

And

$$P(z) = \{(qx,qy),\ (q^{-1}x,qy);\ (q^{-1}x,q^{-1}y),\ (qx,q^{-1}y)\}$$

is the set of points diagonally adjacent to z. 2(5)

We see that $A(z) = N(z) \cup P(z)$. The set of points $S(z) = \{(x,y),\ (qx,y),\ (qx,qy),\ (x,qy)\}$ and $T(z) = \{(x,y),\ (qx,y),\ (x,qy)\}$ are respectively called the tetrad and triad of z. Figure 1 shows the above notations. 2(6)

Any union of tetrads is defined as a discrete domain and is denoted by D ie. $D = \bigcup_{i=1}^{n} S(z_i)$ where n can be infinite. A domain is said to be bounded if we can find some $k > 0$ such that $\max(|x|, |y|) < k$ for all $z \varepsilon D$. 2(7)

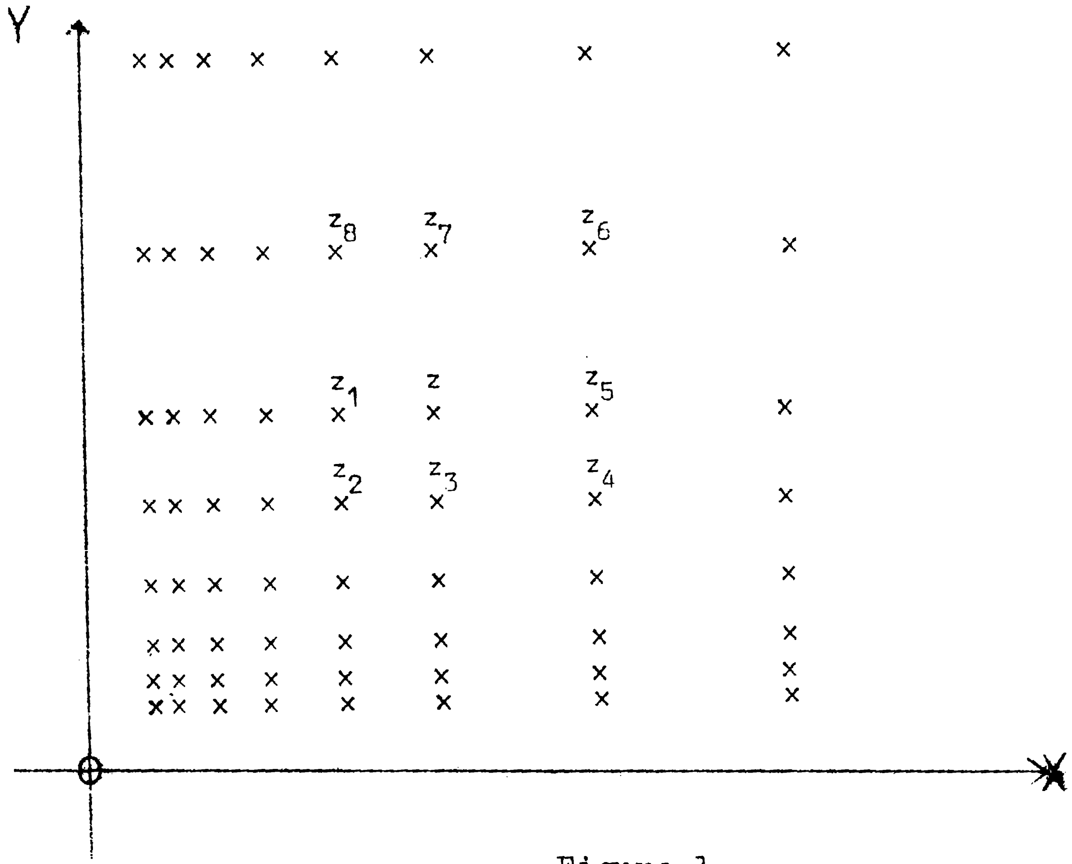

Figure 1

Take $z = (x,y) \in H^1$

$A(z) = \{z_1, z_2, z_3, z_4, z_5, z_6, z_7, z_8\}$

$N(z) = \{z_1, z_3, z_5, z_7\}$

$P(z) = \{z_2, z_4, z_6, z_8\}$

$S(z) = \{z, z_1, z_2, z_3\}$

$T(z) = \{z, z_1, z_3\}$

Since we are considering only H^1, x and y will always be greater than zero. Therefore in that case we need take only max $(x,y) < k$.

A discrete curve in H^1 z_1 to z_n is denoted by the ordered sequence

$$C \equiv < z_1, z_2, \ldots z_i, \; z_{i+1}, \ldots z_n > \qquad 2(8)$$

where z_i, z_{i+1}, $i = 1,2,\ldots n-1$ are directly adjacent points in H^1. If $z_i \neq z_j$ for $i \neq j$, then the discrete curve is said to be simple. 2(9)

If $C = \langle z_1, z_2, \ldots z_n \rangle$ is simple and $z_1 = z_n$ then C is called a simple closed curve. 2(10)

Boundary and interior points

Definition 2(1)

$z \in D$ is an interior point of D if $A(z) \subset D$ 2(11)

Note 2(1)

It can be seen that the minimum number of interior points will be one and the domain will contain nine points. For eg. the domain having z as an interior point will be $A(z) \cup \{z\}$.

Definition 2(2)

All points $z \in D$ which are not interior points of D are called boundary points of D. 2(12)

The set of all interior points of D is called its interior region and denoted by Int(D). 2(13)

Boundary of a domain

Let D be a discrete domain. Boundary of D is defined as $B(D) = D - Int(D)$. 2(14)

In general the boundary of a domain is the union of discrete curves B_1, B_2,...., B_i,....where each B_i contains only boundary points. Discrete curves comprising only boundary points are called boundary curves.

Figures 2 and 3 illustrate the above notations.

Suppose that $z_1, z_2 \in H^1$. Two tetrads $S(z_1)$ and $S(z_2)$ are contiguous if $S(z_1) \cap S(z_2)$ is non empty. 2(15)

Note 2(2)

It follows that if $S(z_1)$ and $S(z_2)$ are contiguous then $z_2 \in A(z_1)$ where $A(z)$ is given by 2(3).

Note 2(3)

If $S(z)$ is a tetrad then the total number of tetrads contiguous to $S(z)$ is 8.

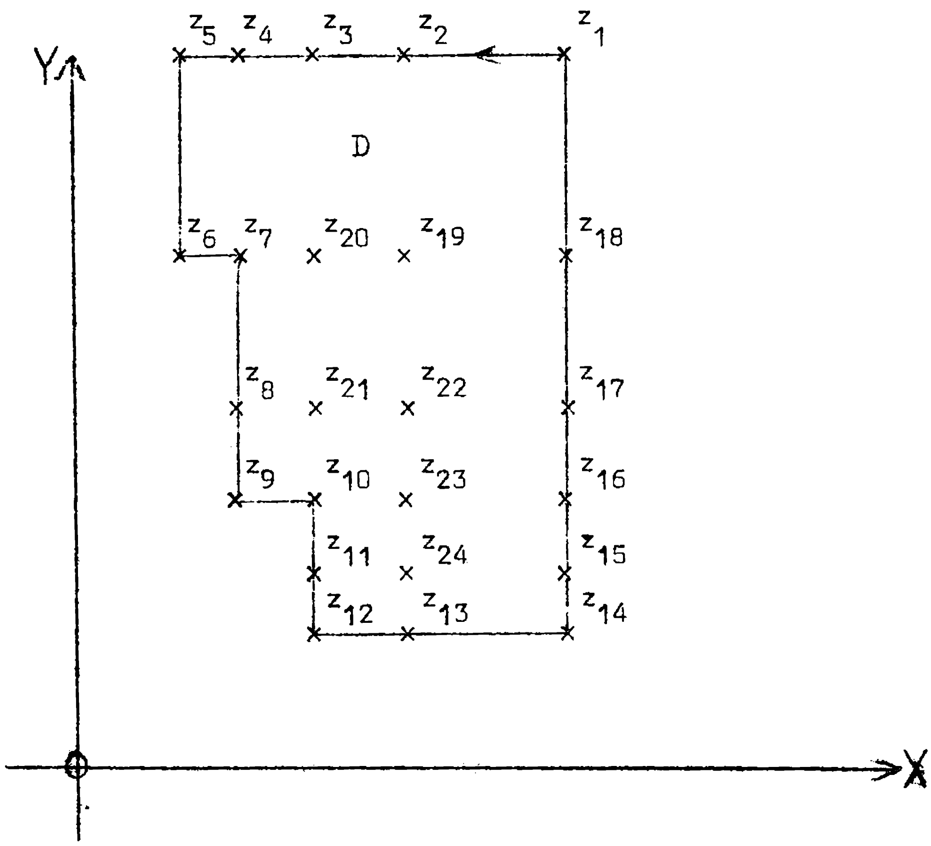

Figure 2

$$D = \{z_1, z_2, z_3, \ldots z_{24}\}$$

$$Int(D) = \{z_{19}, z_{20}, z_{21}, z_{22}, z_{23}, z_{24}\}$$

$$B(D) = \{z_1, z_2, z_3, z_4, z_5, z_6, z_7 \ldots z_{18}\}$$

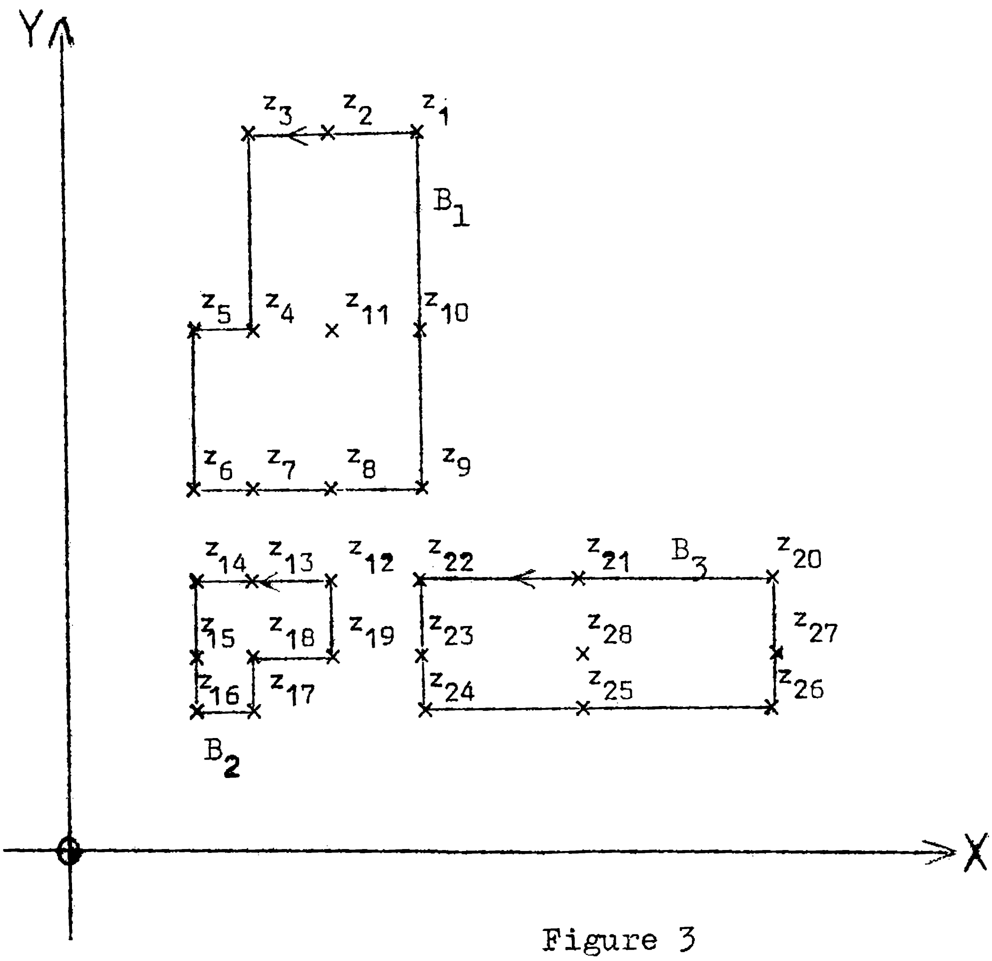

Figure 3

$$D = \{z_1, z_2, z_3, \ldots z_{28}\}$$

$$B_1 = \langle z_1, z_2, z_3, z_4, \ldots z_{10} \rangle$$

$$B_2 = \langle z_{12}, z_{13}, z_{14}, z_{15}, \ldots z_{19} \rangle$$

$$B_3 = \langle z_{20}, z_{21}, z_{22}, z_{23}, \ldots z_{27} \rangle$$

$$B(D) = B_1 \cup B_2 \cup B_3$$

$$\mathrm{Int}(D) = \{z_{11}, z_{28}\}$$

Definition 2(3)

Denote $S(z_i) = S_i$

A connected domain D is a collection of tetrads $\{S_1, S_2, \ldots S_i, S_{i+1}, \ldots S_n\}$ such that S_i and S_{i+1} are contiguous and S_1 is not necessarily equal to S_n.

If $S_1 = S_n$ then the domain is said to be closed. By suitable arrangement one can be made contiguous to another.

Result 2(1)

Let D_1 and D_2 be two connected domains. Then the union is connected if the intersection of D_1 and D_2 is non-empty.

Definition 2(4)

Let D be a connected domain. D is said to be singly connected if D is bounded by only one closed boundary curve.

For eg. H^1 is a singly connected domain. It is bounded by only one closed limiting boundary curve

$$\lim_{m,n \to \pm\infty} \bigcup (q^m x_0, q^n y_0)$$

Note 2(4)

Union of two singly connected domains whose intersection is non-empty need not be singly connected (See Fig.4).

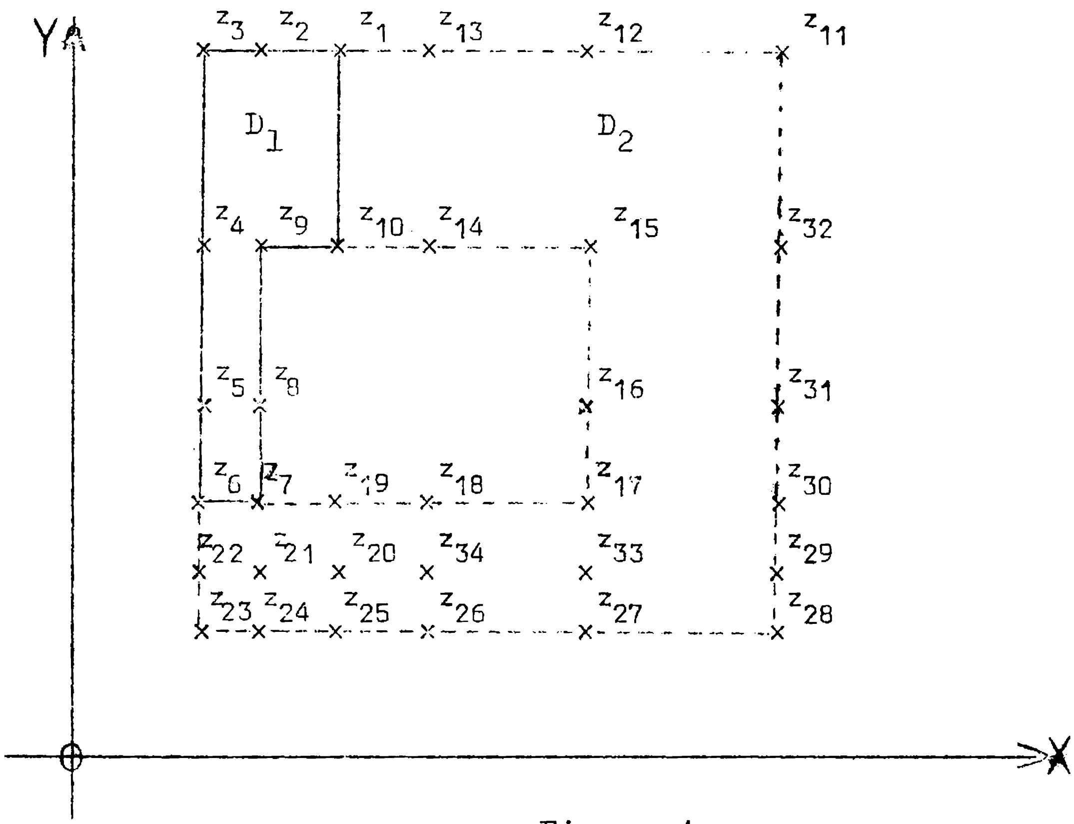

Figure 4

$$D_1 = \{z_1, z_2, \ldots z_{10}\}$$

$$D_2 = \{z_{11}, z_{12}, z_{13}, z_1, z_{10}, z_{14}, z_{15}, z_{16}, z_{17}, z_{18}, z_{19}, z_7, z_6, z_{20}, z_{21}, \ldots z_{34}\}$$

D_1, D_2 are singly connected domains and $D_1 \cap D_2$ is nonempty. But $D_1 \cup D_2$ is not singly connected.

Definition 2(5)

A connected domain D is said to be doubly connected if D is bounded by two closed boundary curves. For example, the domain enclosed between the closed curves B_1 and B_2 where

$$B_1 = \langle (x,y),(qx,y),\ldots.(q^8x,y),(q^8x,qy),(q^8x,q^2y), \ldots.(q^8x,q^6y),(q^7x,q^6y),(q^6x,q^6y),\ldots(x,q^6y), (x,q^5y),(x,q^4y),\ldots(x,y)\rangle \quad \text{and}$$

$$B_2 = \langle (q^3x,q^2y),(q^4x,q^2y),(q^5x,q^2y),(q^5x,q^3y),(q^5x,q^4y), (q^4x,q^4y),(q^3x,q^4y),(q^3x,q^3y),(q^3x,q^2y)\rangle$$

is doubly connected.

Definition 2(6)

A connected domain is said to be multiply connected if it is bounded by two or more closed boundary curves.

For illustration of the above notations see figures 5, 6 and 7.

Let $f : D \to \mathbb{C}$. Then f is called a discrete function. The operators Θ_x and Θ_y are defined as follows:

$$\Theta_x f(z) = \frac{f(z) - f(qx,y)}{(1-q)x} \qquad 2(16)$$

$$\Theta_y f(z) = \frac{f(z) - f(x,qy)}{(1-q)iy} \qquad 2(17)$$

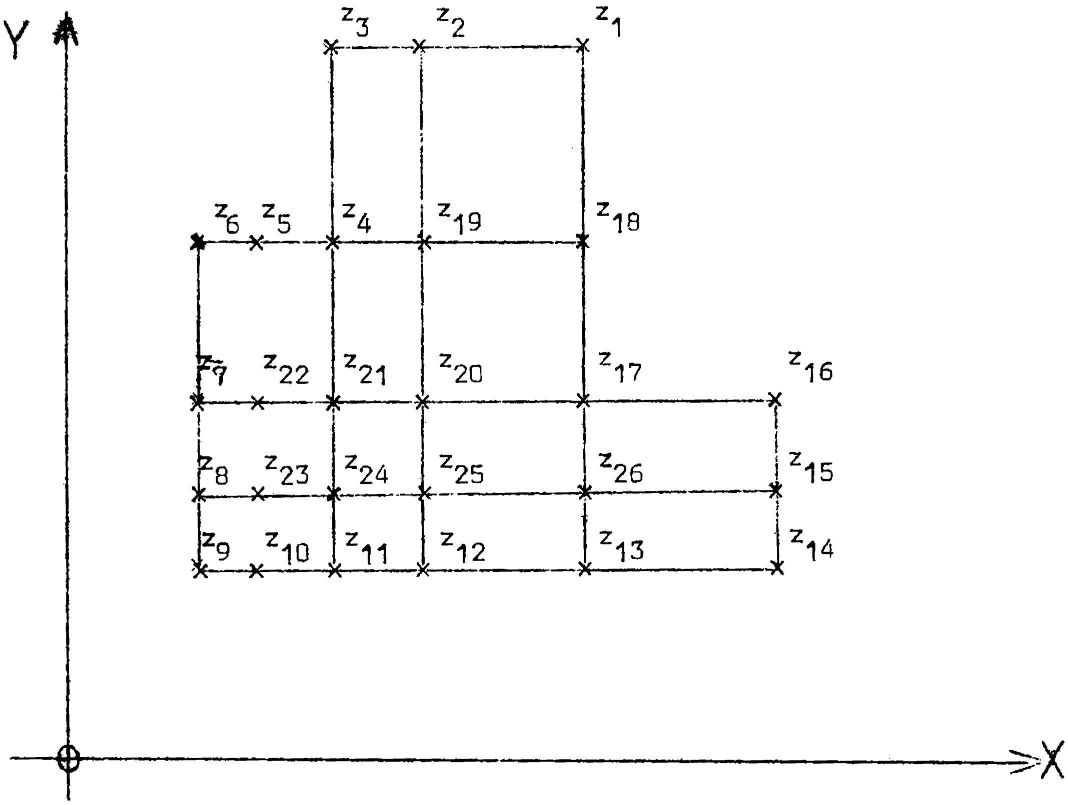

Figure 5

$D = \{z_1, z_2, z_3, \ldots z_{26}\}$

Basic tetrads contiguous to $S(z_1)$ in D are

$S(z_2)$, $S(z_{18})$ and $S(z_{19})$

D is a connected domain and it is singly connected.

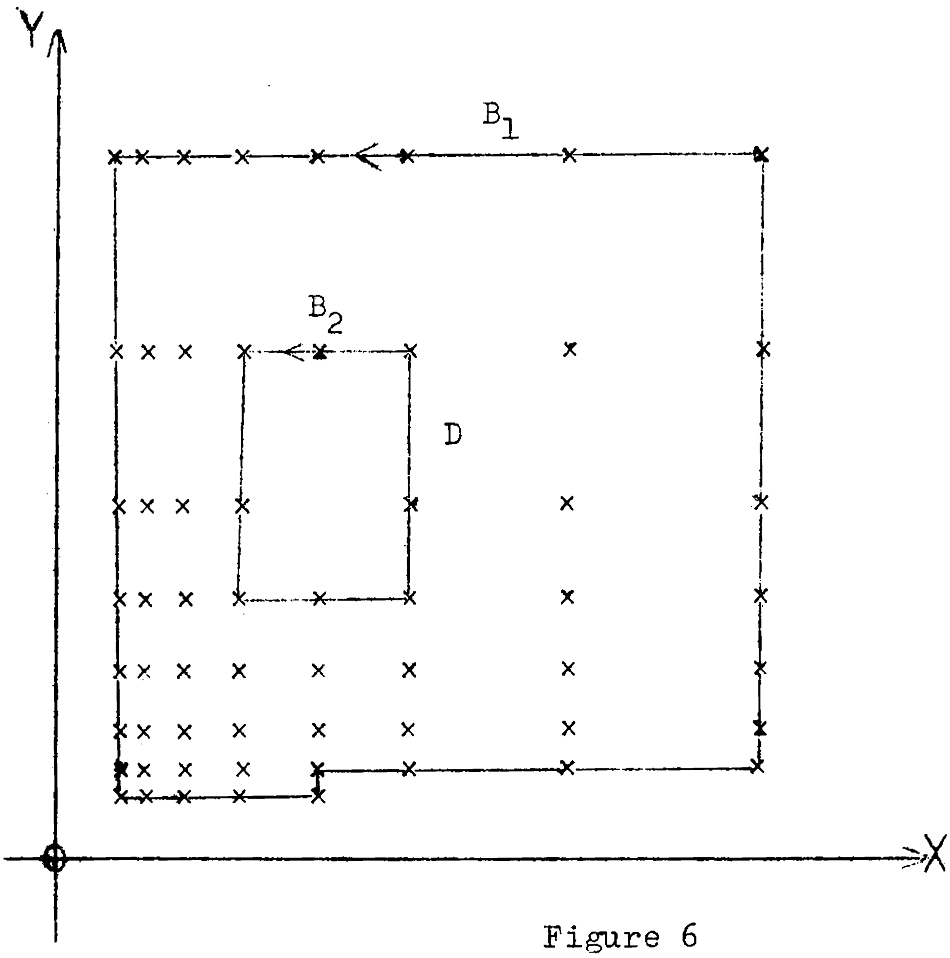

Figure 6

D is a connected domain bounded by two closed boundary curves B_1 and B_2 and so D is a doubly connected domain.

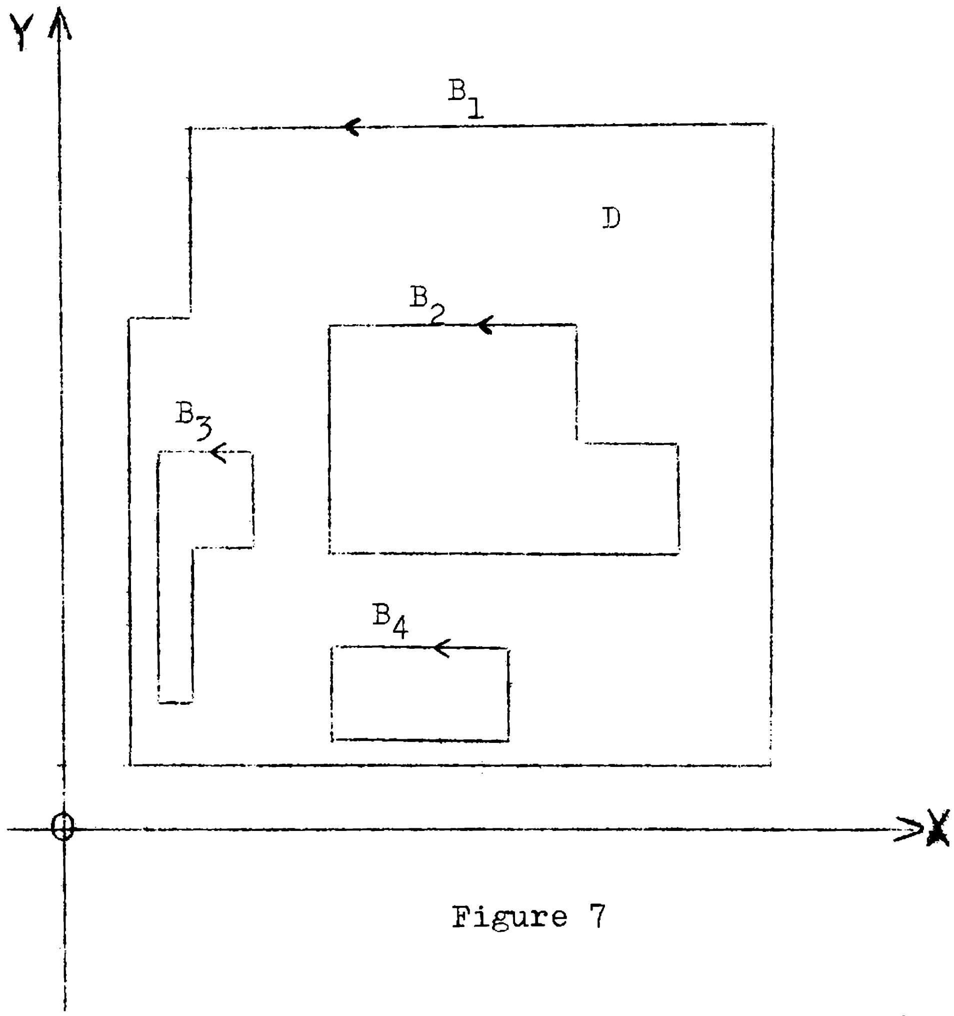

Figure 7

D is a multiply connected domain. B_1, B_2, B_3, B_4 are the four closed boundaries bounding D.

If $\Theta_x f(z) = \Theta_y f(z)$, then f is said to be q-analytic at z and the common operator is denoted by Θ. 2(18)

We define the operators Θ_z and $\Theta_{\bar{z}}$ as follows:

$$\Theta_z f(z) = \tfrac{1}{2}\,[\Theta_x f(z) + \Theta_y f(z)] \qquad 2(19)$$

$$\Theta_{\bar{z}} f(z) = \tfrac{1}{2}\,[\Theta_x f(z) - \Theta_y f(z)] \qquad 2(20)$$

Linearity of the above operations follows from the definitions.

Also simple calculation yields the following properties.

$$\overline{\Theta_z f(z)} = \Theta_{\bar{z}} \overline{f(z)} \qquad 2(21)$$

$$\overline{\Theta_{\bar{z}} f(z)} = \Theta_z \overline{f(z)} \qquad 2(22)$$

The discrete function f is q-analytic on z if and only if $\Theta_{\bar{z}} f(z) = 0$ and in that case $\Theta_z f(z) = \Theta\, f(z)$ 2(23)

Let f, g : $D \longrightarrow \mathbb{C}$, so, $\Theta_{\bar{z}}[f(z)g(z)]$

$$= \tfrac{1}{2}\left\{\Theta_x[f(z)g(z)] - \Theta_y[f(z)g(z)]\right\} \text{ by } 2(20)$$

$$= \tfrac{1}{2}\left\{\left[\frac{f(z)g(z) - f(qx,y)g(qx,y)}{(1-q)x}\right] - \left[\frac{f(z)\,g(z) - f(x,qy)g(x,qy)}{(1-q)iy}\right]\right\}$$

$$= \tfrac{1}{2}\Big\{ f(z)\ [\theta_x g(z) - \theta_y g(z)]$$

$$+ g(qx,y)\ \theta_x f(z) - g(x,qy)\ \theta_y f(z) \Big\}$$

$$= f(z)[\theta_{\bar{z}} g(z)] + \tfrac{1}{2}[g(qx,y)\ \theta_x f(z) - g(x,qy)\ \theta_y f(z)] \qquad 2(24)$$

Using a **similar** argument,

$$\theta_{\bar{z}}[f(z)\ g(z)] \text{ is also } = [\theta_{\bar{z}} f(z)]\ g(z)$$

$$+ \tfrac{1}{2}[f(qx,y)\ \theta_x g(z) - f(x,qy)\ \theta_y g(z)] \qquad 2(25)$$

and

$$\theta_z[f(z)g(z)] = [\theta_z f(z)]g(z) + \tfrac{1}{2}[f(qx,y)\ \theta_x g(z)$$

$$+ f(x,qy)\ \theta_y g(z)] \qquad 2(26)$$

or

$$= [\theta_z g(z)]f(z) + \tfrac{1}{2}[g(qx,y)\ \theta_x f(z)$$

$$+ g(x,qy)\ \theta_y f(z)] \qquad 2(27)$$

Now if both f and g are q-analytic in D, then

$$\theta_{\bar{z}}[f(z)g(z)] = \tfrac{1}{2}[f(qx,y) - f(x,qy)]\ \theta\ g(z) \qquad 2(28)$$

or

$$= \tfrac{1}{2}[g(qx,y) - g(x,qy)]\ \theta\ f(z) \qquad 2(29)$$

It follows that the product fg is q-analytic in D if

$$f(qx,y) = f(x,qy)$$

or

$$g(qx,y) = g(x,qy)$$

Also,

$$\theta_z[f(z)g(z)] = [\theta f(z)]g(z) + \tfrac{1}{2}[f(qx,y) + f(x,qy)]\ \theta g(z) \quad 2(30)$$

or

$$= [\theta g(z)]f(z) + \tfrac{1}{2}[g(qx,y) + g(x,qy)]\ \theta f(z) \quad 2(31)$$

If f is q-periodic in x and y, then $\theta_z[f(z)g(z)] = f(z)\ \theta g(z)$ and if g is q-periodic in x and y, then $\theta_z[f(z)\ g(z)] = g(z)\ \theta f(z)$. 2(32)

2. Hölder type discrete functions

Let D be a discrete domain and $f: D \longrightarrow \not{C}$. Suppose that $z' = (x',y') \in D$ and $|f(z) - f(z')| \leq k\sigma^{\mu}$ where $\sigma = (q^{-1}-1)\max(|x'|,|y'|)$ for every $z \in N(z')$, μ and k are real constants $0 < \mu \leq 1$, then we say that the function f is Hölder-type discrete at z'. Since we are considering the first quadrant, here σ can be taken to be equal to $(q^{-1}-1)\max(x,y)$.

If the above inequality holds for all $z \in D$ such that $N(z) \subset D$ then we say that f is Hölder-type discrete in D. The class of such functions on D will be denoted by $\mathcal{H}(D)$. 2(33)

From the definition it follows that if D is a bounded domain and $f \in \mathcal{H}(D)$ then f is bounded in D. 2(34)

Now, let $f, g \in \mathcal{H}(D)$ where D is a bounded domain.

If f, g are Hölder-type discrete at $z' = (x',y') \in D$, then by definition

$$|f(z) - f(z')| \leq k_1\sigma^{\alpha} \text{ and}$$

$$|g(z) - g(z')| \leq k_2\sigma^{\beta} \text{ for every } z \in N(z')$$

where k_1, k_2, α, β are constants, $0 < \alpha \leq 1$, $0 < \beta \leq 1$, $\sigma = (q^{-1}-1)\max(x',y')$.

Also $|f(z)g(z) - f(z')g(z')| = |f(z)[g(z) - g(z')]$

$+ g(z')[f(z) - f(z')]| \leq |f(z)|\,|g(z) - g(z')|$

$+ |g(z')|\,|f(z) - f(z')| \leq c_1k_1\sigma^{\alpha} + c_2k_2\sigma^{\beta}$

since by 2(34), $|f| \leq c_2$, $|g| \leq c_1$

$= k\sigma^{\alpha} + k'\sigma^{\beta}$

$$\leq \begin{cases} k^{*}\sigma^{\alpha} \text{ when } \sigma \geq 1 \text{ and } \alpha \geq \beta \\ \qquad \text{or } \sigma \leq 1 \text{ and } \alpha \leq \beta \\ k^{*}\sigma^{\beta} \text{ when } \sigma \geq 1 \text{ and } \alpha \leq \beta \\ \qquad \text{or } \sigma \leq 1 \text{ and } \alpha \geq \beta \end{cases}$$

Therefore it follows that if f and g are Hölder-type discrete at z' then the product fg is also a Hölder-type discrete function at z'. 2(35)

<u>Example 2(1)</u>

$\sqrt{x} + i\sqrt{y}$ is Hölder-type in H^1.

3. <u>Generating vector space</u>

Consider a discrete domain D and suppose $g_1, g_2 \in \mathbb{H}(D)$ such that $\mathrm{Im}(\overline{g_1}\ g_2) > 0$ throughout. Then the row vector $g = [g_1 \quad g_2]$ is called a generating vector and the set of all generating vectors $\{g\}$ is the generating space over D denoted by G(D).

It follows that the components of a generating vector cannot be equal for in that case $\mathrm{Im}(\overline{g_1}\ g_1)$ will be equal to zero. Also $[g_1 \quad -g_1]$ will not form a generating vector and neither of the components of a generating vector can be zero.

2(36)

Suppose that $f = [f_1 \quad f_2]'$ where f_1 and f_2 are real valued functions in D. The set of all such column vectors will be denoted by F(D).

Let g be a generating vector and w be any complex valued function defined on D. Then we can show that for any w a unique element $f \in F(D)$ can be found such that $w(z) = (g \cdot f)(z) = g_1(z)f_1(z) + g_2(z)f_2(z)$ for all z in D.

Proof:

Let $w = u + iv$, $g = [g_1 \quad g_2]$ where

$$g_1 = g_1^1 + ig_1^2, \quad g_2 = g_2^1 + ig_2^2$$

$$u + iv = f_1 g_1 + f_2 g_2$$

$$= f_1(g_1^1 + ig_1^2) + f_2(g_2^1 + ig_2^2)$$

Equating real and imaginary parts we have,

$$u = g_1^1 f_1 + g_2^1 f_2$$

$$v = g_1^2 f_1 + g_2^2 f_2$$

ie.,
$$\begin{bmatrix} g_1^1 & g_2^1 \\ g_1^2 & g_2^2 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$$

ie., $f_1 = \frac{1}{g_1^1 g_2^2 - g_2^1 g_1^2} [g_2^2 u - g_2^1 v]$ 2(37)

$f_2 = \frac{1}{g_1^1 g_2^2 - g_2^1 g_1^2} [-g_1^2 u + g_1^1 v]$ 2(38)

But $g_1^1 g_2^2 - g_2^1 g_1^2 > 0$ since $g \in G(D)$. Thus the result follows.

The theory of discrete pseudoanalytic functions is based on assigning the part played by 1 and i to two arbitrary functions $g_1(z)$ and $g_2(z)$. We can say that $\{w(z) \mid z \in D\} = G.F(D)$ where the (.) means the multiplication of a row and a column vector. Thus G.F(D) forms a vector space over R.

4. Discrete pseudoanalytic functions

Let D be a discrete domain. Suppose that $g = [g_1 \; g_2]$ is a generating vector belonging to G(D) and $w \in G.F(D)$, we define the operators

$${}_g\theta_x w(z) = (g. \theta_x f)(z) \qquad 2(39)$$

$${}_g\theta_y w(z) = (g. \theta_y f)(z) \qquad 2(40)$$

where θ_x and θ_y are given by 2(16) and 2(17) respectively.

From the definition it is clear the operators are linear.

Let D be a discrete domain and suppose that w is a complex valued function defined over D, then w is called discrete g-pseudoanalytic of the first kind at z ε D if w ε G.F(D) and ${}_g\Theta_x w(z) = {}_g\Theta_y w(z)$. 2(41)

If this relation holds for all z ε D such that $T(z) \subset D$ then w is called discrete g-pseudoanalytic of the first kind in D. 2(42)

If ${}_g\Theta_x w = {}_g\Theta_y w$, the common derivative is denoted by ${}_g\Theta w$ and is called the **discrete g-derivative** of w. 2(43)

The class of all discrete g-pseudoanalytic functions of the first kind in D is denoted by ${}_1P_D(g)$. Then ${}_1P_D(g)$ forms a vector space over R. 2(44)

For, suppose w ε ${}_1P_D(g)$, then there exists f in F(D) such that w = (g.f). Take α ε R. Then $\alpha w = \alpha(g.f) = (g.\alpha f)$ ε ${}_1P_D(g)$.

<u>Discrete pseudoanalytic functions of the second kind</u>

Suppose that w = (g.f), f ε F(D), g ε G(D). If w ε ${}_1P_D(g)$, then we call $h = f_1 + if_2$ discrete g-pseudoanalytic of the second kind in D. The class of all discrete g-pseudoanalytic functions of the second kind in D be denoted by ${}_2P_D(g)$. 2(45)

Remark

2(1) Each component g_1, g_2 of the generating vector $[g_1\ g_2]$ is itself an element of ${}_1P_D(g)$.

2(2) Now writing $g_1 = g.e_1$, $g_2 = g.e_2$ where $e_1 = [1\ 0]'$, $e_2 = [0\ 1]'$ it follows that ${}_g\theta g_1(z) = {}_g\theta g_2(z) = 0$. In this sense both the components g_1, g_2 can be treated as g-pseudo-constants.

2(3) If w_1 and w_2 ε ${}_1P_D(g)$, then $w_3(z) = a_1w_1(z) + a_2w_2(z)$ where a_1, a_2 are real constants also belong to ${}_1P_D(g)$ and ${}_g\theta w_3(z) = a_1[{}_g\theta w_1(z)] + a_2[{}_g\theta w_2(z)]$

Theorem 2(1)

A complex valued function w will be discrete g-pseudoanalytic of the first kind in a discrete domain D if and only if an f ε F(D) is found such that $\theta_{\bar{z}}f$ is orthogonal to g throughout D.

Proof

(a) Necessity

Suppose that w ε ${}_1P_D(g)$, then

$w = g.f$, f ε F(D), g ε G(D).

By 2(41),

$$ {}_g\Theta_x w(z) = {}_g\Theta_y w(z) $$

ie, $(g.\Theta_x f)(z) = (g.\Theta_y f)(z)$ by 2(39) and 2(40)

ie, $\frac{1}{2}[g.(\Theta_x - \Theta_y)f](z) = 0$

ie, $(g.\Theta_{\bar{z}} f)(z) = 0$ by 2(20) 2(46)

ie, $\Theta_{\bar{z}} f$ is orthogonal to g.

(b) Sufficiency

Suppose that $w = g.f$, $f \in F(D)$, $g \in G(D)$ and $(\Theta_{\bar{z}} f)$ is orthogonal to g, then

$(g.\Theta_{\bar{z}} f)(z) = 0$

ie, $\frac{1}{2}[g.(\Theta_x - \Theta_y)f](z) = 0$

ie, $(g.\Theta_x f - g.\Theta_y f)(z) = 0$

ie, $(g.\Theta_x f)(z) = (g.\Theta_y f)(z)$

By 2(41) $w \in {}_1P_D(g)$.

Thus the theorem is proved.

If further $w \in {}_1P_D(g)$ then ${}_g\Theta w(z) = (g.\Theta_z f)(z)$

2(47)

It may be noted that the g-derivative of an element w of ${}_1P_D(g)$ does not always belong to ${}_1P_D(g)$. However ${}_g\Theta w$ could be an element of ${}_1P_D(g^{(1)})$ where $g^{(1)}$ is also a generating vector. We call $g^{(1)}$, a successor of g and g, a predecessor of $g^{(1)}$. This problem is discussed in a later chapter.

Note 2(5)

A discrete function w(z) is said to be q-periodic in x and y if w satisfy the relation

$$w(x,y) = w(qx,y) = w(x,qy).$$

Such a function is

$$w(z) = \mu(x)\mu(iy)$$

where μ is Pincherle's q-periodic function defined by

$$\mu(x) = x^{\alpha-\beta} \frac{(1-q^{\alpha}x)_{\infty}\,(1-q^{1-\alpha}x^{-1})_{\infty}}{(1-q^{\beta}x)_{\infty}\,(1-q^{1-\beta}x^{-1})_{\infty}}$$ --(See Pincherle [1])

We denote the set of all such functions by $\pi_q(x,y)$

Theorem 2(2)

$_g\Theta w(z) \equiv 0$ if and only if $w(z) = (g.f)(z)$ where f is q-periodic in x and y.

Proof

(a) Necessity

Suppose that $w = g.f$, $g \in G(D)$, $f \in F(D)$ is an element of $_1P_D(g)$.

Then by 2(46) and 2(47),

$$(g.\Theta_{\bar{z}}f)(z) = 0$$

and

$$(g.\Theta_z f)(z) = {}_g\Theta w(z)$$

so that

$$[\bar{g}.(\overline{\Theta_{\bar{z}}f})](z) = 0$$

ie., $(\bar{g}.\Theta_z\bar{f})(z) = 0$ by 2(22)

ie., $(\bar{g}.\Theta_z f)(z) = 0$ since f is real valued.

Thus we have the relations

$$g_1(z)\Theta_z f_1(z) + g_2(z)\Theta_z f_2(z) = {}_g\Theta w(z) \text{ and}$$

$$\overline{g_1(z)}\Theta_z f_1(z) + \overline{g_2(z)}\Theta_z f_2(z) = 0$$

Solving and we get

$$\theta_z f_1(z) = \frac{\overline{g_2(z)}\,{}_g\theta w(z)}{g_1(z)\,\overline{g_2(z)} - \overline{g_1(z)}\,g_2(z)} \qquad 2(48)$$

and

$$\theta_z f_2(z) = \frac{-\overline{g_1(z)}\,{}_g\theta w(z)}{g_1(z)\,\overline{g_2(z)} - \overline{g_1(z)}\,g_2(z)} \qquad 2(49)$$

Now if ${}_g\theta w(z) = 0$, then by 2(48) and 2(49), we have

$$\theta_z f_1(z) = 0 = \theta_z f_2(z)$$

ie., $(\theta_x + \theta_y) f_1(z) = 0$

and

$(\theta_x + \theta_y) f_2(z) = 0$ by 2(19).

ie., $$\frac{f_1(z) - f_1(qx,y)}{(1-q)x} + \frac{f_1(z) - f_1(x,qy)}{(1-q)iy} = 0$$

and

$$\frac{f_2(z) - f_2(qx,y)}{(1-q)x} + \frac{f_2(z) - f_2(x,qy)}{(1-q)iy} = 0$$

ie., $$\left\{\frac{f_1(z) - f_1(qx,y)}{(1-q)x}\right\} + i\left\{-i\left[\frac{f_1(z) - f_1(x,qy)}{(1-q)iy}\right]\right\} = 0$$

and

$$\left\{\frac{f_2(z) - f_2(qx,y)}{(1-q)x}\right\} + i\left\{-i\left[\frac{f_2(z) - f_2(x,qy)}{(1-q)iy}\right]\right\} = 0$$

But f_1 and f_2 are real valued. Therefore, equating real and imaginary parts to zero, we have

$$f_1(z) - f_1(qx,y) = 0$$

$$f_1(z) - f_1(x,qy) = 0$$

$$f_2(z) - f_2(qx,y) = 0$$

and

$$f_2(z) - f_2(x,qy) = 0$$

ie., f_1 and f_2 are q-periodic both in x and y.

(b) Sufficiency

Suppose that $w = (g.f)$, $g \in G(D)$, $f \in F(D)$ is an element of ${}_1P_D(g)$ and f is q-periodic in x and y, then by 2(47), we have,

$$_g\Theta w(z) = (g.\Theta_z f)(z)$$

$$= [g.\tfrac{1}{2}(\Theta_x f + \Theta_y f)](z) \text{ by } 2(19)$$

$$= 0 \text{ since } f \text{ is q-periodic in x and y.}$$

Thus the theorem is proved.

Corollary 2(1)

Solutions of the equation ${}_g\mathcal{D}w(z) = 0$ are called g-pseudo constants. As a consequence of the above theorem it follows that a g-pseudo constant can be represented by g.f where f is q-periodic both in x and y.

Now consider $w \in G.F(D)$ an element of ${}_1P_D(g)$.

Take $w = g.f$, $g \in G(D)$, $f \in F(D)$

$h = I.f$, $I = [1 \quad i]$, so $h \in {}_2P_D(g)$.

Then $\bar{w} = \bar{g}.f$ since f is real valued.

From the above relations we obtain

$$h(z) = u(z)\, w(z) + v(z)\, \bar{w}(z)$$

where

$$u(z) = \frac{\overline{g_2(z)} - i\, \overline{g_1(z)}}{g_1(z)\, \overline{g_2(z)} - \overline{g_1(z)}\, g_2(z)} \qquad 2(50)$$

$$v(z) = \frac{-g_2(z) + i\, g_1(z)}{g_1(z)\, \overline{g_2(z)} - \overline{g_1(z)}\, g_2(z)} \qquad 2(51)$$

The correspondence between w(z) and h(z) is one-to-one. We denote it as

$$p_g\, w(z) = h(z) \qquad 2(52)$$

and

$$p_g^{-1} h(z) = w(z) \qquad 2(53)$$

It follows that

$$p_g(\alpha w_1 + \beta w_2) = \alpha(p_g w_1) + \beta(p_g w_2) \qquad 2(54)$$

where $\alpha, \beta \in R$.

Also we can see that

$$p_g(0) = 0, \quad p_g(g_1) = 1, \quad p_g(g_2) = i \qquad 2(55)$$

Suppose that D is a bounded domain.

We have $p_g w(z) = h(z) = u(z)\, w(z) + v(z)\, \overline{w\ (z)}$

Therefore,

$$|p_g w(z)| \leq |u(z)|\,|w(z)| + |v(z)|\,|\overline{w(z)}|$$

ie., $$\left|\frac{p_g w(z)}{w(z)}\right| \leq |u(z)| + |v(z)|$$

$$\leq k, \text{ since } g_1, g_2 \in \mathcal{H}(D).$$

k depending only on g_1 and g_2.

Remark

2(4) Suppose that $g \in G(D)$. If $h(z) = I.f(z)$ is discrete g-pseudo analytic of the second kind in domain D then $\Theta_{\bar{z}} f$ is orthogonal to g.

2(5) Every element of $\pi_q(x,y)$ is discrete g-pseudoanalytic of the second kind.

We can see that these results are analogous to those of Bers [1].

Note 2(6)

If we take $I = [1 \quad i]$ then it is defined in any domain $D \subset H$ and is a generating vector in D. We obtain the following theorem:

If $w = I.f$ is an element of ${}_1P_D(I)$ then w is q-analytic in D and conversely if w is q-analytic in D, it is an element of ${}_1P_D(I)$.

Proof

Suppose $w = I.f$, is an element of ${}_1P_D(I)$ then by 2(39)

$$
\begin{aligned}
{}_I\theta_x w(z) &= I.\theta_x f(z) \\
&= \theta_x f_1(z) + i\theta_x f_2(z) \\
&= \theta_x w(z)
\end{aligned}
$$

Similarly by 2(40)

$$ {}_I\theta_y w(z) = \theta_y w(z) $$

But ${}_I\theta_x w(z) = {}_I\theta_y w(z)$ since $w \in {}_1P_D(I)$

Therefore we get,

$$ \theta_x w(z) = \theta_y w(z) $$

Then by 2(18) w is q-analytic in D.

Conversely suppose that w is q-analytic in D. Then

$$\theta_x w(z) = \theta_x f_1(z) + i\,\theta_x f_2(z)$$

$$= I.\theta_x f(z)$$

and

$$\theta_y w(z) = \theta_y f_1(z) + i\,\theta_y f_2(z)$$

$$= I.\theta_y f(z)$$

Therefore $I.\theta_x f(z) = I.\theta_y f(z)$ by 2(18)

ie., $w \varepsilon\ {}_1P_D(I)$ by 2(41)

which proves the theorem.

Theorem 2(3)

Let g be a generating vector in D. Suppose $w = g.f$ (f is not q-periodic in x and y) is an element of ${}_1P_D(g)$. Then f_1 and f_2 are the real and imaginary parts of a q-analytic function in D, if and only if $g_2(z) = i\, g_1(z)$.

Proof

Suppose $w = g.f\ \varepsilon\ {}_1P_D(g)$, f is not q-periodic in x and y. Then by 2(46)

$$(g.\theta_{\bar{z}} f)(z) = 0$$

ie., $g_1(z)\Theta_{\bar{z}}f_1(z) + g_2(z)\,\Theta_{\bar{z}}f_2(z) = 0$

ie., $g_1(z)[\Theta_{\bar{z}}f_1(z) + \frac{g_2(z)}{g_1(z)}\,\Theta_{\bar{z}}f_2(z)] = 0$

But $g_1 \neq 0$ since $\mathrm{Im}(\bar{g}_1\, g_2) > 0$

Therefore

$$\Theta_{\bar{z}}f_1(z) + \frac{g_2(z)}{g_1(z)}\,\Theta_{\bar{z}}f_2(z) = 0$$

ie., $\Theta_{\bar{z}}f_1(z) = -[\frac{g_2(z)}{g_1(z)}]\Theta_{\bar{z}}f_2(z)$

Adding $i[\Theta_{\bar{z}}f_2(z)]$ to both sides we obtain

$$\Theta_{\bar{z}}[f_1(z) + if_2(z)] = [\Theta_{\bar{z}}f_2(z)][i - \frac{g_2(z)}{g_1(z)}]$$

Suppose that f_1 and f_2 are the real and imaginary parts of a q-analytic function in D, then by 2(23)

$$\Theta_{\bar{z}}[f_1(z) + i\,f_2(z)] = 0 \text{ in } D$$

Therefore,

$$0 = \Theta_{\bar{z}}f_2(z)\,[i - \frac{g_2(z)}{g_1(z)}]$$

ie., $0 = i - \frac{g_2(z)}{g_1(z)}$ since f_2 is non q-periodic in x and y

ie., $g_2(z) = i\,g_1(z)$.

Conversely

Suppose that $g_2(z) = i\ g_1(z)$ and $w \in {}_1P_D(g)$.

Then,

$$(g.\Theta_{\bar{z}}f)(z) = 0$$

ie., $g_1(z)\Theta_{\bar{z}}f_1(z) + i\ g_1(z)\ \Theta_{\bar{z}}f_2(z) = 0$

ie., $g_1(z)\Theta_{\bar{z}}[f_1(z) + i\ f_2(z)] = 0$

ie., $\Theta_{\bar{z}}[f_1(z) + i\ f_2(z)] = 0$ since $g_1 \neq 0$

ie., $f_1 + i\ f_2$ is q-analytic in D by 2(23).

Thus the theorem is proved.

Result 2(2)

Let $g = [g_1 \quad ig_1]$ be a generating vector in D. Suppose that $w \in {}_1P_D(g)$, then the discrete g-derivative of w is again an element of ${}_1P_E(g)$, where $E = \left\{z \in D \mid T(z) \subset D\right\}$.

Proof

Suppose that $w \in {}_1P_D(g)$.

$${}_g\Theta w(z) = (g.\Theta_x f)(z)$$

$$= (g.\Theta_y f)(z) \text{ by 2(39), 2(40) and 2(43).}$$

Now,

$$(g.\Theta_{\bar{z}}\Theta_x f)(z) = g_1(z)[\Theta_{\bar{z}}\Theta_x f_1(z)] + i\ g_1(z)[\Theta_{\bar{z}}\Theta_x f_2(z)]$$

$$= g_1(z)\Theta_{\bar{z}}[\Theta_x f_1(z) + i\ \Theta_x f_2(z)]$$

By theorem 2(3) f_1 and f_2 are the real and imaginary parts of a q-analytic function in D. Now as shown by Harman [1] the derivative of a q-analytic function in D is also q-analytic in E where $E = \{z \in D | T(z) \subset D\}$. Therefore by 2(23) right hand side is equal to zero in E. Therefore $\Theta_{\bar{z}}(\Theta_x f)$ is orthogonal to g in E. Hence by theorem 2(1) ${}_g\Theta w(z) \in {}_1P_D(g)$.

Theorem 2(4)

If $\{w_n(z)\}$ is a pointwise convergent sequence of discrete g-pseudoanalytic functions of the first kind in D with limit w then,

1) w is discrete g-pseudoanalytic of the first kind in D and 2(56)

2) $\lim_{n\to\infty} {}_g\Theta w_n(z) = {}_g\Theta w(z)$ 2(57)

Proof

1) Suppose that $\{w_n(z)\}$ is pointwise convergent to w, we have to show that $w \in {}_1P_D(g)$.

$$w(z) = \lim_{n\to\infty} [w_n(z)]$$

$$= \lim_{n\to\infty} (g.f_n)(z)$$

$$= (g. \lim_{n\to\infty} f_n)(z)$$

Now,

$$[g(z).\Theta_{\bar{z}} f(z)] = g(z).\Theta_{\bar{z}}[\lim_{n\to\infty} f_n(z)]$$

$$= g(z). \lim_{n\to\infty} \Theta_{\bar{z}} f_n(z)$$

$$= \lim_{n\to\infty} [g(z).\Theta_{\bar{z}} f_n(z)]$$

$= 0$ by theorem 2(1) as $g.f_n \varepsilon\ {}_1P_D(g)$.

Therefore by theorem 2(1) $w \varepsilon\ {}_1P_D(g)$.

2. $\lim_{n\to\infty} {}_g\Theta w_n(z) = \lim_{n\to\infty} [g(z).\Theta_x f_n(z)]$ by 2(41) and 2(43)

$$= g(z).\lim_{n\to\infty} \Theta_x f_n(z)$$

$$= g(z).\lim_{n\to\infty} [\frac{f_n(z) - f_n(qx,y)}{(1-q)x}]$$

$$= g(z).[\lim_{n\to\infty} \frac{f_n(z)}{(1-q)x} - \lim_{n\to\infty} \frac{f_n(qx,y)}{(1-q)x}]$$

$$= g(z).[\frac{f(z) - f(qx,y)}{(1-q)x}]$$

$$= (g.\Theta_x f)(z) \quad \text{by} \quad 2(16)$$

$$= {}_g\Theta w(z) \quad \text{by} \quad 2(41) \text{ and } 2(43)$$

Thus the theorem is proved.

5. Product with Hölder-type discrete functions

Theorem 2(5)

Suppose that $c \neq 0$ is a nonreal constant and $w \in {}_1P_D(g)$ then $c\,w \in {}_1P_D(cg)$. 2(58)

Proof

Suppose that w is an element of ${}_1P_D(g)$

$$w = g.f, \; g \in G(D), \; f \in F(D)$$

Then $cg \in G(D)$ since $Im(\overline{c\,g_1}\; cg_2) > 0$

$$cw = cg.f$$

Now,

$$ {}_{cg}\Theta_x w(z) = (cg.\Theta_x f)(z) $$

$$ = c(g.\Theta_x f)(z) = c[{}_g\Theta_x w(z)] $$

and

$$ {}_{cg}\Theta_y w(z) = (cg.\Theta_y f)(z) $$

$$ = c(g.\Theta_y f)(z) = c[{}_g\Theta_y w(z)] $$

Therefore

$$ {}_{cg}\Theta_x w(z) = {}_{cg}\Theta_y w(z) \text{ since } w \in {}_1P_D(G) $$

Therefore by 2(41) $cw \in {}_1P_D(cg)$

<u>Theorem 2(6)</u>

Let D be a bounded domain and w, an element of ${}_1P_D(g)$. If $p \neq 0$ is a Hölder-type discrete function in D, then $pw \in {}_1P_D(pg)$. 2(59)

<u>Proof</u>

Suppose that $w \in {}_1P_D(g)$

Since $p, g_1, g_2 \in \mathcal{H}(D)$ and D is bounded, pg_1, $pg_2 \in \mathcal{H}(D)$ by 2(35).

Also $\mathrm{Im}(\overline{p\,g_1}\, pg_2) = \mathrm{Im}(|p|^2\, \bar{g}_1 g_2) > 0$ since $g \in G(D)$.

Therefore pg is a generating vector

$$pw = pg.f$$

Now,

$$_{pg}\theta_x w(z) = (pg.\theta_x f)(z)$$

$$= p(z)(g.\theta_x f)(z)$$

$$= p(z)[{}_g\theta_x w(z)] \text{ by } 2(39)$$

and

$$_{pg}\theta_y w(z) = (pg.\theta_y f)(z)$$

$$= p(z)(g.\theta_y f)(z)$$

$$= p(z)[{}_g\theta_y w(z)] \text{ by } 2(40)$$

Therefore

$$_{pg}\theta_x w(z) = {}_{pg}\theta_y w(z) \text{ since by } 2(41),$$

$$_g\theta_x w(z) = {}_g\theta_y w(z)$$

Therefore

$$pw \in {}_1P_D(pg)$$

Hence the theorem is proved.

Theorem 2(7)

Let g be a generating vector in a discrete domain D and w be an element of ${}_1P_D(g)$. Then w satisfies the relation

$$\Theta_{\bar{z}}w(z) = \tfrac{1}{2}\Big\{[f_1(qx,y)\Theta_x g_1(z) - f_1(x,qy)\Theta_y g_1(z)] + [f_2(qx,y)\Theta_x g_2(z) - f_2(x,qy)\Theta_y g_2(z)]\Big\}$$

and the g-derivative satisfies the relation

$$\Theta_{\bar{z}}[{}_g\Theta w(z)] = \tfrac{1}{2}\Big\{[\Theta_y f_1(qx,y)\Theta_x g_1(z) - \Theta_y f_1(x,qy)\Theta_y g_1(z)] + [\Theta_y f_2(qx,y)\Theta_x g_2(z) - \Theta_y f_2(x,qy)\Theta_y g_2(z)]\Big\}$$

Proof

Suppose that w is an element of ${}_1P_D(g)$. Then by 2(46)

$$g_1(z)\Theta_{\bar{z}}f_1(z) + g_2(z)\Theta_{\bar{z}}f_2(z) = 0 \qquad 2(60)$$

Now,

$$\Theta_{\bar{z}}w(z) = \Theta_{\bar{z}}\,[g_1(z)f_1(z) + g_2(z)\,f_2(z)]$$

$$= [\Theta_{\bar{z}}f_1(z)]g_1(z) + [\Theta_{\bar{z}}f_2(z)]g_2(z)$$

$$+ \tfrac{1}{2}[f_1(qx,y)\Theta_x g_1(z) - f_1(x,qy)\Theta_y g_1(z)$$

$$+ f_2(qx,y)\Theta_x g_2(z) - f_2(x,qy)\Theta_y g_2(z)]$$ by 2(24).

$$= 0 + \tfrac{1}{2}[f_1(qx,y)\Theta_x g_1(z) - f_1(x,qy)\Theta_y g_1(z)$$

$$+ f_2(qx,y)\Theta_x g_2(z) - f_2(x,qy)\Theta_y g_2(z)] \qquad 2(61)$$

by 2(60)

By 2(47) we have

$$_g\Theta w(z) = g_1(z)\Theta_z f_1(z) + g_2(z)\Theta_z f_2(z)$$

Now,

$$\Theta_{\bar z}\,[_g\Theta w(z)] = \Theta_{\bar z}[g_1(z)\Theta_z f_1(z) + g_2(z)\Theta_z f_2(z)]$$

$$= [\Theta_{\bar z z}\, f_1(z)]g_1(z) + [\Theta_{\bar z z}\, f_2(z)]g_2(z)$$

$$+ \tfrac{1}{2}[\Theta_z f_1(qx,y)\Theta_x g_1(z) - \Theta_z f_1(x,qy)\Theta_y g_1(z)$$

$$+ \Theta_z f_2(qx,y)\Theta_x g_2(z) - \Theta_z f_2(x,qy)\Theta_y g_2(z)]$$

by 2(24).

$$= \Theta_z[g_1(z)\Theta_{\bar z} f_1(z) + g_2(z)\Theta_{\bar z} f_2(z)]$$

$$- \tfrac{1}{2}[\Theta_{\bar z} f_1(qx,y)\Theta_x g_1(z) - \Theta_{\bar z} f_1(x,qy)\Theta_y g_1(z)$$

$$+ \Theta_{\bar z} f_2(qx,y)\Theta_x g_2(z) - \Theta_{\bar z} f_2(x,qy)\Theta_y g_2(z)]$$

$$+ \tfrac{1}{2}[\Theta_z f_1(qx,y)\Theta_x g_1(z) - \Theta_z f_1(x,qy)\Theta_y g_1(z)$$

$$+ \Theta_z f_2(qx,y)\Theta_x g_2(z) - \Theta_z f_2(x,qy)\Theta_y g_2(z)]$$

by 2(26).

$$= \tfrac{1}{2}[\Theta_y f_1(qx,y)\Theta_x g_1(z) - \Theta_y f_1(x,qy)\Theta_y g_1(z)$$

$$+ \Theta_y f_2(qx,y)\Theta_x g_2(z) - \Theta_y f_2(x,qy)\Theta_y g_2(z)] \qquad 2(62)$$

by 2(46), 2(19) and 2(20).

Hence the theorem follows.

Theorem 2(8)

Suppose that g is a generating vector in a discrete domain D, and w is an element of G.F(D).

If $\Theta_{\bar{z}} w = \tfrac{1}{2}[f_1(qx,y)\Theta_x g_1(z) - f_1(x,qy)\Theta_y g_1(z)$

$$+ f_2(qx,y)\Theta_x g_2(z) - f_2(x,qy)\Theta_y g_2(z)]$$

then w is an element of ${}_1P_D(g)$.

Proof

$$w = g.f = g_1 f_1 + g_2 f_2$$

Suppose that

$$\Theta_{\bar{z}} w(z) = \tfrac{1}{2}[f_1(qx,y)\Theta_x g_1(z) - f_1(x,qy)\Theta_y g_1(z)$$

$$+ f_2(qx,y)\Theta_x g_2(z) - f_2(x,qy)\Theta_y g_2(z)] \qquad 2(63)$$

Now,

$$\theta_{\bar{z}}w(z) = \theta_{\bar{z}}[f_1(z)g_1(z) + f_2(z)g_2(z)]$$

$$= [\theta_{\bar{z}}f_1(z)]g_1(z) + [\theta_{\bar{z}}f_2(z)]g_2(z)$$

$$+ \tfrac{1}{2}[f_1(qx,y)\theta_x g_1(z) - f_1(x,qy)\theta_y g_1(z)$$

$$+ f_2(qx,y)\theta_x g_2(z) - f_2(x,qy)\theta_y g_2(z)] \qquad 2(64)$$

by 2(25).

Equating 2(63) and 2(64) we obtain

$$[\theta_{\bar{z}}f_1(z)]g_1(z) + [\theta_{\bar{z}}f_2(z)]g_2(z) = 0$$

Therefore by Theorem 2(1), $w \in {}_1P_D(g)$.

6. Elliptic system

Suppose that $w = g.f$ is an element of ${}_1P_D(g)$ and $h = I.f$ is an element of ${}_2P_D(g)$.

Then,

$$\theta_{\bar{z}}w(z) = \theta_{\bar{z}}(g.f)(z)$$

$$= \theta_{\bar{z}}[g_1(z)f_1(z) + g_2(z)f_2(z)]$$

$$= g_1(z)\Theta_{\bar z}f_1(z) + g_2(z)\Theta_{\bar z}f_2(z)$$

$$+ \tfrac{1}{2}\Big\{[\Theta_x g_1(z)]f_1(qx,y) - [\Theta_y g_1(z)]f_1(x,qy)$$

$$+ [\Theta_x g_2(z)]f_2(qx,y) - [\Theta_y g_2(z)]f_2(x,qy)\Big\}$$

by 2(24).

$$= (g.\Theta_{\bar z}f)(z) + \tfrac{1}{2}[\Theta_x g(z).f(qx,y)$$

$$- \Theta_y g(z).f(x,qy)]$$

$$= \tfrac{1}{2}[\Theta_x g(z).f(qx,y) - \Theta_y g(z).f(x,qy)]$$

since by theorem 2(1) $g.\Theta_{\bar z}f = 0$.

Now, take

$$\frac{g_2(z)}{ig_1(z)} = \alpha(z) + i\beta(z),$$

α, β are real valued. Since $\mathrm{Im}(\bar g_1\ g_2) > 0$ it follows that $\alpha > 0$. As I.f $\varepsilon\ {}_2P_D(g)$, f satisfies the relation

$$(g.\Theta_{\bar z}f)(z) = 0 \text{ by remark 2(4).}$$

ie., $g_1(z)\Theta_{\bar z}f_1(z) + g_2(z)\Theta_{\bar z}f_2(z) = 0$

ie., $\Theta_{\bar z}f_1(z) + \dfrac{g_2(z)}{g_1(z)}\Theta_{\bar z}f_2(z) = 0$ as $g_1 \neq 0$

ie., $\Theta_{\bar{z}} f_1(z) + [i\alpha(z) - \beta(z)]\Theta_{\bar{z}} f_2(z) = 0$

ie., $[\Theta_x f_1(z) - \Theta_y f_1(z)] + [i\alpha(z) - \beta(z)][\Theta_x f_2(z) - \Theta_y f_2(z)]$

$= 0$ by 2(20).

Equating real and imaginary parts to zero we have,

$$\Theta_x f_1(z) - i\alpha(z)\Theta_y f_2(z) - \beta(z)\Theta_x f_2(z) = 0$$

and

$$i\Theta_y f_1(z) + \alpha(z)\Theta_x f_2(z) - \beta(z) i\Theta_y f_2(z) = 0$$

ie., $\Theta_x f_1(z) = \beta(z)\Theta_x f_2(z) + i\alpha(z)\Theta_y f_2(z)$ 2(65)

and

$$i\Theta_y f_1(z) = -\alpha(z)\Theta_x f_2(z) + i\beta(z)\Theta_y f_2(z) \qquad 2(66)$$

Therefore we see that the real and imaginary parts of h(z) satisfy the equations 2(65) and 2(66) which are analogous to a particular type of Beltrami's equations,

$$u_x = \beta v_x + \gamma v_y$$

$$-u_y = \alpha v_x + \beta v_y$$ (Vekua) [1]

Taking the q-derivatives of 2(65) with respect to y and 2(66) with respect to x we have

$$\theta_y[\theta_x f_1(z)] = \theta_y[\beta(z)\theta_x f_2(z) + i\alpha(z)\theta_y f_2(z)] \qquad 2(67)$$

and

$$\theta_x[i\theta_y f_1(z)] = \theta_x[-\alpha(z)\theta_x f_2(z) + i\beta(z)\theta_y f_2(z)] \qquad 2(68)$$

Now,

$$\theta_y[\theta_x t(z)] = \theta_y[\frac{t(z) - t(qx,y)}{(1-q)x}]$$

$$= \left\{[\frac{t(z) - t(qx,y)}{(1-q)x}] - [\frac{t(x,qy) - t(qx,qy)}{(1-q)x}]\right\} \frac{1}{(1-q)iy}$$

$$= \frac{1}{(1-q)^2 ixy} [t(z) - t(qx,y) - t(x,qy) + t(qx,qy)]$$

$$= \frac{1}{(1-q)x} \left\{[\frac{t(z) - t(x,qy)}{(1-q)iy}] - [\frac{t(qx,y) - t(qx,qy)}{(1-q)iy}]\right\}$$

$$= \theta_x[\frac{t(z) - t(x,qy)}{(1-q)iy}]$$

$$= \theta_x[\theta_y t(z)] \qquad 2(69)$$

We can also easily show that

$$\theta_x[r(z)S(z)] = [\theta_x r(z)]S(z) + r(qx,y)\theta_x S(z) \qquad 2(70)$$

or $\quad = r(z)\Theta_x S(z) + S(qx,y)\,\Theta_x r(z)$ 2(71)

and

$$\Theta_y[r(z)S(z)] = [\Theta_y r(z)]S(z) + r(x,qy)\,[\Theta_y S(z)] \qquad 2(72)$$

or $\quad = r(z)\Theta_y S(z) + S(x,qy)\Theta_y r(z)$ 2(73)

where r and S are any two discrete functions.

Using the symbols $\Theta_x(\Theta_x) = \Theta_{xx}$,

$$\Theta_x(\Theta_y) = \Theta_{xy},\ \Theta_y(\Theta_x) = \Theta_{yx},\ \Theta_y(\Theta_y) = \Theta_{yy}$$

and the relations 2(71) and 2(73) we have

$$\Theta_{xy} f_1(z) = \beta(z)\Theta_{xy} f_2(z) + \Theta_x f_2(x,qy)\Theta_y \beta(z)$$

$$+ \alpha(z) i\Theta_{yy} f_2(z) + i\Theta_y f_2(x,qy)\Theta_y \alpha(z) \qquad 2(74)$$

and

$$i\Theta_{xy} f_1(z) = -\alpha(z)\Theta_{xx} f_2(z) - \Theta_x f_2(qx,y)\Theta_x \alpha(z)$$

$$+ i\beta(z)\Theta_{xy} f_2(z) + i\Theta_y f_2(qx,y)\Theta_x \beta(z) \qquad 2(75)$$

Multiplying 2(75) by i and adding to 2(74) we have

$$0 = -i\alpha(z)\Theta_{xx} f_2(z) + i\alpha(z)\Theta_{yy} f_2(z) - i\Theta_x \alpha(z)\Theta_x f_2(qx,y)$$

$$+ \Theta_y \beta(z)\Theta_x f_2(x,qy) - \Theta_x \beta(z)\Theta_y f_2(qx,y)$$

$$+ i\Theta_y \alpha(z)\Theta_y f_2(x,qy)$$

$$\text{ie., } 0 = \Theta_{xx} f_2(z) + (i)^2\, \Theta_{yy} f_2(z) + [a(z)\Theta_x f_2(qx,y)$$

$$+ b(z)\Theta_x f_2(x,qy)] + [c(z)\Theta_y f_2(qx,y)$$

$$+ d(z)\Theta_y f_2(x,qy)], \qquad 2(76)$$

where

$$a(z) = \frac{\Theta_x \alpha(z)}{\alpha(z)},$$

$$b(z) = \frac{i\Theta_y \beta(z)}{\alpha(z)};$$

$$c(z) = \frac{\Theta_x \beta(z)}{i\alpha(z)}$$

and

$$d(z) = \frac{-\Theta_y \alpha(z)}{\alpha(z)}, \quad \alpha > 0 \text{ by assumption.}$$

The above equation 2(76) forms a discrete analogue of the classical elliptic partial differential equation of the second order:

$$u_{xx} + u_{yy} + A(z)u_x + B(z)u_y + C(z)u = 0$$

Similarly we can show that f_1 also satisfies a system of the form 2(76) with different coefficients.

Theorem 2(9)

Suppose that h is a discrete function in D such that its real and imaginary parts f_1 and f_2 satisfy the system:

$$\Theta_x f_1(z) = \beta(z)\Theta_x f_2(z) + i\alpha(z)\Theta_y f_2(z) \qquad 2(77)$$

$$i\Theta_y f_1(z) = -\alpha(z)\Theta_x f_2(z) + i\beta(z)\Theta_y f_2(z) \qquad 2(78)$$

$\alpha > 0$, α, β are real, $\alpha+i\beta \in \mathbb{H}(D)$, then there exists a generating vector $g = [1 \quad i(\alpha+i\beta)]$ such that $h \in {}_2P_D(g)$.

Proof

Suppose that f_1 and f_2 satisfy the relations 2(77) and 2(78). Multiplying 2(78) by i and adding to 2(77) we have

$$2\Theta_{\bar{z}}[f_1(z)] = \Theta_x f_2(z)[\beta(z) - i\alpha(z)] + \Theta_y f_2(z)[i\alpha(z) - \beta(z)]$$

$$= (\Theta_x - \Theta_y) f_2(z)[\beta(z) - i\alpha(z)]$$

$$= 2\Theta_{\bar{z}} f_2(z)[\beta(z) - i\alpha(z)]$$

ie., $\Theta_{\bar{z}} f_1(z) + [i\alpha(z) - \beta(z)]\Theta_{\bar{z}} f_2(z) = 0$

ie., $\Theta_{\bar{z}} f_1(z) + i[\alpha(z) + i\beta(z)]\Theta_{\bar{z}} f_2(z) \quad = \quad 0$

Take $g = [1 \;\; i\alpha - \beta]$

$\mathrm{Im}(i\alpha - \beta) \quad = \alpha > 0$

Therefore g forms a generating vector.

By remark 2(5) $h \in {}_2P_D(g)$.

Thus the theorem is proved.

CHAPTER 3

INTEGRAL REPRESENTATION OF DISCRETE PSEUDOANALYTIC FUNCTIONS

In this chapter integral representation of discrete pseudoanalytic functions is discussed. Apart from the discrete integral introduced by Harman [1] we need conjugate of the discrete integral, p_g and g-integrals. In sections 1, 2 and 3 we will introduce the above integrals and study their properties. Integrals are used in showing that g-integral of a discrete function is an element of ${}_1P_D(g)$ and p_g-integral of a discrete function is an element of ${}_2P_D(g)$.

1. Conjugate of the discrete integral

Let $z_j = (x_j, y_j)$ be any point in a discrete domain D. Suppose that $z_{j+1} \in N(z_j)$. 3(1)

Let f be a complex valued function defined over D and $c = \langle z_1, z_2, \ldots. z_i, z_{i+1}, \ldots. z_n \rangle$ be any discrete curve in D. Harman [1] defined discrete integral over c by

$$\underset{c}{S} f(t)d(q;t) = \sum_{j=1}^{n-1} \underset{z_j}{\overset{z_{j+1}}{S}} f(t)d(q;t) \qquad 3(2)$$

where

$$\int_{z_j}^{z_{j+1}} f(t)d(q;t) = \begin{cases} (z_{j+1} - z_j)f(z_j) \text{ if} \\ z_{j+1} = (qx_j, y_j) \text{ or} \\ \qquad (x_j, qy_j) \\ (z_{j+1} - z_j)f(z_{j+1}) \text{ if} \\ z_{j+1} = (q^{-1}x_j, y_j) \text{ or} \\ \qquad (x_j, q^{-1}y_j) \end{cases} \quad 3(3)$$

He has shown that this discrete integral has properties analogous to those of Riemann integral.

The conjugate of the discrete integral from z_j to z_{j+1} of f will be

$$\overline{\int_{z_j}^{z_{j+1}} f(t)d(q;t)} = \begin{cases} \overline{(z_{j+1} - z_j)f(z_j)} \text{ if} \\ z_{j+1} = (qx_j, y_j) \text{ or } (x_j, qy_j) \\ \overline{(z_{j+1} - z_j)f(z_{j+1})} \text{ if} \\ z_{j+1} = (q^{-1}x_j, y_j) \text{ or} \\ \qquad (x_j, q^{-1}y_j) \end{cases} \quad 3(4)$$

And

$$\overline{\int_c f(t)d(q;t)} = \sum_{j=1}^{n-1} \overline{\int_{z_j}^{z_{j+1}} f(t)d(q;t)} \quad 3(5)$$

Suppose that $U(z) = \int_a^z f(t)d(q;t)$ 3(6)

and

$$\overline{U(z)} = \overline{\int_a^z f(t)d(q;t)} \qquad 3(7)$$

where a is a fixed point in D, then

$$\Theta_x U(z) = \frac{\int_a^z f(t)d(q;t) - \int_a^{(qx,y)} f(t)d(q;t)}{(1-q)x} \qquad \text{by } 2(16)$$

$$= \frac{\int_a^z f(t)d(q;t) - \int_a^z f(t)d(q;t) - \int_z^{(qx,y)} f(t)d(q,t)}{(1-q)x}$$

$$= - \frac{(q-1)x}{(1-q)x} f(z) \quad \text{by} \quad 3(3)$$

$$= f(z)$$

By similar argument we can show that $\Theta_y U(z) = f(z)$, $\Theta_x \overline{U(z)} = \overline{f(z)}$, $\Theta_y \overline{U(z)} = -\overline{f(z)}$ by 2(16), 2(17), 3(3) and 3(4).

Therefore by 2(19) and 2(20) U and $\bar{U}$ satisfy the following relations:

1) $\Theta_z U(z) = f(z)$ 3(8)

2) $\Theta_{\bar{z}} U(z) = 0$ 3(9)

3) $\Theta_z \overline{U(z)} = 0$ 3(10)

4) $\Theta_{\bar{z}} \overline{U(z)} = \overline{f(z)}$ 3(11)

2. Discrete p_g-integral

Suppose that $g = [g_1 \;\; g_2]$ is a generating vector in D and $c = \langle z_1, z_2, \ldots z_n \rangle$ a discrete curve in D. Suppose w is a complex valued function defined on D. The discrete p_g-integral of w from z_1 to z_n along c is defined by

$$p_g\left(\mathop{S}_{z_1}^{z_n} w(t)\,d_g(q;t)\right) = 2Rl\left(\mathop{S}_{z_1}^{z_n} \frac{\overline{g_2(t)}w(t)}{r(t)}d(q;t)\right)$$

$$-2i\;Rl\left(\mathop{S}_{z_1}^{z_n} \frac{\overline{g_1(t)}w(t)}{r(t)}d(q;t)\right)$$

3(12)

where $r(t) = g_1(t)\;\overline{g_2(t)} - \overline{g_1(t)}\;g_2(t)$ 3(13)

and $\mathop{S}_{z_1}^{z_n} f(t)d(q;t)$ is given by 3(2).

Properties of the p_g-integral

The following elementary properties follow for the p_g-integral.

1) Let $c_1 = \langle z_1, z_2, \ldots z_m \rangle$ and $c_2 = \langle z_m, z_{m+1}, \ldots z_n \rangle$ be two discrete curves in D. Let g be an element of G(D) and w be a complex valued function defined on D. Then,

$$p_g\Big(\int_{c_1} w(t) d_g(q;t) \Big) + p_g\Big(\int_{c_2} w(t) d_g(q;t) \Big) = p_g\Big(\int_{c_1+c_2} w(t) d_g(q;t) \Big) \qquad 3(14)$$

2) Suppose that $c = \langle z_1, z_2, \ldots z_i, z_{i+1}, \ldots z_n \rangle$ be a discrete curve in D, then

$$c^{-1} = \langle z_n, \ldots z_{i+1}, z_i, \ldots z_2, z_1 \rangle$$

and

$$p_g\Big(\int_{c} w(t) d_g(q;t) \Big) = -p_g\Big(\int_{c^{-1}} w(t) d_g(q;t) \Big) \qquad 3(15)$$

3) If α is a scalar constant then,

$$p_g\left(\underset{c}{S} \alpha\, w(t)d_g(q;t) \right) = \alpha\, p_g\left(\underset{c}{S} w(t)d_g(q;t) \right) \qquad 3(16)$$

4) If w_1 and w_2 are two discrete functions then,

$$p_g\left(\underset{c}{S} [w_1(t) + w_2(t)]d_g(q;t) \right) = p_g\left(\underset{c}{S} w_1(t)d_g(q;t) \right)$$

$$+ p_g\left(\underset{c}{S} w_2(t)d_g(q;t) \right) \qquad 3(17)$$

3. Discrete g-integral

Let D be a discrete domain and $c = \langle z_1, z_2, \ldots z_n \rangle$ be a discrete curve in D.

Then the discrete g-integral over c is defined by

$$\overset{z_n}{\underset{z_1}{S}} w(t)d_g(q;t) = g_1(z_n)\mathrm{Rl}\ p_g\left(\overset{z_n}{\underset{z_1}{S}} w(t)d_g(q;t) \right)$$

$$+ g_2(z_n)\mathrm{Im}\ p_g\left(\overset{z_n}{\underset{z_1}{S}} w(t)d_g(q;t) \right)$$

$$= 2g_1(z_n)\mathrm{Rl}\left(\int_{z_1}^{z_n} \frac{\overline{g_2(t)}w(t)}{r(t)}\, d(q;t)\right)$$

$$-2g_2(z_n)\mathrm{Rl}\left(\int_{z_1}^{z_n} \frac{\overline{g_1(t)}w(t)}{r(t)}\, d(q;t)\right) \qquad 3(18)$$

where $r(t)$ is given by 3(13) and $\int_{z_1}^{z_n} f(t)d(q;t)$ is given by 3(2). These p_g and g-integrals are analogous to the integrals introduced by Bers [1].

Properties of the g-integral

1) If α is a scalar constant then,

$$\int_C \alpha w(t)d_g(q;t) = \alpha \int_C w(t)d_g(q;t) \qquad 3(19)$$

2) If w_1 and w_2 are two complex valued functions defined on D, then

$$\int_C [w_1(t) + w_2(t)]d_g(q;t) = \int_C w_1(t)d_g(q;t)$$

$$+ \int_C w_2(t)d_g(q;t) \qquad 3(20)$$

3) If $c = \langle z_1, z_2, \ldots . z_n \rangle$ is a discrete curve in D then,

$$\mathop{S}_{c} w(t)d_g(q;t) \neq - \mathop{S}_{c^{-1}} w(t)d_g(q;t) \qquad 3(21)$$

But if c is closed ie., $z_n = z_1$ then the equality holds. 3(22)

4) Let $c_1 = \langle z_1, z_2, \ldots . z_m \rangle$ and $c_2 = \langle z_m, z_{m+1}, \ldots z_n \rangle$ be two discrete curves in D then,

$$\mathop{S}_{c_1} w(t)d_g(q;t) + \mathop{S}_{c_2} w(t)d_g(q;t) \neq \mathop{S}_{c_1+c_2} w(t)d_g(q;t)$$

where $c_1+c_2 = \langle z_1, z_2, \ldots, z_m, z_{m+1}, \ldots . z_n \rangle$. 3(23)

<u>Theorem 3(1)</u>

Suppose that g is a generating vector in a discrete bounded domain D. Let $c = \langle z_1, z_2, \ldots . z_n \rangle$ be a discrete curve in D. If w is a discrete function in D then,

$$\left| \mathop{S}_{z_1}^{z_n} w(t)d_g(q;t) \right| \leq K \mathop{S}_{z_1}^{z_n} |w(t)| \; |d(q;t)| \qquad 3(24)$$

where

$$\int_{z_j}^{z_{j+1}} f(t)|d(q;t)| = \begin{cases} |z_{j+1}-z_j|f(z_j); \\ \text{if } z_{j+1} = (qx_j,y_j) \text{ or } \\ \qquad (x_j,qy_j) \\ |z_{j+1}-z_j|f(z_{j+1}) \\ \text{if } z_{j+1} = (q^{-1}x_j,y_j) \text{ or } \\ \qquad (x_j,q^{-1}y_j) \end{cases} \quad 3(25)$$

K is a constant depending only on g_1, g_2.

Proof

$$\left|\int_{z_1}^{z_n} w(t)d_g(q;t)\right| = 2\left|g_1(z_n)Rl\left(\int_{z_1}^{z_n} \frac{\overline{g_2(t)}w(t)}{r(t)} d(q,t)\right)\right.$$

$$\left. -g_2(z_n)Rl\left(\int_{z_1}^{z_n} \frac{\overline{g_1(t)}w(t)}{r(t)} d(q;t)\right)\right|$$

where r(t) is given by 3(13)

$$\leq 2\left|g_1(z_n)Rl\left(\int_{z_1}^{z_n} \frac{\overline{g_2(t)}w(t)}{r(t)} d(q;t)\right)\right|$$

$$+2\left|g_2(z_n)Rl\left(\int_{z_1}^{z_n} \frac{\overline{g_1(t)}w(t)}{r(t)} d(q;t)\right)\right|$$

$$= 2 \left|g_1(z_n)\right| \left| Rl \left(\int_{z_1}^{z_n} \frac{\overline{g_2(t)}w(t)}{r(t)} d(q;t) \right) \right|$$

$$+ 2 \left|g_2(z_n)\right| \left| Rl \left(\int_{z_1}^{z_n} \frac{\overline{g_1(t)}w(t)}{r(t)} d(q;t) \right) \right|$$

By 2(34) g_1 and g_2 are bounded in D. Therefore, we can find k_1, k_2 such that $|g_1| < k_1$ and $|g_2| < k_2$.

Therefore the right hand side

$$< 2k_1 \left| Rl \left(\sum_{j=1}^{n-1} \int_{z_j}^{z_{j+1}} \frac{\overline{g_2(t)}w(t)}{r(t)} d(q;t) \right) \right|$$

$$+ 2k_2 \left| Rl \left(\sum_{j=1}^{n-1} \int_{z_j}^{z_{j+1}} \frac{\overline{g_1(t)}w(t)}{r(t)} d(q;t) \right) \right|$$

$$= k_1 \left| \sum_{j=1}^{n-1} \left[\int_{z_j}^{z_{j+1}} \frac{\overline{g_2(t)}w(t)}{r(t)} d(q;t) + \overline{\int_{z_j}^{z_{j+1}} \frac{\overline{g_2(t)}w(t)}{r(t)} d(q;t)} \right] \right|$$

$$+ k_2 \left| \sum_{j=1}^{n-1} \left[\int_{z_j}^{z_{j+1}} \frac{\overline{g_1(t)w(t)}}{r(t)} d(q;t) \right. \right.$$

$$\left. \left. + \int_{z_j}^{z_{j+1}} \overline{\frac{\overline{g_1(t)}w(t)}{r(t)} d(q;t)} \right] \right|$$

$$\leq 2k_1 \sum_{j=1}^{n-1} \left| \int_{z_j}^{z_{j+1}} \frac{\overline{g_2(t)}w(t)}{r(t)} d(q;t) \right|$$

$$+ 2k_2 \sum_{j=1}^{n-1} \left| \int_{z_j}^{z_{j+1}} \frac{\overline{g_1(t)}w(t)}{r(t)} d(q;t) \right|$$

But as shown by Harman [1]

$$\left| \int_c f(t)d(q;t) \right| \leq \int_c |f(t)| \; |d(q;t)| \qquad 3(26)$$

Therefore the right hand side

$$\leq 2k_1 \sum_{j=1}^{n-1} \int_{z_j}^{z_{j+1}} \left| \frac{\overline{g_2(t)}}{r(t)} \right| |w(t)| \; |d(q;t)|$$

$$+2k_2 \sum_{j=1}^{n-1} \int_{z_j}^{z_{j+1}} \left| \frac{\overline{g_1(t)}}{r(t)} \right| |w(t)| \; |d(q;t)|$$

But by 3(13) and 2(34), $\frac{g_2(t)}{r(t)}$ and $\frac{g_1(t)}{r(t)}$ are bounded in D and so we can find $k_3, k_4 > 0$ such that

$$\left|\frac{g_2(t)}{r(t)}\right| < k_3 \text{ and } \left|\frac{g_1(t)}{r(t)}\right| < k_4$$

Therefore the right hand side

$$\leq 2k_1k_3 \sum_{j=1}^{n-1} \int_{z_j}^{z_{j+1}} |w(t)|\,|d(q;t)|$$

$$+ 2k_2k_4 \sum_{j=1}^{n-1} \int_{z_j}^{z_{j+1}} |w(t)|\,|d(q;t)|$$

$$= 2k_1k_3 \int_{z_1}^{z_n} |w(t)|\,|d(q;t)|$$

$$+ 2k_2k_4 \int_{z_1}^{z_n} |w(t)|\,|d(q;t)|$$

$$= K \int_{z_1}^{z_n} |w(t)|\,|d(q;t)|$$

where K depends only on g_1, g_2.

Corollary 3(1)

Suppose that $\max_{t \in D} |w(t)| = M$ and if the curve length $\sum_{j=1}^{n-1} |z_{j+1} - z_j| = b$, then, $\left| \int_C w(t) d_g(q;t) \right| \leq KMb$

where K is a constant depending only on g_1, g_2. 3(27)

Proof

$$\left| \int_C w(t) d_g(q;t) \right| \leq K \int_C |w(t)| \, |d(q;t)| \quad \text{by } 3(24)$$

$$= K \sum_{j=1}^{n-1} \int_{z_j}^{z_{j+1}} |w(t)| \, |d(q;t)|$$

$$\leq K \sum_{j=1}^{n-1} |z_{j+1} - z_j| M$$

ie., $\leq KMb$, K is a constant depending only on g_1, g_2.

Remark 3(1)

The p_g-integral also satisfies a similar theorem and corollary and the proof is similar.

Let us consider discrete functions involving a parameter.

Let S be a given set of complex numbers and D, a discrete domain, such that $w\ (z;\alpha)$ is a discrete function for each $\alpha \varepsilon$ S and $z \ \varepsilon$ D.

If $\lim\limits_{\substack{\alpha \to \alpha_o \\ \alpha \varepsilon S}} w(z;\alpha) = h(z)$, then the limit will be termed 'uniform' if, given $\in > 0$ there exists a $\sigma > 0$ such that if $|\alpha - \alpha_o| < \sigma$ then $|w(z;\alpha) - h(z)| < \in$ for all $z \ \varepsilon$ D, where σ is dependent of z.

Let $g = [g_1 \ g_2]$ be a generating vector in D, and w an element of G.F(D). Then the following theorems are true.

Theorem 3(2)

Suppose that $\lim\limits_{\substack{\alpha \to \alpha_o \\ \alpha \varepsilon S}} w(z;\alpha) = h(z)$ is uniform for $z \ \varepsilon$ D, then

$$\lim_{\substack{\alpha \to \alpha_o \\ \alpha \varepsilon S}} \int_c w(t;\alpha)d_g(q;t) = \int_c h(t)d_g(q;t) \qquad 3(28)$$

where g is a generating vector and w is an element of G.F(D).

Proof

Since the limit is uniform, given $\in > 0$ there exists a $\sigma > 0$ such that if $|\alpha - \alpha_o| < \sigma$ then $|w(z;\alpha) - h(z)| < \frac{\in}{Kb}$ for all $z \ \varepsilon$ D.

By corollary to theorem 3(1) we have

$$\left| \underset{c}{S} [w(t;\alpha) - h(t)] d_g(q;t) \right| \leq \frac{K\in b}{Kb} = \in$$

Therefore we have

$$\lim_{\substack{\alpha \to \alpha_0 \\ \alpha \in S}} \underset{c}{S} [w(t;\alpha) - h(t)] d_g(q;t) = 0$$

ie., $$\lim_{\substack{\alpha \to \alpha_0 \\ \alpha \in S}} \left[\underset{c}{S} w(t;\alpha) d_g(q;t) - \underset{c}{S} h(t) d_g(q;t) \right] = 0$$ by 3(20)

ie., $$\lim_{\substack{\alpha \to \alpha_0 \\ \alpha \in S}} \underset{c}{S} w(t;\alpha) d_g(q;t) = \underset{c}{S} h(t) d_g(q;t)$$

Hence the theorem follows.

Extension of the above theorem is

<u>Theorem 3(3)</u>

Suppose that $g = [g_1 \; g_2]$ is a generating vector, $\sum_{r=0}^{\infty} h_r(z)$ be a series of discrete functions $h_r \in G.F(D)$ converging uniformly for all points of a discrete curve c, then the series may be g-integrated term by term along c (uniformity is required only when c contains an infinite number of points. When c is finite, pointwise convergence is only needed).

<u>Proof</u>

$$w(z;n) = \sum_{r=0}^{n} h_r(z)$$

Uniform convergence of the series is equivalent to the uniform limit of the sequence $\{w(z;n)\}$.

Therefore

$$\lim_{n\to\infty} w(z;n) = \sum_{r=0}^{\infty} h_r(z)$$

By theorem 3(2) we have

$$\mathop{S}_{c} \sum_{r=0}^{\infty} h_r(z)d_g(q;z) = \lim_{n\to\infty} \mathop{S}_{c} \sum_{r=0}^{n} h_r(z)d_g(q;z)$$

$$= \lim_{n\to\infty} \sum_{r=0}^{n} \mathop{S}_{c} h_r(z)d_g(q;z) \quad 3(29)$$

which proves the theorem.

<u>Theorem 3(4)</u>

Suppose that g is a generating vector in a singly connected discrete domain D. Let w be a complex valued function defined on D. Now suppose that $c = \langle a=z_1, z_2, \ldots z_n = z\rangle$, a fixed, is a discrete curve in D and

$$p_1(z) = \mathop{S}_{a}^{z} w(t)d_g(q;t),$$

then $p_1(z)$ is an element of ${}_1P_D(g)$ and ${}_g\Theta p_1(z) = w(z)$.

Proof

Suppose that $p_1(z) = \int_a^z w(t)d_g(q;t)$

$$= 2g_1(z)Rl\left(\int_a^z \frac{\overline{g_2(t)}w(t)}{r(t)}\, d(q;t)\right)$$

$$-2g_2(z)Rl\left(\int_a^z \frac{\overline{g_1(t)}w(t)}{r(t)}\, d(q;t)\right) \text{by } 3(18)$$

where $r(t)$ is given by 3(13)

$$= g_1(z)f_1(z) + g_2(z)f_2(z)$$

where

$$f_1(z) = 2Rl\left(\int_a^z \frac{\overline{g_2(t)}w(t)}{r(t)}\, d(q;t)\right)$$

$$= \int_a^z \frac{\overline{g_2(t)}w(t)}{r(t)}\, d(q;t) + \overline{\int_a^z \frac{\overline{g_2(t)}w(t)}{r(t)}\, d(q;t)}$$

and

$$f_2(z) = -2Rl\left(\int_a^z \frac{\overline{g_1(t)}w(t)}{r(t)} \, d(q;t) \right)$$

$$= -\left[\int_a^z \frac{\overline{g_1(t)}w(t)}{r(t)} \, d(q;t) + \overline{\int_a^z \frac{\overline{g_1(t)}w(t)}{r(t)} \, d(q;t)} \right]$$

Now,

$$\Theta_{\bar{z}} f_1(z) = \Theta_{\bar{z}} \left[\int_a^z \frac{\overline{g_2(t)}w(t)}{r(t)} \, d(q;t) + \overline{\int_a^z \frac{\overline{g_2(t)}w(t)}{r(t)} \, d(q;t)} \right]$$

$$= 0 + \frac{g_2(z)\overline{w(z)}}{\overline{r(z)}} \quad \text{by 3(9) and 3(11)} \qquad 3(30)$$

and

$$\Theta_z f_1(z) = \frac{\overline{g_2(z)}w(z)}{r(z)} + 0 \quad \text{by 3(8) and 3(10)} \qquad 3(31)$$

Similarly we can show that

$$\Theta_{\bar{z}} f_2(z) = \frac{-g_1(z)\overline{w(z)}}{\overline{r(z)}} \qquad 3(32)$$

and

$$\Theta_z f_2(z) = \frac{-\overline{g_1(z)}w(z)}{r(z)} \qquad 3(33)$$

But f_1 and f_2 are real valued and so $f = [f_1\ f_2]'$ is an element of F(D).

Now we have,

$$g(z).\theta_{\bar{z}}f(z) = g_1(z)\theta_{\bar{z}}f_1(z) + g_2(z)\theta_{\bar{z}}f_2(z)$$

$$= \frac{g_1(z)g_2(z)\overline{w(z)}}{\overline{r(z)}} - \frac{g_2(z)g_1(z)\overline{w(z)}}{\overline{r(z)}}$$

by 3(30) and 3(32)

$$= 0$$

Therefore by theorem 2(1) $p_1(z)$ is an element of ${}_1P_D(g)$.

Also,

$${}_g\theta p_1(z) = g(z).\theta_z f(z)$$

$$= g_1(z)\theta_z f_1(z) + g_2(z)\theta_z f_2(z)$$

$$= \frac{g_1(z)\ \overline{g_2(z)}\ w(z) - g_2(z)\ \overline{g_1(z)}\ w(z)}{r(z)}$$

by 3(31) and 3(33)

$$= w(z) \qquad 3(34)$$

Hence the theorem is proved.

<u>Theorem 3(5)</u>

Suppose that g is a generating vector in a singly connected discrete domain D. Let w be a complex valued function defined on D and $c = \langle a=z_1, z_2, z_3, \ldots . z_n = z \rangle$, a fixed, a discrete curve in D.

If $p_2(z) = p_g\left(\overset{z}{\underset{a}{S}} w(t) d_g(q;t) \right)$ then $p_2(z)$ is an element of ${}_2P_D(g)$.

<u>Proof</u>

$$\text{Suppose that } p_2(z) = p_g\left(\overset{z}{\underset{a}{S}} w(t) d_g(q;t) \right)$$

$$= 2\mathrm{Rl}\left(\overset{z}{\underset{a}{S}} \frac{\overline{g_2(t)}w(t)}{r(t)} d(q;t) \right)$$

$$-2i\mathrm{Rl}\left(\overset{z}{\underset{a}{S}} \frac{\overline{g_1(t)}w(t)}{r(t)} d(q;t) \right)$$

$$= f_1(z) + i\, f_2(z)$$

where

$$f_1(z) = 2\mathrm{Rl}\left(\overset{z}{\underset{a}{S}} \frac{\overline{g_2(t)}w(t)}{r(t)} d(q;t) \right)$$

and

$$f_2(z) = -2Ri \left(\int_a^z \frac{\overline{g_1(t)}w(t)}{r(t)} d(q;t) \right)$$

Now,

$$g(z).\Theta_{\bar{z}}f(z) = g_1(z)\Theta_{\bar{z}}f_1(z) + g_2(z)\Theta_{\bar{z}}f_2(z)$$

$$= 0 \text{ by } 3(30) \text{ and } 3(32)$$

Therefore by remark 2(4) $(f_1 + i\, f_2)(z)$ is an element of ${}_2P_D(g)$. ie., $p_2(z)$ is an element of ${}_2P_D(g)$.

Theorem 3(6)

Let D be a singly connected discrete domain and g, a generating vector in D. Suppose that

$$U_1(z) = \int_{a_1}^{z} w(t)d_g(q;t) \text{ and } U_2(z) = \int_{a_2}^{z} w(t)d_g(q;t)$$

where w is a complex valued function defined on D and a_1, a_2 are fixed points in D. Then $U_1(z) = U_2(z) + k(z)$ where $k(z) = (g.f)(z)$, f is a discrete function q-periodic both in x and y. In other words k(z) is a g-pseudoconstant.

Proof

$$\text{Let } k(z) = U_2(z) - U_1(z)$$

$$
\begin{aligned}
{}_g\Theta k(z) &= {}_g\Theta U_2(z) - {}_g\Theta U_1(z) \\
&= w(z) - w(z) \quad \text{by } 3(34) \\
&= 0
\end{aligned}
$$

Hence ${}_g\Theta k(z) = 0.$

By theorem 2(2) $k(z) = (g.f)(z)$ where f is a discrete function q-periodic both in x and y. ie., k(z) is a g-pseudoconstant.

These theorems can be extended to a discrete multiply connected domain also.

In classical analysis Bers [1] has proved the following theorems:

1) If w is pseudoanalytic of the first kind in a singly connected domain D then,

$$\int_c \mathring{w} d_{(F,G)}(z) = 0$$

where $\mathring{w}$ is the (F,G) derivative of w, c is a closed curve in D.

2) If W is a continuous function defined in a singly connected domain D, and if

$$\int_c W d_{(F,G)}(z) = 0,$$

c is a closed curve in D, then there exists an (F,G) pseudo-analytic function w(z) in D such that

$$W(z) = \frac{d_{(F,G)}w(z)}{dz}$$

Bers [1] has also obtained a generalisation of Cauchy's integral formula for pseudoanalytic functions.

In the limiting case as $q \to 1$, the p_g and g-integrals in the discrete system tends to the Bers' integrals.

CHAPTER 4

SOLUTIONS OF A SYSTEM OF PARTIAL q-DIFFERENCE EQUATIONS MODULO-g AND AN ANALOGUE OF BELTRAMI'S SYSTEM

In 1961, Abdi [5] solved partial q-difference equations of n variables under appropriate boundary conditions using q-Laplace transforms. Thereafter nobody worked in this field. Earlier Jackson [3] in 1910 defined q-integration as the inverse of the q-difference operation given by the operator:

$$\theta_x f(x) = \frac{f(qx) - f(x)}{(q-1)x}$$

In this chapter we make use of Jackson's integral to find the solutions of the system of partial q-difference equations modulo-g and an analogue of a Beltrami's system. Some examples are considered.

1. Partial q-difference equations modulo-g

Suppose that g is a generating vector in a discrete domain. By 2(41) $w \in G.F(D)$ is an element of ${}_1P_D(g)$ if

$$ {}_g\theta_x w(z) = {}_g\theta_y w(z) \qquad 4(1)$$

where

$$ {}_g\theta_x w(z) = (g.\theta_x f)(z) \qquad 4(2)$$

and

$$ {}_g\Theta_y w(z) = (g.\Theta_y f)(z) \qquad 4(3) $$

and

$$ {}_g\Theta_x w(z) = {}_g\Theta_y w(z) = {}_g\Theta w(z) \qquad 4(4) $$

Let g be a generating vector in D. Suppose that $w \varepsilon G.F(D)$, $t \varepsilon {}_1P_D(g^{(1)})$ where $g^{(1)}$ is a successor of g. We define a system of partial q-difference equations modulo-g by

$$ (g.\Theta_x f)(z) = t(z) \qquad 4(5) $$

$$ (g.\Theta_y f)(z) = t(z) \qquad 4(6) $$

From 4(5) and 4(6) it follows that

$$ (g.\Theta_x f)(z) = (g.\Theta_y f)(z) = t(z) \qquad 4(7) $$

Suppose that there exists a discrete function $w \varepsilon G.F(D)$ satisfying the above system then by 2(41) w is an element of ${}_1P_D(g)$ and t is the g-derivative of w.

2. Solutions of partial q-difference equations modulo-g

Theorem 4(1)

Any linear combination of two solutions of a system of partial q-difference equations modulo-g is again a solution.

Proof

Let $w_1(z) = (g.\sigma)(z)$ and $w_2(z) = (g.\eta)(z)$ where $\sigma = [\sigma_1\ \sigma_2]'$ and $\eta = [\eta_1\ \eta_2]'$ ε F(D) be two solutions of the system 4(5) and 4(6).

Then we have the equations,

$$(g.\theta_x\sigma)(z) = t(z) \tag{4(8)}$$

$$(g.\theta_y\sigma)(z) = t(z) \tag{4(9)}$$

$$(g.\theta_x\eta)(z) = t(z) \tag{4(10)}$$

and

$$(g.\theta_y\eta)(z) = t(z) \tag{4(11)}$$

Multiplying 4(8) and 4(10) by two arbitrary scalars a and b respectively and adding, we obtain the equations

$$(g.\ (a\theta_x\sigma + b\theta_x\eta))(z) = t_1(z) \tag{4(12)}$$

where

$$t_1(z) = (a+b)t(z)$$

Similarly multiplying 4(9) and 4(11) by a and b respectively and adding, we obtain,

$$(g.\ (a\theta_y\sigma + b\theta_y\eta))(z) = t_1(z) \tag{4(13)}$$

4(12) and 4(13) can be written as

$$(g.\theta_x u)(z) = t_1(z) \qquad 4(14)$$

and

$$(g.\theta_y u)(z) = t_1(z) \qquad 4(15)$$

where $u = a\sigma + b\eta \in F(D)$

which is the same as the system 4(5) and 4(6). This shows that $a\,w_1 + b\,w_2$ is also a solution of the system 4(5) and 4(6).

Thus the theorem is proved.

It is seen that the solution of a system of partial q-difference equations modulo-g is by no means unique. To make the solution definite we prescribe, as in the classical case, certain conditions known as 'boundary conditions'.

In 1910, Jackson [3] defined q-integration as the inverse of the q-difference operation given by the operator:

$$\theta_x f(x) = \frac{f(qx) - f(x)}{(q-1)x} \qquad 4(16)$$

as follows:

$$\theta_x^{-1} f(z) = \frac{1}{(1-q)} \int_{x_0}^{x} f(x)\,d(q;x) \qquad 4(17)$$

and he studied the properties of the inverse operation.

If $\quad \theta_x F(x) = f(x)$ 4(18)

Then,

$$F(0) - F(x) = (q-1)x \sum_{j=0}^{\infty} q^j f(q^j x) \qquad 4(19)$$

and

$$F(\infty) - F(x) = (q-1)x \sum_{j=1}^{\infty} q^{-j} f(q^{-j} x) \qquad 4(20)$$

We can use these properties to find the solutions of the system 4(5) and 4(6).

From 4(5) and 4(6) it follows that

$$(g.\theta_x f)(z) = (g.\theta_y f)(z) = t(z) \qquad 4(21)$$

If $w \varepsilon$ G.F(D) satisfies 4(21) then t is the g-derivative of w.

ie., $t(z) = {}_g\theta w(z)$ 4(22)

Therefore by 2(48)

$$\theta_z f_1(z) = \frac{\overline{g_2(z)}t(z)}{r(z)}, \text{ r(z) given by 3(13)} \qquad 4(23)$$

And by 2(49)

$$\theta_z f_2(z) = \frac{-\overline{g_1(z)}t(z)}{r(z)} \qquad 4(24)$$

But $[f_1 \; f_2]' \varepsilon$ F(D).

Therefore by 2(16), 2(17), 2(19) and 4(23) we have

$$\theta_x f_1(z) = 2Rl[\theta_z f_1(z)]$$

$$= 2Rl[\frac{\overline{g_2(z)}\ t(z)}{r(z)}]$$

$$= \frac{\overline{g_2(z)}\ t(z)}{r(z)} + \frac{g_2(z)\ \overline{t(z)}}{\overline{r(z)}}$$

But $\overline{r(z)} = -r(z)$

Therefore we have the relation,

$$\theta_x f_1(z) = \frac{\overline{g_2(z)}\ t(z) - g_2(z)\ \overline{t(z)}}{r(z)}$$

$$= p_1(z) \quad \text{(say)} \qquad 4(25)$$

Similarly we obtain the relation,

$$\theta_x f_2(z) = Rl[\frac{-2\overline{g_1(z)}\ t(z)}{r(z)}]$$

$$= \frac{g_1(z)\ \overline{t(z)} - \overline{g_1(z)}\ t(z)}{r(z)}$$

$$= p_2(z) \quad \text{(say)} \qquad 4(26)$$

Also

$$-i\theta_y f_1(z) = \mathrm{Im}\left[\frac{2\overline{g_2(z)}\; t(z)}{r(z)}\right]$$

$$= \frac{\overline{g_2(z)}\; t(z) + g_2(z)\; \overline{t(z)}}{ir(z)}$$

ie.,

$$\theta_y f_1(z) = \left[\frac{\overline{g_2(z)}\; t(z) + g_2(z)\; \overline{t(z)}}{r(z)}\right]$$

$$= p_3(z) \quad \text{(say)} \qquad 4(27)$$

And

$$\theta_y f_2(z) = -\left[\frac{\overline{g_1(z)}\; t(z) + g_1(z)\; \overline{t(z)}}{r(z)}\right]$$

$$= p_4(z) \quad \text{(say)} \qquad 4(28)$$

Let D be a rectangular domain. Suppose that $(q^{m+1}x, y)$ and $(x, q^{n+1}y)$ belong to the boundary of D and $w \in G.F(D)$ is known at these points, then $f \in F(D)$ is known at these points. If $w(x,y)$ exists in D then by 4(5), 4(6), 4(19), 4(25) and 4(26) we obtain,

$$w(x,y) = g_1(x,y)\left[(1-q)x \sum_{j=0}^{m} q^j p_1(q^j x, y) + f_1(q^{m+1}x, y)\right]$$
$$+ g_2(x,y)\left[(1-q)x \sum_{j=0}^{m} q^j p_2(q^j x, y) + f_2(q^{m+1}x, y)\right] \qquad 4(29)$$

where p_1, p_2 are given by 4(25), 4(26) respectively.

Similarly using 4(5), 4(6), 4(19), 4(27) and 4(28) we obtain,

$$w(x,y) = g_1(x,y)[(1-q)iy \sum_{j=0}^{n} q^j p_3(x,q^j y) + f_1(x,q^{n+1}y)]$$

$$+ g_2(x,y)[(1-q)iy \sum_{j=0}^{n} q^j p_4(x,q^j y)$$

$$+ f_2(x,q^{n+1}y)] \qquad 4(30)$$

where p_3, p_4 are given by 4(27) and 4(28).

This gives a solution of the system 4(5), 4(6).

Theorem 4(2)

Let g be a generating vector in D and $g^{(1)}$ be a successor of g in D. Let a discrete function t be an element of ${}_1P_D(g^{(1)})$. If w(x,y) is represented by 4(29) or 4(30), then w is an element of ${}_1P_D(g)$.

Proof

Suppose that $t \in {}_1P_D(g^{(1)})$ and w(x,y) is represented by 4(29). We can show that $w \in {}_1P_D(g)$.

In 4(29) we take

$$f_1(x,y) = (1-q)x \sum_{j=0}^{m} q^j p_1(q^j x,y) + f_1(q^{m+1}x,y)$$

where p_1 is given by 4(25).

And

$$f_2(x,y) = (1-q)x \sum_{j=0}^{m} q^j p_2(q^j x,y) + f_2(q^{m+1}x,y)$$

where p_2 is given by 4(26)

$$(g.\theta_x f)(z) = g_1(x,y)\theta_x f_1(x,y) + g_2(x,y)\theta_x f_2(x,y)$$

Now,

$$\theta_x f_1(x,y) = [(1-q)x \sum_{j=0}^{m} q^j p_1(q^j x,y) + f_1(q^{m+1}x,y)$$

$$- (1-q)qx \sum_{j=0}^{m} q^j p_1(q^{j+1}x,y)$$

$$- f_1(q^{m+2}x,y]\frac{1}{(1-q)x} \quad \text{by } 2(16)$$

$$= \sum_{j=0}^{m} q^j p_1(q^j x,y) - \sum_{j=0}^{m} q^{j+1} p_1(q^{j+1}x,y)$$

$$+ \frac{1}{(1-q)x}[f_1(q^{m+1}x,y) - f_1(q^{m+2}x,y)]$$

$$= p_1(x,y) - q^{m+1} p_1(q^{m+1}x,y)$$

$$+ \frac{1}{(1-q)x}[f_1(q^{m+1}x,y) - f_1(q^{m+2}x,y)]$$

Now substituting for $p_1(q^{m+1}x,y)$ by 4(25), $t(q^{m+1}x,y)$ by 4(5) we get

$$\Theta_x f_1(x,y) = p_1(x,y) \qquad 4(31)$$

Similarly we can show that

$$\Theta_x f_2(x,y) = p_2(x,y) \qquad 4(32)$$

Now,

$$(g.\Theta_x f)(z) = g_1(z)\Theta_x f_1(z) + g_2(z)\Theta_x f_2(z)$$

$$= g_1(z)p_1(z) + g_2(z)p_2(z)$$

$$= g_1(z)\left[\frac{\overline{g_2(z)}\ t(z) - g_2(z)\overline{t(z)}}{r(z)}\right]$$

$$+ g_2(z)\left[\frac{g_1(z)\overline{t(z)} - \overline{g_1(z)}t(z)}{r(z)}\right]$$

by 4(25) and 4(26)

$$= t(z) \text{ by } 3(13)$$

By the same argument we can show that $(g.\Theta_y f)(z) = t(z)$. Therefore by 2(41) $w \in {}_1P_D(g)$.

If w is represented by 4(30), then by the same argument we can show that $(g.\theta_x f)(z) = t(z) = (g.\theta_y f)(z)$. Therefore $w \in {}_1P_D(g)$. Thus the theorem is proved.

Now consider H^1 as the domain. Let g be a generating vector in H^1. Suppose that $t : H^1 \to ¢$ and belongs to ${}_1P_{H^1}(g^{(1)})$. The boundary of H^1 is the set of limiting points,

$$\lim_{m,n \to \pm\infty} (q^m x, q^n y) \quad \text{ie, the curves}$$

(x,0), (0,y), (x,∞) and (∞,y).

From 4(5), 4(6), 4(19), 4(20), 4(25), 4(26), 4(27) and 4(28) we obtain

$$w(x,y) = g_1(x,y)[(1-q)x \sum_{j=0}^{\infty} q^j p_1(q^j x,y) + f_1(0,y)]$$

$$+ g_2(x,y)[(1-q)x \sum_{j=0}^{\infty} q^j p_2(q^j x,y) + f_2(0,y)] \quad 4(33)$$

$$w(x,y) = g_1(x,y)[(1-q)x \sum_{j=1}^{\infty} q^{-j} p_1(q^{-j} x,y)$$

$$+ f_1(\infty,y)] + g_2(x,y)\,[(1-q)x \sum_{j=1}^{\infty} q^{-j} p_2(q^{-j} x,y)$$

$$+ f_2(\infty,y)] \quad 4(34)$$

$$w(x,y) = g_1(x,y)[(1-q)iy \sum_{j=0}^{\infty} q^j p_3(x,q^j y) + f_1(x,0)]$$

$$+ g_2(x,y)[(1-q)iy \sum_{j=0}^{\infty} q^j p_4(x,q^j y) + f_2(x,0)] \quad 4(35)$$

$$w(x,y) = g_1(x,y)[(1-q)iy \sum_{j=1}^{\infty} q^{-j} p_3(x,q^{-j} y) + f_1(x,\infty)]$$

$$+ g_2(x,y)[(1-q)iy \sum_{j=1}^{\infty} q^{-j} p_4(x,q^{-j} y) + f_2(x,\infty)] \quad 4(36)$$

Series 4(33) and 4(35) converges when $p_1(q^j x,y)$, $p_2(q^j x,y)$, $p_3(x,q^j y)$ and $p_4(x,q^j y)$ are of order $O(R^{-j})$ where $R > q$. Series 4(34) and 4(36) converges when $p_1(q^{-j}x,y)$, $p_2(q^{-j}x,y)$, $p_3(x,q^{-j}y)$ and $p_4(x,q^{-j}y)$ are of order $O(R^j)$ where $0 < R < q$.

Theorem 4(3)

Let g be a generating vector in H^1 and $g^{(1)}$ be a successor of g. Suppose that t is an element of ${}_1P_{H1}(g^{(1)})$, w is an element of G.F(D), and w is known on the curves $x = 0$, $y = 0$, $x = \infty$, $y = \infty$. If w(x,y) exists in H^1 and is represented by 4(33), 4(34), 4(35) or 4(36), then w is an element of ${}_1P_{H1}(g)$.

Proof

Suppose $t \in {}_1P_{H1}(g^{(1)})$ and w is represented by 4(33).

Take

$$f_1(x,y) = (1-q)x \sum_{j=0}^{\infty} q^j p_1(q^j x,y) + f_1(0,y)$$

$$f_2(x,y) = (1-q)x \sum_{j=0}^{\infty} q^j p_2(q^j x,y) + f_2(0,y)$$

$$\theta_x f_1(x,y) = \left\{ \left[(1-q)x \sum_{j=0}^{\infty} q^j p_1(q^j x,y)\right] - \left[(1-q)qx \sum_{j=0}^{\infty} q^j p_1(q^{j+1} x,y)\right] \right\} \frac{1}{(1-q)x} \quad \text{by } 2(16)$$

(since $\theta_x f_1(0,y) = 0$)

ie., $$= \sum_{j=0}^{\infty} q^j p_1(q^j x,y) - \sum_{j=0}^{\infty} q^{j+1} p_1(q^{j+1} x,y)$$

$$= \sum_{j=0}^{\infty} q^j p_1(q^j x,y) - \sum_{j=0}^{\infty} q^j p_1(q^j x,y) + q^0 p_1(q^0 x,y)$$

$$= p_1(x,y)$$

By the same argument it follows that

$$\theta_x f_2(x,y) = p_2(x,y)$$

Therefore

$$(g.\theta_x f)(z) = g_1(z)p_1(z) + g_2(z)p_2(z)$$

$$= t(z) \quad \text{by } 4(25) \text{ and } 4(26)$$

Similarly we can show that $(g.\theta_y f)(z) = t(z)$. ie, w belongs to ${}_1P_D(g)$. Similar argument follows for other equations.

Thus the theorem is proved.

Example 4(1)

Take $g_1 = 1$, $g_2 = ix$, $t = x$. Then $g = [g_1 \ g_2]$ is a generating vector in H^1 and $t \in {}_1P_{H^1}(g^{(1)})$ where $g^{(1)} = [-i \ x]$. (In a later chapter we will show that if $g = [g_1 \ \ g_2]$ is a generating vector then $g^{(1)} = [\ \frac{g_1}{i} \ \ \frac{g_2}{i}\ ]$ is a successor of g).

From 4(33) we obtain,

$$w(x,y) = [(1-q)x \sum_{j=0}^{\infty} q^j \left(\frac{-ixx - ixx}{-2ix}\right)(q^j x, y)$$

$$+ f_1(0,y)] + 0 + ix\, f_2(0,y)$$

$$= (1-q)x \sum_{j=0}^{\infty} (q^j) q^j x + \quad f_1(0,y) + ix\, f_2(0,y)$$

$$= \frac{(1-q)x^2}{1-q^2} + f_1(0,y) + ix\, f_2(0,y) \qquad 4(37)$$

$$\left(\text{since} \sum_{j=0}^{\infty} q^{2j} = \frac{1}{1-q^2}\right)$$

Similarly from 4(34) we obtain,

$$w(x,y) = 0 + f_1(x,0) + ix(1-q)iy \sum_{j=0}^{\infty} q^j \left(\frac{-x-x}{-2ix}\right)(x, q^j y)$$

$$+ ix\, f_2(x,0)$$

$$= f_1(x,0) + ix(1-q)iy \sum_{j=0}^{\infty} (q^j)\, \frac{1}{i} + ix\, f_2(x,0)$$

$$= ixy + f_1(x,0) + ix\, f_2(x,0) \qquad 4(38)$$

Thus we obtain

$$w(x,y) = \frac{(1-q)x^2}{1-q^2} + f_1(0,y) + ix\ f_2(0,y)$$

$$= ixy + f_1(x,0) + ix\ f_2(0,y)$$

$$= \left(\frac{1-q}{1-q^2}\right)x^2 + ixy + k_1(z) + ix\ k_2(z)$$

$$= \left(\frac{x^2}{1+q}\right) + ixy + k_1(z) + ix\ k_2(z)$$

where k_1 and k_2 are real valued discrete functions q-periodic both in x and y.

Therefore it follows that

$$w(x,y) = \left(\frac{1-q}{1-q^2}\right)x^2 + ixy + k_1(z) + ix\ k_2(z)$$

is an element of ${}_1P_H1(g)$ where $g = [1\ ix]$

For we have

$$\Theta_{\bar{z}}\left[\left(\frac{x^2}{1+q}\right) + k_1(z)\right] + ix\ \left\{\Theta_{\bar{z}}[y + k_2(z)]\right\}$$

$= \Theta_{\bar{z}}\left(\frac{x^2}{1+q}\right) + ix\Theta_{\bar{z}}y$, since k_1 and k_2 are q-periodic in x and y.

$$= \frac{1}{2}\ [\Theta_x(\frac{x^2}{1+q}) - \Theta_y(\frac{x^2}{1+q})] + \frac{ix}{2}[\Theta_x(y) - \Theta_y(y)]$$

by 2(19)

$$= \frac{1}{2}\ [\frac{(1+q)x}{(1+q)} + (ix)\ (-\frac{1}{i})] = 0$$

3. Solution of an analogue of a Beltrami system

Beltrami's system of first order partial differential equation is of the form

$$U_x = \beta V_x + \gamma V_y \qquad 4(39)$$

$$-U_y = \alpha V_x + \beta V_y \qquad 4(40)$$

where U, V are real valued functions of z.

Let us consider a system of the form

$$\Theta_x f_1(z) = \mu(z)\Theta_x f_2(z) + i\sigma(z)\Theta_y f_2(z) \qquad 4(41)$$

$$i\Theta_y f_1(z) = -\sigma(z)\Theta_x f_2(z) + i\mu(z)\Theta_y f_2(z) \qquad 4(42)$$

σ, μ are real, $\sigma > 0$, $\sigma + i\mu \quad \varepsilon\,\mathbb{H}(D)$. 4(43)

Take $v = [1 \; i(\sigma + i\mu)]$ 4(43)

$$\text{Im}[i(\sigma + i\mu)] > 0$$

Therefore v is a generating vector in D. By theorem 2(7) $f_1 + if_2$ is an element of ${}_2P_D(v)$ and by remark 2(4), f satisfies the equation

$$(v.\theta_{\bar{z}}f)(z) = 0 \text{ where } f = [f_1 \; f_2]'$$

ie., $(v.\theta_x f)(z) = (v.\theta_y f)(z)$

Now consider the system

$$(v.\theta_x f)(z) = t(z) \qquad 4(44)$$

$$(v.\theta_y f)(z) = t(z) \qquad 4(45)$$

where $t : D \longrightarrow \not{C}$

Let D be a rectangular domain. Suppose that $(q^{m+1}x,y)$ and $(x,q^{n+1}y)$ belong to the boundary of D and f_1, f_2 are known at these points. Then if $h(x,y) = f_1(x,y) + if_2(x,y)$ exists in D, by 4(19), 4(25), 4(26), 4(44) and 4(45) we have

$$h(x,y) = [(1-q)x \sum_{j=0}^{m} q^j s_1(q^j x,y) + f_1(q^{m+1}x,y)]$$

$$+ i[(1-q)x \sum_{j=0}^{m} q^j s_2(q^j x,y) + f_2(q^{m+1}x,y)] \qquad 4(46)$$

where

$$S_1 = [\frac{-i(\sigma - i\mu)t - i(\sigma + i\mu)\bar{t}}{-2\,i\sigma}] = (\sigma+i\mu)(\frac{t+\bar{t}}{2\sigma}) \qquad 4(47)$$

$$S_2 = \frac{\bar{t} - t}{-2i\sigma} \quad \text{since } g_1 = 1 \text{ and } g_2 = i(\sigma+i\mu) \qquad 4(48)$$

Similarly from 4(25), 4(27), 4(28), 4(44) and 4(45) we obtain

$$h(x,y) = [(1-q)iy \sum_{j=0}^{n} q^j S_3(x,q^j y) + f_1(x,q^{n+1}y)]$$

$$+ i[(1-q)iy \sum_{j=0}^{n} q^j S_4(x,q^j y) + f_2(x,q^{n+1}y)] \qquad 4(49)$$

where

$$S_3 = [\frac{-(i\sigma+\mu)t + (i\sigma-\mu)\bar{t}}{-2i\sigma}] \qquad 4(50)$$

and

$$S_4 = \frac{\bar{t} + t}{2i\sigma} \qquad 4(51)$$

Theorem 4(4)

Suppose that $v = [1 \ i(\sigma + i\mu)]$, $\sigma, \mu \in \mathcal{H}(D)$, $\sigma > 0$, be a generating vector in D and $v^{(1)}$ is a successor of v in D. Let t be an element of ${}_1P_D(v^{(1)})$. If h is represented by 4(46) or 4(49), then h is an element of ${}_2P_D(v)$.

Proof

In 4(46) take

$$f_1(x,y) = (1-q)x \sum_{j=0}^{m} q^j s_1(q^j x,y) + f_1(q^{m+1}x,y)$$

$$f_2(x,y) = (1-q)x \sum_{j=0}^{m} q^j s_2(q^j x,y) + f_2(q^{m+1}x,y)$$

We can easily show that

$$\theta_x f_1(x,y) = s_1(x,y)$$

and

$$\theta_x f_2(x,y) = s_2(x,y)$$

Now,

$$(v.\theta_x f)(x,y) = \theta_x f_1(x,y) + i[\sigma(x,y) + i\mu(x,y)]\theta_x f_2(x,y)$$

$$= s_1(x,y) + i[\sigma(x,y) + i\mu(x,y)]s_2(x,y)$$

$$= t(x,y) \quad \text{by} \quad 4(47) \text{ and } 4(48)$$

ie., $(v.\Theta_x f)(z) = t(z)$

Similarly we can show that

$$(v.\Theta_y f)(z) = t(z)$$

Hence the theorem is proved.

Now consider H^1 as the domain. If $h(x,y)$ exists in H^1 and $h(x,y)$ on the limiting boundaries are known, then we obtain

$$h(x,y) = [(1-q)x \sum_{j=0}^{\infty} q^j s_1(q^j x,y) + f_1(0,y)]$$

$$+ i[(1-q)x \sum_{j=0}^{\infty} q^j s_2(q^j x,y) + f_2(0,y)] \qquad 4(52)$$

$$\text{or } h(x,y) = [(1-q)x \sum_{j=1}^{\infty} q^{-j} s_1(q^{-j} x,y) + f_1(\infty,y)]$$

$$+ i[(1-q)x \sum_{j=1}^{\infty} q^{-j} s_2(q^{-j} x,y) + f_2(\infty,y)] \quad 4(53)$$

or $h(x,y) = [(1-q)iy \sum_{j=0}^{\infty} q^{j} s_3(x,q^{j}y) + f_1(x,0)]$

$$+ i[(1-q)iy \sum_{j=0}^{\infty} q^{j} s_4(x,q^{j}y) + f_2(x,0)] \qquad 4(54)$$

or $h(x,y) = [(1-q)iy \sum_{j=1}^{\infty} q^{-j} s_3(x,q^{-j}y) + f_1(x,\infty)]$

$$+ i[(1-q)iy \sum_{j=1}^{\infty} q^{-j} s_4(x,q^{-j}y) + f_2(x,\infty)] \quad 4(55)$$

according to the boundary conditions.

Theorem 4(5)

Let $v = [1 \; i(\sigma+i\mu)]$, σ, $\mu \in \mathbb{H}(H^1)$ $\sigma > 0$, be a generating vector in H^1 and $v^{(1)}$ be a successor of v in H^1. Let t be an element of ${}_1P_{H^1}(v^{(1)})$. If h is represented by 4(52), 4(53), 4(54), 4(55) then h is an element of ${}_2P_{H^1}(v)$.

Proof is exactly similar to the proof of theorem of 4(4).

Example 4(2)

Taking $v_1 = 1$, $v_2 = ix$, $t = x$ we can easily show that

$$h(x,y) = \frac{x^2}{(1+q)} + iy + k(z)$$

(where $k(z)$ is an element of $\pi_q(x,y)$) is an element of ${}_2P_H1(v)$, $v = [v_1\ v_2]$.

CHAPTER 5

PERIODICITY PROBLEM FOR DISCRETE PSEUDOANALYTIC FUNCTIONS

We discuss concepts like successors and predecessors of generating vectors, generating sequences, periodicity of the generating sequences. It is shown that if $w \varepsilon {}_1P_D(g)$ is not a g-pseudo-constant then g can be embedded in a generating sequence of minimal period one if and only if the first component of the generating vector is equal to the product of the second component and a function of y alone. Bers [1] has not discussed the periodicity problem in detail. But in 1956 Protter [1] studied the problem extensively. He has established sufficient conditions for a generating pair to be embedded in a generating sequence with prescribed minimal periods or a non-periodic generating sequence. It is established in the discrete case that any generating vector can be embedded in a generating sequence of minimal period 2. This shows that if w is an element of ${}_1P_D(g)$ then we can always find a successor v of generating vector g such that ${}_g\Theta w \ \varepsilon \ {}_1P_D(v)$ and the discrete v-derivative of ${}_g\Theta w$ is an element of ${}_1P_D(g)$.

1. Successors, predecessors and generating sequences

Suppose that w is an element of ${}_1P_D(g)$. Further if the g-derivative is an element of ${}_1P_D(g^{(1)})$ where

$g^{(1)}$ is a generating vector in D then $g^{(1)}$ is called a successor of g and g is called a predecessor of $g^{(1)}$. A sequence of generating vectors $\{g^{(v)}\}$, $v = 0, \pm 1, \pm 2, \ldots$ is called a generating sequence if $g^{(v+1)}$ is a successor of $g^{(v)}$. If $g^{(0)} = g$, we say that g is embedded in $\{g^{(v)}\}$.

Equivalent generating vectors

Let g be a generating vector in a discrete domain D.

$$\text{Take } \sigma = [\sigma_1 \quad \sigma_2] \tag{5(1)}$$

where
$$\sigma_1(z) = a_1 g_1(z) + a_2 g_2(z) \tag{5(2)}$$

$$\sigma_2(z) = b_1 g_1(z) + b_2 g_2(z) \tag{5(3)}$$

where a_j, b_j, $j = 1,2$ are real

and
$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} > 0 \tag{5(4)}$$

Theorem 5(1)

Let g be a generating vector in a discrete domain D and w be an element of ${}_1P_D(g)$. Then w is also an element of ${}_1P_D(\sigma)$ where the components of σ are given by

5(2) and 5(3). (We call such generating vectors equivalent generating vectors).

Proof

Suppose $w = g \circ f \;\varepsilon\; {}_1R_D(g)$. Then by 2(46)

$$(g.\theta_{\bar{z}}f)(z) \;=\; 0$$

ie., $g_1(z)\theta_{\bar{z}}f_1(z) + g_2(z)\;\theta_{\bar{z}}f_2(z) = 0$ 5(5)

From 5(2) and 5(3) we obtain

$$g_1(z) \;=\; \frac{a_2\sigma_2(z) - b_2\sigma_1(z)}{b_1a_2 - b_2a_1}$$

and

$$g_2(z) \;=\; \frac{b_1\sigma_1(z) - a_1\sigma_2(z)}{b_1a_2 - b_2a_1}$$

Substituting in 5(5) we obtain

$$\left(\frac{-b_2\sigma_1(z) + a_2\sigma_2(z)}{b_1a_2 - b_2a_1}\right)\theta_{\bar{z}}f_1(z)$$

$$+ \left(\frac{b_1\sigma_1(z) - a_1\sigma_2(z)}{b_1a_2 - b_2a_1}\right)\theta_{\bar{z}}f_2(z) \;=\; 0 \qquad 5(6)$$

By 5(4) it follows that $\mathrm{Im}(\overline{\sigma_1}\ \sigma_2) > 0$.

Therefore $[\sigma_1\ \sigma_2]$ forms a generating vector.

Now,

$$\sigma_1(z)\Theta_{\bar{z}}f^{(1)}(z) + \sigma_2(z)\Theta_{\bar{z}}f^{(2)}(z)$$

$$= \sigma_1(z)\ \Theta_{\bar{z}}\ [\frac{-b_2}{k}f_1(z) + \frac{b_1}{k}f_2(z)]$$

$$+ \sigma_2(z)\ \Theta_{\bar{z}}\ [\frac{a_2}{k}f_1(z) - \frac{a_1}{k}f_2(z)]$$

where $k = b_1a_2 - b_2a_1$

$$= [\Theta_{\bar{z}}f_1(z)]\ [\frac{-b_2\sigma_1(z) + a_2\sigma_2(z)}{k}]$$

$$+ [\Theta_{\bar{z}}f_2(z)][\frac{b_1\sigma_1(z) - a_1\sigma_2(z)}{k}] = 0 \quad \text{by } 5(6)$$

Hence the theorem is proved.

Theorem 5(2)

Let $g = [g_1\ g_2]$ be a generating vector in a discrete domain D. If w is an element of ${}_1P_D(g)$, then

$g^{(1)} = [\ \frac{g_1}{i}\ \ \frac{g_2}{i}\ ]$ is a successor of g.

Proof

Suppose that $w \in {}_1P_D(g)$. Then by 2(40) and 2(43) we have,

$$ {}_g\Theta w(z) = g_1(z)\Theta_y f_1(z) + g_2(z)\Theta_y f_2(z) $$

$$ = [g_1^{(1)}(z)]\, i\Theta_y f_1(z) + [g_2^{(1)}(z)] i\Theta_y f_2(z) \qquad 5(7) $$

where $g_1^{(1)} = \frac{g_1}{i}$, $g_2^{(1)} = \frac{g_2}{i}$.

$$ \mathrm{Im}(\overline{g_1^{(1)}}\; g_2^{(1)}) = \mathrm{Im}(\overline{g_1}\; g_2) > 0 $$

Therefore

$$ g^{(1)} = [\frac{g_1}{i} \quad \frac{g_2}{i}] \text{ forms a generating vector.} \qquad 5(8) $$

Take

$f_1^{(1)} = i\Theta_y f_1$, $f_2^{(1)} = i\Theta_y f_2$, then $[f_1^{(1)} \; f_2^{(1)}]' \in F(D)$.

Therefore by theorem 2(7) and equation 5(7) we have

$$ \Theta_{\bar{z}}\, [{}_g\Theta w(z)] = \tfrac{1}{2}\, [f_1^{(1)}(qx,y)\Theta_x g_1^{(1)}(z) $$

$$ -f_1^{(1)}(x,qy)\Theta_y g_1^{(1)}(z) + f_2^{(1)}(qx,y)\Theta_x g_2^{(1)}(z) $$

$$ -f_2^{(1)}(x,qy)\Theta_y g_2^{(1)}(z)] $$

ie., ${}_g\Theta w = g_1^{(1)} f_1^{(1)} + g_2^{(1)} f_2^{(1)}$ satisfies an equation of the form 2(61). Therefore by theorem 2(8) ${}_g\Theta w(z)$ is an element of ${}_1P_D(g^{(1)})$.

Hence the theorem is proved. By theorem 5(1) any generating vector equivalent to $g^{(1)}$ is also a successor of g.

2. Period of a generating sequence

A generating sequence $\{g^{(v)}\}$ is said to have period $\mu > 0$ if $g^{(v+\mu)}$ is equivalent to $g^{(v)}$ and is said to be non-periodic if no such μ exists. The minimal period of a generating sequence is the smallest period of a generating sequence.

Theorem 5(3)

Suppose that D is a discrete domain. Let $g = [g_1\ g_2]$ be a generating vector in D. Let $w \in G.F(D)$ be an element of ${}_1P_D(g)$ which is not a g-pseudoconstant. Then g is its own successor (ie., has minimal period one) if and only if $g_1(z) = g_2(z)\, p(y)$ where p is a function of y alone.

Proof

Suppose that $f \in F(D)$, $g \in G(D)$ and $w \in {}_1P_D(g)$, where w is not a g-pseudoconstant. Then by 2(39), 2(43) and 2(46) we have,

$$(g.\Theta_x f)(z) = {}_g\Theta w(z)$$

And

$$(g.\Theta_{\bar{z}} f)(z) = 0$$

$$\text{ie., } g_1(z)\Theta_{\bar{z}} f_1(z) + g_2(z)\Theta_{\bar{z}} f_2(z) = 0$$

$$\text{ie., } g_2(z)\left[\frac{g_1(z)}{g_2(z)}\Theta_{\bar{z}} f_1(z) + \Theta_{\bar{z}} f_2(z)\right] = 0$$

But by 2(36), $g_2 \neq 0$.

Therefore

$$\frac{g_1(z)}{g_2(z)}\Theta_{\bar{z}} f_1(z) + \Theta_{\bar{z}} f_2(z) = 0$$

Taking q-derivative with respect to x we have

$$\Theta_x\left[\frac{g_1(z)}{g_2(z)}\Theta_{\bar{z}} f_1(z) + \Theta_{\bar{z}} f_2(z)\right] = 0$$

ie., $\left\{\Theta_x[\Theta_{\bar{z}} f_1(z)]\right\} \frac{g_1(z)}{g_2(z)} + [\Theta_{\bar{z}} f_1(qx,y)]\Theta_x\left[\frac{g_1(z)}{g_2(z)} \right]$

$+ \Theta_x[\Theta_{\bar{z}} f_2(z)] = 0$ by 2(16) 5(9)

Suppose that g is its own successor, then from the definition of successor it follows that the discrete g-derivative ${}_g\Theta w$ is an element of ${}_1P_D(g)$. Hence by 2(46)

$$[g.\Theta_{\bar{z}}(\Theta_x f)](z) = 0$$

ie., $g_1(z)\Theta_{\bar{z}}\, [\Theta_x f_1(z)] + g_2(z)\Theta_{\bar{z}}\, [\Theta_x f_2(z)] = 0$

ie., $g_1(z)\Theta_x[\Theta_{\bar{z}} f_1(z)] + g_2(z)\Theta_x[\Theta_{\bar{z}} f_2(z)] = 0$

ie., $g_2(z) \left\{ \frac{g_1(z)}{g_2(z)}\, \Theta_x[\Theta_{\bar{z}} f_1(z)] + \Theta_x[\Theta_{\bar{z}} f_2(z)] \right\} = 0$

ie., $\frac{g_1(z)}{g_2(z)}\, \Theta_x[\Theta_{\bar{z}} f_1(z)] + \Theta_x[\Theta_{\bar{z}} f_2(z)] = 0$ 5(10)

(since $g_2 \neq 0$)

Substituting in 5(9) we have

$$\Theta_{\bar{z}} f_1(qx,y)\Theta_x\left[\frac{g_1(z)}{g_2(z)} \right] = 0$$

But it is assumed that w is not a g-pseudoconstant. Therefore from Theorem 2(2) it follows that f_1 is a real valued discrete function which is not q-periodic both in x and y.

Hence

$$\Theta_{\bar{z}} f_1(qx,y) \neq 0$$

Therefore

$$\Theta_x\left[\frac{g_1(z)}{g_2(z)} \right] = 0$$

$$\text{ie., } \frac{g_1(z)}{g_2(z)} = p(y)$$

$$\text{ie., } g_1(z) = g_2(z)p(y) \qquad 5(11)$$

Conversely suppose that w is a non g-pseudoconstant element of ${}_1P_D(g)$ and $g_1(z) = g_2(z)p(y)$. Then by 2(46)

$$(g.\Theta_{\bar{z}} f)(z) = 0 \text{ where } g = [g_2 p \;\; g_2]$$

$$\text{ie., } g_2(z)p(y)\Theta_{\bar{z}} f_1(z) + g_2(z)\Theta_{\bar{z}} f_2(z) = 0$$

$$\text{ie., } g_2(z)[p(y)\Theta_{\bar{z}} f_1(z) + \Theta_{\bar{z}} f_2(z)] = 0$$

$$\text{ie., } p(y)\Theta_{\bar{z}} f_1(z) + \Theta_{\bar{z}} f_2(z) = 0 \text{ since } g_2 \neq 0 \qquad 5(12)$$

We have to show that ${}_g\theta w$ ie., $(g.\theta_x f)$ is an element of ${}_1P_D(g)$.

Now,

$$[g.\theta_{\bar{z}}(\theta_x f)](z) = g_2(z)p(y)\theta_{\bar{z}}\theta_x f_1(z) + g_2(z)\theta_{\bar{z}}\theta_x f_2(z)$$

$$= g_2(z)[p(y)\theta_{\bar{z}}\theta_x f_1(z) + \theta_{\bar{z}}\theta_x f_2(z)]$$

$$= g_2(z)\theta_x[p(y)\theta_{\bar{z}} f_1(z) + \theta_{\bar{z}} f_2(z)]$$

$$= 0 \text{ by } 5(12).$$

Therefore by theorem 2(1), ${}_g\theta w \in {}_1P_D(g)$. ie., g is its own successor. Thus the theorem is proved.

Note 5(1)

Suppose that g is an element of G(D) and w a g-pseudoconstant. Then g is its own successor (ie., it has minimal period one). Hence it follows that any generating vector can be embedded in a generating sequence of minimal period one.

Theorem 5(4)

Let g be a generating vector in H^1 with components

$$g_1 = \frac{a(x)}{b(y)}, \quad g_2 = i\,\frac{b(y)}{a(x)}$$

where $a > 0$, $b > 0$ are elements of $\mathcal{H}(H^1)$. Suppose that w is an element of ${}_1P_{H^1}(g)$. Then g can be embedded in a generating sequence of minimal period 2.

Proof

Suppose that $w \in {}_1P_{H^1}(g)$. Then by 2(39), 2(40), 2(43) and 2(46) we have

$$(g.\Theta_x f)(z) = (g.\Theta_y f)(z) = {}_g\Theta w(z) \qquad 5(13)$$

and

$$(g.\Theta_{\bar{z}} f)(z) = 0, \; f \in F(H^1) \qquad 5(14)$$

ie.,

$$g_1(z)\Theta_{\bar{z}} f_1(z) + g_2(z)\Theta_{\bar{z}} f_2(z) = 0$$

But $g_1(z) = \frac{a(x)}{b(y)}$, $g_2(z) = i\,\frac{b(y)}{a(x)}$. Therefore we have

$$\frac{a(x)}{b(y)}\Theta_{\bar{z}} f_1(z) + i\,\frac{b(y)}{a(x)}\Theta_{\bar{z}} f_2(z) = 0$$

ie., $\frac{a(x)}{b(y)}[\Theta_x f_1(z) - \Theta_y f_1(z)] + i\,\frac{b(y)}{a(x)}[\Theta_x f_2(z) - \Theta_y f_2(z)] = 0$

by 2(20).

Equating real and imaginary parts to zero we have,

$$\frac{a(x)}{b(y)}\Theta_x f_1(z) - i\,\frac{b(y)}{a(x)}\Theta_y f_2(z) = 0$$

and

$$\frac{a(x)}{b(y)} i\ \Theta_y f_1(z) + \frac{b(y)}{a(x)}\Theta_x f_2(z) = 0$$

by 2(16), 2(17) and since f_1, f_2 are real valued.

ie., $\frac{a(x)}{b(y)}\Theta_x f_1(z) = i\frac{b(y)}{a(x)}\Theta_y f_2(z)$

and

$$i\frac{a(x)}{b(y)}\Theta_y f_1(z) = -\frac{b(y)}{a(x)}\Theta_x f_2(z)$$

ie., $(a(x))^2\Theta_x f_1(z) = i(b(y))^2\Theta_y f_2(z)$ 5(15)

and

$$i(a(x))^2\Theta_y f_1(z) = -(b(y))^2\Theta_x f_2(z) \qquad 5(16)$$

From 2(43) and 2(47) we have

$$_g\Theta w(z) = (g.\Theta_x f)(z)$$

$$= g_1(z)\Theta_x f_1(z) + g_2(z)\Theta_x f_2(z)$$

$$= \frac{a(x)}{b(y)}\Theta_x f_1(z) + i\frac{b(y)}{a(x)}\Theta_x f_2(z)$$

Take $g_1^{(1)} = \frac{1}{a(x)b(y)}$, $g_2^{(1)} = i\ a(x)\ b(y)$

Then $g^{(1)} = [g_1^{(1)}\ g_2^{(1)}]$ is a generating vector in H^1,

since $\mathrm{Im}(\overline{g_1^{(1)}}\ g_2^{(1)}) = 1 > 0$.

Therefore we can write

$$g^{\Theta w(z)} = [(a(x))^2\Theta_x f_1(z)]g_1^{(1)} + [\frac{\Theta_x f_2(z)}{(a(x))^2}]g_2^{(1)}$$

Taking the partial q-derivative of 5(15) with respect to x we have

$$\Theta_x[(a(x))^2\Theta_x f_1(z)] = \Theta_x[i(b(y))^2\Theta_y f_2(z)]$$

$$= i(b(y))^2\Theta_x[\Theta_y f_2(z)]$$

$$= i(b(y))^2\Theta_y[\Theta_x f_2(z)]$$

$$(\text{since } \Theta_x\Theta_y = \Theta_y\Theta_x)$$

$$= i(b(y))^2\left\{\Theta_y[\frac{\Theta_x f_2(z)}{(a(x))^2}]\right\}(a(x))^2$$

$$= i\left\{\Theta_y\left[\frac{\Theta_x f_2(z)}{(a(x))^2}\right]\right\}[a(x)b(y)]^2$$

or, $\frac{1}{a(x)\ b(y)}\ \Theta_x[(a(x))^2\Theta_x f_1(z)] = ia(x)b(y)\Theta_y[\frac{\Theta_x f_2(z)}{(a(x))^2}]$

or, $\frac{1}{a(x)\ b(y)}\ \Theta_x[(a(x))^2\Theta_x f_1(z)] - ia(x)b(y)\Theta_y[\frac{\Theta_x f_2(z)}{(a(x))^2}]$

$$= 0 \qquad 5(17)$$

5(16) can be written as

$$\frac{i\Theta_y f_1(z)}{(b(y))^2} = -\frac{\Theta_x f_2(z)}{(a(x))^2}.$$

Taking the partial q-derivative with respect to x we have,

$$\Theta_x[\frac{i\Theta_y f_1(z)}{(b(y))^2}] = -\Theta_x\ [\frac{\Theta_x f_2(z)}{(a(x))^2}]$$

ie., $\frac{1}{(b(y))^2}\ i\Theta_y[\Theta_x f_1(z)] = -\Theta_x[\frac{\Theta_x f_2(z)}{(a(x))^2}]$

ie., $\frac{1}{[a(x)\ b(y)]^2}\ i\Theta_y[(a(x))^2\Theta_x f_1(z)] = -\Theta_x[\frac{\Theta_x f_2(z)}{(a(x))^2}]$

$$\text{ie.,}\ \frac{1}{a(x)\ b(y)}\ i\theta_y[(a(x))^2\theta_x f_1(z)] = -a(x)b(y)\theta_x[\frac{\theta_x f_2(z)}{(a(x))^2}]$$

$$\text{ie.,}\ \frac{1}{a(x)\ b(y)}\ i\theta_y[(a(x))^2\theta_x f_1(z)] + a(x)b(y)\theta_x[\frac{\theta_x f_2(z)}{(a(x))^2}]$$

$$= 0 \qquad 5(18)$$

Multiplying 5(18) by i and adding to 5(17) we obtain,

$$\frac{1}{a(x)\ b(y)}\ \theta_x[(a(x))^2\theta_x f_1(z)] \ -a(x)b(y)\ \ i\theta_y[\frac{\theta_x f_2(z)}{(a(x))^2}]$$

$$+ i^2\left\{\theta_y[(a(x))^2\theta_x f_1(z)]\right\}\frac{1}{a(x)\ b(y)}$$

$$+ ia(x)b(y)\theta_x[\frac{\theta_x f_2(z)}{(a(x))^2}] = 0$$

$$\text{ie.,}\ \frac{1}{a(x)\ b(y)}\ \theta_{\bar{z}}[(a(x))^2\theta_x f_1(z)]$$

$$+ ia(x)b(y)\theta_{\bar{z}}[\frac{\theta_x f_2(z)}{(a(x))^2}] = 0 \ \text{ by } 2(20)$$

$$\text{ie.,}\ g_1^{(1)}(z)\theta_{\bar{z}}\ [(a(x))^2\theta_x f_1(z)] + g_2^{(1)}(z)\theta_{\bar{z}}\ [\frac{\theta_x f_2(z)}{(a(x))^2}] = 0$$

ie., $g_1^{(1)}(z)\Theta_{\bar{z}}f_1^{(1)}(z) + g_2^{(1)}(z)\Theta_{\bar{z}}f_2^{(1)}(z) = 0$

where

$$f_1^{(1)} = (a(x))^2\Theta_x f_1, \quad f_2^{(1)} = \frac{\Theta_x f_2}{(a(x))^2}$$

Then by theorem 2(1) ${}_g\Theta w(z)$ is an element of ${}_1P_H 1(g^{(1)})$.

By 2(47) we have,

$$_{g^{(1)}}\Theta[{}_g\Theta w(z)] = (g^{(1)}.\Theta_x f^{(1)})(z)$$

where

$$g^{(1)} = [g_1^{(1)} \; g_2^{(1)}]$$

and

$$f^{(1)} = [f_1^{(1)} \; f_2^{(1)}]'$$

But $g_1^{(1)}(z) = \frac{1}{a(x)b(y)}$, $g_2^{(1)}(z) = ia(x)b(y)$,

$$f_1^{(1)}(z) = (a(x))^2\Theta_x f_1(z) \quad \text{and} \quad f_2^{(1)}(z) = \frac{\Theta_x f_2(z)}{(a(x))^2}$$

Therefore the right hand side of the above equation is equal to

$$\frac{1}{a(x)b(y)}\Theta_x[(a(x))^2\Theta_x f_1(z)] + ia(x)b(y)\Theta_x[\frac{\Theta_x f_2(z)}{(a(x))^2}]$$

$$= \left\{ \frac{\theta_x[(a(x))^2 \theta_x f_1(z)]}{(a(x))^2} \right\} \frac{a(x)}{b(y)}$$

$$+ \left\{ (a(x))^2 \theta_x [\frac{\theta_x f_2(z)}{(a(x))^2}] \right\} i \frac{b(y)}{a(x)}$$

Take $f_1^{(2)}(z) = \dfrac{\theta_x[(a(x))^2 \theta_x f_1(z)]}{(a(x))^2}$

$$f_2^{(2)}(z) = (a(x))^2 \theta_x [\frac{\theta_x f_2(z)}{(a(x))^2}]$$

Then $f^{(2)} = [f_1^{(2)} f_2^{(2)}]' \in F(H^1)$

Therefore the right hand side of the above equation is equal to

$$g_1 \; f_1^{(2)} + g_2 \; f_2^{(2)}$$

Thus we have to show that $g_1 \; f_1^{(2)} + g_2 \; f_2^{(2)}$ is an element of ${}_1P_{H^1}(g)$ where $g = [g_1 \; g_2]$.

Now,

$$\theta_y\left[\frac{-i\theta_x\theta_y f_1(z)}{(b(y))^2}\right] = \theta_y\left\{\theta_x\left[\frac{-i\theta_y f_1(z)}{(b(y))^2}\right]\right\}$$

$$= \theta_y\left\{\theta_x\left[\frac{\theta_x f_2(z)}{(a(x))^2}\right]\right\}$$

$$= \theta_x\left\{\theta_y\left[\frac{\theta_x f_2(z)}{(a(x))^2}\right]\right\} \quad \text{by } 5(16)$$

$$= \theta_x\left\{\frac{\theta_y[\theta_x f_2(z)]}{(a(x))^2}\right\}$$

Multiplying both sides by $i(b(y))^2$ we get

$$i(b(y))^2\theta_y\left[\frac{-i\theta_x\theta_y f_1(z)}{(b(y))^2}\right] = i(b(y))^2\theta_x\left[\frac{\theta_y\theta_x f_2(z)}{(a(x))^2}\right]$$

$$= \theta_x\left[\frac{i(b(y))^2\theta_y\theta_x f_2(z)}{(a(x))^2}\right]$$

$$= \theta_x\left\{\frac{\theta_x[i(b(y))^2\theta_y f_2(z)]}{(a(x))^2}\right\}$$

$$= \theta_x\left\{\frac{\theta_x[(a(x))^2\theta_x f_1(z)]}{(a(x))^2}\right\} \quad \text{by } 5(15)$$

5(19)

But the left hand side of 5(19) is equal to

$$i(b(y))^2 \theta_y \left\{ \frac{\theta_x[-i\theta_y f_1(z)]}{(b(y))^2} \right\}$$

$$\text{ie.,} = i(b(y))^2 \theta_y \left\{ \frac{1}{(b(y))^2} \theta_x\left[\frac{b((y))^2 \theta_x f_2(z)}{(a(x))^2} \right] \right\} \text{ by } 5(16)$$

$$\text{ie.,} = i(b(y))^2 \theta_y \left\{ \theta_x\left[\frac{\theta_x f_2(z)}{(a(x))^2} \right] \right\} \qquad 5(20)$$

Equating 5(19) and 5(20) we obtain

$$\theta_x \left\{ \frac{\theta_x[(a(x))^2 \theta_x f_1(z)]}{(a(x))^2} \right\} = i(b(y))^2 \theta_y \left\{ \theta_x\left[\frac{\theta_x f_2(z)}{(a(x))^2} \right] \right\}$$

Multiplying both sides by $(a(x))^2$ we obtain

$$(a(x))^2 \theta_x \left\{ \frac{\theta_x[(a(x))^2 \theta_x f_1(z)]}{(a(x))^2} \right\} = i[a(x)\ b(y)]^2 \theta_y \left\{ \theta_x\left[\frac{\theta_x f_2(z)}{(a(x))^2} \right] \right\}$$

$$= i(b(y))^2 \theta_y \left\{ (a(x))^2\ \theta_x\left[\frac{\theta_x f_2(z)}{(a(x))^2} \right] \right\}$$

ie., $$\frac{a(x)}{b(y)}\Theta_x\left\{\frac{\Theta_x[(a(x))^2\Theta_x f_1(z)]}{(a(x))^2}\right\} = \frac{ib(y)}{a(x)}\;\Theta_y\left\{(a(x))^2\Theta_x\left[\frac{\Theta_x f_2(z)}{(a(x))^2}\right]\right\}$$

5(21)

By a similar argument we can show that

$$\frac{a(x)}{b(y)}i\Theta_y\left\{\frac{\Theta_x[(a(x))^2\Theta_x f_1(z)]}{(a(x))^2}\right\} = -\frac{b(y)}{a(x)}\Theta_x\left\{(a(x))^2\Theta_x\left[\frac{\Theta_x f_2(z)}{(a(x))^2}\right]\right\}$$

5(22)

Multiplying 5(22) by i and adding to 5(21) we obtain,

$$\frac{a(x)}{b(y)}\Theta_{\bar{z}}\left\{\frac{\Theta_x[(a(x))^2\Theta_x f_1(z)]}{(a(x))^2}\right\} = -\frac{ib(y)}{a(x)}\;\Theta_{\bar{z}}\left\{(a(x))^2\Theta_x\left[\frac{\Theta_x f_2(z)}{(a(x))^2}\right]\right\}$$

by 2(18)

ie., $$\frac{a(x)}{b(y)}\Theta_{\bar{z}}\left\{\frac{\Theta_x[(a(x))^2\Theta_x f_1(z)]}{(a(x))^2}\right\}$$

$$+\frac{ib(y)}{a(x)}\;\Theta_{\bar{z}}\left\{(a(x))^2\Theta_x\left[\frac{\Theta_x f_2(z)}{(a(x))^2}\right]\right\} = 0$$

ie., $g_1(z)\Theta_{\bar{z}}f_1^{(2)}(z) + g_2(z)\Theta_{\bar{z}}f_2^{(2)}(z) = 0$

ie., $(g.\Theta_{\bar{z}}f^{(2)})(z) = 0.$

Therefore by theorem 2(1) $g.f^{(2)}$ is an element of ${}_1P_H1(g)$. Thus g can be embedded in a generating sequence of minimal period 2.

Remark 5(1)

From the proof of the above theorem, it is clear that there may exist successors which are not equivalent to $[\frac{g_1}{i} \;\; \frac{g_2}{i}]$.

Theorem 5(5)

Let g be an element of $G(H^1)$ with components g_1, g_2 and w an element of ${}_1P_H1(g)$ which is not a g-pseudoconstant. Then g can be embedded in a generating sequence of minimal period 2. (In other words, a successor v of generating vector g can be found such that the discrete v-derivative of ${}_g\theta w$ is an element of ${}_1P_H1(g)$.)

Proof

Suppose that $g = [g_1 \;\; g_2] \in G(H^1)$, $w \in {}_1P_H1(g)$ which is not a g-pseudoconstant. Then by theorem 5(2), $g^{(1)} = [\frac{g_1}{i} \;\; \frac{g_2}{i}]$ is a successor of g.

Now take $v = [\frac{g_1}{i} \quad \frac{g_2}{i}] \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$

$= [ig_1 \quad ig_2]$

$= [v_1 \quad v_2]$ where $v_1 = ig$, $v_2 = ig_2$

But $\begin{vmatrix} -1 & 0 \\ 0 & -1 \end{vmatrix} = 1 > 0.$

Therefore by theorem 5(1), v is a generating vector equivalent to $g^{(1)}$. Hence ${}_g\Theta w \in {}_1P_H1(v)$. ie., v is also a successor of g. Now by theorem 5(2), $[g_1 \quad g_2]$ is a successor of v.

Thus it follows that the discrete v-derivative of ${}_g\Theta w \in {}_1P_H1(g)$. This proves that g can be embedded in a generating sequence of minimal period 2.

Note 5(2)

From theorems 5(3) and 5(5) it is clear that if $g_1(z) = p(y) \, g_2(z)$, then ${}_g\Theta w \in {}_1P_D(g)$ and for all other generating vectors if w is a non g-pseudoconstant element of ${}_1P_D(g)$ then we can always find a successor v of g such that the discrete v-derivative of ${}_g\Theta w$ is an element of ${}_1P_D(g)$.

CHAPTER 6

SECOND GENERATION OF DISCRETE PSEUDOANALYTIC FUNCTIONS OF THE FIRST KIND

If w is an element of ${}_1P_D(g)$, w^2, w^3, αw, $\alpha w+\beta$ etc. where α, β are complex constants do not in general belong to ${}_1P_D(g)$. In this chapter we discuss sufficient conditions for the above functions to be an element of ${}_1P_D(g)$. Denoting $\alpha w+\beta$ by w^* and taking the powers $(w^*)^2$ and $(w^*)^3$ we can find some sufficient conditions for a quadratic and a cubic polynomial to be an element of ${}_1P_D(g)$. We believe that under certain conditions on f_1, f_2, α, β etc. an n^{th} degree polynomial will become an element of ${}_1P_D(g)$.

1. Sufficient conditions for the power of a discrete pseudoanalytic function to be discrete pseudo-analytic with the same generating vector

Let g be a generating vector in a discrete domain D and w a non-zero element of ${}_1P_D(g)$.

Let $w = g_1f_1 + g_2f_2$, $[g_1\ g_2] \in G(D)$, $[f_1\ f_2]' \in F(D)$

By squaring we get,

$$w^2 = g_1^2 f_1^2 + g_2^2 f_2^2 + 2g_1 g_2 f_1 f_2$$

$$= g_1(g_1 f_1^2 + g_2 f_1 f_2) + g_2(g_2 f_2^2 + g_1f_1f_2) \quad 6(1)$$

The expressions $g_1f_1^2 + g_2f_1f_2$ and $g_2f_2^2 + g_1f_1f_2$ are real when $\mathrm{Im}(g_1f_1^2 + g_2f_1f_2) = \mathrm{Im}(g_2f_2^2 + g_1f_1f_2) = 0$

ie., when $\mathrm{Im}(g_1f_1 + g_2f_2) = 0$, since f_1, f_2 are real.

Take $g_1 = a_1 + ib_1$, $g_2 = a_2 + ib_2$, then $\mathrm{Im}(g_1f_1 + g_2f_2) = 0$ when $b_1f_1 + b_2f_2 = 0$.

ie., when $\frac{f_1}{f_2} = \frac{-b_2}{b_1}$ 6(2)

Choose f_1, f_2 such that they satisfy relation 6(2).

Take $\sigma = g_1f_1^2 + g_2f_1f_2$

$$= a_1f_1^2 + a_2f_1f_2 \quad \text{by } 6(2)$$

Similarly take

$$\eta = g_2f_2^2 + g_1f_1f_2$$

$$= a_2f_2^2 + a_1f_1f_2 \quad \text{by } 6(2)$$

Now, $g_1\partial_{\bar{z}}\sigma + g_2\partial_{\bar{z}}\eta = g_1\partial_{\bar{z}}[f_1(a_1f_1 + a_2f_2)]$

$$+ g_2\partial_{\bar{z}}[f_2(a_1f_1 + a_2f_2)]$$

$$= g_1(\Theta_{\bar{z}}f_1)(a_1f_1 + a_2f_2) + g_2(\Theta_{\bar{z}}f_2)(a_1f_1 + a_2f_2)$$

$$+ \frac{g_1}{2}[f_1(qx,y)\Theta_x(a_1f_1 + a_2f_2) - f_1(x,qy)\Theta_y(a_1f_1 + a_2f_2)]$$

$$+ \frac{g_2}{2}[f_2(qx,y)\Theta_x(a_1f_1 + a_2f_2) - f_2(x,qy)\Theta_y(a_1f_1 + a_2f_2)]$$

by 2(25) 6(3)

But $w \in {}_1P_D(g)$. Therefore $g_1\Theta_{\bar{z}}f_1 + g_2\Theta_{\bar{z}}f_2 = 0$

Thus

$$g_1\Theta_{\bar{z}}\sigma + g_2\Theta_{\bar{z}}\eta = \frac{g_1}{2}[f_1(qx,y)\Theta_x(a_1f_1 + a_2f_2)$$

$$- f_1(x,qy)\Theta_y(a_1f_1 + a_2f_2)]$$

$$+ \frac{g_2}{2}[f_2(qx,y)\Theta_x(a_1f_1 + a_2f_2)$$

$$- f_2(x,qy)\Theta_y(a_1f_1 + a_2f_2)]$$

$$= \tfrac{1}{2}[\Theta_x(a_1f_1 + a_2f_2)][g_1f_1(qx,y)$$

$$+ g_2f_2(qx,y)] - \tfrac{1}{2}[\Theta_y(a_1f_1 + a_2f_2)]$$

$$[g_1f_1(x,qy) + g_2f_2(x,qy)] \qquad 6(4)$$

We determine some sufficient conditions on f_1, f_2 such that the right hand side of 6(4) vanishes.

Case (1)

$$\text{Suppose } \Theta_x(a_1f_1 + a_2f_2) = \Theta_y(a_1f_1 + a_2f_2) = 0$$

Then $a_1f_1 + a_2f_2$ is q-periodic in x and y. ie., $a_1f_1 + a_2f_2 = p$ where p is a real element of $\pi_q(x,y)$.

ie., $f_1 \text{Rl } g_1 + f_2 \text{Rl } g_2 = p$ 6(5)

Now, by 6(2) we have

$$b_1f_1 + b_2f_2 = 0$$

ie., $f_1 \text{ Im } g_1 + f_2 \text{ Im } g_2 = 0$ 6(6)

From 6(5) and 6(6) we get

$$\begin{bmatrix} \text{Rl } g_1 & \text{Rl } g_2 \\ \text{Im } g_1 & \text{Im } g_2 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} p \\ 0 \end{bmatrix} \qquad 6(7)$$

But $\begin{bmatrix} \text{Rl } g_1 & \text{Rl } g_2 \\ \text{Im } g_1 & \text{Im } g_2 \end{bmatrix}$ is non-singular as for the generating vector g, $\text{Im}(\overline{g_1} \; g_2) > 0$.

From 6(7) we get

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} \text{Rl } g_1 & \text{Rl } g_2 \\ \text{Im } g_1 & \text{Im } g_2 \end{bmatrix}^{-1} \begin{bmatrix} p \\ 0 \end{bmatrix}$$

Thus it follows that for an arbitrary generating vector g, $w^2 \in {}_1P_D(g)$ if f_1, f_2 satisfy the above relation.

Case (2)

Suppose f_1, f_2 are such that

$$g_1(x,y)f_1(qx,y) + g_2(x,y)f_2(qx,y)$$

$$= g_1(x,y)f_1(x,qy) + g_2(x,y)f_2(x,qy) = 0$$

ie., $$\begin{bmatrix} f_1(qx,y) & f_2(qx,y) \\ f_1(x,qy) & f_2(x,qy) \end{bmatrix} \begin{bmatrix} g_1(x,y) \\ g_2(x,y) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

This is valid for all g if the matrix

$$\begin{bmatrix} f_1(qx,y) & f_2(qx,y) \\ f_1(x,qy) & f_2(x,qy) \end{bmatrix}$$ is singular

i.e., $\dfrac{f_1(qx,y)}{f_1(x,qy)} = \dfrac{f_2(qx,y)}{f_2(x,qy)} = \alpha$ (say) 6(8)

α can be a function of x and y. If f_1 and f_2 are related by the following relation:

$$f_2(x,y) = \alpha(xy)f_1(x,y) \qquad 6(9)$$

then 6(8) is satisfied.

For in that case

$$\frac{f_2(qx,y)}{f_2(x,qy)} = \frac{\alpha(qxy)f_1(qx,y)}{\alpha(qxy)f_1(x,qy)}$$

$$= \frac{f_1(qx,y)}{f_1(x,qy)}$$

Thus it follows that if f_1 and f_2 are related by 6(8) and $\dfrac{f_1}{f_2} = -\dfrac{b_2}{b_1}$ then $w^2 \varepsilon \; {}_1P_D(g)$.

Case (3)

Also

$$\frac{\Theta_x(a_1f_1 + a_2f_2)}{\Theta_y(a_1f_1 + a_2f_2)} = \frac{g_1(x,y)f_1(x,qy) + g_2(x,y)f_2(x,qy)}{g_1(x,y)f_1(qx,y) + g_2(x,y)f_2(qx,y)}$$

6(10)

Suppose that

$$\theta_x(a_1f_1 + a_2f_2) = g_1(x,y)f_1(x,qy)$$

$$+ g_2(x,y)f_2(x,qy) = 0 \qquad 6(11)$$

ie., $\theta_x(a_1f_1 + a_2f_2) = 0$ 6(12)

$$a_1(x,y)f_1(x,qy) + a_2(x,y)f_2(x,qy) = 0 \qquad 6(13)$$

and

$$b_1(x,y)f_1(x,qy) + b_2(x,y)f_2(x,qy) = 0 \qquad 6(14)$$

ie., $\begin{bmatrix} a_1(x,y) & a_2(x,y) \\ b_1(x,y) & b_2(x,y) \end{bmatrix} \begin{bmatrix} f_1(x,qy) \\ f_2(x,qy) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Now for the generating vector g, $Im(\bar{g}_1 \; g_2) > 0$. So the

$\begin{bmatrix} a_1(x,y) & a_2(x,y) \\ b_1(x,y) & b_2(x,y) \end{bmatrix}$ cannot be singular.

In that case $f_1 = f_2 = 0$. Similar result follows for the case

$$\theta_y(a_1f_1 + a_2f_2) = g_1f_1(qx,y) + g_2f_2(qx,y) = 0$$

Thus we see that there exists elements of ${}_1P_D(g)$, so that $w^2 \varepsilon {}_1P_D(g)$.

2. Sufficient conditions for αw to be an element of ${}_1P_D(g)$

Let $w \varepsilon {}_1P_D(g)$ and α be a complex constant. Take $w = g_1f_1 + g_2f_2$, $\alpha = \alpha_1 + i\alpha_2$, $[f_1 \ f_2]' \varepsilon F(D)$.

We can show that αw is not in general an element of ${}_1P_D(g)$.

$$\alpha w = (\alpha_1 + i\alpha_2)(g_1f_1 + g_2f_2)$$

$$= \alpha_1g_1f_1 + \alpha_1g_2f_2 + i(\alpha_2g_1f_1 + \alpha_2g_2f_2)$$

Take $g_1 = a_1 + ib_1$ and $g_2 = a_2 + ib_2$

Then,

$$\alpha w = \alpha_1(a_1 + ib_1)f_1 + \alpha_1(a_2 + ib_2)f_2$$

$$+ i[\alpha_2(a_1 + ib_1)f_1 + \alpha_2(a_2 + ib_2)f_2]$$

$$= u_1 + iv_1$$

where

$$u_1 = \alpha_1(a_1f_1 + a_2f_2) - \alpha_2(b_1f_1 + b_2f_2)$$

$$v_1 = \alpha_1(b_1f_1 + b_2f_2) + \alpha_2(a_1f_1 + a_2f_2)$$

Therefore by 2(37) and 2(38) we can express αw as $g_1h_1 + g_2h_2$

where

$$h_1 = \frac{1}{a_1b_2 - a_2b_1}[b_2u_1 - a_2v_1] \quad \text{and}$$

$$h_2 = \frac{1}{a_1b_2 - a_2b_1}[-b_1u_1 + a_1v_1]$$

But as $\text{Im}(\bar{g}_1 \ \ g_2) > 0$ we have $a_1b_2 - a_2b_1 > 0$

Now substituting for u_1 and v_1 we obtain

$$h_1 = \frac{1}{a_1b_2 - a_2b_1}\Big\{b_2[\alpha_1(a_1f_1 + a_2f_2) - \alpha_2(b_1f_1 + b_2f_2)]$$

$$- a_2[\alpha_1(b_1f_1 + b_2f_2) + \alpha_2(a_1f_1 + a_2f_2)]\Big\}$$

$$= \frac{1}{a_1b_2 - a_2b_1} [b_2a_1\alpha_1f_1 + a_2b_2\alpha_1f_2 - b_1b_2\alpha_2f_1$$

$$-b_2^2\alpha_2f_2 - a_2b_1\alpha_1f_1 - a_2b_2\alpha_1f_2 - a_1a_2\alpha_2f_1 - a_2^2\alpha_2f_2]$$

$$= \frac{1}{a_1b_2 - a_2b_1} [(a_1b_2 - a_2b_1)\alpha_1f_1$$

$$-(a_1a_2 + b_1b_2)\alpha_2f_1 - (a_2^2 + b_2^2)\alpha_2f_2]$$

$$= \alpha_1f_1 - (\frac{a_1a_2 + b_1b_2}{a_1b_2 - a_2b_1})\alpha_2f_1 - (\frac{a_2^2 + b_2^2}{a_1b_2 - a_2b_1})\alpha_2f_2 \qquad 6(15)$$

Similarly,

$$h_2 = \frac{1}{a_1b_2 - a_2b_1} \Big\{-b_1[\alpha_1(a_1f_1 + a_2f_2) - \alpha_2(b_1f_1 + b_2f_2)]$$

$$+ a_1[\alpha_1(b_1f_1 + b_2f_2) + \alpha_2(a_1f_1 + a_2f_2)]\Big\}$$

$$= \frac{1}{a_1b_2 - a_2b_1} [-a_1b_1\alpha_1f_1 - b_1a_2\alpha_1f_2 + b_1^2\alpha_2f_1$$

$$+ b_1b_2\alpha_2f_2 + a_1b_1\alpha_1f_1 + a_1b_2\alpha_1f_2 + a_1^2\alpha_2f_1$$

$$+ a_1a_2\alpha_2f_2]$$

$$= \frac{1}{a_1b_2 - a_2b_1} [(a_1b_2 - a_2b_1)\alpha_1 f_2 + (b_1b_2 + a_1a_2)\alpha_2 f_2$$

$$+ (a_1^2 + b_1^2)\alpha_2 f_1]$$

$$= \alpha_1 f_2 + (\frac{b_1b_2 + a_1a_2}{a_1b_2 - a_2b_1})\alpha_2 f_2 + (\frac{a_1^2 + b_1^2}{a_1b_2 - a_2b_1})\alpha_2 f_1 \qquad 6(16)$$

Now

$$g_1\Theta_{\bar{z}}h_1 + g_2\Theta_{\bar{z}}h_2 = g_1\Theta_{\bar{z}}[\alpha_1 f_1 - (\frac{a_1a_2 + b_1b_2}{a_1b_2 - a_2b_1})\alpha_2 f_1$$

$$- (\frac{a_2^2 + b_2^2}{a_1b_2 - a_2b_1})\alpha_2 f_2]$$

$$+ g_2\Theta_{\bar{z}}[\alpha_1 f_2 + (\frac{b_1b_2 + a_1a_2}{a_1b_2 - a_2b_1})\alpha_2 f_2$$

$$+ (\frac{a_1^2 + b_1^2}{a_1b_2 - a_2b_1})\ \alpha_2 f_1]$$

But $g_1\Theta_{\bar{z}}f_1 + g_2\Theta_{\bar{z}}f_2 = 0$ since $w \in {}_1P_D(g)$.

Therefore right hand side is equal to

$$\alpha_2 \Big\{ g_1\Theta_{\bar{z}}[\frac{-(a_1a_2 + b_1b_2)\ f_1 - (a_2^2 + b_2^2)f_2}{a_1b_2 - a_2b_1}]$$

$$+ g_2\Theta_{\bar{z}}[\frac{(b_1b_2 + a_1a_2)f_2 + (a_1^2 + b_1^2)f_1}{a_1b_2 - a_2b_1}]\Big\} \qquad 6(17)$$

$\neq 0$ in general.

Example 6(1)

$$w(x,y) = \left(\frac{1-q}{1-q^3}\right)x^3 + ix^2y \text{ is an element of } {}_1P_D(g)$$

where $g = [1 \quad ix^2]$, D a bounded discrete domain. But αw (where $\alpha = i$) is not an element of ${}_1P_D(g)$.

Proof

$$w(x,y) = \left(\frac{1-q}{1-q^3}\right)x^3 + ix^2y$$

$$= [1 \quad ix^2] \quad \left[\left(\frac{1-q}{1-q^3}\right)x^3 \quad y\right]'$$

$${}_g\theta_{\bar{z}}w(x,y) = \theta_{\bar{z}} \quad \left[\left(\frac{1-q}{1-q^3}\right)x^3\right] + ix^2\theta_{\bar{z}}y$$

$$= \tfrac{1}{2}\left[x^2 + ix^2\left(-\frac{1}{i}\right)\right]$$

$$= 0$$

Now

$$iw(x,y) = i\left[\left(\frac{1-q}{1-q^3}\right)x^3\right] - x^2y$$

$$= [1 \quad ix^2][-x^2y \quad \left(\frac{1-q}{1-q^3}\right)x]'$$

$$ {}_g\theta_{\bar z} iw(x,y) = \theta_{\bar z}(-x^2y) + ix^2\theta_{\bar z}[(\frac{1-q}{1-q^3})x] $$

$$ \neq 0 $$

Therefore

$iw \notin {}_1P_D(g)$.

We can easily show that $iw \in {}_1P_D(ig)$.

Now for the generating vector g, $\mathrm{Im}(\bar{g}_1\, g_2) > 0$ ie., $a_1b_2 - a_2b_1 > 0$.

We try to determine some sufficient conditions on α, f_1, f_2 such that the expression 6(17) vanishes.

Different cases arise

Case (1)

Suppose that $\alpha_2 = 0$ ie, if α is a real constant then $\alpha w \in {}_1P_D(g)$ for all $w \in {}_1P_D(g)$.

Case (2)

Suppose that

$$ \theta_{\bar z}[\frac{-(a_1a_2 + b_1b_2)f_1 - (a_2^2 + b_2^2)f_2}{a_1b_2 - a_2b_1}] $$

$$ = \theta_{\bar z}[\frac{(b_1b_2 + a_1a_2)f_2 + (a_1^2 + b_1^2)f_1}{a_1b_2 - a_2b_1}] = 0 $$

ie., $$\frac{-(b_1b_2 + a_1a_2)f_1 - (a_2^2 + b_2^2)f_2}{a_1b_2 - a_2b_1} = p_1 \qquad 6(18)$$

and

$$\frac{(b_1b_2 + a_1a_2)f_2 + (a_1^2 + b_1^2)f_1}{a_1b_2 - a_2b_1} = p_2 \qquad 6(19)$$

where p_1, p_2 are real discrete functions q-periodic in x and y.

Take

$$\gamma_1 = \frac{a_1a_2 + b_1b_2}{a_1b_2 - a_2b_1} \qquad 6(20)$$

$$\gamma_2 = \frac{(a_2^2 + b_2^2)}{a_1b_2 - a_2b_1} \qquad 6(21)$$

$$\gamma_3 = \frac{(a_1^2 + b_1^2)}{a_1b_2 - a_2b_1} \qquad 6(22)$$

Then 6(18) and 6(19) can be written as

$$-\gamma_1 f_1 - \gamma_2 f_2 = p_1$$

and

$$\gamma_3 f_1 + \gamma_1 f_2 = p_2$$

ie., $$\begin{bmatrix} -\gamma_1 & -\gamma_2 \\ \gamma_3 & \gamma_1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} p_1 \\ p_2 \end{bmatrix}$$

ie., $$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} -\gamma_1 & -\gamma_2 \\ \gamma_3 & \gamma_1 \end{bmatrix}^{-1} \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} \qquad 6(23)$$

Thus for an arbitrary generating vector g and an arbitrary complex constant $\alpha = \alpha_1 + i\alpha_2$ if f_1 and f_2 are given by 6(23) then $\alpha w \in {}_1P_D(g)$.

Case (3)

Suppose that

$$\theta_{\bar{z}}[-\gamma_1 f_1 - \gamma_2 f_2] = -g_2$$

and

$$\theta_{\bar{z}}[\gamma_3 f_1 + \gamma_1 f_2] = g_1$$

ie., $$\tfrac{1}{2}\left\{\theta_x[-\gamma_1 f_1 - \gamma_2 f_2] - \theta_y[-\gamma_1 f_1 - \gamma_2 f_2]\right\}$$

$$= -(a_2 + ib_2) \qquad 6(24)$$

and

$$\tfrac{1}{2}\left\{\theta_x[\gamma_3 f_1 + \gamma_1 f_2] - \theta_y[\gamma_3 f_1 + \gamma_1 f_2]\right\} = (a_1 + ib_1) \qquad 6(25)$$

by 2(20).

Equating real and imaginary parts we get four relations:

$$\Theta_x[-\gamma_1 f_1 - \gamma_2 f_2] = -2a_2 \qquad 6(26)$$

$$i\Theta_y[-\gamma_1 f_1 - \gamma_2 f_2] = -2b_2 \qquad 6(27)$$

$$\Theta_x[\gamma_3 f_1 + \gamma_1 f_2] = 2a_1 \qquad 6(28)$$

$$i\Theta_y[\gamma_3 f_1 + \gamma_1 f_2] = 2b_1 \qquad 6(29)$$

Let $(q^k x_o, q^m y_o)$ be a point fixed in D and denote it by (k_1, k_2). Taking the discrete integral of 6(26) partially with respect to x and 6(27) partially with respect to y we obtain

$$-\gamma_1 f_1 - \gamma_2 f_2 = -2 \int_{(k_1,k_2)}^{(x,k_2)} a_2(t,\eta)\,d(q;t) + q_1 \qquad 6(30)$$

and

$$-\gamma_1 f_1 - \gamma_2 f_2 = -\frac{2}{i} \int_{(k_1,k_2)}^{(k_1,y)} b_2(t,\eta)\,d(q;\eta) + q_2 \qquad 6(31)$$

where q_1 is a real discrete function q-periodic in x only and q_2 is a real discrete function q-periodic in y only.

From 6(30) and 6(31) we get,

$$-\gamma_1 f_1 - \gamma_2 f_2 = -2 \int_{(k_1,k_2)}^{(x,k_2)} a_2(t,\eta) d(q;t) + q_1$$

$$= -\frac{2}{i} \int_{(k_1,k_2)}^{(k_1,y)} b_2(t,\eta) d(q;\eta) + q_2$$

$$= i \int_{(k_1,k_2)}^{(k_1,y)} b_2(t,\eta) d(q;\eta)$$

$$- \int_{(k_1,k_2)}^{(x,k_2)} a_2(t,\eta) d(q;t) + p_1$$

$$= t_1 \text{ (say)} \qquad 6(32)$$

where p_1 is real and q-periodic in both x and y. By a similar argument it follows that

$$\gamma_3 f_1 + \gamma_1 f_2 = -i \int_{(k_1,k_2)}^{(k_1,y)} b_1(t,\eta)\,d(q;\eta)$$

$$+ \int_{(k_1,k_2)}^{(x,k_2)} a_1(t,\eta)\,d(q;t) + p_2$$

$$= t_2 \text{ (say)} \qquad 6(33)$$

where p_2 is a real element of $\pi_q(x,y)$.

From 6(32) and 6(33) we have

$$\begin{bmatrix} -\gamma_1 & -\gamma_2 \\ \gamma_3 & \gamma_1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}$$

$$\text{ie., } \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} -\gamma_1 & -\gamma_2 \\ \gamma_3 & \gamma_1 \end{bmatrix}^{-1} \begin{bmatrix} t_1 \\ t_2 \end{bmatrix} \qquad 6(34)$$

If f_1, f_2 are given by 6(34) then $\alpha w \in {}_1P_D(g)$.

3. <u>Sufficient conditions for $\alpha w + \beta$ to be an element of ${}_1P_D(g)$ where α, β are complex constants</u>

Take $\alpha = \alpha_1 + i\alpha_2$, $\beta = \beta_1 + i\beta_2$,

$g_1 = a_1 + ib_1$, $g_2 = a_2 + ib_2$

Suppose that $w = g.f \in {}_1P_D(g)$.

Now,

$$\alpha w + \beta = (\alpha_1 + i\alpha_2)(g_1f_1 + g_2f_2) + (\beta_1 + i\beta_2)$$

$$= (\alpha_1 + i\alpha_2)[(a_1 + ib_1)f_1 + (a_2 + ib_2)f_2]$$

$$+ (\beta_1 + i\beta_2)$$

$$= u_2 + iv_2$$

where

$$u_2 = \alpha_1a_1f_1 + \alpha_1a_2f_2 - \alpha_2b_1f_1 - \alpha_2b_2f_2 + \beta_1$$

$$v_2 = \alpha_2a_1f_1 + \alpha_2a_2f_2 + \alpha_1b_1f_1 + \alpha_1b_2f_2 + \beta_2$$

Therefore by 2(37) and 2(38) we can express $\alpha w + \beta$ by $g_1 s_1 + g_2 s_2$ where

$$s_1 = \frac{1}{a_1 b_2 - a_2 b_1}\Big\{b_2[\alpha_1 a_1 f_1 + \alpha_1 a_2 f_2 - \alpha_2 b_1 f_1 - \alpha_2 b_2 f_2 + \beta_1]$$

$$-a_2[\alpha_2 a_1 f_1 + \alpha_2 a_2 f_2 + \alpha_1 b_1 f_1 + \alpha_1 b_2 f_2 + \beta_2]\Big\}$$

On simplification

$$s_1 = \alpha_1 f_1 - \gamma_1 \alpha_2 f_1 - \gamma_2 \alpha_2 f_2 + m_1 \qquad 6(35)$$

where

γ_1, γ_2 are given by 6(20) and 6(21) respectively

and

$$m_1 = \frac{\beta_1 b_2 - a_2 \beta_2}{a_1 b_2 - a_2 b_1}$$

By a similar argument

$$s_2 = \alpha_1 f_2 + \gamma_3 \alpha_2 f_1 + \gamma_1 \alpha_2 f_2 + m_2 \qquad 6(36)$$

where γ_1, γ_3 are given by 6(20) and 6(22) respectively

and

$$m_2 = \frac{a_1 \beta_2 - b_1 \beta_1}{a_1 b_2 - a_2 b_1}$$

Now,

$$g_1\Theta_{\bar{z}}s_1 + g_2\Theta_{\bar{z}}s_2 = g_1\Theta_{\bar{z}}[\alpha_1 f_1 - \gamma_1\alpha_2 f_1 - \gamma_2\alpha_2 f_2 + m_1]$$

$$+ g_2\Theta_{\bar{z}}[\alpha_1 f_2 + \gamma_3\alpha_2 f_1 + \gamma_1\alpha_2 f_2 + m_2]$$

$$= g_1\Theta_{\bar{z}}[-\gamma_1\alpha_2 f_1 - \gamma_2\alpha_2 f_2 + m_1]$$

$$+ g_2\Theta_{\bar{z}}[\gamma_3\alpha_2 f_1 + \gamma_1\alpha_2 f_2 + m_2] \quad 6(37)$$

by 2(46)

$\neq 0$ in general.

We try to determine sufficient conditions on f_1, f_2, α, β such that the above expression vanishes.

Suppose that $\beta_1 = \beta_2 = \alpha_2$.

Then the right hand side of 6(37) is equal to

$$\alpha_2 \left\{ g_1\Theta_{\bar{z}}[-\gamma_1 f_1 - \gamma_2 f_2 + (\frac{b_2 - a_2}{a_1 b_2 - a_2 b_1})] \right.$$

$$\left. + g_2\Theta_{\bar{z}}[\gamma_3 f_1 + \gamma_1 f_2 + (\frac{a_1 - b_1}{a_1 b_2 - a_2 b_1})] \right\}$$

Many cases arise:

1. When $\alpha_2 = 0$, already discussed.

2. $\Theta_{\bar{z}}[-\gamma_1 f_1 - \gamma_2 f_2 + m_3] = \Theta_{\bar{z}}[\gamma_3 f_1 + \gamma_1 f_2 + m_4] = 0$

where

$$m_3 = \frac{b_2 - a_2}{a_1 b_2 - a_2 b_1}$$

$$m_4 = \frac{a_1 - b_1}{a_1 b_2 - a_2 b_1}$$

But the expressions inside the brackets are real. Therefore the only possibility is that

$$-\gamma_1 f_1 - \gamma_2 f_2 + m_3 = p_1$$

$$\gamma_3 f_1 + \gamma_1 f_2 + m_4 = p_2$$

p_1, p_2 are real elements of $\pi_q(x,y)$.

Therefore,
$$\begin{bmatrix} -\gamma_1 & -\gamma_2 \\ \gamma_3 & \gamma_1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} p_1 - m_3 \\ p_2 - m_4 \end{bmatrix}$$

ie.,
$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} -\gamma_1 & -\gamma_2 \\ \gamma_3 & \gamma_1 \end{bmatrix}^{-1} \begin{bmatrix} p_1 - m_3 \\ p_2 - m_4 \end{bmatrix}$$

3. $\theta_{\bar{z}}[\gamma_3 f_1 + \gamma_1 f_2 + m_4] = g_1,$ 6(38)

and

$$\theta_{\bar{z}}[-\gamma_1 f_1 - \gamma_2 f_2 + m_3] = -g_2 \qquad 6(39)$$

Then by 2(20)

$$\tfrac{1}{2}\left\{\theta_x[\gamma_3 f_1 + \gamma_1 f_2 + m_4] - \theta_y[\gamma_3 f_1 + \gamma_1 f_2 + m_4]\right\}$$

$$= a_1 + ib_1 \qquad 6(40)$$

and

$$\tfrac{1}{2}\left\{\theta_x[-\gamma_1 f_1 - \gamma_2 f_2 + m_3] - \theta_y[-\gamma_1 f_1 - \gamma_2 f_2 + m_3]\right\}$$

$$= -(a_2 + ib_2) \qquad 6(41)$$

Equating real and imaginary parts we obtain four relations of the form

$$\theta_x[\gamma_3 f_1 + \gamma_1 f_2 + m_4] = 2a_1 \qquad 6(42)$$

$$i\theta_y[\gamma_3 f_1 + \gamma_1 f_2 + m_4] = 2b_1 \qquad 6(43)$$

$$\theta_x[-\gamma_1 f_1 - \gamma_2 f_2 + m_3] = -2a_2 \qquad 6(44)$$

$$i\theta_y[-\gamma_1 f_1 - \gamma_2 f_2 + m_3] = -2b_2 \qquad 6(45)$$

Taking the discrete integral of 6(42) partially with respect to x and 6(43) partially with respect to y we get,

$$\gamma_3 f_1 + \gamma_1 f_2 + m_4 = 2 \underset{(k_1,k_2)}{\overset{(x,k_2)}{S}} a_1(t,\eta)d(q;t) + q_3$$

$$\gamma_3 f_1 + \gamma_1 f_2 + m_4 = \frac{2}{i} \underset{(k_1,k_2)}{\overset{(k_1,y)}{S}} b_1(t;\eta)d(q;\eta) + q_4$$

q_3, real and q-periodic in x only, q_4, real and q-periodic in y only.

Therefore we get

$$\gamma_3 f_1 + \gamma_1 f_2 + m_4 = -i \underset{(k_1,k_2)}{\overset{(k_1,y)}{S}} b_1(t,\eta)d(q;\eta)$$

$$+ \underset{(k_1,k_2)}{\overset{(x,k_2)}{S}} a_1(t,\eta)d(q;t) + p_3$$

$$= d_1 \text{ (say)} \qquad 6(46)$$

p_3, real and belongs to $\pi_q(x,y)$.

Similarly,

$$-\gamma_1 f_1 - \gamma_2 f_2 + m_3 = i \int_{(k_1,k_2)}^{(k_1,y)} b_2(t,\eta)d(q;\eta)$$

$$- \int_{(k_1,k_2)}^{(x,k_2)} a_2(t,\eta)d(q;t) + p_4$$

$$= d_2 \text{ (say)} \qquad 6(47)$$

where p_4, real and belongs to $\pi_q(x,y)$.

Therefore from 6(46) and 6(47) we get

$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} \gamma_3 & \gamma_1 \\ -\gamma_1 & -\gamma_2 \end{bmatrix}^{-1} \begin{bmatrix} d_1 - m_4 \\ d_2 - m_3 \end{bmatrix}$$

4. <u>Sufficient conditions for a quadratic polynomial to be an element of ${}_1P_D(g)$</u>

We have already obtained some sufficient conditions for $\alpha w + \beta$, w^2 to be elements of ${}_1P_D(g)$. Now denote $(\alpha w+\beta)$ by w^*. $(w^*)^2$ will be a quadratic polynomial of w and the sufficient conditions can be seen similarly as in the case of w^2.

Similar arguments can be used to find sufficient conditions for w^3 to be an element ${}_1P_D(g)$. $(w^*)^3$ will be a cubic polynomial of w and its sufficient conditions can be found out as for w^3. Hence we infer that an n^{th} degree polynomial is an element of ${}_1P_D(g)$ under certain conditions.

CONCLUDING REMARKS AND SUGGESTIONS FOR FURTHER STUDY

The theory of pseudoanalytic functions has applications in potential theory, theory of mechanics of continua.

Various problems in mechanics of continua lead to a system of partial differential equations of the form

$$a_1(x)U_x = b_1(y)V_y$$

$$a_2(x)U_y = -b_2(y)V_x$$

Where the coefficients are continuous, $U = U(x,y)$, $V = V(x,y)$ are real valued. Bers and Gelbart [1] have found solutions of the above system.

A discrete analogue of the above system is

$$\alpha_1(x)\theta_x U = \beta_1(y)i\theta_y V$$

$$\alpha_2(x)i\theta_y U = -\beta_2(y)\theta_x V$$

We take the coefficients to be elements of $\mathcal{H}$ (D). In the particular case when $\alpha_1 = \alpha_2 = 1$, $\beta_1 = \beta_2 = \beta > 0$, the solutions are the elements of ${}_2P_D(g)$ where $g = [1 \quad i\beta]$.

A basic result in the theory of pseudoanalytic functions is the similarity principle proved by Bers [3]. Using the similarity principle he has established formal power series expansion of pseudoanalytic functions. He has obtained generalisation of Taylors's theorem, Laurent's theorem etc.

Similar results may be obtained in the discrete case by defining a suitable continuation operator or by finding a discrete analogue of similarity principle.

Discrete quasi-conformality of elements of ${}_2P_D(g)$ can be established by defining concepts of discrete angle and distance.

A discrete hyper-pseudoanalytic theory on the lines of Withalm [1] can be developed for functions defined on H.

APPENDIX

INDEX OF SYMBOLS

In the following, a list is given of the symbols and notations frequently used in the thesis. The page number refers to the page on which the symbol first appears.

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