SOME PROBLEMS OF DISCRETE FUNCTION THEORY

THESIS SUBMITTED BY K. K. VELUKUTTY

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CERTIFICATE

Certified that the work reported in the present thesis is based on the bona fide work done by Mr.K.K. Velukutty, Teacher Fellow under Faculty Improvement Programme, under my guidance in the Department of Mathematics and Statistics, University of Cochin, and has not been included in any other thesis submitted previously for the award of any degree.

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DECLARATION

Certified that the work presented in this thesis is based on the original work done by me under the guidance of Dr. Wazir Hasan Abdi in the Department of Mathematics and Statistics, Cochin University, and has not been included in any other thesis submitted previously for the award of any degree.

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CONTENTS

Page

CHAPTER I

INTRODUCTION

This thesis is a study of discrete analytic functions defined on the lattice $\left\{(q^m x_0, q^n y_0);\ m,\ n \in Z,\ 0<q<1;\ (x_0, y_0) \text{ fixed in } \mathbb{C}\right\}$ in the complex plane. In discrete function theory, the differential operator of the classical complex analysis is replaced by a suitable difference operator. Here a new difference operator to explain analyticity in the above lattice is introduced and an attempt is made to establish a discrete analytic function theory namely q-monodiffricity of functions.

1. Principle of Discretisation

Discretisation of scientific models is initiated much earlier in applied mathematics than the study of discrete analyticity. Ruark [56] and Heisenberg [43,44] are pioneers of this principle. Scientists felt dissatisfied by the over-emphasis of the continuum structure imposed on scientific models. The important difference between the continuum and discrete structures is that infinitesimal is not considered in the latter. In discrete theory, the limit of a quotient of infinitesimals of the continuum structure is replaced by a quotient of finite quantities.

Let us quote Ruark [56], "The differential character of the principal equations of physics implies that physical systems are governed by laws which operate with a precision beyond the limits of verification by experiment. This appears undesirable from an axiomatic standpoint".

The important aspects are the fundamental equations must be capable of describing every feature of the experiment and must not introduce extraneous or undesirable features.

Discrete hodon and chronon are introduced in Physics in recent times. This shows an interest from the side of scientists towards discretisation. Still, there is a task before the scientist to overcome. The differential equations are to be recasted in the form of difference equations.

In Margenau's [52] words, "A word might be said about the reason why physicists are often reluctant to accept discreteness. If it were to be established as the ultimate property of time and space, one or the other of two drastic changes in the theoretical description of nature would have to take place. One is the recasting of all equations of motion in the form of difference equations instead of differential equations, and this is most unpalatable because of the mathematical difficulties attending the solution of difference equations. The other possible modification would involve the elimination of time and space coordinates from scientific description".

Heisenberg is a powerful advocate of this. To simplify the problem, finite geometries of Veblen and others can be utilised or a continuous space time of the Minkowski form in which the events from a discrete lattice may be recommended.

The most general form of a lattice is a sequence of complex numbers, preferrably a dense subset which is also countable. Accepting the postulate of rational description in Physics, the lattice of rational points in the complex plane: $\{(p,q);\ p,\ q \in Q$, the set of rational numbers$\}$ will be the best choice to build a discrete function theory.

In the earliest works of discrete function theory, the arithmetically spaced sequence, in particular the Gaussian integers was considered. Later in the beginning of this decade, a function theory was developed on the set of geometrically spaced sequence. No work is done so far in the general set.

Now discrete function theory has grown to an established branch of Mathematics. The important problem is as E.T. Bell puts, "A major task of Mathematics today is to harmonise the continuous and the discrete to include them in one comprehensive Mathematics and to eliminate obscurity from both". Again a major task of discrete analysts is the unification of known theories.

2. Historical Survey

The theory of discrete functions had its start from R.P. Isaacs's distinguishing paper [45], 'A finite difference function theory' in 1941. He introduced two types of difference operators to describe analyticity in the arithmetically spaced lattice namely monodiffricity of first and second kinds [45,46]. He utilised basic triad and tetrad to define these operators. He studied integration, residues, discrete powers and polynomials. Two of the major difficulties in discrete function theory are (1) the usual product of two discrete analytic functions in a domain is not discrete analytic in that domain and (2) the usual powers of z are not discrete analytic in any domain in the discrete space. Isaacs himself realised these aspects and introduced the analogues.

Later in 1944, Ferrand [33] introduced a discrete function theory basing on another difference operator known as diagonal quotient to describe discrete analyticity called preholomorphicity. She made use of the basic square to define this discrete analyticity.

The development in discrete function theory, was slow for more than a decade from Ferrand's work, though Terracini and Romanov contributed in this decade to discrete function theory. The awakening was made by R.J. Duffin [25] in 1956. He [25-31] modified Ferrand's theory and extended the results to the realm of Applied Mathematics by discussing operational calculus and Hilbert transform. Pioneers of his school of discrete function theory are Duris [29,30], Rohrer [32], Peterson [31] and Kurowski [47-50]. Duffin [26] introduced rhombic lattice to develop potential theory. He also studied Yukawa potential theory in the discrete space of Gaussian integers [27]. Duffin and Duris [30] studied discrete product and discrete partial differential equations.

The Russian school of discrete function theory of which the leading names are Abdullaev [4-7], Babadzanov [5-7], Chumakov [21], Silic [57] and Fuksman [34], has improved the theory by introducing different lattice, construction of a discrete analytic function and so on. In particular, Chumakov [21] developed semi-discrete function theory and Silic [57] investigated physical models in discrete function theory.

Hayabara [41,42], Deeter and Lord [23,54] developed operational calculus for discrete functions.

The school led by Deeter, whose distinguishing figures are Berzsenyi [12-14], Perry and Mastin [24] has studied discrete functions in Isaacs's direction. Perry studied generalised discrete functions.

Abdullaev, Babadzanov and Hayabara developed discrete theory of higher dimensions. Kurowski [47-50] introduced a function theory in the semi-discrete lattice. Transform techniques were analysed by many like Duffin [25,28] and Bednar [11]. Mastin [24], Ferrand [33] and Isaacs [45,46] constructed theories of conformal representation. Tu [60-62] discussed discrete derivative equations in three papers. The discrete theory was extended by Hundhausen to harmonic analysis. Deeter [22] and Berzsenyi [14] gave comprehensive bibliography of discrete function theory.

All the works so far explained are mainly in the set of Gaussian integers. Harman [38-40] developed a discrete function theory in the geometric lattice in 1972, by utilising the q-difference theory developed by Jackson, Hahn and Abdi. Differentiation, integration, convolution product, polynomial theory and conformal mapping were discussed in his thesis. He modified the continuation operators of Duffin, Kurowski and Abdullaev using q-difference theory and incorporated the convolution product with it. As against the classical case, the fundamental theorem of algebra does not hold good in discrete function theory. Isaacs, Terracini and Harman investigated the roots of discrete polynomials.

Later Zeilberger [63-69] introduced a few results such as discrete powers and entire functions in the set of Gaussian integers. Recently Subhash Kak [59] extended Duffin's theory of Hilbert transform to the realm of electronics. Mugler [54] also studied exponential function.

3. Background of q-Monodiffricity

In classical analysis, analyticity of a function in a domain means its differentiability in that domain. In discrete function theory the same concept is taken over; but the continuous derivative is replaced by its counterpart, the discrete derivative. Usually the discrete analyticity is expressed in terms of a difference operator. A triad, a tetrad or a basic square of lattice points is considered to evolve such an operator. The important concepts of discrete analyticity are Monodiffricity of first and second kinds, Preholomorphicity, Rhombic analyticity, semi-discrete analyticity of first and second kinds, and q- and p-analyticities. The first two are defined in the arithmetic lattice: $\left\{(mh, nh);\ m, n \in Z,\ h > 0 \text{ fixed}\right\}$ and the third in the lattice of Gaussian integers. It is clear from the name that rhombic analyticity is defined in the rhombic lattice and semi-discrete analyticity in the semi-discrete lattice: $\{(x,y);\ x \in R,\ y = nh,\ n \in Z,\ h > 0 \text{ fixed}\}$. q- and p-analyticities are defined in the geometric lattice:

$$H = \{(q^m x_0, q^n y_0);\ m,\ n \in Z,\ 0 < q < 1 \text{ fixed } x_0 > 0,\ y_0 > 0 \text{ fixed}\}. \tag{1.1}$$

The corresponding difference operators are respectively,

$$M_1 f(z) \equiv (1-i)f(z) + if(z+h) - f(z+ih), \tag{1.2}$$

$$M_2 f(z) \equiv f(z+ih) - f(z-ih) - i[f(z+h) - f(z-h)], \tag{1.3}$$

$$M_3 f(z) \equiv f(z) + if(z+1) + i^2 f(z+1+i) + i^3 f(z+i), \tag{1.4}$$

$$N_1 f(z) \equiv (z_2 - z_4)[f(z_3) - f(z_1)] - (z_3 - z_1)[f(z_2) - f(z_4)]$$

where z_1, z_2, z_3 and z_4 are the vertices of a rhombus in the lattice, (1.5)

$$S_1 f(z) \equiv f(z) - f(z+ih) + ih\frac{\partial f(z)}{\partial y}$$ where $\frac{\partial f(z)}{\partial y}$ is the usual continuous partial derivative of $f(z)$ w.r.t. y, (1.6)

$$S_2 f(z) \equiv f(z+ih) - f(z-ih) - 2ih\frac{\partial f(z)}{\partial y}, \tag{1.7}$$

$$R_q f(z) \equiv \bar{z}f(z) - xf(x,qy) + iyf(qx,y) \tag{1.8}$$

and

$$R_p f(z) \equiv \bar{z}f(z) - xf(x,py) + iyf(px,y) \tag{1.9}$$

Equality of any of the expressions to zero at some lattice point gives the concerned analyticity at that point.

Also these operators are derived by assuming the discrete analogue of Cauchy-Riemann relations and Cauchy's integral formula.

Curve and domain in the discrete sense are defined in terms of directly or diagonally adjacent points to suit the mode of discretisation. If $\{(x_m, y_n);\ m, n \in Z\}$ is the lattice structure, any point in the set $\{(x_{s+1}, y_t), (x_s, y_{t+1}), (x_{s-1}, y_t), (x_s, y_{t-1})\}$ is a directly adjacent point of (x_s, y_t) and any point in the set $\{(x_{s+1}, y_{t+1}), (x_{s-1}, y_{t+1}) (x_{s-1}, y_{t-1}), (x_{s+1}, y_{t-1})\}$ is a diagonally adjacent point of (x_s, y_t). A sequence of points is a discrete curve if each set of consecutive points in the sequence are directly (diagonally) adjacent points. Accordingly the path of integration is defined. (1.10)

There are two standard ways of defining discrete integration. If z_j and z_{j+1} are directly adjacent points, monodiffric type:

$$\int_{z_j}^{z_{j+1}} f(z)\delta z = \begin{cases} f(z_j)(z_{j+1}-z_j) \text{ if } z_{j+1} = z_j + h \text{ or } z_j + ih \\ -\int_{z_{j+1}}^{z_j} f(z)\delta z \text{ if } z_{j+1} = z - h \text{ or } z_j - ih \end{cases} \qquad (1.11)$$

and preholomorphic type:

$$\int_{z_j}^{z_{j+1}} f(z)dz = \frac{f(z_j) + f(z_{j+1})}{2}(z_{j+1}-z_j). \qquad (1.12)$$

The first definition is used in semi-discrete theory and q-analytic theory and the second in rhombic analytic theory.

Using the definition of discrete analyticity by the difference operator, a discrete analytic function in a certain domain can be continued discrete analytically to the entire discrete plane.

In preholomorphic theory, the continuation of such a function can be done from the coordinate axes and in monodiffric theory of first kind, the continuation is possible to the upper half plane from the x-axis. The same method is utilised in q-analytic theory also. But using the q-difference theory, we get that a q-analytic function can be continued from any of the axes to the entire discrete geometric space. Similar continuation is seen in p-analytic theory.

To overcome the difficulty that the product of two discrete analytic functions in any domain is not discrete analytic in general, different discrete products are attempted. In monodiffric and preholomorphic theories the discrete product arises from double dot line integrals which read

$$\int_z^{z+k} f(z) : g(z)dz = \begin{cases} f(z+k)[g(z+h) - g(z)] & \text{if } k = 1 \text{ or } i \\ -\int_{z+k}^{z} f(z) : g(z)dz & \text{if } k = -1 \text{ or } -i \end{cases}$$

in monodiffric theory specialising to the Gaussian integers due to Berzsenyi (1.13)

and

$$\int_z^{z+k} f(z) : g(z)dz = [f(z+k) + f(z)][g(z+k) + g(z)]k,$$

where k = 1, -1, i or -i in preholomorphic theory due to Duffin. (1.14)

If $*$ is the discrete product,

$$f*g(z) = \int_C f(z-\xi):g(\xi)d\xi$$ where C is an admissible curve in the concerned domain in monodiffric theory and (1.15)

$$f*g(z) = \int_0^z f(z-\xi):g(\xi)d\xi$$ in preholomorphic theory. (1.16)

Kurowski defined discrete product in terms of the continuation operator:

$$f*g(z) = \sum_{k=0}^{\infty} i^k \binom{y}{k} \Delta_1^k[f(x,0)\ g(x-1,0)]$$

$$= \sum_{k=0}^{\infty} i^k \binom{y}{k}\ g(z-1+k-ik)\ \Delta_1^k f(x,0) \tag{1.17}$$

where $\Delta_1 f(z) = f(z+1) - f(z)$.

In the same way, discrete product in q-analytic theory is defined as

$$f*g(z) = \mathcal{C}_y[f(x,0)\ g(x,0)]$$

$$= \sum_{j=0}^{\infty} \frac{(1-q)^j}{(1-q)_j} (iy)^j\, \mathcal{D}_x^j[f(x,0)\ g(x,0)]$$ where

$$(1-q)_j = (1-q)(1-q^2)\dots(1-q^j) \text{ and } \mathcal{D} f(x,0) = \frac{f(x,0)-f(qx,0)}{(1-q)x}$$

(1.18)

Similar product is defined in p-analytic theory.

The second task is eliminated by introducing discrete powers to replace the usual powers. The following are the discrete powers:

$$z^{(n)} = \sum_{j=0}^{n} \binom{n}{j} (x)_j \, i^{n-j} \, (y)_{n-j} \text{ where } (x)_j = x(x-1)\ldots(x-j+1),$$

in monodiffric theory due to Isaacs. (1.19)

$$z^{(n+1)} = (n+1)\int_0^z z^{(n)}dz; \; z^{(0)} = 1 \text{ in preholomorphic theory due}$$

to Duffin. (1.20)

Similar discrete powers as that of Duffin are defined in monodiffric theory by Berzsenyi, rhombic analytic theory by Duffin and semi-discrete theory by Kurowski.

$$z^{(n)} = \sum_{j=0}^{n} \frac{(1-q)^j}{(1-q)_j} (iy)^j \, \vartheta_x^j f(x,0) \text{ in q-analytic theory}$$

due to Harman. (1.21)

Similar discrete powers are introduced in p-analytic theory. We note that Harman derived the discrete powers using the continuation operator.

We cannot avoid some mention of q-difference theory because a discrete function theory developed on the geometric lattice will be firmly dependent on the q-difference theory.

Fermat, Euler, Gauss, Laplace, Heine and Babbage were the early pioneers of q-difference theory.

In this century, an extensive study of q-difference theory was made by Jackson, Hahn [37] and Abdi [1-3]. Al-Salam [8,9] and Andrews [10] improved the q-basic theory in recent times. Milne-Thomson's [53] 'Calculus of finite differences' is a prerequisite to study q-basic theory as well as discrete function theory.

Jackson introduced q-analogues of derivative and integration as

$$\Theta f(x) = \frac{f(x) - f(qx)}{(1-q)x}, \quad |q| \neq 1 \text{ and} \tag{1.22}$$

$$\Theta^{-1} f(x) = \frac{1}{1-q} \int f(x)\, d(q,x). \tag{1.23}$$

Accordingly,

$$\int_0^x f(x)d(q,x) = (1-q)x \sum_{j=0}^{\infty} q^j f(q^j x),$$

$$\int_x^{\infty} f(x)d(q,x) = (1-q)x \sum_{j=1}^{\infty} q^{-j} f(q^{-j} x),$$

and $\int_0^{\infty} f(x)d(q,x) = (1-q)x \sum_{j=-\infty}^{\infty} q^j f(q^j x)$ define integration as a sum. **(1.24)**

We also note the following notations in q-basic theory.

$$(1+x)_n = (1+x)(1+qx)\ldots(1+q^{n-1}x); \ (1+x)_0 = 1$$

$$\binom{n}{r}_q = \frac{(1-q)_n}{(1-q)_r(1-q)_{n-r}}$$

$$[n]! = \frac{1-q^n}{1-q} \cdot \frac{1-q^{n-1}}{1-q} \cdots \frac{1-q^2}{1-q} \cdot \frac{1-q}{1-q} = \frac{(1-q)_n}{(1-q)^n}. \tag{1.25}$$

The solution of f(x) = f(qx) is called q-periodic function. This function plays the role of a constant in the q-difference theory. Pincherle found a solution as

$$\phi(x) = x^{\alpha-\beta} \prod_{n=0}^{\infty} \frac{(1-q^{\alpha+n}x)(1-q^{1-\alpha-n}x^{-1})}{(1-q^{\beta+n}x)(1-q^{1-\beta-n}x^{-1})}. \tag{1.26}$$

The following are also q-periodic functions.

$$\text{Sin } \frac{2\pi \log x}{\log q} \text{ due to Harman and } \tan(\pi \log_q x). \tag{1.27}$$

The first has infinite number of zeroes and has no poles. But the second has infinite number of zeroes as well as poles as the Pincherle's function.

With such a basic foundation, a new version of discrete function theory is envisaged in this thesis.

Finally, quoting from Berzsenyi [14] "At present research in the theory of analyticity in the discrete is steadily gaining recognition. In view of the fact that computational complexities can be overcome with the aid of computers, this area of Mathematics provides a workable model for the numerical analysis of analytic functions. In fact, one may prophesize the advent of the day when the direct application of discrete analyticity will replace the discretising of many of the continuous models in classical analysis".

4. Summary of Results Established

This research starts from the investigation of functions which are both q- and p-analytic in certain domain in the discrete geometric space. The solution is named bianalytic function. The continuation of such a function from two adjacent rays is examined. Then the problem is generalised as investigation of functions having p- and q-residues equal. It is found that such functions satisfy the notion of monodiffricity of second kind in the geometric lattice. Such functions are now named q-monodiffric functions.

Monodiffricity of second kind was totally neglected so far. Further, writers like Duffin [25] and Harman [38] mistook the idea that monodiffricity of second kind and preholomorphicity are equivalent. This assertion is disproved along the pages of this thesis.

In the second chapter, q-monodiffric differentiation is discussed in detail, q-monodiffric constant which is the general solution of the derivate equation: first derivate is equated to zero, is studied.

In discrete function theory, the concept of construction of an entire discrete analytic function from its discrete analyticity in a known domain, using the difference operator defined to describe the discrete analyticity is important. We have explained the construction of bianalytic function and q-monodiffric function. Bifunctions and q-monodiffric constants are well studied. They stand to replace the concepts of functions and complex numbers respectively of the classical complex function theory. The condition that the usual product of two q-monodiffric functions in a given domain is also q-monodiffric function there is also analysed.

Among the three approaches to analytic function theory, the second is dealt in the third chapter whereas the third in the fourth chapter. Here two types of integrals are defined. Either of them will not stand as a counterpart to the classical integral. But both of them taken together represent the theory of integration in q-monodiffric theory and plays the same role of classical integration. Fundamental concepts of integration like Cauchy's integral formula and theorem are developed in the q-monodiffric sense. Meromorphic function along with pole and polar residue is studied. The relation between these integrals is also obtained.

The second fundamental difficulty arose in the formulation of the discrete function theory is solved by introducing discrete powers in the q-monodiffric sense. Again this leads to the third approach of a discrete analytic function namely representation of it in the form of an infinite series in terms of discrete powers. Unlike the previous theories, results like n^{th} discrete power of z has exactly n zeroes hold in this theory. Some estimates of discrete powers are evolved. Using these estimates convergence of infinite series is discussed. Also a comparison test to decide the convergence of infinite series is found.

The late sections of the fourth chapter deals with polynomials and zeroes of them. Mainly three types of polynomials: polynomials defined over complex numbers, biconstants and q-monodiffric constants are studied.

Quadratic polynomials of each type are exercised in detail with roots of unity in the q-monodiffric sense: in other words, the zeroes of the equation $z^{(n)} = 1$ are obtained.

Lastly, special polynomials are discussed. A theory to classify the discrete polynomials is obtained and some special polynomials are classified in this line. Another way of describing a set of discrete polynomials is from the generating function. Such a study is also completed in the fifth chapter. Simple and complete sequence of such a type is described. Properties are also discussed.

CHAPTER II

BASIC PROPERTIES OF q-MONODIFFRIC FUNCTIONS

In this chapter, we study the properties of a class of functions which are both q- and p-analytic called bianalytic functions. Continuation of such a function from two adjacent straight rays to entire H is given. This leads to the study of more general class of functions having the q- and p-residues equal namely q-monodiffric functions. The condition of q-monodiffricity of the usual product of two q-monodiffric functions in certain domains is found. The solution of the derivate equation: the first derivate of a q-monodiffric function in a domain equated to zero, called q-monodiffric constant is investigated.

1. Bianalytic Functions

A theory of discrete analytic functions was developed by Harman on the geometric lattice $H = \left\{ (q^m x_0, q^n y_0);\ m,\ n \in Z,\ 0 < q < 1,\ \text{fixed},\ (x_0, y_0)\ \text{fixed}\ x_0 > 0,\ y_0 > 0 \right\}$. In what follows $z \in H$, $z = (x,y) = (q^m x_0, q^n y_0)$, $\bar{z} = (x,-y)$ and $p = q^{-1}$. Two operators R_q and R_p are defined with,

$$R_q f(z) = \bar{z} f(x,y) - x f(x,qy) + iy f(qx,y) \tag{2.1}$$

and

$$R_p f(z) = \bar{z} f(x,y) - x f(x,py) + iy f(px,y) \tag{2.2}$$

where $f : H \longrightarrow \emptyset$.

$R_q f(z)$ and $R_p f(z)$ are respectively called the q- and p-residues of the function at z. If the q-residue (p-residue) of f is zero at z, f is said to be q-analytic (p-analytic) at z.

In establishing his theory Harman used two discrete derivatives of f at z with respect to q as

$$\mathcal{D}_x f(z) = \frac{f(x,y) - f(qx,y)}{(1-q)x} \quad \text{and} \quad \mathcal{D}_y f(z) = \frac{f(x,y) - f(x,qy)}{(1-q)iy}. \tag{2.3}$$

But when $\mathcal{D}_x f(z) = \mathcal{D}_y f(z)$, we write $\mathcal{D} f(z)$ for both and it is easy to see that f becomes q-analytic at z. Similarly, with respect to p,

$$\mathcal{O}_x f(z) = \frac{f(x,y) - f(px,y)}{(1-p)x} \quad \text{and} \quad \mathcal{O}_y f(z) = \frac{f(x,y) - f(x,py)}{(1-p)iy}. \tag{2.4}$$

Accordingly, when $\mathcal{O}_x f(z) = \mathcal{O}_y f(z)$, we write $\mathcal{O} f(z)$ for both and then f is p-analytic at z.

<u>Definitions</u>. In order to develop the concept of bianalytic functions, we will need the following definitions.

The set of points $\lambda(z) = \{(q^s x, q^s y),\ s \in Z\}$ is called the straight ray through z and the set $\lambda^*(z) = \{(q^{-s} x, q^s y),\ s \in Z\}$ the distorted ray. (2.5)

It turns out that Λ is a set of collinear lattice points in the Euclidean sense and Λ^* a set of lattice points lying on a branch of a rectangular hyperbola.

The basic set of $z \in H$ is defined as $T(z) = \{(qx,y),\ (x,qy),\ (q^{-1}x,y),\ (x,q^{-1}y),\ (x,y)\}$. A subset S of H is called a region if

$S = \bigcup_{i=1}^{N} T(z_i)$, $i = 1,2,\ldots,N$. N can be also infinite. If $T(z) \subset S$, z is an interior point of S. The set of interior points in S is denoted by D and is called a domain. Thus ∂D, the compliment of D in S becomes the boundary of D. Accordingly $S = D \cup \partial D$ and $D \cap \partial D = \emptyset$. Now if D is treated as a region, it has an interior D^1 and $D - D^1 = \partial D^1$. Similarly for $n \in Z^+$, D^n is defined as the interior of D^{n-1}. We get $\partial D^n = D^{n-1} - D^n$ and $D^n = S - \bigcup_{i=0}^{n} D^i$ where D^0 means D. (2.6)

We will call a function $f : H \longrightarrow \emptyset$ bianalytic in D if it is both q- and p-analytic there. In fact, it satisfies the equation $R_q f(z) = R_p f(z) = 0$ everywhere in D.

(2.7)

It is seen that $f(z) = \alpha z + \beta$; $\alpha,\beta \in \emptyset$ is a trivial example of a bianalytic function in entire H. In the sequel, the set of bianalytic functions in D is denoted by $B(D)$.

Now let f be bianalytic at z, then by definition $\mathcal{D}_x f(z) = \mathcal{D}_y f(z)$.

$$\text{So } \mathcal{D}_x f(z) = \frac{f(x,y) - f(qx,y)}{(1-q)x}$$

$$= \frac{f(qx,y) - f(x,y)}{(1-q^{-1})qx}$$

$$= \Theta_x f(qx,y)$$

$$= \Theta f(qx,y).$$

Similarly $\mathcal{D}_y f(z) = \Theta f(x,qy)$.

Combining, $\Theta f(x,qy) = \Theta f(qx,y)$.

Thus we have:

<u>Lemma</u>. Let $f \in B(D)$ and $(x,y) \in D$. Then,

$\mathcal{D} f(x,qy) = \mathcal{D} f(qx,y)$ and $\Theta f(x,qy) = \Theta f(qx,y)$.

Further as a consequence of the lemma,

$\mathcal{D} f(q^{-1}x,qy) = \mathcal{D} f(x,y) = \mathcal{D} f(qx,q^{-1}y)$.

By iteration, $\mathcal{D} f(q^{-r}x,q^{r}y) = \mathcal{D} f(x,y) = \mathcal{D} f(q^{r}x,q^{-r}y)$

Similarly, $\Theta f(q^{-r}x,q^{r}y) = \Theta f(x,y) = \Theta f(q^{r}x,q^{-r}y)$.

Thus

if $f \in B(D)$, its derivatives along the distorted ray are invariant. (2.8)

Conversely suppose f is q-analytic in D, and has q-derivative k(z) everywhere on $\Lambda^*(z) \cap D$.
Then by a simple calculation, $\Theta f(q^{-(s+1)}x, q^s y) = k(z)$ everywhere on $\Lambda^*(z') \cap D$ where $z' = (q^{-1}x, y)$.

Similarly, if the p-derivative of f is $k_1(z)$ invariant everywhere on $\Lambda^*(z) \cap D$, then $\mathcal{V} f(q^{-(s-1)}x, q^s y) = k_1(z)$ for every lattice point on $\Lambda^*(z'') \cap D$ where $z'' = (qx, y)$.
In other words,

If f is q-analytic (p-analytic) in D and $\mathcal{V} f(\Theta f)$ is invariant on $\Lambda^*(z) \cap D$ for $z \in D$, then f is bianalytic in the interior of D. (2.9)

Summing up we have:

Theorem. A necessary and sufficient condition for f to be bianalytic in the interior of a domain D is invariance of its derivatives on each distorted ray.

2. Construction of a Bianalytic Function

Given a function defined on two adjacent straight rays $\{(q^s x_1, q^s y_1);\ s \in Z\}$ and $\{(q^{s+1}x_1, q^s y_1);\ s \in Z\}$. We want to construct f which is bianalytic on entire H. Suppose $f(q^s x_1, q^s y_1) = a_s$, $f(q^{s+1}x_1, q^s y_1) = b_s$. Then $k_s = \dfrac{a_s - b_s}{q^s(1-q)x_1}$ is the q-derivative of f at $(q^s x_1, q^s y_1)$.

Let r be a positive integer, we denote $f(q^{s-r}x_1, q^{s+r}y_1)$, $f(q^{s+r}x_1, q^{s-r}y_1)$, $f(q^{s-r+1}x_1, q^{s+r}y_1)$ and $f(q^{s+r}x_1, q^{s-r+1}y_1)$ by α_r, α_{-r}, β_r and β_{-r} respectively.

Then due to invariance of the derivative on the distorted ray passing through $(q^s x_1, q^s y_1)$,

$$\frac{\alpha_{r-1} - \beta_r}{i(1-q)q^{s+r-1}y_1} = k_s = \frac{\alpha_r - \beta_r}{(1-q)q^{s-r}x_1}$$

i.e., $\alpha_r = \beta_r + k_s(1-q)q^{s-r}x_1$

and

$$\alpha_{r-1} = \beta_r + k_s(1-q)q^{s+r-1}iy_1.$$

Then $\alpha_r = \alpha_{r-1} + k_s(1-q)\ [q^{s-r}x_1 - iq^{s+r-1}y_1]$

$$= a_s + k_s(1-q)[q^s(q^{-r} + q^{-(r-1)} + \dots + q^{-1})x_1$$

$$-iq^s(q^{r-1} + q^{r-2} + \dots + q^1 + q^0)y_1]$$

$$= a_s + k_s(1-q)q^s[q^{-1}\frac{1-q^{-r}}{1-q^{-1}}x_1 - i\frac{1-q^r}{1-q}y_1]$$

$$= a_s + k_s(1-q^r)q^s(q^{-r}x_1 - iy_1). \tag{2.10}$$

Now $\dfrac{\alpha_{-r} - \beta_r}{i(1-q)q^{s-r}y_1} = k_s = \dfrac{\alpha_{-(r-1)} - \beta_r}{(1-q)q^{s+r-1}x_1}.$

Using the same argument, we have

$$\alpha_{-r} = a_s + k_s(1-q^r)q^s(-x_1 + iq^{-r}y_1) \qquad (2.11)$$

$f(q^{s-r+1}x_1, q^{s+r}y_1)$ and $f(q^{s+r-1}x_1, q^{s-r}y_1)$ can be continued from $(q^{s-1}x_1, q^s y_1)$ at which the values of the function and the q-derivative are found from the following.

$$\frac{f(q^{s-1}x_1, q^{s-1}y_1) - f(q^{s-1}x_1, q^s y_1)}{(q^{s-1} - q^s)iy_1} = k_{s-1}$$

and

$$f(q^{s-1}x_1, q^s y_1) = \frac{f(q^{s-1}x_1, q^s y_1) - f(q^s x_1, q^s y_1)}{(q^{s-1} - q^s)x_1}$$

as $f(q^{s-1}x_1, q^s y_1) = a_{s-1} - k_{s-1}q^{s-1}(1-q)iy_1$

and $\mathcal{D} f(q^{s-1}x_1, q^s y_1) = \dfrac{a_{s-1} - k_{s-1}(1-q)iy_1 - a_s}{q^{s-1}(1-q)x_1}$.

If $f(q^{s-1}x_1, q^s y_1)$ and $\mathcal{D} f(q^{s-1}x_1, q^s y_1)$ are denoted respectively by c_s and d_s, from (2.10) and (2.11) we have

$$f(q^{s-1-r}x_1, q^{s+r}y_1) = c_s + d_s(1-q^r)q^s(q^{-(r+1)}x_1 - iy_1) \qquad (2.12)$$

and

$$f(q^{s+r-1}x_1, q^{s-r}y_1) = c_s + d_s(1-q^r)q^{s-1}(-x_1 + iq^{-r}y_1). \qquad (2.13)$$

(2.10), (2.11), (2.12) and (2.13) together give the continuation of f to entire H.

From (2.10),

$$f(q^{s-1}x_1, q^{s+1}y_1) = k_s q^s(1-q)(q^{-1}x_1 - iy_1) + a_s$$

and

$$f(q^{s-2}x_1, q^s y_1) = k_{s-1}q^{s-1}(1-q)(q^{-1}x_1 - iy_1) + a_{s-1}.$$

If $f(q^{s-1}x_1, q^s y_1)$ is denoted by f, from difference quotients,

$$\frac{f - a_s}{(q^{s-1} - q^s)x_1} = \frac{f - a_s - k_s q^s(1-q)(q^{-1}x_1 - iy_1)}{(q^s - q^{s+1})iy_1} = k_s$$

and

$$\frac{f - a_{s-1}}{(q^s - q^{s-1})iy_1} = \frac{f - a_{s-1} - k_{s-1}q^{s-1}(1-q)(q^{-1}x_1 - iy_1)}{(q^{s-1} - q^{s-2})x_1} = k_{s-1}.$$

They reduce to two relations,

$$f = a_{s-1} + k_{s-1}(q-1)q^{s-1}iy_1$$

and $$f = a_s + k_s q^{s-1}(1-q)x_1.$$

We get $a_s + k_s q^{s-1}(1-q)x_1 = a_{s-1} + k_{s-1}q^{s-1}(q-1)iy_1$.

Thus $x_1(a_s-b_s) + (a_{s-1}- b_{s-1})iqy_1 + (a_s-b_{s-1})qx_1 = 0$ is the condition of existence of the continuation to $(q^{s-1}x_1, q^s y_1)$. (2.14)

The theorem of the above section guarantees existence and uniqueness of continuation to all $(x,y) \varepsilon H$. Thus we have:

Theorem. If $f(q^s x_1, q^s y_1) = a_s$ and $f(q^{s+1}x_1, q^s y_1) = b_s$, $s \varepsilon Z$ are given subject to the condition

$x_1(a_s-b_s) + (a_{s-1}-b_{s-1})iqy_1 + (q_s-a_{s-1})qx_1 = 0$,

then f is determined uniquely and belongs to B(H).

Immediate consequences of the above theorem are

Corollary 1. If $f(q^s x_1, q^s y_1) = a_s$ and $\mathcal{V} f(q^s x_1, q^s y_1) = k_s$, $s \varepsilon Z$ are given with the condition

$a_s + x_1 k_s(1-q)q^{s-1} = a_{s-1} + iy_1 k_{s-1}q^{s-1}(1-q)$, then f is determined uniquely and belongs to B(H).

Corollary 2. If $f(q^s x_1, q^s y_1) = a_s$ and $\mathcal{G} f(q^s x_1, q^s y_1) = p_s$, $s \varepsilon Z$ are given with the condition

$a_s + p_s q^{s-1}(1-q)x_1 = a_{s-1} + p_s q^{s-1}(q-1)iy_1$, then f is determined uniquely and belongs to B(H).

<u>Example 1</u>. A simple example of bianalytic function is obtained from the above corollary 1, with

$a_s = q^s x_1 + i q^s y_1$, $s \in Z$.

We get $k_s = 1$, $s \in Z$.

From (2.10), (2.11), (2.12) and (2.13),

$$f(q^{s-r}x_1, q^{s+r}y_1) = q^{s-r}x_1 + iq^{s+r}y_1$$

$$f(q^{s+r}x_1, q^{s-r}y_1) = q^{s+r}x_1 + iq^{s-r}y_1$$

$$f(q^{s-1-r}x_1, q^{s+r}y_1) = q^{s-1-r}x_1 + iq^{s+r}y_1$$

$$f(q^{s+r-1}x_1, q^{s+r}y_1) = q^{s+r-1}x_1 + iq^{s-r}y_1$$

Thus $f(q^m x_1, q^n y_1) = q^m x_1 + iq^n y_1$ is the continuation of the function to the entire H.

<u>Example 2</u>. Fixing $k_s = s$, $s \in Z$, we get a nontrivial example of bianalytic function.

If $s \in Z^+$, $a_s = a_o - x_1\left(\frac{1-q^s}{1-q} + sq^s\right) - iy_1\left(\frac{1-q^s}{1-q} + sq^s + s\right)$

$$a_{-s} = a_o + x_1\left(\frac{1-q^{-(s-1)}}{1-q} - q^{-s}(s-1)\right) + iy_1\left(\frac{1-q^{-(s-1)}}{1-q} - q^{-s}(s-1) - s\right).$$

Thus substituting these values in (2.10), (2.11), (2.12) and (2.13), we get a bianalytic function in entire H.

<u>Example 3</u>. Another example is obtained directly from the above theorem, with fixing

$$a_s = b_s + 1$$

$$a_s = a_{s-1} + \frac{q^{-1}x_1 + iy_1}{x_1} .$$

If $s \varepsilon Z^+$,

$$a_s = a_o + s \frac{q^{-1}x_1 + iy_1}{x_1}$$

$$a_{-s} = a_o - s \frac{q^{-1}x_1 + iy_1}{x_1}$$

$$b \ = a_o - 1 + s \frac{q^{-1}x_1 + iy_1}{x_1}$$

$$b_{-s} = a_o - 1 - s \frac{q^{-1}x_1 + iy_1}{x_1} .$$

Thus substituting these values in (2.10), (2.11), (2.12) and (2.13), we get a bianalytic function in entire H.

3. q-Monodiffric Functions

We introduced earlier a class of functions which are both q- and p-analytic. Here a more general class of functions is found by defining discrete analyticity in another way and introducing a new operator.

<u>Definitions</u>. The basic tetrad associated with z is defined as $\tau(z) = \{(qx,y),\ (x,qy),\ (q^{-1}x,y),\ (x,q^{-1}y)\}$. If $z \in H$, the points on the set $\{(q^{-1}x,q^{-1}y),\ (q^{-1}x,qy),\ (qx,qy),\ (qx,q^{-1}y)\}$ are diagonally adjacent to z while elements of the set $\{(qx,y),\ (q^{-1}x,y),\ (x,qy),\ (x,q^{-1}y)\}$ are called directly adjacent points of z. (2.15)

A sequence of lattice points in H: $\{z_o, z_1, \ldots, z_r, z_{r+1}, \ldots, z_n\}$ is a discrete curve C if z_r and z_{r+1} are diagonally adjacent for every $r = 0,1,\ldots,n-1$. C is denoted by $\langle z_o, z_1, \ldots, z_r, z_{r+1}, \ldots, z_n\rangle$. C is closed if $z_o = z_n$. C is simple if $z_r \neq z_s$ for $r \neq s$ and $r = 0,1,\ldots,n-1$; $s = 1,2,\ldots,n$. C is simply closed curve if C is simple and closed. (2.16)

The smallest simply closed curve around z is $\langle (qx,y),\ (x,qy),\ (q^{-1}x,y),\ (x,q^{-1}y),\ (qx,y)\rangle$ called the basic quadrilateral associated with z. (2.17)

$$\text{If } C_1 \equiv \langle z_o, z_1, \ldots, z_n\rangle \text{ and}$$

$$C_2 \equiv \langle z_n, z_{n+1}, \ldots, z_m\rangle \text{ then}$$

$$C_1 + C_2 \equiv \langle z_o, z_1, \ldots, z_n, z_{n+1}, \ldots, z_m\rangle \text{ and}$$

$$C_1^{-1} \equiv \langle z_n, z_{n-1}, \ldots, z_1, z_o\rangle. \qquad (2.18)$$

The straight ray, $\{(q^s x, q^s y),\ s \in Z\}$ and the distorted ray, $\{(q^{-s} x, q^s y),\ s \in Z\}$ through z defined in (2.5) are discrete curves.

It may be noted that while a simply closed curve encloses certain lattice points which constitute a domain with the given curve as the boundary, the converse is not true. For example, $D = \{(x,y),\ (qx,y)\}$, which is the interior of $S = T(z) \cup T(z')$ where $z = (x,y)$ and $z' = (qx,y)$ is not enclosed by a curve as the boundary.

Discrete monodiffricity can be defined by means of difference quotients as well as by the so called q- and p-residues.

Let $f : H \to \emptyset$ and $z \in D$. The difference operators δ_x and δ_y are defined as

$$\delta_x f(x,y) = \frac{f(q^{-1}x,y) - f(qx,y)}{(q^{-1}-q)x}$$

and

$$\delta_y f(x,y) = \frac{f(x,q^{-1}y) - f(x,qy)}{(q^{-1}-q)iy}. \tag{2.19}$$

If $\delta_x f(x,y) = \delta_y f(x,y)$, then f is q-monodiffric at z and $\delta_x f(z) = \delta_y f(z)$ is the q-monodiffric derivate of f at z and is denoted by $\delta f(z)$. The above equality reduces to

$x[f(x,q^{-1}y) - f(x,qy)] - iy\,[f(q^{-1}x,y) - f(qx,y)] = 0.$

In other words, f is q-monodiffric at z if and only if

$$Mf(z) \equiv x[f(x,q^{-1}y) - f(x,qy)] - iy[f(q^{-1}x,y) - f(qx,y)] = 0 \tag{2.20}$$

It is easy to verify that if f is q-monodiffric at z, then

$$Mf(z) \equiv R_p f(z) - R_q f(z) = 0 \tag{2.21}$$

The set of q-monodiffric functions in D is denoted by $\mathcal{M}(D)$. B(D) is the set of bianalytic function in D. B(D) is a proper subset of $\mathcal{M}(D)$. (2.22)

As M is a linear operator, it easily follows that

<u>Theorem 1</u>. $\mathcal{M}(D)$ is a vector space over $\emptyset$.

Let $f(z) = u(x,y) + iv(x,y) \in \mathcal{M}(D)$.

So

$$\delta_x[u(x,y) + iv(x,y)] = \delta_y[u(x,y) + iv(x,y)]$$

i.e., $\delta_x u(x,y) = i\delta_y v(x,y)$

and $\delta_x v(x,y) = -i\delta_y u(x,y)$ for every $z \in D$.

This is the q-monodiffric analogue of Cauchy-Riemann relations.

Conversely, if $\delta_x u(x,y) = i\delta_y v(x,y)$ and

$\delta_x v(x,y) = -i\delta_y u(x,y)$ for every $z \in D$,

by addition, $M(u + iv) = 0$ in D.

i.e., $f \in \mathcal{M}\mathcal{G}(D)$.

Hence we have:

Theorem 2. The necessary and sufficient condition for $f = u + iv \in \mathcal{M}\mathcal{G}(D)$ is $\delta_x u(x,y) = i\delta_y v(x,y)$ and $\delta_x v(x,y) = -i\delta_y u(x,y)$ for $\forall\ z \in D$.

Remark. It can be noted that in classical function theory, the Cauchy-Riemann relations are not sufficient for analyticity while the discrete analogue is also sufficient for q-monodiffricity.

If $f \in \mathcal{M}\mathcal{G}(D)$, then according to the definition of q-monodiffricity, D must be interior of some S. Thus δf exists in D, while it is q-monodiffric only in D^1, the interior of D.

In other words,

Theorem 3. $f \in \mathcal{M}\mathcal{G}(D) \Longrightarrow \delta f \in \mathcal{M}\mathcal{G}(D^1)$.

Let the sequence $\{f_n\}$ be such that $f_n \in \mathcal{M}\mathcal{G}(D)$ and $\lim_{n\to\infty} f_n(z) = f(z)$, $\forall\ z \in D$. Then $Mf(z) = \lim_{n\to\infty} Mf_n(z)$ and $\delta f(z) = \lim_{n\to\infty} \delta f_n(z)$, for every $z \in D$.

Thus we have:

<u>Theorem 4</u>. Given the sequence $\{f_n\}$, $f_n \in \mathcal{M}(D)$, $\lim_{n\to\infty} f_n(z) = f(z)$ for $\forall\, z \in D$, then $f \in \mathcal{M}(D)$ and $\delta f(z) = \lim_{n\to\infty} \delta f_n(z)$ for $\forall\, z \in D$. Again $\delta f(z) \in \mathcal{M}(D^1)$.

Now let $\sum_{j=1}^{\infty} g_j = f$ and $g_j \in \mathcal{M}(D)$. Then according to the above theorem, $f_n = \sum_{j=1}^{n} g_j$ constitutes a sequence and $f \in \mathcal{M}(D)$ and $\lim_{n\to\infty} f_n = f$.

This leads to the theorem:

<u>Theorem 5</u>. If $g_j \in \mathcal{M}(D)$ and $\sum_{j=1}^{\infty} g_j = f$ then $f \in \mathcal{M}(D)$ and $\delta f(z) = \sum_{j=1}^{\infty} \delta g_j(z)$ for $\forall\, z \in D$.

<u>Examples</u>:

(1) <u>Basic q-monodiffric functions</u>. $x + iy$, $(x+iqy)(x+iq^{-1}y)$ and $(x+iq^2y)(x+iy)(x+iq^{-2}y)$ are q-monodiffric functions in H. Also their reciprocals: $(x+iy)^{-1}$, $(x+iqy)^{-1}(x+iq^{-1}y)^{-1}$ and $(x+iq^2y)^{-1}(x+iy)^{-1}(x+iq^{-2}y)^{-1}$ belong to $\mathcal{M}(H)$.

In general, if n is a positive integer,

$$\prod_{j=0}^{n-1} (x+iq^{n-2j-1}y) \text{ and } \prod_{j=0}^{n-1} (x+iq^{n-2j-1}y)^{-1} \qquad (2.23)$$

are q-monodiffric in H.

We also note that the discrete powers in q-monodiffric theory are evolved from this set of functions.

(2) Bifunctions. Let f, g $\varepsilon \mathcal{M}$(D), then f $\oplus$ g is defined as

$(f \oplus g)(z) = f(z) + (-1)^{m+n} g(z)$ for $z = (q^m x_o, q^n y_o) \ \varepsilon \ D$ (2.24)

f $\oplus$ g behaves as f + g on H_2 and as f - g on H_1. The set of bifunctions in D is denoted by $\beta(D)$. (2.25)

We can easily see that

$$M(f \oplus g)(z) = Mf(z) + (-1)^{m+n-1} Mg(z). \qquad (2.26)$$

Due to the q-monodiffricity of f and g in D, f $\oplus$ g is q-monodiffric in D and also

$$\delta(f \oplus g)(z) = \delta f(z) + (-1)^{m+n-1} \delta g(z). \qquad (2.27)$$

Thus we have:

For f,g $\varepsilon \mathcal{M}$(D), f $\oplus$ g $\varepsilon \mathcal{M}$ (D) and $\delta(f \oplus g) = \delta f \oplus \delta(-g)$. Also $\delta(f \oplus g) \ \varepsilon \ \beta(D)$. (2.28)

Let f,g $\varepsilon \ \beta(D)$ such that $f = f_1 \oplus f_2$ and $g = g_1 \oplus g_2$.

Then $(f \oplus g)(z) = f(z) + (-1)^{m+n} g(z)$

$$= f_1(z) + (-1)^{m+n} f_2(z) + (-1)^{m+n}[g_1(z) + (-1)^{m+n} g_2(z)]$$

$$= f_1(z) + g_2(z) + (-1)^{m+n}(f_2(z) + g_1(z)).$$

Hence $\beta(D)$ is closed under $\oplus$. (2.29)

Then let us solve for F from $f \oplus F = g$ where f and g are bifunctions.

We get $f(z) + (-1)^{m+n}F(z) = g(z)$.

$F(z) = g_2(z) - f_2(z) + (-1)^{m+n}(g_1(z) - f_1(z))$ is a bifunction and is unique. (2.30)

Similarly $F \oplus f = g$ has also a unique solution in $\beta(D)$. (2.31)

Thus we have:

$$(\beta(D), \oplus) \text{ is a quasi group} \tag{2.32}$$

4. q-Monodiffric Constants

Consider a simple equation of the first order $\delta f(z) = 0$.

i.e., $$\frac{f(q^{-1}x,y) - f(qx,y)}{(q^{-1} - q)x} = \frac{f(x,q^{-1}y) - f(x,qy)}{(q^{-1} - q)iy} = 0$$

or $f(q^{-1}x,y) = f(qx,y)$ and $f(x,q^{-1}y) = f(x,qy)$, which also leads to $f(x,y) = f(q^2x,y) = f(x,q^2y)$.

Similarly by iteration,

$$f(x,y) = f(q^{2r}x, q^{2s}y);\ r,\ s \in Z. \tag{2.33}$$

Thus we observe that a q-monodiffric constant, which is the solution of the equation $\delta f(z) = 0$ in H, is fully determined by fixing its value on any rectangle $\{(x_0,y_0),$ $(qx_0,y_0),\ (x_0,qy_0),\ (qx_0,qy_0)\}$ by $f(x_0,y_0) = \alpha_1, f(qx_0,y_0) = \alpha_2,$ $f(x_0,qy_0) = \alpha_3$ and $f(qx_0,qy_0) = \alpha_4$.

Then the solution of the equation $\delta f(z) = 0$ for entire H can be given as

$$f(q^m x_0, q^n y_0) = \begin{cases} \alpha_1 & \text{if m is even, n is even} \\ \alpha_2 & \text{if m is odd, n is even} \\ \alpha_3 & \text{if m is even, n is odd} \\ \alpha_4 & \text{if m is odd, n is odd.} \end{cases} \tag{2.34}$$

In other words, the q-monodiffric constant α can be represented as a 4-tuple $\alpha = (\alpha_1,\alpha_2,\alpha_3,\alpha_4)$. Thus if α and β are two q-monodiffric constants their addition and multiplication is defined componentwise in the usual way.
Then it is easy to see

<u>Theorem 1</u>.

a) The set of q-monodiffric constants, $\{\alpha,\ \alpha \in \not{C}^4 \cap \mathcal{M}_0(H)\}$ form an abelian group with respect to addition.

b) It is a vector space of dimension four over the complex field.

c) It is a commutative algebra over $\emptyset$ with divisors of zero.

We say that $(x,y) = (q^m x_0, q^n y_0) \in H$ is an odd or even point according as $m + n$ is odd or even. If the set of odd points is denoted by H_1 and the set of even points by H_2, then $H = H_1 \cup H_2$ and $H_1 \cap H_2 = \emptyset$. Hence we get a partition of H. (2.35)

A special case of q-monodiffric constant namely biconstant can be given as $(\alpha_1, \alpha_2, \alpha_2, \alpha_1)$. A biconstant takes one value on H_1 and another on H_2. (2.36)

Biconstants form a vector space, of dimension two over $\emptyset$. A uniconstant $\alpha = (\alpha_1, \alpha_1, \alpha_1, \alpha_1)$ is a trivial example of a q-monodiffric constant which amounts to $\alpha \in \emptyset$, $\alpha = \alpha_1$.

Let $f \in \mathcal{MO}(D)$ and α be a q-monodiffric constant. Then $\alpha f \in \mathcal{MO}(D)$ implies

$\delta_x[\alpha f](x,y) = \delta_y[\alpha f](x,y), (x,y) \in D$ which gives the solution

$\alpha_1 = \alpha_4$ and $\alpha_2 = \alpha_3$. This implies,

<u>Theorem 2</u>. If f is q-monodiffric function in D and α, a q-monodiffric constant then αf is q-monodiffric in D if and only if α is a biconstant.

The bifunction of two constant functions is a biconstant, but that of two biconstants is again a biconstant. In general, the bifunction of two q-monodiffric constants is a q-monodiffric constant.

i.e., $(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \oplus (\beta_1,\beta_2,\beta_3,\beta_4) =$

$$(\alpha_1 + \beta_1, \alpha_2 - \beta_2, \alpha_3 - \beta_3, \alpha_4 + \beta_4)$$

5. Product of q-Monodiffric Functions

In all the earlier discrete function theories, the usual product of two discrete analytic functions in a domain, in general, does not turn out to be discrete analytic. For example, z is discrete analytic according to all the theories, but $z^2 = z\cdot z$ is not. Here we show that under certain very general conditions, the ordinary product of two q-monodiffric functions in D is also q-monodiffric in the given domain.

Let f, g, $f\cdot g \in \mathcal{M}_b(D)$. Consider the basic tetrad to $z \in D$. We denote $f(qx,y)$, $f(x,qy)$, $f(q^{-1}x,y)$ and $f(x,q^{-1}y)$ by f_1, f_2, f_3 and f_4 respectively and $\delta f(x,y)$ is denoted by f'.

So $M[fg](z) = x[f_4g_4-f_2g_2] -iy[f_3g_3-f_1g_1]$ which after some calculation is equal to

$$\frac{ixy}{2}(q^{-1}-q)[g'(f_4 + f_2- f_3- f_1) + f'(g_4+ g_2 -g_3 -g_1)].$$

Hence for $Mfg(z) = 0$, the following cases arise:

Case 1: $f' = g' = 0 \Longrightarrow f$ and g are q-monodiffric constants.

Case 2: $f' = 0,\ g' \neq 0 \Longrightarrow f_4 + f_2 = f_1 + f_3$ and

$$f_4 = f_2;\ f_1 = f_3$$

$$\Longrightarrow f \text{ is a biconstant.}$$

Case 3: $f_4 + f_2 - f_3 - f_1 = 0$ and $g_4 + g_2 - g_3 - g_1 = 0$

$$\Longrightarrow \frac{f'}{g'} = \frac{f_1 - f_3}{g_1 - g_3} = \frac{f_4 - f_2}{g_4 - g_2} = \frac{f_1 - f_3 + f_4 - f_2}{g_1 - g_3 + g_4 - g_2}$$

$$= \frac{f_1 - f_2}{g_1 - g_2} = \frac{f_4 - f_3}{g_4 - g_3} = \frac{0}{0}.$$

$\Longrightarrow f$ and g are biconstants.

Case 4: $f' = 0,\ g' = 0,\ f_4 + f_2 - f_1 - f_3 \neq 0$ and

$g_4 + g_2 - g_1 - g_3 \neq 0.$

$$\Longrightarrow \frac{f'}{g'} = \frac{f_2 + f_4 - f_1 - f_3}{g_2 + g_4 - g_1 - g_3}$$

$$\Longrightarrow \frac{f_4 - f_3}{g_4 - g_3} = \frac{f_4 - f_2}{g_4 - g_2} = \frac{f_2 + f_4 - f_1 - f_3}{g_2 + g_4 - g_1 - g_3} = \frac{f_1 - f_3 + f_4 - f_2}{g_1 - g_3 + g_4 - g_2}$$

$$= \frac{f_4 - f_3}{g_4 - g_3} = \frac{f_1 - f_2}{g_4 - g_2} = \frac{f_4 - f_3}{g_1 - g_2} = \frac{f_1 - f_4}{g_1 - g_4} = \frac{f_1 - f_2}{g_1 - g_2}$$

$\Longrightarrow g_1 - g_2 = g_4 - g_3$ and $f_1 - f_2 = f_3 - f_4$

$\Longrightarrow f_1 = f_3,\ f_2 = f_4$ and $g_1 = g_3,\ g_2 = g_4$

$\Longrightarrow$ f and g are q-monodiffric constants.

This is a contradiction. Case 4 does not exist. Thus we have:

<u>Theorem</u>. If f, g $\varepsilon \mathcal{M}(D)$ and also fg $\varepsilon \mathcal{M}(D)$ then either both f and g are q-monodiffric constants or one of them is a biconstant.

6. Construction of q-Monodiffric Functions

In monodiffric and q-analytic theories, the continuation of a discrete function from the boundary of a finite domain to the interior of it has been dealt. A similar result is evolved in q-monodiffric theory. For this we need two more definitions.

If D is a finite domain in H, $\partial D \cup \partial D^1$ is known as the annular boundary of D^1. (2.37)

Now if $D = \{(x,y)\}$, the annular boundary of D^1 is $\{(x,y),(qx,y),(x,qy),(q^{-1}x,y),\ (x,q^{-1}y)\}$; but $D^1 = \emptyset$. Thus we get the null set has an annular boundary. Again if $D^1 = \{(x,y)\}$, the annular boundary of D^1 is $\{(q^m x, q^n y);\ |m| + |n| = 1 \text{ or } 2\}$. (2.38)

A domain $D \varepsilon H$ is a packed domain if boundary of D is a simply closed curve C satisfying Int $C = D$. (2.39)

Our claim is that if a q-monodiffric function $f \varepsilon \mathcal{M}(D)$ is defined on the annular boundary of D^1 of a packed domain D, then there exists a function $g \varepsilon \mathcal{M}(D)$ such that $f = g$ on the annular boundary of D^1.

We can easily verify that the result is true if D^1 contains atmost one point. Now let us assume that the result is true in the case of D^1 having (n-1) points.

Consider a packed domain D for which D^1 has n points. Let $C = \langle z_1, z_2, \ldots, z_{r-1}, z_r, z_{r+1}, \ldots z_1 \rangle$ be the boundary of D. Without loss of generality we assume that there exists z_{r-1}, z_r, z_{r+1} in C such that $\langle z_{r-1}, z_r \rangle$ is along a straight ray and $\langle z_r, z_{r+1} \rangle$ is along a distorted ray and an z' in D^1 satisfying $z_{r-1}, z_r, z_{r+1}, z'$ is the basic quadrilateral of a point $z^* \varepsilon \partial D^1$.

Then consider the domain $D_1 = D - \{z^*\}$. Due to the q-monodiffric condition at z^*, f is known at z'. Hence f is defined on the annular boundary of D_1. Also D_1 contains only (n-1) points. Thus the claim is true in the case of D_1 due to the assumption. Since $D = \{z^*\} \cup D_1$, we get that f is defined in D.

Hence by induction we have:

Theorem 1. If a q-monodiffric function $f \in \mathcal{M}(D)$ is defined on the annular boundary of D^1 of a packed domain D, then there exists a $g \in \mathcal{M}(D)$ such that $f = g$ on the annular boundary of D^1.

Duffin, Kurowski, Berzsenyi and Harman introduced discrete continuation of a function in the concerned theories. Now we introduce the continuation of a q-monodiffric function from the $\mathcal{C}$-belt or from the pair of so-called straight or distorted belts. We take $n \in Z^+$, $0 \leq r \leq n$.

Definitions. $\{(q^m x_o, q^n y_o);\ m \in Z,\ n = 0, 1\}$ is known as the $\mathcal{C}$-belt. (2.40)

$$(q^{m+n-2r}x_o, y_o),\ (q^{m+n-(2r+1)}x_o, qy_o),\ (q^{m+n-2r}x_o, qy_o)$$

and $(q^{m+n-(2r+1)}x_o, y_o)$, belonging to the $\mathcal{C}$-belt, are respectively denoted by $\lambda_{m,n,2r}$, $\lambda_{m,n,2r+1}$, $\mu_{m,n,2r}$ and $\mu_{m,n,2r+1}$. (2.41)

The discrete curves $C_{m,-n}$ and $C_{m,n}$ are defined as follows:

$$C_{m,-n} \equiv \langle \lambda_{m,n,0},\ \lambda_{m,n,1}, \dots, \lambda_{m,n,2r},\ \lambda_{m,n,2r+1}, \dots, \lambda_{m,n,2n} \rangle \tag{2.42}$$

$$C_{m,n} \equiv \langle \mu_{m,n,0},\ \mu_{m,n,1}, \dots, \mu_{m,n,2r},\ \mu_{m,n,2r+1}, \dots, \mu_{m,n,2n} \rangle \tag{2.43}$$

$\left\{(q^m x_o, q^n y_o);\ m \in Z,\ n = m, m+1\right\}$ is called the straight belt and $\left\{(q^m x_o, q^n y_o);\ m \in Z,\ n = -m,\ -(m+1)\right\}$ is called the distorted belt. (2.44)

The following relations on λ's and μ's can easily be seen

a) $\lambda_{m,n,2r+1} = \mu_{m-1,n,2r} = \mu_{m,n-1,2r}$

b) $\mu_{m,n,2r+1} = \lambda_{m-1,n,2r} = \lambda_{m,n-1,2r}$

c) $\lambda_{m,n,r} = \lambda_{m-s,n+s,r} = \lambda_{m+s,n-s,r}$

d) $\mu_{m,n,r} = \mu_{m-s,n+s,r} = \mu_{m+s,n-s,r}$

e) $\lambda_{m,n,r} = \lambda_{m-2s,n,r-2s} = \lambda_{m,n-2s,r-2s}$

f) $\mu_{m,n,r} = \mu_{m-2s,n,r-2s} = \mu_{m,n-2s,r-2s}$,

for $s \in Z^+$, $r \geq s \geq 0$. (2.45)

<u>Construction from $\mathcal{C}$-belt</u>. Let $f \in \mathcal{M}(H)$ and f known on the $\mathcal{C}$-belt. Now f being q-monodiffric at $(q^m x_o, y_o)$, $f(q^m x_o, q^{-1} y_o)$ is determined uniquely. Then using the belt $\left\{(q^m x_o, q^n y_o);\ m \in Z,\ n = 0,\ -1\right\}$, $f(q^m x_o, q^{-2} y_o)$ is obtained uniquely, similarly and so on. Thus $f(q^m x_o, q^{-n} y_o)$ is

uniquely determined for any finite n. In the same way $f(q^m x_0, q^n y_0)$ is determined uniquely.

Thus we have:

$f \in \mathcal{M}(H)$ is uniquely continued from the $\mathcal{C}$-belt to the entire H. (2.46)

Let $f \in \mathcal{M}(H)$ and f be known on the $\mathcal{C}$-belt. We assert that $f(q^m x_0, q^{-n} y_0)$ can be expressed as a sum in terms of $f(\lambda_{m,n,r})$. Similarly $f(q^m x_0, q^n y_0)$ is considered. This result is verified when n = 1,2. Let us assume the result is true for the first (n-1) positive integers for n. Then f being q-monodiffric at $(q^m x_0, q^{-(n-1)} y_0)$, we get:

$$f(q^m x_0, q^{-n} y_0) = f(q^m x_0, q^{-(n-2)} y_0)$$

$$+ \frac{iq^{m+n-1} y_0}{x_0} f(q^{m-1} x_0, q^{-(n-1)} y_0)$$

$$- \frac{iq^{-(m+n-1)} y_0}{x_0} f(q^{m+1} x_0, q^{-(n-2)} y_0) . \qquad (2.47)$$

Now we can note that $f(q^m x_0, q^{-(n-2)} y_0)$, $f(q^{m-1} x_0, q^{-(n-1)} y_0)$ and $f(q^{m+1} x_0, q^{-(n-1)} y_0)$ are expressible as a sum in terms of $f(\lambda_{m,n,r})$ due to the assumption and (2.45).

Totally what we have proved is the assertion in the case of n.

Hence by induction we have:

Theorem 2. If $f \in \mathcal{M}(H)$, $f(q^m x_o, q^{-n} y_o)$ is expressible as a sum in terms of $f(\lambda_{m,n,r})$ and $f(q^m x_o, q^n y_o)$ in terms of $f(\mu_{m,n,r})$.

In the light of the above theorem, we write

$$f(q^m x_o, q^{-n} y_o) = \sum_{r=0}^{2n} \alpha_{m,n,r} f(\lambda_{m,n,r}) \qquad (2.48)$$

and

$$f(q^m x_o, q^n y_o) = \sum_{r=0}^{2n} \beta_{m,n,r} f(\mu_{m,n,r}) \qquad (2.49)$$

We can also note that $\alpha_{m,n,r}$ **and** $\beta_{m,n,r}$ are independent of f. They will be functions of m,n and r only.

Using the q-monodiffric conditions at $(q^m x_o, q^{-(n-1)} y_o)$ and $(q^m x_o, q^{n-1} y_o)$ we get

$$\sum_{r=0}^{2n} \alpha_{m,n,r} f(\lambda_{m,n,r}) = \sum_{r=0}^{2(n-2)} \alpha_{m,n-2,r} f(\lambda_{m,n-2,r})$$

$$+ \frac{iq^{-(m+n-1)} y_o}{x_o} \sum_{r=0}^{2(n-1)} \alpha_{m-1,n-1,r} f(\lambda_{m-1,n-1,r})$$

$$- \frac{iq^{-(m+n-1)} y_o}{x_o} \sum_{r=0}^{2(n-1)} \alpha_{m+1,n-1,r} f(\lambda_{m+1,n-1,r}) \qquad (2.50)$$

and

$$\sum_{r=0}^{2n} \beta_{m,n,r}\, f(\mu_{m,n,r}) = \sum_{r=0}^{2(n-2)} \beta_{m,n-2,r}\, f(\mu_{m,n-2,r})$$

$$- \frac{iq^{n-m-1} y_o}{x_o} \sum_{r=0}^{2(n-1)} \beta_{m-1,n-1,r}\, f(\mu_{m-1,n-1,r})$$

$$+ \frac{iq^{n-m-1} y_o}{x_o} \sum_{r=0}^{2(n-1)} \beta_{m+1,n-1,r}\, f(\mu_{m+1,n-1,r}). \qquad (2.51)$$

Thus using (2.5) and comparing the coefficients, we have the following theorem

Theorem 3.

$$\alpha_{m,n,r} = \alpha_{m,n-2,r-2} + \frac{iq^{-(m+n-1)} y_o}{x_o} \alpha_{m-1,n-1,r-2}$$

$$- \frac{iq^{-(m+n-1)} y_o}{x_o} \alpha_{m+1,n-1,r}$$

and

$$\beta_{m,n,r} = \beta_{m,n-2,r-2} - \frac{iq^{n-m-1} y_o}{x_o} \beta_{m-1,n-1,r-2}$$

$$+ \frac{iq^{-n-m-1} y_o}{x_o} \beta_{m+1,n-1,r}.$$

Examples. Now we will see the construction of a few simple q-monodiffric entire functions.

ε_j^r; $j \in Z$, $r = 0, 1$ are defined as follows:

$$\varepsilon_j^r \ (q^m x_o, q^n y_o) = \begin{cases} 1 \text{ if } m = j \text{ and } n = r \\ 0 \text{ otherwise on } \mathcal{C}\text{-belt.} \end{cases} \tag{2.52}$$

Continuation of ε_j^r to entire H can be given as

$$\varepsilon_j^o(q^m x_o, q^{-n} y_o) = \begin{cases} \alpha_{m,n,m+n-j} \text{ if } m+n-j \text{ is even and} \\ \qquad 0 \leq m+n-j \leq 2n \\ 0 \text{ otherwise.} \end{cases}$$

$$\varepsilon_j^o(q^m x_o, q^n y_o) = \begin{cases} \beta_{m,n,m+n-j} \text{ if } m+n-j \text{ is odd and} \\ \qquad 0 \leq m+n-j \leq 2n. \\ 0 \text{ otherwise.} \end{cases} \tag{2.53}$$

$$\varepsilon_j^1(q^m x_o, q^{-n} y_o) = \begin{cases} \alpha_{m,n,m+n-j} \text{ if } m+n-j \text{ is odd and} \\ \qquad 0 \leq m+n-j \leq 2n. \\ 0 \text{ otherwise.} \end{cases}$$

$$\varepsilon_j^1(q^m x_o, q^n y_o) = \begin{cases} \beta_{m,n,m+n-j} \text{ if } m+n-j \text{ is even and} \\ \qquad 0 \leq m+n-j \leq 2n. \\ 0 \text{ otherwise.} \end{cases} \tag{2.54}$$

<u>Straight and distorted belts</u>. In a similar way, $f \in \mathcal{M}(H)$ can be continued uniquely to entire H from the straight and

distorted belts. If f is known on the points $\left\{(q^m x_o, q^n y_o);\ m \in Z,\ n = m, m+1\right\} \bigcup \left\{(q^m x_o, q^n y_o);\ m \in Z,\ n = -m, -(m+1)\right\}$, by a similar procedure as (2.54), the continuation to any point is possible. Likewise the uniqueness also is guaranteed.

CHAPTER III

DISCRETE q-INTEGRATION AND CAUCHY'S PROBLEM

In this chapter discrete integration is developed. Integrals of the first and second types i.e, the line integral and the inverse of q-monodiffric derivate respectively are defined. The integral of the first type is expressed as a finite sum of the function values at certain points which form a curve in H while the integral of the second type is the solution of a pair of partial q-difference equations which again is expressed as a sum of function values at certain points in the given domain. In the second case, if the integral is known on the annular boundary of a domain, the function is determined for the entire domain. Considering H as a domain, it is true that if the integral of the second type is known on the annular boundary of it namely the lattice points on the axes, it is fully determined in H.

For convenience, the integrals of first and second types are called $\mathcal{S}$- and $\mathcal{J}$-integrals and thus symbolically $\mathcal{J} = \delta^{-1}$.

The $\mathcal{S}$-integral possesses many important results analogous to classical integral connected to singularity, pole and contour, but it lacks that the integral of a q-monodiffric function in a domain is q-monodiffric there whereas the second holds this though it is handicapped by many properties of contour integration. Thus both the integrals taken together represent the theory of integration in q-monodiffric study of functions and plays the same role of classical integration.

Both of these analogues of integration reduce to the Riemannian integration in the limit case as $q \longrightarrow 1$.

1. Integral of the First Type

The following definitions are essential for the development of the theory of the integral of the first type.

Singularity of a function. A function $f : H \longrightarrow \mathbb{C}$ satisfying

$$\begin{cases} Mf(z_r) = a_r; \ a_r \neq 0, \ a_r \in \mathbb{C} \text{ and finite} \\ \qquad\qquad r = 1,2,\dots,n \text{ and } z_r \in D \\ Mf(z) = 0; \ z \in D, \ z \neq z_r, \ r = 1,2,\dots,n \end{cases}$$

is called a q-monodiffric function in D with singularities at $z_1, z_2, \dots, z_n$. (3.1)

q-Meromorphic function. Let $g \in \mathcal{M}_q(D)$, $z_1, z_2, \dots, z_n \in D$ and $f : H \longrightarrow \mathbb{C}$ satisfying

$$f(z) = \begin{cases} g(z); \ z \in D, \ z \neq z_1, z_2, \dots, z_n \\ \infty \text{ if } z = z_1, z_2, \dots, z_n. \end{cases}$$

Then $f(z)$ is said to be q-meromorphic in D with $z_1, z_2, \ldots$ and z_n as poles. (3.2)

We note $f(z)$ is q-monodiffric at $z_1, z_2, \ldots$ and z_n, but not at points on the basic quadrilaterals of $z_1, z_2, \ldots$ and z_n. Thus if $f(z)$ is q-meromorphic in D with the poles at $z_1, z_2, \ldots$ and z_n, then $f(z) \in \mathcal{M}(\mathbb{D})$ where

$$\mathbb{D} = D - \left\{\tau(z_i);\ i = 1, 2, \ldots, n\right\}. \qquad (3.3)$$

We note that $\frac{1}{z - z_0}$, $z_0 \in H$, is not q-monodiffric anywhere in H. Hence $\frac{1}{z - z_0}$ is not a meromorphic function.

If f is a q-meromorphic function in D having every $z \in \mathbb{D}$ as a pole, then $f \in \mathcal{M}(D - \not{\mathcal{B}})$ where $\not{\mathcal{B}}$ is the region for which $\mathbb{D}$ is the interior.

If f is a q-meromorphic function having every point of a straight ray (distorted ray) a pole, then $f \in \mathcal{M}(H - \not{\mathcal{B}})$ where $\not{\mathcal{B}}$ is the region consisting of the straight ray (distorted ray) together with its adjacent rays.

The result is true if we consider rays like $\left\{(q^m x_1, q^n y_1);\ m \in Z,\ n \text{ fixed}\right\}$.

Let us consider q-monodiffric constants: $\left\{\alpha : (\alpha_1, \alpha_2, \alpha_3, \alpha_4)\right\}$. If α_1 or α_4 is infinite we get $\alpha \in \mathcal{M}(H_2)$. If α_2 or α_3 is infinite we get $\alpha \in \mathcal{M}(H_1)$.

Again if α_1 and α_2 (or α_3 and α_4) are infinite, α is q-monodiffric nowhere in H and hence it is not q-meromorphic.

(3.4)

<u>$\mathcal{S}$-integral</u>. The integral of the first type is defined along diagonal-wise path.

If z_j and z_{j+1} are two diagonally adjacent points in H, then the line integral from z_j to z_{j+1} is

$$\mathop{\mathcal{S}}_{z_j}^{z_{j+1}} f(z)d(z:q) = \frac{f(z_{j+1}) + f(z_j)}{2}(z_{j+1} - z_j).$$

If $C \equiv \langle z_0, z_1, \ldots, z_r, z_{r+1}, \ldots, z_n \rangle$ is a discrete curve in H,

$$\mathop{\mathcal{S}}_{C} f(z)d(z:q) = \sum_{j=1}^{n} \frac{f(z_j) + f(z_{j-1})}{2}(z_j - z_{j-1})$$

(3.5)

<u>Linearity</u>. Let f, g: $H \to \emptyset$, $C_1 \equiv \langle z_0, z_1, \ldots, z_n \rangle$ and $C_2 \equiv \langle z_n, z_{n+1}, \ldots, z_m \rangle$

Then for $C = C_1$ or C_2,

a) $\mathop{\mathcal{S}}_{C^{-1}} f(z)d(z:q) = -\mathop{\mathcal{S}}_{C} f(z)d(z:q)$

b) $\mathop{\mathcal{S}}_{C} (f+g)(z)d(z:q) = \mathop{\mathcal{S}}_{C} f(z)d(z:q) + \mathop{\mathcal{S}}_{C} g(z)d(z:q)$

c) $\int_C (\alpha f)(z)d(z:q) = \alpha \int_C f(z)d(z:q), \alpha \in \not{C}$

d) $\int_{C_1+C_2} f(z)d(z:q) = \int_{C_1} f(z)d(z:q) + \int_{C_2} f(z)d(z:q).$

We can see by direct calculation that

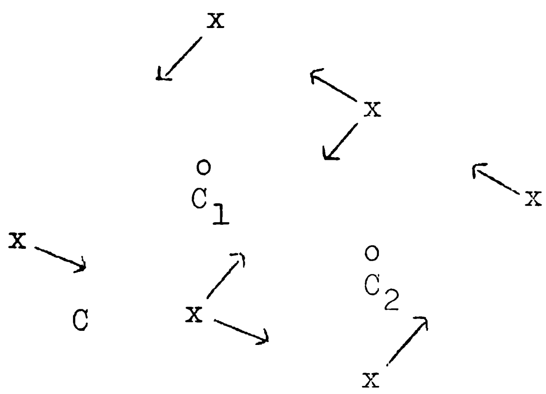
if $C \equiv \langle (qx,y),(x,qy),(q^{-1}x,y),(x,q^{-1}y),(qx,y) \rangle$

$$\int_C f(z)d(z:q) = \frac{q^{-1}-q}{2} Mf(x,y)$$

$$= 0 \text{ if } f \text{ is q-monodiffric at } (x,y). \qquad (3.7)$$

Now if C is a simply closed curve in H_1 which encloses only two points, then these points will be in H_2. Then if C_1 and C_2 are the basic quadrilaterals of these points by simplification we get

$$\int_{C_1} f(z)d(z:q) + \int_{C_2} f(z)d(z:q) = \int_C f(z)d(z:q) \qquad (3.8)$$



Then by actual integration we get that if $C \varepsilon H_1$ is a simply closed curve and C encloses $z_1, z_2, \ldots$ and z_n belonging to H_2 as interior points,

then

$$\sum_{j=1}^{n} \oint_{C_j} f(z)d(z:q) = \oint_{C} f(z)d(z:q)$$ where C_j is the basic quadrilateral of z_j. (3.9)

Thus if C is a simply closed curve belonging to H_1, than C can be replaced by basic quadrilaterals $C_j^{'}$s of every interior point z_j of C belonging to H_2.

Hence if f is q-monodiffric at every z_j, we get $\oint_C f(z)d(z:q) = 0$.

Similarly simply closed curves belonging to H_2 is considered. Thus using the properties of a packed domain we have:

<u>Theorem 1</u>. If D is a packed domain, $f : H \longrightarrow \emptyset$ has no singularity in $H_i \cap D$ and $C \varepsilon H_j$, $i \neq j$, then $\oint_C f(z)d(z:q) = 0$.

Let two curves $C_1 \equiv \langle z_1, z_2, \ldots, z_{n-1}, z_n \rangle$ and $C_2 \equiv \langle z_1^{'}, z_2^{'}, \ldots, z_{n-1}^{'}, z_n \rangle$ belonging to H_j lie wholly in a packed domain D. Also let $C_1 \cap C_2 = \{z_1, z_n\}$. Then $C_1 + C_2^{-1}$

is a simply closed curve in D belonging to H_j. Let us assume f has no singularity in $H_i \cap D$, $i \neq j$. Then by the above theorem,

$$f \in \mathcal{M}(D) \implies \mathfrak{S}_{C_1+C_2^{-1}} f(z)d(z:q) = 0$$

$$\implies \mathfrak{S}_{C_1} f(z)d(z:q) = \mathfrak{S}_{C_2} f(z)d(z:q).$$

Thus we have:

<u>Theorem 2</u>. If D is a packed domain and $f : H \to \mathbb{C}$ has no singularity in $H_i \cap D$ and z_1 and $z_n \in H_j \cap D$ where i and j are different, then

$$\mathfrak{S}_{z_1}^{z_n} f(z)d(z:q) \text{ is path independent.}$$

Let us assume $f : H \to \mathbb{C}$ satisfies that $\mathfrak{S}_C f(z)d(z:q) = 0$ for every closed curve C in S of a packed domain D. This result is true in the cases of every basic quadrilateral in S. Hence $\mathfrak{S}_C f(z)d(z:q) = 0$ where C is the basic quadrilateral of $(x,y) \in D$, implies $Mf(x,y) = 0$.

Thus we have:

If the $\mathfrak{S}$-integral of $f : H \to \mathbb{C}$ along every simply closed curve in S of a packed domain D is zero, then $f \in \mathcal{M}(D)$. (3.10)

Let $(x,y) \varepsilon D$ be a singularity of $f : H \longrightarrow \emptyset$ in the packed domain D. C_1 is the basic quadrilateral of (x,y). Then $\int_{C_1} f(z)d(z:q) = \frac{q^{-1}-q}{2} Mf(x,y)$ by (3.7).

Let $(x,y) \varepsilon H_i$ and $C \varepsilon H_j$, $i \neq j$ be a simply closed curve in S. If $(x,y) \varepsilon$ Int C, is the only singularity of f in Int C, we see $\int_C f(z)d(z:q) = \frac{q^{-1}-q}{2} Mf(x,y)$ due to the replacement of C into basic quadrilaterals as we saw in (3.7) and (3.9), but if $(x,y) \varepsilon H_i$ and $C \varepsilon H_i$, we get $\int_C f(z)d(z:q) = 0$.

Thus we have:

If (x,y) is the singularity of f in a packed domain D and C is a simply closed curve in S such that $(x,y) \varepsilon$ Int C, then

$$\int_C f(z)d(z:q) = \begin{cases} \frac{q^{-1}-q}{2} Mf(x,y) \text{ if } (x,y) \varepsilon H_i \text{ and } C \varepsilon H_j \; i \neq j. \\ 0 \text{ if } (x,y) \varepsilon H_i \text{ and } C \varepsilon H_i. \end{cases} \tag{3.11}$$

Using the above result, theorem 1 of this section and the principle of replacement of a curve by basic quadrilaterals, we arrive at the result:

<u>Theorem 3</u>. Suppose $z_{11}, z_{12}, \ldots$ and $z_{1m} \varepsilon H_1$ and $z_{21}, z_{22}, \ldots$ and $z_{2n} \varepsilon H_2$ are the singularities of f in a packed domain D

belonging to the interior of a simply closed curve C in S then,

$$\mathfrak{S}_C f(z)d(z:q) = \begin{cases} \dfrac{q^{-1}-q}{2} \displaystyle\sum_{j=1}^{m} Mf(z_{1j}) & \text{if } C \in H_2 \\ \dfrac{q^{-1}-q}{2} \displaystyle\sum_{j=1}^{n} Mf(z_{2j}) & \text{if } C \in H_1. \end{cases}$$

Let f be a meromorphic function in a packed domain D with $z_1 = (x_1, y_1)$ as the only pole in D. Suppose C is a simply closed curve in D having $z_1 \in$ Int C.

C can be replaced by basic quadrilaterals. As we saw earlier, $\mathfrak{S}$-integral of f over C is the sum of $\mathfrak{S}$-integral of f over these basic quadrilaterals.

If $z_1 \in H_i$ and $C \in H_j$, $i \neq j$, the basic quadrilaterals of the directly adjacent points of z_1 are not included in the replacement. Hence due to the q-monodiffricity of f at the other points, we get

$$\mathfrak{S}_C f(z)d(z:q) = 0 \text{ if } z_1 \in H_i \text{ and } C \in H_j,\ i \neq j. \qquad (3.12)$$

On the other hand, if $z_1 \in H_j$ and $C \in H_i$, we see that the basic quadrilateral of z_1 is not and the basic quadrilaterals of the adjacent points of z_1 are included in the replacement of C as the basic quadrilaterals belong to H_j if $C \in H_i$, $i \neq j$.

Due to the q-monodiffricity of f at the points other than the points directly adjacent to z_1, we get $\underset{C}{S} f(z)d(z:q)$ depends on the residues of the points directly adjacent to z_1. It is also easy to see that $\underset{C}{S} f(z)d(z:q)$ is the sum of residues at (qx_1,y_1), $(q^{-1}x_1,y_1)$, (x_1,qy_1) and $(x_1,q^{-1}y_1)$, the directly adjacent points of z_1. (3.13)

Hence $\underset{C}{S} f(z)d(z:q)$

$$= \frac{q^{-1}-q}{2}[Mf(qx_1,y_1) + Mf(x_1,qy_1) + Mf(q^{-1}x_1,y_1)+ Mf(x_1,q^{-1}y_1)]$$

$$= q^{-1}\bar{z}_1\ f(q^{-1}x_1,q^{-1}y_1) + q\bar{z}_1 f(qx_1,qy_1)$$

$$+ (q^{-1}x_1-iqy_1)f(q^{-1}x_1,qy_1)+(-qx_1+iq^{-1}y_1)f(qx_1,q^{-1}y_1)$$

$$+ iy_1[f(q^{-2}x_1,y_1)-f(q^2x_1,y_1)]+x_1[f(x_1,q^{-2}y_1)-f(x_1,q^2y_1)]$$

$$= P f(z_1) \text{ (say)}.$$

We name $P(z_1)$ as the polar residue of f at z_1. (3.14)

Thus we have:

Theorem 4. D is a packed domain and C is a simply closed curve in D. $z_{11}, z_{12}, \ldots$ and $z_{1m} \in H_1 \cap D$ and $z_{21}, z_{22}, \ldots$

and $z_{2n} \varepsilon H_2 \cap D$ are the only poles of a q-meromorphic function in D such that z_{1j} does not belong to the set of adjacent points of z_{2k}.

$$\text{Then } \underset{C}{S} f(z)d(z:q) = \begin{cases} \sum_{j=1}^{n} f(z_{1j}) \text{ if } C \varepsilon H_2 \\ \sum_{j=1}^{m} f(z_{2j}) \text{ if } C \varepsilon H_1. \end{cases}$$

2. q-Monodiffricity of the S-Integral

If z_1 and $z \varepsilon S \cap H_i$ where S is the region of a packed domain D and $f \varepsilon \mathcal{M}(D)$, then

$$\underset{z_1}{\overset{z_n}{S}} f(z)d(z:q) = \underset{C}{S} f(z)d(z:q) \text{ where } C \equiv \langle z_1, z_2, \ldots, z_n \rangle$$

is a curve in S.

Uniqueness of $\underset{z_1}{\overset{z_n}{S}} f(z)d(z:q)$ is guaranteed by the path independence of the line integral in S.

Let α and $z = (x,y) \varepsilon D$ such that $\alpha \varepsilon H_i$ and $z \varepsilon H_j$, $i \neq j$ and $f \varepsilon \mathcal{M}(D)$ where D is a domain in H packed by the curve C.

Then

$$M\left[\int_{\alpha}^{z} f(z)d(z;q)\right]$$

$$= x\left[\int_{\alpha}^{z_{0,-1}} f(z)d(z;q) - \int_{\alpha}^{z_{0,1}} f(z)d(z;q)\right]$$

$$-iy\left[\int_{\alpha}^{z_{-1,0}} f(z)d(z;q) - \int_{\alpha}^{z_{1,0}} f(z)d(z;q)\right], \text{ where}$$

$$z_{i,j} = (q^i x, q^j y)$$

$$= x \int_{z_{0,1}}^{z_{0,-1}} f(z)d(z;q) - iy \int_{z_{1,0}}^{z_{-1,0}} f(z)d(z;q) \tag{3.15}$$

$$= x\left[\frac{f(x,qy) + f(q^{-1}x,y)}{2}((q^{-1}-1)x + (q-1)iy)\right.$$

$$\left. + \frac{f(q^{-1}x,y) + f(x,q^{-1}y)}{2}((1-q^{-1})x + (q^{-1}-1)iy)\right]$$

$$-iy\left[\frac{f(qx,y) + f(x,qy)}{2}((1-q)x + (q-1)iy)\right.$$

$$\left. + \frac{f(x,qy) + f(q^{-1}x,y)}{2}((q^{-1}-1)x + (1-q)iy)\right]$$

$$= \frac{1-q^{-1}}{2}(x-iy)\left\{x[f(x,q^{-1}y)-f(x,qy)] - iqy[f(q^{-1}x,y)-f(qx,y)]\right\}$$

$= 0$ if and only if $f(z)$ is a q-monodiffric constant as

$Mf(z) = 0$ and

$x[f(x,q^{-1}y) - f(x,qy)] - iqy[f(q^{-1}x,y) - f(qx,y] = 0$

are simultaneously possible if and only if

$f(x,y) = f(q^{2r}x,y) = f(x,q^{2s}y)$; $r, s \in Z$.

Thus we have:

<u>Theorem 1</u>. $f \in \mathcal{M}(D)$ and $\underset{\alpha}{\overset{z}{S}} f(z)d(z:q)$ is q-monodiffric in D where D is a packed domain in H$\Longleftrightarrow$f is a q-monodiffric constant in D.

<u>S-integral of a q-monodiffric constant</u>

Let $w = (\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ be a q-monodiffric constant and $z = (q^m x_o, q^n y_o)$; m is odd and n is even.

$$\underset{z_{N,N}}{\overset{z}{S}} wd(z:q) = -\underset{z}{\overset{z_{N,N}}{S}} wd(z:q)$$

$$= -\sum_{r=0}^{N-1} \frac{\alpha_2+\alpha_3}{2}(q^{r+1}-q^r)z$$

$$= (q-1)\frac{\alpha_2+\alpha_3}{2} z\left(\sum_{r=0}^{N-1} q^r\right)$$

$$= (1-q^N)\frac{\alpha_2+\alpha_3}{2}z. \qquad (3.16)$$

Similarly other possible cases of m and n are discussed. Thus we have:

$$\int_{z_{N,N}}^{z} wd(z:q) = \begin{cases} (1-q^{N})\,\dfrac{\alpha_2+\alpha_3}{2}z & \text{if } m+n \text{ is odd} \\ (1-q^{N})\,\dfrac{\alpha_1+\alpha_4}{2}z & \text{if } m+n \text{ is even.} \end{cases} \tag{3.17}$$

Expressing the line integral of a q-monodiffric constant as a bifunction we have:

$$\int_{z_{N,N}}^{z} wd(z:q) = \left[\frac{\alpha_1+\alpha_2+\alpha_3+\alpha_4}{4}z \oplus \frac{\alpha_1+\alpha_4-\alpha_2-\alpha_3}{4}z\right](1-q^{N}). \tag{3.18}$$

Similarly

$$\int_{z_{-N,N}}^{z} wd(z:q) = \begin{cases} \dfrac{\alpha_2+\alpha_3}{2}[(q^{-N}-1)x + i(q^{N}-1)y] & \text{if } m+n \text{ is odd} \\ \dfrac{\alpha_1+\alpha_4}{2}[(q^{-N}-1)x + i(q^{N}-1)y] & \text{if } m+n \text{ is even} \end{cases}$$

$$= \frac{\alpha_1+\alpha_2+\alpha_3+\alpha_4}{4}[(q^{-N}-1)x + i(q^{N}-1)y]$$

$$\oplus \frac{\alpha_1+\alpha_4-\alpha_2-\alpha_3}{4}[(q^{-N}-1)x + i(q^{N}-1)y] \tag{3.19}$$

and

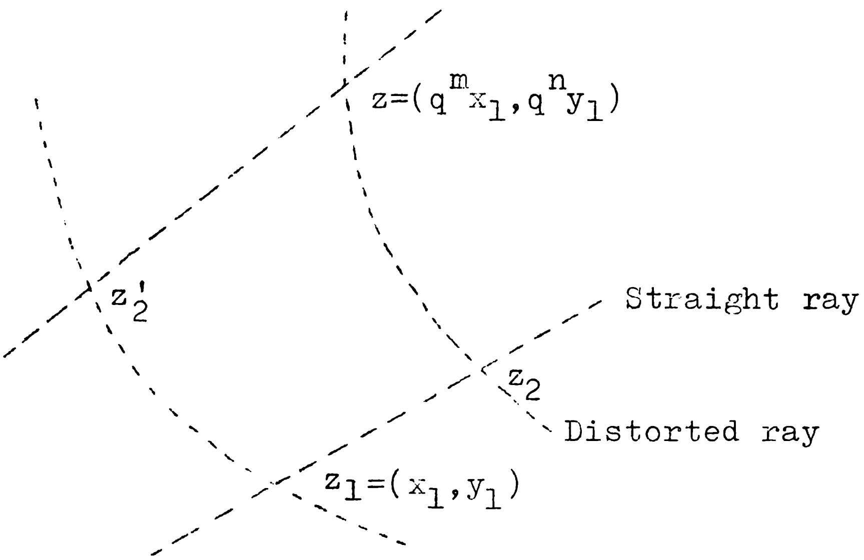
$$\underset{z_{N,-N}}{\overset{z}{\mathcal{S}}} wd(z:q) = \begin{cases} \frac{\alpha_2+\alpha_3}{2}[(1-q^{-N})x + i(1-q^{N})y] & \text{if } m+n \text{ is odd.} \\ \frac{\alpha_1+\alpha_4}{2}[(1-q^{-N})x + i(1-q^{N})y] & \text{if } m+n \text{ is even} \end{cases}$$

$$= \frac{\alpha_1+\alpha_2+\alpha_3+\alpha_4}{4}[(1-q^{-N})x + i(1-q^{N})y]$$

$$\oplus \frac{\alpha_1+\alpha_4-\alpha_2-\alpha_3}{4}[(1-q^{-N})x + i(1-q^{N})y]. \qquad (3.20)$$

Thus concluding from the above two results we get the $\mathcal{S}$-integral of a q-monodiffric constant along the curves: straight ray and distorted ray are bifunctions. (3.21)

Thus making use of the above two $\mathcal{S}$-integrals along the curves: straight ray and distorted ray, $\overset{z}{\underset{z_1}{\mathcal{S}}} wd(z:q)$ where z, $z_1 \varepsilon H_i$ is found as follows:



The straight ray through z_1 and the distorted ray through z join at z_2. Similarly the other pair join at z_2'. Then

$$z_2 = (q^{\frac{m+n}{2}} x_1, q^{\frac{m+n}{2}} y_1)$$

and
$$z_2' = (q^{\frac{m-n}{2}} x_1, q^{\frac{n-m}{2}} y_1). \tag{3.22}$$

From the above figure,

$$\int_{z_1}^{z} wd(z:q) = \int_{z_1}^{z_2} wd(z:q) + \int_{z_2}^{z} wd(z:q)$$

$$= \int_{z_1}^{z_2'} wd(z:q) + \int_{z_2'}^{z} wd(z:q).$$

Also $\int_{z_1}^{z} d(z:q)$ is a bifunction.

Thus combining the earlier results we have:

<u>Theorem 2</u>. If $w = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is a q-monodiffric constant and z, $z_1 \in H_i$ such that $z_1 = (x_1, y_1)$ and $z = (q^m x_1, q^n y_1)$ then

$$\int_{z_1}^{z} wd(z:q) = \frac{\alpha_1+\alpha_2+\alpha_3+\alpha_4}{2}[\pm(1-q^N)z \pm ((q^{-N}-1)x + i(q^N-1)y)]$$

$$\oplus \frac{\alpha_1+\alpha_4-\alpha_2-\alpha_3}{4}[\pm(1-q^N) \pm ((q^{-N}-1)x + i(q^N-1)y)],$$

where $N = \frac{m+n}{2}$.

<u>Extended integral of the first type.</u> In general we study functions defined in H as $f : H \to \mathbb{C}$. The nature of such functions in certain cases is important in the limit points namely the lattice points on the axes.

Accordingly we define $\bar{H} = H \cup H_x \cup H_y$ where

$H_x = \{(q^m x_o, 0);\ m \in Z\}$ and $H_y = \{(0, q^n y_o);\ n \in Z\}$.

Some functions like q-monodiffric constants and bifunctions do not exist in the limit points. On the contrary, quite a lot of functions belonging to $\mathcal{M}(H)$ are well defined in the limit sets: H_x and H_y also. Thus it is essential to have a study of such functions in $\bar{H}$ and in particular the line integral of such functions in the limiting case also deserves some notice. (3.23)

Let $f \in \mathcal{M}(D)$ and $C \equiv$

$\langle (q^n x, y), (q^{n-1}x, qy), \ldots, (q^{n-1}x, q^{2k-1}y), (q^n x, q^{2k}y) \rangle \in D$.

Then from $\displaystyle\int_{(q^n x, y)}^{(q^n x, q^{2k}y)} f(z)d(z:q)$, using

$$\int_{(0,y)}^{(0,q^{2k}y)} f(z)d(z:q) = \lim_{n\to\infty} \int_{(q^n x, y)}^{(q^n x, q^{2k}y)} f(z)d(z:q),$$

we get,

$$\int_{(0,y)}^{(0,q^{2k}y)} f(z)d(z:q) = (q-1)iy\, f(0,y) + (q^{2k}-q^{2k-1})iyf(0,q^{2k}y) + \sum_{r=1}^{\infty}(q^{r+1}-q^{r-1})iyf(0,q^r y) \ . \qquad (3.24)$$

Also,

$$\int_{(0,y)}^{(0,0)} f(z)d(z;q) = (q-1)iy[f(0,y)+(q+1)\sum_{r=1}^{\infty} q^{r-1}f(0,q^r y)]$$

with the assumption f(0,0) is finite. (3.25)

As in the above example, we get,

$$\int_{(x,0)}^{(q^{2k}x,0)} f(z)d(z;q) = (q-1)x[f(x,0)+q^{2k-1}f(q^{2k}x,0) + (q+1)\sum_{r=1}^{\infty} q^{r-1}f(q^r x,0)] \quad (3.26)$$

and

$$\int_{(x,0)}^{(0,0)} f(z)d(z;q) = (q-1)x[f(x,0)+(q+1)\sum_{r=1}^{\infty} f(q^r x,0)]$$

with the assumption f(0,0) is finite. (3.27)

3. Integral of the Second Type

As in the classical theory of both continuous and discrete functions, integration is also viewed as the inverse of derivation.

Definition. Let $f \in \mathcal{M}(D)$ and $z \in S$. Any solution of the pair of q-difference equations

$$F(q^{-1}x,y) - F(qx,y) = (q^{-1}-q)xf(x,y) \qquad \ldots (1)$$

and

$$F(x,q^{-1}y) - F(x,qy) = (q^{-1}-q)iyf(x,y) \qquad \ldots (2)$$

is denoted as $F(x,y) = \mathcal{J} f(z)$ and it is called an $\mathcal{J}$-integral of $f(z)$. (3.28)

It is easy to note that $\mathcal{J} f(z)$ is defined in a set D^{-1} whose interior is S. Accordingly D^{-n} is defined as the set whose interior is $D^{-(n-1)}$.

Thus if $f \in \mathcal{M}(H)$, $F \in \mathcal{M}(\bar{H})$.

Also in the limiting case, the annular boundary of H reduces to the points on the axes.

In earlier theories, integration is treated uniquely to represent the 'inverse of differentiation' and 'summation', but in this theory two different concepts are developed. Then the relation between them is evolved in $\bar{H}$.

<u>q-Monodiffricity of $\mathcal{J}$-integral</u>

From the definition of $\mathcal{J} f(z)$, we get

$$\frac{F(q^{-1}x,y) - F(qx,y)}{(q^{-1}-q)x} = \frac{F(x,q^{-1}y) - F(x,y)}{(q^{-1}-q)iy} = f(x,y)$$

for $(x,y) \in S$.

Thus we get:

If $f \in \mathcal{M}(D)$ and $\mathcal{J} f(z)$ exists, then $\mathcal{J} f(z) \in \mathcal{M}(S)$. (3.29)

Linearity of $\mathcal{J}$-integral. From the definition of F(x,y) it follows that

$$\text{if } f,g \in \mathcal{M}(D), \quad \mathcal{J}(f+g)(z) = \mathcal{J}f(z) + \mathcal{J}g(z) \text{ and}$$
$$\mathcal{J}(\alpha f)(z) = \alpha \mathcal{J}f(z) \text{ in } S,\ \alpha \in \emptyset. \tag{3.30}$$

The value of $\mathcal{J}f(z)$ at $(x_1,y_1) \in H$ is denoted by $F(x_1,y_1)$.

Let D be a packed domain such that $f \in \mathcal{M}(D)$. Also we know F(x,y) for (x,y) belonging to the annular boundary of D namely $\partial D \cup \partial S$. Then if F(x,y) exists in D, using (3.28)(1),

$$F(x,y) = F(q^{2n+2}x,y) + \sum_{j=0}^{n} q^{2j+1}f(q^{2j+1}x,y)x\,(q^{-1}-q)$$

where $(q^{2n+2}x,y)$ belongs to the annular boundary of D. (3.31)

Then

$$F(x,q^{-1}y)-F(x,qy)$$
$$= F(q^{2n+2}x,q^{-1}y) + \sum_{j=0}^{n} q^{2j+1}f(q^{2j+1}x,q^{-1}y)x(q^{-1}-q)$$
$$-[F(q^{2n+2}x,qy) + \sum_{j=0}^{n} q^{2j+1}f(q^{2j+1}x,qy)x(q^{-1}-q)]$$
$$= F(q^{2n+2}x,q^{-1}y)-F(q^{2n+2}x,qy)+\sum_{j=0}^{n} q^{2j+1}[f(q^{2j+1}x,q^{-1}y)$$
$$-f(q^{2j+1}x,qy)]x(q^{-1}-q)$$

$$= F(q^{2n+2}x, q^{-1}y) - F(q^{2n+2}x, qy) + \sum_{j=0}^{n} [f(q^{2j}x, y)$$

$$-f(q^{2j+2}x, y)]iy(q^{-1}-q)$$

$$= iy(q^{-1}-q)f(q^{2n+2}x, y) + [f(x, y) - f(q^{2n+2}x, y)]iy(q^{-1}-q)$$

$$= iy(q^{-1}-q)f(x, y).$$

Thus we see that 3.31 satisfies (2) in 3.28 also.

Also

$$F(x, y) = F(x, q^{2m+2}y) + \sum_{j=0}^{m} q^{2j+1} f(x, q^{2j+1}y) iy(q^{-1}-q)$$

where $(x, q^{2m+2}y)$ belongs to the annular boundary of D. (3.32)

Similarly 3.32 also satisfies both (1) and (2) in 3.28. Thus 3.31 and 3.32 are solutions of 3.28.

If $f \in \mathcal{M}(D)$ and $F(x,y)$ exists in D, then $F(x, y) = \mathcal{J} f(z)$ can be continued uniquely to D from the annular boundary of D. (3.33)

The above result can be extended to the case $D = H$. In this case, the annular boundary reduces to the lattice points on the axes.

Thus

If $f \in \mathcal{M}(H)$ and $F(x,y)$ exists in H and $F(x,y)$ is known on the axes, then we can continue $F(x,y)$ to entire H from the axes. (3.34)

Thus

$$F(x,y) = (q^{-1}-q)x \sum_{j=0}^{\infty} q^{2j+1} f(q^{2j+1}x,y) + F(0,y),$$

$$F(x,y) = (q^{-1}-q)iy \sum_{j=0}^{\infty} q^{2j+1} f(x,q^{2j+1}y) + F(x,0),$$

$$F(x,y) = (q^{-1}-q)x \sum_{j=0}^{\infty} q^{-(2j+1)} f(q^{-(2j+1)}x,y) + F(\infty,y)$$

and

$$F(x,y) = (q^{-1}-q)iy \sum_{j=0}^{\infty} q^{-(2j+1)} f(x,q^{-(2j+1)}y) + F(x,\infty)$$

give $F(x,y)$ determined in H if $F(0,y) = \phi_1(y)$, $F(x,0) = \psi_1(x)$, $F(\infty,y) = \phi_2(y)$ or $F(x,\infty) = \psi_2(x)$ is known. If $f(q^{2j}x,y)$ and $f(x,q^{2j}y)$ are of order $O(q^{-j})$, first two series are convergent in H. Third and fourth serii are convergent depending on $f(q^{-(2j+1)}x,y)$ and $f(x,q^{-(2j+1)}y)$ are of order $O(q^{3j})$. (3.35)

Uniqueness of $\mathcal{G}$-integral. It is a trivial example that if $F(z)$ is an $\mathcal{G}$-integral of $f(z)$, $F(z) + w$ where w is a q-monodiffric constant, is also an $\mathcal{G}$-integral of $f(z)$, in certain domain.

Conversely, if $F_1(z)$ and $F_2(z)$ are $\mathcal{J}$-integrals of $f(z)\ \varepsilon \mathcal{M}(D)$, let $F_2(z) = F_1(z) + g(z)$.

Then from the governing q-difference equations (3.28), we get

$g(q^{-1}x,y) = g(qx,y)$ and $g(x,q^{-1}y) = g(x,qy)$, for every $(x,y)\ \varepsilon\ S$.

Thus we have:

$F_1(z)$ and $F_2(z)$ are $\mathcal{J}$-integrals of $f(x,y)\ \varepsilon \mathcal{M}(D)$

$\Longleftrightarrow F_2(z) = F_1(z) + w$ in S where w is a q-monodiffric constant. (3.36)

4. Relation between the Integrals

The integral of the first type of f $\varepsilon \mathcal{M}(D)$ where D is a packed domain is expressible in terms of integral of the second type of f. For this purpose we define two curves $C^h(x,y)$ and $C^v(x,y)$. Let D contain them.

$$C^h(x,y) \equiv \langle (x,y),(qx,qy),(q^2x,y),\ldots,(q^{2n}x,y),(q^{2n+1}x,qy),\ldots\rangle \tag{3.37}$$

and

$$C^v(x,y) \equiv \langle (x,y),(qx,qy),(x,q^2y),\ldots,(x,q^{2n}y),(qx,q^{2n+1}y),\ldots\rangle . \tag{3.38}$$

Then

$$\int_{C^h(x,y)} f(z)d(z:q) = \sum_{j=0}^{\infty} \frac{f(q^{2j}x,y)+f(q^{2j+1}x,qy)}{2}[(q^{2j+1}-q^{2j})x + (q-1)iy]$$

$$+ \sum_{j=0}^{\infty} \frac{f(q^{2j+1}x,qy)+f(q^{2j+2}x,y)}{2}[(q^{2j+2}-q^{2j+1})x+(q-1)iy]$$

$$= \tfrac{1}{2}f(x,y)[(q-1)x + (q-1)iy]$$

$$+ \tfrac{1}{2} \sum_{j=0}^{\infty} f(q^{2j+2}x,y)(q^{2j+3}-q^{2j+1})x$$

$$+ \tfrac{1}{2} \sum_{j=0}^{\infty} f(q^{2j+1}x,qy)(q^{2j+2}-q^{2j})x$$

$$= \tfrac{1}{2}f(x,y)(q-1)z + \tfrac{1}{2} \sum_{j=0}^{\infty} f(q^{2j+2}x,y)(q^{-1}-q)q^{2j+2}x$$

$$+ \tfrac{1}{2} \sum_{j=0}^{\infty} f(q^{2j+1}x,qy)(q^{-1}-q)q^{2j+1}x.$$

Thus we have:

$$\int_{C^h(x,y)} f(z)d(z:q) + F(0,y) + F(0,qy)$$

$$= \tfrac{1}{2}(q-1)zf(x,y) + \tfrac{1}{2}F(qx,y) + \tfrac{1}{2}F(x,qy). \tag{3.39}$$

By similar calculations we get that

$$\int_{C^v(x,y)} f(z)d(z:q) + F(x,0) + F(qx,0)$$

$$= \tfrac{1}{2}(q-1)zf(x,y) + \tfrac{1}{2}F(qx,y) + \tfrac{1}{2}F(x,qy). \tag{3.40}$$

5. Standard Integrals

A few standard integrals are worked out in this section.

Example 1. If c is a complex constant, then $\mathcal{J}\, c$

$$= (q^{-1}-q)x \sum_{j=0}^{\infty} q^{2j+1}c + F(0,y)$$

$$= (q^{-1}-q)iy \sum_{j=0}^{\infty} q^{2j+1}c + F(x,0).$$

Thus $\mathcal{J}\, c = cx + F(0,y) = ciy + F(x,0)$.

Hence $\mathcal{J}\, c = cz + w$ where w is a q-monodiffric constant.

$$(3.41)$$

Example 2. Let $u(z) = (\alpha_1, \alpha_2, \alpha_2, \alpha_1)$ be a biconstant.

We can also represent as $u(x,y) = \frac{1+(-1)^{m+n}}{2}\alpha_1 + \frac{1-(-1)^{m+n}}{2}\alpha_2$.

Then proceeding as in the above example,

$$\mathcal{J}\, u(z) = \begin{cases} \alpha_2 z \text{ if } z \in H_2 \\ \alpha_1 z \text{ if } z \in H_1. \end{cases}$$

Thus we get the solution as

$\mathcal{J}(\alpha_1, \alpha_2, \alpha_2, \alpha_1) = [\frac{1+(-1)^{m+n}}{2}\alpha_2 + \frac{1-(-1)^{m+n}}{2}\alpha_1]z + w$ where w is a q-monodiffric constant. (3.42)

Incidentally we found that the $\mathcal{J}$-integral of a biconstant is a bifunction:

$$\mathcal{J}(\alpha_1, \alpha_2, \alpha_2, \alpha_1) = (\alpha_2 + \alpha_1)z \oplus (\alpha_2 - \alpha_1)z + w. \qquad (3.43)$$

Example 3. Let w be a q-monodiffric constant namely $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$.

Calculating $\mathcal{J}$ w

$$\mathcal{J}w = \begin{cases} \alpha_2 x + \alpha_3 iy \text{ if m is even, n is even} \\ \alpha_4 x + \alpha_1 iy \text{ if m is even, n is odd} \\ \alpha_1 x + \alpha_4 iy \text{ if m is odd, n is even} \\ \alpha_3 x + \alpha_2 iy \text{ if m is odd, n is odd.} \end{cases}$$

We can also note that the above solution is the solution of the equation $\delta^2 f(z) = 0$.

Example 4. Let $f, g \in \mathcal{M}(D)$.

Then $(f \oplus g)(z) = f(z) + (-1)^{m+n} g(z) \in \mathcal{M}(D)$.

Calculating $\mathcal{J}$-integral

$\mathcal{J}(f \oplus g)(z) = \mathcal{J}f(z) \oplus \mathcal{J}(-g(z)) + w$ where w is a q-monodiffric constant. (3.45)

CHAPTER IV

BASIC PROPERTIES OF DISCRETE POSITIVE POWERS

Discrete powers are introduced in discrete function theory to replace the usual powers of the classical analysis. z^n is not analytic in any discrete theory whereas $z^{(n)}$ is defined to suit discrete analyticity in every such theory. In this chapter the q-monodiffric analogue of z^n is introduced and properties are discussed. $z^{(n)}$ is q-monodiffric in H.

Infinite series of discrete powers whose coefficients are from complex numbers is discussed. Criterion for convergence of such series and a comparison test comparing the discrete series with a known classical counterpart are found. This provides a sufficient condition for an infinite discrete series to represent a q-monodiffric function.

Polynomial theories of discrete powers defined over complex numbers, biconstants and q-monodiffric constants are studied. They are called respectively discrete polynomials, bipolynomials and qm-polynomials. An attempt is made to investigate zeroes of these polynomials. Quadratic polynomials are studies in detail.

1. Discrete Powers

Isaacs, Duffin and Harman defined discrete powers. Here a more general class of discrete powers is found by defining in another way thus removing some difficulties occurring in the earlier literature of discrete powers and polynomial theory.

qm-binomial coefficients $C_{n,j}$ are defined as

$$C_{n,j} = \prod_{r=0}^{j-1} \frac{q^{-(n-r)} - q^{n-r}}{q^{-1} - q} \qquad n, j \in Z^+, \; j \quad n \qquad (4.1)$$

and

$$C_{n,0} = 1$$

$$(j)_q! = \prod_{r=0}^{j-1} \frac{q^{-(r+1)} - q^{r+1}}{q^{-1} - q} \quad \text{for } j \in Z^+$$

and

$$(0)_q! = 1 \text{ is called the qm-factorial.} \qquad (4.2)$$

q-monodiffric discrete powers for any nonnegative integral index is defined as

$$z^{(n)} = \prod_{j=0}^{n-1} (x + i q^{-(n-1)+2j} y) = \prod_{j=0}^{n-1} (q^{-(n-1)+2j} x + i y)$$

$$\text{for } z \in H, \; n \in Z^+ \text{ and } Z^{(0)} = 1. \qquad (4.3)$$

We write equivalently

$z^{(n)} = [x+iy]_n, \; [x+iy]_0 = 1.$ This notation is analogous to the q-basic $(x+iy)_n$. (4.4)

A discrete product for such powers is introduced as

$$z^{(m)} * z^{(n)} = [x+iy]_m^* [x+iy]_n = [x+iy]_{m+n} = z^{(m+n)}. \qquad (4.5)$$

In another way writing,

$$z^{(m+n)} = [x+iy]_{m+n}$$

$$= \prod_{j=0}^{m+n-1} (x+iq^{-(m+n-1)+2j}y)$$

$$= \prod_{j=0}^{m-1} (x+iq^{-(m+n-1)+2j}y) \prod_{j=m}^{m+n-1} (x+iq^{-(m+n-1)+2j}y)$$

$$= [x+iq^{-n}y]_m [x+iq^{m}y]_n .$$

Also

$$z^{(m+n)} = [x+iq^{-m}y]_n [x+iq^{n}y]_m .$$

Thus we have:

Theorem 1. $z^{(m)} * z^{(n)} = (x+iq^{-n}y)^{(m)}(x+iq^{m}y)^{(n)}$

$$= (x+iq^{-m}y)^{(n)}(x+iq^{n}y)^{(m)} .$$

Now we solve the simplest of polynomial equations namely $z^{(n)} = 0$. The proof directly follows from $z^{(n)} = [x+iy]_n$ as both the real and imaginary parts of each j^{th} factor $(x+iq^{-(n-1)+2j}y)$ vanish.

Theorem 2. The polynomial equation $z^{(n)} = 0$ has n and only n zeroes which are at origin.

It is easy to note that $Mz^{(n)} = 0$ for every $z \in H$ and $\delta_x z^{(n)} = \delta_y z^{(n)} = (n)_q z^{(n-1)}$

where $(n)_q = \dfrac{q^{-n}-q^n}{q^{-1}-q}$ for $n \in Z^+$.

Thus we have:

Theorem 3. $z^{(n)} \in \mathcal{M}\mathcal{G}(H)$ and $\delta z^{(n)} = (n)_q z^{(n-1)}$ for $n \in Z^+$ and $\delta z^{(0)} = 0$.

The following simple results are immediate.

$$\text{a)}\ \lim_{q\to 1} z^{(n)} = z^n \ ; \quad \lim_{q\to 1} \delta z^{(n)} = nz^{n-1} \tag{4.6}$$

$$\text{b)}\ \lim_{z\to 0} z^{(n)} = 0,\ n>0 \ ; \quad (\lambda z)^{(n)} = \lambda^n z^{(n)} \text{ for } \lambda \in \emptyset. \tag{4.7}$$

The discrete power of z to the index n can also be expressed as a sum of (n+1) terms. For this purpose we need the following result.

A homogeneous expression of the form $\sum_{j=0}^{n} \alpha_j x^{n-j}(iy)^j$, $\alpha_0 = 1$ and $\alpha_j \in R$ is q-monodiffric in H if and only if $\alpha_j = C_{n,j}$.

As the product $\prod_{j=0}^{n-1}(x+iq^{-(n-1)+2j}y)$ is homogeneous in x and y and coefficient of x^n is 1, using the above result we have

<u>Theorem 4</u>. $z^{(n)} = \prod_{j=0}^{n-1}(x+iq^{-(n-1)+2j}y) = \sum_{j=0}^{n} C_{n,j}x^{n-j}(iy)^j$

for $n \in Z^+$.

Let us investigate a few estimates of $z^{(n)}$.

1) $$z^{(2k)} = \prod_{j=0}^{2k-1}(x+iq^{-2k+2j+1}y)$$

$$= \prod_{j=0}^{k-1}(x+iq^{2k-2j-1}y)(x+iq^{-2k+2j+1}y).$$

Thus

$$z^{(2k)} = \prod_{j=0}^{k-1} q^{-2k+2j+1}(x+iq^{2k-2j-1}y)(q^{2k-2j-1}x+iy)$$

$$= \prod_{j=0}^{k-1} q^{2k-2j-1}(x+iq^{-2k+2j+1}y)(q^{-2k+2j+1}x+iy).$$

Using $|z_1 z_2| = |z_1||z_2|$ and simplifying we get

$$q^{k^2} \left|z^{2k}\right| \leq \left|z^{(2k)}\right| \leq q^{-k^2} \left|z^{2k}\right| . \tag{4.9}$$

Similarly

$$q^{k(k+1)} \left|z^{2k+1}\right| \leq \left|z^{(2k+1)}\right| \leq q^{-k(k+1)} \left|z^{2k+1}\right| . \tag{4.10}$$

Combining we have

$$q^{[\frac{n}{2}][\frac{n+1}{2}]} \quad |z|^n \leq |z^{(n)}| \leq q^{-[\frac{n}{2}][\frac{n+1}{2}]} \quad |z|^n$$

where [s] means the integral part of s. (4.11)

2) From $z^{(n)} = \prod_{j=0}^{n-1} (x+iq^{-(n-1)+2j}y)$, by combining the

factors $(x+iq^{-(n-1)+2j}y)$ and $(x+iq^{n-1-2j}y)$ and simplifying

we get $|z^{(n)}| \geq |z|^n$. (4.12)

Likewise $|z^{(n)}| = |(x+iqy)^{(n-1)}| \, |x + iq^{-(n-1)}y|$

i.e., $|z^{(n)}| \geq |x+iqy|^{n-1} \, |x+iq^{-(n-1)}y|$ (4.13)

and $|z^{(n)}| \geq |x+iq^{n-1}y| \, |x+iq^{-1}y|^{n-1}$. (4.14)

The following result is an inequality between $z^{(n)}$ and $z^{(n+1)}$.

$$\left|\frac{z^{(n+1)}}{z^{(n)}}\right| = \left|\frac{\prod_{j=0}^{n}(x+iq^{-n+2j}y)}{\prod_{j=0}^{n-1}(x+iq^{-(n-1)+2j}y)}\right|$$

$$= \frac{\sqrt{|x+iq^{n}y|\ |x+iq^{n}y|\ |x+iq^{(n-2)}y|\ |x+iq^{n-2}y|\ |x+iq^{n-4}y| \ldots |x+iq^{-n}y|}}{|x+iq^{n-1}y|\ |x+iq^{n-3}y| \ldots |x+iq^{-n+1}y|}\,.$$

Using the fact that $\dfrac{\sqrt{|x+iq^{n}y|\ |x+iq^{n-2}y|}}{|x+iq^{n-1}y|} > 1$

and so on we get

$$\left|\frac{z^{(n+1)}}{z^{(n)}}\right| \geq \sqrt{|x+iq^{n}y|\ |x+iq^{-n}y|}\,. \tag{4.15}$$

Similarly $\left|\dfrac{z^{(n+1)}}{z^{(n)}}\right| \leq \dfrac{|x+iq^{n}y|\ |x+iq^{-n}y|}{\sqrt{|x+iq^{n-1}y|\,|x+iq^{-(n-1)}y|}}\,.$ (4.16)

Thus we have:

$$|x+iq^{n}y|\,|x+iq^{-n}y|\ |z^{(n)}| \leq |z^{(n+1)}| \leq \frac{|x+iq^{n}y|\ |x+iq^{-n}y|}{\sqrt{|x+iq^{n-1}y|\,|x+iq^{-(n-1)}y|}}\,|z^{(n)}|\,. \tag{4.17}$$

Also writing in another way

$$|z^{(n)}| \leq \frac{|z^{(n+1)}|}{\sqrt{|x+iq^{n}y|\ |x+iq^{-n}y|}} \leq \sqrt{\frac{|x+iq^{n}y|\ |x+iq^{-n}y|}{|x+iq^{(n-1)}y|\,|x+iq^{-(n-1)}y|}}\,|z^{(n)}|\,. \tag{4.18}$$

2. Infinite Series

Using the definition of $z^{(r)}$, we study q-monodiffric functions of the type $\sum_{r=0}^{\infty} a_r z^{(r)}$ where $a_r \in \emptyset$ in a certain domain. Thus we get the study of q-monodiffric functions in terms of Weiestrassian approach of an analytic function.

Let us consider $\sum_{r=0}^{\infty} a_r z^{(r)}$ which we write as

$$\sum_{r=0}^{\infty} b_r u_r \text{ where } u_r = q^{\frac{r^2}{4}} z^{(r)} \text{ and } b_r = a_r q^{\frac{-r^2}{4}}. \tag{4.19}$$

Then

$$\frac{u_{r+1}}{u_{r-1}} = \frac{u_{r+1}}{u_r} \cdot \frac{u_r}{u_{r-1}} = q^r(x+iq^r y)(x+iq^{-r}y). \tag{4.20}$$

Also
$$\lim_{r\to\infty}\left|\frac{u_{r+1}}{u_r}\right| = \lim_{r\to\infty} \frac{q^{\frac{(r+1)^2}{4}}}{q^{\frac{r^2}{4}}} \left|\frac{z^{(r+1)}}{z^{(r)}}\right|$$

$$\leq \lim_{r\to\infty} q^{\frac{2r+1}{4}} \sqrt{x} \frac{|x+iq^{-r}y|}{\sqrt{|x+iq^{-(r-1)}y|}},$$

using an estimate of $\dfrac{z^{(r+1)}}{z^{(r)}}$

$$= \lim_{r\to\infty} q^{\frac{2r+1}{4} - r + \frac{(r-1)}{2}} \sqrt{xy}$$

i.e.,
$$\lim_{r\to\infty}\left|\frac{u_{r+1}}{u_r}\right| \leq q^{-\frac{1}{2}}\sqrt{xy}. \tag{4.21}$$

Also,

$$\left|\,\left|\frac{u_{r+1}}{u_r}\right| - \left|\frac{u_r}{u_{r-1}}\right|\,\right| = \left|\, q^{\frac{2r+1}{4}} \left|\frac{z^{(r+1)}}{z^{(r)}}\right| - q^{\frac{2r-1}{4}} \left|\frac{z^{(r)}}{z^{(r-1)}}\right|\,\right|$$

$$= \left|\, q^{\frac{2r+1}{4}} \left|\frac{z^{(r+1)}}{z^{(r-1)}}\right| \left|\frac{z^{(r-1)}}{z^{(r)}}\right| - q^{\frac{2r-1}{4}} \left|\frac{z^{(r)}}{z^{(r-1)}}\right| \left|\frac{z^{(r)}}{z^{(r+1)}}\right|\,\right|$$

$$= q^r \left|x+iq^r y\right| \left|x+iq^{-r}y\right| \left|\, q^{\frac{2r+1}{4}} \left|\frac{z^{(r-1)}}{z^{(r)}}\right| - q^{\frac{2r-1}{4}} \left|\frac{z^{(r)}}{z^{(r+1)}}\right|\,\right|.$$

But $\left|\frac{z^{(r)}}{z^{(r+1)}}\right| \leq \frac{1}{|x+iq^n y|} \leq \frac{1}{x}$.

$$\text{Thus } \lim_{r\to\infty} \left|\,\left|\frac{u_{r+1}}{u_r}\right| - \left|\frac{u_r}{u_{r-1}}\right|\,\right| = 0 . \tag{4.22}$$

Combining the above two results, we get

$$\lim_{r\to\infty} \left|\frac{u_{r+1}}{u_r}\right| = \lim_{r\to\infty} \left|\frac{u_r}{u_{r-1}}\right|. \tag{4.23}$$

Then substituting (4.23) in (4.20), we get

$$\lim_{r\to\infty} \left|\frac{u_{r+1}}{u_r}\right| = \sqrt{xy} . \tag{4.24}$$

Now to test the convergence of

$$\sum_{r=0}^{\infty} a_r z^{(r)} = \sum_{r=0}^{\infty} b_r u_r ,$$

$$\lim_{r\to\infty} \left|\frac{b_{r+1}u_{r+1}}{b_r u_r}\right| = \lim_{r\to\infty} \left|\frac{b_{r+1}}{b_r}\right| \sqrt{xy} .$$

Hence expressing b_r's in terms of a_r's, we get:

$\sum_{r=0}^{\infty} a_r z^{(r)}$ is absolutely convergent in $D \subset H$ if

$$\lim_{r\to\infty} \left|\frac{a_{r+1}}{a_r}\right| q^{-\frac{2r+1}{4}} \sqrt{xy} \leq 1$$

for $(x,y) \in D$. (4.25)

Thus

$$\sum_{r=0}^{\infty} a_r z^{(r)} \in \mathcal{M}(D) \text{ if } \lim_{r\to\infty} \left|\frac{a_{r+1}}{a_r}\right| q^{-\frac{2r+1}{4}} \sqrt{xy} \leq 1$$

for $(x,y) \in D$. (4.26)

Using the estimate $|z^{(n)}| \leq q^{-[\frac{n}{2}][\frac{n+1}{2}]} |z^n|$,

root test also gives a similar result namely

if $\lim_{r\to\infty} |a_r|^{\frac{1}{r}} q^{-(\frac{r+1}{4})} |z| \leq 1$ for every $z \in D$ then

$$\sum_{r=0}^{\infty} a_r z^{(r)} \in \mathcal{M}(D). \qquad (4.27)$$

In fact, due to the fact that an estimate of $z^{(n)}$ is used in the latter, the domain of convergence from the ratio test is more accurate and larger compared to that from the root test.

$$\sum_{r=0}^{\infty} \frac{z^{(r)}}{(r)_q!} \text{ and } \sum_{r=0}^{\infty} q^{r^2} z^{(r)}$$ are discrete entire in the sense that they are q-monodiffric in H, while $\sum_{r=0}^{\infty} q^{\frac{r(r+1)}{4}} z^{(r)}$ is q-monodiffric in a domain in H bounded by the limit points and a distorted ray which is given by the relation $q^{m+n+1} x_0 y_0 < 1$ where $(x,y) = (q^m x_0, q^n y_0)$. (4.28)

If we take $x_0 = y_0 = 1$, $\sum_{r=0}^{\infty} q^{\frac{r(r+1)}{4}} z^{(r)}$ is q-monodiffric at (x,y) if $m+n+\frac{1}{2} > 0$. Thus we get the domain of convergence of $\sum_{r=0}^{\infty} q^{\frac{r(r+1)}{4}} z^{(r)}$ is bounded by the limit points and the distorted ray through $(x_0, y_0) = (1,1)$.

Also, $$\sum_{r=0}^{\infty} q^{-\left(\frac{r+1}{2}\right)} \frac{z^{(r)}}{(r)_q!} \text{ and } \sum_{r=0}^{\infty} \frac{z^{(r)}}{\sqrt{(r)_q!}}$$ are q-monodiffric in certain domains bounded by limit points and distorted rays. (4.30)

Due to the condition on domain of convergence of $\sum_{r=0}^{\infty} a_r z^{(r)}$ namely $\lim_{r\to\infty} \left|\frac{a_{r+1}}{a_r}\right| q^{\frac{-2r+1}{4}} \sqrt{xy} < 1$, we get any such domain of convergence is bounded by limit points and a distorted ray. Thus the circle of convergence in the classical analysis is replaced by a distorted ray of convergence in q-monodiffric theory. (4.31)

<u>Comparison test</u>. Now we introduce a test to fix the domain of q-monodiffricity of an infinite series in discrete powers.

The domain in the complex plane satisfying $|x+iy| \leq a$ is denoted by $\mathcal{D}_a$ while the domain in H satisfying $\sqrt{xy} \leq a$ by D_a. (4.32)

Let $\sum\limits_{r=0}^{\infty} a_r z^r$ is absolutely convergent in $\mathcal{D}_a$.

Then $\lim\limits_{r\to\infty} \left|\frac{a_{r+1}}{a_r}\right| \cdot a < 1$.

Considering $\sum\limits_{r=0}^{\infty} a_r q^{\frac{r^2}{4}} z^{(r)}$, we get

$$\lim_{r\to\infty} \left|\frac{a_{r+1}}{a_r}\right| \frac{q^{\frac{(r+1)^2}{4}}}{q^{\frac{r^2}{4}}} \left|\frac{z^{(r+1)}}{z^{(r)}}\right| = \lim_{r\to\infty} \left|\frac{a_{r+1}}{a_r}\right| \sqrt{xy}$$

$$\leq \lim_{r\to\infty} \left|\frac{a_{r+1}}{a_r}\right| a < 1.$$

Hence we get that $\sum\limits_{r=0}^{\infty} a_r z^r$ is analytic in

$\mathcal{D}_a \Rightarrow \sum\limits_{r=0}^{\infty} a_r q^{\frac{r^2}{4}} z^{(r)} \in \mathcal{M}(D_a)$.

Similarly the converse also is true. It is stated as

$$\sum_{r=0}^{\infty} a_r z^{(r)} \varepsilon \mathcal{M}(D_a) \Rightarrow \sum_{r=0}^{\infty} a_r q^{-\frac{r^2}{4}} z^r$$

is analytic in $\mathbb{D}_a$.

Thus combining both the results we have:

<u>Theorem</u>. $\sum_{r=0}^{\infty} a_r z^{(r)} \varepsilon \mathcal{M}(D_a) \Longleftrightarrow \sum_{r=0}^{\infty} a_r q^{-\frac{r^2}{4}} z^r$

is analytic in $\mathbb{D}_a$.

To illustrate, $\sum_{r=0}^{\infty} z^r$ is analytic in $\mathbb{D}_1$ and $\sum_{r=0}^{\infty} q^{\frac{r^2}{4}} z^{(r)} \varepsilon \mathcal{M}(D_1)$. We get the distorted ray of convergence of $\sum_{r=0}^{\infty} q^{\frac{r^2}{4}} z^{(r)}$ is given by the relation $m+n = 1$ if $x_0 = y_0 = 1$.

We can also note that $\mathbb{D}_a$ represents a circle with radius a in the complex plane while D_a represents a set of lattice points in H enclosed by $x = 0$, $y = 0$ and the hyperbola $xy = a^2$ in the first quadrant of the complex plane. Also, $\mathbb{D}_a$ is a finite domain whereas D_a is infinite for all finite a. If a is infinite, $\mathbb{D}_a$ enlarges to $\emptyset$ and D_a to H.

(4.33)

3. Discrete Polynomial Theory

A discrete polynomial is defined as

$p(z) = \sum_{j=0}^{n} a_j z^{(j)}$, $a_j \in \not{C}$, $a_n \neq 0$. a_j's are called the coefficients of the polynomial. (4.34)

A discrete polynomial is q-monodiffric in H.

+ and * are defined in the set of polynomials as

$a_j z^{(j)} + b_j z^{(j)} = (a_j + b_j) z^{(j)}$ and $z^{(j)} * z^{(k)} = z^{(j+k)}$.

Degree of the polynomial $\sum_{j=0}^{n} a_j z^{(j)}$, $a_n \neq 0$ is n. Degree of p(z) is denoted by d(p(z)). (4.35)

The set of polynomials of the form $\left\{ \sum_{j=0}^{n} a_j z^{(j)},\ a_j \in \not{C},\ a_n \neq 0,\ n \in Z^{0+} \right\}$ is denoted by $\not{C}^*[z]$ where as the integral domain of polynomials:

$\left\{ \sum_{j=0}^{n} a_j z^{j},\ a_j \in \not{C},\ a_n \neq 0,\ n \in Z^{0+} \right\}$ by $\not{C}[z]$. (4.36)

$\not{C}^*[z]$ is a vector space over $\not{C}$ for which the nonnegative discrete powers $\{z^{(n)}\}$ form a basis. $(\not{C}^*[z],+,*)$ is an integral domain. Also $\not{C}^*[z]$ is an algebra. We can imbed this integral domain in a field called the rational field of discrete functions. (4.37)

Due to the isomorphism of $\emptyset^*[z]$ and $\emptyset[z]$, we get the following results.

a) Let $p_1, p_2 \ \varepsilon\ \emptyset^*[z]$ and $p_2 \neq 0$. Then there exist unique polynomials p_3 and $p_4\ \varepsilon\ \emptyset^*[z]$ where

$$d(p_4) < d(p_2) \text{ or } p_4 \equiv 0 \text{ such that } p_1(z) \equiv p_3(z) * p_2(z) + p_4(z). \tag{4.38}$$

b) $\emptyset^*[z]$ is an Euclidean ring (4.39)

c) $\emptyset^*[z]$ is a unique factorization domain (4.40)

d) $\sum_{j=0}^{n} a_j z^{(j)}\ \varepsilon\ \emptyset^*[z]$ can be uniquely expressed as

$(z-\alpha_1) * (z-\alpha_2) * \ldots * (z-\alpha_n)$ where $\alpha_j\ \varepsilon\ \emptyset$ satisfying

$$\sum_{j=0}^{n} a_j z^j = (z-\alpha_1)(z-\alpha_2)\ldots.(z-\alpha_n). \tag{4.41}$$

<u>Zeroes of the polynomial.</u> z_1 is a zero of the polynomial

$$p(z) = \sum_{j=0}^{n} a_j z^{(j)} \text{ if } p(z_1) = 0 \tag{4.42}$$

Consider $\sum_{j=0}^{n} a_j z^{(j)}$, $a_n \neq 0$ which we can express as

$\sum_{j=0}^{n} a_j z^{(j)} = \emptyset_1(x,y) + i\emptyset_2(x,y)$; (x_1,y_1) is a zero of

$\sum_{j=0}^{n} a_j z^{(j)}$ means $\emptyset_1(x_1,y_1) = 0$ and $\emptyset_2(x_1,y_1) = 0$.

Trivially both the equations are of degree n in x and y. The solution of x in terms of y from one equation is substituted in the other and solving we get that there exist at most n^2 zeroes for the given n^{th} degree polynomial

$\sum_{j=0}^{n} a_j z^{(j)}$, $a_n \neq 0$, if $\phi_1(x_1,y_1)$ and $\phi_2(x_1,y_1)$ are prime to each other. (4.43)

Now we prove $\phi_1(x,y)$ and $\phi_2(x,y)$ are prime to each other. If not, without loss of generality, let $y = ax + b$ be a common factor of them. Then $y = ax + b$ is a factor of $\sum_{j=0}^{n} a_j z^{(j)}$, $a_n \neq 0$ also.

Then replacing y by ax + b in

$\sum_{j=0}^{n} a_j \left[\sum_{r=0}^{j} C_{j,r} x^{j-r}(iy)^r\right]$, we get a polynomial in x only of degree n, whose coefficient must vanish identically. Considering coefficient of x^n, we get $\sum_{j=0}^{n} C_{n,j}(ia)^j = 0$.

If $a \neq 0$, we replace a by α/β and thus $(\alpha + i\beta)^{(n)} = 0$ which implies $\alpha = 0$, $\beta = 0$ due to the fact $z^{(n)} = 0 \implies z = 0$. If $a = 0$, from the above relation $1 = 0$. Thus both the cases reduce to contradiction.

Thus we have:

<u>Theorem</u>. A discrete polynomial of degree n cannot have more than n^2 zeroes.

Due to the fact that $z^{(n)} = z^n$ if and only if z is a purely real or imaginary point, we get the following result.

α (or $i\alpha$) where α is a real number is a zero of $\sum_{j=0}^{n} a_j z^{(j)}$ if and only if α (or $i\alpha$) is a zero of $\sum_{j=0}^{n} a_j z^j$.

(4.44)

Then using the above result and the classical result that $\sum_{r=0}^{n} a_r z^r$, $a_r \in R$, $a_n \neq 0$ and n odd has atleast one real zero, we have:

$\sum_{r=0}^{n} a_r z^{(r)}$, $a_r \in R$, $a_n \neq 0$, n odd, has atleast one real zero. (4.45)

Consider the quadratic polynomial $z^{(2)} + bz + c$. Without loss of generality we take b, c to be real. To investigate zeroes,

$$(x + iqy)\,(x + iq^{-1}y) + b(x + iy) + c = 0$$

$$\Rightarrow \begin{cases} x^2 - y^2 + bx + c = 0 \\ (q + q^{-1})xy + by = 0 \end{cases}$$

Zeroes of $z^{(2)} + bz + c$ is the set:

$$\left\{ \left(\frac{-b \pm\sqrt{b^2-4c}}{2}, 0\right), \quad \left(\frac{-b}{q^{-1}+q}, \frac{\pm\sqrt{c(q^{-1}+q)^2-b^2(q^{-1}+q-1)}}{q^{-1}+q}\right) \right\}. \quad (4.46)$$

Thus we have the following conclusions.

a) If $b^2-4c > 0$ and $c(q^{-1}+q)^2-b^2(q^{-1}+q-1) > 0$.

$z^{(2)} + bz + c$ has four zeroes namely

$$\left\{ \left(\frac{-b \pm\sqrt{b^2-4c}}{2}, 0\right) ; \quad \left(\frac{-b}{q^{-1}+q}, \frac{\pm\sqrt{c(q^{-1}+q)^2-b^2(q^{-1}+q-1)}}{q^{-1}+q}\right) \right\}.$$

b) If $b^2 = 4c$ and $c(q^{-1}+q)^2-b^2(q^{-1}+q-1) > 0$, $z^{(2)} + bz + c$ has three zeroes namely $\left\{ \left(-\frac{b}{2}, 0\right) ; \left(\frac{-b}{q^{-1}+q}, \frac{\pm\sqrt{c}(q^{-1}+q-2)}{q^{-1}+q}\right) \right\}$.

c) If $b^2-4c > 0$ and $c(q^{-1}+q)^2 = b^2(q^{-1}+q-1)$ then the zeroes are

$$\left\{ \left(\frac{-b \pm\sqrt{b^2-4c}}{2}, 0\right) ; \quad \left(\frac{-b}{q^{-1}+q}, 0\right) \right\}.$$

d) If $b^2-4c > 0$ and $c(q^{-1}+q)^2-b^2(q^{-1}+q-1) < 0$,
$z^{(2)} + bz + c$ has two zeroes namely $\left\{ \left(\frac{-b \pm\sqrt{b^2-4c}}{2}, 0\right) \right\}$.

e) If $b^2-4c < 0$ and $c(q^{-1}+q)^2-b^2(q^{-1}+q-1) > 0$,

$z^{(2)} + bz + c$ has two zeroes namely

$$\left\{ \left(\frac{-b}{q^{-1}+q}, \frac{\pm\sqrt{c(q^{-1}+q)^2-b^2(q^{-1}+q-1)}}{q^{-1}+q}\right) \right\}.$$

f) Then the four cases:

(1) $b^2-4c < 0$; $c(q^{-1}+q)^2-b^2(q^{-1}+q-1) < 0$,

(2) $b^2-4c = 0$; $c(q^{-1}+q)^2-b^2(q^{-1}+q-1) = 0$,

(3) $b^2-4c = 0$; $c(q^{-1}+q)^2-b^2(q^{-1}+q-1) < 0$

and

(4) $b^2-4c < 0$; $c(q^{-1}+q)^2-b^2(q^{-1}+q-1) = 0$.

do not exist.

Hence

The number of zeroes of the q-monodiffric polynomial $z^{(2)}+ bz + c$; $b, c \in R$ ranges from two to four. (4.46)

It is interesting to note that the zeroes of the polynomial $(z+1)*(z+1) = z^{(2)}+ 2z + 1$ are three in number:

$$\left\{(-1,0),\ \left(\frac{-2}{q^{-1}+q},\pm\frac{\sqrt{(q^{-1}+q)^2-4(q^{-1}+q-1)}}{q^{-1}+q}\right)\right\}. \tag{4.47}$$

So we give a note to the result (4.44). α is a purely real or imaginary zero of the polynomial $\sum_{j=0}^{n} a_j z^{(j)}$, $a_j \in \emptyset$ repeated r times does not imply that α is a zero of $\sum_{j=0}^{n} a_j z^j$ repeated r times, but strictly implies that α is a zero of $\sum_{j=0}^{n} a_j z^j$. (4.48)

Also the zeroes of $z^{(2)} + \alpha + i\beta$; α , $\beta \in R$ are

$$\left\{ \left(\frac{\pm\sqrt{-\alpha \pm\sqrt{\alpha^2 + \frac{4\beta^2}{(q^{-1}+q)^2}}}}{2} , \frac{-\sqrt{2\beta}}{\pm(q^{-1}+q)\sqrt{-\alpha \pm \sqrt{\alpha^2 + \frac{4\beta^2}{(q^{-1}+q)^2}}}} \right) \right\}.$$

<u>q-nth roots of unity.</u> Now we solve the polynomial equation $z^{(n)} = 1$. The zeroes are called the q-nth roots of unity. They are denoted by $1^{(1/n)}$.

$$z^{(n)} = 1 \Longrightarrow x^n - C_{n,2}x^{n-2}y^2 + C_{n,4}x^{n-4}y^4 - \dots = 1 \qquad (4.50)$$

$$\text{and} \quad C_{n,1}x^{n-1}y - C_{n,3}x^{n-3}y^3 + C_{n,5}x^{n-5}y^5 - \dots = 0 . \qquad (4.51)$$

Let $y = mx$ where m is real be a solution of the above homogeneous equation.

Then (4.51) has atmost n solutions with $m = m_i$.

Then, for the possible values of m,

$$\prod_{j=0}^{n-1} (x+iq^{-(n-1)+2j}y) = 1 \Longrightarrow x = \left[\frac{1}{\prod_{j=0}^{n-1}(1+iq^{-(n-1)+2j}m_i)}\right]^{1/n}$$

But $\left[\frac{1}{\prod_{j=0}^{n-1}(1+iq^{-(n-1)+2j}m_i)}\right]^{1/n}$ gives

atmost two real values if n is even

one real value if n is odd.

Thus we have:

$z^{(n)}-1$ has atmost 2n zeroes if n is even and n zeroes if n is odd. (4.52)

<u>Examples.</u>

1. There are only two q-square roots of unity: $\{(\pm 1, 0)\}$. (4.53)

2. There are three q-cube roots of unity:

$$\left\{(1,0),\ \left(\frac{-1}{\sqrt[3]{(2+q^{-2}+q^{2})(q^{-2}+q^{2})}},\ \frac{\pm\sqrt{1+q^{-2}+q^{2}}}{\sqrt[3]{(2+q^{-2}+q^{2})(q^{-2}+q^{2})}}\right)\right\}. \tag{4.54}$$

3. There are eight q-fourth roots of unity.

$$\left\{(\pm 1,0),(0\ \pm 1),\ \left(\frac{\pm 1}{\sqrt[4]{2-C_{4,2}}},\ \frac{\pm 1}{\sqrt[4]{2-C_{4,2}}}\right)\right\}. \tag{4.55}$$

4. There are five fifth roots of unity:

$$\left\{(1,0),\left(\frac{1}{\sqrt[5]{1-C_{5,2}m^{2}+C_{5,4}m^{4}}},\ \frac{m}{\sqrt[5]{1-C_{5,2}m^{2}+C_{5,4}m^{4}}}\right)\right. \text{ where}$$

$$\left. m=\pm\sqrt{\frac{C_{5,2}\pm\sqrt{C_{5,2}-4C_{5,1}}}{2}}\right\}. \tag{4.56}$$

Then likewise q-nth roots of -1, i and -i are immediate. The same theorem holds in these cases also.

$z^{(n)} - \alpha$ where α is real or purely imaginary has atmost 2n zeroes if n is even and n zeroes if n is odd.

(4.57)

4. Bipolynomials

Duffin introduced bipolynomials in his basic paper. In Duffin's theory if $g(z)$ and $h(z)$ are polynomials in m and n where $(m,n) = z$ is in Gaussian lattice,

$$f(z) = \begin{cases} g(z) & \text{if } m + n \text{ is even} \\ h(z) & \text{if } m + n \text{ is odd} \end{cases}$$

is a bipolynomial. In general Duffin's bipolynomial is not discrete analytic. If $g(z) = h(z)$, the bipolynomial reduces to a polynomial which is not necessarily discrete analytic. Using this definition and preholomorphic integration, the preholomorphic discrete powers which are discrete entire are defined. Zeilberger used bipolynomials to study some problems in entire functions.

The bipolynomial in this theory is defined as $\sum_{j=0}^{n} u_j z^{(j)}$ where u_j's are biconstants and $u_n \neq 0$ and multiplication is pointwise. We note that bipolynomials are bifunctions. Again the derivative of a bipolynomial is bipolynomial. A simple example of a bipolynomial is a biconstant. (4.58)

$\sum\limits_{j=0}^{n} u_j z^{(j)}$ can be expressed as $\sum\limits_{j=0}^{n} a_j z^{(j)}$ in H_1

and $\sum\limits_{j=0}^{n} b_j z^{(j)}$ in H_2. Any solution (zero at the concerned lattice point) of any of these two q-monodiffric polynomials is a solution of the bipolynomial. (4.59)

As in the case of discrete polynomial here also, we get an upper bound for the zeroes of a bipolynomial. Using the definition of zero of a bipolynomial, we have the result as

A bipolynomial $\sum\limits_{j=0}^{n} u_j z^{(j)}$, $u_n \neq 0$ has atmost 4n zeroes.

<u>Example 1.</u> $\sum\limits_{j=0}^{4} u_j z^{(j)}$; $u_4 = (1,0,0,1)$, $u_3 = (0,0,0,0)$

$u_2 = (0,1,1,0)$, $u_1 = (0,b,b,0)$, $u_o = (1,c,c,1)$

where $b,c \in R$ has eight zeroes:

$$\left\{(\pm 1,0),(0,\pm 1),\ \left(\frac{-b \pm \sqrt{b^2-4c}}{2},0\right),\ \left(\frac{-b}{q^{-1}+q},\ \frac{\pm\sqrt{c(q^{-1}+q)^2-b^2(q^{-1}+q-1)}}{q^{-1}+q}\right)\right\}$$

but some of them may not exist depending on b and c.

In particular if $b = 0$, $c = 1$, the zeroes of the bipolynomial are reduced to four:

$\left\{(\pm 1,0),(0,\pm 1)\right\}$; but $(+1,0)$ and $(-1,0)$ are repeated twice.

Example 2. $\sum_{j=0}^{2} u_j z^{(j)}$, $u_2 = (1,1,1,1)$, $u_1 = (b_1,b_2,b_2,b_1)$ and $u_o = (c_1,c_2,c_2,c_1)$ has eight zeroes.

$$\left\{ \left(\frac{-b_1 \pm \sqrt{b_1^2-4c_1}}{2}, 0\right), \left(\frac{-b_2 \pm \sqrt{b_2^2-4c_2}}{2}, 0\right) \right.$$

$$\left(\frac{-b_1}{q^{-1}+q} \quad \frac{\pm\sqrt{c_1(q^{-1}+q)^2-b_1^2(q^{-1}+q-1)}}{q^{-1}+q}\right),$$

$$\left. \left(\frac{-b_2}{q^{-1}+q}, \frac{\pm\sqrt{c_2(q^{-1}+q)^2-b_2^2(q^{-1}+q-1)}}{q^{-1}-q}\right) \right\}.$$

But if $b_1 = b_2 = 0$ and $c_1 = 1$, $c_2 = -1$, the zeroes of the bipolynomial are only four $\left\{(\pm 1,0), (0,\pm 1)\right\}$.

Likewise the general quadratic bipolynomial $\sum_{j=0}^{2} u_j z^{(j)}$; $u_2 = (1,1,1,1)$, $u_1 = (b_1,b_2,b_2,b_1)$ and $u_o = (c_1,c_2,c_2,c_1)$ where $b_1,b_2,c_1,c_2 \in R$ has the solution set:

$$\left\{ \left(\frac{-b_1 \pm \sqrt{b_1^2-4c_1}}{2}, 0\right), \left(\frac{-b_1}{q^{-1}+q}, \frac{\pm\sqrt{c_1(q^{-1}+q)^2-b_1^2(q^{-1}+q-1)}}{q^{-1}+q}\right), \right.$$

$$\left. \left(\frac{-b_2 \pm \sqrt{b_2^2-4c_2}}{2}, 0\right), \left(\frac{-b_2}{q^{-1}+q}, \frac{\pm\sqrt{c_2(q^{-1}+q)^2-b_2^2(q^{-1}+q-1)}}{q^{-1}+q}\right) \right\}. \quad (4.61)$$

Some of these zeroes may not exist depending on the coefficients b_1, b_2, c_1 and c_2.

5. qm-Polynomials

A generalisation of discrete polynomials is discussed in this section. The coefficients of the polynomial are taken from the not-associative algebra of q-monodiffric constants.

Let $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ be a q-monodiffric constant and $f(z) = u(x+iy) + iv(x,y) : H \longrightarrow \emptyset$. Then the discrete product of α and $f(z)$ is defined as

$$\alpha * f(z) = (\alpha_3, \alpha_4, \alpha_1, \alpha_2) u(x,y) + i(\alpha_2, \alpha_1, \alpha_4, \alpha_3)\, v(x,y)$$

where the product on the right hand side is the point wise multiplication.

If α, β are q-monodiffric constants and $f, g : H \longrightarrow \emptyset$, then

$$(\alpha + \beta) * f = \alpha * f + \beta * f$$

$$\text{and} \qquad \alpha * (f + g) = \alpha * f + \alpha * g. \tag{4.63}$$

Let $f \in \mathcal{M}_{\mathcal{G}}(D)$.

Then

$$\delta_x(\alpha * f(z)) = \begin{cases} \alpha_4 u_x(x,y) + i\alpha_1 v_x(x,y) & \text{if } m \text{ is even, } n \text{ is even} \\ \alpha_3 u_x(x,y) + i\alpha_2 v_x(x,y) & \text{if } m \text{ is odd, } n \text{ is even} \\ \alpha_2 u_x(x,y) + i\alpha_3 v_x(x,y) & \text{if } m \text{ is even } n \text{ is odd} \\ \alpha_1 u_x(x,y) + i\alpha_4 v_x(x,y) & \text{if } m \text{ is odd, } n \text{ is odd.} \end{cases}$$

and

$$\delta_y(\alpha * f(z)) = \begin{cases} \alpha_1 u_y(x,y) + i\alpha_4 v_y(x,y) & \text{if } m \text{ is even, } n \text{ is even} \\ \alpha_2 u_y(x,y) + i\alpha_3 v_y(x,y) & \text{if } m \text{ is odd, } n \text{ is even} \\ \alpha_3 u_y(x,y) + i\alpha_2 v_y(x,y) & \text{if } m \text{ is even, } n \text{ is odd} \\ \alpha_4 u_y(x,y) + i\alpha_1 v_y(x,y) & \text{if } m \text{ is odd, } n \text{ is odd.} \end{cases}$$

Using the Cauchy-Riemann relations:

$u_x = iv_y$ and $u_y = iv_x$, we get

$\delta_x(\alpha * f(z)) = \delta_y(\alpha * f(z))$ for $z \in D$.

Thus we have:

<u>Theorem 1</u>. $f(z) \in \mathcal{M}(D) \Longrightarrow \alpha * f(z) \in \mathcal{M}(D)$ and $\delta(\alpha * f(z)) = (\alpha_2,\alpha_1,\alpha_4,\alpha_3) * \delta_x f(z) = (\alpha_3,\alpha_4,\alpha_1,\alpha_2) * \delta_y f(z)$ where $\alpha = (\alpha_1,\alpha_2,\alpha_3,\alpha_4)$ is a q-monodiffric constant.

Since a q-monodiffric constant is q-monodiffric in H, we can define the discrete product $*$ of two q-monodiffric constants.

Let $\alpha \equiv (\alpha_1,\alpha_2,\alpha_3,\alpha_4) \equiv (a_1+ib_1, a_2+ib_2, a_3+ib_3, a_4+ib_4)$ and $\beta \equiv (\beta_1,\beta_2,\beta_3,\beta_4) \equiv (c_1+id_1, c_2+id_2, c_3+id_3, c_4+id_4)$.

Then

$$\alpha * \beta = (\alpha_1,\alpha_2,\alpha_3,\alpha_4) * (\beta_1,\beta_2,\beta_3,\beta_4)$$

$$= (a_3+ib_3,a_4+ib_4,a_1+ib_1,a_2+ib_2)(c_1,c_2,c_3,c_4)$$

$$+ i(a_2+ib_2,a_1+ib_1,a_4+ib_4,a_3+ib_3)(d_1,d_2,d_3,d_4)$$

$$= (c_1(a_3+ib_3)+id_1(a_2+ib_2),\ c_2(a_4+ib_4)+id_2(a_1+ib_1),$$

$$c_3(a_1+ib_1)+id_3(a_4+ib_4),\ c_4(a_2+ib_2)+id_4(a_3+ib_3)). \tag{4.64}$$

Hence we get the set of q-monodiffric constants is closed under $*$. Also $*$ is not commutative, but distributive over + and again we get by direct simplification, that $*$ is not associative in the case of q-monodiffric constants. Further there does not exist a q-monodiffric constant e such that $\alpha * e = e * \alpha = \alpha$ for every α.

To find divisors of zero, let $\alpha * \beta = 0$.

We get

$$c_1a_3-d_1b_2 = 0 \ ; \ c_1b_3 + d_1a_2 = 0;$$

$$c_2d_4-d_2b_1 = 0 \ ; \ c_2b_4 + d_2a_1 = 0;$$

$$c_3a_1-d_3b_4 = 0 \ ; \ c_3b_1 + d_3a_4 = 0;$$

$$c_4a_2-d_4b_3 = 0 \ ; \ c_4b_2 + d_4a_3 = 0.$$

i.e., $\frac{c_1}{d_1} = \frac{b_2}{a_3} = -\frac{a_2}{b_3}$, $\frac{c_2}{d_2} = \frac{b_3}{a_4} = -\frac{a_1}{b_4}$;

$\frac{c_3}{d_3} = \frac{b_4}{a_1} = -\frac{a_4}{b_1}$; $\frac{c_4}{d_4} = \frac{b_3}{a_2} = -\frac{a_3}{b_2}$

i.e., $\frac{c_2}{d_2} = -\frac{d_3}{c_3} = \frac{b_1}{a_4} = -\frac{a_1}{b_4}$; $\frac{c_1}{d_1} = -\frac{d_4}{c_4} = \frac{b_2}{a_3} = -\frac{a_2}{b_3}$.

Hence we have:

Theorem 2. The set of q-monodiffric constants is a not-associative ring without identity under + and $*$.
$\alpha = (a_1+ib_1, a_2+ib_2, a_3+ib_3, a_4+ib_4)$ is a divisor of zero if and only if $a_1a_4 + b_1b_4 = 0 = a_2a_3 + b_2b_3$.

An expression of the form $\sum_{r=0}^{n} a_r * z^{(r)}$ where a_r are q-monodiffric constants is called a qm-polynomial. a_r's are called the coefficients of the polynomial. n is the degree of the polynomial if $a_n \neq (0,0,0,0)$. (4.65)

A lattice point $(q^m x_0, q^n y_0) \varepsilon H$ is a zero of the polynomial $\sum_{r=0}^{n} a_r * z^{(r)}$ if it vanishes at $(q^m x_0, q^n y_0)$.

Example 1. Now we investigate zeroes of $\alpha * z$. For this, with usual notations,

$$\alpha * z = (a_3+ib_3, a_4+ib_4, a_1+ib_1, a_2+ib_2)x$$

$$+ i(a_2+ib_2, a_1+ib_1, a_4+ib_4, a_3+ib_3)y$$

$= (w_1, w_2, w_3, w_4)$ where atleast one of w_i's is zero.

We get that the zero will satisfy atleast one pair of the following equations.

(1) $a_3x - b_2y = 0$; $b_3x + a_2y = 0$,

(2) $a_4x - b_1y = 0$; $b_4x + a_1y = 0$,

(3) $a_1x - b_4y = 0$; $b_1x + a_4y = 0$,

(4) $a_2x - b_3y = 0$; $b_2x + a_3y = 0$.

(0,0) is a trivial solution. If there exists any other zero, atleast one of the following is true.

$$\frac{x}{y} = \frac{a_3}{b_2} = -\frac{b_3}{a_2} \; ; \; \frac{x}{y} = -\frac{a_4}{b_1} = \frac{b_4}{a_1} \; ;$$

$$\frac{x}{y} = \frac{a_1}{b_4} = \frac{b_1}{a_4} \; ; \; \frac{x}{y} = \frac{a_2}{b_3} = -\frac{b_2}{a_3} \, .$$

Hence we get that $\alpha * z$ has a zero which is non-zero if and only if $a_2a_3 + b_2b_3 = 0$ or $a_1a_4 + b_1b_4 = 0$.

Thus, if α is a divisor of zero, $\alpha * z$ has infinite number of zeroes.

The converse is not true. For example, $\alpha * z$ where $\alpha = (1+i, 3-i, 2+6i, 2+i)$ which is not a divisor of zero has infinite number of zeroes.

Example 2. Consider $\alpha * z + \beta$ where

$\alpha = (a_1+ib_1, a_2+ib_2, a_3+ib_3, a_4+ib_4)$ and

$\beta = (c_1+id_1, c_2+id_2, c_3+id_3, c_4+id_4)$.

As in the previous example, we get four pairs of linear equations but non-homogeneous as

(1) $a_3x - b_2y = c_1$; $b_3x + a_2y = d_1$,

(2) $a_4x - b_1y = c_2$; $b_4x + a_1y = d_2$,

(3) $a_1x - b_4y = c_3$; $b_1x + a_4y = d_3$,

(4) $a_2x - b_3y = c_4$; $b_2x + a_3y = d_4$.

We get three possibilities as

1. If α is not a divisor of zero and $a_1a_4 + b_1b_4 \neq 0$ and $a_2a_3 + b_2b_3 \neq 0$, there are four zeroes for the polynomial.

2. If α is not a divisor of zero and one of $a_1a_4 + b_1b_4 = 0$ and $a_2a_3 + b_2b_3 = 0$ is satisfied, there are two or infinite number of zeroes for the polynomial depending on the other coefficients.

3. If α is a divisor of zero, there is no zero or there are an infinite number of zeroes for the polynomial.

Example 3. Let us solve $\alpha * z^{(2)} + \beta * z + \gamma$ for zeroes. We restrict the parameters of the q-monodiffric constants to be real.

Let $\alpha = (a_1, a_2, a_3, a_4)$, $\beta = (b_1, b_2, b_3, b_4)$

and $\gamma = (c_1, c_2, c_3, c_4)$.

Then

$$\alpha * z^{(2)} + \beta * z + \gamma \equiv (a_3, a_4, a_1, a_2)(x^2 - y^2)$$

$$+ i(a_2, a_1, a_4, a_3)(q^{-1}+q)xy + (b_3, b_4, b_1, b_2)x$$

$$+ i(b_2, b_1, b_4, b_3)y + (c_1, c_2, c_3, c_4) = (w_1, w_2, w_3, w_4)$$

where atleast one of w_j's is zero.

Thus we have four pairs of equations. A solution of any such pair is a zero of the polynomial.

(1) $a_3(x^2-y^2) + b_3x + c_1 = 0$; $a_2(q^{-1}+q)xy + b_2y = 0$,

(2) $a_4(x^2-y^2) + b_4x + c_2 = 0$; $a_1(q^{-1}+q)xy + b_1y = 0$,

(3) $a_1(x^2-y^2) + b_1x + c_3 = 0$; $a_4(q^{-1}+q)xy + b_4y = 0$,

(4) $a_2(x^2-y^2) + b_2x + c_4 = 0$; $a_3(q^{-1}+q)xy + b_3y = 0$.

Solving the first pair we get,

$$\left(\frac{-b_3 \pm \sqrt{b_3^2 - 4a_3c_1}}{2a_3},\ 0 \right) \quad \text{and}$$

$$\left(\frac{-b_2}{a_2(q^{-1}+q)},\ \pm\sqrt{\frac{b_2b_3}{a_2(q^{-1}+q)} - c_1 - \frac{a_3b_2^2}{a_2^2(q^{-1}+q)^2}} \right)$$

are the zeroes of the polynomial.

Similarly twelve zeroes are found from the other three pairs. Hence there are sixteen zeroes for the polynomial $\alpha * z^{(2)} + \beta * z + \gamma$.

Likewise we can find the zeroes of any polynomial. Still, all the zeroes found, may not be lattice points.

CHAPTER V

SPECIAL POLYNOMIALS

In earlier chapters we made an attempt to establish a theory of discrete functions. We now introduce a few special polynomials and their classifications to illustrate the theory of q-monodiffric functions. Further, references to mention are Boas and Buck [16] and Rainville [55].

1. q-Type Classification for Discrete Polynomials

Since the set of discrete polynomials form a unique factorisation domain in + and *, we have:

Let $\left\{\phi_n(z)\right\}$ be a simple sequence of polynomials such that $\phi_n(z) = \sum_{i=0}^{n} a_{n,i} z^{(i)}$ where $a_{n,n} \neq 0$. Consider $T_o(z) * [\delta\phi_1(z)] = \phi_o(z)$. Thus $T_o(z)$ will be a constant namely $\frac{a_{0,0}}{a_{1,1}}$. Then we take $\sum_{k=0}^{1} T_k(z)*[\delta^{k+1}\phi_2(z)] = \phi_1(z)$.

That is, $T_o(z) * \delta\phi_2(z) + T_1(z)*\delta^2\phi_2(z) = \phi_1(z)$.

Thus $T_1(z)$ is a unique polynomial

$$\frac{a_{1,1}^2 - a_{0,0}a_{2,2}}{2a_{1,1}a_{2,2}} z^{(1)} + \frac{a_{1,0}a_{1,1} - a_{0,0}a_{2,1}}{(2)_q a_{1,1}a_{2,2}}$$

of degree atmost 1.

Similarly $\sum_{k=0}^{2} T_k(z)*[\delta^{k+1}\phi_3(z)] = \phi_2(z)$ also gives a unique $T_2(z)$ of degree atmost two and so on. For any finite n,

$$\sum_{k=0}^{n} T_k(z)*[\delta^{k+1}\phi_{n+1}(z)] = \phi_n(z) \text{ determines } T_n(z)$$

uniquely of degree atmost n as $T_k(z)$ for $k < n$ is already fixed uniquely by the same method.

Thus we have:

If $\{\phi_n(z)\}$ is a simple sequence of polynomials $T_o(z) * [\delta\phi_1(z)] = \phi_o(z)$ and $\sum_{k=0}^{n} T_k(z) * [\delta^{k+1}\phi_{n+1}(z)] = \phi_n(z)$, $n \geq 1$ defines $T_n(z)$ uniquely of degree $\leq n$. (5.1)

In other words, using the fact that $\delta^r\phi_n(z) = 0$ for any $r > n$.

For a simple sequence of polynomials $\{\phi_n(z)\}$ there exists a unique derivate operator of the form $J(z,\delta) = \sum_{k=0}^{\infty} T_k(z)* \delta^{k+1}$ in which $T_k(z)$ is a discrete polynomial of degree $\leq k$ for which $J(z,\delta)\phi_n(z) = \phi_{n-1}(z)$, $n \geq 1$.

The polynomial sequence $\{\phi_n(z)\}$ is associated with the operator $J(z,\delta)$. $J(z,\delta)$ is unique for any given simple sequence $\{\phi_n(z)\}$. It is possible to classify the simple q-monodiffric polynomials sequences accordingly. (5.2)

The simple sequence $\{\phi_n(z)\}$ is q-j type, if the degree of $T_k(z)$ will not exceed j for any k and q-infinite type if there does not exist any such j. (5.3)

Now we take $T_k(z) = \sum_{r=0}^{k} c_{k,r} z^{(r)}$. So if $c_{k,r} = 0$ for all $r \geq 1$, then $\{\phi_n(z)\}$ is of q-zero type, if $c_{k,r} = 0$ for all $r \geq j+1$, then $\{\phi_n(z)\}$ is of q-j type. If there does not exist some j satisfying, $c_{k,r} = 0$ for all $r \geq j+1$, $\{\phi_n(z)\}$ is of q-infinite type. In particular, if $\{\phi_n(z)\}$ is of q-zero type, then $J(z,\delta) = \sum_{k=0}^{\infty} \alpha_k \delta^{k+1}$ where α_k are constants. (5.4)

The simple sequence of polynomials

$$\left\{ 1, \prod_{r=0}^{n-1} \frac{1}{(n-r)_q[1+(n-r-1)_q]} z^{(n)}, n = 1,2,\ldots \right\}$$ is

of q-one type classification having the operator

$J(z,\delta) = \delta + z * \delta^2$. (5.5)

A q-two type simple sequence of polynomials is given by

$$\left\{ 1, z^{(1)}, \prod_{r=0}^{n-2} \frac{1}{(n-r)_q[1+(n-r-1)_q-(n-r-1)_q(n-r-2)_q]} z^{(n)}, n \geq 2 \right\}$$

whose operator is $\delta + z * \delta^2 + z^{(2)} * \delta^3$ (5.6)

If $\{\phi_n(z)\}$ is a simple set of q-zero type polynomials and $\sum_{k=0}^{\infty} \alpha_k \delta^{k+1}[\phi_{n+1}(z)] = \phi_n(z)$ for $n \geq 1$,

then taking derivate on both sides successively r times, we get

$$\sum_{k=0}^{\infty} \alpha_k \delta^{k+1}[\delta^r \phi_n(z)] = \delta^r \phi_{n-1}(z) \text{ for } n \geq 1.$$

Thus we have:

If $\{\phi_n(z)\}$ is a simple sequence of q-zero discrete polynomials and r is a **positive integer, then** $\{\delta^r \phi_n(z)\}$ also **is a simple** sequence of q-zero type polynomials. (5.7)

Let $\{\phi_n(z)\}$ and $\{\psi_n(z)\}$ be two simple sequences of discrete polynomials and $\{b_k\}$ a sequence of complex numbers satisfying

$$\psi_n(z) = \sum_{k=0}^{n} b_k \phi_{n-k}(z) \text{ where } b_k \text{ is independent of } n.$$

Let $J(z,\delta)$ be the operator to which $\phi_n(z)$ belongs.

Then $J(z,\delta)\, \psi_n(z) = \sum_{k=0}^{n} b_k J(z,\delta)\, \phi_{n-k}(z)$

$$= \sum_{k=0}^{n-1} b_k\, \phi_{n-k-1}(z)$$

$$= \psi_{n-1}(z).$$

Thus $\{\psi_n(z)\}$ and $\{\phi_n(z)\}$ belong to the same operator $J(z,\delta)$. Conversely, suppose that $\{\phi_n(z)\}$ and $\{\psi_n(z)\}$ belong to the same operator $J(z,\delta)$. Since $\{\phi_n(z)\}$ and $\{\psi_n(z)\}$ are simple sets of polynomials, we get

$$\psi_n(z) = \sum_{k=0}^{n} \alpha_{k,n} \phi_{n-k}(z)$$ where $\alpha_{k,n}$ are constants. Applying

the operator $J(z,\delta)$ on both sides

$$\psi_{n-1}(z) = \sum_{k=0}^{n-1} \alpha_{k,n} \phi_{n-k-1}(z).$$

Replacing n by n+1, $\psi_n(z) = \sum_{k=0}^{n} \alpha_{k,n+1} \phi_{n-k}(z)$.

From the two equivalent expressions of $\psi_n(z)$ we get

$\alpha_{k,n} = \alpha_{k,n+1}$ which is possible only if $\alpha_{k,n}$ is independent of n for any k. Thus we see that

$$\psi_n(z) = \sum_{k=0}^{n} b_k \phi_{n-k}(z).$$

Concluding,

$\{\phi_n(z)\}$ and $\{\psi_n(z)\}$ are two simple sets of discrete polynomials belonging to the same operator $J(z,\delta)$ if and only if there exists a relation of the form

$$\psi_n(z) = \sum_{k=0}^{n} b_k \phi_{n-k}(z)$$

where the sequence $\{b_k\}$ is independent of n. (5.8)

For example, take $\left\{D_n(z,0)\right\} = \left\{\frac{z^{(n)}}{(n)_q!}\right\}$

and $\left\{D_n(z,1)\right\} = \left\{\sum_{r=0}^{n} \frac{z^{(r)}}{(r)_q!}\right\}$ are of q-zero type. (5.9)

2. A Simple Sequence of Discrete Polynomials

We define a simple sequence of discrete polynomials: $N_o(z) = 1$ and

$$N_n(z) = \frac{z * (z-1) * (z-2) * \dots * (z-n+1)}{(n)_q!} \text{ for } n \geq 1. \quad (5.10)$$

By direct calculation we get that

$N_o(z) = 1$

$N_1(z) = z^{(1)}$

$N_2(z) = \frac{z^{(2)} - z^{(1)}}{(2)_q!}$

and

$N_3(z) = \frac{z^{(3)} - 3z^{(2)} + 2z^{(1)}}{(3)_q!}$.

Also $T_o(z) = 1$, $T_1(z) = \frac{1}{(2)_q!}$, $T_2(z) = (\frac{3}{(3)_q!} - \frac{2}{(2)_q!})z^{(1)} +$

Thus

$$c_{0,0} = 1 \; ; \quad c_{1,1} = 0, \quad c_{1,0} = \frac{1}{(2)_q!} \; ;$$

$$c_{2,2} = 0 \; ; \quad c_{2,1} = \frac{3}{(3)_q!} - \frac{2}{(2)_q!} \; , \quad c_{2,0} = \frac{1}{(3)_q!} \; .$$

Proceeding recursively, the relation $\sum_{k=0}^{\infty} T_k(z) * \delta^{k+1}[N_n(z)] = N_{n-1}(z)$ yields the result that the coefficient of $z^{(n)}$ in $T_n(z)$ namely $c_{n,n} = 0$. However the coefficient of $z^{(n)}$ in $T_{n+1}(z)$ is uniquely determined. It satisfies the recurrence relation

$$\frac{1}{(n+1)_q!} + \frac{c_{1,0}}{(n)_q!} + \frac{c_{2,1}}{(n-1)_q!} + \frac{c_{3,1}}{(n-2)_q!} \cdots$$

$$+ c_{n,n-1} + c_{n+1,n} = \frac{1}{(n+1)_q!} \; . \qquad (5.11)$$

Suppose, $T_r(z)$ is of degree atmost $(r-2)$ for all $r > n$, a fixed integer. Then we get $c_{r+1,r+1} = 0$, $c_{r+1,r} = 0$ for any $r > n$.

Hence

and in general

$$\frac{c_{1,0}}{(n+s)_q!} + \frac{c_{2,1}}{(n+s-1)_q!} + \frac{c_{3,2}}{(n+s-2)_q!} + \dots + \frac{c_{n,n-1}}{(s+1)_q!} = 0$$

$$\text{for any } s \geq 0. \qquad (5.12)$$

We write the above set of homogeneous linear equations in the form $\sum_{j=1}^{n} a_{ij}x_j = 0$

where

$x_j = c_{j,j-1}$, $i = 1,2,\dots,n$ and $a_{ij} = \dfrac{1}{(n+i-j+1)_q!}$.

The matrix (a_{ij}) where $i, j = 1,2,\dots, n$ is of rank n as the rows of it are linearly independent.

To prove this, consider the following rows:

$$R_1 = \left(\frac{1}{(s+1)_q!}, \frac{1}{(s+2)_q!}, \dots, \frac{1}{(s+n)_q!}\right),$$

$$R_2 = \left(\frac{1}{(s+2)_q!}, \frac{1}{(s+3)_q!}, \dots, \frac{1}{(s+n+1)_q!}\right),$$

$$R_n = \left(\frac{1}{(s+n)_q!}, \frac{1}{(s+n+1)_q!}, \dots, \frac{1}{(s+2n-1)_q!}\right).$$

If $\alpha_1 R_1 + \alpha_2 R_2 + \alpha_3 R_3 + \ldots\ldots + \alpha_n R_n = 0$,

we get

$$\frac{\alpha_1}{(s+1)_q!} + \frac{\alpha_2}{(s+2)_q!} + \ldots + \frac{\alpha_n}{(s+n)_q!} = 0,$$

$$\frac{\alpha_1}{(s+2)_q!} + \frac{\alpha_2}{(s+3)_q!} + \ldots + \frac{\alpha_n}{(s+n+1)_q!} = 0,$$

$$\frac{\alpha_1}{(s+n)_q!} + \frac{\alpha_2}{(s+n+1)_q!} + \ldots + \frac{\alpha_n}{(s+2n-1)_q!} = 0.$$

Adding all the equations and arranging the terms,

$$\frac{\alpha_1}{(s+1)_q!} + \frac{\alpha_1+\alpha_2}{(s+2)_q!} + \ldots\ldots + \frac{\alpha_1+\alpha_2+\ldots\ldots+\alpha_n}{(s+n)_q!}$$

$$+ \frac{\alpha_2+\alpha_3+\ldots\ldots\alpha_n}{(s+n+1)_q!} + \ldots + \frac{\alpha_n}{(s+2n-1)_q!} = 0.$$

As this result is true for any integer $s \geq 0$, the coefficient of each term in the above identity is zero. Thus

$\alpha_1 = \alpha_2 = \ldots\ldots = \alpha_n = 0.$

Thus the system of equations has only the trivial solution $c_{j,j-1} = 0$ for all $j = 1,2,\ldots\ldots,n$. This is contradictory to the fact. Hence we get that $c_{n+s,n+s-1} \neq 0$ for some $s > 0$. Then considering the same aspect taking

$n+s$ in the place of n, we continue to get another $c_{j,j-1} \neq 0$ where $j > n+s$ and so on. Thus we get $c_{j,j-1}$ cannot be zero for every $j > n$, a fixed integer.

Thus we conclude:

$\{N_n(z)\}$ is a simple sequence of discrete polynomials having q-infinite type classification. (5.13)

3. Polynomials from Generating Functions

Discrete polynomials can be studied through the generating functions also. To illustrate this aspect we introduce a new sequence of discrete polynomials.

Let t be the continuous complex variable and $z \in H$. Then the discrete polynomial $P_n(z,\lambda)$ of n^{th} degree is defined from the relation

$$(1-t)^{\lambda}\left(\sum_{n=0}^{\infty}\frac{t^n z^{(n)}}{(n)_q!}\right) = \sum_{n=0}^{\infty} P_n(z,\lambda)t^n,$$ λ is not a positive integer. (5.14)

Since $\sum_{n=0}^{\infty}\frac{t^n z^{(n)}}{(n)_q!}$ is entire in z and t and due to the validity of series expansion of $(1-t)^{\lambda}$ in powers of t, the above relation is valid for $|t| < 1$ and any $z \in H$. (5.15)

Comparing the coefficients, we get

$$P_n(z,\lambda) = \sum_{r=0}^{n-1} \lambda \frac{(\lambda-1)\ldots(\lambda+r+1-n)}{(n-r)!\,(r)_q!} z^{(r)} + \frac{z^{(n)}}{(n)_q\,!}.$$

The coefficient of $z^{(n)}$ being $\frac{1}{(n)_q!}$, $P_n(z,\lambda)$ is strictly of degree n. Also any discrete polynomial of degree n over $\emptyset$ is representable as a linear sum of $\{P_n(z,\lambda)\}$. Thus $\{P_n(z,\lambda)\}$ where $n \in Z^{0+}$ is a simple and complete set of discrete polynomials. This sequence of polynomials serves the purpose of basis for the vector space of discrete polynomials over $\emptyset$. (5.16)

Another simple set of polynomials is defined as

$$D_0(z) = 1,$$

$$D_n(q^{-1}z) = D_n(qz) + (q^{-1}-q)z * D_{n-1}(z), \; n \geq 1 . \tag{5.17}$$

From the above relation we get that $D_1(z) = z^{(1)} + \alpha$ where $\alpha \in \emptyset$. Two simple forms of $D_n(z)$ are obtained by fixing $\alpha = 1$ or 0.

Thus

$$D_n(z,1) = \sum_{r=0}^{n} \frac{z^{(r)}}{(r)_q!} \text{ and } D_n(z,0) = \frac{z^{(n)}}{(n)_q\,!}. \tag{5.18}$$

Taking the derivate of $P_n(z,\lambda)$, $\delta P_n(z,\lambda) = P_{n-1}(z,\lambda)$. Also $P_n(z,-1) = \sum_{r=0}^{n} \frac{z^{(r)}}{(r)_q!}$. Thus $\lim_{n\to\infty} P_n(z,-1) = \sum_{r=0}^{\infty} \frac{z^{(r)}}{(r)_q!}$ satisfies the derivate equation

$\delta f(z) = f(z)$. Further this function is discrete entire. Also we make note that $P(z,\lambda)$ is q-zero type. (5.19)

$$P_n(z,-1) = D_n(z,1) \text{ as well as}$$

$$P_n(z,-1) = \sum_{r=0}^{n} D_r(z,0).$$ $D_n(z)$ satisfies the derivate equation $\delta D_n(z) = D_{n-1}(z)$. $D_n(z)$ is characterised by $\sum_{r=0}^{n} \alpha_{n-r} \frac{z^{(r)}}{(r)_q!}$ where $\alpha_0 = 1$ and $\alpha_j \varepsilon\ \emptyset$ for $j \geq 1$. The special cases $\left\{D_n(z,1)\right\}$ and $\left\{D_n(z,0)\right\}$ are obtained by fixing $\alpha_j = 1$ for all j in the case of former and $\alpha_j = 0$ for all $j \geq 1$ in the other. (5.20)

$$P_n(z,-2) = \sum_{r=0}^{n-1} \frac{2\cdot 3\cdot 4\cdots(n+1-r)}{(n-r)!} \frac{z^{(r)}}{(r)_q!} + \frac{z^{(n)}}{(n)_q!}$$

$$= \sum_{r=0}^{n-1} \frac{(n+1-r)z^{(r)}}{(r)_q!} + \frac{z^{(n)}}{(n)_q!}.$$

Thus we have:

$$P_n(z,-2) = \sum_{r=0}^{n} D_r(z,1)$$

$$P_n(z,-2) = \sum_{r=0}^{n} (n+1-r)D_r(z,0)$$

and

Further extending the result, if λ is a positive integer,

$$P_n(z,-\lambda) = \sum_{r=0}^{n-1} \frac{(n+1-r)(n+2-r)\cdots(n+\lambda-1-r)}{(r-1)\,!} \; \frac{z^{(r)}}{(r)_q\,!} + \frac{z^{(n)}}{(n)_q!}$$

and

$$P_n(z,-\lambda) - P_{n-1}(z,-\lambda) = P_n(z,-(\lambda-1)). \qquad (5.22)$$

Thus if λ is a positive integer,

$$P_n(z,-\lambda) = \sum_{r=2}^{\lambda} P_{n-1}(z,-r) + P_n(z,-1). \qquad (5.23)$$

Using the result, $P_n(z,-2) = \sum_{r=0}^{n} D_r(z,1)$ we introduce a matrix form to the system of above equations as

$$P_i(z,-2) = \sum_{j=0}^{n} a_{ij} D_j(z,1) \text{ where } a_{ij} = \begin{cases} 1 \text{ if } i \geq j \\ 0 \text{ if } i < j. \end{cases} \qquad (5.24)$$

The matrix (a_{ij}) is triangular and invertible whose inverse is given by (b_{ij})

$$\text{where } b_{ij} = \begin{cases} 1 \text{ if } i = j \\ -1 \text{ if } i = j+1 \\ 0 \text{ otherwise}. \end{cases}$$

(b_{ij}) is also a triangular matrix. (5.25)

Then $D_i(z,1) = \sum_{j=0}^{n} b_{ij} P_j(z,-2)$. (5.26)

Thus there exists unique linear expressions over the real numbers for $D_r(z,1)$ in terms of $P_j(z,-2), j = 0,1,2,$ and vice versa. (5.27)

Also $P_i(z,-2) = \sum_{j=0}^{n} c_{ij} D_j(z,0)$ where

$$c_{ij} = \begin{cases} j+1 \text{ if } i \geq j \\ 0 \text{ if } j > i. \end{cases} \quad (5.28)$$

(c_{ij}) is triangular and invertible.

Thus $D_i(z,0) = \sum_{j=0}^{n} d_{ij} P_j(z,-2)$ where $(d_{ij}) = (c_{ij})^{-1}$. (5.29)

4. Conclusion

An attempt is made to establish a theory of discrete functions in the complex plane. Classical analysis q-basic theory, monodiffric theory, preholomorphic theory and q-analytic theory have been utilised to develop concepts like differentiation, integration and special functions.

To mention a few of further extensions of the theory we introduce:

Duffin, Zeilberger and Mugler had attempted to study entire functions. An attempt is made here to study meromorphic functions also. Integral transform theory and operational calculus in the q-monodiffric sense are essential parts of a function theory. Boas [15] and Buschman [17-20] are some guidelines in this direction.

q-monodiffric constants play the role of scalars in q-monodiffric theory. The classical complex number is replaced by a more general concept of number in this theory. The q-monodiffric constants form a not associative ring. Hilbert space structure for the q-monodiffric functions over the q-monodiffric constants will make the theory interesting. Results introduced in Gilbert and Hille [35,36] and Souchek and Yip Li [58] are important and extension of the theory in the q-monodiffric sense is possible.

It is hoped that the q-monodiffric theory is more suitable than any other concepts in evolving the basis of discrete function theory in the complex plane.

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