STOCHASTIC PROCESSES AND THEIR APPLICATIONS

TIME DEPENDENT SOLUTIONS FOR SOME QUEUEING AND INVENTORY MODELS

THESIS SUBMITTED FOR THE DEGREE OF
DOCTOR OF PHILOSOPHY

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1988

CERTIFICATE

Certified that the work reported in the present thesis is based on the bonafide work done by Sri.T.P.Madhusoodanan under my guidance in the Department of Mathematics and Statistics, Cochin University of Science and Technology, and has not been included in any other thesis submitted previously for the award of any degree.

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DECLARATION

This thesis contains no material which has been accepted for the award of any other degree or diploma in any University and, to the best of my knowledge and belief, it contains no material previously published by any other person, except where due reference is made in the text of the thesis.

18.2.88

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ACKNOWLEDGEMENT

I wish to express my gratitude to Professor A. Krishnamoorthy for his guidance and suggestions and to Professor T.Thrivikraman, Head of the Department for his advice and inspiration throughout the work. Also, I wish to express my sincere thanks to Mr.M.J.Jacob, for his help, suggestions and criticisms. I also thank all my friends for their help and co-operation and Mr. Jose for his excellent typing. Finally, I thank Cochin University for extending financial support.

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Chapter 1

INTRODUCTION

The objective of this thesis is to study the time dependent behaviour of some complex queueing and inventory models. It contains a detailed analysis of the basic stochastic processes underlying these models. In the theory of queues, analysis of time dependent behaviour is an area very little developed compared to steady state theory. Time dependence seems certainly worth studying from an application point of view, but unfortunately, the analytic difficulties are considerable. Closed form solutions are complicated even for such simple models as M/M/1. Outside M/M/1, time dependent solutions have been found only in special cases and involve most often double transforms which provide very little insight into the behaviour of the queueing systems themselves. In inventory theory also there is not much results available giving the time dependent solution of the system size probabilities. Our emphasis is on explicit results free from all types of transforms and the method used may be of special interest to a wide variety of problems having regenerative structure.

In this thesis we consider different queueing models with Poisson arrivals, general service, with and without vacations to the server and derive transient solutions for all the models. Also we study some (s,S) inventory systems under the assumptions of random lead times, random quantity replenished, bulk demands and vacations to the server. In the following chapters we analyse each of these models. Each chapter contains a self introduction and some important references. In that sense, each chapter is self contained. In this chapter we give a brief general introduction to the subject and some related topics.

1.1 QUEUEING THEORY

The question may arise whether after about 80 years of research in queueing theory, it is still possible to make a substantial contribution to the theory and to come up with some new results. To account for our positive answer to this question, and to place the present work in its proper context, we first give a short historical survey of the origin and development of queueing theory and subsequently discuss the current state of affairs.

Historically, Johannsen's " Waiting Times and Number of Calls" (an article published in 1907 and reprinted in Post Office Electrical Engineers Journal, London, October 1910) seems to be the first paper in the subject. But the method used in this paper is not mathematically exact and therefore, from the point of view of exact treatment, the paper that has historic importance is A.K. Erlang's " The theory of Probabilities and Telephone Conversations", published in 1909. In this paper, he lays the foundation for the place of Poisson (and hence exponential) distribution in congestion theory. His papers written in the next 20 years contain some of the most important concepts and techniques. That include the notion of statistical equilibrium, the method of writing down balance of state equations (later called Chapman-Kolmogorov equation). Erlang delay probability and the phase method of Erlang.

Until 1940, the majority of the contribution to queueing theory was made by people active in the field of telephone traffic problems. After the Second World War, the field of operations research rapidly developed and queueing applications were also found in production planning, inventory control and maintenance problems. In this period, much theoretically oriented research on queueing problems were done.

In the fifties and sixties, the theory reached a very high mathematical level (see Cohen (1969) and Takacs (1962)). Advanced mathematical techniques like transform methods, Weiner Hopf decomposition and function theoretic tools were developed and refined. This research resulted in a number of elegant mathematical solutions.

In particular, noting the inadequacy of the equilibrium theory in many queue situations, Pollaczek in 1934 began investigations of the behaviour of the system during a finite interval. Since then he has done considerable work in the analytical behavioural study of queueing systems. The trend towards the analytical study of the basic stochastic processes of the system has continued, and queueing theory has proved to be a fertile field for researchers who wanted to do fundamental research on stochastic processes involving mathematical models.

The processes involved are not simple and for the time dependent analysis, more sophisticated mathematical procedures are necessary. For instance, for the queue with Poisson arrivals and exponential service times, under statistical equilibrium, the balance of state equations are simple and the limiting distribution of the queue size is obtained by recursive arguments and induction. But for the time dependent solution, the use of transforms is necessary.

The time dependent solution was first given by Bailey (1954b) and Ledermann and Reuter (1954). While Bailey used the method of generating functions for the differential equations, Ledermann and Reuter used spectral theory for their solution.

To analyse the case of M/G/1 queues, Kendall (1953) has used the method of regeneration points due to Palm. The method of supplementary variables investigated by Cox(1955) was already used in L.Koston's thesis in 1942. It is extensively discussed in the book by Gnedenko and Kovalenko (1968).

The study of bulk queues is considered to be originated with the pioneering work of Bailey (1954a). In a way the study of bulk queues may be said to have begun with Erlang's investigation of $M/E_k/1$; for its solution contains implicitly the solution of the model $M^k/M/1$. Bailey studied the stationary behaviour of a single server queue having simple Poisson input, intermittently available server and service in batches of fixed maximum size. The results of this study are given in terms of probability generating functions, the evaluation of which requires determining the zeroes of a polynomial. This study was followed by a series of papers involving the treatment of queueing processes with group arrivals or batch service. Gaver (1959) seems to be the first to take up specifically queues involving group arrivals. For more details on bulk queues, one may refer to Medhi (1984). For a detailed treatment of queueing systems and for further references, one can refer any one of the standard books on the subject like, Saaty (1961), Takacs (1962), Cohen (1969), Prabhu (1965, 1980), Gnendenko and Kovalenko (1968), Cooper (1972), Gross and Harris (1974), Kleinrock (1975) and Asmussen (1987).

One important name worth mentioning in the study of time dependent behaviour of queueing systems is that of Takacs. He has studied the transient behaviour of M/G/1 and E_K/G/1 models. Also the influence of his study of virtual waiting time process has been tremendous in the development of queueing theory. For more details, one may refer to Takacs (1962).

Queueing systems in which the service process is subject to interruptions resulting from unscheduled break-downs of servers, scheduled off periods, arrival of customers with pre-emptive or non-pre emptive priorities or the server working on primary and secondary customers arise naturally. The impact of these service interruptions on the performance of a queueing system will depend on the specific interaction between the interruption process and the service process.

Queueing models with interruptions and their connection to priority models were first studied by White and Christie (1958), who considered the case with exponential service, on-time and off-time distributions. Their results were extended by Gaver (1962), Keilson (1962), Avi-Itzhak and Naor (1962) and Thiruvengadam (1963) to models with general service time and off-time distributions but exponential on-times. When the on periods are not exponential, the problem became very difficult and one such model is studied by Federgruen and Green (1986). A detailed analysis of single server queueing system with server failure is given in Gnedenko and Kovalenko (1968).

Another variation of the interruption model is the vacation model. In this the queueing system incurs a start-up delay whenever an idle period ends or the server takes vacation periods. The vacation model includes server working on primary and secondary customers also. Analysis of queueing systems with vacations to the server is motivated by the study of cyclic queues and Miller (1964) was the first to study such a system. Miller analysed a system in which the server goes for a vacation (a 'rest period') of random length whenever it becomes idle. He also considered a system in which the server behaves normally but the first customer arriving to an empty system has a special service time. These types of systems and similar ones were also examined by Welsch (1964), Avi-Itzhak, Maxwell and Miller (1965), Cooper (1970), Pakes (1973), Lemoine (1975), Levy and Yechiali (1975), Heyman (1977), Van der Duyn Schouten (1978), Shantikumar (1980, 1982) and Scholl and Kleinrock (1983).

All the above models (having rest periods, set-up time, starter, interruptions etc.) can be jointly called as vacation models. While the queue with interruption has preemptive priority for vacation, other types of vacations have least priority among all work with vacation taken when the system is empty. Variations of vacation models are available with single and multiple vacations and exhaustive and non-exhaustive service disciplines.

When the system becomes empty, server starts a vacation and the server keeps on taking vacations until, on return from a vacation, at least one customer is present. This is called a multiple vacation system. The server taking exactly one vacation at the end of each busy period, is called a single vacation system. We say that a vacation model has the property of exhaustive service in case each time the server becomes available, he works in a continuous manner until the system becomes empty. Systems with a vacation period beginning after every service completion, (or after any vacation period if the queue is empty) is known as the single service discipline. There is another non-exhaustive service discipline which is a generalization of both exhaustive and single service disciplines known as the Bernoulli schedule discipline defined as follows. After each service completion, the server takes a vacation with probability p and starts a new service with probability 1-p. If the system is empty, after a service completion or vacation completion, server always takes a vacation and after any vacation if customers are present server resumes service.

Vacation systems with exhaustive service discipline are analysed by several authors. See for example, Levy and Yechiali (1975), Heyman (1977), Courtois (1980), Shantikumar (1980), Scholl and Kleinrock (1983), Lee (1984), Fuhrmann (1984), Doshi (1985), Servi (1986a), Levy and Kleinrock (1986) and Keilson and Servi (1986b). Systems without exhaustive service discipline are considered by Ali and Neuts (1984), Neuts and Ramalhoto (1984), Fuhrmann and Cooper (1985), Keilson and Servi (1986 b,c) and Servi (1986 a). The case of Bernoulli schedule discipline is introduced by Keilson and Servi (1986a) and further studied by Servi (1986b).

The main results in the vacation systems is the delay analysis by decomposition. The 'stochastic decomposition property' of M/G/1 queueing system with vacation says that the (stationary) number of customers present in the system at a random point in time is distributed as the sum of two independent random variables. One is the (stationary) number of customers present in the corresponding standard M/G/1 queue (i.e. without vacation) at a random point in time and the other is the number of arrivals in the forward recurrence time of vacation period. For more details on queueing systems with vacations one may refer to Doshi (1986).

All the above models assumes the existence of stationary distribution and studied some aspects of the queue length and waiting time distribution. Some aspects of the dynamic behaviour of M/G/1 queues with vacations is studied by Keilson and Servi (1986c). The time dependent solution for a finite capacity M/G/1 queueing system with vacations to the server is given by Jacob and Krishnamoorthy (1987). They have introduced a new method namely the convolution product of matrices, whose elements are the transition probability density functions, to arrive at the solution. Using renewal theoretic arguments, they have given explicit expressions for the time dependent system size probabilities at arbitrary epochs and also the probability distribution of the virtual waiting time in the queue at time t. Time dependent solution for a finite capacity $M/G^{a,b}/1$ queueing system with vacations to the server is given by Jacob and Madhusoodanan (1988), using the theory of regenerative processes. In this thesis we extend these results to a number of variations of the M/G/1 queue.

1.2. INVENTORY THEORY

By inventory, we mean the measured amount of some items which varies in quantity over time in response to a 'demand' process, which operates to diminish the stock, and a 'replenishment' process, which operates to increase it. Usually the demand is not subject to control, but the timing and magnitude of the replenishment can be regulated.

The real need for analysis of an inventory system was first recognized in industries that had a combination of production scheduling problems and inventory problems. That is, situations where items were produced in lots and then stored at a factory warehouse. The earliest derivation of what is often called the " simple lot size formula" was obtained by Ford Harris in 1915. The same formula have been developed, independently by many researchers since then. It is often referred to as " Wilson's formula" , since it was derived by R.H.Wilson as an integral part of an inventory control scheme.

During World War II, a useful stochastic model was developed as the Christmas tree model. Shortly thereafter, a stochastic version of the simple lot size model was developed by Whitin, whose book published in 1953 was the first book in English which dealt in any detail with stochastic inventory models. The paper by the economists, Arrow, Harris and Marschak (1951) was one of the first to provide a rigorous mathematical analysis of a simple type of inventory model. It was followed by the often quoted and rather abstract papers by the mathematicians Dvoretzky, Kiefer and Wolfowitz (1952, 1953). Since then a number of papers by mathematicians have appeared.

A valuable review of the problems in probability theory of storage systems is given by Gani (1957). A systematic account of probabilistic treatment in the study of inventory systems using renewal theoretic arguments is given by Arrow, Karlin and Scarf (1958). Hadley and Whitin (1963) deals with the application of mathematical models to practical situations. The cost analysis of different inventory systems is given by Naddor (1966). Tijms (1972) gives a detailed analysis of inventory systems under (s,S) policy. A practical treatment of the (s,S) inventory systems can be found in the recent books by Silver and Peterson (1985) and Tijms (1986).

Gaver (1959) analyses the case of an (s,S) inventory system with compound Poisson demand and random lead times. Some aspects of (s,S) inventory system with arbitrary interarrival time of demands and random lead times is discussed by Finch (1961). Veinott (1966) gives a detailed review of the status of mathematical inventory theory upto 1965. Gross and Harris (1971) and Gross, Harris and Lechner (1971) deal with one for one ordering inventory policies with state dependent lead times. Sivazlian (1974) considers an (s,S) inventory system with arbitrary interarrival time of demands and zero lead time and Srinivasan (1979) extended these results to allow the lead time to follow arbitrary distribution. The case of an (s,S) inventory system with bulk demands and constant lead time is analysed by Sahin (1979). Also Sahin (1983) discussed the problem of an (s,S) inventory system with compound renewal demand and random lead times and he obtained the binomial moments of the inventory deficit.

A continuous review (s,S) inventory system in random environment is analysed by Feldman (1978). Richards (1979) analyses an (s,S) inventory system with compound Poisson demand. Algorithms for a continuous review (s,S) inventory system in which the demand is according to a versatile Markovian point process is given by Ramaswami (1981). An inventory system with two ordering levels and random lead times is analysed by Thangaraj and Ramanarayanan (1983). Approximation for the single-product production-inventory problem with compound Poisson demand and two possible production rates where the product is continuously added to inventory is given by De Kok, Tijms and Van der Duyn Schouten(1984). Using Markov decision drift processes, Hordijk and Van der Duyn Schouten (1986) examines the optimality of (s,S) policy in a continuous review inventory model with constant lead time when the demand process is a superposition of a compound Poisson process and a continuous deterministic process.

An important variation of the inventory problem is the perishable commodity inventory system. For details of the work in this area, one may refer to the excellent survey given by Nahmias (1982). The continuous review perishable inventory system can be identified with queueing systems with impatient customers as viewed by Kaspi and Perry (1983, 1984).

In the case of random lead times, the concept of vacations to the server during dry period is introduced in inventory systems by Daniel and Ramanarayanan (1987 a,b).

Usha, Ramanarayanan and Jacob (1987) analyses the case of finite backlog of demands and vacations to the server. Several other models with vacations to the server, bulk demands, varying ordering levels etc. can be found in Jacob (1987).

1.3. NOTATIONS

Here we introduce the following notations that are frequently used in this thesis.

$*$ denotes the convolution operator.

$f^{*n}(x)$ denotes the n-fold convolution of f(x) with itself.

For a distribution function F(x), $\bar{F}(x) = 1-F(x)$, the survival probability.

Now we define the convolution of two matrices as follows.

If $A(t) = [a_{ij}(t)]$ is a matrix of order m x p, and

$B(t) = [b_{ij}(t)]$ is a matrix of order p x n, then

$(A*B)(t) = [c_{ij}(t)]$ is a matrix of order m x n whose elements are given by

$$c_{ij}(t) = \int_0^t \sum_{k=1}^{p} a_{ik}(u)\ b_{kj}(t-u)du$$

For a square matrix $A(x)$ of order m, let $A^{*0}(x)$ be the identity matrix of order m and for $n \geq 1$, let $A^{*n}(x)$ be the n-fold convolution of $A(x)$ with itself.

1.4 RENEWAL THEORY

Renewal processes are the simplest regenerative stochastic processes. To define a renewal process, let $\{ X_n, n=1,2, \ldots \}$ be a sequence of non-negative independent random variables. Assume that $\Pr\{X_n=0\} < 1$, and that the random variables are identically distributed with distribution function $F(.)$. Since X_n is non-negative, it follows that $E\, X_n$ exists.

Let $S_o = 0, \quad S_n = X_1+X_2+ \ldots + X_n$ for $n \geq 1$,

and let $F_n(x) = \Pr\{ S_n \leq x \}$ be the distribution function of S_n.

Since X_i's are i.i.d., $F_n(x) = F^{*n}(x)$.

Define the random variable

$$N(t) = \operatorname{Sup} \{ n \mid S_n \leq t \}$$

The process $\{N(t), t \geq 0\}$ is called a renewal process.

If for some n, $S_n = t$, then the nth renewal is said to occur at t; S_n gives the time of the nth renewal and is called the nth renewal epoch. The random variable $N(t)$ gives the number of renewals in the interval $(o,t]$.

The function $M(t) = E[N(t)]$ is called the renewal function of the process with distribution function F. It is easy to see that

$$N(t) \geqslant n \Longleftrightarrow S_n \leq t$$

Thus the distribution of $N(t)$ is given by

$$\Pr\left\{N(t) = n\right\} = F^{*n}(t) - F^{*n+1}(t)$$

and the expected number of renewals is given by

$$M(t) = \sum_{n=1}^{\infty} F^{*n}(t)$$

Its derivative

$$m(t) = \frac{dM(t)}{dt} = \sum_{n=1}^{\infty} f^{*n}(t)$$

is the renewal density function, assuming the density function $f(t)$ exists. To better understand the meaning of $m(t)$, let us consider the increment of $M(t)$.

$$\delta M(t) = M(t+\delta t) - M(t)$$

$$= \sum_{n=1}^{\infty} [\ F^{*n}(t+\delta t) - F^{*n}(t)]$$

$$= \sum_{n=1}^{\infty} \Pr\left\{t < S_n \leq t+\delta t\right\}$$

On the other hand, we have for $\delta t \longrightarrow 0$,

$\Pr\left\{\text{more than one renewal point in } (t,t+\delta t)\right\} \longrightarrow O(\delta t)$

For $\delta t \rightarrow 0$, we get

$$\delta M(t) = \Pr\left\{S_1 \text{ or } S_2 \text{ or } S_3 \text{ or } \ldots \text{ lies in } (t,t+\delta t)\right\}$$

$$\longrightarrow m(t)\delta t$$

This interpretation of renewal density is important in practical applications.

Now, suppose that the first interoccurrence time X_1 has a distribution G which is different from the common distribution F of the remaining interoccurrence times X_2, X_3, X_4, ...

As before let us define

$$S_0 = 0, \quad S_n = \sum_{i=1}^{n} X_i$$

and $N_D(t) = \text{Sup}\{n \mid S_n \leq t\}$

The stochastic process $\{N_D(t),\ t \geq 0\}$ is called a Delayed or Modified renewal process.

Here we have

$$\Pr\{N_D(t) = n\} = G*F^{*n-1}(t) - G*F^{*n}(t)$$

so that the modified renewal function is

$$M_D(t) = E[N_D(t)]$$

$$= \sum_{n=0}^{\infty} G*F^{*n}(t)$$

The modified renewal density function is given by

$$m_D(t) = M_D'(t)$$

$$= \sum_{n=0}^{\infty} g*f^{*n}(t)$$

under the additional assumptions that the density functions $g(x) = G'(x)$ and $f(x) = F'(x)$ exists.

Now, consider a stochastic process $\{X(t), t \geq 0\}$ with state space $\{0,1,2,\ \ldots\}$, having the property that there exist time points at which the process (probabilistically) restarts itself. That is, suppose that with probability one, there exists a time T_1, such that the continuation of the process beyond T_1 is a probabilistic replica of the whole process starting at 0. Note that this property implies the existence of further times T_2, T_3, ..., having the same property as T_1. Such a stochastic process is known as a regenerative process.

From the above, it follows that $\{T_1, T_2, \ldots\}$ forms a renewal process; and we shall say that a cycle is completed every time a renewal occurs. It is easy to see that a renewal process is regenerative, and T_1 represents the time of the first renewal.

For details on renewal theory, one may refer to Cox (1962), Feller (1983), Ross (1970) or Cinlar (1975).

1.5. SUMMARY OF THE WORK INCLUDED IN THE THESIS

In this thesis we study the time dependent behaviour of some complex queueing and inventory models. In our analysis, renewal theory plays an important role. In each model, identifying the regeneration points and using matrix convolutions, we obtain the required transition probability density functions.

The thesis is divided into seven chapters. In the second chapter, we consider a finite capacity general bulk service queueing system. The arrival of customers is according to a homogeneous Poisson process and the service times are generally distributed independent random variables whose distribution depends on the size of the batch being served. Customers are served in batches according to the general bulk service rule (See Neuts (1967)). Using renewal theoretic arguments we derive the probability density function of the busy period and the time dependent system size probabilities at arbitrary time points. Also we derive the probability distribution of the virtual waiting time in the queue at any time t.

The next chapter is devoted to the study of an infinite capacity $M^X/G/1$ queueing system with vacations to the server. The arrival of customers is according to a compound Poisson process and service is done one by one with service time following a general distribution. Under exhaustive service discipline, server takes vacations for a random period having a general distribution. The server keeps on taking vacations until on return from a vacation, at least one customer is present. Here also we derive the probability density function of the busy period, time dependent system size probabilities at arbitrary epochs and the probability distribution for the virtual waiting time in the queue, using renewal theoretic arguments.

A finite capacity $M/G^{a,b}/1$ queueing system with vacations to the server is analysed in chapter four. The general bulk service rule is modified to allow the arriving customers to enter for service, if the maximum service capacity is not attained, without altering the service time. Even if service of a batch is in progress with less than 'b' customers, all the arrivals may not be interested to join for partial service. So we assume that an arriving customer enter for partial service with probability p and wait for full service with probability 1-p, till the service capacity is attained. Server goes for vacation whenever he finds less than 'a' customers in the system and this is a multiple vacation. The vacation periods are independent and identically distributed random variables following a general distribution. Using the theory of regenerative processes, explicit expressions for the busy period, the time dependent system size probabilities and the probability distribution of the virtual waiting time are derived.

In chapter five, we consider a finite capacity $M/G^{B}/1$ vacation system with Bernoulli schedules. The arrival of customers is according to a homogeneous Poisson process and the service is in batches of maximum capacity B.

The service times are generally distributed random variables which depend upon the size of the batch being served. The Bernoulli vacation model is defined as follows. After each service completion the server starts a new service, if a customer is present, with probability p and takes a vacation with probability 1-p. If the system is empty after a service completion or a vacation completion, the server always takes a vacation. The decision about taking a vacation after each service completion or vacation completion are independent. If the server finds at least one customer upon return from a vacation, he starts service of the batch. Also the vacations are independent and identically distributed random variables following a general distribution. Here also we derive explicit expressions for the probability density function of a busy period, time dependent system size probabilities and the probability distribution of the virtual waiting time in the queue, using the theory of regenerative processes.

In the next two chapters, we analyse some continuous review (s,S) inventory systems with random lead times. Chapter six is devoted to the study of a single item (s,S) inventory system in which the quantity replenished is random. The demand is for one unit at a time and the interarrival time of demands and lead times are independent sequences of independent and identically distributed random variables following general distributions. Whenever the inventory level drops to s, an order is placed for S-s units. However, the quantity replenished is a random variable that can assume values s+1, ..., S-s. Explicit expressions for the probability mass function of the stock level at arbitrary epochs are derived, using renewal theoretic arguments. An expression for the total cost over a period of time of length t is obtained. Then we consider the special case of zero lead time and derive the time dependent as well as the stationary distribution of the inventory level. Using this, the associated optimization problem is discussed in detail. Finally, some numerical examples are given.

In the last chapter, we consider an (s,S) inventory system with random lead times depending on the size of the order. The time between successive demands and quantities demanded at these points are independent sequences of independent and identically distributed random variables following general distributions. Whenever the inventory level falls to or below s, an order is placed to fill the inventory. The lead times are also independent random variables following general distributions. Whenever the inventory becomes dry, the server goes for a vacation for a random length of time which follows a general distribution. Using the theory of regenerative process, we derive explicit expressions for the inventory level probabilities at arbitrary epochs.

Chapter 2

A FINITE CAPACITY $M/G^{a,b}/1$ QUEUEING SYSTEM

2.1. INTRODUCTION

The study of bulk queues is considered to be originated with the pioneering work of Bailey (1954a). He studied the stationary behaviour of a single server queue with Poisson input, intermittently available server and service in batches of fixed maximum size. This motivated several authors to study queueing systems with group arrivals or batch service. A queueing system with general bulk service rule was first considered by Neuts (1967). He considered a queueing system with Poisson input, general service time and service in batches of minimum size a and maximum size b.

In the literature of queueing theory, very little work has been done in the case of transient solution of non-Poisson queues. Recently, Jacob and Krishnamoorthy(1987) considered this problem and have given explicit expressions for the time dependent system size probabilities of an M/G/1 queueing system with vacations to the server. They have also computed the distribution of virtual waiting time in the queue at arbitrary time points.

Results given in this chapter find place in a paper accepted for publication in 'Naval Research Logistics'.

In this chapter, we give the time dependent system size probabilities for the finite capacity $M/G^{a,b}/1$ queueing system at arbitrary epochs using renewal theoretic arguments. In section 2.2, we give the description of the model. The basic results regarding the transition probability density functions and the renewal function are given in Section 2.3. Here, we also derive the probability density function of an idle period and busy period. In section 2.4, we derive explicit expressions for the system size probabilities at arbitrary epochs. In the last section, we give the probability distribution of the virtual waiting time in the queue at any time point t.

2.2 DESCRIPTION OF THE MODEL

The arrival of customers into the system is according to a homogeneous Poisson process with rate μ. The successive service times are independent random variables, following general distributions, depending on the size of the batch being served. Let $G_i(.)$ denote the distribution function of the service time when i is the batch size, which has a density function $g_i(.)$. There is only one server and the service is in batches according to the general bulk service rule. That is, a minimum of 'a' customers are needed to start a service and a maximum of 'b' customers can be served at a time, where 'a' and 'b' are positive integers. We also assume that the queueing system has a waiting room of finite capacity 'b', so that the service starts with all the customers waiting for service at that time.

Let us assume that there are $r(a \leq r \leq b)$ customers present in the waiting room at time zero, when the process starts. All the r customers enter for service at time zero itself. At the time of completion of service of these r units, if the number of customers in the waiting room is less than a, the server has to wait until there are a customers, whereupon all a enter for service. If there are a or more in the waiting room, on completion of a service, all the customers present are taken for service. The process continues in this fashion. Also we assume that, all the arrivals that takes place when the waiting room is full are lost. It is very easy to see that the time points at which the server starts service after an idle period will be renewal points.

2.3. BASIC RESULTS

For $j = 0,1, \ldots, b-1$, let $\mu_j(x)$ denote the probability that there are exactly j arrivals during the interval $(o,x]$ and let $\mu_b(x)$ denote the probability that there are at least b arrivals in the interval $(o,x]$.

Then for $j = 0, 1, \ldots, b-1$, we have

$$\mu_j(x) = \frac{e^{-\mu x}(\mu x)^j}{j!}$$

and (2.1)

$$\mu_b(x) = \sum_{j=b}^{\infty} \frac{e^{-\mu x}(\mu x)^j}{j!}$$

Now, we are in a position to find out the probability density function of an idle period, conditional on the system size at the service completion point. This is the time interval for (a-i) arrivals, if there are i units present in the system at time zero. This is given by the sum of (a-i) independent and identically distributed exponential variates and its distribution is Erlang of order (a-i) with density function given by

$$E_{a-i}(x) = \frac{\mu^{a-i}\, x^{a-i-1}}{\Gamma(a-i)}\, e^{-\mu x} \quad \text{for } i = 0, 1, \ldots, a-1 \qquad (2.2)$$

where Γ is the gamma function.

For $a \le i \le b$, $o \le j \le b$, let $f_{ij}(x)dx$ be the probability that starting at time zero, the service of a batch of size i units is over in the interval (x,x+dx) and there are j accepted arrivals during the interval (o,x].

Then,

$$f_{ij}(x) = g_i(x)\ \mu_j(x) \tag{2.3}$$

Now, for i = a, a+1, ..., b

$$\underline{f}_i(x) = (f_{ia}(x),\ f_{i(a+1)}(x), \ldots, f_{ib}(x))$$

is a row vector of order (b-a+1).

Also we define two matrices $\mathbb{F}$ and $\mathbb{G}$ as follows.

$$\mathbb{F}(x) = \begin{bmatrix} f_{aa}(x) & \cdots & f_{ab}(x) \\ \vdots & & \\ f_{ba}(x) & \cdots & f_{bb}(x) \end{bmatrix}$$

$$\mathbb{G}(x) = \begin{bmatrix} f_{ao}(x) & \cdots & f_{a(a-1)}(x) \\ \vdots & & \\ f_{bo}(x) & \cdots & f_{b(a-1)}(x) \end{bmatrix}$$

Here, $\mathbb{F}$ is a square matrix of order (b-a+1) and $\mathbb{G}$ is a matrix of order (b-a+1) x a.

Then, $(\underline{f}_i * \sum_{n=o}^{\infty} \mathbb{F}^{*n})(x)$ will be a row vector of order (b-a+1) and $\left(\underline{f}_i * \sum_{n=0}^{\infty} \mathbb{F}^{*n} * \mathbb{G}\right)(x)$ will be a row vector of order a.

In both these cases, the vector $\underline{f}_i$ is convoluted with the column vectors of the matrices

$$\sum_{n=0}^{\infty} \mathbb{F}^{*n} \quad \text{and} \quad \left(\sum_{n=0}^{\infty} \mathbb{F}^{*n} * \mathbb{G}\right) \text{ respectively.}$$

For $\eta = a, a+1, \ldots, b$, let

$$M_i^{\eta}(x) = (\eta-a+1)\text{th coordinate of } \left(\underline{f}_i * \sum_{n=0}^{\infty} \mathbb{F}^{*n}\right)(x) \tag{2.4}$$

and for $\eta = 0, 1, \ldots, a-1$, let

$$K_i^{\eta}(x) = (\eta+1)\text{th coordinate of } \left(\underline{f}_i * \sum_{n=0}^{\infty} \mathbb{F}^{*n} * \mathbb{G}\right)(x) \tag{2.5}$$

Now we can find out the probability density function of a busy period. For $i=r,a$ and $\eta = 0, 1, \ldots, a-1$, let Z_i^{η} be the length of a busy period beginning with i units and ending with η units left. Let us denote the probability density function of Z_i^{η} by $F_i^{\eta}(.)$. Then for $i=r,a$ and $\eta = 0, 1, \ldots, a-1$,

$$F_i^{\eta}(x) = K_i^{\eta}(x) + f_{i\eta}(x) \tag{2.6}$$

The renewal points of the process are those time points at which the service starts after an idle period. Let Z be the time between two such consecutive renewal points. Then the probability density function of Z is given by

$$k(t) = \int_0^t \sum_{\eta=0}^{a-1} F_a^{\eta}(u) \; E_{a-\eta}(t-u)\,du \tag{2.7}$$

Let X be the length of the initial busy cycle (i.e., the sum of the initial busy period and idle period). Then the probability density function of X is given by

$$h(t) = \int_0^t \sum_{\eta=0}^{a-1} F_r^{\eta}(u) \; E_{a-\eta}(t-u)\,du \tag{2.8}$$

Then the renewal density function of the delayed renewal process is given by

$$M(u) = \sum_{n=0}^{\infty} (h*k^{*n})(u) \tag{2.9}$$

It should be noted that the probability density function of an idle period, conditional on the initial state of the system, is given by equation (2.2) and the probability density function of a busy period, conditional on the final state of the system, is given by equation (2.6).

Now, the state space of the process is given by

$$S = \{(i,j) \mid a \le i \le b,\ o \le j \le b\} \cup \{(o,j) \mid o \le j \le a-1\}$$

When $a \le i \le b$, $o \le j \le b$, (i,j) represents the state that a batch of size i units is being served and there are j units in the waiting room. For $j = 0,1,\ldots,a-1$, (o,j) denote the state that there are j units in the waiting room and the server is idle.

2.4. TRANSIENT PROBABILITIES OF THE SYSTEM SIZE

Let $P_{ij}(t)$ be the probability that the system is in the state (i,j) at time t. Then by considering all mutually exclusive and exhaustive cases, we can write the following relations. By our assumption, the first busy period is started with the service of r units.

For $j = 0,1, \ldots, a-1$, we have

$$P_{oj}(t) = \int_o^t \sum_{i=o}^{j} F_r^i(u)\ \mu_{j-i}(t-u)\,du$$

$$+ \int_o^t M(u) \int_u^t \sum_{i=o}^{j} F_a^i(v-u)\ \mu_{j-i}(t-v)\,dv\ du \qquad (2.10)$$

For $j = 0, 1, \ldots, b$, we have

$$P_{rj}(t) = [1-G_r(t)]\,\mu_j(t) + \int_0^t M_r^r(u)\,[1-G_r(t-u)]\mu_j(t-u)\,du + \int_0^t M(u)\int_u^t M_a^r(v-u)[1-G_r(t-v)]\mu_j(t-v)\,dv\,du \qquad (2.11)$$

For $j = 0, 1, \ldots, b$, we have

$$P_{aj}(t) = \int_0^t M_r^a(u)\,[1-G_a(t-u)]\,\mu_j(t-u)\,du + \int_0^t M(u)\,[1-G_a(t-u)]\,\mu_j(t-u)\,du + \int_0^t M(u)\int_u^t M_a^a(v-u)\,[1-G_a(t-v)]\mu_j(t-v)\,dv\,du \qquad (2.12)$$

For $i = a+1, a+2, \ldots, r-1, r+1, \ldots, b$ and $j = 0, 1, \ldots, b$, we have

$$P_{ij}(t) = \int_0^t M_r^i(u)\,[1-G_i(t-u)]\mu_j(t-u)\,du + \int_0^t M(u)\int_u^t M_a^i(v-u)[1-G_i(t-v)]\mu_j(t-v)\,dv\,du \qquad (2.13)$$

2.5. VIRTUAL WAITING TIME IN THE QUEUE

The virtual waiting time in the queue at time t is defined as the waiting time of a unit in the queue if it were to arrive at time t (see Takacs (1962) for details). Let W_t be the virtual waiting time in the queue at time t. The probability distribution of W_t is computed here, conditional on the state of the system at time t. This is sufficient because, we have already computed the system size probabilities and hence this will give us the explicit expressions for the virtual waiting time at time t. It should be noted that even if the waiting room is full, the virtual customer can join the queue and enter for the next service along with the waiting customers.

We consider the following cases seperately.

Case (i): The state is (i,j) at time t, $a \le i \le b$, $a-1 \le j \le b$ so that the server is working.

Then

$$\Pr\{W_t \le x\} = \int_0^t M_r^i(u)\; G_i(t+x-u)\,du + \int_0^t M(u) \int_u^t M_a^i(v-u)\; G_i(t+x-v)\,dv\,du \tag{2.14}$$

Case (ii): The state is (i,j) at time t, $a \leq i \leq b$, $o \leq j \leq a-2$.

Then

$$\Pr\{W_t \leq x\} = \int_0^t M_r^i(u)\, G_i(t+x-u) \sum_{k=a-j-1}^{b} \mu_k(t+x-u)\,du$$

$$+ \int_0^t M(u) \int_u^t M_a^i(v-u)\, G_i(t+x-v)$$

$$\times \sum_{k=a-j-1}^{b} \mu_k(t+x-v)\,dv\,du \qquad (2.15)$$

Case (iii): The state is (o,j) at time t, $o \leq j \leq a-1$, so that the server is idle.

Then

$$\Pr\{W_t \leq x\} = \sum_{k=a-j-1}^{b} \mu_k(x) \qquad (2.16)$$

Chapter 3

AN M/G/1 QUEUE WITH GROUP ARRIVALS AND VACATIONS TO THE SERVER

3.1. INTRODUCTION

Several authors have analysed the case of M/G/1 queueing system with bulk arrivals. Gaver (1959) seems to be the first to take up queues with bulk arrivals. In a way, the study of bulk queues may be said to have begun with Erlang's investigation of the model $M/E_k/1$; for its solution contains implicitly the solution of the model $M^k/M/1$, a Poisson queue where arrivals are in groups of size k. Gaver considers the case of a queueing system with compound Poisson input and general service time and finds transform of the steady state queue length distribution. Bhat (1968) analyses this system in detail in his monograph. Chaudhry (1979) discusses limiting queue size distribution at three different epochs; random epoch, epoch just before an arrival and epoch just after a departure, using supplementary variable technique.

In all the queueing models referred to above, an idle server remains alert awaiting a new arrival and will commence service immediately upon the customer's arrival.

The effect of vacation periods in queueing models is studied by several authors. Scholl and Kleinrock(1983) analyses an M/G/1 queueing system with vacations to the server. Assuming steady state exists, Fuhrmann and Cooper (1985) shows that for a class of M/G/1 queueing system with generalized vacations to the server, the 'decomposition property' holds. An M/G/1 model in which the server is required to search for customers is analysed by Neuts and Ramalhoto (1984). Keilson and Servi (1986b) analyses the case of blocking probability for M/G/1 vacation system with occupancy level dependent schedules. For more details on queueing systems with vacations to the server, one may refer to Doshi (1986).

All the above mentioned articles analyse the case of steady state distribution. Jacob and Krishnamoorthy(1987) using renewal theoretic arguments gives transient solution for a finite capacity M/G/1 queueing system with vacations to the server. Time dependent solution for a finite capacity $M/G^{a,b}/1$ queueing system with vacations to the server is given by Jacob and Madhusoodanan (1988). In this chapter, we extend these results to an infinite capacity M/G/1 queueing system with group arrivals and vacations to the server.

Here we consider a service facility with only one server. Customers arrive in groups of size $\{G_n\}$ (n=1,2, ...) with distribution

$$\Pr\{G_n = j\} = p_j, \quad j = 1,2, \ldots \tag{3.1}$$

These group arrivals occur according to a Poisson process with parameter μ. Service is one by one and the service times are independent and identically distributed random variables with distribution function G(.) having density function g(.). Whenever the system becomes empty, server goes for vacation for a random length of time. Vacation periods are independent and identically distributed random variables with distribution function H(.) having density function h(.).

Let us suppose that, at time zero the system starts with 'a' ($>$ o) units in the waiting room. The server takes all the 'a' units to the service station and serves them one by one. When all the 'a' units are served, server goes back to the waiting room. If there is at least one unit waiting, server takes all of them to the service station and starts service. If there is nobody waiting for service, server goes for vacation for a random duration. On completion of this vacation, if there is at least one unit present in the system server starts serving them. On the otherhand, if there is no unit in the waiting room, the server extends his vacation for one more period having the same probability distribution. This process is continued until there is at least one unit in the system waiting for service. According to the terminology of Doshi (1986), this vacation is known as multiple vacation. Since the server serves the customers in a continuous manner until all the customers are exhausted, the service discipline is exhaustive. Thus we have an $M^X/G/1$ multiple vacation system with exhaustive service discipline.

3.2. BASIC RESULTS

Let $A(t)$ denote the number of arrivals during $(o,t]$.

Let $\quad \varphi(z) = \sum_{i=1}^{\infty} p_i z^i$ with $o < \varphi'(1) < \infty$

and $\quad p_j^{(k)} =$ the coefficient of z^j in $[\varphi(z)]^k$

This says that $p_j^{(k)}$ is the k-fold convolution of $\{p_j\}$ with itself

and $\quad p_j^{(o)} = 0$ for $j > 0$

$\qquad\quad = 1$ for $j = 0$

Therefore, for $j = 0,1,2, \ldots$

$$\mu_j(t) = \Pr\{A(t) = j\}$$

$$= \sum_{k=o}^{j} \frac{e^{-\mu t}(\mu t)^j}{j!} p_j^{(k)} \tag{3.2}$$

For $i = 1,2, \ldots, j = 0,1,2, \ldots$, we define the transition probability density functions as follows. Let $f_{ij}(x)dx$ be the probability that starting at time zero, the service of i units is over in the interval $(x, x+dx)$ and there are j arrivals during the interval $(o,x]$.

So we get

$$f_{ij}(x) = g^{*i}(x)\ \mu_j(x) \tag{3.3}$$

Let us define an infinite vector

$$\underline{f}_i(x) = (f_{i1}(x),\ f_{i2}(x),\ \ldots) \text{ for } i = 1,2, \ldots$$

and $$\underline{f}_o(x) = (f_{10}(x),\ f_{20}(x),\ \ldots)^T$$

where T denotes transpose.

Now we can define a matrix $\mathbb{F}$ given by

$$\mathbb{F}(x) = \begin{bmatrix} f_{11}(x) & f_{12}(x) & \cdots \\ f_{22}(x) & f_{22}(x) & \cdots \\ \cdot\cdot & \cdot\cdot & \cdots \\ \cdot\cdot & \cdot\cdot & \cdots \end{bmatrix}$$

For i = 1,2, ..., $(\underline{f}_i * \sum_{n=0}^{\infty} \mathbb{F}^{*n})(x)$ will be a row vector of infinite order, where the vector $\underline{f}_i$ is convoluted with the columns of the matrix $\sum_{n=0}^{\infty} \mathbb{F}^{*n}$. Now for $i, \eta = 1,2,\ldots,$ let

$$M_i^{\eta}(x) = \eta\text{th coordinate of the vector} (\underline{f}_i * \sum_{n=0}^{\infty} \mathbb{F}^{*n})(x) \quad (3.4)$$

Also let,

$$K_i(x) = (\underline{f}_i * \sum_{n=0}^{\infty} \mathbb{F}^{*n} * \underline{f}_o)\ (x) \quad (3.5)$$

Now we can find out the probability distribution of a busy period. Let $F_i(x)$ denote the probability density function of a busy period initiated by i customers.

Then for i = 1,2, ..., we have

$$F_i(x) = f_{io}(x) + K_i(x) \quad (3.6)$$

For $j = 1,2, \ldots,$ let $b_j(x)dx$ be the probability that after a busy period, server goes for vacation at time zero and after one or more vacations, the next busy period starts in the interval $(x,x+dx)$ when there are j units present in the system.

Then for $j = 1,2, \ldots,$

$$b_j(x) = \int_0^x \sum_{m=0}^{\infty} h^{*m}(u)\ \mu_0(u)\ h(x-u)\ \mu_j(x-u)du \qquad (3.7)$$

The renewal points of the process are those time points at which the server goes for vacation after a busy period. Let Z be the time between two such renewal points. Then the probability density function of Z is given by

$$k(t) = \int_0^t \sum_{j=1}^{\infty} b_j(u)\ F_j(t-u)du \qquad (3.8)$$

Then the renewal density function of the delayed renewal process is given by

$$M(u) = \sum_{n=0}^{\infty} (F_a * k^{*n})(u) \qquad (3.9)$$

In the next section, we derive explicit expressions for the time dependent system size probabilities at arbitrary epochs.

3.3. SYSTEM SIZE PROBABILITIES

Let $P_i(t)$ denote the probability that at time t, there are i customers in the system including the one being served. Considering all the mutually exclusive and exhaustive cases, we have the following equations.

$$P_o(t) = \int_o^t M(u) \int_u^t \sum_{m=o}^{\infty} h^{*m}(v-u)[1-H(t-v)]\mu_o(t-v)dv \; du \quad (3.10)$$

For i = 1,2, ..., a, we have

$$P_i(t) = \sum_{j=1}^{i} [G^{*a-j}(t) - G^{*a-j+1}(t)] \; \mu_{i-j}(t)$$

$$+ \int_o^t \sum_{k=1}^{i-1} M_a^k(u) \sum_{j=1}^{k} [G^{*k-j}(t-u) - G^{*k-j+1}(t-u)]$$

$$\times \; \mu_{i-j}(t-u)du$$

$$+ \int_o^t \sum_{k=i}^{\infty} M_a^k(u) \sum_{j=1}^{i} [G^{*k-j}(t-u) - G^{*k-j+1}(t-u)]$$

$$\times \; \mu_{i-j}(t-u)du$$

$$+ \int_{o}^{t} M(u) \int_{u}^{t} \sum_{k=1}^{i-1} b_k(v-u) \sum_{j=1}^{k} [G^{*k-j}(t-v)-G^{*k-j+1}(t-v)]$$

$$x\ \mu_{i-j}(t-v)dv\ du$$

$$+ \int_{o}^{t} M(u) \int_{u}^{t} \sum_{k=i}^{\infty} b_k(v-u) \sum_{j=1}^{i} [G^{*k-j}(t-v)-G^{*k-j+1}(t-v)]$$

$$x\ \mu_{i-j}(t-v)dv\ du$$

$$+ \int_{o}^{t} M(u) \int_{u}^{t} \sum_{\gamma=1}^{\infty} b_\gamma(v-u) \int_{v}^{t} \sum_{k=1}^{i-1} M_\gamma^k(w-v)$$

$$x \sum_{j=1}^{k} [G^{*k-j}(t-w)-G^{*k-j+1}(t-w)]\mu_{i-j}(t-w)dw\ dv\ du$$

$$+ \int_{o}^{t} M(u) \int_{u}^{t} \sum_{\gamma=1}^{\infty} b_\gamma(v-u) \int_{v}^{t} \sum_{k=i}^{\infty} M_\gamma^k(w-v)$$

$$x \sum_{j=1}^{i} [G^{k-j}(t-w)-G^{k-j+1}(t-w)]\mu_{i-j}(t-w)dw\ dv\ du$$

$$+ \int_{o}^{t} M(u) \int_{u}^{t} \sum_{m=o}^{\infty} h^{*m}(v-u)\ \mu_o(v-u)\ [1-H(t-v)]$$

$$x\ \mu_i(t-v)dv\ du \qquad (3.11)$$

For i = a+1, a+2, ..., we have

$$P_i(t) = \sum_{j=1}^{a} [G^{*a-j}(t) - G^{*a-j+1}(t)]\, \mu_{i-j}(t)$$

$$+ \int_0^t \sum_{k=1}^{i-1} M_a^k(u) \sum_{j=1}^{k} [G^{*k-j}(t-u) - G^{*k-j+1}(t-u)]$$

$$\times \quad \mu_{i-j}(t-u)\,du$$

$$+ \int_0^t \sum_{k=i}^{\infty} M_a^k(u) \sum_{j=1}^{i} [G^{*k-j}(t-u) - G^{*k-j+1}(t-u)]$$

$$\times \quad \mu_{i-j}(t-u)\,du$$

$$+ \int_0^t M(u) \int_u^t \sum_{k=1}^{i-1} b_k(v-u) \sum_{j=1}^{k} [G^{*k-j}(t-v) - G^{*k-j+1}(t-v)]$$

$$\times \quad \mu_{i-j}(t-v)\,dv\,du$$

$$+ \int_0^t M(u) \int_u^t \sum_{k=i}^{\infty} b_k(v-u) \sum_{j=1}^{i} [G^{*k-j}(t-v) - G^{*k-j+1}(t-v)]$$

$$\times \quad \mu_{i-j}(t-v)\,dv\,du$$

$$+ \int_{o}^{t} M(u) \int_{u}^{t} \sum_{\gamma=1}^{\infty} b_{\gamma}(v-u) \int_{v}^{t} \sum_{k=1}^{i-1} M_{\gamma}^{k}(w-v)$$

$$x \sum_{j=1}^{k} [G^{*k-j}(t-w)-G^{*k-j+1}(t-w)]\mu_{i-j}(t-w)dw\ dv\ du$$

$$+ \int_{o}^{t} M(u) \int_{u}^{t} \sum_{\gamma=1}^{\infty} b_{\gamma}(v-u) \int_{v}^{t} \sum_{k=i}^{\infty} M_{\gamma}^{k}(w-v)$$

$$x \sum_{j=1}^{i} [G^{*k-j}(t-w)-G^{*k-j+1}(t-w)]\mu_{i-j}(t-w)dw\ dv\ du$$

$$+ \int_{o}^{t} M(u) \int_{u}^{t} \sum_{m=o}^{\infty} h^{*m}(v-u)\mu_{o}(v-u)[1-H(t-v)]\mu_{i}(t-v)dv\ du \tag{3.12}$$

3.4 VIRTUAL WAITING TIME IN THE QUEUE

The virtual waiting time in the queue at time t is defined as the waiting time of a customer in the queue if it were to arrive at time t. Let W_t be the virtual waiting time at time t. Here we compute the probability distribution of W_t conditional on the state of the system at time t and it is enough because the system size probabilities are known. Here we assume that service is in the order of their arrivals.

We consider the following cases seperately.

Case (i): At time t, there are i, i=1,2, ... units in the system and the server is working.

Then

$$\Pr\left\{ W_t \leq x \right\} = \int_0^t \sum_{j=0}^{i-1} \sum_{k=i-j}^{\infty} M_a^k(u)\ \mu_j(t-u)\, G^{*k+j}(t+x-u)\,du$$

$$+ \int_0^t M(u) \int_u^t \sum_{\gamma=1}^{\infty} b_\gamma(v-u) \int_v^t \sum_{j=0}^{i-1} \sum_{k=i-j}^{\infty} M_\gamma^k(w-v)$$

$$\times\ \mu_j(t-w)\ G^{*k+j}(t+x-w)\,dw\,dv\,du \qquad (3.13)$$

Case (ii): At time t, there are i, i=1,2, ..., units in the system and the server is on vacation.

Then

$$\Pr\left\{ W_t \leq x \right\} = \int_0^t M(u) \int_u^t \sum_{m=0}^{\infty} h^{*m}(v-u)\ \mu_0(v-u)$$

$$\times \int_t^{t+x} h(w-v)\, G^{*i}(t+x-w)\,dw\,dv\,du \qquad (3.14)$$

<u>Case (iii)</u>: At time t, the system is empty and the server is on vacation.

Then

$$\Pr\{W_t \leq x\} = \int_0^t M(u) \int_u^t \sum_{m=0}^{\infty} h^{*m}(v-u)[H(t+x-v)-H(t-v)]dv\, du \tag{3.15}$$

<u>Remark</u>:

The queueing system $M^Y/G/1$ without vacations to the server can be obtained as a special case of this model by taking the vacation period distribution as the distribution of the idle period. That is, the same exponential distribution of the interarrival time of customers.

Chapter 4

A FINITE CAPACITY M/G/1 QUEUEING SYSTEM WITH BULK SERVICE AND VACATIONS TO THE SERVER

4.1. INTRODUCTION

In this chapter we consider an M/G/1 queueing system with bulk service and server going for vacation whenever there is less than a prespecified number of customers in the system. The general bulk service rule [see Neuts (1967)] is modified to allow the arriving customers to enter for service without altering the service time of the batch being served. This type of procedure is usually adopted in the case of entrance to cinema hall etc. When at least a prespecified number of customers are available, show starts and the arriving customers are allowed to enter the hall if it is not full (this number may be found out by considering the cost associated with one show so that the profit is increased by the entrance of new arrivals). Arrival of these customers will not alter the service time. But among the arriving customers, some may need full service and they wait until the next show begins. The server is allowed to take vacations whenever the system size is less than the minimum number of units needed to start a service.

Motivated by the study of cyclic service queues, an extensive study has been made on single server queueing systems with vacations to the server. Courtois (1980) analyses a finite capacity M/G/1 queueing system with delays. Under exhaustive service discipline, Lee (1984) analyses a finite capacity M/G/1 queueing system with vacations to the server and Scholl and Kleinrock (1983) analyse an infinite capacity M/G/1 queue with vacations. The case of non-exhaustive service disciplines are analysed by Neuts and Ramalhoto (1984) and Fuhrmann and Cooper(1985). Keilson and Servi (1986b) obtain the blocking probability of an M/G/1 vacation system with occupancy level dependent schedules. For a detailed survey on vacation systems, one may refer to Doshi (1986).

All the papers mentioned above deal with the situation where steady state exists. In this chapter, we extend the results of last two chapters to obtain the time dependent solution for the modified $M/G^{a,b}/1$ queueing system with vacations to the server. The model is described as follows. Customers arrive to the system according to a homogeneous Poisson process of rate μ. The customers are served in batches according to the general bulk service rule.

That is, a minimum of 'a' customers are needed to start a service and a maximum of 'b' customers can be served at a time. Service times are independent and identically distributed random variables having distribution function G(.) and density function g(.). The system is having a waiting room of finite capacity 'b', so that each service is started with all the units that are waiting for service at that time.

Whenever the size of the batch being served is less than 'b', the arriving units are allowed to enter for service until it reaches the maximum capacity 'b'. This service rule is known as general bulk service with accessible batches. Let p be the probability that an arriving customer enter for partial service and let (1-p) be the probability that he waits until the next service begins. Any time immediately after a service, if the server finds at least 'a' customers waiting, he takes all of them for service. If he finds less than 'a' customers in the system, he goes for vacation for a random length of time having probability distribution function H(.) and density function h(.). If the server returns from a vacation to find less than 'a' customers in the waiting room, he begins another vacation independent of the previous vacation and having the same probability distribution.

Now, let us assume that when the service room is full, all the arriving customers enter the waiting room. When the waiting room is full all the new arrivals are lost even if the service room is not full. Also, all the arrivals that are taking place when the server is under vacation enter the waiting room until it is full. Finally, we assume that the system starts with $r(a \leq r \leq b)$ units waiting in the system at time zero.

4.2. BASIC RESULTS

For $j = 0,1,2, \ldots,$ let $\mu_j(x)$ be the probability that there are exactly j arrivals during an interval of length x.

Then

$$\mu_j(x) = \frac{e^{-\mu x}(\mu x)^j}{j!} \tag{4.1}$$

For $a \leq i \leq b$ and $o \leq j \leq b$, let $f_{ij}(x)dx$ be the probability that starting with i units at time zero, the service of a batch is over in the interval $(x,x+dx)$ and there are j units waiting in the system at this service completion point.

Then for $a \leq i \leq b-1$ and $o \leq j \leq b-1$, we have

$$f_{ij}(x) = g(x) \left\{ \sum_{k=o}^{b-i-1} \mu_{j+k}(x) \binom{j+k}{k} p^k (1-p)^j \right.$$

$$\left. + \sum_{\gamma=o}^{j} \mu_{b-i+j}(x) \binom{\gamma+b-i}{b-i} p^{b-i}(1-p)^{\gamma} \right\} \qquad (4.2)$$

For $a \leq i \leq b-1$, we have

$$f_{ib}(x) = g(x) \left\{ \sum_{k=o}^{b-i-1} \sum_{j=b+k}^{\infty} \mu_j(x) \binom{b+k-1}{k} p^k(1-p)^b \right.$$

$$\left. + \sum_{\gamma=o}^{b} \sum_{j=2b-i}^{\infty} \mu_j(x) \binom{\gamma+b-i}{b-i} p^{b-i} (1-p)^{\gamma} \right\} \qquad (4.3)$$

For $o \leq j \leq b-1$, we have

$$f_{bj}(x) = g(x) \mu_j(x) \qquad (4.4)$$

and

$$f_{bb}(x) = g(x) \sum_{j=b}^{\infty} \mu_j(x) \qquad (4.5)$$

For $a \leq i \leq b$ and $o \leq j \leq b$, let $h_{ij}(x)dx$ be the probability that starting at time zero, service of a batch is not over in $(o,x]$, there are i units in the service room and j units in the waiting room at time x.

Then for $a \le i \le b-1$ and $o \le j \le b-1$, we have

$$h_{ij}(x) = [1-G(x)] \sum_{k=a}^{i} \mu_{j+i-k}(x) \binom{j+i-k}{i-k} p^{i-k}(1-p)^{j} \quad (4.6)$$

for $a \le i \le b-1$, we have

$$h_{ib}(x) = [1-G(x)] \sum_{k=a}^{i} \sum_{j=b+i-k}^{\infty} \mu_j(x) \binom{b+i-k-1}{i-k} p^{i-k}(1-p)^{b} \quad (4.7)$$

For $o \le j \le b-1$, we have

$$h_{bj}(x) = [1-G(x)] \left\{ \sum_{k=a}^{b-1} \sum_{\gamma=o}^{j} \mu_{j+b-k}(x) \binom{\gamma+b-k}{b-k} p^{b-k} \times (1-p)^{\gamma} + \mu_j(x) \right\} \quad (4.8)$$

and

$$h_{bb}(x) = [1-G(x)] \left\{ \sum_{k=a}^{b-1} \sum_{j=2b-k}^{\infty} \sum_{\gamma=o}^{b-1} \mu_j(x) \times \binom{b-k+\gamma}{b-k} p^{b-k} (1-p)^{\gamma} + \sum_{j=b}^{\infty} \mu_j(x) \right\} \quad (4.9)$$

Now for $a \le i \le b$, let

$$\underline{f}_i(x) = (f_{ia}(x), \ldots, f_{ib}(x))$$

It is a vector of order (b-a+1).

Let $\mathbb{F}$ be a square matrix of order (b-a+1) given by

$$\mathbb{F}(x) = \begin{bmatrix} f_{aa}(x) & \cdots & f_{ab}(x) \\ \vdots & & \\ f_{ba}(x) & \cdots & f_{bb}(x) \end{bmatrix}$$

and $\mathbb{G}$ a matrix of order (b-a+1) x a given by

$$\mathbb{G}(x) = \begin{bmatrix} f_{ao}(x) & \cdots & f_{a(a-1)}(x) \\ \vdots & & \\ f_{bo}(x) & \cdots & f_{b(a-1)}(x) \end{bmatrix}$$

Then for $a \le i \le b$, $(\underline{f}_i * \sum_{n=o}^{\infty} \mathbb{F}^{*n})(x)$ will be a vector of order (b-a+1) and

$(\underline{f}_i * \sum_{n=o}^{\infty} \mathbb{F}^{*n} * \mathbb{G})(x)$ will be a vector of order a.

For $a \le i$, $\eta \le b$, let

$$M_i^{\eta}(x) = (\eta-a+1)\text{th coordinate of } (\underline{f}_i * \sum_{n=0}^{\infty} F^{*n})(x) \qquad (4.10)$$

and for $a \le i \le b$ and $0 \le \eta \le a-1$, let

$$K_i^{\eta}(x) = (\eta+1)\text{th coordinate of } (\underline{f}_i * \sum_{n=0}^{\infty} F^{*n} * G)(x) \qquad (4.11)$$

Thus we obtain the probability density function of a busy period starting with i units and ending with η units left, as

$$F_i^{\eta}(x) = f_{i\eta}(x) + K_i^{\eta}(x) \text{ for } a \le i \le b \text{ and } 0 \le \eta \le a-1 \qquad (4.12)$$

For $0 \le i \le a-1$ and $a \le j \le b$, let $c_{ij}(x)dx$ be the probability that after a busy period, the server goes for vacation at time zero when there are i units waiting and after one or more vacations, the next busy period starts in the interval $(x, x+dx)$ with j units in the system.

Then for $0 \le i \le a-1$ and $a \le j \le b-1$, we have

$$c_{ij}(x) = \int_0^x \sum_{m=0}^{\infty} h^{*m}(u) \sum_{k=0}^{a-i-1} \mu_k(u)\; h(x-u)\mu_{j-k-i}(x-u)\,du \qquad (4.13)$$

and

$$c_{ib}(x) = \int_0^x \sum_{m=0}^{\infty} h^{*m}(u) \sum_{k=0}^{a-i-1} \mu_k(u)\ h(x-u)$$

$$\times \sum_{j=b-k-i}^{\infty} \mu_j(x-u)\,du \qquad (4.14)$$

Now we look at the time points at which the busy periods starts and the time between two such consecutive points is called a busy cycle.

For $a \leq i,k \leq b$, let $d_{ik}(x)dx$ be the probability that a busy period is started at time zero with i units in the system and the next busy period is started in the interval $(x,x+dx)$ with k units in the system.

Then the probability density function of this busy cycle is given by

$$d_{ik}(x) = \int_0^x \sum_{\eta=0}^{a-1} F_i^{\eta}(u)\ c_{\eta k}(x-u)\,du \qquad (4.15)$$

Now we define a vector of order $(b-a+1)$ as

$$\underline{d}_r(x) = (d_{ra}(x), \ldots, d_{rb}(x))$$

and a square matrix $\mathbb{D}$ of order (b-a+1) as

$$\mathbb{D}(x) = \begin{bmatrix} d_{aa}(x) & \cdots & d_{ab}(x) \\ \vdots & & \\ d_{ba}(x) & \cdots & d_{bb}(x) \end{bmatrix}$$

Then $(\underline{d}_r * \sum_{n=o}^{\infty} \mathbb{D}^{*n})(x)$ will be a vector of order (b-a+1).

For $a \leq \eta \leq b$, let

$$D_\eta(x) = (\eta-a+1)\text{th coordinate of } (\underline{d}_r * \sum_{n=o}^{\infty} \mathbb{D}^{*n})(x) \quad (4.16)$$

The state space of the system is given by

$$S = \{(i,j) \mid a \leq i \leq b,\ o \leq j \leq b\} \cup \{(o,j) \mid o \leq j \leq b\}$$

For $a \leq i \leq b$ and $o \leq j \leq b$, (i,j) denotes the state that a batch of i units is being served and there are j units waiting at that time. For $o \leq j \leq b$, (o,j) denotes the state that server is under vacation and there are j units waiting in the system.

4.3. THE SYSTEM SIZE PROBABILITIES

For $i = o, a, a+1, \ldots, b$ and $j = o, 1, \ldots, b$, let $P_{ij}(t)$ be the probability that the state of the system is (i,j) at time t. Considering all the mutually exclusive cases, the following relations can be obtained. Note that the first busy period starts with r units.

Then for $r \leq i \leq b$ and $o \leq j \leq b$, we have

$$P_{ij}(t) = h_{ij}(t) + \int_o^t \sum_{\eta=a}^{i} M_r^{\eta}(u)\ h_{ij}(t-u)du$$

$$+ \int_o^t \sum_{\eta=a}^{i} D_{\eta}(u)\ h_{ij}(t-u)du$$

$$+ \int_o^t \sum_{\eta=a}^{b} D_{\eta}(u) \int_u^t \sum_{k=a}^{i} M_{\eta}^{k}(v-u) h_{ij}(t-v)dv\ du \quad (4.17)$$

For $a \leq i \leq r-1$ and $o \leq j \leq b$, we have

$$P_{ij}(t) = \int_o^t \sum_{\eta=a}^{i} M_r^{\eta}(u)\ h_{ij}(t-u)du$$

$$+ \int_o^t \sum_{\eta=a}^{i} D_{\eta}(u)\ h_{ij}(t-u)du$$

$$+ \int_o^t \sum_{\eta=a}^{b} D_{\eta}(u) \int_u^t \sum_{k=a}^{i} M_{\eta}^{k}(v-u) h_{ij}(t-v)dv\ du \quad (4.18)$$

For $o \leq j \leq a-1$, we have

$$P_{oj}(t) = \int_{o}^{t} \sum_{k=o}^{j} F_r^k(u)\ \mu_{j-k}(t-u)\,du$$

$$+ \int_{o}^{t} \sum_{\eta=a}^{b} D_\eta(u) \int_{u}^{t} \sum_{k=o}^{j} F_\eta^k(v-u)\ \mu_{j-k}(t-v)\,dv\ du \tag{4.19}$$

For $a \leq j \leq b-1$, we have

$$P_{oj}(t) = \int_{o}^{t} \sum_{\eta=o}^{a-1} F_r^\eta(u) \int_{u}^{t} \sum_{m=o}^{\infty} h^{*m}(v-u) \sum_{k=o}^{a-\eta-1} \mu_k(v-u)$$

$$x\ [1-H(t-v)]\ \mu_{j-k-\eta}(t-v)\,dv\ du$$

$$+ \int_{o}^{t} \sum_{\eta=a}^{b} D_\eta(u) \int_{u}^{t} \sum_{i=o}^{a-1} F_\eta^i(v-u)$$

$$x \int_{v}^{t} \sum_{m=o}^{\infty} h^{*m}(w-v) \sum_{k=o}^{a-i-1} \mu_k(w-v)$$

$$x\ [1-H(t-w)]\mu_{j-k-i}(t-w)\,dw\ dv\ du \tag{4.20}$$

Finally,

$$P_{ob}(t) = \int_{o}^{t} \sum_{i=o}^{a-1} F_r^i(u) \int_{u}^{t} \sum_{m=o}^{\infty} h^{*m}(v-u) \sum_{k=o}^{a-i-1} \mu_k(v-u)$$

$$x \quad [1-H(t-v)] \sum_{j=b-k-i}^{\infty} \mu_j(t-v)\,dv\,du$$

$$+ \int_{o}^{t} \sum_{\eta=a}^{b} D_\eta(u) \int_{u}^{t} \sum_{i=o}^{a-1} F_\eta^i(v-u) \int_{v}^{t} \sum_{m=o}^{\infty} h^{*m}(w-v)$$

$$x \sum_{k=o}^{a-i-1} \mu_k(w-v) \; [1-H(t-w)]$$

$$x \sum_{j=b-k-i}^{\infty} \mu_j(t-w)\,dw\,dv\,du \qquad (4.21)$$

4.4 VIRTUAL WAITING TIME IN THE QUEUE

The virtual waiting time in the queue at time t is defined as the length of time a (virtual) customer arriving at time t has to wait before starting his service. Let W_t be the virtual waiting time in the queue at time t. The probability distribution of W_t is computed here, conditional on the state of the system at time t. It should be noted that, we have already computed the system size probabilities and hence this will give us the explicit expressions for the virtual waiting time in the queue at time t. Further, it is to be noted that even if the waiting room is full, the virtual customer can join the queue and enter for the next service along with the waiting customers.

We consider the following cases seperately.

Case (i): The state of the system is (i,j) at time t, $a \le i < b$ and $a-1 \le j < b$, so that the server is working.

Then

$$\Pr\{W_t \le x\} = p + (1-p)\int_0^t \sum_{\eta=a}^{i} M_r^{\eta}(u) \int_t^{t+x} \sum_{k=j}^{b} f_{\eta k}(z-u)\,dz\,du$$

$$+ (1-p)\int_0^t \sum_{\eta=a}^{i} D_{\eta}(u) \int_t^{t+x} \sum_{k=j}^{b} f_{\eta k}(z-u)\,dz\,du$$

$$+ (1-p)\int_0^t \sum_{\eta=a}^{b} D_{\eta}(u) \int_u^t \sum_{\gamma=a}^{i} M_{\eta}^{\gamma}(v-u)$$

$$\times \int_t^{t+x} \sum_{k=j}^{b} f_{\eta k}(z-v)\,dz\,dv\,du \qquad (4.22)$$

Case (ii): The state is (b,j) at time t, $a-1 \le j \le b$, server is working.

Then

$$\Pr\{W_t \le x\} = \int_0^t \sum_{\eta=a}^{b} M_r^{\eta}(u) \int_t^{t+x} \sum_{k=j}^{b} f_{\eta k}(z-u)\,dz\,du$$

$$+ \int_0^t \sum_{\eta=a}^{b} D_\eta(u) \int_t^{t+x} \sum_{k=j}^{b} f_{\eta k}(z-u)\,dz\,du$$

$$+ \int_0^t \sum_{\eta=a}^{b} D_\eta(u) \int_u^t \sum_{\gamma=a}^{b} M_\eta^{\gamma}(v-u)$$

$$\times \int_t^{t+x} \sum_{k=j}^{b} f_{\gamma k}(z-v)\,dz\,dv\,du \qquad (4.23)$$

Case (iii): The state is (i,b) at time t, $a \le i < b$. Server is working

Then

$$\Pr\{W_t \le x\} = \int_0^t \sum_{\eta=a}^{i} M_r^{\eta}(u) \int_t^{t+x} f_{\eta b}(z-u)\,dz\,du$$

$$+ \int_0^t \sum_{\eta=a}^{i} D_\eta(u) \int_t^{t+x} f_{\eta b}(z-u)\,dz\,du$$

$$+\int_0^t \sum_{\eta=a}^{b} D_\eta(u) \int_u^t \sum_{k=a}^{i} M_\eta^k(v-u)$$

$$\times \int_t^{t+x} f_{kb}(z-v)\,dz\,dv\,du \tag{4.24}$$

Case (iv): The state is (i,j) at time t, $a \le i < b$, $o \le j \le a-2$. Server is working.

Then

$$\Pr\{W_t \le x\} = p+(1-p)\int_0^t \sum_{\eta=a}^{i} M_r^\eta(u) \int_t^{t+x} \sum_{k=a}^{b} f_{\eta k}(z-u)\,dz\,du$$

$$+ (1-p)\int_0^t \sum_{\eta=a}^{i} D_\eta(u) \int_t^{t+x} \sum_{k=a}^{b} f_{\eta k}(z-u)\,dz\,du$$

$$+ (1-p)\int_0^t \sum_{\eta=a}^{b} D_\eta(u) \int_u^t \sum_{\gamma=a}^{i} M_\eta^\gamma(v-u)$$

$$\times \int_t^{t+x} \sum_{k=a}^{b} f_{\gamma k}(z-u)\,dz\,dv\,du$$

$$+ (1-p)\int_0^t \sum_{\eta=a}^{i} M_r^\eta(u) \int_t^{t+x} \sum_{k=j+1}^{a-1} f_{\eta k}(v-u)$$

$$\times \int_v^{t+x} \sum_{m=o}^{\infty} h^{*m}(w-v) \sum_{\ell=o}^{a-k-1} \mu_\ell(w-v)$$

$$\times H(t+x-w) \sum_{\gamma=a-k-\ell}^{\infty} \mu_\gamma(t+x-w)\,dw\,dv\,du$$

$$+ (1-p) \int_o^t \sum_{\eta=a}^{i} D_\eta(u) \int_t^{t+x} \sum_{k=j+1}^{a-1} f_{\eta k}(v-u)$$

$$\times \int_v^{t+x} \sum_{m=o}^{\infty} h^{*m}(w-v) \sum_{\ell=o}^{a-k-1} \mu_\ell(w-v)$$

$$\times H(t+x-w) \sum_{\gamma=a-k-\ell}^{\infty} \mu_\gamma(t+x-w)\,dw\,dv\,du$$

$$+ (1-p) \int_o^t \sum_{\eta=a}^{b} D_\eta(u) \int_u^t \sum_{r=a}^{i} M_\eta^r(v-u) \int_t^{t+x} \sum_{k=j+1}^{a-1} f_{rk}(w-v)$$

$$\times \int_w^{t+x} \sum_{m=o}^{\infty} h^{*m}(y-w) \sum_{\ell=o}^{a-k-1} \mu_\ell(y-w)\, H(t+x-y)$$

$$\times \sum_{\gamma=a-k-\ell}^{\infty} \mu_\gamma(t+x-y)\,dy\,dw\,dv\,du \qquad (4.25)$$

Case (v): The state is (b,j) at time t, $o \le j \le a-2$.

Then

$$\Pr\left\{ W_t \le x \right\} = \int_o^t \sum_{\eta=a}^{b} M_r^\eta(u) \int_t^{t+x} \sum_{k=a}^{b} f_{\eta k}(z-u)\,dz\,du$$

$$+ \int_o^t \sum_{\eta=a}^{b} D_\eta(u) \int_t^{t+x} \sum_{k=a}^{b} f_{\eta k}(z-u)\,dz\,du$$

$$+\int_0^t \sum_{\eta=a}^{b} D_\eta(u) \int_u^t \sum_{\gamma=a}^{b} M_\eta^\gamma(v-u)$$

$$x \int_t^{t+x} \sum_{k=a}^{b} f_{\gamma k}(z-v)\,dz\;dv\;du$$

$$+\int_0^t \sum_{\eta=a}^{b} M_r^\eta(u) \int_t^{t+x} \sum_{k=j+1}^{a-1} f_{\eta k}(v-u)$$

$$x \int_v^{t+x} \sum_{m=o}^{\infty} h^{*m}(w-v) \sum_{i=o}^{a-k-1} \mu_i(w-v)$$

$$x\;\; H(t+x-w) \sum_{\gamma=a-k-i}^{\infty} \mu_\gamma(t+x-w)\,dw\;dv\;du$$

$$+\int_o^t \sum_{\eta=a}^{b} D_\eta(u) \int_t^{t+x} \sum_{k=j+1}^{a-1} f_{\eta k}(v-u)$$

$$x \int_v^{t+x} \sum_{m=o}^{\infty} h^{*m}(w-v) \sum_{i=o}^{a-k-1} \mu_i(w-v)$$

$$x\; H(t+x-w) \sum_{\gamma=a-k-i}^{\infty} \mu_\gamma(t+x-w)\,dw\;dv\;du$$

$$+\int_{o}^{t}\sum_{\eta=a}^{b} D_{\eta}(u)\int_{u}^{t}\sum_{\ell=a}^{b} M_{\eta}^{\ell}(v-u)\int_{t}^{t+x}\sum_{k=j+1}^{a-1} f_{\ell k}(w-v)$$

$$x\int_{w}^{t+x}\sum_{m=o}^{\infty} h^{*m}(y-w)\sum_{i=o}^{a-k-1}\mu_i(y-w)$$

$$x\ H(t+x-y)\sum_{\gamma=a-k-i}^{\infty}\mu_{\gamma}(t+x-y)\,dy\ dw\ dv\ du \qquad (4.26)$$

Case (vi): The state is (o,j) at time t, $a-1 \leq j \leq b$. Server is under vacation.

Then

$$Pr\{W_t \leq x\} = \int_{o}^{t}\sum_{\eta=o}^{a-1} F_r^{\eta}(u)\int_{u}^{t}\sum_{m=o}^{\infty} h^{*m}(v-u)\sum_{i=o}^{a-\eta-1}\mu_i(v-u)$$

$$x\ [H(t+x-v)-H(t-v)]dv\ du$$

$$+\int_{o}^{t}\sum_{\eta=a}^{b} D_{\eta}(u)\int_{u}^{t}\sum_{k=o}^{a-1} F_{\eta}^{k}(v-u)\int_{v}^{t}\sum_{m=o}^{\infty} h^{*m}(w-v)$$

$$x\sum_{i=o}^{a-k-1}\mu_i(w-v)\ [H(t+x-w)-H(t-w)]dw\ dv\ du$$

(4.27)

Case (vii): The state is (o,j) at time t, $o \leq j \leq a-2$.
Server is under vacation.

Then

$$\Pr\{W_t \leq x\} = \int_o^t \sum_{\eta=o}^{a-1} F_r^{\eta}(u) \int_u^t \sum_{m=o}^{\infty} h^{*m}(v-u) \sum_{i=o}^{a-\eta-1} \mu_i(v-u)$$

$$\times \ [H(t+x-v)-H(t-v)] \sum_{k=a-i-\eta}^{\infty} \mu_k(t+x-v)\,dv\,du$$

$$+ \int_o^t \sum_{\eta=a}^{b} D_{\eta}(u) \int_u^t \sum_{\gamma=o}^{a-1} F_{\eta}^{\gamma}(v-u) \int_v^t \sum_{m=o}^{\infty} h^{*m}(w-v)$$

$$\times \sum_{i=o}^{a-\gamma-1} \mu_i(w-v)\ [H(t+x-w)-H(t-w)]$$

$$\times \sum_{k=a-i-\gamma}^{\infty} \mu_k(t+x-w)\,dw\,dv\,du \qquad (4.28)$$

Remark

If we take p = o in the above analysis, we will get the usual $M/G^{a,b}/1$ queueing system with vacations to the server. If we take p = 1, we will get the case of M/G/1 bulk service system with accessible batches in which all the arrivals join for service until the capacity is attained.

Chapter 5

A FINITE CAPACITY $M/G^B/1$ VACATION SYSTEM WITH BERNOULLI SCHEDULES

5.1. INTRODUCTION

Queueing systems with vacations to the server have been studied by several authors. Levy and Yechiali (1975), Courtois (1980), Scholl and Kleinrock(1983), Lee (1984), Fuhrmann (1984) and Doshi (1985) analysed the case of vacation models with exhaustive service, in which each time the server becomes available, he works in a continuous manner until the system becomes empty. The case of vacation queueing models without exhaustive service have been considered by Neuts and Ramalhoto (1984), Fuhrmann and Cooper (1985) and Keilson and Servi (1986b). An excellent survey on queueing systems with vacations to the server has been given by Doshi (1986).

The case of Bernoulli schedules in vacation systems is first analysed by Keilson and Servi (1986a). Using oscillating random walk, they have proved that for a GI/G/1 vacation system with Bernoulli schedules, the 'decomposition property' holds for the ergodic waiting time. Servi(1986b) analysed the average delay in M/G/1 cyclic service queue with Bernoulli schedules.

In all the queueing models referred to above, the existence of stationary distribution is assumed. Under exhaustive service discipline, Jacob and Krishnamoorthy(1987) gives the transient solution for a finite capacity M/G/1 queueing system with vacations to the server, using renewal theoretic arguments. The case of a finite capacity $M/G^{a,b}/1$ queueing system with vacations to the server is analysed by Jacob and Madhusoodanan (1988).

Here we consider a single server queueing model in which customers arrive at a counter according to a homogeneous Poisson process of rate μ. They are served in batches with maximum size B and minimum size 1. Service times are independent random variables having distribution function $G_i(.)$, if i is the size of the batch being served and let $g_i(.)$ be the corresponding probability density function. The waiting room is of finite capacity B, so that each service starts with all the units that are waiting for service at that time. All the arrivals that are taking place when the waiting room is full are lost.

Server goes for vacation whenever the system becomes empty and this vacation is called 'compulsory vacation'. When a batch has just been served out and other customers are present, server goes for vacation according to Bernoulli schedule. That is, the server accepts the next batch with probability p and commences a vacation with probability 1-p. Hence there are two types of vacations. One is the vacation started with probability one when the system is empty, which is repeated until there is at least one customer present on return from a vacation (according to the terminology of Doshi (1986), this is a multiple vacation). The other is the vacation started with probability 1-p, where the system is not empty and this vacation period cannot be extended for more than one period (i.e., this is a single vacation). In any case, on completion of a vacation, if customers are present, service is resumed The vacation periods are independent and identically distributed random variables having general distribution with distribution function H(.) and density function h(.). Setting p=o, we obtain the single service discipline in which a vacation period begins after every service completion or after any vacation if the queue is empty. If we set p=1, we get the exhaustive service discipline. Thus the single service discipline and exhaustive service discipline are special cases of Bernoulli schedule vacation system.

5.2. BASIC RESULTS

Let us suppose that at time zero, service starts with 'a' units ($1 \le a \le B$) in the system.

Now for $j = 0,1, \ldots, B-1$, let $\mu_j(x)$ denote the probability that there are exactly j arrivals during the interval (o,x] and let $\mu_B(x)$ denote the probability that there are at least B arrivals during the interval (o,x].

Then for $j = 0,1, \ldots, B-1$, we have

$$\mu_j(x) = \frac{e^{-\mu x}(\mu x)^j}{j!}$$

and (5.1)

$$\mu_B(x) = \sum_{j=B}^{\infty} \frac{e^{-\mu x}(\mu x)^j}{j!}$$

For $1 \le i \le B$ and $o \le j \le B$, let $f_{ij}(x)dx$ be the probability that at time zero, the service of a batch of size i units is over in the interval (x,x+dx) and there are j accepted arrivals during the interval (o,x].

Then

$$f_{ij}(x) = g_i(x)\ \mu_j(x) \tag{5.2}$$

Now we can define the transition probabilities as follows.

For $1 \le j$, $k \le B$, let $h_{jk}(x)dx$ be the probability that service of a batch of size j is started at time zero, the service of the next batch of size k units is started in the interval $(x, x+dx)$ given that on completion of service of the first batch, the waiting room was not empty.

Then for $1 \le j \le B$ and $1 \le k < B$, we have

$$h_{jk}(x) = f_{jk}(x)p + \int_0^x \sum_{i=1}^{k} f_{ji}(u)(1-p)h(x-u)\mu_{k-i}(x-u)du$$

and for $1 \le j \le 3$, we have

$$h_{jB}(x) = f_{jB}(x)p + \int_0^x \sum_{i=1}^{B} f_{ji}(u)(1-p)h(x-u)$$

$$\times \sum_{k=B-i}^{B} \mu_k(x-u)du \tag{5.3}$$

For $1 \le i \le B$, define a vector of order B given by

$$\underline{h}_i(x) = (h_{i1}(x),\ h_{i2}(x),\ \ldots,\ h_{iB}(x)\ )$$

and another vector of order B given by

$$\underline{f}_o(x) = (f_{10}(x),\ f_{20}(x), \ldots, f_{B0}(x))^T,$$

where T denotes transpose.

Define a square matrix $\mathbb{H}$ of order B given by

$$\mathbb{H}(x) = \begin{bmatrix} h_{11}(x) & \cdots & h_{1B}(x) \\ \vdots & & \\ h_{B1}(x) & \cdots & h_{BB}(x) \end{bmatrix}$$

Then for $1 \leq i \leq B$, $(\underline{h}_i * \sum_{n=0}^{\infty} \mathbb{H}^{*n})(x)$ will be a vector of order B.

For $\eta = 1, 2, \ldots, B$, let

$$M_i^{\eta}(x) = \eta\text{th coordinate of } (\underline{h}_i * \sum_{n=0}^{\infty} \mathbb{H}^{*n})(x) \qquad (5.4)$$

and

$$K_i(x) = (\underline{h}_i * \sum_{n=0}^{\infty} \mathbb{H}^{*n} * \underline{f}_o)(x) \qquad (5.5)$$

For this queueing system, we define busy period as the length of time from the instant at which the server starts service after a compulsory vacation until the server starts the next compulsory vacation. The time between the starting points of two consecutive busy periods is called a busy cycle.

Thus we obtain the probability density function of a busy period generated by i customers, $1 \leq i \leq B$, as

$$F_i(x) = f_{io}(x) + K_i(x) \tag{5.6}$$

Now we look at the time points at which the busy periods start and obtain the probability density function of the time between two such consecutive points. That is, the probability density function of a busy cycle.

For $1 \leq i, j \leq B$, let $d_{ij}(x)dx$ be the probability that a busy period is started at time zero with i units in the system and the next busy period is started in the interval $(x, x+dx)$ with j units in the system.

Then

$$d_{ij}(x) = \int_o^x F_i(x) \int_u^x \sum_{m=o}^{\infty} h^{*m}(v-u)\ \mu_o(v-u)$$

$$\times\ h(x-v)\ \mu_j(x-v)\,dv\,du \tag{5.7}$$

Now define a vector of order B given by

$$\underline{d}_a(x) = (d_{a1}(x), \ldots, d_{aB}(x))$$

and a square matrix $\mathbb{D}$ of order B given by

$$\mathbb{D}(x) = \begin{bmatrix} d_{11}(x) & \cdots & d_{1B}(x) \\ \vdots & & \\ d_{B1}(x) & \cdots & d_{BB}(x) \end{bmatrix}$$

Then $(\underline{d}_a * \sum_{n=o}^{\infty} \mathbb{D}^{*n})(x)$ will be a vector of order B.

For $1 \leq \eta \leq B$, let

$$D_\eta(x) = \eta\text{th coordinate of } (\underline{d}_a * \sum_{n=o}^{\infty} \mathbb{D}^{*n})(x) \qquad (5.8)$$

Finally, the state space of the system is given by

$$S = \left\{(i,j) \;\middle|\; o \leq i\,,\, j \leq B\right\}$$

For $1 \leq i \leq B$ and $o \leq j \leq B$, the state (i,j) denotes that a batch of i units is being served and there are j units waiting at that time. Also for $o \leq j \leq B$, (o,j) denotes the state that the server is under vacation and there are j units waiting.

5.3. THE SYSTEM SIZE PROBABILITIES

For $o \le i, j \le B$, we can define $P_{ij}(t)$ as the probability that the state of the system is (i,j) at time t. Note that the first busy period is started with the service of 'a' units. Considering all the mutually exclusive and exhaustive cases, the following relations can be cobtained.

For $o \le j \le B$, we have

$$P_{aj}(t) = [1-G_a(t)]\,\mu_j(t)$$
$$+ \int_o^t M_a^a(u)\,[1-G_a(t-u)]\,\mu_j(t-u)\,du$$
$$+ \int_o^t D_a(u)\,[1-G_a(t-u)]\,\mu_j(t-u)\,du$$
$$+ \int_o^t \sum_{\eta=1}^{B} D_\eta(u) \int_u^t M_\eta^a(v-u)\,[1-G_a(t-v)]$$
$$\times\ \mu_j(t-v)\,dv\,du \qquad (5.9)$$

For i = 1,2, ..., a-1, a+1, ..., B and $o \le j \le B$, we have

$$P_{ij}(t) = \int_o^t M_a^i(u)\,[1-G_i(t-u)]\,\mu_j(t-u)\,du$$

$$+ \int_o^t D_i(u)\ [1-G_i(t-u)]\ \mu_j(t-u)\,du$$

$$+ \int_o^t \sum_{\eta=1}^{B} D_\eta(u) \int_u^t M_\eta^i(v-u)[1-G_i(t-v)]$$

$$x\ \ \mu_j(t-v)\,dv\ du \qquad (5.10)$$

Also,

$$P_{OB}(t) = \int_o^t \sum_{k=1}^{B} f_{ak}(u)(1-p)[1-H(t-u)] \sum_{i=B-k}^{B} \mu_i(t-u)\,du$$

$$+ \int_o^t \sum_{\eta=1}^{B} M_a^\eta(u) \int_u^t \sum_{k=1}^{B} f_{\eta k}(v-u)(1-p)[1-H(t-v)]$$

$$x \sum_{i=B-k}^{B} \mu_i(t-v)\,dv\ du$$

$$+ \int_o^t \sum_{\eta=1}^{B} D_\eta(u) \int_u^t \sum_{k=1}^{B} f_{\eta k}(v-u)(1-p)[1-H(t-v)]$$

$$x \sum_{i=B-k}^{B} \mu_i(t-v)\,dv\ du$$

$$+ \int_o^t \sum_{\eta=1}^{B} D_\eta(u) \int_u^t \sum_{j=1}^{B} M_\eta^j(v-u) \int_v^t \sum_{k=1}^{B} f_{jk}(w-v)(1-p)$$

$$x\ [1-H(t-w)] \sum_{i=B-k}^{B} \mu_i(t-w)\,dw\ dv\ du$$

$$+\int_{o}^{t} F_a(u)\int_{u}^{t}\sum_{m=o}^{\infty} h^{*m}(v-u)\mu_o(v-u)$$

$$\times\ [1-H(t-v)]\ \mu_B(t-v)\,dv\ du$$

$$+\int_{o}^{t}\sum_{\eta=1}^{B} D_\eta(u)\int_{u}^{t} F_\eta(v-u)\int_{v}^{t}\sum_{m=o}^{\infty} h^{*m}(w-v)\mu_o(w-v)$$

$$\times\ [1-H(t-w)]\mu_B(t-w)\,dw\ dv\ du \qquad (5.11)$$

For $j = 1,2, \ldots, B-1$, we have

$$P_{oj}(t) = \int_{o}^{t}\sum_{k=1}^{j} f_{ak}(u)(1-p)[1-H(t-u)]\mu_{j-k}(t-u)\,du$$

$$+\int_{o}^{t}\sum_{\eta=1}^{B} M_a^\eta(u)\int_{u}^{t}\sum_{k=1}^{j} f_{\eta k}(v-u)(1-p)[1-H(t-v)]$$

$$\times\ \mu_{j-k}(t-v)\,dv\ du$$

$$+\int_{o}^{t}\sum_{\eta=1}^{B} D_\eta(u)\int_{u}^{t}\sum_{k=1}^{j} f_{\eta k}(v-u)(1-p)[1-H(t-v)]$$

$$\times\ \mu_{j-k}(t-v)\,dv\ du$$

$$+\int_{o}^{t} \sum_{\eta=1}^{B} D_{\eta}(u) \int_{u}^{t} \sum_{i=1}^{B} M_{\eta}^{i}(v-u) \int_{v}^{t} \sum_{k=1}^{j} f_{ik}(w-v)(1-p)$$

$$x\ [1-H(t-w)]\ \mu_{j-k}(t-w)dw\ dv\ du$$

$$+\int_{o}^{t} F_{a}(u) \int_{u}^{t} \sum_{m=o}^{\infty} h^{*m}(v-u)\mu_{o}(v-u)[1-H(t-v)]$$

$$x\ \mu_{j}(t-v)dv\ du$$

$$+\int_{o}^{t} \sum_{\eta=1}^{B} D_{\eta}(u) \int_{u}^{t} F_{\eta}(v-u) \int_{v}^{t} \sum_{m=o}^{\infty} h^{*m}(w-v)\mu_{o}(w-v)$$

$$x\ [1-H(t-w)]\mu_{j}(t-w)dw\ dv\ du \qquad (5.12)$$

Finally,

$$P_{oo}(t) = \int_{o}^{t} F_{a}(u) \int_{u}^{t} \sum_{m=o}^{\infty} h^{*m}(v-u)\mu_{o}(v-u)$$

$$x\ [1-H(t-v)]\mu_{o}(t-v)dv\ du$$

$$+\int_{o}^{t} \sum_{\eta=1}^{B} D_{\eta}(u) \int_{u}^{t} F_{\eta}(v-u) \int_{v}^{t} \sum_{m=o}^{\infty} h^{*m}(w-v)\mu_{o}(w-v)$$

$$x\ [1-H(t-w)]\mu_{o}(t-w)dw\ dv\ du \qquad (5.13)$$

5.4 VIRTUAL WAITING TIME IN THE QUEUE

Let W_t be the virtual waiting time in the queue at time t. The probability distribution of W_t is computed here, conditional on the state of the system at time t. It should be noted that we have already computed the system size probabilities and hence this will give us explicit expressions for the virtual waiting time. Further, it is to be noted that even if the waiting room is full, the virtual customer can join the queue and enter for the next service along with the waiting customers.

We consider the following cases seperately.

Case (i): The state is (i,j) at time t, $o < i \leq B$ and $o \leq j \leq B$, so that the server is working.

Then

$$\Pr\{W_t \leq x\} = \int_0^t M_a^i(u)\,[G_i(t+x-u)-G_i(t-u)]p\,du$$

$$+\int_0^t M_a^i(u) \int_t^{t+x} G_i(v-u)(1-p)H(t+x-v)\,dv\,du$$

$$+\int_0^t D_i(u)\,[G_i(t+x-u)-G_i(t-u)]p\,du$$

$$+\int_0^t D_i(u) \int_t^{t+x} G_i(v-u)(1-p)\, H(t+x-v)\,dv\,du$$

$$+\int_0^t \sum_{\eta=1}^{B} D_\eta(u) \int_u^t M_\eta^i(v-u)[G_i(t+x-v)-G_i(t-v)]p\,dv\,du$$

$$+\int_0^t \sum_{\eta=1}^{B} D_\eta(u) \int_u^t M_\eta^i(v-u) \int_t^{t+x} G_i(w-v)(1-p)$$

$$\times H(t+x-w)\,dw\,dv\,du \qquad (5.14)$$

Case (ii): At time t, the state is (o,j), $1 \le j \le B$. Server is under vacation

Then

$$\Pr\{W_t \le x\} = \int_0^t F_a(u) \int_u^t \sum_{m=o}^{\infty} h^{*m}(v-u)\mu_o(v-u)$$

$$\times \quad [H(t+x-v)-H(t-v)]\,dv\,du$$

$$+\int_0^t \sum_{\eta=1}^{B} D_\eta(u) \int_u^t F_\eta(v-u) \int_v^t \sum_{m=o}^{\infty} h^{*m}(w-v)\mu_o(w-v)$$

$$\times \quad [H(t+x-w)-H(t-w)]\,dw\,dv\,du$$

$$+ \int_{o}^{t} \sum_{k=1}^{j} f_{ak}(u)(1-p)[H(t+x-u)-H(t-u)]du$$

$$+ \int_{o}^{t} \sum_{\eta=1}^{B} M_{a}^{\eta}(u) \int_{u}^{t} \sum_{k=1}^{j} f_{\eta k}(v-u)(1-p)$$

$$x\ [H(t+x-v)-H(t-v)]dv\ du$$

$$+ \int_{o}^{t} \sum_{\eta=1}^{B} D_{\eta}(u) \int_{u}^{t} \sum_{k=1}^{j} f_{\eta k}(v-u)(1-p)$$

$$x\ [H(t+x-v)-H(t-v)]dv\ du$$

$$+ \int_{o}^{t} \sum_{\eta=1}^{B} D_{\eta}(u) \int_{u}^{t} \sum_{i=1}^{B} M_{\eta}^{i}(v-u) \int_{v}^{t} \sum_{k=1}^{j} f_{ik}(w-v)$$

$$x\ (1-p)\ [H(t+x-w)-H(t-w)]dw\ dv\ du \qquad (5.15)$$

<u>Case (iii)</u>: At time t, the state is (o,o).
Server is under vacation.

Then

$$Pr\left\{ W_t \le x \right\} = \int_{o}^{t} F_a(u) \int_{u}^{t} \sum_{m=o}^{\infty} h^{*m}(v-u)\mu_o(v-u)$$

$$x\ [H(t+x-v)-H(t-v)]dv\ du$$

$$+ \int_{o}^{t} \sum_{\eta=1}^{B} D_{\eta}(u) \int_{u}^{t} F_{\eta}(v-u) \int_{v}^{t} \sum_{m=o}^{\infty} h^{*m}(w-v)\mu_o(w-v)$$

$$x\ [H(t+x-w)-H(t-w)]dw\ dv\ du \qquad (5.16)$$

Chapter 6

AN (s,S) INVENTORY SYSTEM WITH RANDOM REPLENISHMENT

6.1. INTRODUCTION

Inventory systems based on (s,S) policy has been studied quite extensively by several authors. A systematic account of the probabilistic treatment in the study of inventory systems using renewal theoretic arguments has been given by Arrow, Karlin and Scarf (1958). A valuable review of storage systems was given by Gani (1957). Hadley and Whitin (1963) deals with the applications of mathematical models to practical situations. Tijms(1972) gives a detailed analysis of (s,S) inventory systems and chapter 3 of his monograph deals with the probabilistic analysis. Sivazlian (1974) considers the case of arbitrary interarrival time distribution and zero lead time. Srinivasan (1979) analyses the case of arbitrary inter-arrival time distribution and random lead time. The same problem with two ordering levels have been considered by Thangaraj and Ramanarayanan (1983). Sahin (1979) considers the case of bulk demands and constant lead times. In all these cases, the probability mass function of on hand inventory is derived and the associated optimization problems discussed.

In this chapter we consider a continuous review (s,S) inventory system with time between unit demands and lead times random variables following general distributions. In all the cases discussed in the past, whenever the level drops to s or below, an order is placed for a quantity so as to fill the inventory, and after a lead time the entire quantity ordered for is received. But in practice, this may not be the case. If the items have very high demand in the market, we may not get the full amount we have ordered. The item will be replenished according to the availability with the supplier. So it will be reasonable to assume that the quantity replenished is random which lies between a fixed lower level and the quantity ordered. Here we consider this problem and using renewal theoretic arguments, we derive the probability mass function for the stock level at an arbitrary time point t.

In section 6.2 we give the assumptions underlying the model and the notations used. Basic results regarding the transition probabilities are given in Section 6.3. In Section 6.4, we derive expressions for the probability distribution of the inventory level at arbitrary epochs. We find the expected value of total inventory carrying cost and the total expected profit due to sales over a time interval of length t, in Section 6.5. Section 6.6 deals with the case of zero lead time, and there the probability distribution of the inventory level is derived under such a condition. The steady state solution is also obtained in this section. Using this, the associated optimization problem is discussed in Section 6.7. Finally, we give some numerical results.

6.2. THE MODEL

Let us consider a warehouse of maximum capacity S and suppose that initially the warehouse is full. Due to incoming demands, the stock level goes on decreasing. The demands are assumed to occur for one unit at a time and the time interval between consecutive demands form a sequence of independent and identically distributed random variables. Whenever the stock level drops to s, the reorder level, an order is placed for S-s units. We assume that $S > 2s$ to avoid perpetual shortage. The lead time for materialization of an order is assumed to be arbitrarily distributed random variable, independent of the stock level and the time between demands. Further, lead times are assumed to be independent and identically distributed random variables. The market considered here is assumed to be competitive enough to rule out backlogging of demands and the demands that emanate during the stock out period are deemed to be lost. Since the market is competitive and there are sufficient demand for the item, the supplier may not be ready to supply (or may not be in a position to supply) the whole amount we have ordered. Thus, eventhough the order is for a fixed quantity S-s, the replenishment is by a random quantity. The quantity replenished is assumed to lie between s+1 and S-s. The probability that replenishment is by a quantity i is p_i for $s+1 \leq i \leq S-s$ such that $\sum_{i=s+1}^{S-s} p_i = 1$. Then the stock level can be described by a discrete-valued stochastic process $\{ I(t),\ t \geq 0 \}$ with $I(o) = S$. The epochs corresponding to the arrival of demands constitute a renewal process.

Notations

$I(t)$	stock level at arbitrary time t
$\pi(n,t)$	$Pr \{ I(t)=n \mid I(o)=S \}$
$F(.)$, $f(.)$	Cumulative Distribution Function (CDF) and probability density function (pdf) of the inter occurrence time of successive demands.

$G(.)$, $g(.)$	CDF and pdf of lead time
$\bar{F}(.)$, $\bar{G}(.)$	$1-F(.)$, $1-G(.)$ respectively
$f_{s,i}(x)dx$	probability that due to the first demand after order realization, the inventory level drops to i in the interval $(x,x+dx)$, given that the order is placed at time zero when the level was s. $(s \leq i \leq S-1)$.
${}_{s}f_{s}(x)dx$	probability that an order is placed in the interval $(x,x+dx)$, given that the previous order is placed at time zero.

6.3. TRANSITION PROBABILITIES

It is to be noted that the successive time points at which the inventory level drops to s will form a renewal process. Since the initial inventory level is S, this will form a delayed renewal process. We can derive the renewal density function for this delayed renewal process, by using the transition probabilities $f_{s,i}$.

Considering all the mutually exclusive and exhaustive cases, we can write expressions for the transition probabilities as follows.

$$f_{s,S-1}(x) = G(x)p_{S-s}f(x) \tag{6.1}$$

For $S-s \leq i \leq S-2$, we have

$$f_{s,i}(x) = G(x)\, p_{i-s+1}\, f(x) + \int_0^x \sum_{j=1}^{S-i-1} f^{*j}(u)[G(x)-G(u)] \times p_{i+j-s+1}\, f(x-u)\,du \qquad (6.2)$$

For $2s \leq i \leq S-s-1$, we have

$$f_{s,i}(x) = G(x)\, p_{i-s+1}\, f(x) + \int_0^x \sum_{j=1}^{s-1} f^{*j}(u)\, [G(x)-G(u)] \times p_{i+j-s+1}\, f(x-u)\,du + \int_0^x \sum_{m=0}^{\infty} f^{*m+s}(u)[G(x)-G(u)]p_{i+1} f(x-u)\,du \qquad (6.3)$$

For $s+1 \leq i \leq 2s-1$, we have

$$f_{s,i}(x) = \int_0^x \sum_{j=2s-i}^{s-1} f^{*j}(u)[G(x)-G(u)]p_{i+j-s+1} f(x-u)\,du + \int_0^x \sum_{m=0}^{\infty} f^{*m+s}(u)[G(x)-G(u)]p_{i+1} f(x-u)\,du \qquad (6.4)$$

Finally,

$$f_{s,s}(x) = \int_0^x \sum_{m=0}^{\infty} f^{*m+s}(u) \; [G(x)-G(u)]p_{s+1}f(x-u)du \tag{6.5}$$

Using equations (6.1) - (6.5) we can write

$${}_sf_s(x) = \int_0^x \sum_{i=s+1}^{S-1} f_{s,i}(u)f^{*i-s}(x-u)du+f_{s,s}(x) \tag{6.6}$$

Hence the renewal density function of the process is given by

$$R(x) = \sum_{m=0}^{\infty} {}_sf_s^{*m}(x) \tag{6.7}$$

and the renewal density function of the delayed renewal process is given by

$$M(u) = \int_0^u f^{*S-s}(v) \; R(u-v)dv \tag{6.8}$$

6.4 MAIN RESULTS

The principal quantity of interest is the probability mass function of the inventory level at any arbitrary point t on the time axis. That is, $Pr\{I(t)=n|I(o)=S\}$ for n=0,1,2,..,S.

Suppose that we consider the sequence of events consisting of the times at which the inventory level reaches s (the reorder level). Let Y_1 denote the time elapsed from origin until the first event occured. Y_2 be the time elapsed between first and second events and so on. The sequence of random variables $\{Y_k\}$, k=1,2, ... form a delayed renewal process [see Cox (1962)]. In each of the following expressions, we will make use of the renewal density M(u) given by (6.8).

Now considering all the mutually exclusive and exhaustive cases, we will get the following equations for the inventory level probabilities.

$$\pi(S,t) = \bar{F}(t) + \int_0^t M(u)\ \bar{F}(t-u)\ G(t-u) p_{S-s}\ du \qquad (6.9)$$

For $S-s < n < S$, we have

$$\pi(n,t) = [F^{*S-n}(t) - F^{*S-n+1}(t)] + \int_0^t M(u) \sum_{j=1}^{S-n} [F^{*j}(t-u) - F^{*j+1}(t-u)] \times G(t-u)\ p_{n+j-s}\ du \qquad (6.10)$$

$$\pi(S-s,t) = [F^{*s}(t) - F^{*s+1}(t)]$$

$$+ \int_0^t M(u) \sum_{j=1}^{s-1} [F^{*j}(t-u) - F^{*j+1}(t-u)]$$

$$\times \; G(t-u)\, p_{S-2s+j} \; du$$

$$+ \int_0^t M(u) \int_u^t \sum_{m=0}^{\infty} f^{*m+s}(v-u)$$

$$\times \; [G(t)-G(v)]\, p_{S-s} \, \bar{F}(t-v)\, dv \; du \qquad (6.11)$$

For $2s < n < S-s$, we have

$$\pi(n,t) = [F^{*S-n}(t) - F^{*S-n+1}(t)]$$

$$+ \int_0^t M(u) \, \bar{F}(t-u) \, G(t-u)\, p_{n-s} \; du$$

$$+ \int_0^t M(u) \int_u^t \sum_{j=1}^{s-1} \sum_{k=0}^{S-n-j} f^{*j}(v-u)$$

$$\times \; G(v-u)\, p_{n+j+k-s} \, [F^{*k}(t-v) - F^{*k+1}(t-v)]\, dv \; du$$

$$+ \int_0^t M(u) \int_u^t \sum_{m=0}^{\infty} f^{*m+s}(v-u)\, [G(t)-G(v)]\, p_n \, \bar{F}(t-v)\, dv \; du$$

$$+\int_{o}^{t} M(u) \int_{u}^{t} \sum_{m=o}^{\infty} f^{*m+s}(v-u)$$

$$\times \int_{v}^{t} \sum_{j=1}^{S-s-n} [G(w)-G(v)]p_{n+j}$$

$$\times [F^{*j}(t-w)-F^{*j+1}(t-w)]dw\ dv\ du \qquad (6.12)$$

For $s+1 \leq n \leq 2s$, we have

$$\pi(n,t) = [F^{*S-n}(t)-F^{*S-n+1}(t)]$$

$$+\int_{o}^{t} M(u) \int_{u}^{t} \sum_{j=1}^{s-1} \sum_{k=s-j}^{S-n-j} f^{*j}(v-u)$$

$$\times G(v-u)p_{n+j+k-s}[F^{*k}(t-v)-F^{*k+1}(t-v)]dv\ du$$

$$+\int_{o}^{t} M(u) \int_{u}^{t} \sum_{m=o}^{\infty} f^{*m+s}(v-u)[G(t)-G(v)]p_{n}\ \overline{F}(t-v)dv\ du$$

$$+\int_{o}^{t} M(u) \int_{u}^{t} \sum_{m=o}^{\infty} f^{*m+s}(v-u) \int_{v}^{t} \sum_{j=1}^{S-s-n} [G(w)-G(v)]p_{n+j}$$

$$\times [F^{*j}(t-w)-F^{*j+1}(t-w)]dw\ dv\ du \qquad (6.13)$$

$$\pi(s,t) = \int_0^t M(u)\ \bar{G}(t-u)\ \bar{F}(t-u)du \qquad (6.14)$$

For $o < n < s$, we have

$$\pi(n,t) = \int_0^t M(u)\ G(t-u)[F^{*s-n}(t-u)-F^{*s-n+1}(t-u)]du \qquad (6.15)$$

Finally,

$$\pi(o,t) = \int_0^t M(u)\ \bar{G}(t-u)\ F^{*s}(t-u)du \qquad (6.16)$$

6.5. INVENTORY CARRYING COST AND CUMULATIVE PROFIT

If the cost of carrying is constant and depends only on the amount of commodity in the inventory, it can be taken as unity per unit time, without loss of generality. Then the total inventory carrying cost $C(t)$ during the interval (o,t), where o is our reference point at which the stock level drops to s, is given by

$$C(t) = \int_0^t I(u)du$$

Taking expectations on both sides we get,

$$E\,[C(t)] = \sum_{n=1}^{S} n \int_{o}^{t} \pi(n,u)\,du \tag{6.17}$$

where $\pi(n,u)$ is explicitly given by (6.9)-(6.16).

Let α be the profit that is realized in the sale of each item and we denote by $P(t)$ the cumulative profit due to sales in the interval (o,t), where o is our reference point coinciding with an order point. We now proceed to obtain explicit expression for the expected value of $P(t)$.

Now we define $\psi(n,t)$, for $s \leq n \leq S-1$, as

$\psi(n,t)$ = Probability that due to a demand in $(t,t+dt)$, inventory level drops to n and no ordering takes place in the interval $(o,t]$, given that an order was placed at time zero.

Considering all the different cases, we obtain the following equations.

For $S-s \leq n \leq S-1$, we have

$$\psi(n,t) = \sum_{i=1}^{S-n} f^{*i}(t)\; p_{n-s+i}\, G(t) \tag{6.18}$$

For $s+1 \leq n \leq S-s-1$, we have

$$\psi(n,t) = \int_0^t \sum_{j=1}^{s-1} \sum_{k=1}^{S-n-j} f^{*j}(u)\, G(u)\, p_{n+j+k-s} f^{*k}(t-u)\, du$$

$$+ \int_0^t \sum_{m=0}^{\infty} f^{*m+s}(u) \int_u^t \sum_{k=1}^{S-s-n} G(v-u)\, p_{n+k}$$

$$\times \quad f^{*k}(t-v)\, dv\, du \qquad (6.19)$$

Also

$$\psi(s,t) = {}_sf_s(t) \qquad (6.20)$$

For $o \leq n < s$, we have

$$\psi(n,t) = f^{*s-n}(t)\, \bar{G}(t) \qquad (6.21)$$

Now we define $\rho(n,t)$ as the probability that there occurs a demand during the interval $(t,t+dt)$ by which the stock level drops to n conditional upon the stock level has dropped to s in an arbitrarily small interval preceeding time zero.

Then $\rho(n,t)$ can be written as

$$\rho(n,t) = \psi(n,t) + \int_0^t R(u)\, \psi(n,t-u)\, du \qquad (6.22)$$

where $\psi(n,t)$ is given by (6.18)-(6.21) and $R(u)$ is given by (6.7).

Now we can find out the expected value of the cumulative profit due to sales in the interval (o,t), with o chosen to coincide with an ordering point, as

$$E[P(t)] = \alpha \sum_{n=o}^{S-1} \int_{o}^{t} \rho(n,u)\,du \qquad (6.23)$$

Thus the probability mass function $\pi(n,t)$ and the coincidence function $\rho(n,t)$ enables us to determine the expected value of the inventory carrying cost and the cumulative profit due to sales over any period of time. The cumulative cost of reorders can also be dealt with directly by evaluating the expected value of the number of reorders.

6.6 THE MODEL WITH ZERO LEAD TIME

As a particular case, if we assume that the lead time is zero in the above model, then $\{I(t),\ t \geq o\}$ will be a discrete valued continuous parameter stochastic process taking values s+1, ..., S. Here the sequence of random variables $\{Y_k\}$, k = 1,2, ... forms a delayed renewal process in which the distribution function of Y_1 is given by

$$\Pr\{Y_1 \leq y\} = \int_{o}^{y} f^{*S-s}(u)\,du$$

and the distribution function of Y_k for $k = 2,3, \ldots,$ is given by

$$\Pr\{Y_k \le y\} = \int_0^y \sum_{i=s+1}^{S-s} f^{*i}(u)\, p_i \, du$$

Let its density function be denoted by h(.). The probability that the kth order, k=1,2, ..., will be placed in the interval (t,t+dt) is

$$\Pr\{t < Y_1+Y_2+ \ldots +Y_k \le t+dt\}$$

$$= (f^{*S-s} * h^{*k-1})(t), \quad k=1,2, \ldots$$

Then the renewal density of the delayed renewal process is given by

$$M(u) = \int_0^u f^{*S-s}(v) \sum_{k=1}^{\infty} h^{*k-1}(u-v)\, dv \qquad (6.24)$$

Now considering all the different cases, we can get the probability mass function of the inventory level I(t) as follows.

$$\pi(S,t) = \bar{F}(t) + \int_0^t M(u)\bar{F}(t-u)\, p_{S-s}\, du \qquad (6.25)$$

For $2s+1 < n < S$, we have

$$\pi(n,t) = [F^{*S-n}(t) - F^{*S-n+1}(t)]$$

$$+ \int_0^t M(u) \sum_{j=0}^{S-n} p_{n+j-s}[F^{*j}(t-u) - F^{*j+1}(t-u)]du$$

(6.26)

For $s < n \leq 2s+1$, we have

$$\pi(n,t) = [F^{*S-n}(t) - F^{*S-n+1}(t)]$$

$$+ \int_0^t M(u) \sum_{j=2s+1-n}^{S-n} p_{n+j-s}$$

$$\times [F^{*j}(t-u) - F^{*j+1}(t-u)]du \qquad (6.27)$$

Now we can find out the stationary distribution of the inventory level probabilities using Laplace transforms.

Let $\hat{\pi}(n,z) = \int_0^\infty e^{-zt} \pi(n,t)\,dt$

$$\hat{f}(z) = \int_0^\infty e^{-zt} f(t)\,dt$$

$$\hat{h}(z) = \int_0^\infty e^{-zt} h(t)dt$$

Then

$$\hat{M}(z) = \int_0^\infty e^{-zt} M(t)dt$$

$$= \frac{1}{z} [\hat{f}(z)]^{S-s} [1-\hat{h}(z)]^{-1} \qquad (6.28)$$

Then by taking Laplace transforms on both sides of equations (6.25)-(6.27) we get

$$\hat{\pi}(S,z) = \frac{1}{z} \left\{[1-\hat{f}(z)] + p_{S-s}[1-\hat{f}(z)][\hat{f}(z)]^{S-s}[1-\hat{h}(z)]^{-1}\right\} \qquad (6.29)$$

For $n = 2s+2, \ldots, S-1$, we have

$$\hat{\pi}(n,z) = \frac{1}{z} \left\{[\hat{f}(z)]^{S-n}[1-\hat{f}(z)] + \sum_{j=0}^{S-n} p_{n+j-s}[\hat{f}(z)]^{S-s+j} \times [1-\hat{f}(z)] [1-\hat{h}(z)]^{-1}\right\} \qquad (6.30)$$

For $n = s+1, \ldots, 2s+1$, we have

$$\hat{\pi}(n,z) = \frac{1}{z} \left\{[\hat{f}(z)]^{S-n} [1-\hat{f}(z)] + \sum_{j=2s+1-n}^{S-n} p_{n+j-s}[\hat{f}(z)]^{S-s+j} [1-\hat{f}(z)][1-\hat{h}(z)]^{-1}\right\} \qquad (6.31)$$

Let P_n be the probability that exactly n units, n=s+1,...S are in the inventory in the steady state.

Then

$$P_n = \Pr\{I=n\} = \lim_{t\to\infty} \pi(n,t)$$

$$= \lim_{z\to o} z\,\hat{\pi}(n,z)$$

Thus

$$P_S = \lim_{z\to o} [1-\hat{f}(z)] + \lim_{z\to o} p_{S-s}[\hat{f}(z)]^{S-s} \frac{1-\hat{f}(z)}{1-\hat{h}(z)}$$

$$= \lim_{z\to o} p_{S-s}[\hat{f}(z)]^{S-s} \frac{\hat{f}'(z)}{\hat{h}'(z)} \text{ [Using L'Hospital's rule]}$$

$$= p_{S-s} \frac{EX}{EY} \qquad (6.32)$$

where X is the random variable denoting interoccurrence time of successive demands and Y the random variable denoting time between successive orders.

For n=2s+2, ..., S-1, we have

$$P_n = \lim_{z\to o}[\hat{f}(z)]^{S-n}[1-\hat{f}(z)] + \lim_{z\to o} \sum_{j=o}^{S-n} p_{n+j-s}[\hat{f}(z)]^{S-s+j} \frac{1-\hat{f}(z)}{1-\hat{h}(z)}$$

$$= \lim_{z\to o} \sum_{j=o}^{S-n} p_{n+j-s}[\hat{f}(z)]^{S-s+j} \frac{\hat{f}'(z)}{\hat{h}'(z)} \text{[Using L'Hospital's rule]}$$

$$= \sum_{j=o}^{S-n} p_{n+j-s} \frac{EX}{EY}$$

$$= \frac{EX}{EY} \sum_{j=n-s}^{S-s} p_j \tag{6.33}$$

For n = s+1, ..., 2s+1, we have

$$P_n = \lim_{z\to o} [\hat{f}(z)]^{S-n} [1-\hat{f}(z)] + \lim_{z\to o} \sum_{j=2s+1-n}^{S-n} p_{n+j-s}$$

$$\times [\hat{f}(z)]^{S-s+j} \frac{1-\hat{f}(z)}{1-\hat{h}(z)}$$

$$= \sum_{j=2s+1-n}^{S-n} p_{n+j-s} \frac{EX}{EY}$$

$$= \frac{EX}{EY} \sum_{j=s+1}^{S-s} p_j$$

$$= \frac{EX}{EY} \tag{6.34}$$

Thus the steady state probabilities are given as follows.

For $n = 2s+2, \ldots, S$, we have

$$p_n = \frac{EX}{EY} \sum_{j=n-s}^{S-s} p_j \tag{6.35}$$

and for $n = s+1, \ldots, 2s+1$, we have

$$p_n = \frac{EX}{EY} \tag{6.36}$$

In this case, if we take $p_{S-s}=1$ and all other p_j's equal to zero, we will get the results of Sivazlian(1974).

6.7 AN OPTIMIZATION PROBLEM

Our objective function is the steady state total expected cost per unit time. We have to choose the decision variables s and $S-s = Q$ so as to minimize the objective function.

Let R denote the expected quantity replenished

i.e. $$R = \sum_{n=s+1}^{S-s} n\, p_n$$

Also $$R = \frac{EY}{EX}$$

Let $\bar{D}$ denote the expected number of demands per unit time.

Then

$$\bar{D} = \frac{\text{Expected quantity replenished}}{\text{Expected time elapsed between successive orders}}$$

$$= \frac{R}{E\,Y}$$

Therefore

$$EY = \frac{R}{\bar{D}}$$

Therefore the expected number of orders placed per unit time is

$$\frac{1}{E\,Y} = \frac{\bar{D}}{R} \qquad (6.37)$$

Expected inventory level at any instant of time is

$$E[I] = \sum_{n=s+1}^{s+Q} n\, P_n$$

$$= s + \sum_{n=1}^{Q} n\, P_{s+n}$$

$$= s + \sum_{n=1}^{Q} n \sum_{j=n}^{Q} \frac{1}{R}\, p_j$$

$$= s + \frac{1}{R} \sum_{n=1}^{Q} \sum_{j=n}^{Q} n\, p_j \qquad (6.38)$$

The total expected cost per unit time is given by

$$F(s,Q) = \frac{K + cR}{EY} + h\, E[I]$$

where K is the fixed order cost, c is the variable procurement cost per unit and h is the holding cost per unit per time.

Using equations (6.37) and (6.38) we get

$$F(s,Q) = \frac{K\bar{D}}{R} + c\bar{D} + hs + \frac{h}{R} \sum_{n=1}^{Q} \sum_{j=n}^{Q} n\, p_j \qquad (6.39)$$

where s is a nonnegative integer and Q is a positive integer. Equation (6.39) gives the objective function. Since the lead time is zero, it is very easy to see that the optimum value of s is $s^* = o$. Then the optimum value of Q is obtained by minimizing the function

$$F(Q) = \frac{K\bar{D}}{R} + c\bar{D} + \frac{h}{R} \sum_{n=1}^{Q} \sum_{j=n}^{Q} n\, p_j \qquad (6.40)$$

over the set of positive integers.

Since R is a function of Q, the first and last terms are functions of Q and we can show that there exists optimum values of Q that minimizes the cost function F(Q). We will explain it by considering the special case that all the p_j's are equal. When all the p_j's are equal and equal to p, we get

$$p = \frac{1}{Q}$$

Then the expected quantity replenished is given by

$$R = \sum_{i=1}^{Q} i \cdot \frac{1}{Q} = \frac{1}{Q} \frac{Q(Q+1)}{2}$$

$$= \frac{Q+1}{2}$$

Therefore

$$F(Q) = \frac{2K\bar{D}}{Q+1} + c\bar{D} + \frac{2h}{Q+1} \sum_{n=1}^{Q} \sum_{j=n}^{Q} n \cdot \frac{1}{Q}$$

$$= \frac{2K\bar{D}}{Q+1} + c\bar{D} + \frac{2h}{Q(Q+1)} \sum_{n=1}^{Q} n(Q-n+1)$$

$$= \frac{2K\bar{D}}{Q+1} + c\bar{D} + \frac{h}{3}(Q+2) \qquad (6.41)$$

It is easy to see that this function is convex and the optimum value of Q, Q^* exists which minimizes (6.41).

Then Q^* is approximately given by the integer nearest to $\sqrt{\frac{6K\bar{D}}{h}} - 1$.

That is

$$Q^* \cong \sqrt{\frac{6\ K\bar{D}}{h}} - 1 \qquad (6.42)$$

To illustrate this, we give a numerical example, by assuming different values for h, K and $\bar{D}$, in table 1.

Instead of considering this, if we are following the same procedure as that of Sivazlian (1974), we will get

$$Q^*(Q^*+1) \leq \frac{6K\bar{D}}{h} \leq (Q^*+1)(Q^*+2) \qquad (6.43)$$

In table 2 we give the numerical values obtained for this case.

If we take $\frac{6K\bar{D}}{h} = 80$, then $Q^* = 8$.

If $\frac{6K\bar{D}}{h} = 42$, we get $Q^* = 5$ or 6 and table 1 suggests that Q^* could be taken as 5.

Table 1: Optimum values of Q for different values of h, K and $\bar{D}$

$\bar{D}$	Q^*								
	K=10			K=15			K=20		
	h=0.2	h=0.5	h=1.0	h=0.2	h=0.5	h=1.0	h=0.2	h=0.5	h=1.0
0.2	7	4	2	8	5	3	10	6	4
0.4	10	6	4	12	7	5	14	9	6
0.6	12	7	5	15	9	6	18	11	7
0.8	14	9	6	18	11	7	21	13	9
1.0	16	10	7	20	12	8	23	14	10
1.2	18	11	7	22	14	9	26	16	11
1.4	19	12	8	24	15	10	28	17	12
1.6	20	13	9	26	16	11	30	19	13
1.8	22	14	9	27	17	12	32	20	14
2.0	23	14	10	29	18	12	34	21	14

Table 2

Q^*	0	1	2	3	4	5	6	7	8	9	10	...
$Q^*(Q^*+1)$	0	2	6	12	20	30	42	56	72	90	110	...
$(Q^*+1)(Q^*+2)$	2	6	12	20	30	42	56	72	90	110	132	...

Remark:

The model analysed in Section 6.3 can be extended to allow vacation to the server whenever the system becomes empty. In that case also, one can write explicit expressions for the inventory level probabilities at arbitrary epochs, but the optimization part seems difficult.

Chapter 7

AN (s,S) INVENTORY SYSTEM WITH BULK DEMANDS AND VACATIONS TO THE SERVER

7.1. INTRODUCTION

In this chapter we consider a continuous review (s,S) inventory system with random lead times. We assume that the time between demand points and quantity demanded at these points are independent sequences of independent and identically distributed random variables. The lead times are generally distributed random variables depending upon the size of the order. Whenever the inventory becomes dry, server goes for vacation of random duration having a general distribution. No backlogging of demands is allowed and all demands occurring when the server is on vacation are lost.

The probabilistic analysis of (s,S) inventory system using renewal theoretic arguments is considered by several authors. For instance, Arrow, Karlin and Scarf(1958) and Tijms(1972) contain detailed treatment of these models. Srinivasan (1979) considers the case of unit demands and random lead times. The case of bulk demands and constant lead times is considered by Sahin (1979). Gaver (1959) analyses the case of compound Poisson demand and random lead times. Ramanarayanan and Jacob (1987) analyse the case of general bulk demands and random lead times.

The effect of vacation periods in inventory theory is first analysed by Daniel and Ramanarayanan(1987 a,b). Usha, Ramanarayanan and Jacob (1987) consider the case of unit demand, finite backlog of demands, random lead times and vacations to the server. Here we consider the case of bulk demands, random lead times and vacations to the server and derive the time dependent solution of the inventory level at an arbitrary time point t.

7.2. ASSUMPTIONS OF THE MODEL

1. The maximum capacity of the inventory is S.
2. An order is placed to fill the inventory whenever the inventory level falls to or below s.
3. The inter-occurrence time of demands are independent and identically distributed random variables with distribution function F(.) and density function f(.).
4. The quantity demanded at each demand point is a random variable which can vary from 1 to b where $s < b < S-2s-1$ with probability p_k for demanding k units, such that $\sum_{k=1}^{b} p_k=1$.

5. The lead times are independent random variables depending on the size of the order. If an order is placed at level i, $o \leq i \leq s$, the lead time has distribution function $G_i(.)$ and density function $g_i(.)$.

6. Whenever the inventory becomes dry, server goes for a vacation of random duration having distribution function H(.) and density function h(.).

7. The inter-occurrence time of demands, quantities demanded, lead times and vacation periods are all independent.

8. Arriving demands are lost during the inventory dry period and the vacation period of the server.

<u>Notations</u>

$\varphi(z)$	$\sum_{k=1}^{b} p_k z^k$
p_k^n	coefficient of z^k in $[\varphi(z)]^n$
$q(v)$	$\sum_{n=1}^{\infty} f^{*n}(v)$

$\pi_i(t)$ The conditional probability that the inventory level is i at time t, given that at time zero the inventory level was S and the demand process starts. $(i = 0,1,2, \ldots, S)$.

$h_{i,j}(x)dx$ The conditional probability that due to some demands, inventory level drops to j in the interval $(x,x+dx)$ given that inventory level was i at time zero and there is no replenishment during the interval $(o,x]$ $(i > j \geq 0)$.

Now consider the time points at which the first demand after each order realization occurs and look at the inventory level at these points. S is the level at time zero and if $\eta(S-s-b \leq \eta \leq S-1)$ is the level after the first transition, $f_{S,\eta}(x)$ denotes the probability density function of the transition time. Similarly if η is the level at one such time point and γ is the level at the next such time point, $f_{\eta,\gamma}(x)(S-s-b \leq \eta, \gamma \leq S-1)$ denotes this transition time probability density function. Transitions can occur with a vacation period or without a vacation period during lead time. Let ${}_1f_{i,j}(x)$ denote the transition time probability density function with a vacation period and let ${}_2f_{i,j}(x)$ denote the transition time probability density function without a vacation period.

Let

$$\underline{f}_S(x) = (f_{S,S-s-b}(x), f_{S,S-s-b+1}(x), \ldots, f_{S,S-1}(x))$$

It is a vector of order b+s.

Now we define a square matrix $\mathbf{F}$ of order (b+s) given by

$$\mathbf{F}(x) = \begin{bmatrix} f_{S-s-b,S-s-b}(x) & \cdots & f_{S-s-b,S-1}(x) \\ \vdots & & \\ f_{S-s,S-s-b}(x) & \cdots & f_{S-1,S-1}(x) \end{bmatrix}$$

Then $(\underline{f}_S * \sum_{n=0}^{\infty} \mathbf{F}^{*n})(x)$ will be a vector of order b+s.

For η = S-s-b, ..., S-1, let

$$F_\eta(x) = (\eta - S+s+b+1)\text{th coordinate of}$$

$$(\underline{f}_S * \sum_{n=0}^{\infty} \mathbf{F}^{*n})(x) \qquad (7.1)$$

7.3. TRANSITION TIME PROBABILITIES

First we find out the expressions for $h_{i,j}$'s.

For i = s+1, ..., S and j = 1,2, ...,i, we have

$$h_{i,j}(x) = \sum_{n=1}^{\infty} \sum_{k=s+1}^{s+b} \int_{o}^{x} f^{*n-1}(u) P_{i-k}^{n-1} f(x-u) p_{k-j} \, du \quad (7.2)$$

For i = s+1, ..., S, we have

$$h_{i,o}(x) = \sum_{n=1}^{\infty} \sum_{k=s+1}^{s+b} \int_{o}^{x} f^{*n-1}(u) P_{i-k}^{n-1} f(x-u) \sum_{j=k}^{b} p_j \, du \quad (7.3)$$

For i = 1,2, ...,s and j = 1,2, ...,i, we have

$$h_{i,j}(x) = \sum_{n=1}^{\infty} \sum_{k=s+1}^{s+b} \int_{o}^{x} f^{*n-1}(u) P_{i-k}^{n-1} f(x-u) p_{k-j} \, du \quad (7.4)$$

and for i = 1,2, ..., s, we have

$$h_{i,o}(x) = \sum_{n=1}^{\infty} \sum_{k=1}^{b} \int_{o}^{x} f^{*n-1}(u) P_{i-k}^{n-1} f(x-u) \sum_{j=k}^{b} p_j \, du \quad (7.5)$$

Now the following relations for transition time probabilities can be easily obtained.

For $S-s-b \leq \eta \leq S-1$, we have

$$_1f_{S,\eta}(x) = \int_0^x \sum_{i=o}^{s} h_{S,i}(u) \int_u^x h_{i,o}(v-u) \int_v^x q(w-v)$$

$$\times [H(x-w)-H(w)] [G_i(x-w)-G_i(v)]$$

$$\times f(x-w) p_{S-i-\eta} \, dw \, dv \, du \qquad (7.6)$$

For $S-b \leq \eta \leq S-1$, we have

$$_2f_{S,\eta}(x) = \int_0^x \sum_{i=o}^{s} h_{S,i}(u) \int_u^x \sum_{j=1}^{i} h_{i,j}(v-u) G_i(x-v)$$

$$\times f(x-v) \, p_{S-i+j-\eta} \, dv \, du \qquad (7.7)$$

Now for η satisfying $S-b \leq \eta \leq S-1$, we have

$$f_{S,\eta}(x) = {_1f_{S,\eta}(x)} + {_2f_{S,\eta}(x)} \qquad (7.8)$$

and for $S-s-b \leq \eta \leq S-b-1$, we have

$$f_{S,\eta}(x) = {_1f_{S,\eta}(x)} \qquad (7.9)$$

Now for $S-s-b \leq \eta$, $\gamma \leq S-1$, we have

$$ {}_1f_{\eta,\gamma}(x) = \int_0^x \sum_{i=o}^{s} h_{\eta,i}(u) \int_u^x h_{i,o}(v-u) \int_v^x q(w-v) $$

$$ \times [H(x-w)-H(w)] [G_i(x-w)-G_i(v)] f(x-w) $$

$$ \times p_{S-i-\gamma} \, dw \, dv \, du \tag{7.10} $$

For $S-s-b \leq \eta \leq S-1$ and $S-b \leq \gamma \leq S-1$ we have

$$ {}_2f_{\eta,\gamma}(x) = \int_0^x \sum_{i=1}^{s} h_{\eta,i}(u) \int_u^x \sum_{j=1}^{i} h_{i,j}(v-u) G_i(x-v) $$

$$ \times f(x-v) \, p_{S-i+j-\gamma} \, dv \, du \tag{7.11} $$

Now for $S-s-b \leq \eta \leq S-1$ and $S-b \leq \gamma \leq S-1$, we have

$$ f_{\eta,\gamma}(x) = {}_1f_{\eta,\gamma}(x) + {}_2f_{\eta,\gamma}(x) \tag{7.12} $$

and for $S-s-b \leq \eta \leq S-1$ and $S-s-b \leq \gamma \leq S-b-1$, we have

$$ f_{\eta,\gamma}(x) = {}_1f_{\eta,\gamma}(x) \tag{7.13} $$

From equations (7.8) and (7.9) we get all the entries of vector $\underline{f}_S$ and from equations (7.12) and (7.13), we get all the entries of the matrix $\mathbf{F}$. Now we derive the expressions for inventory level probabilities using the above equations.

7.4. INVENTORY LEVEL PROBABILITIES

Considering all the mutually exclusive and exhaustive cases, we can derive the following expressions for the inventory level probabilities.

$$\pi_S(t) = \bar{F}(t) + \sum_{i=1}^{s} \int_o^t h_{S,i}(u)\ \bar{F}(t-u)\ G_i(t-u)du$$

$$+ \int_o^t h_{S,o}(u) \int_u^t q(v-u)[H(t-u)-H(v)]G_o(t-v)\bar{F}(t-v)dv\ du$$

$$+ \int_o^t h_{S,o}(u) \int_u^t q(v-u)\bar{H}(t-v)\ G_o(t-v)dv\ du$$

$$+ \sum_{\eta=S-s-b}^{S-1} \sum_{i=1}^{s} \int_o^t F_\eta(u) \int_u^t h_{\eta,i}(v-u)G_i(t-v)\bar{F}(t-v)dv\ du$$

$$+ \sum_{\eta=S-s-b}^{S-1} \int_o^t F_\eta(u) \int_u^t h_{\eta,o}(v-u) \int_v^t q(w-v)[H(t-v)-H(w)]$$

$$\times\ G_o(t-v)\ \bar{F}(t-w)dw\ dv\ du$$

$$+ \sum_{\eta=S-s-b}^{S-1} \int_0^t F_\eta(u) \int_u^t h_{\eta,o}(v-u) \int_v^t q(w-v)\overline{H}(t-w)$$

$$\times\ G_o(t-v)\,dw\ dv\ du \qquad (7.14)$$

For $1 \leq i \leq s$, we have

$$\pi_{S-i}(t) = \int_0^t h_{S,S-i}(u)\ \overline{F}(t-u)\,du$$

$$+ \int_0^t F_{S-i}(u)\ \overline{F}(t-u)\,du$$

$$+ \int_0^t h_{S,i}(u) \int_u^t h_{i,o}(v-u) \int_v^t q(w-v)\,[H(t-v)-H(w)]$$

$$\times\ G_i(t-v)\ \overline{F}(t-w)\,dw\ dv\ du$$

$$+ \int_0^t h_{S,i}(u) \int_u^t h_{i,o}(v-u) \int_v^t q(w-v)\overline{H}(t-w)\,G_i(t-v)\,dw\ dv\ du$$

$$+ \sum_{\eta=S-i+1}^{S-1} \int_0^t F_\eta(u) \int_u^t h_{\eta,S-i}(v-u)\overline{F}(t-v)\,dv\ du$$

$$+ \sum_{\eta=S-s-b}^{S-1} \int_0^t F_\eta(u) \int_u^t h_{\eta,i}(v-u) \int_v^t h_{i,o}(w-v) \int_w^t q(y-w)$$

$$\times\ [H(t-w)-H(y)]\ G_i(t-w)\ \overline{F}(t-y)\,dy\ dw\ dv\ du$$

$$+\sum_{\eta=S-s-b}^{S-1} \int_0^t F_\eta(u) \int_u^t h_{\eta,i}(v-u) \int_v^t h_{i,o}(w-v)$$

$$x \int_w^t q(y-w)\ G_i(t-w)\overline{H}(t-y)dy\ dw\ dv\ du \qquad (7.15)$$

For $S-s-b \leq i \leq S-s-1$, we have

$$\pi_i(t) = \int_0^t h_{S,i}(u)\ \overline{F}(t-u)du$$

$$+\int_0^t F_i(u)\ \overline{F}(t-u)du$$

$$+\sum_{\eta=i+1}^{S-1} \int_0^t F_\eta(u) \int_u^t h_{\eta,i}(v-u)\ \overline{F}(t\ \ v)dv\ du \qquad (7.16)$$

For $s+1 \leq i \leq S-s-b-1$, we have

$$\pi_i(t) = \int_0^t h_{S,i}(u)\ \overline{F}(t-u)du$$

$$+\sum_{\eta=i}^{S-1} \int_0^t F_\eta(u) \int_u^t h_{\eta,i}(v-u)\ \overline{F}(t-v)dv\ du \qquad (7.17)$$

Also

$$\pi_s(t) = \int_0^t h_{S,s}(u)\, \bar{F}(t-u)\, \bar{G}_s(t-u)\,du$$

$$+ \sum_{\eta=S-s-b}^{S-1} \int_0^t F_\eta(u) \int_u^t h_{\eta,s}(v-u)\bar{F}(t-v)\bar{G}_s(t-v)\,dv\,du \tag{7.18}$$

For $1 \leq i \leq s-1$, we have

$$\pi_i(t) = \sum_{j=i}^{s} \int_0^t h_{S,j}(u) \int_u^t h_{j,i}(v-u)\bar{F}(t-v)\bar{G}_j(t-v)\,dv\,du$$

$$+ \sum_{\eta=S-s-b}^{S-1} \sum_{j=i}^{s} \int_0^t F_\eta(u) \int_u^t h_{\eta,j}(v-u) \int_v^t h_{j,i}(w-v)$$

$$\times\ \bar{F}(t-w)\ \bar{G}_j(t-w)\,dw\,dv\,du \tag{7.19}$$

Finally,

$$\pi_o(t) = \int_o^t h_{S,o}(u) \int_u^t q(v-u)\ \bar{G}_o(t-v)\,dv\,du$$

$$+ \sum_{i=1}^{s} \int_o^t h_{S,i}(u) \int_u^t h_{i,o}(v-u) \int_v^t q(w-v)\bar{G}_i(t-w)\,dw\,dv\,du$$

$$+ \sum_{\eta=S-s-b}^{S-1} \int_o^t F_\eta(u) \int_u^t h_{\eta,o}(v-u) \int_v^t q(w-v)$$

$$\times\ \bar{G}_o(t-w)\,dw\,dv\,du$$

$$+ \sum_{\eta=S-s-b}^{S-1} \sum_{i=1}^{s} \int_{o}^{t} F_{\eta}(u) \int_{u}^{t} h_{\eta,i}(v-u)$$

$$x \int_{v}^{t} h_{i,o}(w-v) \int_{w}^{t} q(y-w)\overline{G}_i(t-y)dy\ dw\ dv\ du \quad (7.20)$$

7.5 CONCLUDING REMARKS

We have studied in this thesis, the time dependent behaviour of some queueing models of the M/G/1 type with bulk arrival, bulk service and vacations to the server. Also we have studied the transient behaviour of inventory models with random replenishment, bulk demands and vacations to the server. The expressions derived are explicit but complicated and not easily yielding for practical purposes.

The most important problem one can think of is to develop an algorithm to solve the given results numerically. To the application point of view, this is quite worthwhile work. For developing the algorithm, possibly one can effectively use some fast transform techniques (see, Elliott and Rao (1982)), because here we cannot use the usual procedure of Laplace transforms.

It should be noted that the bulk service queueing systems we have solved include the so called general case of Erlang input as special case. The $E_k/G/1$ queueing system can be identified as an $M/G^k/1$ model as it is done by Takacs (1961). Similarly, other Erlang input models with and without vacations to the server can be solved by imbedding it in some M/G/1 bulk service queue.

In vacation models, one important result is the stochastic decomposition property of the system size or waiting time. One can think of extending this to the transient case. For the decomposition property of the waiting time, one can use the distribution of the virtual waiting time, since it can be defined as the unfinished work (see Kleinrock (1975)).

One can use our method of matrix convolutions to derive the time dependent solution for the GI/M/1 models also. Here the regeneration points will be the arrival points. Another problem one can think of is the cyclic service M/G/1 queue. Suppose there are n queues attended by a single server in a cyclic order. For a particular queue, the time server attending other queues will be equivalent to a vacation. In this case, the vacations need not be identically distributed. Here also one can derive the time dependent solution by using the method we have used in this thesis. Variations with gated service etc. can also be looked into.

Also one can study the case of $M/G/\infty$ queueing system. Suppose in an inventory system with Poisson demand and random lead times, order is placed whenever the level becomes s and full backlogging is allowed. Then the number of outstanding orders at a particular time will be equivalent to the number of servers busy in the $M/G/\infty$ queueing system. Solving other multi-server queues will also be an interesting problem.

In inventory theory, one can study the case of multi-item, multi-echelon problems, using our method. Also one can study the problem with perishable commodities. Inventory system with perishable commodities is equivalent to a queueing system with impatient customers (see, Kaspi and Perry (1983, 1984)). So we can study M/G/1 queues with impatient customers and use the results in inventory theory.

Consider an inventory system having two types of demands with one having priority over the other. Before the inventory level attains a certain level, both types of demands are met. After attaining this level, the higher priority demands are met and the other type of demands backlogged. For this model, one can derive the inventory level probabilities and this can be extended to the case of more than two types of demands.

The technique used to derive the time dependent solution of queueing system may be of special interest to any stochastic system having regenerative structure. For example, reliability theory is one such area, where we can effectively use this method to derive the time dependent solution.

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