STUDIES ON TOPOLOGY AND ITS APPLICATIONS

SOME PROBLEMS IN SET TOPOLOGY RELATING GROUP OF HOMEOMORPHISMS AND ORDER

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DECLARATION

This thesis contains no material which has been accepted for the award of any other Degree or Diploma in any University and, to the best of my knowledge and belief, it contains no material previously published by any other person, except where due reference is made in the text of the thesis.

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CERTIFICATE

Certified that the work reported in the present thesis is based on the bona fide work done by Sri. P.T. Ramachandran, under my guidance in the Department of Mathematics and Statistics, University of Cochin, and has not been included in any other thesis submitted previously for the award of any degree.

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INTRODUCTION

In this thesis we investigate some problems in set theoretical topology related to the concepts of the group of homeomorphisms and order. Many problems considered are directly or indirectly related to the concept of the group of homeomorphisms of a topological space onto itself. Order theoretic methods are used extensively.

Chapter-1 deals with the group of homeomorphisms. This concept has been investigated by several authors for many years from different angles. It was observed that non-homeomorphic topological spaces can have isomorphic groups of homeomorphisms. Many problems relating the topological properties of a space and the algebraic properties of its group of homeomorphisms were investigated. The group of isomorphisms of several algebraic, geometric, order theoretic and topological structures had also been investigated. A related concept of the semigroup of continuous functions of a topological space also received attention.

J. DEGROOT [14] proved that any group is isomorphic to the group of homeomorphisms of a topological space.

A related, although possibly more difficult, problem is to determine the subgroups of the group of permutations of a fixed set X, which can be represented as the group of homeomorphisms of a topological space (X,T) for some topology T on X. This problem appears to have not been investigated so far.

In Chapter-1 we discuss some results along this direction. These include the result that no nontrivial proper normal subgroup of the group of permutations of a fixed set X can be represented as the group of homeomorphisms of a topological space (X,T) for some topology T on X.

Homogeneity and rigidity are two topological properties closely related to the group of homeomorphisms. In 1979 PAUL BANKSTON [4] defined an anti-property for any topological property and discussed the anti-properties of many topological properties like compactness, Lindelöfness, sequential compactness and others related to compactness. Later I.L. REILLY and M.K. VAMANAMOORTHY [29] obtained the anti-properties corresponding to several separation axioms and compactness properties. D.B. GAULD, I.L. REILLY and M.K. VAMANAMOORTHY [11] proved that there is only one non-trivial anti-normal space. Rigid spaces are investigated by several authors like J. DE GROOT [14], V. KANNAN and M. RAJAGOPALAN ([21], [22], [23]).

In Chapter-2, we investigate anti-homogeneous spaces and give several characterizations for them. In particular we prove that a space is anti-homogeneous if and only if it is hereditarily rigid. To prove these results, we use the concept of a pre-order (a reflexive, transitive relation) associated with a topology. This association was studied earlier by A.K. STEINER [35], FRANCOIS LORRAIN [27], and SUSAN J. ANDIMA and W.J.THRON[1]. We discuss the concepts of homogeneity, anti-homogeneity and rigidity for pre-ordered sets also. It is then proved that a topological space is anti-homogeneous if and only if the associated pre-ordered set is anti-homogeneous. The main order theoretic tool used is a structure theorem for semi-well ordered sets (linearly ordered sets in which every non-empty subset has either a first element or a last element).

Chapter-3 deals with the Čech closure spaces, which is a generalization of the concept of topological spaces. EDUARD ČECH, J. NOVAK, R. FRIC and many others have earlier studied this concept and many topological concepts were extended to the Čech closure spaces. In this chapter we try to extend some results discussed for topological spaces in the earlier chapters to Čech closure spaces. These include the characterization of completely homogeneous spaces and many results related to the preorder associated with a topology.

It is well known that the set of all topologies on a fixed set forms a complete lattice with the natural order of set inclusion (see [6]) and that this lattice is not distributive in general (see [38]). In 1966, A.K. STEINER [33] showed that the lattice of topologies on a set with more than two elements is not even modular. The lattice of topologies is atomic. In 1964, OTTO FRÖLICH [9] determined the dual atoms of this lattice and proved that it is also dually atomic.

In 1958, JURIS HARTMANIS [18] proved that the lattice of topologies on a finite set is complemented and raised the question about the complementation in the lattice of topologies on an arbitrary set. H. GAIFMAN[10] proved that the lattice of topologies on a countable set is complemented. Finally in 1966, A.K. STEINER [33] proved that the lattice of topologies on an arbitrary set is complemented. VAN ROOIJ [39] gave a simpler proof independently in 1968. HARTMANIS noted that even in the lattice of topologies on a set with three elements, only the least and the greatest element have unique complements. PAUL S. SCHNARE [30] proved that every element in the lattice of topologies on a set X, except the least and the greatest elements have at least n-1 complements when X is finite such that $|X| = n \geq 2$ and have infinitely many complements when X is infinite.

In Chapter-4 we conduct an analogous investigation of the lattice of closure operators on a fixed set X with special attention to complementation. The atoms and the dual atoms of the lattice are determined first. The complementation problem is solved in the negative using this. The lattice is dually atomic but not atomic when X is infinite. It is then proved that no element in it has more than one complement. Finally, some sublattices of this lattice and the fixed points of the automorphisms of this lattice are discussed.

CHAPTER-1

GROUP OF HOMEOMORPHISMS

In this chapter we discuss some problems related to the group of homeomorphisms.

1.1 PRELIMINARY RESULTS

Topologies on an arbitrary set with the trivial subgroup as the group of homeomorphisms have been constructed by many authors (See [14], [22]).

The group of homeomorphisms of a discrete or an indiscrete topological space coincides with the group of all permutations.

1.1.1 NOTATIONS

X denotes an arbitrary set

S(X) denotes the group of all permutations of X

We first consider the problem of representing some nontrivial proper subgroups of S(X) as the groups of homeomorphisms of a topological space (X,T) for some topology T on X.

1.1.2 THEOREM

Let H be a subgroup of S(X) containing two elements only. Then there exists a topology T on X such that the group of homeomorphisms of (X,T) is H.

PROOF

Since H contains only two elements. One element is the identity map on X and the other element is a permutation p of X which is of order two in the group $S(X)$. Thus p is a product of disjoint transpositions. Then there exists a class $\{A_i : i \in I\}$ of disjoint subsets of X each containing precisely two elements such that p permutes elements of A_i for every $i \in I$ and keeps all other elements fixed. Let $Y = X \setminus \bigcup_{i \in I} A_i$. Now well order Y and take the topology on it containing precisely Y and its open initial segments. Note that we assume axiom of choice. Also well order I. On $X \setminus Y$ take a topology containing precisely $X \setminus Y$ and sets of the form $B_j = \bigcup_{i < j} A_i$ for some $j \in I$. Now take the topological sum of Y and $X \setminus Y$. This gives a topology T on X such that the group of homeomorphisms of (X,T) is H.

1.1.3 NOTE

A topological space (X,T) is homogeneous if for any x,y in X, there exists a homeomorphism h of (X,T) onto itself such that $h(x) = y$.

J. GINSBURG [12] proved that every finite homogeneous space is a product of a discrete space and an indiscrete space. From this we can easily deduce the following results which will be used later.

(a) If the cardinality of X is prime, any homogeneous topology on X is either discrete or indiscrete.

(b) If (X,T) is a finite homogeneous space, a partition of X forms a base for the topology T.

(c) There exists a transposition of X which is a homeomorphism of (X,T) onto itself whenever $|X| \geqslant 2$.

1.1.4 REMARK

Theorem 1.1.2 proves that every subgroup of S(X) of order 2 can be represented as the group of homeomorphisms of a topological space (X,T) for some topology T on X. But this is not true for subgroups of S(X) of order 3 as shown by the following counter example. Let $X = \{a,b,c\}$ and $H = \{I, (a,b,c), (a,c,b)\}$ where I is the identity map on X. Then there exists no topology T on X such that the group of homeomorphisms of (X,T) is H. For, otherwise (X,T) is homogeneous and by Remark 1.1.3(a), (X,T) is discrete or indiscrete in which case the group of homeomorphisms is S(X) and not H.

Generalizing this observation we can also prove

1.1.5 THEOREM

Let X be a finite set $\{a_1, a_2, \ldots, a_n\}$, $n \geqslant 3$. Let H be the group of permutations of X generated by the cycle $p = (a_1, a_2, \ldots, a_n)$. Then H cannot be represented as the group of homeomorphisms of (X,T) for any topology T on X.

PROOF

Otherwise, let T be a homogeneous topology on X. Then by Remark 1.1.3 (c) there exists a transposition of X which is a homeomorphism of (X,T). But that is not an element of H. This contradiction proves the result.

The group H mentioned in Remark 1.1.4 is in fact the alternating group of permutations of X. Generalizing this we can prove

1.1.6 THEOREM

If X is a finite set such that $|X| \geqslant 3$, then there exists no topology T on X such that the group of homeomorphisms of (X,T) is the alternating group of permutations of X.

PROOF

Otherwise, (X,T) is homogeneous since all 3-cycles are elements of the alternating group of X. Then by Remark 1.1.3 (c), there exists a transposition of X, which is an element of the group of homeomorphisms, which is in fact the alternating group of X. This contradiction proves the result.

1.2 NORMAL SUBGROUPS OF S(X)

When $|X| \leq 2$, the only sub-group of S(X) are the trivial subgroup and itself. And both can be represented as the group of homeomorphisms of a topological space (X,T) for some topology T on X.

When $|X| = 3$, it can be verified that all subgroups of S(X) other than the alternating group can be represented as the group of homeomorphisms of a topological space (X,T) for some topology T on X. The alternating group is the group H mentioned in Remark 1.1.4 and cannot be represented as the group of homeomorphisms of a topological space as explained there.

By theorem 1.1.6, the alternating group of permutations of a set X with three or more elements cannot be represented as the group of homeomorphisms of any topological space (X,T). Recall that the normal subgroups of $S(X)$ for a finite set X such that $|X| = 3$ or $|X| \geqslant 5$ are precisely the trivial subgroup, the alternating group and $S(X)$ itself.

When X is the set $\{a,b,c,d\}$ with four elements, the normal subgroups of $S(X)$ are precisely the trivial subgroup, $\{I,(a,b)(c,d), (a,c)(b,d), (a,d)(b,c)\}$, $A(X)$ the alternating group and $S(X)$ where I is the identity map on X. But it can be verified using Note 1.1.3 that neither $A(X)$ nor $\{I, (a,b)(c,d), (a,c)(b,d), (a,d)(b,c)\}$ can be represented as the group of homeomorphisms of any topological space on X. Thus when X is any finite set, no proper nontrivial normal subgroup of $S(X)$ can be represented as the group of homeomorphisms of any topological space on X.

Our aim is to extend this result to the infinite case.

1.2.1 NOTATIONS

Let p be a permutation of X. Then let

$$M(p) = \{x \in X : p(x) \neq x\}$$

A(X) denotes the group of all permutations p of X such that M(p) is finite and p can be written as a product of an even number of transpositions.

If α is any cardinal number let

$$H_\alpha = \{p \in S(X) : |M(p)| < \alpha\}$$

We use the following lemma proved by BAER [2].

1.2.2 LEMMA

The normal subgroups of the group S(X) of permutations of X are precisely the trivial subgroup, A(X), S(X) and the subgroups of S(X) of the form H_α for some infinite cardinal number α, $\alpha \leq |X|$.

1.2.3 LEMMA

Let X be any infinite set and T any topology on X such that A(X) is a subgroup of the group of homeomorphisms of (X,T). Then

(a) (X,T) is homogeneous

(b) Supersets of nonempty open sets of (X,T) are open

(c) If (X,T) is not indiscrete, every finite subset is closed

(d) If (X,T) is not discrete, intersection of nonempty open sets of (X,T) is nonempty.

(e) If (X,T) is not discrete, no nonempty finite set is open.

PROOF

(a) Let a and b be two distinct points of (X,T). Now choose two more distinct points c and d other than a and b. Consider the permutation $p = (a,b)\ (c,d)$. It is a homeomorphism since it is an element of A(X), and it maps a to b. Hence (X,T) is homogeneous.

(b) Let A be a nonempty open set of (X,T) and $A \subset B$. If $A = X$ or $A = B$, the result is evident. Otherwise choose an element a of A and an element b of $B \setminus A$. Also choose two distinct points c and d other than a and b both from either A or $B \setminus A$. Now consider the permutation $p = (a,b)\ (c,d)$. Then p is a homeomorphism of (X,T) since p is an element of A(X). Then $A \cup \{b\} = A \cup p(A)$ is open. Thus $B = \bigcup_{b \in B \setminus A} (A \cup \{b\})$ and hence is open. Hence the result.

(c) Since (X,T) is not indiscrete, there exists a proper nonempty open set A of (X,T). Let b be an element of $X \setminus A$. Then $X \setminus \{b\}$ is open by (b). Thus $\{b\}$ is closed.

Then every singleton subset of (X,T) is closed, since (X,T) is homogeneous by (a). Hence every finite subset being a finite union of singleton subsets is closed.

(d) Let A and B be two nonempty open subsets of (X,T). To prove that $A \cap B \neq \varphi$. Otherwise choose an element a from A and an element b from B. Then $a \neq b$. Choose two distinct points c and d other than a and b both from either A or B or $X \setminus (A \cup B)$ whichever is infinite. Now consider p = (a,b) (c,d). Then p is a homeomorphism since p is an element of A(X). Thus p(A) is open. Then $\{b\} = p(A) \cap B$ is open. Since (X,T) is homogeneous, every singleton subset is open and hence (X,T) is discrete, which is a contradiction to the hypothesis. Hence the result.

(e) If (X,T) is indiscrete, the result is obvious. Otherwise every finite subset of (X,T) is closed by (c). If a nonempty finite subset F is also open, then both F and $X \setminus F$ are open. This contradicts (d) which proves the result.

1.2.4 REMARK

Lemma 1.2.3 shows that if A(X) is a subgroup of the group of homeomorphisms of a topological space (X,T) which is not discrete, then the nonempty open sets of (X,T) form a filter.

1.2.5 LEMMA

Let (X,T) be an infinite topological space in which nonempty open sets form a filter. Let A be a proper closed subset of (X,T). Then every permutation of X which moves only elements of A is a homeomorphism of (X,T).

PROOF

Let p be a permutation of X which moves only the elements of A. If U is a nonempty open set

$$p(U) \supset U \cap A^c$$

and $U \cap A^c$ is open and nonempty by hypothesis. Thus p is an open map. Similarly we can prove that p is a continuous map. Hence p is a homeomorphism.

1.2.6 LEMMA

Let (X,T) be an infinite topological space which is neither discrete nor indiscrete such that the group of homeomorphisms H of (X,T) is a normal subgroup of S(X) containing A(X). If K is a proper closed subset of (X,T), then $|K| < |X|$.

PROOF

On the contrary let $|K| = |X|$. By lemma 1.2.5, which can be applied in view of Remark 1.2.4, any permutation of X which moves every element of K and keeps every element of $X \setminus K$ fixed is a homeomorphism of (X,T). Here $|M(p)| = |K| = |X|$. Then by Lemma 1.2.2 every permutation of X is a homeomorphism of (X,T) since H is normal.

Without loss of generality we may assume that $|K| = |X \setminus K|$ for otherwise take a suitable subset and that subset is also closed by Lemma 1.2.3(b). Now consider a permutation t of X which maps K onto $X \setminus K$ and $X \setminus K$ onto K. Such permutation exists since $|K| = |X \setminus K|$. Now t is a homeomorphism of (X,T) onto itself by the last paragraph. Hence $t(K) = X \setminus K$ is closed. Now K and $X \setminus K$ are open which contradicts Lemma 1.2.3(d). Hence the result.

1.2.7 LEMMA

Let (X,T) be an infinite topological space which is neither discrete nor indiscrete such that the group H of homeomorphisms of (X,T) is a normal subgroup of S(X) containing A(X). Let K be a proper closed subset of (X,T). Then every permutation p such that $|M(p)| \leq |K|$ is a homeomorphism and every subset M of X such that $|M| \leq |K|$ is closed.

PROOF

Since K is a proper closed subset of (X,T), by Lemma 1.2.5 which may be applied in view of Lemma 1.2.3 a permutation t of X which moves every element of K and leaves every element of $X \setminus K$ fixed is a homeomorphism of (X,T) such that $|M(t)| = |K|$. Then by Lemma 1.2.2, every permutation p of X such that $|M(p)| \leq |K|$ is a homeomorphism of (X,T) since H is normal in S(X).

Now to prove that every subset M of X such that $|M| \leq |K|$ is closed. Without loss of generality, we may assume that $|M| = |K|$ for otherwise take a suitable subset of K and subsets of K are also closed by Lemma 1.2.3. Since $|K| = |M|$ and $|X \setminus K| = |X \setminus M|$ as $|K| < |X|$ by Lemma 1.2.6, there exists a permutation p of X which maps K onto M, M onto K and keeps every other element fixed. Then

$$|M(p)| \leq |K| + |M|$$

If K is finite, M is closed by Lemma 1.2.3(c) since $|K| = |M|$. Therefore we may assume that K is infinite. Then

$$|M(p)| \leq |K| + |M| = |K|$$

Thus p is a homeomorphism by first paragraph. Then $M = p(K)$ is closed. Hence the result.

1.2.8 LEMMA

Let (X,T) be an infinite topological space, which is neither discrete nor indiscrete such that the group H of homeomorphisms of (X,T) is a nontrivial normal subgroup of S(X). Then $T = T_\alpha$ for some infinite cardinal α such that $\alpha \leq |X|$ when $T_\alpha = \{\varphi\} \cup \{A \subset X : \text{Card } (X \setminus A) < \alpha\}$

PROOF

Since H is a nontrivial normal subgroup S(X), it contains A(X) by Lemma 1.2.2. Then by Lemma 1.2.3, every finite subset of (X,T) is closed. Let α be the smallest infinite cardinal number of a subset of (X,T) which is not closed.

Now to prove that $T = T_\alpha$. $T \subset T_\alpha$ for otherwise there exists $U \in T$ but $U \notin T_\alpha$. i.e. Card $(X \setminus U) \geqslant \alpha$. Now let M be any subset of X such that $|M| = \alpha$, then M is closed in (X,T) by Lemma 1.2.7, since $X \setminus U$ is closed in (X,T) and $|M| \leqslant |X \setminus U|$. This contradicts the definition of α. Also $T_\alpha \subset T$. For otherwise if $U \in T_\alpha$, $U \neq \varphi$, Card $(X \setminus U) < \alpha$. Then $X \setminus U$ is closed in (X,T) by the definition of α. Thus $U \in T$. Hence the result.

1.2.9 LEMMA

The group of homeomorphisms of a topological space (X,T) where T is either discrete, indiscrete or of the form

$T_\alpha = \{\varphi\} \cup \{A \subset X : \text{Card } (X \setminus A) < \alpha\}$ for some infinite cardinal number α, $\alpha \leq |X|$ is $S(X)$.

PROOF

It can be observed that the group of homeomorphisms of a discrete or indiscrete space coincides with the group of permutations.

Now let p be a permutation of (X, T_α) for some infinite cardinal number α, $\alpha \leq |X|$. Let U be a nonempty open set in (X, T_α). Then Card $(X \setminus U) < \alpha$. Then Card $(X \setminus p(U)) < \alpha$ since p is a permutation. Thus p is an open map. Similarly we can prove that p is continuous. Thus p is a homeomorphism. Hence the result.

1.2.10 THEOREM

Let X be an infinite set. Then no nontrivial proper normal subgroup of S(X) can be represented as the group of homeomorphisms of (X,T) for any topology T on X.

PROOF

Let T be a topology on X where the group of homeomorphisms of (X,T) is a nontrivial normal subgroup of S(X). Then T is either discrete, indiscrete or of the form T_α for some infinite cardinal number $\alpha \leq |X|$, by Lemma 1.2.8. Then the group of homeomorphisms of (X,T) is S(X) by Lemma 1.2.9. Hence the result.

1.2.11 DEFINITION

A topological space (X,T) is completely homogeneous if the group of homeomorphisms of (X,T) coincides with the group of permutations of X.

1.2.12 REMARKS

It is not difficult to see that a finite completely homogeneous space is either discrete or indiscrete. We may use the method analogous to the proof of Lemma 1.2.3(c). R.E. LARSON [26] determined the completely homogeneous spaces. His result given below easily follows from the Lemmas 1.2.8, 1.2.9 and the above remark.

1.2.13 THEOREM

A topological space (X,T) is completely homogeneous if and only if the topology T is either discrete, indiscrete or of the form

$$T_\alpha = \{\varphi\} \cup \{A \subset X : \text{Card}\,(X \setminus A) < \alpha\}$$

for some infinite cardinal number $\alpha \leqslant |X|$.

CHAPTER 2

ANTI-HOMOGENEITY AND HEREDITARY RIGIDITY

In this chapter we study the anti-homogeneous spaces in the sense of PAUL BANKSTON [4] and give several characterizations. In particular we prove that anti-homogeneity is equivalent to hereditary rigidity.

2.1 PRELIMINARIES

2.1.1 DEFINITION

A topological space (X,T) is rigid if the identity map is the only homeomorphism of (X,T) onto itself.

2.1.2 NOTE

PAUL BANKSTON [4] defined a topological property " anti-P" corresponding to any topological property P as follows. If P is any topological property, the spectrum of P, denoted by Spec (P), is the class of all cardinal numbers α such that any topological space on a set of cardinal number α has the property P. Now a topological space (X,T) is said to have the property anti-P if a subspace of it has the property P only if the cardinality of the subspace is an element of Spec (P).

The spectrum of the topological property of homogeneity is $\{0,1\}$. Thus a topological space (X,T) is anti-homogeneous if and only if no subspace of (X,T) containing more than one point is homogeneous.

2.1.3 DEFINITIONS

A reflexive, transitive relation R on a set X is called a pre-order on X. The ordered pair (X,R) is called a pre-ordered set. If $Y \subset X$, then $R \cap (Y \times Y)$ is a pre-order on Y and is called the pre-order induced by R on Y. Unless otherwise specified every subset of a pre-ordered set is assumed to be pre-ordered by this pre-order. If (X,R) and (Y,S) are pre-ordered sets, a one-one function f from X onto Y is called an order isomorphism if f(a)Sf(b) if and only if aRb. A pre-ordered set (X,R) is homogeneous if for every a,b in X, there exists an order isomorphism f of (X,R) onto itself such that f(a)=b. It is anti-homogeneous if no subset containing more than one point is homogeneous. It is rigid if the identity map is the only order isomorphism of (X,R) onto itself.

2.1.4 NOTE

The following facts from [1] will be used later.

Let T be a topology on a set X. Then we can associate with it a pre order $\leq$ on X such that $a \leq b$ if and only if every open set of (X,T) containing b contains a. This pre-order $\leq$ associated with the topology T is denoted by $\rho(T)$.

Let (X,T) and (Y,S) be two topological spaces. Then every homeomorphism from (X,T) onto (Y,S) is an order isomorphism from (X, $\rho(T)$) onto (Y, $\rho(S)$).

2.1.5 REMARKS

Using 2.1.4 we have

(a) Let (Z, $\leq$) be the set of all integers with the usual order. Then the pre-order $\leq$ is the pre-order $\rho(T)$ associated with the unique topology T on Z whose elements are φ, Z and the subsets of Z of the form $\{x \in Z : x \leq m\}$ for some m in Z. Both the topological space (Z,T) and the associated pre-ordered set (Z, $\leq$) are homogeneous.

(b) A topological space (X,T) is rigid if (X, $\rho(T)$) is rigid.

(c) If a topological space (X,T) is homogeneous, then the associated pre-ordered set is also homogeneous.

2.2 SEMI-WELL ORDERED SETS

2.2.1 DEFINITIONS

A linearly ordered set is well ordered if every non empty subset has a first element. It is co-well ordered if every nonempty subset has a last element. It is semi-well ordered if every nonempty subset has either a first or a last element.

2.2.2 NOTATION

Let A and B be two disjoint linearly ordered sets with linear orders R and S respectively. Then by A+B, we denote the set $A \cup B$ with the linear order

$$R \cup S \cup \{(a,b) : a \varepsilon A,\ b \varepsilon B\}$$

on it.

Now we shall prove the following theorem on the structure of semi-well ordered sets.

2.2.3 THEOREM

Every semi-well ordered set X can be written in the form A+B where A is a well-ordered set and B is a co-well ordered set.

PROOF

Let

$A = \{x \in X :$ x has no immediate predecessor or x is the n^{th} successor of an element having no immediate predecessor for some positive integer $n\}$, and

$B = X \setminus A$

To prove that for any element b of B, either b has no immediate sucessor, or b is the n^{th} predecessor of an element having no immediate sucessor for some positive integer n.

On the contrary let there be an element x_o in B having an immediate successor and which is not the n^{th} predecessor of an element having no immediate successor for any positive integer n. Then x_o has n^{th} sucessor for every positive integer n for otherwise x_o will be the n^{th} predecessor of an element having no immediate sucessor for some positive integer n. Denote the n^{th} sucessor of x_o by x_n for every positive integer n.

Also since x_o is not an element of A, x_o has an immediate predecessor (say x_{-1}) and x_o is not the n^{th}

sucessor of an element having no immediate predecessor. Then x_0 has n^{th} predecessor for every positive integer n, for otherwise x_0 will be the n^{th} sucessor of an element having no immediate predecessor. Denote the n^{th} predecessor of x_0 by X_{-n} for every positive integer n.

Now consider the set $\{x_i : i \varepsilon Z\}$ where Z is the set of all integers. It is a nonempty subset of X having neither a first element nor a last element. This contradicts the fact that X is semi-well ordered and hence the assertion.

Now, to prove that for every x in A, y in B, $x \leqslant y$. Suppose not. Then there exists a in A and b in B such that $b < a$. Now since a is in A, either a has no immediate predecessor, or a is the n^{th} sucessor of an element having no immediate predecessor for some positive integer n. Without loss of generality we may assume that a has no immediate predecessor, for otherwise, take in place of a the element p in A which has no immediate predecessor and which is the n^{th} predecessor of a for some positive integer n and $b < p$ for otherwise b will be the m^{th} predecessor of p for some positive integer m and then b is an element of A which contradicts the hypothesis.

Similarly we can assume without loss of generality that b has no immediate successor. Now take the set M of all elements of X greater than b and less than a. It is a nonempty subset of X having neither a first element nor a last element, since b has no immediate sucessor and a has no immediate predecessor. This contradicts the fact that X is semi-well ordered and hence the assertion.

To prove that A is well ordered. Otherwise, it has a nonempty subset M having no first element. But then M has a last element (say, r), X being semi-well ordered. Also note that M is infinite. Then either r has no immediate predecessor or it is the n^{th} sucessor of an element q of A having no immediate predecessor. Without loss of generality we may assume that r has no immediate predecessor for otherwise delete all elements of M greater than q, and M is still nonempty for we are only deleting a finite number of elements. Now $M' = M \setminus \{r\}$ is a nonempty subset of X having neither a first element nor a last element. This contradicts the fact that X is semi-well ordered and hence the assertion.

Dually we can prove that B is co-well ordered.

Thus X = A+B, where A is well ordered and B is co-well ordered.

2.2.4 THEOREM

Every semi-well ordered set is hereditarily rigid.

PROOF

Let X be a semi-well ordered set. Then by Theorem 2.2.3, it can be written in the form A+B where A is well ordered and B is co-well ordered. Let f be an order isomorphism of X onto itself.

To prove that f maps every element x of A to x itself. Otherwise, let a be the first element of A such that $f(a) \neq a$. But then every element x of A smaller than a will be mapped to x itself, but a is mapped to another element $f(a) > a$. Then no element of A can be mapped to a for f is an order isomorphism. Clearly no element of B can be mapped to a. This is a contradiction since f is onto. Hence f maps every element x of A to x itself.

Dually we can prove that f maps every element y of B to y itself. Thus f is the identity map. Since every subset of a semi-well ordered set is semi-well ordered, hereditary rigidity follows.

2.3 ANTI-HOMOGENEOUS PRE-ORDERED SETS

In this section, we characterize the anti-homogeneous pre-orders.

2.3.1 LEMMA

Every anti-homogeneous pre-ordered set is linearly ordered.

PROOF

Let $(X, \leq)$ be an anti-homogeneous pre-ordered set. Let a,b be elements of X. If $a \leq b$ and $b \leq a$, then $\{a,b\}$ is homogeneous. Then $a=b$, since $(X, \leq)$ is anti-homogeneous. Thus $\leq$ is anti-symmetric.

Now, let a and b be elements of X, $a \neq b$. To prove that either $a \leq b$ or $b \leq a$. Otherwise $\{a,b\}$ is homogeneous, which contradicts the fact that $(X, \leq)$ is anti-homogeneous. Hence the result.

2.3.2 LEMMA

A nonempty linearly ordered set having neither a first element nor a last element contains a subset order isomorphic to the set Z of all integers with usual order.

PROOF

Let X be a nonempty linearly ordered set containing neither a first element nor a last element. Let x_0 be an element of X. Since x_0 is not the last element, we can choose x_1 in X such that $x_0 < x_1$. Similarly, we can choose x_i, for each positive integer i such that

$$x_0 < x_1 < x_2 < x_3 < \dots$$

Similarly, since X has no first element, we can choose x_i, for each negative integer i such that

$$\dots < x_{-3} < x_{-2} < x_{-1} < x_0$$

Now $A = \{x_i : i \in Z\}$ where Z is the set of all integers, is order isomorphic to the set Z of all integers with the usual order. Hence the result.

2.3.3 THEOREM

Let X be a pre-ordered set. Then the following are equivalent

(a) X is anti-homogeneous

(b) X is a linearly ordered set containing no subset isomorphic to the set of integers with the usual order.

(c) X is semi-well ordered.

(d) X is of the form A+B, where A is well ordered and B is co-well ordered.

(e) X is hereditarily rigid.

PROOF

(a) $\Longrightarrow$ (b) by Lemma 2.3.1 and Remark 2.1.5 (a)

(b) $\Longrightarrow$ (c) by Lemma 2.3.2

(c) $\Longrightarrow$ (d) by Theorem 2.2.3

(d) $\Longrightarrow$ (e) by Theorem 2.2.4

(e) $\Longrightarrow$ (a) by the fact that no set containing more than one point can be both homogeneous and rigid.

2.4 ANTI-HOMOGENEOUS TOPOLOGICAL SPACES

In this section we characterize the anti-homogeneous topological spaces.

2.4.1 THEOREM

Let (X,T) be a topological space. Let R be the pre-order associated with T. Let Y be a subset of X with the induced topology T'. Then the pre-order R' associated with T' is the same as the pre-order S on Y induced by R.

PROOF

Let a and b be elements of Y such that aSb. Then aRb since S is the pre-order induced by R. Thus every open set in (X,T) containing b contains a. Let A be open in (Y,T') and b an element of A. Then A = $B \cap Y$ for some B open in (X,T). Also b is an element of B. Then a is an element of B since aRb. Thus a is an element of A = $B \cap Y$. Thus every open set in (Y,T') containing b contains a. Thus aR'b.

Now let aR'b, for some a and b in Y. To prove that aSb. On the contrary assume that aSb is false. Then aRb is false. Then there exists an open set D in (X,T) such that b is an element of D and a is not an element of D. This leads to a contradiction since then aR'b, $D \cap Y$ is open in (Y,T'), $b \in D \cap Y$ and $a \notin D \cap Y$. Hence the result.

2.4.2 THEOREM

Let (X,T) be an anti-homogeneous space and $\leq$ be the pre-order associated with T. Then the pre-ordered set (X, $\leq$) is anti-homogeneous.

PROOF

Let a and b be elements of X. If $a \leqslant b$ and $b \leqslant a$, then the topology on the subset $\{a,b\}$ induced by T is $\{\varphi, \{a,b\}\}$ and is homogeneous. Therefore a=b, since (X,T) is anti-homogeneous. Thus $\leqslant$ is anti-symmetric.

Also if a and b are elements of X, $a \neq b$, either $a \leqslant b$ or $b \leqslant a$ for otherwise the topology on the subset $\{a,b\}$ induced by T is $\{\varphi, \{a\}, \{b\}, \{a,b\}\}$ which is homogeneous which is a contradiction. Thus $(X, \leqslant)$ is a linearly ordered set.

Furthermore, $(X, \leqslant)$ does not contain a subset order isomorphic to the set of all integers with usual order, for otherwise the subset will be a homogeneous subspace with more than one point of the anti-homogeneous topological space (X,T) by Theorem 2.4.1. and Remark 2.1.5(a). Thus $(X, \leqslant)$ is an anti-homogeneous pre-ordered set by Theorem 2.3.3.

The following theorem gives several characterizations of anti-homogeneous spaces.

2.4.3 THEOREM

Let (X,T) be a topological space. Let $\leqslant$ be the pre-order associated with the topology T on X. Then the following are equivalent.

(a) (X,T) is anti-homogeneous

(b) $(X,\leqslant)$ is anti-homogeneous

(c) $(X,\leqslant)$ is semi-well ordered

(d) $(X,\leqslant)$ is hereditarily rigid

(e) (X,T) is hereditarily rigid

PROOF

(a) $\Longrightarrow$ (b) by Theorem 2.4.2

(b) $\Longrightarrow$ (c) by Theorem 2.3.3

(c) $\Longrightarrow$ (d) by Theorem 2.2.4

(d) $\Longrightarrow$ (e) by Remark 2.1.5(b)

(e) $\Longrightarrow$ (a) since no topological space with more than one point can be both homogeneous and rigid.

2.5 HEREDITARILY HOMOGENEOUS SPACES

A related problem is to give a satisfactory characterization of hereditarily homogeneous spaces. We know the following facts.

2.5.1 THEOREM

Every completely homogeneous space is hereditarily homogeneous.

PROOF

Every completely homogeneous space is homogeneous and every subspace of a completely homogeneous space is completely homogeneous (see Theorem 1.2.13) and hence the result follows.

Also,

2.5.2 THEOREM

A finite topological space is hereditarily homogeneous if and only if it is either discrete or indiscrete.

PROOF

The sufficiency part follows from Theorem 2.5.1.

Now to prove that every finite hereditarily homogeneous space is either discrete or indiscrete. Otherwise, let (X,T) be a finite hereditarily homogeneous space, which is neither discrete nor indiscrete. Then by 1.1.2(c) there exists a partition of X which forms a base since (X,T) is homogeneous. Choose $\{a\}$ and $\{b,c\}$ subsets of two distinct elements of the partition where $b \neq c$. That is possible since (X,T) is neither discrete nor indiscrete. Now the topology induced by (X,T) on the subspace $\{a,b,c\}$ is $\{\varphi, \{a\}, \{b,c\}, \{a,b,c\}\}$ and hence that subspace is not homogeneous, a contradiction. Hence the result.

2.5.3 THEOREM

If X is any infinite set, then there exists a hereditarily homogeneous topology on X which is not completely homogeneous.

PROOF

Let $X = A \cup B$, where $A \cap B = \varphi$ and $|A| = |B|$. Now let

$$T = \{U \subset X : |U \cap B| < |X|\}$$

Then T can be verified to be a hereditarily homogeneous topology which is not completely homogeneous by Theorem 1.2.13.

2.5.4 THEOREM

A hereditarily homogeneous topology is either indiscrete or T_1.

PROOF

Let (X,T) be a hereditarily homogeneous space. Then the pre-ordered set (X, ρ(T)) is also hereditarily homogeneous by Remark 2.1.5(c) and Theorem 2.4.1. Let ρ(T) = R. Then R is symmetric, for otherwise let $(x,y) \in R$ but $(y,x) \notin R$. Then the subset $\{x,y\}$ of (X,R) is not homogeneous. This is a contradiction. Thus R is an equivalence relation.

To prove that either there is only one equivalence class or each equivalence class contains only one element. Otherwise, let $\{a\}$ and $\{b,c\}$ be subsets of two distinct equivalence classes, where $b \neq c$. Now the subset $\{a,b,c\}$ of (X,R) is not homogeneous, a contradiction.

If there is only one equivalence class, i.e. xRy for every x,y in X, the topology T is indiscrete. On the other hand if each equivalence class contain only one element, the topology T is T_1 for otherwise there exists a,b in (X,T) such that every open set containing b containsa which implies aRb.

2.5.5 NOTE

A topological space is anti-rigid in the sense of PAUL BANKSTON [4], if and only if no subspace containing more than one point is rigid since the spectrum of rigidity is $\{0,1\}$ (see 1.1.2).

2.5.6 THEOREM

A hereditarily homogeneous space is anti-rigid.

PROOF

No topological space containing more than one point can be both homogeneous and rigid. Hence the result.

2.5.7 REMARK

The converse of the Theorem 2.5.6 is false since the set $X = \{a,b,c\}$ with the topology $T = \{\varphi, \{a\}, \{b,c\}, \{a,b,c\}\}$ is not hereditarily homogeneous even though it is anti-rigid.

2.5.8 REMARK

The problem of determining the hereditarily homogeneous spaces remains yet to be solved.

CHAPTER-3

Čech CLOSURE SPACES

In this chapter we discuss some problems related to Čech closure spaces. We determine the completely homogeneous Čech closure spaces. We investigate the reflexive relation associated with a closure operator in detail.

3.1 PRELIMINARIES

3.1.1 NOTATION

P(X) denotes the power set of a set X.

3.1.2 DEFINITION

A Čech closure operator V on a set X is a function V from P(X) into P(X) such that

(1) $V(\varphi) = \varphi$

(2) $A \subset V(A)$ for every A in P(X)

(3) $V(A \cup B) = V(A) \cup V(B)$ for every A,B in P(X)

For brevity we call it a closure operator on X. Also (X,V) is called a closure space.

3.1.3 DEFINITIONS

In a closure space (X,V), a subset A of X is said to be closed if $V(A)=A$. A subset A of X is open if $X\setminus A$ is closed in (X,V). The set of all open sets of (X,V) forms a topology on X called the topology associated with the closure operator V.

Let T be a topology on a set X. Then a function V from $P(X)$ into $P(X)$ defined by $V(A)=\bar{A}$ for every A in $P(X)$, where $\bar{A}$ is the closure of A in (X,T), is a closure operator on X called the closure operator associated with the topology T.

A closure operator V on a set X is called topological if $V(V(A))=V(A)$ for every A in $P(X)$.

3.1.4 REMARKS

Note that a closure operator on a set X is topological if and only if it is the closure operator associated with a topology on X.

Also note that different closure operators can have the same associated topology.

3.1.5 DEFINITIONS

Let (X,V) and (Y,V') be closure spaces. A one-one function f from X onto Y is called a closure isomorphism if

$$f(V(A)) = V'(f(A)) \text{ for every } A \text{ in } P(X)$$

The closure isomorphisms of a closure space (X,V) onto itself form a group under the operation of composition of functions which is called the group of closure isomorphisms of the closure space.

3.1.6 DEFINITION

Let V_1 and V_2 be closure operators on a set X. Then let

$V_1 \leqslant V_2$ if and only if $V_2(A) \subset V_1(A)$ for every A in P(X).

This relation is a partial order in the set of all closure operators on X.

3.1.7 DEFINITION

A subset A of a closure space (X,V) is dense if $V(A)=X$.

3.1.8 DEFINITION

A closure operator V on X is T_1 if $V(\{x\}) = \{x\}$ for every x in X.

3.2 COMPLETELY HOMOGENEOUS CLOSURE SPACES

3.2.1 DEFINITION

A closure space (X,V) is called completely homogeneous if the group of closure isomorphisms of (X,V) coincides with the group of permutations of X.

3.2.2 LEMMA

Let (X,V) be a completely homogeneous closure space. Then every subset of X is either closed or dense in (X,V).

PROOF

Suppose not. Then there exists a proper non-empty subset A of X such that A is neither closed nor dense. Then $A \neq V(A) \neq X$.

Let $x \in V(A) \setminus A$ and $y \in X \setminus V(A)$.

The permutation $p = (x,y)$ is a closure isomorphism. Then $p(V(A)) = V(A)$. Therefore since x is an element of V(A), y is an element of V(A). This is a contradiction since y is an element of $X \setminus V(A)$, and hence the result.

3.2.3 THEOREM

A closure space (X,V) is completely homogeneous if and only if V is the closure operator associated with a completely homogeneous topology on X.

PROOF

Let (X,V) be completely homogeneous. From lemma 3.2.2 we see that V(A)=A or V(A)=X for every subset A of X. Then V(V(A)) = V(A) for every A in P(X). Thus V is a topological closure operator. But it can be seen that every closure isomorphism of (X,V) onto itself, where V is a topological closure operator associated with a topology T, is a homeomorphism of (X,T) onto itself. Thus (X,T) is completely homogeneous.

Now let (X,V) be a closure operator associated with a completely homogeneous topology T on X. Then every homeomorphism of (X,T) is a closure isomorphism of (X,V). Hence (X,V) is completely homogeneous.

3.3 REFLEXIVE RELATIONS ASSOCIATED WITH CLOSURE OPERATORS

3.3.1 NOTE

The following theory was developed by SUSAN J. ANDIMA and W.J. THRON in [1] for topological spaces and pre-orders.

As mentioned in 2.1.4, we can associate with each topology T on X, a pre-order R= ρ(T) on X such that aRb if and only if every open set in (X,T) containing b contains a.

Now, let R be a pre-order on a set X. Then let

$$\{\overline{x}\} = \{y \in X : xRy\}$$

and
$$\{\hat{x}\} = \{y \in X : yRx\}$$

We define $\mu(R)$ to be the smallest topology on X in which $\{\overline{x}\}$ is closed for every x in X. Also we define $\nu(R)$ to be the smallest topology on X in which $\{\hat{x}\}$ is open for every x in X. Then $\rho(T)=R$ for some topology T on X if and only if $\mu(R) \subset T \subset \nu(R)$.

It can also be proved that $R_1 \subset R_2$ if and only if $\nu(R_2) \subset \nu(R_1)$ for any two pre-orders R_1 and R_2.

But for any two pre-orders R_1 and R_2 such that $R_1 \subset R_2$, $\mu(R_1)$ and $\mu(R_2)$ are not comparable in general.

A property P of a topological space is a topological property if it is preserved by homeomorphisms. A property K of a pre-ordered set is an order property if it is preserved by order isomorphisms.

A topological property P is called an order induced topological property if there is an order property K such that a topological space (X,T) has the property P if and only if the pre-ordered set (X, $\rho(T)$) has the order property K.

If P is a topological property then a topology T on a set X is called maximal (minimal) P if no topology finer (coarser) than T has the property P.

If K is an order property a pre-order R on a set X is called maximal (minimal) K if no pre-order on X larger (smaller) than R has the property K.

If P is a topological property induced by an order property K, then a topology T on X is maximal P if and only if T = $\nu(R)$ and R a pre-order on X, which is minimal K.

Also if R is a pre-order on X which is maximal K and if T = $\mu(R)$, then T is minimal P. Moreover, if T is minimal P, R = $\rho(T)$, then T = $\mu(R)$. But we cannot say in general that a topology T is minimal P only if T = $\mu(R)$ and R is maximal K.

Now we extend the theory to the case of closure spaces.

3.3.2 DEFINITION

Let V be a closure operator on a set X. Define a relation R on X such that aRb if and only if $b \in V(\{a\})$. This relation is reflexive and called the reflexive relation associated with the closure operator V and is denoted by ρV.

3.3.3 DEFINITIONS

Let R be a reflexive relation on a set X. We can define the closure operators μR and νR on X as follows:

$$\mu R(A) = \begin{cases} \{y : xRy \text{ for some } x \text{ in } A\} & \text{if } A \text{ is finite} \\ X & \text{otherwise} \end{cases}$$

$$\nu R(A) = \{ y : xRy \text{ for some } x \text{ in } A \} \text{ for every } A \text{ in } P(X)$$

3.3.4 THEOREM

If R is a reflexive relation on a set X, then $R = \rho V$ for some closure operator V on X if and only if $\mu R \leq V \leq \nu R$. In particular $\rho \mu R = \rho \nu R = R$.

PROOF

Let $\mu R \leqslant V \leqslant \nu R$. To prove that $\rho V = R$.

$$(x,y) \in \rho V \iff y \in V(\{x\})$$

$y \in V(\{x\}) \Rightarrow y \in \mu R(\{x\}) \Rightarrow (x,y) \in R$

$\Rightarrow y \in \nabla R(\{x\}) \Rightarrow y \in V(\{x\})$

Thus $(x,y) \in \rho V \Leftrightarrow (x,y) \in R$

Hence $\rho V = R$

Now let $\rho V = R$. To prove that $\mu R \leq V \leq \nabla R$.

Since $\rho V = R$

$y \in V(\{x\}) \Leftrightarrow (x,y) \in R \Leftrightarrow y \in \mu R(\{x\})$

$\Leftrightarrow y \in \nabla R(\{x\})$

When A is a finite subset of X,

$$y \in V(A) \Leftrightarrow y \in \bigcup_{a \in A} V(\{a\})$$

$\Leftrightarrow y \in V(\{a\})$ for some $a \in A$

$\Leftrightarrow y \in \mu R(\{a\})$ for some $a \in A$

$\Leftrightarrow y \in R(\{a\})$ for some $a \in A$

$\Leftrightarrow y \in \mu R(A)$

$\Leftrightarrow y \in \nabla R(A)$

Thus when A is finite

$$V(A) = \mu R(A) = \nabla R(A)$$

Since $\mu R(B) = X$ when B is infinite

$$\mu R \leq V$$

When B is an infinite subset of X

$$y \in \vee R(B) \Rightarrow y \in \vee R(\{b\}) \text{ for some } b \in B$$

$$\Rightarrow y \in V(\{b\}) \text{ for some } b \in B$$

$$\Rightarrow y \in V(B)$$

Thus $\vee R(B) \subset V(B)$ and hence $V \leqslant \vee R$. Hence the result.

3.3.5 THEOREM

Let R_1 and R_2 be reflexive relations on a set X. Then the following are equivalent

(a) $R_1 \subset R_2$

(b) $\vee R_2 \leqslant \vee R_1$

(c) $\mu R_2 \leqslant \mu R_1$

Also $R_1 = R_2$ if and only if $\mu R_1 = \mu R_2$ if and only if $\vee R_1 = \vee R_2$.

PROOF

(a) $\Longrightarrow$ (b)

$$\vee R_1(A) = \{ y : xR_1y \text{ for some } x \in A \}$$

$$\subset \{ y : xR_2y \text{ for some } x \in A \}$$

$$= \vee R_2(A) \text{ for every } A \text{ in } p(X)$$

Thus $\vee R_2 \leqslant \vee R_1$

(b) $\Longrightarrow$ (c) :

When A is a finite subset of X, $\mu R_1(A) = \vee R_1(A)$ and $\mu R_2(A) = \vee R_2(A)$. When A is infinite $\mu R_1(A) = \mu R_2(A) = X$. Hence (c) follows from (b).

(c) $\Longrightarrow$ (a) : Let $\mu R_2 \leqslant \mu R_1$

Then $(x,y) \in R_1 \Longrightarrow y \in \mu R_1(\{x\})$

$\Longrightarrow y \in \mu R_2(\{x\})$

$\Longrightarrow (x,y) \in R_2$

Thus $R_1 \subset R_2$

The last assertion follows easily from the equivalence of (a), (b) and (c).

3.3.6 DEFINITIONS

Let R be a reflexive relation on a set X. Then (X,R) is defined as a pseudo ordered set.

Let (X,R) and (Y,S) be pseudo ordered sets. Let f be a one-one function from X onto Y. It is called an isomorphism if for a,b in X, aRb if and only if

$f(a)\ S\ f(b)$. If there is an isomorphism from a pseudo ordered set (X,R) onto a pseudo ordered set (Y,S) they are isomorphic.

A property K of a pseudo ordered set is called a pseudo order property if it is preserved by isomorphisms.

3.3.7 REMARK

A pseudo ordered set can be interpreted as a digraph. Interpret the points of X as vertices and there is an edge from a vertex a into a different vertex b if and only if a is related to b. We are allowing neither loops nor multiple edges (See [17]).

3.3.8 DEFINITIONS

A property P of a closure space is called a closure property if it is preserved by closure isomorphisms.

A closure property P is called a pseudo order induced closure property if there exists a pseudo order property K such that a closure space (X,V) has the property P if and only if the pseudo ordered set $(X, \rho V)$ has the property K.

3.3.9 DEFINITIONS

Let P be a closure property. A closure operator V on a set X is called minimal (respectively maximal) P if for no closure operator V' on X such that $V' < V$ (respectively $V < V'$), (X,V') has the property P.

Let K be a pseudo order property. A reflexive relation R on a set X is called minimal (respectively maximal) K if for no reflexive relation R' on X such that $R' \subset R$ (respectively $R \subset R'$) and $R' \neq R$, (X,R') has the property K.

3.3.10 THEOREM

Let P be a closure property induced by a pseudo order property K. Then a closure operator V on a set X is maximal P if and only if $V = \nu R$ for some reflexive relation R on X which is minimal K. Also V is minimal P if and only if $V = \mu R$ for some reflexive relation R on X, which is maximal K.

PROOF

Let V be a closure operator on X which is maximal P. Let $\rho V = R$. Here (X,R) has the order property K.

We have $V = \nabla R$, for otherwise $(X, \nabla R)$ will also have the property P and $V < \nabla R$ by Theorem 3.3.4. Also R is maximal K, for otherwise let R' be a reflexive relation on X such that (X,R') has the property K and $R' \subset R$, $R \neq R$. Then $\nabla R < \nabla R'$ by theorem 3.3.5. Clearly $(X, \nabla R')$ has the property P and $V = \nabla R < \nabla R'$. This contradicts the fact that V is maximal P. Hence the result.

Now let R be minimal K and $V = \nabla R$. Let $V < V'$ for some closure operator V' on X. Then V' will not have the property P. For, otherwise $(X, \nabla R')$ will also have the property P where $R' = \rho V'$. But then

$$\nabla R \leq V < V' \leq \nabla R'$$

Then by Theorem 3.3.5, $R' \subset R$, $R' \neq R$. Also (X,R') has the property K. This contradicts the fact that R is minimal K. Thus V is maximal P.

The second assertion can be proved dually in view of Theorem 3.3.5.

3.3.11 THEOREM

When X is a finite set, then ρ is a one-one map from the set of all closure operators on X onto the set of all reflexive relations on X such that $V_1 \leq V_2$ if and only if $\rho V_2 \subset \rho V_1$ for every V_1, V_2 closure operators on X.

PROOF

Let R be a reflexive relation on X. Then for the closure operator ∇R, $\rho \nabla R = R$. Thus ρ is onto.

Also

$$V_1 \leq V_2 \Leftrightarrow V_2(A) \subset V_1(A) \text{ for every } A \text{ in } P(X)$$

$$\Leftrightarrow V_2(\{x\}) \subset V_1(\{x\}) \text{ for every } x \text{ in } X$$

(since X is finite)

$$\Leftrightarrow \rho V_2 \subset \rho V_1$$

The fact that ρ is one-one also follows from this.

CHAPTER-4

LATTICE OF CLOSURE OPERATORS

We denote the set of all closure operators on a fixed set X by L(X). It is a complete lattice with the partial order $\leq$ defined on it as in 3.1.6. In this chapter we study some properties of this lattice.

4.1 INFRA AND ULTRA CLOSURE OPERATORS

4.1.1 DEFINITIONS

The closure operator D on X defined as $D(A)=A$ for every A in P(X) is called the discrete closure operator.

The closure operator I on X defined by

$$I(A) = \varphi \text{ if } A = \varphi$$
$$= X \text{ otherwise}$$

is called the indiscrete closure operator.

4.1.2 REMARKS

Note that D and I are the closure operators associated with the discrete and the indiscrete topologies on X respectively.

Moreover D is the unique closure operator whose associated topology is discrete.

Also I and D are the smallest and the largest elements of L(X) respectively.

4.1.3 DEFINITIONS

A closure operator on X, other than I, is called an infra closure operator if the only closure operator on X, strictly smaller than it, is I.

A closure operator on X, other than D, is called an ultra closure operator if the only closure operator on X, strictly larger than it, is D.

4.1.4 REMARK

Note that the infra closure operators and the ultra closure operators are precisely the atoms and the dual atoms respectively of the lattice L(X).

4.1.5 NOTATION

For a, b in X, $a \neq b$, let

$$
\begin{aligned}
V_{a,b}(A) &= \varphi && \text{if } A = \varphi \\
&= X \setminus \{b\} && \text{if } A = \{a\} \\
&= X && \text{otherwise}
\end{aligned}
$$

$V_{a,b}$ can be verified to be a closure operator on X.

Now we can characterize the infra closure operators.

4.1.6 THEOREM

A closure operator on X is an infra closure operator if and only if it is of the form $V_{a,b}$ for some a,b in X, $a \neq b$.

PROOF

If V is a closure operator on X strictly smaller than $V_{a,b}$, then $V(\{a\})$ will be strictly larger than $X \setminus \{b\}$ and hence equal to X. Also V(A) = X for every A in P(X) other than φ and $\{a\}$. Hence V = I. Thus all closure operators of the form $V_{a,b}$ are infra closure operators.

Now let V be any closure operator on X other than I. Then there exists a non-empty subset A of X such that $V(A) \neq I(A) = X$. Now choose an element a of A and an element b of $X \setminus V(A)$. Then b is not an element of $V(\{a\})$ since b is not an element of V(A). Also $V_{a,b}(M) = X$ for every nonempty subset M of X other than $\{a\}$. Then $V_{a,b} \leqslant V$. Thus all infra closure operators are of the form $V_{a,b}$ for some a,b in X, $a \neq b$.

4.1.7 NOTE

A topology T on X which is not discrete, is called an ultra topology if the discrete topology is the only topology strictly larger than T. In [9] O. FRÖLICH proved that the ultra topologies on X are precisely the topologies of the form $P(X \setminus \{a\}) \cup \mathfrak{U}$ where $a \in X$ and $\mathfrak{U}$ is an ultra filter on X which does not contain $\{a\}$.

The closure operator V associated with an ultratopology $P(X \setminus \{a\}) \cup \mathfrak{U}$ is given by

$$\begin{aligned} V(A) &= A \text{ if } A = \varphi,\ a \in A \text{ or } X \setminus A \in \mathfrak{U} \\ &= A \cup \{a\} \quad \text{otherwise} \end{aligned}$$

4.1.8 THEOREM

A closure operator on X is an ultra closure operator if and only if it is the closure operator associated with some ultra topology on X.

PROOF

Let $P(X \setminus \{a\}) \cup \mathfrak{U}$ be an ultra topology on X and V the associated closure operator. Let V' be a closure operator on X strictly larger than V. Then there exists a subset A of X such that $V'(A) \subset V(A)$, but $V'(A) \neq V(A)$. Then $V(A) = A \cup \{a\}$ and $V'(A) = A$, which means that $X \setminus A$ is open in (X,V') and not open in (X,V). Also every open set in (X,V) is open in (X,V'). Thus the associated topology of V' is strictly larger than the ultra topology and hence is discrete. Then $V' = D$ by Remark 4.1.2. Hence the closure operator associated with an ultra topology is an ultra closure operator.

To prove that every ultra closure operator is the closure operator associated with an ultra topology. Let V be a closure operator on X other than D. It suffices to prove that there exists a closure operator associated with an ultratopology larger than V. Since $V \neq D$, there exists an element a of X such that $\{a\}$ is not open in (X,V). Now, consider

$$\mathcal{F} = \{A \subset X : a \in A \text{ and } a \notin V(X \setminus A)\}$$

$\mathcal{F}$ can be verified to be a filter on X. Here $\{a\}$ is not an element of $\mathcal{F}$, for $a \in V(X \setminus \{a\})$ by the choice of a. Then $\mathcal{F} \cup \{X \setminus \{a\}\}$ is a family with finite intersection property. For, otherwise there will be an F in $\mathcal{F}$ such that $F \cap (X \setminus \{a\}) = \varphi$. Then $F \subset \{a\}$. Then $\{a\} \in \mathcal{F}$, for $F \in \mathcal{F}$, a contradiction. By Zorn's lemma, we have an ultrafilter $\mathcal{U}$ on X containing $\mathcal{F} \cup \{X \setminus \{a\}\}$. Clearly $\mathcal{U}$ does not contain $\{a\}$. Now consider the ultratopology $P(X \setminus \{a\}) \cup \mathcal{U}$. Let V' be the closure operator associated with it. Then $V \leq V'$. For, otherwise there exists a nonempty subset M of X such that $V'(M) \not\subset V(M)$. But then there exists an element a of X such that $a \in V'(M)$ but $a \notin V(M)$. Since $a \notin V(M)$, $X \setminus M \in \mathcal{F} \subset \mathcal{U}$. Then $V'(M) = M$, a contradiction. Hence the result.

4.1.9 REMARKS

In the course of the proof of Theorems 4.1.6 and 4.1.8, we also proved that every element of L(X) other than I is larger than or equal to an atom and every element of L(X) other than D is smaller than or equal to a dual atom.

4.1.10 DEFINITIONS

Let x be an element of X. Then the set $\mathcal{U}(x) = \{ A \subset X : x \in A \}$ is an ultrafilter on X. Such ultrafilters are called principal ultrafilters. An ultratopology $P(X \setminus \{a\}) \cup \mathcal{U}$ is called a principal ultratopology or a nonprincipal ultratopology according as $\mathcal{U}$ is principal or not. The closure operator associated with an ultratopology is called principal ultra closure operator or nonprincipal ultra closure operator according as the ultratopology is principal or not.

4.1.11 THEOREM

Infra closure operators are less than or equal to any nonprincipal ultra closure operator.

PROOF

Let $V_{x,y}$ be an infra closure operator and V a non principal ultra closure operator. Since $V_{x,y}(A)=X$ for all A in P(X) other than φ and $\{x\}$, we need show only that

$$V(\{x\}) \subset V_{x,y}(\{x\}) = X\setminus\{y\}$$

But since all nonprincipal ultratopologies are T_1 (see [35]), $V(\{x\}) = \{x\}$ for all x in X. Hence the result.

4.1.12 NOTATION

We denote the principal ultra closure operator associated with the principal ultra topology $P(X\setminus\{a\}) \cup \mathcal{U}(b)$ by $T_{a,b}$.

4.1.13 THEOREM

An infra closure operator $V_{x,y}$ is less than or equal to a principal ultra closure operator $T_{a,b}$ if and only if $x \neq b$ or $y \neq a$. Also $V_{a,b}$ and $T_{b,a}$ are incomparable.

PROOF

At first we prove that $V_{a,b}$ and $T_{b,a}$ are incomparable. We have $V_{a,b}(\{a\}) = X \setminus \{b\}$ and $T_{b,a}(\{a\}) = \{a,b\}$. Then

$$T_{b,a}(\{a\}) \not\subset V_{a,b}(\{a\})$$

Thus $V_{a,b} \not\leq T_{b,a}$. Also $V_{a,b} \not> T_{b,a}$ for $V_{a,b}$ is an atom and $T_{b,a}$ is a dual atom of the lattice $L(X)$. Hence $V_{a,b}$ and $T_{b,a}$ are incomparable.

Now when A is a non empty subset of X, other than $\{x\}$,

$$T_{a,b}(A) \subset X = V_{x,y}(A)$$

Also when $x \neq b$,

$$T_{a,b}(\{x\}) = \{x\} \subset X \setminus \{y\} = V_{x,y}(\{x\})$$

and when $y \neq a$

$$T_{a,b}(\{x\}) \subset \{x,a\} \subset X \setminus \{y\} = V_{x,y}(\{x\})$$

Thus

$$V_{x,y} \leqslant T_{a,b} \text{ if } x \neq b \text{ or } y \neq a$$

4.2 COMPLEMENTATION IN THE LATTICE L(X)

In this section we study the complementation in the lattice L(X).

4.2.1 THEOREM

The lattice L(X) is complemented when X is finite.

PROOF

To every closure operator we can associate a reflexive relation ρV as explained in 3.3.2. Then ρ is a one-one map from L(X) onto the lattice of reflexive relations with the partial order of set inclusion such that $V_1 \leqslant V_2$ if and only if $\rho V_2 \subset \rho V_1$ for V_1 and V_2 in L(X) by Theorem 3.3.11. Thus ρ is a dual isomorphism and also the lattice of reflexive relations with the partial order of set inclusion is complemented. Hence L(X) is complemented.

But we have

4.2.2 THEOREM

No nonprincipal ultra closure operator has a complement.

PROOF

On the contrary let V be a nonprincipal ultra closure operator with a complement V' in the lattice L(X). Since V' is not indiscrete, there exists an infra closure operator $V_{x,y} \leq V'$ by 4.1.9. But $V_{x,y} \leq V$ by Theorem 4.1.11. This contradicts the fact that V and V' are complements in the lattice L(X) and hence the result.

4.2.3 REMARK

When X is infinite we can prove that there exists a non principal ultra closure operator on X since then, assuming axiom of choice, we can prove that there exists a non principal ultrafilter. Thus by the previous theorem L(X) is not complemented when X is infinite.

But some elements of L(X) do have complements as proved below in

4.2.4 THEOREM

The infra closure operators $V_{x,y}$ and the principal ultra closure operators $T_{y,x}$ are lattice complements of each other in L(X) for any x,y in X, $x \neq y$.

PROOF

Here $V_{x,y}$ is an atom and $T_{y,x}$ is a dual atom of the lattice L(X) and they are incomparable by Theorem 4.1.13. Hence the result follows.

4.2.5 THEOREM

Let V and V' be closure operators on X which are not discrete such that an ultra closure operator is greater than or equal to V if and only if it is greater than or equal to V'. Then V = V'.

PROOF

Suppose $V \neq V'$. Then without loss of generality we assume that there exists a nonempty subset A of X such that $V(A) \not\subset V'(A)$. Thus there exists $a \in V(A)$ such that $a \notin V'(A)$. Now let

$$\mathcal{F} = \{ F \subset X : a \in F \text{ and } a \notin V(X \setminus F) \}$$

and

$$\mathcal{F}' = \{ F \subset X : a \in F \text{ and } a \notin V'(X \setminus F) \}$$

We can easily verify that $\mathcal{F}$ and $\mathcal{F}'$ are filters on X such that $A \in \mathcal{F}'$ and $A \notin \mathcal{F}$.

Now consider $\mathcal{B} = \mathcal{F} \cup \{A\}$. It is a family with finite intersection property for otherwise there exists $F \in \mathcal{F}$ such that $F \cap A = \varphi$ and then $X \setminus A \in \mathcal{F}$ as $F \subset X \setminus A$ which is a contradiction since $a \notin X \setminus A$ as $A \in \mathcal{F}'$. Then $\mathcal{B}$ is contained in some ultra filter $\mathcal{U}$ on X which does not contain $\{a\}$ for $a \notin A$ and $A \in \mathcal{U}$.

Let S be the ultra closure operator on X associated with the ultra topology $P(X \setminus \{a\}) \cup \mathcal{U}$. But $S(M) \subset M \cup \{a\}$ for every M in P(X) and whenever $a \notin V(M)$, $X \setminus M \in \mathcal{F} \subset \mathcal{U}$ and hence $S(M) = M$. Thus $S(M) \subset V(M)$ for every M in P(X). That is $V \leqslant S$.

But $V' \nleqslant S$ for $S(A) = A \cup \{a\}$ as $A \neq \varphi$, $a \notin A$ and $X \setminus A \in \mathcal{U}$ and $a \notin V'(A)$. This contradicts the hypothesis and hence the result.

4.2.6 REMARKS

Let V be any closure operator on X which is not discrete. Then V is the greatest lower bound of the set of all ultra closure operators greater than or equal to V, by the previous theorem. Thus L(X) is dually atomic. Eventhough L(X) is atomic when X is finite, it is not atomic when X is infinite, for then there will be many nonprincipal ultra closure operators, all of them greater than every infra closure operator.

4.2.7 THEOREM

If V' is the complement of V in the lattice L(X), then every ultra closure operator is greater than or equal to one of V and V'.

PROOF

On the contrary assume that there exists an ultra closure operator U associated with the ultra topology $P(X \setminus \{a\}) \cup \mathfrak{U}$ such that $U \ngeq V$ and $U \ngeq V'$. Since $U \ngeq V$, $U(A) \not\subset V(A)$ for some nonempty subset A of X. But $U(A) \subset A \cup \{a\}$. Therefore $a \notin V(A)$. Also $A \in \mathfrak{U}$ for otherwise $X \setminus A \in \mathfrak{U}$ and hence $U(A) = A \subset V(A)$, a contradiction. Similarly there exists a nonempty subset B of X such that $a \notin V'(B)$ and $B \in \mathfrak{U}$. Let $b \in A \cap B \neq \varphi$ since $A \in \mathfrak{U}$, $B \in \mathfrak{U}$. Then $a \notin V(A \cap B)$ and $a \notin V'(A \cap B)$.

Thus $V_{b,a} \leqslant V$ and $V_{b,a} \leqslant V'$. This contradicts the fact that V' is the lattice complement of V in L(X) and hence the result.

4.2.8 THEOREM

In the lattice L(X) of closure operators on X, no element has more than one complement.

PROOF

Clearly I and D have unique complements in L(X). Let a closure operator V on X other than I and D have lattice complements V_1 and V_2 in L(X). Then both the set of ultra closure operators greater than or equal to V_1 and the set of ultra closure operators greater than or equal to V_2 are the same as the set of all ultra closure operators which are not greater than or equal to V, by Theorem 4.2.7. But then $V_1 = V_2$ by Theorem 4.2.5. Hence the result.

4.2.9 REMARK

The lattice of topologies on a set X is not even modular when $|X| \geqslant 3$. Also it is not self dual when $|X| > 3$ (See [33]).

But the lattice of closure operators on a finite set X is dually isomorphic to the lattice of reflexive relations on X as shown in 4.2.1. The latter lattice is isomorphic with the lattice of all subsets of the set $\{(x,y) : x \in X, y \in X, x \neq y\}$ and hence distributive and self dual. Thus the lattice of closure operators on a finite set X is also distributive and self dual. But when X is infinite, the lattice of closure operators is not self dual. Since then the number of dual atoms will be $2^{2^{|X|}}$, as in the case of the lattice of topologies, but the number of atoms is equal to $|X|$.

An important problem is to determine whether the lattice of closure operators on an infinite set is distributive or even modular or not. We could not yet solve it.

4.3 SOME SUB-LATTICES OF L(X)

In this section we discuss some sublattices of L(X).

4.3.1 NOTE

The lattice of topologies has two important sublattices, namely, the lattice of principal topologies and the lattice of T_1 topologies generated by the set of all principal ultratopologies and the set of all non-principal ultratopologies respectively. A topology T on X is principal if and only if T is the intersection of all principal ultratopologies finer than T if and only if arbitrary intersection of open sets of (X,T) is open in it if and only if $\overline{\bigcup_\alpha A_\alpha} = \bigcup_\alpha \overline{A_\alpha}$ for every collection $\{A_\alpha\}$ of subsets of X where $\overline{A}$ denotes the closure $A \subset X$, in (X,T). Also ∇ is a dual isomorphism from the lattice of pre-orders on X onto the lattice of principal topologies on X (See [1], [27]).

In the case of L(X), we have

4.3.2 THEOREM

The set of all T_1 closure operators on X form a sublattice of L(X) generated by the set of all non-principal ultra closure operators.

PROOF

Nonprincipal ultra closure operators are T_1 since nonprincipal ultratopologies are T_1 (see [33]).

Let V be any T_1 closure operator. It is the infimum of all ultra closure operators greater than or equal to it. Also since V is T_1, no principal ultra closure operator can be greater than or equal to V. Hence V is the infimum of all non principal ultra closure operator greater than or equal to it. Then the result follows by noting that the T_1 closure operators on X form a subinterval of L(X) containing all closure operators greater than or equal to the closure operator C_0 associated with the cofinite topology on X.

4.3.3 NOTE

The natural generalization of the concept of a principal topology is that of a quasi-discrete closure operator in the sense of [8]. A closure operator V on X is called quasi-discrete if $V(\bigcup_\alpha A_\alpha) = \bigcup_\alpha V(A_\alpha)$ for every collection $\{A_\alpha\}$ of subsets of X. Also it is proved in [8], that $\vee$ is a dual isomorphism from the lattice of reflexive relations onto the lattice of quasi-discrete closure operators. When X is finite, every closure operator on X is quasi-discrete. But when X is infinite, the quasi-discrete closure operators on X do not form a sublattice of L(X) as shown below.

Let $X = A \cup B$, where A and B are disjoint subsets of X having the same cardinality. Define $V_1 : P(X) \longrightarrow P(X)$ such that

$$\begin{aligned} V_1(M) &= \varphi && \text{if } M = \varphi \\ &= M \cup A && \text{if } M \cap A = \varphi \text{ and } M \cap B \neq \varphi \\ &= M \cup B && \text{if } M \cap A \neq \varphi \text{ and } M \cap B = \varphi \\ &= X && \text{if } M \cap A \neq \varphi \text{ and } M \cap B \neq \varphi \end{aligned}$$

Then V_1 can be verified to be quasi-discrete closure operator on X. Also define $V_2 : P(X) \longrightarrow P(X)$ such that

$$\begin{aligned} V_2(M) &= \varphi && \text{if } M = \varphi \\ &= M \cup A && \text{if } M \cap A \neq \varphi \text{ and } M \cap B = \varphi \\ &= M \cup B && \text{if } M \cap A = \varphi \text{ and } M \cap B \neq \varphi \\ &= X && \text{if } M \cap A \neq \varphi \text{ and } M \cap B \neq \varphi \end{aligned}$$

Then V_2 can be verified to be quasi-discrete closure operator on X.

The lattice join $V_1 \vee V_2$ of V_1 and V_2 is the closure operator C_o on X defined by

$$\begin{aligned} C_o(A) &= A \text{ if } A \text{ is finite} \\ &= X \text{ otherwise} \end{aligned}$$

But C_0 is not quasi-discrete. Hence the quasi-discrete closure operators on X do not form a sublattice of L(X).

4.3.4 **THEOREM**

The function μ is a dual isomorphism from the lattice of reflexive relations on X onto the sublattice of L(X) containing all closure operators on X less than or equal to the closure operator C_0 associated with the cofinite closure operator.

PROOF

If R is any reflexive relation, μR is a closure operator on X less than or equal to C_0. Also, it can easily be seen that every closure operator on X less than or equal to C_0 is of the form μR for some reflexive relation R on X. By Theorem 3.3.5, μ is one-one and $R_1 \subset R_2$ if and only if $\mu R_2 \leqslant \mu R_1$. Hence the result.

4.3.5 NOTE

In view of the fact that the largest element of L(X) and the dual atoms of it are topological, we would like to ask the following question. What are the topologies on X such that all closure operators on X greater than the closure operator associated with it are topological?

A more general problem is to determine the intervals in the lattice $L(X)$ containing only topological closure operators. Some partial solutions to the first problem are given below.

4.3.6 THEOREM

Let A be a proper nonempty subset of X and

$$T = \{ M \subset X : \text{either } X \setminus A \subset M \text{ or } M \subset X \setminus A \}$$

Then every closure operator greater than the closure operator associated with the topology T are topological.

PROOF

Let V be the closure operator on X associated with T. Then

$$V(B) = B \text{ if } B \subset A \text{ or } A \subset B$$

$$= B \cup A \text{ otherwise}$$

Let V' be any closure operator such that $V < V'$. Let N be a nonempty proper subset of X such that

$$N \neq V'(N)$$

Since $N \neq V'(N) \subset V(A)$, $V(N) = N \cup A$

Then $V'(N) = N \cup A'$ for some $A' \subset A$

Thus $V'(V'(N)) = V'(N \cup A') = V'(N) \cup V'(A')$

$= V'(N) \cup A'$ (Since $A' \subset V'(A') \subset V(A') = A'$)

$= V'(N)$

Thus V' is topological. Hence the result.

4.3.6 DEFINITION (See [1])

A topological space (X,T) is called a T_F space if given a finite subset F of X and x in $X \setminus F$, either there exists an open set containing x which is disjoint with F or there exists an open set containing F which does not contain x.

4.3.7 REMARK

In [1], it is proved that a topological space (X,T) is T_F if and only if $\rho(T)$ is a partial order of length atmost 1. It can be seen that a partially ordered set $(X, \leq)$ is of length atmost 1 if and only if it does not contain three distinct elements x,y and z such that $x < y < z$.

4.3.8 THEOREM

If every closure operator on X greater than the closure operator associated with a topology T on X, is topological, then (X,T) is T_F.

PROOF

Suppose not. Let $R = \rho(T)$. Then (X,R) is a partially ordered set containing three distinct elements x,y and z such that xRy and yRz. Let $R' = R \setminus \{(x,z)\}$. Now $V \leqslant \vee R$ since $R = \rho(T)$ is the same as ρV associated with the closure operator V associated with T. Also $\vee R \leqslant \vee R'$ by Theorem 3.3.5. Thus $V \leqslant \vee R'$. Also $\vee R'$ is not topological since

$$\vee R'(\{x\}) = \{x,y\} \text{ and}$$

$$\begin{aligned} \vee R'(\{x,y\}) &= \vee R'(\{x\}) \cup \vee R'(\{y\}) \\ &= \{x,y\} \cup \{y,z\} \\ &= \{x,y,z\} \neq \vee R'(\{x\}) \end{aligned}$$

Hence the result.

4.3.9 REMARK

The original problem mentioned in 4.3.5 remains to be solved.

4.4 GROUP OF AUTOMORPHISMS OF L(X)

In this last section we would like to discuss a problem related to the group of automorphisms of the lattice L(X). This is to determine the points of L(X) which are left fixed by every automorphism of the lattice L(X).

4.4.1 NOTE

Let C_o be a function from P(X) into P(X) defined by

$$C_o(A) = A \text{ if } A \text{ is finite}$$
$$= X \text{ otherwise}$$

Then C_o is a closure operator on X. It is actually the closure operator associated with the cofinite topology and is called the cofinite closure operator. All infra closure operators are smaller than C_o since $C_o(\{x\}) = \{x\}$ for every x in X. Also if V is a closure operator larger than every infra closure operator, $V(\{x\}) = \{x\}$ for every x in X. Then V(F) = F for every finite subset F of X. Thus $C_o \leq V$. Hence C_o is the least upper bound of all infra closure operators.

4.2.2 THEOREM

The elements I,D and C_o are left fixed by every automorphism of L(X).

PROOF.

The fact that I and D are left fixed by every automorphism of L(X) follows since they are respectively the smallest and the largest elements.

C_0 is left fixed by every automorphism since it is the least upper bound of all infra closure operators on X, infra closure operators are mapped into infra closure operators by any automorphism of L(X) being atoms and automorphisms preserve least upper bound of arbitrary subsets.

The following theorem gives a sufficiency condition.

4.4.3 THEOREM

If a closure operator V on X is left fixed by every automorphism of L(X), then (X,V) is completely homogeneous.

PROOF

Let f be any permutation of X. If V is an element of L(X), consider the function $T_f V$ from P(X) into P(X) defined by

$$T_f V(A) = f^{-1}(V\,(f(A)))$$

$T_f V$ can be verified to be a closure operator on X. Also the function T_f from L(X) into L(X) defined by $T_f(V) = T_f V$ can be verified to an automorphism of L(X).

If V is a closure operator on X, left fixed by every automorphism of L(X), then clearly $T_f V = V$ for every permutation f of X.

Then $V(A) = f^{-1}(V(f(A)))$ for every A subset of X.

Then $f(V(A)) = V(f(A))$

Then f is a closure isomorphism of (X,V) for every permutation f of X. Thus (X,V) is completely homogeneous.

4.4.4 REMARK

According to Theorem 3.2.3 the completely homogeneous closure operators are precisely the closure operators associated with completely homogeneous topologies. Also, by Theorem 1.2.13, a topology is completely homogeneous if and only if it is either discrete, indiscrete or of the form

$$T_{\alpha} = \{\varphi\} \cup \{A \subset X : \text{Card}\ (X \setminus A) < \alpha\}$$

for some infinite cardinal number $\alpha \leqslant |X|$. Then in view of Theorem 4.4.3, every closure operator left fixed by every automorphism of L(X) are closure operators associated with topologies among these. Also by Theorem 4.4.2, three of these are shown to be left fixed by every automorphism of L(X). Now the problem reduces to determine whether the remaining completely homogeneous closure operators are left fixed by every automorphisms of L(X) or not. This problem remains yet to be solved.

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