Some Problems in Convexity and Topology

STUDIES ON FUZZY MATROIDS AND RELATED TOPICS

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Certificate

This is to certify that thesis entitled ***Studies on Fuzzy Matroids and Related Topics*** *is a bonafide record of the research work carried out by* ***Smt. Shiny Philip*** *under our supervision in the Department of Mathematics, Cochin University of Science and Technology. The results embodied in the thesis have not been included in any other thesis submitted previously for the award of any degree or diploma.*

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Declaration

I hereby declare that the work presented in this thesis entitled **Studies on Fuzzy Matroids and Related Topics** is based on the original work done by me under the supervision of Dr. R. S. Chakravarti and Dr. T. Thrivikraman, in the Department of Mathematics, Cochin University of Science and Technology, Kochi-22, Kerala; and no part thereof has been presented for the award of any other degree or diploma.

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Shiny Philip

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Chapter 1

Introduction

1.1 History and Development

1.1.1 Fuzzy set theory

The German Mathematician George Cantor (1845–1918) described a crisp set as a well defined collection of objects. A crisp set is defined in such a way as to dichotomize the individuals in some given universe of discourse (region of consideration) into two groups—members and nonmembers. A sharp, unambiguous distinction exists between the members of the class or category represented by the crisp set. However, many classification concepts we commonly employ and express in natural language describe sets that do not exhibit this characteristic. Instead their boundaries seen vague, and the transition from member to nonmember appears gradual rather than abrupt. Real situations are very often not crisp and deterministic and they cannot be described precisely. Such situations in our real life which are characterized by vagueness or imprecision cannot be answered just in 'yes' or 'no'. Thus an answer to capture the concept of imprecision in a way that would differentiate imprecision from uncertainty, the very simple idea put forward by the American Cyberneticist C. A. Zadeh [68] in 1965 as the generalization of the concept of the characteristic function of a set to allow for immediate grades of membership was the genesis of the concept of a fuzzy set. This in fact laid the foundation of fuzzy set theory. A fuzzy set can be defined mathematically by assigning to each possible individual in the universe of discourse, a value representing its grade of membership in the fuzzy set. This grade corresponds the degree to which that individual is similar or compatible with the concept represented by the fuzzy set. The membership grades are very often represented by real number values ranging in the closed interval between 0 and 1. The nearer the value of an element to unity, the higher the grade of its membership. The term fuzzy in the sense used here seems to have been first introduced by Zadeh [67] in 1962. This paper was followed in 1965 by the technical exposition [68] of just such a mathematics now termed the 'Theory of fuzzy sets'.

Although the range of values between 0 and 1, both inclusive, is the most commonly used, for representing membership grades, an arbitrary set with some natural total or partial ordering can in fact be used. Elements of this set are not required to be numbers as long as the ordering among them can be interpreted as representing various strengths of membership degree. Thus the membership set can be any set that is at least partially ordered and the most frequently used membership set is a lattice. J. A. Goguen [18] in 1967 introduced the notion of a fuzzy set with a lattice as the membership set. Fuzzy sets defined with a lattice as the membership set are called L- Fuzzy sets or L- sets where L is intended as an abbreviation for lattice. In this thesis, we consider the fuzzy set as a function from a nonempty set X to the interval [0, 1].

Owing to the fact that set theory is the corner stone of modern mathematics, a new and more general frame work of mathematics was established. Fuzzy mathematics is just a kind of mathematics developed in this framework. Because of this, a fuzzy set theory has a wide scope of applicability than classical set theory in solving various problems.

Applications appear in computer science, artificial intelligence, decision analysis, information science, system science, control engineering, expert systems, pattern recognition, management science, operations research, robotics etc.

Fuzzy set theory, a developing subject in mathematics is making in roads into different disciplines of mathematics also. Among various branches of mathematics, convexity was one of the many subjects where the notion of fuzzy set was applied.

1.1.2 Convexity theory

Convexity theory has been accepted to be of increasing importance in recent years in the study of extremum problems in many areas of applied mathematics. The concept of convexity which was mainly defined and studied in R^n in the pioneering works of Newton, Minkowski and others as described in [2], now finds a place in several other mathematical structures such as vector spaces, posets, lattices, metric spaces and graphs. This development is motivated by not only the need for an abstract theory of convexity generalizing the classical theorems in R^n due to Helly, Caratheodory etc; but also by the necessity to unify geometric aspects of all these mathematical structures. Though convex sets are defined in various settings, the most useful definition is based on the notion of betweenness. When X is a space in which such a notion is defined, a subset C of X is called convex provided that for any two points, x and y of C, C includes all the points between x and y.

The theory of convexity can be sorted into two kinds. One deals with concrete convexity and the other that deals with abstract convexity. In concrete situations it was considered by R. T. Rockafellar [48], Kelly and Weiss [25], S. R. Lay [29] and many others. In this thesis, we are mainly concentrating on abstract convexity.

A set X together with a collection $\mathcal{C}$ of distinguished subsets of X called convex sets form a convexity space or aligned space if the following axioms are satisfied.

C_1: $\emptyset \in \mathcal{C}$, $X \in \mathcal{C}$

C_2: $\mathcal{C}$ is closed under arbitrary intersections

C_3: $\mathcal{C}$ is closed for the union of totally ordered sub collections.

$\mathcal{C}$ is called an alignment or convexity on X. The convex hull of a set S is defined as Co $(S) = \cap\{A \in \mathcal{C} | S \subseteq A\}$. Those families of sets which satisfy C_1

and C_2 are known as Moore Families or closure systems. Axioms C_1 and C_2 were first used by F. W. Levi [30] in 1951 and later on by Eckhoff [12], Jamison [21], Kay and Womble [24] and Sierksma [54]. The term "alignment" is due to Jamison [21]. Hammer [19] has shown that for Moore families the axiom C_3 is equivalent to the "domain finiteness" condition which states that for each $S \subseteq X$, Co $(S) = \cup\{\text{Co}(T) | T \subseteq S, |T| < \infty\}$ ($|T|$ denotes the cardinality of T). Alternative terminologies for convexity space are "algebraic closure systems" [8], the closed sets associated with finitary hull operators and domain finite convexity spaces ([12, 19, 24, 54, 53, 55]).

The study of abstract hull operators has several valid ways of approach, convexity being only one. Convex hull operator plays a fundamental role in the convexity theory. It can be easily shown [21] that any hull operator and the alignment it generates agree on all finite sets. Thus in combinatorial studies (as those around the three classical convex invariants) which involve only hulls of finite sets, one can assume with out loss of generality that one has an alignment.

Topologies and alignments usually represent two different aspects of geometry. If topology is the rubber sheet geometry, in which structures can be stretched and deformed because only limiting behavior is important, then alignments are rigid sheet geometries basically combinatorial in nature. There is of course an important overlap of these two types of geometry.

In this discussion, an effort is made to stress the importance of the finitary property (that the hull operator is determined in general by its action on finite sets) and avoid forcing alignment into the mould of topology.

Since 1950's the theory of convexity spaces has branched and grown into several related theories. An elegant survey has been done by Van de Vel [60] whose work has been acclaimed as remarkable.

Regarding the application part of convexity theory, interesting problems attempted to include the determination of computational complexity of convex hulls and computational complexity of the evaluation of convex invariants, problems of pattern recognition, optimization etc. A bibliography on digital and computational convexity has been prepared by Ronse [49].

The traditional definition of convexity (in terms of line segments) in Euclidean space are not applicable to discrete spaces, so it is necessary to investigate a more abstract axiomatic formulation in terms of alignment axioms. This results in discrete models of aligned spaces that hold the exchange axiom. These correspond to affine geometries which give rise to matroids.

1.1.3 Matroid theory

Matroids are an abstraction of several combinatorial objects—among them graphs and matrices. The word 'matroid' was coined by Whitney in 1935 in his landmark paper "on the abstract properties of linear dependence" [65]. In defining a matroid, Whitney tried to capture fundamental properties of dependence that are common to graphs and matrices. Almost simultaneously, Birkhoff showed that a matroid can be interpreted as a geometric lattice, Maclane showed that matroids have a geometric representation in terms of points, planes, dimension 3 spaces etc. Since then, it has been recognized that matroids arise naturally in combinatorial optimization and can be used as a framework for approaching a diverse variety of combinatorial problems. Often the term combinatorial geometry is used instead of simple matroids.

A matroid can be defined in several equivalent ways. Each of them is based on an axiom system. The primitive object of each axiom system can be identified with either the primitive or some derived objects of every other axiom system. The existence of various equivalent approaches giving rise to equivalent axiomatizations is one of the main features of the theory of matroids. There is no one preferred or customary definition; in that respect, matroids differ from many other mathematical structures such as groups and topologies. Significant definitions of a matroid include those in terms of independent sets, bases, circuits, flats, closure operators and rank functions.

In combinatorics, a branch of mathematics, a matroid or independence structure is a structure that captures the essence of a notion of "independence" that generalises linear independence in vector spaces. One of the most valuable definitions is that in terms of independence. In this definition, a finite matroid M is a pair (E, I) where E is a finite set and I is a collection of subset of E (called the independent sets) with the following properties.

1. The empty set is independent (alternatively at least one subset of E is independent)

2. Every subset of an independent set is independent.

3. If A and B are two independent sets and A has more elements than in B, then $\exists$ an element in A which is not in B and when added to B still gives an independent set.

A subset of E that is not independent is called dependent. A maximum independent set - *i.e.*, an independent set which becomes dependent on adding any element of E is called a basis for the matroid. It is a basic result of a matroid theory, directly analogous to similar theorem of linear algebra that any two bases of a matroid M have the same number of elements. This number is called the rank of M.

Matroids are important combinatorial structures both from the point of view of theory and applications. Matroid theory is one of the areas that straddles across several branches of discrete mathematics such as combinatiorics, graph theory, finite fields, algebra and coding theory. Matroids are a unifying concept in which some problems in graph theory, design theory, coding theory and combinatorial optimization becomes simpler to understand. One of the subjects to which applications appear is the electrical network theory.

1.2 Summary of the thesis

The thesis contains 5 chapters.

The first chapter briefly describes the history and development of the subject, the summary of the thesis and the pre-requisites including some basic definitions and results which are required in the subsequent chapters.

Motivated by the theories for convex structures and fuzzy sets, we try to develop fuzzy convexity theory parallel to that of a convexity theory in an abstract setting. In [60], matroid has been defined as a convexity space satisfying the exchange law or equivalently flats of the independent structure form matroid. The definition of matroid given in [58] is not equivalent to the definition given in [60] as it does not satisfy the non-degeneracy law of the collection of independent sets. In the 2nd chapter, we introduce the notion of a fuzzy matroid analogous to that in [60]. In section 1, we define fuzzy matroid. Section 2 deals with some elementary properties like preservation of a fuzzy matroid under fuzzy convexity preserving (FCP) and fuzzy convex to convex (FCC) functions in the sense of [51].

In chapter 3, as a continuation of the study done in chapter 2, we define the notion of fuzzy independent structures analogues to that in [60]. In section 2, we study some properties of fuzzy independent structures which includes the fuzzy transitive law of dependence and the properties of a hull operator which helps to establish the relation between fuzzy matroids and fuzzy independent structures. Also it is proved that for a fuzzy convexity space $(X, \mathcal{C})$, Co

$(A) = \vee\{\mathrm{Co}\,(F) | F \subseteq A,\ F$ is finite$\}$ whose importance according to us is in lifting results from finite to the general case. At the end of the section, it is shown that fuzzy matroids form fuzzy independent structures as in the crisp case. But the converse need not be true and it is illustrated by a counter example.

In the 4th chapter, section 1 describes various types of fuzzy matroids derived from vector space and section 2 discusses some of their properties.

The classical numbers of Helly, Caratheodory and Radon play a central role in abstract convexity. Each of them is defined as a degree of independence tolerated by a convex structure. Various notions of dependence of a non empty finite set in a crisp convexity space and based on this, some characterization of the classical convex invariants namely Helly number, Caratheodory number, Radon number and Exchange number are available in literature (*c.f.* [60]).

In the 5th chapter, we try to extend different types of dependence defined in [60] to the fuzzy context and reveal some interesting inter-relations among them. In the first section we introduce the fuzzy analogues of the properties of the crisp convex structure like join hull commutativity and cone union property as given in [60]. After that in section 2, different types of dependence in a fuzzy convexity space are defined and explored. Finally in the 3rd section, we get certain interrelationship among different types of dependence in the fuzzy context.

The thesis ends with a conclusion of the work done and further scope of study. Some of the results contained in this thesis have been presented in seminars or published/accepted for publication in journals as below.

1. Shiny Philip, A Note on Fuzzy Matroid (to appear in The Journal of Fuzzy Mathematics).

2. Shiny Philip, A Note on Fuzzy Independent Structures (Proceedings of the National Seminar on "Fuzzy Mathematics and Graph Theory", St. Teresa's College, Ernakulam, July 23–25, 2009).

3. Shiny Philip, A Study on Various Fuzzy Matroids from Vector Spaces (Communicated).

4. Shiny Philip, On Various Notions of Dependence in a Fuzzy Convexity Space. (to appear in the Oriental Journal of Applied Mathematics).

5. Shiny Philip, Relation between Fuzzy Matroids and Fuzzy Independent Structures, Bulletin of Kerala Mathematics Association, Vol.5, No.1 (2008, December) 63–80.

6. Shiny Philip, Some Results on Fuzzy Matroid (Proceedings of The International Seminar on Recent Trends in Topology and its Applications, St. Joseph's College, Irinjalakuda, March 2009).

1.3 Basic definitions and properties

We give below the essential preliminaries needed.

1.3.1 Fuzzy subset and its properties

A fuzzy subset A on a nonempty set X is a function from X to $I = [0, 1]$.

The set of all fuzzy subsets of X is denoted by I^X.

A fuzzy point a_α, $\alpha > 0$ is a fuzzy subset defined as

$$a_\alpha(y) = \begin{cases} \alpha, & \text{if } y = a \\ 0, & \text{otherwise.} \end{cases}$$

Support of a fuzzy subset A is

$$\text{supp } A = \{x \in X | A(x) > 0\}.$$

A fuzzy subset is said to be finite if its support is finite.

The set of all finite fuzzy subsets of X is denoted by I^X_{fin}.

If $\mathcal{A} \subset I^X$ we define $\vee\mathcal{A}$ a fuzzy subset with $(\vee\mathcal{A})(x) = \sup \mathcal{A}(x)$ for all $x \in X$ as the union and $\wedge\mathcal{A}$ a fuzzy subset with $(\wedge\mathcal{A})(x) = \inf \mathcal{A}(x)$ for all $x \in X$ as the intersection.

Also for a fuzzy subset A, its complement A' is defined as $A'(x) = (A(x))'$ for all $x \in X$.

In particular, the complement a'_α of a fuzzy point a_α is defined as

$$a'_\alpha(y) = \begin{cases} 1 - \alpha, & \text{if } y = a \\ 1, & \text{otherwise} \end{cases}$$

Given two fuzzy sets $A, B \in I^X$, we say that A is a subset of B and write $A \subseteq B$ (or $A \leq B$) if $A(x) \leq B(x)\ \forall x \in X$. In particular, $a_\alpha \subseteq B$ (or $a_\alpha \leq B$) if $a_\alpha(x) \leq B(x)\ \forall x \in X$.

$$\text{i.e., } a_\alpha \subseteq B \text{ (or } a_\alpha \leq B) \quad \text{if } B(a) \geq \alpha.$$

When $a_\alpha \subseteq B$ (or $a_\alpha \leq B$), we say that the fuzzy point a_α belongs to B and is denoted also by $a_\alpha \in B$.

Also $A \backslash B = A \wedge B'$.

For details regarding fuzzy subset and its properties, see [27].

Definition 1.3.1. *[27] Let $f : X \to Y$ be an ordinary map, then we have the maps $f : I^X \to I^Y$ and : $f^{-1} : I^Y \to I^X$ where $\forall A \in I^X$,*

$$f(A)(y) = \vee\{A(x)\,|x \in X,\ f(x) = y\}, \quad \forall y \in Y$$

[This means that $f(A)(y) = 0$ if there is no x with $f(x) = y$] and for all $B \in I^Y$,

$$f^{-1}(B)(x) = B(f(x)), \quad \forall x \in X.$$

Note 1.3.2. *[27] f and f^{-1} preserves arbitrary joins.*

We recall the following definition, available in literature (cf. [51])

Definition 1.3.3. *A family $\mathcal{C}$ of subsets of X is called a convexity if*

(i) $\phi, X \in \mathcal{C}$

(ii) If $C \subset \mathcal{C}$ is nonempty, then $\cap C \in \mathcal{C}$.

(iii) If $C \subset \mathcal{C}$ is nonempty and totally ordered by inclusion, then $\cup C \in \mathcal{C}$

The pair $(X, \mathcal{C})$ is a convex structure or a convexity space. Members of $\mathcal{C}$ are called convex sets.

Analogously, we have the definition in the fuzzy context.

Definition 1.3.4. *A family $\mathcal{C}$ of fuzzy subsets of X is called a fuzzy convexity if*

(i) $\underline{0}, \underline{1} \in \mathcal{C}$

(ii) If $F \subseteq \mathcal{C}$ is nonempty, then $\wedge F \in \mathcal{C}$.

(iii) If $F \subseteq \mathcal{C}$ is nonempty and totally ordered by inclusion, then $\vee F \in \mathcal{C}$

The pair $(X, \mathcal{C})$ is a fuzzy convex structure or a fuzzy convexity space. Members of $\mathcal{C}$ are called fuzzy convex sets.

Note 1.3.5. *$\underline{a}$ denotes a constant function whose value at x is 'a' $\forall x \in X$.*
If F is any fuzzy subset, then convex hull of F,

$$Co(F) = \wedge\{C \in \mathcal{C} | F \subseteq C\}.$$

Definition 1.3.6. *[51] Let $X_1 = (X, \mathcal{C}_1)$ and $X_2 = (X, \mathcal{C}_2)$ be two fuzzy convexity spaces. Let $f : X_1 \to X_2$ be a function. Then f is said to be*
(i) a fuzzy convexity preserving function (FCP), if for each fuzzy convex set C in X_2, $f^{-1}(C)$ is a fuzzy convex set in X_1
(ii) a fuzzy convex to convex function (FCC), if for each fuzzy convex set C in X_1, the fuzzy subset $f(C)$ is convex in X_2.

Proposition 1.3.7. *[51] Let X_1 and X_2 be two fuzzy convexity spaces. Let $f : X_1 \to X_2$ be a function. Then*
(i) f is an FCP function iff $f(Co(F)) \subseteq Co(f(F))$ for every finite fuzzy subset F in X_1,
(ii) f is an FCC function iff $f(Co(F)) \supseteq Co(f(F))$ for every finite fuzzy subset F in X_1.

Proposition 1.3.8. *[60] Let X be a join-hull commutative (JHC) space and $F \subseteq X$ be a finite set. If X has decomposable segments and F has at least two points, then for all $x \in Co(F)$,*

$$Co(F) = \cup_{a \in F} Co(\{x\} \vee (F \backslash a)).$$

Definition 1.3.9. *[27] A binary relation R on a set X is defined as any subset of $X \times X$. If $R \subset X \times X$ and $(x, y) \in R$, we say 'x is related to y' or 'x' is related to y under R and often write xRy.*

A binary relation R on a set X is said to be

(i) reflexive if for every $x \in X$, xRx

(ii) symmetric if for every $x, y \in R$,

$$xRy \Rightarrow yRx$$

(iii) transitive if for every $x, y \in X$, xRy *and* $yRz \Rightarrow xRz$

A binary relation which is reflexive, symmetric and transitive is called an equivalence relation and is denoted by $\equiv$.

For each $x \in X$, *the* R*-equivalence class in defined as the set* $[x] = \{y \in X | yRx\}$.

Chapter 2

Fuzzy Matroid

CONTENTS

Some of the results of this chapter will appear in the *Journal of Fuzzy Mathematics (2010)*

In [60], matroids have been defined as a convexity space satisfying the exchange laws or equivalently flats of the independent structure form matroids. The definition of matroids given in [58] is not equivalent to the definition given in [60] as the former does not satisfy the non-degeneracy laws of the collection of independent sets. In this chapter, we introduce the notion of a fuzzy matroid analogous to that in [60] and study some elementary properties like preservation under fuzzy convexity preserving (FCP) and fuzzy convex to convex (FCC) functions in the sense of [51].

2.1 Fuzzy matroid

Analogous to the notion of matroid [60], we introduce the concept of fuzzy matroid.

Definition 2.1.1. *A family $\mathcal{C}$ of fuzzy subsets of X is called a fuzzy convexity on X if*

(i) $\underline{0}, \underline{1} \in \mathcal{C}$

(ii) If $F \subseteq \mathcal{C}$ is nonempty, then $\wedge F \in \mathcal{C}$

(iii) If $F \subseteq \mathcal{C}$ is nonempty and totally ordered by fuzzy inclusion, then $\vee F \in \mathcal{C}$.

The pair $(X, \mathcal{C})$ is a fuzzy convexity space. The members of $\mathcal{C}$ are called fuzzy convex sets.

Note 2.1.2. *$\underline{a}$ denotes a constant function whose value at x is 'a' $\forall x \in X$.*
If A is any fuzzy subset of X, then convex hull of A,

$$Co(A) = \wedge\{C \in \mathcal{C} | A \subseteq C\}.$$

Fuzzy exchange law 2.1.3

If $A \subseteq I^X$ and if $p_\alpha, q_\beta \notin \text{Co}(A)$, then

$$p_\alpha \in \text{Co}(q_\beta \vee A) \Rightarrow q_\beta \in \text{Co}(p_\alpha \vee A).$$

Definition 2.1.4. *A fuzzy convexity space which satisfies the fuzzy exchange law is called a fuzzy matroid.*

Example 2.1.5. *Let $X = \mathbb{N} \cup \{0\}$ where $\mathbb{N}$ is the set of natural numbers. Consider the fuzzy convexity C on X as $C = \{\underline{0}, \underline{1}, A\}$ where $A = 1_{\frac{1}{2}} \vee 2_{\frac{3}{4}}$. Let $\underline{0} \neq B \in I^X$. Let $a_\alpha, b_\beta \notin Co(B)$ and $a_\alpha \in Co(b_\beta \vee B)$. If $Co(B) \neq A$ (i.e., $Co(B) = \underline{1}$), there is nothing to prove.*

Let $Co(B) = A$, i.e., $B \leq A$. Then we proceed as follows. $a_\alpha, b_\beta \notin Co(B) = A$ corresponds to the following cases.

(i) $a_\alpha = 1_\alpha, \alpha > \frac{1}{2}, b_\beta = 1_\beta, \beta > \frac{1}{2}$

(ii) $a_\alpha = 1_\alpha, \alpha > \frac{1}{2}, b_\beta = 2_\beta, \beta > \frac{3}{4}$

(iii) $a_\alpha = 2_\alpha, \alpha > \frac{3}{4}, b_\beta = 1_\beta, \beta > \frac{1}{2}$

(iv) $a_\alpha = 2_\alpha, \alpha > \frac{3}{4}, b_\beta = 2_\beta, \beta > \frac{3}{4}$

(v) $a_\alpha = x_\alpha, b_\beta = y_\beta, x, y \neq 1, 2$

Consider the case (i). Here

$$Co(b_\beta \vee B) = \underline{1} = Co(a_\alpha \vee B).$$

Let $\beta \leq \alpha$ and $a_\alpha \in \text{Co}(b_\beta \vee B)$

i.e., $\beta \leq \alpha \leq (\text{Co}(b_\beta \vee B))(a) = (\text{Co}(a_\alpha \vee B))(b)$

i.e., $b_\beta \in \text{Co}(a_\alpha \vee B)$

i.e., $a_\alpha \in \text{Co}(b_\beta \vee B) \Rightarrow b_\beta \in \text{Co}(a_\alpha \vee B)$.

Let $\beta > \alpha$ and $a_\alpha \in \text{Co}(b_\beta \vee B)$

To show $b_\beta \in \text{Co}(a_\alpha \vee B)$, assume the contrary

i.e.. assume if possible, $b_\beta \notin \text{Co}(a_\alpha \vee B)$

$$i.e.,\ \beta > \text{Co}(a_\alpha \vee B)(b) = \underline{1} = \text{Co}(b_\beta \vee B)(a) \geq \alpha.$$

i.e., $(b_\beta \vee B)(a) \leqslant \text{Co}(b_\beta \vee B)(a) < \beta$.

i.e., $(b_\beta \vee B)(a) < \beta$.

When $a = b$, $\beta \vee B(a) < \beta$.

Here $B(a) = B(b) \leq A(b) < \beta$ *as* $b_\beta \notin A$.

i.e. $\beta < \beta$ which is not true.

That is, we have when $\alpha < \beta$, $a_\alpha \in \text{Co}(b_\beta \vee B) \Rightarrow b_\beta \in \text{Co}(a_\alpha \vee B)$

i.e., fuzzy exchange law is satisfied.

In the remaining cases also, we have $\text{Co}(b_\beta \vee B) = \underline{1} = \text{Co}(a_\alpha \vee B)$.

Following same lines, we can show that the fuzzy exchange law is satisfied.

i.e., $(X, \mathcal{C})$ is a fuzzy matroid.

2.2 Some properties of a fuzzy convexity space under a FCP and FCC mapping

In this section, we study some elementary properties like preservation of a fuzzy matroid under fuzzy convexity preserving (FCP) and fuzzy convex to convex (FCC) functions. Also it is proved that a FCP and FCC image of fuzzy join hull commutative (JHC) space is JHC.

Definition 2.2.1. *A fuzzy convexity space is join hull commutative (JHC) if the* *following holds: if $A \in I^X$ is any nonzero fuzzy convex set and if $a_\alpha \in I^X$, then*

$$Co(a_\alpha \vee F) = \vee_{x_\beta \in F} Co(a_\alpha \vee x_\beta)$$

Proposition 2.2.2. *A FCP and FCC image of a fuzzy JHC space is JHC.*

Proof. Let X and Y be fuzzy convexity spaces and let $f : X \to Y$ be a FCP and FCC surjection where X is assumed to be a fuzzy JHC space. Let $p_\alpha \in Y$ and $C \subseteq Y$ be a fuzzy convex set. Fix a fuzzy point $p'_\beta \in f^{-1}(p_\alpha)$. Then f maps the fuzzy subset Co $(p'_\beta \vee f^{-1}(C))$ onto Co $(p_\alpha \vee C)$

$$i.e., \quad f(\text{Co}(p'_\beta \vee f^{-1}(C))) = \text{Co}(p_\alpha \vee C)$$

For,

$$\begin{aligned} f(\text{Co}(p'_\beta \vee f^{-1}(C))) &= \text{Co}(f(p'_\beta \vee f^{-1}(C))) \text{ by Proposition 1.3.7} \\ &= \text{Co}(p_\alpha \vee C) \qquad (2.2.1) \end{aligned}$$

since $f(p'_\beta) = p_\alpha$, $f(f^{-1}(C)) = C$, $f(A \vee B) = f(A) \vee f(B)$.

Since X is JHC,

$$\text{Co}(p'_\beta \vee f^{-1}(C)) = \vee Co(p'_\beta \vee c_r).\ c_r \in f^{-1}(C) \qquad (2.2.2)$$

where $f^{-1}(C)$ is convex in X corresponding to a fuzzy convex set C in Y. That is

$$f(\mathrm{Co}(p'_\beta \vee f^{-1}(C))) = f(\vee_{c_r \in f^{-1}(C)} \mathrm{Co}(p'_\beta \vee c_r)), \text{ taking } fof(2.2.2) \quad (2.2.3)$$

$$= \vee_{c_r \in f^{-1}(C)} f(\mathrm{Co}(p'_\beta \vee c_r)) \text{ since } f(A \vee B) = f(A) \vee f(B)$$

$$= \vee_{c_r} \mathrm{Co}(f(p'_\beta \vee c_r)) \text{ since } f \text{ is FCC and FCP}$$

$$= \vee_{c_{r_1} = f(c_r) \in C} \mathrm{Co}(p_\alpha \vee C_{r_1}) \quad (2.2.4)$$

From (2.2.1) and (2.2.4),

$$f(\mathrm{Co}(p'_\beta \vee f^{-1}(C))) = \mathrm{Co}(p_\alpha \vee C) = \vee_{C_{r_1}} \mathrm{Co}(p_\alpha \vee C_{r_1}).$$

That is Y is JHC. □

Proposition 2.2.3. *A FCP and FCC image of a fuzzy matroid is a fuzzy matroid.*

Proof. Let X and Y be two fuzzy convexity space and suppose that X is a fuzzy matroid. Let $f : X \to Y$ be an FCP, FCC surjection. Suppose $A \subseteq Y$ and let $p_\alpha, q_\beta \in Y$ be such that $p_\alpha, q_\beta \notin \mathrm{Co}(A)$ and $p_\alpha \in \mathrm{Co}(q_\beta \vee A)$. We have to show $q_\beta \in \mathrm{Co}(p_\alpha \vee A)$. For this, we fix a fuzzy point $q'_\beta \in f^{-1}(q_\beta)$.

$$q'_\beta \in f^{-1}(q_\beta) = q_\beta \circ f$$

$$\Leftrightarrow q'_\beta(x) \leq (q_\beta \circ f)(x)\ \forall x \in X$$

$$\Leftrightarrow \beta \leq (q_\beta \circ f)(q') = q_\beta(f(q'))$$

$$\Leftrightarrow f(q') = q.$$

Claim 1. $f(q'_\beta) = q_\beta$ if $f(q') = q$.

By definition,

$$\begin{aligned} f(q'_\beta)(q) &= \sup\{q'_\beta(x) | f(x) = q\} \\ &= \beta \ \text{ since } f(q') = q \end{aligned}$$

For $y \neq q$,

$$f(q'_\beta)(y) = 0 \ \text{ since } q'_\beta(x) = 0 \ \forall x \neq q'$$

i.e., $f(q'_\beta) = q_\beta$.

Hence there is a fuzzy point $p'_\alpha \in f^{-1}(p_\alpha)$ such that $p'_\alpha \in \text{Co}(q'_\beta \vee f^{-1}(A))$.

Claim 2. $p'_\alpha, q'_\beta \notin \text{Co}(f^{-1}(A))$.

On the contrary, assume $q'_\beta \in \text{Co}(f^{-1}(A))$

i.e., $q'_\beta \vee f^{-1}(A) \leq \text{Co}(f^{-1}(A)) \vee f^{-1}(A) = \text{Co}(f^{-1}(A))$

i.e., $\text{Co}(q'_\beta \vee f^{-1}(A)) \leq \text{Co}(f^{-1}(A))$.Then,

$$\begin{aligned} f(\text{Co}(q'_\beta \vee f^{-1}(A))) &\leq f(\text{Co}(f^{-1}(A))) \\ &\leq \text{Co}(f(f^{-1}(A))) \ \text{ since } f \text{ is FCP.} \\ &= \text{Co}(A) \end{aligned}$$

i.e., $p_\alpha = f(p'_\alpha) \in f(\text{Co}(q'_\beta \vee f^{-1}(A))) \leq \text{Co}(A)$.

which is a contradiction to $p_\alpha \notin \text{Co}(A)$.

Therefore, our assumption is wrong

$$\textit{i.e.}, \ q'_\beta \notin \text{Co}(f^{-1}(A)).$$

Similarly, we can show that $p'_\alpha \notin \text{Co}(f^{-1}(A))$.

Since $p'_\alpha, q'_\beta \notin \text{Co}(f^{-1}(A))$, by the property of the fuzzy matroid X, we have

$$p'_\alpha \in \text{Co}(q'_\beta \vee f^{-1}(A)) \Rightarrow q'_\beta \in \text{Co}(p'_\alpha \vee f^{-1}(A)).$$

$$\begin{aligned} \text{i.e., } q_\beta = f(q'_\beta) &\in f(\text{Co}(p'_\alpha \vee f^{-1}(A))) \\ &\leq \text{Co}(p_\alpha \vee A) \\ \text{i.e., } q_\beta &\in \text{Co}(p_\alpha \vee A) \end{aligned}$$

i.e., Y is a fuzzy matroid. □

Chapter 3

Relation between fuzzy matroids and fuzzy independent structures

CONTENTS

Almost all the results of this chapter appeared as a research paper in *Bulletin of Kerala Mathematics Association* (2008) [45]

In [60], (crisp convexity theory), matroid has been defined as a convexity space satisfying the exchange law and it is shown that an independent structure (a collection of independent sets satisfying finitary, non degenaracy and replacement laws on a non-empty set) can be constructed from a matroid and conversely. In chapter 2, the above concept of a matroid was extended by us to the fuzzy context. Here we continue the study and try to define fuzzy independent structures. In this chapter, we show that fuzzy matroids form fuzzy independent structures, but the converse need not be true.

3.1 Fuzzy independent structures

Analogous to the notion of independent structures [60], we introduce the concept of fuzzy independent structures.

Definition 3.1.1. *Let (X, C) be an arbitary fuzzy convexity space. A non zero fuzzy subset F is called convexly independent (or independent) provided $x_\alpha \notin Co(F \backslash x_\alpha)$ for all $x_\alpha \in F$, $\alpha > \frac{1}{2}$; we say F is convexly $*$-independent if $x_\alpha \notin Co(F \backslash x_\alpha)$ for all $x_\alpha \in F$, $\alpha > 0$.*

Definition 3.1.2. *The collection ϵ of independent sets of a fuzzy matroid is said to satisfy the non-degeneracy law iff all fuzzy singletons are in ϵ.*

Definition 3.1.3. *The collection ϵ of independent sets of a fuzzy matroid is said to satisfy the finitary law if the following holds: A fuzzy subset $F \in \epsilon$ iff all non-zero finite fuzzy subsets of F are in ϵ.*

Definition 3.1.4. *The collection ϵ of independent sets of a fuzzy matroid is said to satisfy the replacement law provided for every finite fuzzy subsets $A, B \in \epsilon$ with card (supp B) $>$ card (supp A), there is a fuzzy point $b_\beta \in B \backslash A$ such that $A \vee b_\beta \in \epsilon$.*

Definition 3.1.5. *A pair (X, ϵ) consisting of X and a family ϵ of nonzero fuzzy independent sets of X satisfying laws given in the definitions 3.1.2, 3.1.3 and 3.1.4 is called a fuzzy independence structure and the members of ϵ are independent sets.*

Definition 3.1.6. *A fuzzy point p_α depends on a fuzzy subset A, if $p_\alpha \in A$ or if there is a finite fuzzy independent subset $B \subseteq A$ such that $B \vee p_\alpha$ is dependent (i.e., it does not belong to ϵ).*

Definition 3.1.7. *A flat is a fuzzy subset containing each fuzzy point which depends on it.*

Example 3.1.8. *Let* $X = \{1, 2, 3\}$, $I = [0, 1]$.
Consider the fuzzy convexity $\mathcal{C}$ *on* X *as* $\mathcal{C} = \{\underline{0}, \underline{1}, G \in I^X | G \leq F\}$ *where* $F = 1_{0.6} \vee 2_{0.7}$.

Let $F_1 = 1_{0.52}$, $1_{0.51} \in F_1$, $1'_{0.51}(x) = \begin{cases} 0.49 & \text{if } x = 1 \\ 1 & \text{otherwise.} \end{cases}$

Then $1_{0.51} \notin Co(F_1 \backslash 1_{0.51})$ *where* $F_1 \backslash 1_{0.51} = F_1 \wedge 1'_{0.51}$.

Similarly, we can see that $x_\alpha \notin Co(F_1 \backslash x_\alpha)$ *for all* $x_\alpha \in F_1$, $\alpha > \frac{1}{2}$.

That is $F_1 = 1_{0.52}$ is convexly independent. In a similar way, we can see that the fuzzy points $x_\alpha \in F$, $\alpha > \frac{1}{2}$ are convexly independent.

Also the fuzzy points 1_α, $\alpha \leq 0.6$, 2_β, $\beta \leq 0.7$ depend on F as they belong to F.

Consider the fuzzy point $2_{0.8} \notin F$. Then

$$1_{0.51} \in \text{Co}((1_{0.52} \vee 2_{0.8}) \backslash 1_{0.51}) = \underline{1}$$

where $1_{0.52}$ is a convexly independent set $\subseteq F$ and $1_{0.51} \in 1_{0.52} \vee 2_{0.8}$.
i.e. the fuzzy subset $1_{0.52} \vee 2_{0.8}$ is dependent.

i.e. the fuzzy point $2_{0.8}$ depends on F.
Similarly, we can see that the fuzzy points 1_{α_1}, $\alpha_1 > 0.6$, 2_{β_1}, $\beta_1 > 0.7$ and 3_1 depend on F.
i.e., $1_1 \vee 2_1 \vee 3_1$ (i.e., $\underline{1}$) is a fuzzy subset containing all the fuzzy points depending on it. i.e., $\underline{1}$ is a flat.
Here F is not a flat as it does not contain all fuzzy points depending on that.

Example 3.1.9. *Let* $X = \{1, 2, 3\}$.
Consider the fuzzy convexity $\mathcal{C}$ *on* X *as* $\mathcal{C} = I^X$.
Let $\epsilon = \{F \subseteq I^X | F(x) > \frac{1}{2}\}$.

By non-degeneracy law , $x_1 \in \epsilon$ where $x \in X$.
By finitary law , all x_α's, $\alpha > \frac{1}{2} \in \epsilon$.
Clearly $\underline{1} \in \epsilon$ as $x_\alpha \notin \mathrm{Co}(\underline{1} \backslash x_\alpha)$, $x_\alpha \in \underline{1}$, $\alpha > \frac{1}{2}$.
Let $A, B \in \epsilon$ and # supp $(A) >$ #supp(B).
Let $b_\beta \in A \backslash B = A \wedge B'$.

$$\textit{Then } B \vee b_\beta \in \epsilon \text{ since } z_\gamma \notin \mathrm{Co}((B \vee b_\beta) \backslash z_\gamma)$$

where $z_\gamma \in B \vee b_\beta$, $\gamma > \frac{1}{2}$.
Then the collection of independent sets is $\epsilon = \{F \in I^X | F(x) > \frac{1}{2}\}$ and (X, ϵ) from an independence structure.

3.2 Some properties of fuzzy independent structures

Here we discuss some relations between fuzzy matroids and fuzzy independent structures. It is proved that fuzzy matroids form fuzzy independent structures as in the crisp case. But the converse need not be true and it is illustrated by a counter example.

Proposition 3.2.1 (Fuzzy transitive law of dependence). *Let A be a finite fuzzy subset of a fuzzy independence structure. If x_α depends on A and y_β depends on $A \vee x_\alpha$, then y_β depends on A.*

Proof. We consider the non-trivial case for establishing this result. Let $x_\alpha \notin A$ and $y_\beta \notin x_\alpha \vee A$. As y_β depends on $A \vee x_\alpha$, there is an independent fuzzy subset $F \subseteq A \vee x_\alpha$ such that $F \vee y_\beta$ is dependent. We fix one with card (supp F) maximal. As x_α depends on A, there is an independent set $G \subseteq A$ with $G \vee x_\alpha$ dependent set. If card (supp G) $<$ card (supp F) by the replacement law, there exists some fuzzy point $b_\gamma \in F \wedge G'$ such that $G \vee b_\gamma \in \epsilon$. Also we can see that $b_\gamma \notin A$. Since if $b_\gamma \in A$, $\exists$ an independent set $G \subseteq A$ such that $G \vee b_\gamma$ is dependent which is a contradiction.

$$\therefore \text{card (supp } G) \geq \text{card (supp } F).$$

If y_β does not depend on A, $G \vee y_\beta$ is independent.
As card (supp $(G \vee y_\beta)$) $>$ card (supp F) by replacement law, $\exists$ a fuzzy point say $z_\delta \in (G \vee y_\beta) \wedge F'$ such that $F \vee z_\delta \in \epsilon$. But $z_\delta \neq y_\beta$, since $F \vee y_\beta$ is dependent and $z_\delta \notin A$.

i.e., y_β depends on A. □

Definition 3.2.2. *The collection ϵ of independent sets of a fuzzy matroid is said to satisfy the strong replacement law provided for any two fuzzy subsets $A, B \in \epsilon$ with card (supp B) $>$ card (supp A), there is a fuzzy point $b_\beta \in B \backslash A$ with $A \vee b_\beta \in \epsilon$.*

Proposition 3.2.3. *For a fuzzy convexity space (X, C),*

$$Co(A) = \vee\{Co(F) | F \subseteq A, F \text{ is finite}\}$$

Proof. Case 1. Let A be a finite fuzzy subset. We have $F \subseteq A$.

$$i.e. \quad Co(F) \subseteq Co(A) \quad \forall F$$

$$i.e. \quad \vee Co(F) \leq Co(A) \tag{3.2.1}$$

Let $x_\alpha \in Co(A)$.

$$\begin{aligned}
&i.e. && Co(A)(x) \geq \alpha. \\
&i.e. && (Co(\vee F))(x) \geq \alpha \\
&i.e. && Co(F)(x) \geq \alpha \text{ for some } F \subseteq \vee F = A \\
&i.e. && (\vee Co(F))(x) \geq (Co(F))(x) \geq \alpha. \\
&i.e. && (\vee Co(F))(x) \geq \alpha. \\
&i.e. && x_\alpha \in \vee Co(F). \\
&i.e. && Co(A) \leq \vee Co(F)
\end{aligned} \tag{3.2.2}$$

From (3.2.1) & (3.2.2), we have $Co(A) = \{\vee Co(F) | F \subseteq A,\ F \text{ is finite}\}$.

Case 2. Let A be an infinite fuzzy subset. *i.e.*, card (supp A) is infinite. Here we apply transfinite induction on card (supp A). We assume that the result is true for all fuzzy subsets which has smaller cardinality of its support than the cardinality of the supp (A). For each $a \in X$, let $P(a)$ be a fuzzy subset whose support is the set of all points in X less than 'a' where

$$P(a)(x) = \begin{cases} A(x), & \forall x \in \text{supp } P(a) \\ 0, & \text{otherwise} \end{cases}$$

Well-order supp (A) such that card (supp $P(a)$) $<$ card (supp A). The fuzzy subsets $Co(P(a))$, $a \in X$ form a chain in $\mathcal{C}$. Let $C = \vee Co(P(a)) \in \mathcal{C}$, since $\mathcal{C}$ being a fuzzy convexity, nested union of fuzzy convex sets is convex.

$A = \vee P(a) \leq \vee Co(P(a)) = C$

i.e., $A \leq C$

i.e., $Co(A) \leq Co(C) = C$

$$i.e., Co(A) \leq C \tag{3.2.3}$$

Also $P(a) \subseteq A$.

i.e., $Co(P(a)) \subseteq Co(A)\ \forall a \in X$

$$i.e., \ C = \vee Co(P(a)) \leq Co(A). \quad i.e., \ C \leq Co(A) \tag{3.2.4}$$

From (3.2.3) & (3.2.4), $C = Co(A)$.

By induction, $Co(P(a)) = \vee\{Co(F)|F \leq P(a), F \text{ finite}\}$, since

card (supp $P(a)$) $<$ card (supp A).

i.e., $Co(A) = C = \vee Co(P(a)) = \vee\{\vee Co(F)|F \leq P(a), F \text{ finite}\}$

$= \vee\{Co(F)|F \leq A, F \text{ finite}\}$, since $F \leq P(a) \leq A$.

$= \vee\{Co(F)|F \subseteq A, F \text{ finite}\}$ □

Proposition 3.2.4. *Let X be a set. Let $h : I^X_{fin} \to I^X$ be an operator satisfying the following conditions.*

(i) $h(\underline{0}) = \underline{0}$;

(ii) $F \leq h(F)$ *for each finite fuzzy subset* F;

(iii) For $F, G \in I^X_{ffin}$, $F \leq G \Rightarrow h(F) \leq h(G)$;

(iv) $h(h(F)) = h(F)$;

(v) $h(\vee F_i) = \vee_i h(F_i)$ *for any family of fuzzy subsets* $\{F_i, i \in I\}$ *which is totally ordered by inclusion. Then there is precisely one convexity on* I^X *with the hull operator equal to* h *on* I^X_{fin},

$$i.e., \quad \mathcal{C} = \{F|h(F) = F\}.$$

Conversely, the hull operator of any convexity on I^X satisfies the conditions (i–v).

Proof. (i) Necessity part

By (i), $h(\underline{0}) = 0$ *i.e.*, $\underline{0} \in \mathcal{C}$.

By (ii),

$$\underline{1} \leq h(\underline{1}) \tag{3.2.5}$$

but

$$h(\underline{1}) \leq \underline{1} \quad \text{always} \tag{3.2.6}$$

From (3.2.5) & (3.2.6),

$$h(\underline{1}) = \underline{1} \quad i.e., \ \underline{1} \in \mathcal{C}.$$

To show $\wedge F_i \in \mathcal{C}$ whenever $F_i \in \mathcal{C}$, we have to show $h(\wedge F_i) = \wedge F_i$.

$$F_i \in \mathcal{C} \Rightarrow h(F_i) = F_i \tag{3.2.7}$$

By (ii),

$$\wedge F_i \leq h(\wedge F_i) \tag{3.2.8}$$

We know $\wedge F_i \leq F_i \ \forall i$.

By (iii),

$$h(\wedge F_i) \leq h(F_i) = F_i \ \ \forall \, i \ \text{ by (3.2.7)}$$

$$i.e., \ h(\wedge F_i) \leq \wedge F_i \tag{3.2.9}$$

From (3.2.8) & (3.2.9), $h(\wedge F_i) = \wedge F_i$ *i.e.*, $\wedge F_i \in \mathcal{C}$.

To show $\vee F_i \in \mathcal{C}$ for any nested union of $F_i \in \mathcal{C}$, we have to show $h(\vee F_i) = \vee F_i$.

By (v), $h(\vee F_i) = \vee h(F_i) = \vee F_i$ by (3.2.7).

i.e. $\mathcal{C}$ is a fuzzy convexity and $(X, \mathcal{C})$ is a fuzzy convexity space.

Uniqueness follows from proposition 3.2.3.

(ii) Sufficiency part

Since $\underline{0} \in \mathcal{C}$, $h(\underline{0}) = \underline{0}$

i.e. condition (i).

If $F \in \mathcal{C}$, $h(F) = F$, *i.e.* $F \leq h(F)\ \forall F \in \mathcal{C}$.

i.e. Condition (ii).

If $F, G \in \mathcal{C}$, then $h(F) = F$, $h(G) = G$.

$$F \leq G \Rightarrow h(F) \leq h(G).$$

i.e. condition (iii).

$$\text{If } F \in \mathcal{C}, h(F) = F, \text{ then } h(h(F)) = h(F).$$

If F_i's are totally ordered and $\in \mathcal{C}$, $h(F_i) = F_i$.

Since $\mathcal{C}$ is a convexity, $F_i \in \mathcal{C} \Rightarrow \vee F_i \in \mathcal{C}$.

i.e., $h(\vee F_i) = \vee F_i = \vee h(F_i)$.

i.e., the hull operator associated with the convexity satisfies conditions (i-v). ☐

Lemma 3.2.5. *For a fuzzy matroid, the collection ϵ of independent sets satisfy the non-degeneacy law.*

Proof. Here we show that a_1's are convexly independent.

Case 1. When $\alpha < \frac{1}{2}$, $1 - \alpha > \frac{1}{2}$

$a_{1-\alpha} > a_{\frac{1}{2}} > a_\alpha$

i.e., $a_\alpha < a_{1-\alpha} \leq Co(a_{1-\alpha}) = Co(a_1 \backslash a_\alpha)$

i.e., a_1 is convexly dependent.

Case 2. When $\alpha = \frac{1}{2}$, $a_{\frac{1}{2}} \in Co(a_{\frac{1}{2}}) = Co(a_1 \backslash a_\alpha)$

i.e., a_1 is convexly dependent.

Case 3. When $\alpha > \frac{1}{2}$, $1 - \alpha < \frac{1}{2}$

$$a_{1-\alpha} < a_{\frac{1}{2}}. \quad Co(a_{1-\alpha}) < Co(a_{\frac{1}{2}}) \tag{3.2.10}$$

Also $a_{\frac{1}{2}} < a_\alpha$,

$$i.e., \quad Co(a_{\frac{1}{2}}) < Co(a_\alpha) \tag{3.2.11}$$

From (3.2.10) and (3.2.11),

$$Co(a_{1-\alpha}) < Co(a_{\frac{1}{2}}) < Co(a_\alpha) \tag{3.2.12}$$

We show $Co(a_{1-\alpha}) < a_\alpha$. For this, suppose the contrary

i.e. $Co(a_{1-\alpha}) \geq a_\alpha$.

i.e. $Co(a_\alpha) \leq Co(a_{1-\alpha})$ which is a contradiction to (3.2.12).

i.e., $Co(a_{1-\alpha}) < a_\alpha$.

i.e., $a_\alpha > Co(a_{1-\alpha}) = Co(a_1 \backslash a_\alpha)$.

i.e., a_1 is convexly independent. □

Lemma 3.2.6. *The collection ϵ of independent sets of a fuzzy matroid satisfies the finitary law.*

Proof. Let a fuzzy subset A be convexly independent.

i.e., $x_\alpha \notin Co(A\backslash x_\alpha)\ \forall\ x_\alpha \in A, \alpha > \frac{1}{2}$

i.e., $x_\alpha \notin Co(F\backslash x_\alpha)\ \forall\ F \leq A$, and $x_\alpha \in F, \alpha > \frac{1}{2}$

i.e., F is convexly independent.

i.e., A is convexly independent $\Rightarrow$ a subset F of A is convexly independent.

To show A is convexly independent whenever F is convexly independent, on the contrary we assume that A is convexly dependent.

i.e., $x_\alpha \in Co(A\backslash x_\alpha)$ for some $x_\alpha \in A$, $\alpha > \frac{1}{2}$.

i.e., $x_\alpha \in Co(F\backslash x_\alpha)$ for some $F \subseteq A$ and $x_\alpha \in F$, $\alpha > \frac{1}{2}$,

since $Co(A\backslash x_\alpha) = \vee\{Co(F\backslash x_\alpha) | F\backslash x_\alpha \leq A\backslash x_\alpha, F\backslash x_\alpha \text{ finite}\}$ by proposition 3.2.3.

i.e., F is convexly dependent.

i.e., a fuzzy subset F of A is convexly independent $\Rightarrow$ A is convexly independent.

Hence the finitary law. □

Lemma 3.2.7. *The collection of convexly $\star$ independent fuzzy subsets of a fuzzy matroid satisfies the strong replacement law.*

Proof. Let A, B be two fuzzy independent subsets with card (supp B) $>$ card (supp A). To prove the strong replacement law, on the contrary we assume that

$$x_\alpha \vee A \text{ is dependent } \forall\, x_\alpha \in B. \tag{3.2.13}$$

Let $b_\beta \in B$ be fixed. Then either $b_\beta \in Co(A)$ or $b_\beta \notin Co(A)$.

<u>Step 1</u> We show $B \subseteq Co(A)$.

Let $b_\beta \notin Co(A)$. Then $b_\beta \notin Co(A \backslash a_\gamma)$ where $a_\gamma \in A$. We know

$$\begin{aligned}(b_\beta \vee A)\backslash a_\gamma &= (b_\beta \vee A) \wedge a'_\gamma \\ &= (b_\beta \wedge a'_\gamma) \vee (A \wedge a'_\gamma) \\ &= b_\beta \vee (A \wedge a'_\gamma)\end{aligned}$$

By (3.2.13), $b_\beta \vee A$ is dependent. *i.e.*, for some $a_\gamma \in b_\beta \vee A$,

$$a_\gamma \in Co((b_\beta \vee A)\backslash a_\gamma) = Co(b_\beta \vee (A \wedge a'_\gamma)).$$

Hence $a_\gamma \notin Co(A \backslash a_\gamma)$, since A is independent.

i.e., if $b_\beta \notin Co(A)$, we have an $a_\gamma \in A$, so that $a_\gamma \in Co(b_\beta \vee (A \wedge a'_\gamma))$.

Since $b_\beta \notin Co(A \backslash a_\gamma)$ and $a_\gamma \notin Co(A \backslash a_\gamma)$, by the exchange law $b_\beta \in Co(a_\gamma \vee (A \wedge a'_\gamma)) \subseteq Co(A)$.

i.e. in all cases we have $b_\beta \in Co(A)$. *i.e.* $B \subseteq Co(A)$.

If A is an infinite fuzzy subset *i.e.* if card (supp A) is infinite, then A and B can be rearranged such that it is finite (preserving (3.2.13)).

We know $B \subseteq Co(A) = \vee\{Co(F(b_\beta)) | F(b_\beta) \subseteq A, F(b_\beta) \text{ is finite}\}$, by proposition 3.2.3

i.e. $\forall\, b_\beta \in B$, there is a finite set $F(b_\beta) \subseteq A$ with $b_\beta \in Co(F(b_\beta))$.

If card (supp A) is infinite, the set of all finite subsets of supp (A) is infinite.

i.e. card $(2_{\text{fin}}^{\text{supp}\,(A)})$ = card (supp A). Then we can see that card (supp B) > card $(2_{\text{fin}}^{\text{supp}\,(A)})$. Hence some finite set $F \subseteq A$ occurs as $F(b_\beta)$ for infinitely many $b_\beta \in B$. Take this F instead of A and the corresponding infinite subset of B instead of B. Then reduce B further to a finite size larger than card (supp F).

Again assume from now on that card (supp A) < card (supp B) < ∞. Among all possible pairs A, B of this kind satisfying (3.2.13), consider one for which the number n = card (supp B) − card (supp $(B \wedge A)$) is smallest. Here $n > 0$.

Fix $b_\beta \in B \backslash A$. Let $F \subseteq A$ be minimal with respect to the property that $b_\beta \in Co(F)$.

<u>Step 2</u> We show that $F \not\subseteq B$.

On the contrary we assume that $F \subseteq B$. Then $F \vee b_\beta \subseteq B \vee b_\beta = B$ for all $b_\beta \in B$.

i.e. $F \vee b_\beta \leq B$.

i.e. Since B is independent, $F \vee b_\beta$ is independent by finitary law which is a contradiction to (3.2.13).

$$F \not\subseteq B.$$

Hence there is a point $a_\gamma \in F \backslash B$ where $F(a) \leq 1 - \gamma$.

Now $b_\beta \notin Co(F \backslash a_\gamma)$ by the minimality of F and $a_\gamma \notin Co(F \backslash a_\gamma)$ by the independence of A since $F \leq A$ and A is independent.

<u>Step 3</u> We show that $Co(a_\gamma \vee (F \backslash a_\gamma)) = Co(F)$.

We know

$$\begin{aligned} a_\gamma \vee (F \wedge a'_\gamma) &= (a_\gamma \vee F) \wedge (a_\gamma \vee a'_\gamma) \\ &= F \wedge (a_\gamma \vee a'_\gamma) \\ &\leq F. \end{aligned}$$

$$i.e., \; Co(a_\gamma \vee (F \wedge a'_\gamma)) \leq Co(F) \tag{3.2.14}$$

Next we show $F \leq a_\gamma \vee (F \wedge a'_\gamma)$ iff $F(a) \leq 1 - \gamma$.

Let $F \leq a_\gamma \vee (F \wedge a'_\gamma)$

i.e. when $x = a$, $F(a) \leq \gamma \vee (F(a) \wedge (1 - \gamma))$

i.e. $F(a) \leq \gamma$ or $F(a) \leq F(a) \wedge (1 - \gamma)$

i.e. $F(a) \leq F(a) \wedge (1 - \gamma)$ as $F(a) \leq \gamma$ is not true.

i.e. $F(a) \leq F(a) \wedge (1 - \gamma) \leq F(a)$.

i.e. $F(a) = F(a) \wedge (1 - \gamma)$.

i.e. $F(a) \leq 1 - \gamma$.

$$i.e. \quad F \leq a_\gamma \vee (F \wedge a'_\gamma) \Rightarrow F(a) \leq 1 - \gamma \tag{3.2.15}$$

Conversely, let $F(a) \leq 1 - \gamma$.

$$i.e., \quad (a_\gamma \vee (F \wedge a'_\gamma))(a) = \gamma \vee (F(a) \wedge (1 - \gamma)) = \gamma \vee F(a) = F(a)$$

Also $(a_\gamma \vee F \wedge a'_\gamma)(x) = F(x) \; \forall \, x \neq a$

$$i.e. \text{ when } F(a) \leq 1 - \gamma, \quad F = a_\gamma \vee (F \wedge a'_\gamma) \tag{3.2.16}$$

From (3.2.15) & (3.2.16) we have

$$F \leq a_\gamma \vee (F \wedge a'_\gamma) \text{ where } F(a) \leq 1-\gamma$$

$$i.e. \quad Co(F) \leq Co(a_\gamma \vee (F \wedge a'_\gamma)) \tag{3.2.17}$$

From (3.2.14) & (3.2.17), we have the equality

$$i.e. \quad b_\beta \in Co(F) = Co(a_\gamma \vee (F \wedge a'_\gamma))$$

Then by the fuzzy exchange property of the matroid

$$a_\gamma \in Co(b_\beta \vee (F \backslash a_\gamma)) \leq Co(b_\beta \vee (A \backslash a_\gamma)).$$

Step 4 We show $b_\beta \notin Co(A\backslash a_\gamma)$.

On the contrary, we assume

$$b_\beta \in Co(A\backslash a_\gamma) \tag{3.2.18}$$

We know, $A\backslash a_\gamma \leq Co(A\backslash a_\gamma)$

$$b_\beta \vee (A\backslash a_\gamma) \leq b_\beta \vee Co(A\backslash a_\gamma) = Co(A\backslash a_\gamma) \quad \text{by (3.2.18)}$$

i.e. $a_\gamma \in Co(b_\beta \vee (A\backslash a_\gamma)) \leq Co(A\backslash a_\gamma)$

i.e. $a_\gamma \in Co(A\backslash a_\gamma)$ which is a contradiction to the independence of A.

$\therefore$ Our assumption is wrong.

i.e., $b_\beta \notin Co(A\backslash a_\gamma)$.

Step 5 We show $A' = b_\beta \vee (A\backslash a_\gamma)$ is independent.

i.e. we have to show that

$$b_\beta \notin Co(A'\backslash b_\beta) \tag{3.2.19}$$

and

$$a'_\gamma \notin Co(A'\backslash a'_\gamma) \tag{3.2.20}$$

where $a'_\gamma \neq b_\beta$.

For proving (3.2.19), on the contrary we assume that $b_\beta \in Co(A'\backslash b_\beta)$.

$$\begin{aligned} b_\beta \in Co(A'\backslash b_\beta) &= Co((b_\beta \vee (A\backslash a_\gamma))\backslash b_\beta) \\ &= Co((b_\beta \vee (A\backslash a_\gamma)) \wedge b'_\beta). \end{aligned}$$

Also $b_\beta \notin A\backslash a_\gamma$ as $b_\beta \notin A$.

Hence $b_\beta \vee (A\backslash a_\gamma) \leq B$.

i.e., $Co((b_\beta \vee (A\backslash a_\gamma)) \wedge b'_\beta) \leq Co(B \wedge b'_\beta)$.

i.e., $b_\beta \in Co((b_\beta \vee (A\backslash a_\gamma)) \wedge b'_\beta) \leq Co(B \wedge b'_\beta)$.

i.e., $b_\beta \in Co(B \wedge b'_\beta)$ which is not true since B is independent.

∴ Our assumption is wrong.

i.e., $b_\beta \notin Co(A'\backslash b_\beta)$. *i.e.*, (3.2.19)

For proving (3.2.20), on the contrary, we assume that for some $a'_\gamma \in A'$ different from b_β, it is true that

$$\begin{aligned} a'_\gamma &\in Co(A'\backslash a'_\gamma) \\ &= Co((b_\beta \vee (A\backslash a_\gamma))\backslash a'_\gamma) \\ &= Co(b_\beta \vee ((A\backslash a_\gamma)\backslash a'_\gamma)) \end{aligned}$$

By the fuzzy exchange law,

$$b_\beta \in Co(a'_\gamma \vee ((A\backslash a_\gamma)\backslash a'_\gamma)) \leq Co(A\backslash a_\gamma)$$

i.e., $b_\beta \in Co(A\backslash a_\gamma)$ which is a contradiction to $b_\beta \notin Co(A\backslash a_\gamma)$.

$\therefore$ our assumption is wrong.

i.e. $a'_\gamma \notin Co(A' \backslash a'_\gamma)$

i.e. A' is independent.

Step 6 We show $Co(A') = Co(A \vee b_\beta) = Co(A)$.

We know $b_\beta \in Co(A)$.

i.e. $b_\beta \vee A \leq A \vee Co(A) = Co(A)$.

$$i.e.\ Co(b_\beta \vee A) \leq Co(A) \tag{3.2.21}$$

Also $A \leq A \vee b_\beta$.

$$Co(A) \leq Co(b_\beta \vee A) \tag{3.2.22}$$

From (3.2.21) & (3.2.22),

$$Co(A) = Co(A \vee b_\beta) \tag{3.2.23}$$

We know $A \in Co(A')$ or $A \leq Co(A')$.

$$Co(A) \leq Co(A') \tag{3.2.24}$$

Also $A \backslash a_\gamma \leq A$, $A' = b_\beta \vee (A \backslash a_\gamma) \leq b_\beta \vee A$.

$$i.e. Co(A') \leq Co(b_\beta \vee A) \tag{3.2.25}$$

From (3.2.23), (3.2.24) & (3.2.25)

$$Co(A \vee b_\beta) = Co(A) \leq Co(A') \leq Co(b_\beta \vee A)$$

$$i.e. \quad Co(A') = Co(A \vee b_\beta) = Co(A)$$

i.e. each point of B is dependent on A' as well. This establishes (3.2.13) for A', B whereas the independent set A' has one more point in common with support of B in its support.

Then card (supp B) − card (supp $(B \backslash A)$) $< n$ which is a contradiction with the definition of n.

$\therefore$ our first assumption (3.2.13) is wrong.

i.e., we have the strong replacement law. $\square$

Remark 3.2.8. *The collection ϵ of independent fuzzy subsets of a fuzzy matroid satisfies the replacement law.*

From the Lemmas 3.2.5, 3.2.6 and remark 3.2.8 we have the following result.

Theorem 3.2.9. *Let $(X, \mathcal{C})$ be a fuzzy matroid. Then the collection ϵ of all independent fuzzy subsets of X satisfies the non-degeneacy, finitary and replacement laws and hence it is a fuzzy independence structure.*

Proposition 3.2.10. *Let (X, ϵ) be a fuzzy independence structure. Then the flats of (X, ϵ) form a fuzzy convex structure such that the convex hull of a fuzzy subset is the collection of all fuzzy points depending on it.*

Proof. We define an operator $h : I^X_{fin} \to I^X$ as follows:
$h(\underline{0}) = \underline{0}$ and for a non zero fuzzy subset $F \in I^X_{\text{fin}}$, let $h(F)$ be the set of all fuzzy points depending on F.

Let $p_\alpha \in h(p_{\alpha_1} \vee p_{\alpha_2} \vee \cdots \vee p_{\alpha_n})$ and p_{α_i}, $i = 1, 2, \ldots, n$ depends on the finite fuzzy subset F.

Let $G_0 = F$, $G_1 = F \vee p_{\alpha_1}$, $G_2 = F \vee (p_{\alpha_1} \vee p_{\alpha_2}), \ldots,$
$G_n = F \vee (p_{\alpha_1} \vee p_{\alpha_2} \vee \cdots \vee p_{\alpha_n})$.

p_{α_1} depends on $F = G_0$.

p_{α_2} depends on $F = G_0 \Rightarrow p_{\alpha_2}$ depends on $F \vee p_{\alpha_1} = G_1$ by finitary law.

Similarly p_{α_3} depends on $G_2 \cdots, p_{\alpha_n}$ depends on G_{n-1}.

Also $p_\alpha \in h(p_{\alpha_1} \vee \cdots \vee p_{\alpha_n}) \Rightarrow p_\alpha$ depends on $p_{\alpha_1} \vee p_{\alpha_2} \vee \cdots \vee p_{\alpha_n}$.

$\therefore p_\alpha$ depends on $F \vee (p_{\alpha_1} \vee \cdots \vee p_{\alpha_n}) = G_n$ by finitary law.

i.e. p_α depends on $G_n = G_{n-1} \vee p_{\alpha_n}$ where as p_{α_n} depends on G_{n-1}.

Then by proposition 3.2.1, p_α depends on $G_{n-1} = G_{n-2} \vee p_{\alpha_{n-1}}$.

Similarly we can see that p_α depends on $G_{n-2}, \ldots, p_\alpha$ depends on $G_1 = F \vee p_{\alpha_1}$.

Again since p_{α_1} depends on F and p_α depends on

$$G_1 = F \vee p_{\alpha_1}, \quad p_\alpha \text{ depends on } F \tag{3.2.26}$$

Hence $p_\alpha \in F$.

Also $p_\alpha \in F \Rightarrow p_\alpha$ depends on F. *i.e.* $p_\alpha \in h(F)$.

i.e. $F \leq h(F)$.

Let $F \leq G$

$$i.e.\ p_\alpha \in F \Rightarrow p_\alpha \in G$$

$$i.e.\ p_\alpha \in h(F) \Rightarrow p_\alpha \in h(G) \quad i.e.\ h(F) \leq h(G)$$

$$i.e.\ h(F) \leq h(G) \quad \text{whenever} \quad F \leq G.$$

Next we show $h(h(F)) = h(F)$.

$$\text{Clearly} \quad h(F) \leq h(h(F)) \tag{3.2.27}$$

Let $p_\alpha \in h(h(F))$ *i.e.* p_α depends on the set of all points depending on F. Then we have proved that p_α depends on F.

i.e. $p_\alpha \in h(F)$.

$$i.e. \quad h(h(F)) \leq h(F) \tag{3.2.28}$$

From (3.2.27) & (3.2.28), we have $h(h(F)) = h(F)$.

Now we prove $h(\vee_i F_i) = \vee_i h(F_i)$.

Let $p_\alpha \in h(\vee F_i)$

i.e. p_α depends on $\vee F_i$.

i.e. p_α depends on $F_1 \vee F_2 \vee \ldots$.

Since F_i depends on F_i and p_α depends on $\vee F_i$, by proposition 3.2.1, p_α depends on F_i.

i.e., $p_\alpha \in h(F_i)$

i.e., $p_\alpha \in \vee h(F_i)$

i.e.,

$$h(\vee F_i) \leq \vee h(F_i). \tag{3.2.29}$$

Clearly

$$\vee h(F_i) \leq h(\vee F_i). \tag{3.2.30}$$

From (3.2.29) and (3.2.30), $h(\vee F_i) = \vee h(F_i)$

i.e., the operator satisfies all the conditions of Prop. 3.2.4. $\therefore$ there is a well defined convex structure with h as a restricted hull operator. *i.e.*, for a non-zero fuzzy subset $A \in I^X$,

$$\begin{aligned} Co(A) &= \vee\{h(F) | F \subseteq A, F \text{ is finite}\} \\ &= \{x_\alpha | x_\alpha \text{ depends on } F\} \\ &= \{x_\alpha | x_\alpha \text{ depends on } A\}, \text{since } F \subseteq A. \end{aligned}$$

i.e., corresponding fuzzy subsets which are convex are exactly flats. □

Proposition 3.2.11. *The fuzzy convex structure formed by the flats of* (X, ϵ) *satis-*

fies the fuzzy exchange law for two fuzzy points p_α, q_β with $p = q$ and $\beta < \alpha$.

Proof. Let $p_\alpha, q_\beta \notin Co(A)$ and let $p_\alpha \in Co(q_\beta \vee A)$ where $A \in I^X$.

$$p_\alpha \notin Co(A) \Rightarrow p_\alpha \notin A \text{ i.e., } A(p) < \alpha. \tag{3.2.31}$$

Similarly

$$q_\beta \notin Co(A) \Rightarrow A(q) < \beta. \tag{3.2.32}$$

<u>Step 1</u> We claim $p_\alpha \notin q_\beta \vee A$.

For proving this, assume the contrary that $p_\alpha \in q_\beta \vee A$.

$$p_\alpha \in q_\beta \vee A \Rightarrow \begin{cases} \alpha \leq \beta \vee A(p). & \text{if } x = p = q \\ \alpha \leq A(p), & \text{if } x = p \neq q \\ 0 \leq A(x), & \text{if } x \neq p \neq q. \end{cases}$$

i.e., when $x = p \neq q$, we have $A(p) \geq \alpha$ which is a contradiction to (3.2.31).

$\therefore$ Our assumption is wrong.

i.e., $p_\alpha \notin q_\beta \vee A$.

<u>Step 2</u> We claim $Co(q_\beta \vee A)(p) > \beta$.

By step 1,we have $p_\alpha \notin q_\beta \vee A$.

$\therefore p_\alpha(x) > (q_\beta \vee A)(x)$ for some x. (x can be only p)

i.e., $\alpha > q_\beta(p) \vee A(p)$.

i.e., $\alpha > \beta$ when $q = p$ and $\alpha > A(p)$ always.

$$\text{i.e., when} \quad x = p = q, \quad \beta < \alpha. \tag{3.2.33}$$

$\beta < \alpha$ and $p_\alpha \in \text{Co}\ (q_\beta \vee A) \Rightarrow \beta < \alpha \leq Co(q_\beta \vee A)(p)$

i.e., $Co(q_\beta \vee A)(p) > \beta$.

Step 3 We claim $(q_\beta \vee A)(p) \leq (p_\alpha \vee A)(q)$ when $p = q$.

We assume the contrary that $(q_\beta \vee A)(p) > (p_\alpha \vee A)(q)$ when $p = q$

i.e., $\beta \vee A(p) > \alpha \vee A(q) = \alpha$ when $p = q$

i.e., $\beta \vee A(p) > \alpha$ which is not true.

i.e.,

We have $(q_\beta \vee A)(p) \leq (p_\alpha \vee A)(q)$ when $p = q$

i.e., $(q_\beta \vee A)(p) \leq (p_\alpha \vee A)(p)$.

Also when $p = q \neq x$, we have $(q_\beta \vee A)(x) \leq (p_\alpha \vee A)(x)$.

i.e., $(q_\beta \vee A)(x) \leq (p_\alpha \vee A)(x)\ \forall x$

i.e., $q_\beta \vee A \leq p_\alpha \vee A$.

i.e., $Co(q_\beta \vee A) \leq Co(p_\alpha \vee A)$

i.e., $\beta < Co(q_\beta \vee A)(p) \leq Co(p_\alpha \vee A)(p)$

i.e., $Co(p_\alpha \vee A)(p) > \beta$.

Step 4 We claim $q_\beta \in Co(p_\alpha \vee A)$ when $p = q$.

Suppose the contrary. *i.e.*, $Co(p_\alpha \vee A)(q) < \beta$ when $p = q$.

i.e., $(p_\alpha \vee A)(q) < \beta$ when $p = q$.

i.e., $\alpha \vee A(q) < \beta$

i.e., $\alpha < \beta$ which is a contradiction to (3.2.33).

$\therefore$ Our assumption is wrong.

i.e., when $p = q$, we have $q_\beta \in Co(p_\alpha \vee A)$

i.e., $p_\alpha \in Co(q_\beta \vee A) \Rightarrow q_\beta \in Co(p_\alpha \vee A)$ when $p = q$.

i.e., the exchange law is satisfied when the support of the fuzzy points p_α, q_β are same. □

Remark 3.2.12. *In the crisp case, flats of* (X, ϵ) *form a fuzzy matroid. In the fuzzy case, this need not be true. In the above proposition, when* $\beta > \alpha$, *flats of* (X, ϵ) *do not necessarily form a fuzzy matroid as they do not satisfy the fuzzy exchange law. For illustrating this, consider the following counter example.*

Counter example 3.2.13. *Let* $X = \{1, 2, 3\}$, $\alpha, \beta \in (0, 1]$.
Then $\epsilon = \{F \subseteq I^X | F(x) > \frac{1}{2}, x \in X\}$, (X, ϵ) *form a fuzzy independence structure.*

Flats of $\epsilon = \{G \in \epsilon | G \leq F \in \epsilon\}$, which form a fuzzy convex structure.
Consider the fuzzy subset $A = 1_{\frac{1}{2}} \vee 2_{\frac{1}{4}}$.
Let $p_\alpha = 3_{\frac{1}{4}}$, $q_\beta = 3_{\frac{3}{4}}$. Here $p_\alpha, q_\beta \notin \text{Co}(A)$. Also $p = q$ where $\beta = \frac{3}{4} > \frac{1}{4} = \alpha$.

$$3_{\frac{1}{4}} = p_\alpha \in Co(q_\beta \vee A) = Co(3_{\frac{3}{4}} \vee 1_{\frac{1}{2}} \vee 2_{\frac{1}{4}})$$
$$= \{\text{ all fuzzy subsets } \leq 3_{\frac{3}{4}} \vee 1_{\frac{1}{2}} \vee 2_{\frac{1}{4}}\}$$

But $3_{\frac{3}{4}} = q_\beta \notin Co(p_\alpha \vee A) = Co(3_{\frac{1}{4}} \vee 1_{\frac{1}{2}} \vee 2_{\frac{1}{4}})$

$$= \{\text{all fuzzy pts depending on } 3_{\frac{1}{4}} \vee 1_{\frac{1}{2}} \vee 2_{\frac{1}{4}}\}$$

i.e., $p_\alpha \in Co(q_\beta \vee A)$ doesn't imply $q_\beta \in Co(p_\alpha \vee A)$
i.e., exchange law is not satisfied
i.e., flats of the fuzzy independence structure do not form a matroid.

Chapter 4

A study on various fuzzy matroids from vector spaces

CONTENTS

Convex structures are unambiguously determined by their hull operator and it is possible to classify convex structure by the properties of the hull operator . As a result, matroid is defined as a convexity space satisfying the exchange law or equivalently, flats of the independent structure form matroids. In [60], matroids from vector spaces are discussed and based on this, some characterizations are obtained. In this chapter, the notions of fuzzy linear, fuzzy affine and fuzzy projective matroids are introduced and we present some of their properties.

4.1 Fuzzy matroids from vector spaces

Analogous to different types of matroids derived from vector spaces as given in [60], we introduce certain fuzzy matroids and discuss some of their properties.

Definition 4.1.1. *Let V be a vector space over a field F. A non-zero fuzzy subset A of V is called a fuzzy linear set provided for each pair of points $x, y \in V$ and for each $\alpha, \beta \in F$,*

$$A(\alpha x + \beta y) \geq A(x) \wedge A(y).$$

Proposition 4.1.2. *Let V be a vector space over a field F and $V_0 = V \setminus \{0\}$. The collection of all traces on V_0 of all fuzzy linear subsets of V [i.e. the collection of all fuzzy linear subsets of V obtained by deleting '0' from their support] is a fuzzy linear convexity C of V_0.*

Proof. Let $A_i \in C$ for all i. We show that $\wedge A_i$ is a fuzzy linear set. Since A_i's are fuzzy linear sets,

$$\begin{aligned} A_i(\alpha x + \beta y) &\geq A_i(x) \wedge A_i(y) \quad \forall i \in I \\ \text{then } (\wedge_i A_i)(\alpha x + \beta y) &= \wedge_i (A_i(\alpha x + \beta y)) \\ &\geq \wedge [A_i(x) \wedge A_i(y)] \\ &= (\wedge_i A_i(x)) \wedge (\wedge_i A_i(y)) \end{aligned}$$

i.e., $\wedge_i A_i$ is a fuzzy linear set $\in \mathcal{C}$.

Next, we show that nested union of fuzzy linear sets is a fuzzy linear set. i.e., we show $\vee_i A_i$ is a fuzzy linear set whenever $\{A_i | i \in I\}$ is a family of totally ordered fuzzy linear sets $\in \mathcal{C}$. We know

$$\begin{aligned} (\vee A_i)(\alpha x + \beta y) &= \vee_i A_i(\alpha x + \beta y) \\ &\geq \vee_i ((A_i(x) \wedge A_i(y))) \end{aligned} \tag{4.1.1}$$

Next we show that

$$\vee_i (A_i(x) \wedge A_i(y)) \geq (\vee A_i)(x) \wedge (\vee A_i)(y)$$

Let

$$z = (\vee A_i)(x) \wedge (\vee A_i)(y)$$
$$\text{i.e., } z \leq (\vee A_i)(x) \text{ and } z \leq (\vee A_i)(y)$$

Let $\epsilon > 0$ be any +ve number.

Then there exists j, k such that $z - \epsilon < A_j(x)$, $z - \epsilon < A_k(y)$.

Being totally ordered, either $A_j \leq A_k$ or $A_k \leq A_j$.

Assume w.l.o.g, $A_j \leq A_k$.

$$\therefore\ z - \epsilon < A_k(x) \text{ and } z - \epsilon < A_k(y)$$

$$\therefore\ z - \epsilon < A_k(x) \wedge A_k(y) \leq \vee(A_i(x) \wedge A_i(y))$$

This is true for all $\epsilon > 0$

$$\therefore\ z \leq \vee_i(A_i(x) \wedge A_i(y))$$

$$\text{i.e., } \vee_i(A_i(x) \wedge A_i(y)) \geq (\vee A_i)(x) \wedge (\vee A_i)(y)$$

$$\text{i.e., By (4.1.1), } (\vee A_i)(\alpha x + \beta y) \geq \vee_i(A_i(x) \wedge A_i(y)) \geq (\vee A_i)(x) \wedge (\vee A_i)(y)$$

Consider a fuzzy linear set F on V defined as

$$F(x) = \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise} \end{cases}$$

Then supp$F = \{0\}$.
If we remove '0' from supp F, we obtain zero function $\underline{0}$ which clearly $\in \mathcal{C}$.

Also we know every fuzzy linear set $F \subseteq \underline{1}$ and since nested union of fuzzy linear sets is a fuzzy linear set, $\underline{1} \in \mathcal{C}$.

Therefore $\mathcal{C}$ is a fuzzy linear convexity on V_0. □

Note 4.1.3. *We use 'lin' instead of 'Co' to describe the corresponding hull operator. Let $p_\alpha, q_\beta \notin lin(A)$. Here the fuzzy exchange law means*

$$(lin(q_\beta \vee A))(\alpha p + \beta q) \geq \alpha \wedge \beta \Rightarrow (lin(p_\alpha \vee A))(\alpha p + \beta q) \geq \alpha \wedge \beta.$$

Definition 4.1.4. *Fuzzy linear matroid is a fuzzy linear convexity space which satisfies the fuzzy exchange law.*

Definition 4.1.5. *Let V be a vector space over a field F. A non-zero fuzzy subset A of V is called a fuzzy affine set provided for each pair of points $x, y \in V$ and for each $\alpha, \beta \in F$, $A(\alpha x + \beta y) \geq A(x) \wedge A(y)$ where $\alpha + \beta = 1$.*

Proposition 4.1.6. *Let V be a vector space over a field F. Let $V_0 = V \backslash \{0\}$. The collection of all traces on V_0 of all fuzzy affine sets of V [i.e. the collection of all fuzzy affine sets of V obtained by deleting '0' from their support] is a fuzzy affine convexity C of V_0.*

Proof. Proof is similar to that of Proposition 4.1.2. □

Remark 4.1.7. *Clearly fuzzy affine sets are fuzzy linear sets. Let the affine hull of a fuzzy subset A be denoted by aff A.*

Let $p_\alpha, q_\beta \notin \text{aff}(A)$. Hence the fuzzy exchange law means

$$(\text{aff}(q_\beta \vee A))(\alpha p + \beta q) \geq \alpha \wedge \beta \Rightarrow (\text{aff}(p_\alpha \vee A))(\alpha p + \beta q) \geq \alpha \wedge \beta$$

where $\alpha + \beta = 1$.

Definition 4.1.8. *Fuzzy affine matroid is a fuzzy affine convexity space which satisfies the fuzzy exchange law.*

Definition 4.1.9. *Let (X, C) be a fuzzy convexity space and let R be a equivalence relation on X. Consider the quotient set X/R which consists of all R-equivalence classes and the quotient function $q : X \to X/R$ which assigns to a point of X, its R-equivalence class.*

Define $C/R = \{F \in I^{X/R} / q^{-1}(F)$ is a fuzzy convex set on $X\}$.
We can show that C/R is a convexity on X/R.

For,

$$\text{clearly,} \quad \underline{0}, \underline{1} \in C/R.$$
$$\text{Let} \quad \{F_i | i \in I\} \subseteq C/R$$

Then $\wedge_i F_i$ and nested union of F_i are in C/R whenever $F_i \in C/R$ since by definition 1.3.1

$$q^{-1}(\wedge F_i)(x) = (\wedge F_i)(q(x))$$
$$= (\wedge F_i)[x]$$
$$\text{and} \quad (q^{-1}(\vee F_i))(x) = (\vee F_i)(q(x)) = (\vee F_i)[x]$$

That is fuzzy convex sets of X/R are the images of R-saturated fuzzy convex sets of X. The resulting fuzzy convex structure $(X/R, C/R)$ is called a fuzzy quotient space of X where C/R is called a fuzzy quotient convexity.

Definition 4.1.10. *Consider a fuzzy linear matroid in a vector space V over a field K. Let $V_0 = V \backslash \{0\}$. Define a equivalence relation on V_0 as follows. For any $x, y \in V_0$,*

$$x \equiv y \Leftrightarrow \exists t \in k \backslash \{0\}, y = tx$$

and $\mu(tx) \geq \mu(x)$ where μ is a fuzzy subset on V_0.

Let P denote the fuzzy quotient space. Here the quotient function $q : V_0 \rightarrow P$ is FCP by the definition of the quotient function. It is also FCC since the image of a fuzzy linear set is fuzzy linear under a fuzzy quotient map.

For, here we show that $q(\mu)$ is a fuzzy linear set. *i.e.*, we show that $(q(\mu))[\alpha x + \beta y] \geq (q(\mu))[x] \wedge (q(\mu))[y]$.

By definition 1.3.1,

$$(q(\mu))[\alpha x + \beta y] = \vee_{\alpha x+\beta y}\{\mu(\alpha x + \beta y)/q(\alpha x + \beta y) = [\alpha x + \beta y]\}$$
$$\geq \vee(\mu(x) \wedge \mu(y)) \text{ Since } \mu \text{ is a fuzzy linear set} \qquad (4.1.2)$$

Next we show that

$$\vee(\mu(x) \wedge \mu(y)) \geq (\vee\mu(x)) \wedge (\vee\mu(y)) \qquad (4.1.3)$$

Let $z = (\vee\mu(x)) \wedge (\vee\mu(y))$

i.e., $z \leq \vee\mu(x)$ and $z \leq \vee\mu(y)$

i.e., $z \leq \mu(x_1)$ for some $\mu(x_1)$ and

$z \leq \mu(y_1)$ for some $\mu(y_1)$

i.e., $z \leq \mu(x_1) \wedge \mu(y_1)$

i.e., $z \leq \vee(\mu(x) \wedge \mu(y))$

Hence the inequality (4.1.3).

By (4.1.2) and (4.1.3),

$$(q(\mu))[\alpha x + \beta y] \geq (\vee\mu(x)) \wedge (\vee\mu(y)).$$

That is

$$q(\mu)[\alpha x + \beta y] \geq (q(\mu))[x] \wedge (q(\mu))[y]$$

P is a fuzzy matroid by Proposition 2.2.3, called the fuzzy projective matroid over K and its fuzzy convex sets are called fuzzy projective sets.

Let the hull of a fuzzy subset A be denoted by Pr (A).

4.2 Properties of Fuzzy Matroids

Proposition 4.2.1. *A fuzzy convexity space* $(X, \mathcal{C})$ *is JHC* $\Leftrightarrow$ *for each non-zero finite fuzzy subset F and for each* $a_\alpha \notin Co(F)$,

$$Co(a_\alpha \vee F) \leq \vee_{x_\beta}\{Co(a_\alpha \vee x_\beta) | x_\beta \in Co\ (F)\}$$

Proof. Assume that the fuzzy convexity space $(X, \mathcal{C})$ is JHC and F is a non-zero finite fuzzy subset F. then

$$\text{Co}(a_\alpha \vee F) \leq \text{Co}(a_\alpha \vee \text{Co}(F)) = \vee_{x_\beta}\{\text{Co}(a_\alpha \vee x_\beta) | x_\beta \in \text{Co}(F)\}. by JHC.$$

$$\text{i.e., } \text{Co}(a_\alpha \vee F) \leq \vee_{x_\beta}\{\text{Co}(a_\alpha \vee x_\beta) | x_\beta \in \text{Co}(F)\}. \qquad (4.2.1)$$

Conversely, assume that (4.2.1) is true for a non-zero finite fuzzy subset F. If F is not a finite fuzzy subset, then by Proposition 3.2.3

$$\begin{aligned}\text{Co}(a_\alpha \vee F) &= \vee\{\text{Co}(C) | C \leq a_\alpha \vee F. C \text{ finite}\} \\ &\leq \vee_{x_\beta}\{\text{Co}(a_\alpha \vee x_\beta) | x_\beta \in F \in \text{Co}\ (F)\}\end{aligned}$$

i.e., if F is any non-zero fuzzy subset, particularly, if F is any non-zero fuzzy convex set (4.2.1) is true.

To show the reverse inequality corresponding to (4.2.1), assume

$$y_\gamma \in \vee_{x_\beta \in F} \mathrm{Co}(a_\alpha \vee x_\beta)$$

$$\text{i.e.,} \quad y_\gamma \in \text{ some } \mathrm{Co}(a_\alpha \vee x_\gamma),\ x_\gamma \in F$$

$$\text{i.e.,} \quad y_\gamma \in \mathrm{Co}(a_\alpha \vee F)$$

$$\text{i.e.,} \quad \vee_{x_\beta \in F} \mathrm{Co}(a_\alpha \vee x_\beta) \leq \mathrm{Co}(a_\alpha \vee F) \tag{4.2.2}$$

From (4.2.1) and (4.2.2), we have $\mathrm{Co}(a_\alpha \vee F) = \vee\, \mathrm{Co}(a_\alpha \vee x_\beta)$.
i.e., $(X, \mathcal{C})$ is JHC. □

Proposition 4.2.2. *A fuzzy linear matroid is join-hull commutative.*

Proof. Let A be a fuzzy linear set and $x_\beta \in A$. Let $a_\alpha \in I^X$.
Then $a_\alpha \vee x_\beta \in a_\alpha \vee A$ for every $x_\beta \in A$.
i.e., lin $(a_\alpha \vee x_\beta) \in \mathrm{lin}(a_\alpha \vee A)$.
i.e., $\vee_{x_\beta}\mathrm{lin}(a_\alpha \vee x_\beta) \in \mathrm{lin}(a_\alpha \vee A)$ for every $x_\beta \in A$

$$\text{i.e.,} \quad \mathrm{lin}\,(a_\alpha \vee A) \geq \vee_{x_\beta}\mathrm{lin}\,(a_\alpha \vee x_\beta). \tag{4.2.3}$$

To show the reverse inequality, assume that $z_\delta \in$ lin $(a_\alpha \vee A)$ where z is a linear combination of the members in support in $\mathrm{lin}(a_\alpha \vee A)$.
i.e., z is a linear combination of the members in support of $(a_\alpha \vee A)$.
i.e., z is a linear combination of the members in support of $(a_\alpha \vee x_\beta)$ for some $x_\beta \in A$ where x is a linear combination of the members in support of A since A is a fuzzy linear set.
i.e., $z_\delta \in$ lin $(a_\alpha \vee x_\beta)$ for some $x_\beta \in A$.
i.e., $z_\delta \in \vee$lin $(a_\alpha \vee x_\beta)$ where $x_\beta \in A$.

$$\text{i.e.,} \quad \mathrm{lin}\,(a_\alpha \vee A) \leq \vee_{x_\beta}\mathrm{lin}\,(a_\alpha \vee x_\beta). \tag{4.2.4}$$

From (4.2.3) and (4.2.4), we have the equality.
i.e., fuzzy linear matroid is join-hull commutative. □

Proposition 4.2.3. *A fuzzy affine matroid is join-hull commutative.*

Proof. Proof is similar to that of Proposition 4.2.2. □

Proposition 4.2.4. *A fuzzy affine projective matroid is join-hull commutative.*

Proof. A fuzzy projective matroid is JHC by Propositions 2.2.3 and 4.2.2. □

Definition 4.2.5. *A fuzzy subset of the type Co (F), where F is a finite fuzzy subset is called a fuzzy polytope.*

Proposition 4.2.6. *If $A_1, A_2, \ldots, A_n$ are non-zero fuzzy convex sets in a JHC convexity space, then*

$$Co(\vee_{i=1}^{n} A_i) = \vee\{Co(a_{\alpha_1} \vee a_{\alpha_2} \vee \cdots a_{\alpha_n})\ \forall i = 1, \ldots, n.\ a_{\alpha_i} \in A_i\}$$

Proof. It is enough to prove the result for two non-zero fuzzy polytopes $F = \text{Co}(F)$ and $G = \text{Co}(G)$ with # supp $F = n$, # supp $G = m$, finite. We prove the result by induction on $n + m$. The result is true if $n + m = 2$ (i.e., if $n = 1 = m$). Assume $n + m > 2$ and the result is true for values $< n + m$. For $n = 1$, the result is true by the definition of JHC..
Let $n > 1$ and let

$$x_\beta \in \text{Co}(F \vee G) = \text{Co}(\text{Co}(F) \vee \text{Co}(G)).$$

Let $q_{\beta_1} \in F = \text{Co}(F)$ be a fixed fuzzy point, then

$$\begin{aligned} x_\beta &\in \text{Co}(F \vee G) \\ &= \text{Co}(q_{\beta_1} \vee (G \vee (F \backslash q_1))) \\ &\in \text{Co}[q_{\beta_1} \vee \text{Co}(G \vee (F \backslash q_1))] \\ &\in \text{Co}[q_{\beta_1} \vee a_\gamma \vee b_\delta] \end{aligned}$$

where $a_\gamma \in \text{Co}(F \backslash q_1)$, $b_\delta \in \text{Co}(G)$ by induction hypothesis.
Here $a_\gamma \vee q_{\beta_1} \in \text{Co}(a_\gamma \vee q_{\beta_1}) \in \text{Co}(F)$.
i.e.,

$$\begin{aligned} x_\beta &\in \text{Co}((q_{\beta_1} \vee a_\gamma) \vee b_\delta) \\ &\in \text{Co}(\text{Co}(F)) \vee b_\delta) \\ &\in \text{Co}(a_\alpha \vee b_\delta) \text{ where } a_\alpha \in \text{Co}(F) \\ &\subseteq \vee\text{Co}(a_\alpha \vee b_\delta). \end{aligned}$$

i.e.,

$$\text{Co}(F \vee G) \subseteq \vee\{\text{Co}(a_\alpha \vee b_\delta) | a_\alpha \in F, b_\delta \in G\}.$$

The reverse inequality is obvious.
Hence the result. □

As a result of the above proposition we have the following definition.

Definition 4.2.7. *A fuzzy convexity space is weakly JHC if for each pair of fuzzy convex sets $A, B \in I^X$ where $A \wedge B \neq \underline{0}$,*

$$Co(A \vee B) = \vee\{Co(a_\alpha \vee b_\beta) | a_\alpha \in A, b_\beta \in B\}$$

Remark 4.2.8. *By Proposition 4.2.6, we can see that a JHC fuzzy convexity space is a weakly JHC space.*

By the finitary law, the union of independent sets totally ordered by fuzzy inclusion is independent. By Zorn's lemma, three exists a maximal element. This motivates to intriduce the following definition in the fuzzy context.

Definition 4.2.9. *Basis of a flat A in a fuzzy convexity space is any maximal fuzzy convexly independent set B in the support of A.*

Number of elements in B is called the rank of A and is denoted by $d(A)$. i.e., $d(A) = \#B$.

Proposition 4.2.10. *Let (X, C) be a weakly JHC matroid. Let $A, B \in I^X$ be two non-disjoint flats. If $d(C)$ denotes the rank of a flat C, then the equality $d(A \vee B) + d(A \wedge B) = d(A) + d(B)$ need not be true.*

Counter example 4.2.11. *Let $X = \{1, 2, 3\}$, $I = [0, 1]$.*
Consider the convexity $\mathcal{C}$ on X as $\mathcal{C} = \mathcal{P}(X)$.
Consider a fuzzy convex structure on X as

$$\mathcal{C}_1 = \{\underline{0}, \underline{1}, 1_{0.2}, 1_{\frac{3}{4}}, 1_{0.2} \vee 2_{0.6}\}$$

which can be shown to be a fuzzy matroid.

For, first we show that the fuzzy exchange law is satisfied.
Let $p_\alpha, q_\beta \notin \text{Co}(A)$ and $p_\alpha \in \text{Co}(q_\beta \vee A)$.
Case 1. Let $\text{Co}(A) = 1_{0.2}$.

(i) Let $\text{Co}(q_\beta \vee A) = 1_{\frac{3}{4}}$.
then A has to be 1_α for some $\alpha \leq 0.2$ and q_β has to be 1_β for some β such that $(0.2 < \beta \leq \frac{3}{4})$.

$$\text{i.e., } p_\alpha \in \text{Co}(q_\beta \vee A) = 1_{\frac{3}{4}} \Rightarrow q_\beta \in \text{Co}(p_\alpha \vee A)$$

(ii) Let Co $(q_\beta \vee A) = 1_{0.2} \vee 2_{0.6}$ then

$$p_\alpha \in \text{Co}(q_\beta \vee A) = 1_{0.2} \vee 2_{0.6}$$
$$\Rightarrow q_\beta \leq 2_\beta,\ \beta \leq 0.6$$
$$\text{i.e.,}\quad q_\beta \in \text{Co}(p_\alpha \vee A).$$

(iii) Let Co $(q_\beta \vee A) = \underline{1}$ then

$$p_\alpha \in \text{Co}(q_\beta \vee A) = \underline{1} \Rightarrow q_\beta \in \text{Co}(p_\alpha \vee A)$$

Case 2. Let Co $(A) = 1_{\frac{3}{4}}$.

$$p_\alpha, q_\beta \notin \text{Co}(A) \Rightarrow p_\alpha, q_\beta \notin A.$$

Here A has to be 1_α for some α such that $0.2 < \alpha \leq \frac{3}{4}$

then

$$p_\alpha \in \text{Co}(q_\beta \vee A) = \underline{1} \Rightarrow q_\beta \in \text{Co}(p_\alpha \vee A) = \underline{1}.$$

Case 3. Let Co $(A) = 1_{0.2} \vee 2_{0.6}$,

then A has to be $1_\alpha \vee 2_\beta$ for some α, β such that $\alpha \leq 0.2$, $\beta \leq 0.6$

i.e., $p_\alpha \in \text{Co}(q_\beta \vee A) = \underline{1} \Rightarrow q_\beta \in \text{Co}(p_\alpha \vee A) = \underline{1}$.

Also C_1 is a weakly JHC matroid.

For,

consider the fuzzy convex sets $A = 1_{0.2}$, $B = 1_{\frac{3}{4}}$.

then $\text{Co}(A \vee B) = \text{Co}(B) = B$.

i.e., $\vee\{\text{Co}(a_\alpha \vee b_\beta) | a_\alpha \in A, b_\beta \in B\} = B = \text{Co}(A \vee B)$ where $A \wedge B \neq \underline{0}$.

Next consider the fuzzy convex sets B and $C = 1_{0.2} \vee 2_{0.6}$ where $B \wedge C = 1_{0.2} \neq \underline{0}$.

then Co $(B \vee C) = \text{Co}(1_{\frac{3}{4}} \vee 2_{0.6}) = \underline{1}$.

$\vee\{\text{Co}(b_\beta \vee c_\gamma)|b_\beta \in B, c_\gamma \in C\} = \underline{1} = Co(B \vee C)$

Similarly we get the same equality if we consider the other pairs of fuzzy convex sets in $\mathcal{C}_1$.

i.e.,$(X, \mathcal{C}_1)$ is a weakly JHC fuzzy matroid.

Now consider two non-disjoint flats $A = 1_{\frac{3}{4}}$ and $C = 1_{0.2} \vee 2_{0.6}$

then $A \vee C = 1_{\frac{3}{4}} \vee 2_{0.6}$, $\text{Co}(A \vee C) = \underline{1}$.

$d(\text{Co}(A \vee C)) = d(\underline{1}) = 3$, since # (maximal independent set in supp $\underline{1}$ (i.e. X)) is 3.

Similarly, $d(A \wedge C) = 1$, $d(A) = 1$, $d(C) = 2$.

i.e., $d(\text{Co}(A \vee C)) + d(A \wedge C) = 3 + 1 = 4 \neq 3 = 1 + 2 = d(A) + d(C)$

i.e., there is no equality.

Remark 4.2.12. *(1) In Proposition 4.2.10, if $(X, \mathcal{C})$ is a fuzzy JHC space, then the equality need not be true.*

This follows from the remark 4.2.8 and the Counter eg 4.2.11.

(2) Even for disjoint flats, the equality in Proposition 4.2.10 need not be true as shown by the following Counter eg. 4.2.13.

Counter example 4.2.13. *Consider $\mathcal{C}$ and $\mathcal{C}_1$ as in Counter eg. 4.2.11.*

Let $A = 1_{\frac{3}{4}}$ and $B = 2_{0.6}$ be two disjoint flats.

$A \vee B = 1_{\frac{3}{4}} \vee 2_{0.6}$, $Co(A \vee B) = \underline{1}$.

$d(Co(A \vee B)) = 3$, $d(A) = 1 = d(B)$.

$$i.e., d(Co(A \vee B)) + d(A \wedge B) > d(A) + d(B).$$

That is no equality.

Note 4.2.14. *In Proposition 4.2.10 the equality holds only when $A \subseteq B$ or $B \subseteq A$.*

Chapter 5

Various notions of dependence in a Fuzzy Convexity Space

CONTENTS

Some of the results of this chapter will appear in the *Oriental Journal of Applied Mathematics*.

Various notions of dependence of a non-empty finite set in a crisp convexity space and based on this, some characterization of the classical convex invariants namely Helly number, Caratheodary number, Radon number and exchange number are available in literature (c.f. [60]). In this chapter, we try to extend different types of dependence defined in [60] to the fuzzy context and study their interrelations.

5.1 Certain properties of a fuzzy convex structure and related definitions

Here we introduce the fuzzy analogues of the properties of the crisp convex structure like cone-union property given in [60].

Definition 5.1.1. *A fuzzy convex structure satisfies the cone-union property (CUP) if the following holds: If $F, F_1, F_2, \ldots, F_n \in I^X$ are fuzzy convex sets with $F \leq \vee_{i=1}^{n} F_i$ and if $a_\alpha \in I^X$, then*

$$Co(a_\alpha \vee F) \leq \vee_{i=1}^{n} Co(a_\alpha \vee F_i)$$

Definition 5.1.2. *Let X be a set and let $I : X \times X \to I^X$ be a function with the following properties.*

1. *fuzzy extensive law: $a, b \in I(a, b)$*
   [i.e. $I(a, b)(a) > 0$, $I(a, b)(b) > 0$].
2. *fuzzy symmetry law: $I(a, b) = I(b, a)$.*

Then I is called a fuzzy interval operator on X. The resulting pair (X, I) is called a fuzzy interval space and $I(a, b)$ is called the fuzzy interval between a and b for each $a, b \in X$.

Definition 5.1.3. *Let (X, C) be a fuzzy convex structure and let I be a fuzzy interval on X. Then C is said to be generated by I if $I(a, b) \in C$ for all $a, b \in X$.*

The pair (X, C) is a fuzzy convex structure generated by I.

Definition 5.1.4. *A fuzzy convex structure generated by the fuzzy interval operator is said to satisfy the Ramification property if for all $b, c, d \in X$, $c \notin I(b, d)$ and $d \notin I(b, c) \Rightarrow I(b, c) \wedge I(b, d) = b_\alpha$ for some $\alpha > 0$.*

[Here $c \notin I(b, d)$ means $I(b, d)(c) = 0$]

Definition 5.1.5. *A fuzzy interval $I(a, b)$ on a fuzzy convex structure is decomposable provided for each $x \in I(a, b)$, $I(a, x) \vee I(x, b) = I(a, b)$ and $I(a, x) \wedge I(x, b) = x_\alpha$ for some $\alpha > 0$.*

Example 5.1.6. *Let $X = \{3, 5\}$.*

Consider the fuzzy interval operator defined on X as follows:

$I(3, 3) = 3_\alpha$, $I(5, 5) = 5_\beta$, $I(3, 5) = I(5, 3) = 3_\alpha \vee 5_\beta$ *where α, β are chosen elements in* $(0, 1]$.

Here for $3 \in I(3, 5)$, $I(3, 3) \vee I(3, 5) = 3_\alpha \vee (3_\alpha \vee 5_\beta) = 3_\alpha \vee 5_\beta = I(3, 5)$ *and* $I(3, 3) \wedge I(3, 5) = 3_\alpha \wedge (3_\alpha \vee 5_\beta) = 3_\alpha$.

Also for $5 \in I(3, 5)$, $I(3, 5) \vee I(5, 5) = I(3, 5)$ *and* $I(3, 5) \wedge I(5, 5) = 5_\beta$.

Similarly $I(5, 5) \vee I(5, 3) = I(5, 3)$ *and* $I(5, 5) \wedge I(5, 3) = 5_\beta$,

where $5 \in I(5, 3)$,

$I(5, 3) \vee I(3, 3) = I(5, 3)$ *and* $I(5, 3) \wedge I(3, 3) = 3_\alpha$

where $3 \in I(5, 3)$.

i.e., the fuzzy interval $I(a, b)$ is decomposable.

Example 5.1.7. *Let $X = R$.*

Consider a fuzzy interval operative $I : R \times R \to I^R$ defined as $I(a, b) = [a, b]_{\frac{1}{2}}$.

Let $b, c, d \in R$.

Let $c \notin I(b,d)$ and $d \notin I(b,c)$.

i.e. $I(b,d)(c) = 0$ and $I(b,c)(d) = 0$.

We show that $I(b,d) \wedge I(b,c) = b_{\frac{1}{2}}$.

$$[I(b,d) \wedge I(b,c)](c) = I(b,d)(c) \wedge I(b,c)(c) = 0$$

$$[I(b,d) \wedge I(b,c)](d) = I(b,d)(d) \wedge I(b,c)(d) = 0.$$

Let $x \neq b, c, d$.

$$[I(b,d) \wedge I(b,c)](x) = [I(b,d)](x) \wedge [I(b,c)](x). \tag{5.1.1}$$

If $x \in I(b,c)$, then $x \notin I(b,d)$.

For proving this, suppose on the contrary that $x \in I(b,d)$. *i.e.* $x \in [b,d]$.

i.e. $x \in [b,c]$ as $x \in I(b,c)$ and $x \in [b,d]$ which means that either $b \leq c \leq d$ or $b \leq d \leq c$.

Also $b \leq c \leq d$ means $c \in [b,d]$.

i.e. $c \in I(b,d)$ which is not possible by our assumption that $c \notin I(b,d)$.

Similarly, $b \leq d \leq c \Rightarrow d \in I(b,c)$ which is not true.

i.e. $x \notin I(b,d)$.

i.e. if $x \in I(b,c)$, then $x \notin I(b,d)$.

In a similar way we can see that, if $x \in I(b,d)$, then $x \notin I(b,c)$.

So, by (5.1.1), $[I(b,d) \wedge I(b,c)](x) = 0$.

When $x = b$, $[I(b,d) \wedge I(b,c)](x) = b_{1/2}$

i.e., when $c \notin I(b,d)$ and $d \notin I(b,c)$.

$$I(b,d) \wedge I(b,c) = b_{1/2}$$

Hence the ramification property.

5.2 Types of dependence

In this section, we introduce the fuzzy analogues of different types of dependence introduced in [60] and study their interrelations.

Definition 5.2.1. *Let (X, C) be a fuzzy convex structure. A nonzero finite fuzzy subset F of X is Helly dependent (or H-dependent) provided $\wedge_{a_\alpha \in F} Co(F \backslash a_\alpha) \neq \underline{0}$ where $F \backslash a_\alpha = F \wedge a'_\alpha$, $a'_\alpha = 1 - a_\alpha$ and otherwise it is Helly independent.*

The Helly number of X is the smallest 'n' such that for each non zero finite fuzzy subset F of X with cardinality in its support at least $n + 1$ is Helly dependent.

Definition 5.2.2. *Let (X, C) be a fuzzy convex structure. A non-zero finite fuzzy subset F of X is Radon dependent (or R-dependent) if there exists a Radon partition $\{F_1, F_2\}$ of F (i.e. $F_1 \wedge F_2 = \underline{0}$, $F_1 \vee F_2 = F$) such that $Co(F_1) \wedge Co\ (F_2) \neq \underline{0}$ and if for every partition $\{F_1, F_2\}$ of F, $Co(F_1) \wedge Co(F_2) = \underline{0}$, F is called R-independent.*

The Radon number of X is the smallest 'n' such that for each non zero finite fuzzy subset F of X with cardinality in its support at least $n + 1$ is Radon dependent.

Definition 5.2.3. *For a fuzzy convex structure (X, C), a nonzero finite fuzzy subset F of X is Caratheodory dependent (or C-dependent) provided $Co(F) \subseteq \vee_{a_\alpha \in F} Co(F \backslash a_\alpha)$ and otherwise it is C-independent.*

The Caratheodory number of X is the smallest 'n' such that for each non zero finite fuzzy subset F of X with cardinality in its support at least $n + 1$ is Caratheodory dependent.

Definition 5.2.4. *For a fuzzy convex structure (X, C), a nonzero finite fuzzy subset F of X is Exchange dependent (or E-dependent) provided for each $p_\alpha \in F$,*

$$Co(F\backslash p_\alpha) \leq \vee_{a_\beta \in F}\{Co(F\backslash a_\beta) | a_\beta \in F\backslash p_\alpha, a \neq p\}$$

otherwise it is E-independent.

The Exchange number of X is the smallest 'n' such that for each non zero finite fuzzy subset F of X with cardinality in its support at least $n+1$ is Exchange dependent.

The fuzzy analogues of the classical convex invariants like Helly number, Caratheodory number, Radon number and Exchange number form an important area of study in fuzzy convexity theory. As a background of future research in this area, we have defined the above concepts in the fuzzy context. In this thesis, we are not doing further work in this area.

Example 5.2.5. *Let $X = \{1, 3, 5\}$.*
Consider the fuzzy convexity C on X as $C = \{\underline{0}, \underline{1}, G\}$ where $G \leq F = 5_{0.6} \vee 3_{0.4}$

(i) Let $a_\alpha \in F$
then $F\backslash a_\alpha \neq \underline{0}$ for all $a_\alpha \in F$.
i.e., $Co(F\backslash a_\alpha) \neq \underline{0}$ for all $a_\alpha \in F$
i.e., $\wedge_{a_\alpha \in F} Co(F\backslash a_\alpha) \neq \underline{0}$
i.e., F is H-dependent.

(ii) Let $a_\alpha = 5_{0.4} \in F$.
Then $F\backslash a_\alpha = F\backslash 5_{0.4} = F$.
i.e., $Co\ (F\backslash a_\alpha) = Co\ (F)$.
For other $a_\alpha \in F$, $Co\ (F\backslash a_\alpha) \leq Co\ (F)$.
i.e., $\vee_{a_\alpha \in F} Co\ (F\backslash a_\alpha) = Co\ (F)$.

i.e., $Co\ (F) \leq \vee_{a_\alpha \in F} Co\ (F \backslash a_\alpha)$ *is true for the finite fuzzy subset* F.

i.e., F *is* C*-dependent.*

(iii) *Consider the Radon partion* $\{F_1, F_2\}$ *of* $F = 5_{0.6} \vee 3_{0.4}$ *where* $F_1 = 5_{0.6}$, $F_2 = 3_{0.4}$ *(Here this is the only one partition of* F.*) then*

$$Co(F_1) \wedge Co(F_2) = 5_{0.6} \wedge 3_{0.4} = \underline{0}.$$

i.e. F *is* R*-independent.*

But if we consider the fuzzy convexity $\mathcal{C}$ *as* $\mathcal{C} = \{\underline{0}, \underline{1}, F\}$, *then* F *is* R*-dependent, since* $Co\ (F_1) = F = Co(F_2)$ *and* $Co\ (F_1) \wedge Co\ (F_2) = F \neq 0$.

(iv) *Let* $p_\alpha \in F$.

Then $F \backslash p_\alpha \leq F$ *so that*

$$Co(F \backslash p_\alpha) \leq Co(F) \quad \forall p_\alpha \in F \tag{5.2.1}$$

$p_\alpha \in F$ *means* $p_\alpha = 5_{\alpha_1}$, $\alpha_1 \leq 0.6$ *or* $p_\alpha = 3_{\alpha_2}$, $\alpha_2 \leq 0.4$.

Case 1: *Let* $p_\alpha = 5_{\alpha_1}$, $\alpha_1 \leq 0.6$.

$$a_\beta \in F \backslash p_\alpha,\ a \neq p \Rightarrow a_\beta = 3_{\beta_1},\ \beta_1 \leq 0.4.$$

When $\beta_1 = 0.4$, $1 - \beta_1 = 0.6$.

i.e. $F \backslash 3_{\beta_1} = F$.

i.e. $Co\ (F \backslash 3_{\beta_1}) = Co\ (F)$.

Also $Co\ (F \backslash a_\beta) \leq Co\ (F)$ *for the remaining* a_β*'s except* $a_\beta = 3_{\beta_1} = 3_{0.4}$.

i.e. $\vee_{a_\beta} Co\ (F \backslash a_\beta) = Co\ (F)$

i.e. when $p_\alpha = 5_{\alpha_1}, \alpha_1 \leq 0.6,$

$$Co\ (F\backslash p_\alpha) \leq \vee_{a_\beta}\{Co\ (F\backslash a_\beta)|a_\beta \in F\backslash p_\alpha, a \neq p\}$$

Case 2: *Let* $p_\alpha = 3_{\alpha_2}, \alpha_2 \leq 0.4.$

$a_\beta \in F\backslash p_\alpha,\ a \neq p \Rightarrow a_\beta = 5_{\beta_2},\ \beta_2 \leq 0.6.$

When $\beta_2 = 0.4,\ 1 - \beta_2 = 0.6.$ *then* $F\backslash 5_{\beta_2} = F.$

i.e., $Co\ (F\backslash 5_{\beta_2}) = Co\ (F).$

Also $Co\ (F\backslash a_\beta) \leq Co\ (F)$ *for the remaining* a_β*'s except* $a_\beta = 5_{\beta_2} = 5_{0.4}.$

i.e., $\vee_{a_\beta} Co\ (F\backslash a_\beta) = Co\ (F).$

i.e., when $p_\alpha = 3_{\alpha_2},\ \alpha_2 \leq 0.4,\ Co\ (F\backslash p_\alpha) \leq_{a_\beta} \{Co\ (F(a_\beta)|a_\beta \in F\backslash p_\alpha, a \neq p\}.$

i.e., $\forall p_\alpha \in F$,

$$Co\ (F\backslash p_\alpha) \leq \vee_{a_\beta}\{Co\ (F\backslash a_\beta)|a_\beta \in F\backslash p_\alpha, a \neq p\}$$

That is F *is E-dependent.*

Remark 5.2.6. *In the crisp case, if* (X, C) *is a JHC space having the Ramification property and if* $F \subseteq X$ *is a finite R-independent set, then each pair of subsets* $F_1, F_2 \subseteq F$*, we have the equality* $Co(F_1)\bigcap Co(F_2) = Co(F_1 \bigcap F_2)$*. But in the fuzzy case, the above equality need not be true. As an illustration of this fact, consider the following counter example.*

Counter example 5.2.7. *Let* $X = \{1, 2, 3\}$.

Define the fuzzy interval opeator I as $I(1, 1) = 1_{\frac{1}{2}}, I(2, 2) = 2_{\frac{1}{4}}, I(3, 3) = 3_{\frac{1}{6}}, I(1, 2) = I(2, 1) = 1_{\frac{1}{3}} \vee 2_{\frac{1}{3}}, I(1, 3) = I(3, 1) = 1_{\frac{1}{4}} \vee 3_{\frac{1}{4}}, I(2, 3) = I(3, 2) = 2_{\frac{1}{5}} \vee 3_{\frac{1}{5}}.$

Fuzzy convexity generated by I is

$$C = \{\underline{0}, \underline{1}, 1_{\frac{1}{2}}, 2_{\frac{1}{4}}, 3_{\frac{1}{6}}, 1_{\frac{1}{3}} \vee 2_{\frac{1}{3}}, 1_{\frac{1}{4}} \vee 3_{\frac{1}{4}}, 2_{\frac{1}{5}} \vee 3_{\frac{1}{5}}, 1_{\frac{1}{3}}, 1_{\frac{1}{4}}, 2_{\frac{1}{5}}, 3_{\frac{1}{5}}\}$$

Let $F = 1_{\frac{1}{3}} \vee 2_{\frac{1}{5}} \vee 3_{\frac{1}{7}}$ which is finite.

Consider $F_1, F_2 \subseteq F$ where $F_1 = 1_{\frac{1}{5}} \vee 2_{\frac{1}{6}} \vee 3_{\frac{1}{7}}$, $F_2 = 1_{\frac{1}{3}}$

Then

$$\text{Co}\,(F_1 \wedge F_2) = 1_{\frac{1}{4}} \tag{5.2.2}$$

$$\text{Co}\,(F_1) \wedge \text{Co}\,(F_2) = 1_{\frac{1}{3}} \tag{5.2.3}$$

From (5.2.2) and (5.2.3),

$$\text{Co}\,(F_1 \wedge F_2) \leq \text{Co}\,(F_1) \wedge \text{Co}\,(F_2)$$

i.e., no equality as in the crisp case.

Here F is R-independent since $\text{Co}(F_1) \wedge \text{Co}(F_2) = \underline{0}$ for the possible partitions.

1. $\{F_1 = 1_{\frac{1}{3}}, F_2 = 2_{\frac{1}{5}} \vee 3_{\frac{1}{7}}\}$
2. $\{F_1 = 2_{\frac{1}{5}}, F_2 = 1_{\frac{1}{3}} \vee 3_{\frac{1}{7}}\}$
3. $\{F_1 = 3_{\frac{1}{7}}, F_2 = 1_{\frac{1}{3}} \vee 2_{\frac{1}{5}}\}$

Also $I(1,2) \wedge I(1,3) = 1_{\frac{1}{4}}$ where $3 \notin I(1,2)$ $2 \notin I(1,3)$,
$I(2,3) \wedge I(1,3) = 3_{\frac{1}{5}}$ where $1 \notin I(2,3)$ and $2 \notin I(1,3)$ and
$I(2,3) \wedge I(1,2) = 2_{\frac{1}{5}}$ where $1 \notin I(2,3)$ and $3 \notin I(1,2)$
i.e., $(X, \mathcal{C})$ has the Ramification property.

Now we show that $(X, \mathcal{C})$ is JHC.

i.e., we have to show that for a fuzzy convex set F,

$$\mathrm{Co}(a_\alpha \vee F) = \vee_{x_\beta \in F}\mathrm{Co}(a_\alpha \vee x_\beta) \tag{5.2.4}$$

Let $a_\alpha \in I^X$, then $a_\alpha = 1_\alpha, 2_\beta$ or 3_ν.

Consider $F = 1_{\frac{1}{2}} \in \mathcal{C}$.

Let $a_\alpha = 1_\alpha$.

When $\alpha \leq \frac{1}{2}$,

$$\mathrm{Co}(a_\alpha \vee F) = \mathrm{Co}(F) = F = \vee_{a_\beta \in F}\,\mathrm{Co}(a_\alpha \vee x_\beta)$$

When $\alpha > \frac{1}{2}$,

$$\mathrm{Co}(a_\alpha \vee F) = \underline{1} = \vee_{x_\beta}\mathrm{Co}(a_\alpha \vee x_\beta)$$

i.e., when $a_\alpha = 1_\alpha$, (5.2.4) is true for the fuzzy convex set $1_{\frac{1}{2}}$ corresponding to the cases (1) $\alpha \leq \frac{1}{2}$ (2) $\alpha > \frac{1}{2}$.

Also (5.2.4) is true when $a_\alpha = 2_\beta$ or 3_γ, Since

$$\mathrm{Co}(a_\alpha \vee F) = \underline{1} = \vee_{x_\beta}\mathrm{Co}(a_\alpha \vee x_\beta)$$

The same argument follows for the fuzzy subsets $2_{\frac{1}{4}}, 3_{\frac{1}{6}} \in \mathcal{C}$.

Consider $F = 1_{\frac{1}{3}} \vee 2_{\frac{1}{3}} \in \mathcal{C}$.

Let $a_\alpha = 1_\alpha$.

When $\alpha \leq \frac{1}{3}$, $\mathrm{Co}(a_\alpha \vee F) = F = \vee_{x_\beta}\mathrm{Co}(a_\alpha \vee x_\beta)$

When $\alpha > \frac{1}{3}$, $\mathrm{Co}(a_\alpha \vee F) = \underline{1} = \vee_{x_\beta}\mathrm{Co}(a_\alpha \vee x_\beta)$.

i.e. when $a_\alpha = 1_\alpha$, (5.2.4) is true for the fuzzy convex set $1_{\frac{1}{3}} \vee 2_{\frac{1}{3}}$ corresponding to the cases (1) $\alpha \leq \frac{1}{3}$ (2) $\alpha > \frac{1}{3}$.

A similar proof follows when $a_\alpha = 2_\beta$ or 3_ν.

Also (5.2.4) is true for $1_{\frac{1}{4}} \vee 3_{\frac{1}{4}},\ 2_{\frac{1}{5}} \vee 3_{\frac{1}{5}} \in \mathcal{C}$.

i.e., we have to show that for a fuzzy convex set F,

$$\mathrm{Co}(a_\alpha \vee F) = \vee_{x_\beta \in F}\mathrm{Co}(a_\alpha \vee x_\beta) \tag{5.2.4}$$

Let $a_\alpha \in I^X$, then $a_\alpha = 1_\alpha, 2_\beta$ or 3_ν.

Consider $F = 1_{\frac{1}{2}} \in \mathcal{C}$.

Let $a_\alpha = 1_\alpha$.

When $\alpha \le \frac{1}{2}$,

$$\mathrm{Co}(a_\alpha \vee F) = \mathrm{Co}(F) = F = \vee_{a_\beta \in F}\,\mathrm{Co}(a_\alpha \vee x_\beta)$$

When $\alpha > \frac{1}{2}$,

$$\mathrm{Co}(a_\alpha \vee F) = \underline{1} = \vee_{x_\beta}\mathrm{Co}(a_\alpha \vee x_\beta)$$

i.e., when $a_\alpha = 1_\alpha$, (5.2.4) is true for the fuzzy convex set $1_{\frac{1}{2}}$ corresponding to the cases (1) $\alpha \le \frac{1}{2}$ (2) $\alpha > \frac{1}{2}$.

Also (5.2.4) is true when $a_\alpha = 2_\beta$ or 3_γ, Since

$$\mathrm{Co}(a_\alpha \vee F) = \underline{1} = \vee_{x_\beta}\mathrm{Co}(a_\alpha \vee x_\beta)$$

The same argument follows for the fuzzy subsets $2_{\frac{1}{4}}, 3_{\frac{1}{6}} \in \mathcal{C}$.

Consider $F = 1_{\frac{1}{3}} \vee 2_{\frac{1}{3}} \in \mathcal{C}$.

Let $a_\alpha = 1_\alpha$.

When $\alpha \le \frac{1}{3}$, $\mathrm{Co}(a_\alpha \vee F) = F = \vee_{x_\beta}\mathrm{Co}(a_\alpha \vee x_\beta)$

When $\alpha > \frac{1}{3}$, $\mathrm{Co}(a_\alpha \vee F) = \underline{1} = \vee_{x_\beta}\mathrm{Co}(a_\alpha \vee x_\beta)$.

i.e. when $a_\alpha = 1_\alpha$, (5.2.4) is true for the fuzzy convex set $1_{\frac{1}{3}} \vee 2_{\frac{1}{3}}$ corresponding to the cases (1) $\alpha \le \frac{1}{3}$ (2) $\alpha > \frac{1}{3}$.

A similar proof follows when $a_\alpha = 2_\beta$ or 3_ν.

Also (5.2.4) is true for $1_{\frac{1}{4}} \vee 3_{\frac{1}{4}},\ 2_{\frac{1}{5}} \vee 3_{\frac{1}{5}} \in \mathcal{C}$.

Next consider $F = 1_{\frac{1}{3}} \in \mathcal{C}$.

Let $a_\alpha = 1_\alpha$.

When $\alpha \leq \frac{1}{3}$,

$$\text{Co}(a_\alpha \vee F) = F = \vee_{x_\beta}\text{Co}(a_\alpha \vee x_\beta).$$

when $\frac{1}{3} < \alpha \leq \frac{1}{2}$,

$$\text{Co}(a_\alpha \vee F) = \text{Co}(1_{\frac{1}{2}}) = \vee_{x_\beta}\text{Co}(a_\alpha \vee x_\beta).$$

When $\alpha > \frac{1}{2}$,

$$\text{Co}(a_\alpha \vee F) = \underline{1} = \vee_{x_\beta}\text{Co}(a_\alpha \vee x_\beta)$$

i.e. (5.2.4) is true for the fuzzy subset $F = 1_{\frac{1}{3}} \in \mathcal{C}$.

The same equality follows when $a_\alpha = 2_\beta$ or 3_ν.

For the remaining fuzzy convex sets also, (5.2.4) can be shown to be true following similar arguments.

Proposition 5.2.8. *Let $(X, \mathcal{C})$ be a fuzzy convex structure and let F be a non zero fuzzy subset of X. Then for every $x_\gamma \in Co(F)$,*

$$Co(F) = \vee_{a_{\alpha_1} \in F} Co(x_\gamma \vee (F \backslash a_{\alpha_1})) \text{ where } \alpha_1 \neq 1.$$

Proof. Let $a_{\alpha_1} \in F$.

Then we have $F \backslash a_{\alpha_1} \leq F \leq \text{Co}(F)$,

i.e., $x_\gamma \vee (F \backslash a_{\alpha_1}) \leq \text{Co}(F)$

i.e., $\text{Co}(x_\gamma \vee (F \backslash a_{\alpha_1})) \leq \text{Co}(F)$ for each $a_{\alpha_1} \in F$

$$\vee_{a_{\alpha_1} \in F}\text{Co}(x_\gamma \vee (F \backslash a_{\alpha_1})) \leq \text{Co}(F) \tag{5.2.5}$$

Case 1. Let $F(a) = \alpha \leq \frac{1}{2}$

Then $a_{\alpha_1} \in F$ where $\alpha_1 \leq \alpha \leq \frac{1}{2}$.

i.e. $\alpha_1 \leq \frac{1}{2}$.

(i) Let $x_\gamma \in F \in \text{Co}\,(F)$

When $F(a) = \alpha \leq \frac{1}{2}$,

$$F \backslash a_{\alpha_1} = F \quad \text{since}$$
$$(F \backslash a_{\alpha_1})(a) = F(a) \wedge (1 - \alpha_1) = F(a)$$
$$(F \backslash a_{\alpha_1})(x) = F(x). \quad x \neq a.$$

Also $x_\gamma \vee (F \backslash a_{\alpha_1}) = F$.

i.e., $\text{Co}(x_\gamma \vee (F \backslash a_{\alpha_1}) = \text{Co}(F)$ for all $a_{\alpha_1} \in F$

i.e., $\vee_{a_{\alpha_1} \in F} \text{Co}(x_\gamma \vee (F \backslash a_{\alpha_1})) = \text{Co}(F)$

(ii) Let $x_\gamma \notin F$, but $x_\gamma \in \text{Co}(F)$,

$$\text{i.e.} \quad x_\gamma(x) = \gamma > F(x).$$

Here also $F \backslash a_{\alpha_1} = F$, since $F(a) = \alpha \leq \frac{1}{2}$.

i.e., $x_\gamma \vee (F \backslash a_{\alpha_1}) \geq F$

i.e., $\text{Co}(x_\gamma \vee (F \backslash a_{\alpha_1})) \geq \text{Co}(F)\ \forall a_{\alpha_1} \in F$

$$\text{i.e.} \quad \vee_{a_{\alpha_1}} \text{Co}(x_\gamma \vee (F \backslash a_{\alpha_1})) \geq \text{Co}(F) \tag{5.2.6}$$

From (5.2.5) & (5.2.6), we have the equality.

Case 2. Let $F(a) = \alpha > \frac{1}{2}\ (\alpha \neq 1)$.

Then $\exists$ at least one $a_{\alpha_1} \in F$, $\alpha_1 \leq 1 - \alpha$, such that $F \backslash a_{\alpha_1} = F$.

(i) Let $x_\gamma \in F \in \text{Co}(F)$

$$\vee_{a_{\alpha_1} \in F} \text{Co}(x_\gamma \vee (F \backslash a_{\alpha_1})) = \text{Co}(F)$$

(ii) Let $x_\gamma \notin F$, but $x_\gamma \in \text{Co}(F)$ then $x_\gamma \vee (F \backslash a_{\alpha_1})) \geq F$

i.e. Co $(x_\gamma \vee (F \backslash a_{\alpha_1})) \geq \text{Co}(F)$

$$\vee_{a_{\alpha_1}} \text{Co}(x_\gamma \vee (F \backslash a_{\alpha_1})) \geq \text{Co}(F) \tag{5.2.7}$$

From (5.2.5) and (5.2.7), we have the equality. □

Proposition 5.2.9. *If a fuzzy convex structure* $(X, \mathcal{C})$ *has decomposable segments and* F *is a finite fuzzy subset having at least two points in its support, then for all* $x_\gamma \in Co(F)$,

$$Co(F) = \vee_{a_{\alpha_1} \in F} Co(x_\gamma \vee (F \backslash a_{\alpha_1})).$$

Proof. The result follows from the Proposition 1.3.8 which corresponds to the crisp case (*i.e.* when $\alpha_1 = 1$) and the Proposition 5.2.8. □

5.3 Inter-relations between different types of dependence

Proposition 5.3.1. *For a non-zero finite fuzzy subset of a fuzzy convex structure, R dependence implies H-dependence.*

Proof. Let $(X, \mathcal{C})$ be a fuzzy convex structure.

Let $F \in I^X$ be a non zero finite fuzzy subset.

Assume that F is R-dependent. *i.e.* $\exists$ a radon partition $\{F_1, F_2\}$ of F such that $\text{Co}(F_1) \wedge \text{Co}(F_2) \neq \underline{0}$.

Let $a_\alpha \in F$.

Then $F \backslash a_\alpha = F \wedge a'_\alpha$ where $a'_\alpha : \begin{cases} a \to 1 - \alpha \\ x \to 1 \end{cases}, x \neq a.$

We have $F_1 \leq F \wedge a'_\alpha$ or $F_2 \leq F \wedge a'_\alpha$ for each $a_\alpha \in F$.

For,

$$\begin{aligned}(F \wedge a'_\alpha)(x) &= F(x) \wedge a'_\alpha(x). \quad x \in X \\ &= F(x) \wedge (1 - a_\alpha)(x)\end{aligned}$$

Case 1: Let $x \neq a$,

$$\begin{aligned}(F \wedge a'_\alpha)(x) &= F(x) \wedge 1 \\ &= (F_1 \vee F_2)(x) \wedge 1 \geq F_1(x) \text{ and} \\ &\geq F_2(x)\end{aligned}$$

Case 2: Let $x = a$,

$$\begin{aligned}(F \wedge a'_\alpha)(a) &= F(a) \wedge (1 - \alpha) \\ &= (F_1 \vee F_2)(a) \wedge (1 - \alpha) \\ &\geq \alpha \wedge (1 - \alpha) \\ &\geq F_2(a) \quad \text{if } F_1(a) > 0 \text{ or} \\ &\geq F_1(a) \quad \text{if } F_2(a) > 0.\end{aligned}$$

i.e., $F_1 \leq F \wedge a'_\alpha$ or $F_2 \leq F \wedge a'_\alpha$ for each $a_\alpha \in F$.

$$\begin{aligned}\text{i.e.,} \quad \mathrm{Co}(F_1) &\leq \wedge_{a_\alpha \in F} \mathrm{Co}(F \wedge a'_\alpha) \text{ or} \\ \mathrm{Co}(F_2) &\leq \wedge_{a_\alpha \in F} \mathrm{Co}(F \wedge a'_\alpha) \\ \text{i.e.,} \quad \mathrm{Co}(F_1) \wedge \mathrm{Co}(F_2) &\leq \wedge_{a_\alpha \in F} \mathrm{Co}(F \wedge a'_\alpha) \\ \text{i.e.,} \quad \wedge_{a_\alpha \in F} \mathrm{Co}(F \wedge a'_\alpha) &\geq \mathrm{Co}(F_1) \wedge \mathrm{Co}(F_2) \neq 0\end{aligned}$$

by R-independence of F.

$$\text{i.e.,}\quad \wedge_{a_\alpha \in F}\text{Co}(F \wedge a'_\alpha) \neq \underline{0}.$$

$$\text{i.e. } F \text{ is } H\text{-dependent.}$$

□

Remark 5.3.2. *For a nonzero finite fuzzy subset of a fuzzy convex structure, H-dependence need not imply R-dependence by the following example.*

Consider Counter example 5.2.7, where $F = 1_{\frac{1}{3}} \vee 2_{\frac{1}{5}} \vee 3_{\frac{1}{7}}$.

Let $a_\alpha \in F$, then

$$\begin{aligned} a_\alpha &= 1_\alpha,\ \alpha \leq \frac{1}{3} \text{ or} \\ &= 2_\beta,\ \beta \leq \frac{1}{5} \text{ or} \\ &= 3_\gamma,\ \gamma \leq \frac{1}{7}. \end{aligned}$$

Here $F \backslash a_\alpha = 1\frac{1}{3} \vee 2\frac{1}{5} \vee 3\frac{1}{7}$.

$\text{Co}(F \backslash a_\alpha) = \underline{1}$ for all $a_\alpha \in F$.

i.e., $\wedge_{a_\alpha \in F} Co(F \backslash a_\alpha) = \underline{1} \neq 0$.

i.e. F is H-dependent.

In the Counter example 5.2.7, it is verified that F is R-independent and (X, C) is JHC. *i.e.*, H dependence need not imply R-dependence.

Remark 5.3.3. *In the crisp case, if (X, C) is JHC and has the ramification property, then R-dependence $\Leftrightarrow$ H dependence.*

But in the fuzzy case, there is no such equivalence.

Lemma 5.3.4. *Let (X, C) be a fuzzy convex structure. Then a nonzero finite fuzzy subset F of X with $F(p) = \alpha$, $\alpha \in (\frac{1}{2}, 1]$ is C-dependent.*

Proof. Let $F(p) = \alpha > \frac{1}{2}$.

Then $p_{\alpha_1} \in F$.

i.e., $\alpha_1 \leq \alpha$.

When $\alpha_1 = \alpha > \frac{1}{2}$, $1 - \alpha < \frac{1}{2}$.

i.e, $p_{1-\alpha} \in p_\alpha \in F$.

Then $F \backslash p_{1-\alpha} = F$, since when $x = p$.

$$\begin{aligned}(F \backslash p_{1-\alpha})(p) &= F(p) \wedge \alpha \\ &= \alpha \wedge \alpha \\ &= \alpha = F(p)\end{aligned}$$

When $x \neq p$

$$\begin{aligned}(F \backslash p_{1-\alpha})(x) &= F(x) \wedge 1 \\ &= F(x).\end{aligned}$$

i.e., $\mathrm{Co}(F \backslash p_{1-\alpha}) = \mathrm{Co}(F)$.

i.e., $\vee_{x_\beta \in F} \mathrm{Co}(F \backslash x_\beta) = \mathrm{Co}(F)$.

i.e., F is C-dependent. □

Proposition 5.3.5. *If a fuzzy convex structure $(X, \mathcal{C})$ has a CUP, then for every nonzero finite fuzzy subset, E-dependence $\Rightarrow$ C-dependence.*

Proof. We assume that $(X, \mathcal{C})$ has a CUP and F is a non zero finite fuzzy subset which is E-dependent.

Let $p_\alpha \in F$.

Case 1. Let $\alpha \leq \frac{1}{2}$.

We show that $F \leq p_\alpha \vee (F \wedge p'_\alpha)$ iff $F(p) \leq 1 - \alpha$.

For, let $F \leq p_\alpha \vee (F \wedge p'_\alpha)$.

When $x = p$, $F(p) \leq \alpha \vee [F(p) \wedge (1-\alpha)]$

i.e., $F(p) \leq \alpha$ or $F(p) \leq F(p) \wedge (1-\alpha)$

i.e., $F(p) \leq F(p) \wedge (1-\alpha)$ as $F(p) \leq \alpha$ is not true.

i.e., $F(p) \leq F(p) \wedge (1-\alpha) \leq F(p)$.

i.e., $F(p) = F(p) \wedge (1-\alpha)$

i.e., $F(p) \leq 1-\alpha$.

Conversely, let $F(p) \leq 1-\alpha$.

$$\begin{aligned} \text{i.e.,} \quad (p_\alpha \vee (F \wedge p'_\alpha))(p) &= \alpha \vee [F(p) \wedge (1-\alpha)] \\ &= \alpha \vee F(p) = F(p). \end{aligned}$$

Also, $(p_\alpha \vee (F \wedge p'_\alpha))(x) = F(x)$ for all $x \neq p$.

i.e., $p_\alpha \vee (F \wedge p'_\alpha) = F$.

i.e., $F \leq p_\alpha \vee (F \wedge p'_\alpha)$ where $F(p) \leq 1-\alpha$.

$$\text{i.e., Co }(F) \leq \text{Co}(p_\alpha \vee (F \wedge p'_\alpha)) \leq \text{Co}(p_\alpha \vee \text{Co}(F \wedge p'_\alpha)) \tag{5.3.1}$$

Also $p_\alpha \vee \text{Co}(F \wedge p'_\alpha) \leq \text{Co}(F)$.

$$\text{Co}(p_\alpha \vee \text{Co}(F \wedge p'_\alpha)) \leq \text{Co}(F). \tag{5.3.2}$$

From (5.3.1) & (5.3.2),

$$\text{Co}(F) = \text{Co}(p_\alpha \vee \text{Co}(F \wedge p'_\alpha)) \tag{5.3.3}$$

By E-dependence of F, for each $p_\alpha \in F$,

$$\text{Co}(F \wedge p'_\alpha) \leq \vee_{a_\beta \in F}\{\text{Co}(F \wedge a'_\beta) | a_\beta \in F \wedge p'_\alpha, a \neq p\}$$

By CUP,

$$\mathrm{Co}(p_\alpha \vee \mathrm{Co}(F \wedge p'_\alpha)) \leq \vee_{a_\beta}\{\mathrm{Co}(p_\alpha \vee \mathrm{Co}(F \wedge a'_\beta))|a_\beta \in F \wedge p'_\alpha,\ a \neq p\}$$

By (5.3.3),

$$Co(F) \leq \vee_{a_\beta}[\mathrm{Co}(p_\alpha \vee \mathrm{Co}(F \wedge a'_\beta))|a \neq p] \tag{5.3.4}$$

We know $p_\alpha \in F\backslash a_\beta = F \wedge a'_\beta \in Co(F \wedge a'_\beta)$.

i.e., $p_\alpha \vee \mathrm{Co}(F \wedge a'_\beta) = \mathrm{Co}(F \wedge a'_\beta)$.

i.e., Co $(p_\alpha \vee \mathrm{Co}(F \wedge a'_\beta)) = \mathrm{Co}(F \wedge a'_\beta)$.

By (5.3.4),

$$\begin{aligned}\mathrm{Co}(F) &\leq \vee_{a_\beta \in F}\{\mathrm{Co}(F \wedge a'_\beta)|a \neq p\}\\ &\leq \vee_{a_\beta \in F}\mathrm{Co}(F \wedge a'_\beta).\end{aligned}$$

i.e., F is C-dependent.

In particular when $F(p) = \alpha$ and $F(p) \leq 1 - \alpha$. (*i.e.* when $\alpha \leq \frac{1}{2}$), E-dependence $\Rightarrow$ C-dependence.

Case 2. Let $F(p) = \alpha > \frac{1}{2}$.

By lemma 5.3.4, F is C-dependent when $F(p) = \alpha > \frac{1}{2}$.

In particular when F is E dependent, F is C dependent.

Hence the result. □

Proposition 5.3.6. *Let $(X, \mathcal{C})$ be a fuzzy convex structure which is JHC and has decomposable segments. Then for a nonzero finite fuzzy subset of $(X, \mathcal{C})$, H-dependence implies E-dependence.*

Proof. Assume that F is H-dependent, *i.e*, $\wedge_{a_\beta \in F}\mathrm{Co}(F\backslash a_\beta) \neq \underline{0}$

then $[\wedge_{a_\beta \in F}\mathrm{Co}(F\backslash a_\beta)](x) = \gamma \neq 0$ for some γ

i.e, $x_\gamma \in \wedge_{a_\beta}\text{Co}(F\backslash a_\beta)$

$$i.e. \quad x_\gamma \in \text{Co}(F\backslash a_\beta) \text{ for each } a_\beta \in F. \tag{5.3.5}$$

Also $F\backslash p_\alpha\backslash a_\beta \leq F\backslash a_\beta$ where $a_\beta \in F\backslash p_\alpha$, $a \neq p$.

$$\begin{aligned} i.e.\, x_\gamma \vee (F\backslash p_\alpha\backslash a_\beta) &\leq x_\gamma \vee (F\backslash a_\beta) \\ &\leq \text{Co }(F\backslash a_\beta) \\ \text{since} \quad x_\gamma \vee (F\backslash a_\beta) &\leq x_\gamma \vee \text{Co}(F\backslash a_\beta) \\ &= \text{Co}(F\backslash a_\beta) \text{ by (5.3.5)} \end{aligned}$$

i.e,

$$\text{Co}(x_\gamma \vee (F\backslash p_\alpha\backslash a_\beta) \leq \text{Co}(F\backslash a_\beta)$$

for all $x_\gamma \in \text{Co}(F\backslash a_\beta) \in \text{Co}(F)$.

$$\text{i.e.} \quad \vee_{a_\beta}\text{Co}(x_\gamma \vee (F\backslash p_\alpha\backslash a_\beta) \leq \vee_{a_\beta}\text{Co}(F\backslash a_\beta)$$

i.e, for each $p_\alpha \in F$, by prop. 5.2.9

$$\text{Co}(F\backslash p_\alpha) \leq \vee_{a_\beta}\{\text{Co}(F\backslash a_\beta)|a_\beta \in F\backslash p_\alpha, a \neq p\}.$$

i.e., F is E-dependent. □

Conclusion

Since the publication of the classical paper on fuzzy sets by L. A. Zadeh in 1965, the theory of fuzzy mathematics has gained more and more recognition from many researchers in a wide range of scientific fields. Among various branches of pure and applied mathematics, convexity was one of the areas where the notion of fuzzy set was applied. Many researchers have been involved in extending the notion of abstract convexity to the broader framework of fuzzy setting. As a result, a number of concepts have been formulated and explored. However, many concepts are yet to be fuzzified. The main objective of this thesis was to extend some basic concepts and results in convexity theory to the fuzzy setting.

The concept like matroids, independent structures, classical convex invariants like Helly number, Caratheodory number, Radon number and Exchange number form an important area of study in crisp convexity theory. In this thesis, we try to generalize some of these concepts to the fuzzy setting.

Further extending the concepts in crisp convexity theory, we have introduced fuzzy JHC space, CUP etc as the properties of a fuzzy convexity space. We have proved many results in this direction. Further, we have extended and explored the notion of various types of dependence as a background for doing further work in the theory of classical convex invariants to the fuzzy context. Also we have obtained one result (Prop 3.2.3) whose importance according to us is in lifting results from the finite to the general case. Another important result that we have proved is that fuzzy matroids form fuzzy independent structures as in the crisp case but the converse need not be true. Finally, we have defined different types of fuzzy matroids derived from vector spaces and discussed some of their properties.

Still there are results in crisp theory related to the topics covered in this thesis which are to be investigated in the fuzzy setting. There are lots of ideas still left in convexity, for which fuzzy analogues are not defined and explored.

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Index

List of Symbols

I^X	Set of all fuzzy subsets of X
Supp A	Support of fuzzy subset A
I^X_{fin}	Set of all finite fuzzy subsets of X
$\vee$	join
$\wedge$	meet
A'	Complement of A
$(X, \mathcal{C})$	convexity space
$\mathcal{C}$	convexity
$\underline{a}$	constant function whose value is 'a', anywhere
Co (F)	fuzzy convex hull of F
xRy	x is related to y under R
$\equiv$	equivalence relation
$[x]$	fuzzy R-equivalence class
$A \backslash B$	A difference B for fuzzy subsets A, B
Card (A)	Cardinality of A
ϵ	Collection of independent sets
# A	Number of elements in A
$h(A)$	Set of all fuzzy points depending on A
$d(A)$	Rank of A
(X, I)	Interval space on X
$I(a, b)$	fuzzy interval between a and b
lin (A)	Linear hull of A
aff (A)	Affine hull of A
Pr (A)	Projective hull of A
$\mathcal{P}(X)$	set of all subsets of X