RELIABILITY MODELLING AND ANALYSIS IN DISCRETE TIME

SOME CONCEPTS AND MODELS USEFUL IN THE ANALYSIS OF DISCRETE LIFE TIME DATA

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By

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CERTIFICATE

Certified that the thesis entitled "SOME CONCEPTS AND MODELS USEFUL IN THE ANALYSIS OF DISCRETE LIFE TIME DATA" is a bonafide record of work done by Smt. Priya. P. Menon under my guidance in the Department of Statistics, Cochin University of Science and Technology and that no part of it has been included any where previously for the award of any degree or title.

Cochin University of Science and Technology

October 30, 2000

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DECLARATION

This thesis contains no material which has been accepted for the award of any other Degree or Diploma in any University and to the best of my knowledge and belief, it contains no material previously published by any other person, except where due references are made in the text of the thesis.

Priya

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October 30, 2000

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CHAPTER I

INTRODUCTION

1.1 Reliability models in discrete time

In most of the studies relating to life testing and reliability, life time is treated as continuous and accordingly continuous probability distributions are proposed as models. There is an extensive literature on reliability modelling, inference relating to reliability characteristics, various notions of aging and applications to specific problems in continuous time. However there is comparatively less discussion on reliability when life time is treated as discrete. There are examples of discrete random variables that arise naturally in life length studies like the number of cycles to failure, or the number of failures in a given time interval. Further, many of the sophisticated equipments used in the manufacturing process require very accurate measuring devices to record their failures in continuous time. In situations where such measuring instruments are very costly or their availability cannot be ensured it may be desirable to go in for failure times that are in completed units of time (Xekalaki (1983)). The latter procedure is more desirable, provided the loss of accuracy in replacements of continuous measurements to discrete ones is more than compensated by the gain in terms of other considerations such as money, ease of analysis, saving in time etc. Discrete distributions naturally arise when records are taken in completed units of time. The fact that many of the discrete distributions, can be closely approximated by continuous distributions adds to the utility of the former as models of life length. Also there is a well developed methodology to separately find the distribution of the integer parts and fractional parts of continuous random variables, that often permit inference on parameters based on count data to be translated to those based on continuous measurements with a reasonable estimate of the margin of error on account of the translation.

Motivated by the relevance and usefulness of discrete models the present study aims at establishing some results that have applications in the modelling and analysis of life time data in the discrete time domain. In the following sections we present a brief review of the concepts and results that will be helpful in carrying out this objective.

1.2 Basic reliability concepts

Cox (1972), Kalbfleisch and Prentice (1980) and Lawless (1982) in their books briefly touch upon some theoretical aspects of reliability modelling in the discrete time domain.

Let X denote a discrete random variable in the support of $I^{+}=\{0,1,2, \ldots\}$ denoting the time to failure of an equipment or device. Denoting by,

$$R(\mathrm{x}) = \mathrm{P}\ (X \geq x) \tag{1.1}$$

the survival function of X and by $f(x)$ the probability mass function of X, these authors define the failure rate of X as

$$\begin{aligned} h(x) &= \mathrm{P}(X=x\ |X \geq x) \\ &= f(x)/R(x). \end{aligned} \tag{1.2}$$

It is shown that the failure rate determines the life distribution uniquely through the formula,

$$R(x) = \prod_{y=0}^{x-1} (1-h(y)) \tag{1.3}$$

The expression (1.3) demonstrates that if the functional form of $h(x)$ is known, the life distribution can be determined. This points out to the use of the functional form of $h(x)$ in modelling failure time data and also for characterizing the underlying life distribution.

This fact was exploited by Xekalaki (1983) to show that if X is a random variable taking values in the set $\{0, 1, 2, ..., m\}$, $m \in (0,1,2......) \cup \{+\infty\}$. Then,

$$h(x) = (a+bx)^{-1} \tag{1.4}$$

iff X has geometric distribution with

$$P(X = x) = pq^x,\ 0<p<1,\ q = 1-p \tag{1.5}$$

for b=0,

Waring distribution with

$$P(X= x) = (a-b)\,\frac{(b)_x}{(a)_{x+1}},\quad a>b>0 \tag{1.6}$$

where $(a)_x = a\,(a+1)...(a+x-1)$ is the Pochammer symbol, for $b>0$ and negative hypergeometric distribution with

$$P(X=x) = \frac{\binom{k+n-x-1}{n-x}}{\binom{k+n}{n}},\ k>0,\ x=0,\ 1,\ 2,...,\ n \tag{1.7}$$

for $b<0$.

Hitha (1990) has shown that the continuous approximations of the geometric, Waring and negative hypergeometric distributions are respectively the exponential, Pareto II and beta distributions which have the same form for failure rates in continuous time.

A second concept that is extensively used in modelling is the mean residual life (MRL),

$$r(x) = \mathrm{E}(X - x \mid X > x)$$

$$= R(x+1)^{-1} \sum_{x+1}^{\infty} R(y). \tag{1.8}$$

Like the failure rate, MRL also determines the distribution of X as,

$$R(x) = \prod_{u=1}^{x-1} \frac{r(u-1)-1}{r(u)} (1-f(0)) \tag{1.9}$$

where $f(0)$ is determined such that $\sum f(x) = 1$ (Nair and Hitha, (1989)).

The relationship between failure rate and MRL is given by,

$$1 - h(x+1) = \frac{r(x)-1}{r(x+1)}, \; x = 0, 1, 2, \ldots \tag{1.10}$$

Nair (1983) has used the function $r(x)$ to define the notion of memory of life distributions and also to classify them as possessing no memory, negative memory and positive memory at a point x, according as $r(x) = r(x+1)$, $r(x)<r(x+1)$ and $r(x)>r(x+1)$. Since a distribution can have different types of memory at various points of its support, a consolidated measure of memory for the entire support was obtained by considering a weighted average of the measures at various points which is given by,

$$M = \frac{2E^2(X) + E(X) - E(X^2)}{E(X^2) + E(X)} \tag{1.11}$$

The distribution itself has lack of memory, negative memory and positive memory according as M is zero, negative or positive. In a subsequent paper Nair (1989) has shown that geometic, Waring and negative hypergeometric laws in that order are the only discrete distributions that possess lack of memory, constant negative memory and constant positive memory at each point of its support.

Various characterizations of life distributions based on the functional form of r(x) is possible. For example, Nair and Hitha (1991) have shown that the mean residual life of X, $r(x)$ is of the form,

$$r(x) = Ax+B \tag{1.12}$$

if and only if X has geometric distribution for A=0, or Waring distribution for A>0 and negative hypergeometric distribution for A<0.

The case when A=0 was established earlier in Shanbhag (1970). Salvia and Bollinger (1982) besides obtaining (1.3) proved that for increasing failure rate distributions,

$$E(x) \leq \frac{1 - h_0}{h_0}$$

and

$$R(x) \leq (1-h_0)^k = e^{-h_0 x},$$

where $h_0 = h(0)$ and $f(x)$ defines a proper probability mass function if and only if $\sum_{j=0}^{\infty} h(j)$ diverges to $+\infty$. They also demonstrated some limiting behaviour through the equations

$$r = (R-1)^{-1}, f = R^{-1} \text{ and } h = (r+1)$$

where,

$$r = \lim r(x),\ h = \lim h(x),\ f = \lim \frac{f(x)}{f(x+1)} \text{ and } R = \lim \frac{R(x)}{R(x+1)}.$$

A concept that is closely related to that of MRL is the vitality function defined by Kupka and Loo (1989) as

$$v(x) = \mathrm{E}(X|X>x) \tag{1.13}$$

Evidently

$$r(x) = v(x) - x \tag{1.14}$$

and

$$h(x+1)r(x+1) = v(x+1) - v(x). \tag{1.15}$$

It has been mentioned earlier that functional forms of $h(x)$ or $r(x)$ can be used to characterize life distibutions and some results in this connection were reviewed. In the majority of distributions, however simple functional forms do not exist for $r(x)$ or $h(x)$. Common distributions like binomial, Poisson, negative binomial, hypergeometric etc belong to this category. Therefore instead of postulating functional forms of $r(x)$ and $h(x)$ some researchers have attempted to derive relationship between $r(x)$ and $h(x)$ that render unique representation of $f(x)$.

Osaki and Li (1988) generalized the Shanbhag (1970) result by providing a characterization of the negative binomial distribution. They proved that,

if X is a positive discrete random variable with pmf given by

$$f(x) = \binom{x+r-1}{r-1} p^r q^x,\ x = 0, 1, 2, \ldots, p, q>0 \tag{1.16}$$

if and only if,

$$E(X|X>m) = \mu + (m+1-r)\frac{h(x+1)}{p} \tag{1.17}$$

for all integers $m \geq r-1$, where $\mu = r/p$ and $h(k)$ is the failure rate at k. Following this Ahmed (1991) established that

$$E(X|X \geq x) = np + q\ x\ h(x) \tag{1.18}$$

and

$$E(X|X \geq x) = \lambda + x\ h(x) \tag{1.19}$$

are respectively the characteristic properties of the binomial and Poisson distributions.

Generalising these results for the Ord family defined by

$$\frac{f(x+1)-f(x)}{f(x)} = \frac{-(x+d)}{b_0+b_1x+b_2x^2} \tag{1.20}$$

Nair and Sankaran (1991) obtained the result,

$$E(X|X \geq x) = \mu + (a_0+a_1x + a_2x^2)\, h(x) \tag{1.21}$$

where,

$$d = \frac{a_1-a_2-\mu}{2a_1+1}, \quad b_i = \frac{a_i}{2a_2+1}, \quad i = 0,\ 1,\ 2$$

as a characterization. The results of Osaki and Li (1988) and Ahmed (1991) are particular cases of Nair and Sankaran (1991). Further extension of these results are given in Ruiz and Navarro (1994). They define,

$$\frac{\rho(k+1)}{\rho(k)} = \frac{h(x_k+1)}{h(x_k)}(1-h(x_k))$$

for a discrete random variable X with $v(x)<\infty$ and Q:D→R, a real function wth $q(x_b) = x_b - c$ for $b<\infty$, or $\lim_{k\to b}[q(x_k)\rho(k)] = 0$ other wise.

Then the conditions,

1. $\frac{\rho(k+1)-\rho(k)}{\rho(k)} = \frac{x_k - c + q(x_k+1) - q(x_k)}{q(x_k+1)}$

2. $v(x_k) = c + q(x_k)\, h(x_k)$

are equivalent.

If $\lim_{k\to a+}[q(x_k)h(x_k)] = 0$ then $c = \mu$. Taking

a. $x_k = k$ and $q(k) = k/\alpha$, we obtain the results given by Osaki and Li (1988), Shanbhag (1970) for geometric and negative binomial distributions.

b. $x_k = k$ and $q(k) = \frac{1-\rho}{k}$ or $q(k) = k$ we obtain the results in Ahmed (1991) for binomial and Poisson distributions.

c. $x_k = k$, $q(k) = ak^2 + bk + c$, we obtain the result given by Nair and Sankaran (1991).

For $c = 0$, $\frac{v(x)}{h(x)}$ uniquely determines $F(x)$.

If one defines $Y_x = \{X - x | X > x\}$, then Y_x is also a random variable and represents life time remaining to an individual or device when it has survived age x. The survival function of Y_x is

$$R(y;\, x) = \frac{R(x+y+1)}{R(x+1)},\; y \geq 0 \tag{1.22}$$

and the distribution of Y_x is called the residual life distribution (RLD) of X beyond age x. It can be seen that,

$$r(x) = \mathrm{E}(Y_x)$$

implying that the mean of the RLD is the MRL. Once this interpretation is accepted a detailed understanding of the characteristics of residual life requires an analysis of the residual life distribution. We may then require the higher moments of residual life as well.

Accordingly, the second factorial moment of residual life is

$$M(x) = \mathrm{E}\ Y_x(Y_x - 1)$$

$$= \mathrm{E}(Y_x^2) - \mathrm{E}(Y_x) \qquad (1.23)$$

If we denote E (Y_x^2) as $r_2(x)$

$$M(x) = r_2(x) - r(x) \qquad (1.24)$$

In terms of the survival function

$$\mathrm{E}(Y_x^2) = r_2(x) = 1 + \frac{1}{R(x+1)} \sum_1^\infty (2n+1)R(x+n+1)$$

and (1.25)

$$\mathrm{E}(Y_x) = r(x) = 1 + \frac{1}{R(x+1)} \sum_1^\infty R(x+n+1)$$

Thus

$$M(x) = \frac{2}{R(x+1)} \sum_{n=1}^\infty nR(x+n+1). \qquad (1.26)$$

Navarro, Franco and Ruiz (1998) have given a general method to obtain a distribution function $F(x)$ through the k^{th} moment of residual life, defined by,

$$r_k(x) = \mathrm{E}[(X-x)^k \mid X \geq x] \text{ for } k = 1,2,3,\ldots \qquad (1.27)$$

They study characterizations based on relations between failure rate functions, left censored moment function,

$$v_k(x) = \mathrm{E}[X^k|X>x] \tag{1.28}$$

Their results are

1. If $F(x)$ is the distribution function of a discrete r.v. X with mass in $\{x_i,\ i = a, a+1,\ldots, b\}$ where $x_i < x_{i+1,}\ a$ can be $-\infty$, b can be $+\infty$ then $r_k(x)$ uniquely determines $F(x)$ through the inversion formula.

$$F(x) = 1 - \prod_{x_i \leq x} \frac{r_k(x_i)}{r_k(x_{i+})},$$

for all $x < x_b$, $r_k(x+) = \lim_{t \to x+} r_k(t)$.

2. Taking $r_k(x) = \alpha_k$, where $\alpha_k = \mathrm{E}(X^k)$ they have characterized the geometric distribution.

They have also characterized a distribution function through a relation between failure rate $h(x)$ and left censored moments $r_k(x)$ both in continuous and discrete cases.

1.3 Some notions of aging used in the present study

All the concepts described above can be used in the exposition of the manner in which life length is affected by the advancement of age. In other words, we can define various notions of aging based on failure rate, MRL, or vitality function that tells us what happens to the life length of a device (whether decreases, increases or remains steady) as it ages. Classification of life distributions based on different criteria of aging is available in continuous time, but their counterparts for the discrete domain have not been fully investigated. Klesfsjo (1982) has provided discrete analogues to some of the classes discussed in literature in the continuous case. A brief review of the classes that are mentioned in the sequel is presented in the next section.

First we present the simplest concept of aging based on the monotone character of failure rate.

Definition 1.2.1

A discrete random variable X or the corresponding survival function $R(x)$ belongs to the increasing failure rate or IFR (decreasing failure rate or DFR) class if, the failure rate $h(x)$ is an increasing (decreasing) function of x, for all x in I^+.

Characterization of geometric distribution and discrete IFR (DFR) disctributions using order statistics are established in Neweichi and Govindarajulu (1979). Hitha (1991) has given certain necessary and sufficient conditions for a distribution to be IFR (DFR). The results are

1. X is IFR (DFR) if and only if $\frac{R(x+y)}{R(x)}$ is a decreasing (increasing) function of x, for all y in I^{+}.
2. X is IFR (DFR) if and only if $H(x, y)$ is an increasing (decreasing) function of x for all y in I^{+}, where $H(x,y)$ is the cumulated failure rate in the interval $[x, x + y\text{-}1]$ defined by,

$$H(x, y) = \sum_{t=x}^{x+y-1} h(t) \tag{1.29}$$

Studies in the same direction are given in Roy and Gupta (1991) who proved the following results.

1. The probability mass function $f(x)$ belongs to both IFR and DFR class if and only if

$$f(x) = \begin{cases} (1-c)^{x} c & x = 0, 1, 2, \ldots, k-1 \\ (1-c)^{k} & x = k \end{cases} \tag{1.30}$$

   where c is any arbitrary parameter between 0 and 1.
2. If $R(x)$ is the survival function of a life distribution belonging to IFR class then for $0<q<1$,

$$D(x) = R(x) - q^{x}, \; x = 1, 2, \ldots .$$

   changes sign atmost once and the change occurs from positive to negative.

3. For a survival function $R(x)$ of any life distribution belonging to IFR class, given any integer $x_0 \in K$, $K = \{0,1,2,\ldots, k\}$

$$R(x) \geq [R(x_0)]^{x/x_0}, \quad \text{for } x = 0,\ldots, x_0$$
$$\leq (R(x_0))^{x/x_0}, \quad \text{for } x = x_0+1,\ldots$$

and the bound is sharp.

4. A survival function $R(x)$ of a discrete life distribution is IFR if and only if for every fixed integer $x_0 \in K$,

$$\log \frac{R(x)}{R(x_0)} \leq L(x\text{-}x_0) \text{ for all } x \in \mathrm{K},$$

where L is a constant depending on $h(x_0)$.

A similar classification of life distributions in terms of the monotone nature of MRL function is also possible.

Definition 1.2.2

A discrete r.v X or its distribution belongs to the DMRL (IMRL) class if $r(x) \geq r(x+1)$[$r(x) \leq r(x+1)$]for every x in I^+.

Ebrahami (1986) has discussed the class of discrete decreasing and increasing mean residual life functions. The increasing or decreasing nature of failure rates can be subsumed into the concept of bath-tub failure rates in which the failure rate at first (increases) decreases then remain constant and there upon starts decreasing (increasing). The papers by Guess and Park (1988) and Miejie (1994) discuss the properties of the class of discrete life distributions with bath-tub shaped failure rates. Guess and Park (1988) in their paper develops a general approach to modelling discrete bath-tub and upside down bath-tub mean residual life functions.

Hitha (1991) has shown that a sufficient condition for X to be DMRL(IMRL) is that $\frac{R(x+y)}{R(x)}$ is a decreasing (increasing) function of x, for all y in I^+.

It has also been shown that

1. A necessary and sufficient condition for X to be DMRL is that, the variance residual life,

$$V(Y_x) \leq r(x)\ (r(x)-1)$$

2. X is DMRL (IMRL) if, $r(x)h(x)$ is not less than (not greater than) unity.

3. If $b(x) = \frac{V(Y_x)}{r(x)(r(x)-1)}$, among the class of distributions with strictly increasing (decreasing) MRL, Waring (negative hypergeometric) is the only member for which $b(x)$ is a constant.

Salvia (1996) has given an upper and lower bound for MRL for devices with monotone failure rates given by

1. If $\{h_k\}$ is IFR , then $(1\text{-}h_k) \leq r_k \leq (1 - h_0)/h_0$

2. If $\{h_k\}$ is DFR and if E(X) exists , then $\frac{1-h_0}{h_0} \leq r_k \leq \frac{r_k - 1}{1-h_0}$.

It is known that if X is IFR (DFR), then X is DMRL (IMRL). However, the converse need not be true as exemplified in Hitha (1991). The additional condition required on IFR distributions to make it DMRL remains to be established.

1.4 Equilibrium distributions

Another concept that is used in the sequel is of an equilibrium distribution. Corresponding to every discrete distribution of a random variable X in the support of I^+, we can define a random variable Y, with probability mass function,

$$g(y) = \frac{P(X>y)}{\mu}, \; y = 0, 1, 2, \ldots \tag{1.31}$$

and μ = E(X) which is finite. This is called the distribution based on the partial sums of X (Johnson, Kotz and Kemp, (1992)) or the equilibrium distribution corresponding to X. In the context of reliability, Gupta (1979) has shown that the failure rate of Y is the reciprocal of the mean residual life function of X and when the equilibrium distribution belongs to the class of modified power series distributions, the geometric law is the only one satisfying the property $E(X) = E(Y)$. Continuing the work along these lines Nair and Hitha (1989) obtained mutual characterizations of the distributions of X and Y. Some characterizations given by them are

(1) A necessary and sufficient conditions for X to be geometric (p), Waring (a,b), negative hypergeometric (k, n) is that Y is geometric(p), Waring$(a,b+1)$ negative hypergeometric $(k+1,n-1)$.

(2) If $h(x)$, $k(x)$ are the failure rates of X and Y, the property $h(x)=ck(x)$ for all integers $x \geq 0$ and a constant C, characterizes geometric (p) for $C=1$, Waring (a,b) for $C>1$ and negative hypergeometric(k,n) for $0<C<1$.

(3) The MRL and failure rate of X is such that $r(x)\ h(x) = C$ for all integers $x \geq 0$ and a constant $C>0$ if and only if X is geometric (p) for $C=1$, Waring (a,b) for $C>1$ and negative hypergeometric (k,n) for $0<C<1$.

(4) The relationship $r(x) = ka(x)$ where $a(x)$ is the MRL of Y is satisfied for all integers $x \geq 0$ and a constant k if and only if X is geometric (for $k=1$) or Waring (for $k<1$) or negative hypergeometric (for $k>1$).

(5) If X has a particular type of memory, at a given point, then Y also has the same type of memory at that point.

1.5 Mixture models

Let $\underline{X} = (X_1, X_2, \ldots, X_n)$ be a random vector with a family $\{f(x,\theta),\ \theta \in \Theta\}$ of probability distributions, where Θ is a subset of R^T, $R = (-\infty, \infty)$ and T is an arbitrary fixed positive integer. Let $G(\underline{\theta})$ be a T dimensional cumulative distribution function. Then

$$f(x) = \int_{\Theta} f(x,\underline{\theta})\, dG(\underline{\theta}) \qquad (1.32)$$

is called a mixture density function. In (1.32), G(.) is called the mixing distribution. The definition given above is quite general in nature and the special case when $G(.)$ is discrete and assigns positive probability to only a finite number of points (θ_i, i = 1, 2,...,k) is often found useful in applications. When this is the case we get what is called a finite mixture density which can be written in the form,

$$f(x) = p_1 f_1(x) + p_2 f_2(x) + \ldots + p_k f_k(x) \qquad (1.33)$$

Where

$$f_i(x) \geq 0,\ \sum f_i(x) = 1\ ,\ p_i \geq 0\ (i = 1, 2, \ldots, k),\ \sum_{i=1}^{k} p_i = 1.$$

The constant p_i's are called mixing weights and f_i's are called component densities.

Nelson (1982) points out that units manufactured in different production periods may have different life distributions due to difference in design, raw materials, handling etc and it is necessary to identify production period, customer, environment etc that has poor units for remedial action on that part of the population. When the population decomposes into different sub populations the appropriate model of life time data is a mixture of distributions. Cox (1979) has analysed data on failure times using a mixture of exponential models by classifying the data on failure times into two subpopulations depending on whether the cause of failure was identified or not. Mendenhall and Hader (1958), Kao (1959), Fowlkes (1979) Everitt and Hand (1981) and Mendelbaum and Harris (1982) provide examples of life lengths that are distributed in the form of finite mixtures. Thus there is a strong case for a detailed study of mixture distributions in the context of reliability analysis.

The survival function, failure rate and MRL for the mixture distribution represented by (1.33) are

$$R(x) = p_1\, R_1(x) + p_2\, R_2(x) + \dots + p_k\, R_k(x) \tag{1.34}$$

$$h(x) = \frac{f(x)}{R(x)} = \frac{p_1 f_1(x) + p_2 f_2(x) + \dots + p_k f_k(x)}{p_1 R_1(x) + p_2 R_2(x) + \dots + p_k R_k(x)} \tag{1.35}$$

$$r(x) = \frac{\sum_{x+1}^{\infty}[p_1R_1(t) + p_2R_2(t)+\ldots+p_kR_k(t)]}{p_1R_1(x+1) + p_2R_2(x+1)+\ldots+p_kR_k(x+1)} \tag{1.36}$$

Identification of models in the case of mixture distributions can be facilitated in the same way as general models through characterization theorems involving reliability concepts. However characterization of mixture distributions through reliability concepts is a rarely visited area of investigation. The only result that is known to us is in the continuous case.

Nassar and Mahmoud (1985) gave a necessary and sufficient condition for the random variable X to be distributed as a mixture of two exponential distributions. They showed that

$$f(x) = \alpha\lambda_1 e^{-\lambda_1 x} + (1-\alpha)\,\lambda_2 e^{-\lambda_2 x},\ \ \lambda_1,\ \lambda_2 > 0,\ 0<\alpha<1 \tag{1.37}$$

if and only if, for all $y > 0$,

$$\mathrm{E}[X|X>y] = y + \left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right) - \frac{h(y)}{\lambda_1\lambda_2}. \tag{1.38}$$

In our knowledge , no characterization of discrete distributions in terms of reliability concepts have appeared in literature and therefore an attempt in this direction is taken up in a subsequent chapter.

1.6 Partial moments

The r^{th} partial moment of a random variable about a point is defined as,

$$p_r(t) = \mathrm{E}[(X - t)^+]^r, \; r = 0, 1, 2, \ldots \tag{1.39}$$

where $(X\text{-}t)^+ = \max\,(0, X\text{-}t)$.

The properties of partial moments can be used to characterize probability distributions in the same way as truncated moments are employed. The random variable $(X\text{-}t)^+$ used in defining partial moments are meaningful in the study of personal incomes. Those incomes which fall short of tax exemption level t is of no consequence in the computation of taxes and therefore they are as good as treated to be zero. Thus the study of partial moments is useful in analysing measurements that exceed a threshold level without truncating the distribution at t.

Chong (1977) has characterized the exponential distribution by the property.

$$\mathrm{E}(X\text{-}t\text{-}s)^+ \, \mathrm{E}(X) = \mathrm{E}(X\text{-}t)^+ \, \mathrm{E}(X\text{-}s)^+ \tag{1.40}$$

of the partial means. Gupta and Gupta (1983) have mentioned the definition and some properties of partial moments in the discrete case. A detailed study does not seem to have taken place.

1.7 Summary of the present study

The present study consists of five chapters. In Chapter II we take up the derivation of some general results useful in reliability modelling that involves two component mixtures. Expression for the failure rate, mean residual life and second moment of residual life of the mixture distributions in terms of the corresponding quantities in the component distributions are investigated. Some applications of these results are also pointed out. The role of the geometric, Waring and negative hypergeometric distributions as models of life lengths in the discrete time domain has been discussed already. While describing various reliability characteristics, it was found that they can be often considered as a class. The applicability of these models in single populations naturally extends to the case of populations composed of sub-populations making mixtures of these distributions worth investigating. Accordingly the general properties, various reliability characteristics and characterizations of these models are discussed in chapter III. Inference of parameters in mixture distribution is usually a difficult problem because the mass function of the mixture is a linear function of the component masses that makes manipulation of the likelihood equations, leastsquare function etc and the resulting computations very difficult. We show that one of our characterizations help in inferring the parameters of the geometric mixture without involving computational hazards. As mentioned in the review of results in the previous sections, partial moments were not studied extensively in literature especially in the case of discrete distributions. Chapters IV and V deal with descending and ascending partial factorial moments. Apart from studying their properties, we prove characterizations of distributions by functional forms of partial moments and establish recurrence relations between successive moments for some well known families. It is further demonstrated that partial moments are equally efficient and convenient compared to many of the conventional tools to resolve practical problems in reliability modelling and analysis. The study concludes by indicating some new problems that surfaced during the course of the present investigation which could be the subject for a future work in this area.

CHAPTER II

RELIABILITY CONCEPTS IN DISCRETE MIXTURE MODELS

2.1 Introduction

Eversince the concept of mixture distribution was introduced by Newcomb in 1886 as a model of outliers, the interest in mixture distributions have increased considerably as models appropriate to a wide variety of data situations in different scientific investigations. In Section 1.5 we have pointed out various practical problems that lead to a mixture of distributions for life length and cited case studies and examples in this direction. In addition to known physical characteristics that suggest such laws, cases where interrelationships within the system is not known to indicate the appropriate model it may some times happen that there would be a pronounced lack of fit to single models such as Poisson, negative binomial etc in which case mixture of two or more distributions would be more appropriate. Further, for data sets with multiple modes, mixtures are preferred to multimodal single distributions, because of the flexibility in the former in terms of more parameters. In this chapter we discuss some general results involving the basic reliability concepts mentioned in Section 1.2 when the life distribution is a mixture of two discrete distributions.

2.2 Failure rate and residual life

Let X be a discrete random variable in the support of the set of non-negative integers I^+ with probability mass function of the form

$$f(x) = p\, f_1(x) + (1-p)\, f_2(x),\ 0<p<1. \tag{2.1}$$

Then the survival function of X is

$$R(x) = \mathrm{P}(X \geq x) = p\, R_1(x) + (1-p)\, R_2(x) \tag{2.2}$$

where $R_i(x)$ is the survival function corresponding to $f_i(x)$, $i = 1, 2$. The mixture and component distributions possess failure rates specified by

$$h(x) = \frac{pf_1(x) + (1-p)f_2(x)}{pR_1(x) + (1-p)R_2(x)} \tag{2.3}$$

and

$$h_i(x) = \frac{f_i(x)}{R_i(x)},\ i = 1, 2. \tag{2.4}$$

It follows that

$$R(x)\, h(x) = p\, R_1(x)\, h_1(x) + (1-p)\, R_2(x)\, h_2(x). \tag{2.5}$$

Under the assumption that $E(X)<\infty$, the MRL functions arising from $f(x), f_1(x)$ and $f_2(x)$ are

$$r(x) = \frac{\sum_{t=x+1}^{\infty}(pR_1(t)+(1-p)R_2(t))}{pR_1(x+1)+(1-p)R_2(x+1)} \tag{2.6}$$

$$r_i(x) = \frac{\sum_{x+1}^{\infty} R_i(t)}{R_i(x+1)}. \tag{2.7}$$

From (2.6) and (2.7)

$$R(x+1)\ r(x) = p\ R_1(x+1)\ r_1(x) + (1-p)\ R_2(x+1)\ r_2(x). \tag{2.8}$$

Also from the relationship between MRLF and vitality function,

$$R(x+1)(v(x)-x) = p\ R_1(x+1)(v_1(x)-x) + (1-p)\ R_2(x+1)(v_2(x)-x)$$

or

$$R(x+1)\ v(x) = p\ R_1(x+1)\ v_1(x) + (1-p)\ R_2(x+1)\ v_2(x). \tag{2.9}$$

Eliminating R_1, R_2 and R from equation (2.2), (2.5) and (2.9)

$$v(x)\ [h_1(x+1)-h_2(x+1)]-\ h(x+1)[v_1(x)-v_2(x)]$$

$$=h_1(x+1)v_2(x) - h_2(x+1)v_1(x) \tag{2.10}$$

Similarly

$$r(x)\ [h_1(x+1)-h_2(x+1)]-\ h(x+1)[r_1(x)-r_2(x)]$$

$$=h_1(x+1)r_2(x) - h_2(x+1)r_1(x) \tag{2.11}$$

The primary use of the last two equations is to work down identities connecting $v(x)$ and $h(x)$ of mixtures when the form of the component densities are known. Equation (2.11) can be converted into an equation connecting $r(x)$, $r_1(x)$ and $r_2(x)$ if one uses the relationship between failure rate and MRLF given in (1.10) viz.

$$1-h(x+1) = \frac{r(x)-1}{r(x+1)} \qquad (2.12)$$

Rewriting (2.11) as

$$r(x)\,[1-h_2(x+1)-(1-h_1(x+1))]+(1-h(x+1))[r_1(x)-r_2(x)]$$

$$=(1-h_2(x+1))r_1(x) - (1-h_1(x+1))r_2(x)$$

and applying (2.12), we have

$$r(x)\left[\frac{r_2(x)-1}{r_2(x+1)}-\frac{r_1(x)-1}{r_1(x+1)}\right]+\frac{r(x)-1}{r(x+1)}[r_1(x)-r_2(x)]$$

$$=\frac{r_2(x)-1}{r_2(x+1)}r_1(x)-\frac{r_1(x)-1}{r_1(x+1)}\,r_2(x). \qquad (2.13)$$

The second factorial moment of residual life of a random variable X was obtained in equation (1.26) as

$$M(x) = 2[R(x+1)]^{-1} \sum_{n=1}^{\infty} nR(x+n+1)$$

Denoting by $M_1(x)$ and $M_2(x)$ the second factorial moments of the component distributions we can write

$$R_1(x+1)M_1(x) = 2 \sum_{n=1}^{\infty} nR_1(x+n+1)$$

and

$$R_2(x+1)M_2(x) = 2 \sum_{n=1}^{\infty} nR_2(x+n+1).$$

Thus

$$M(x)R(x+1) = 2\sum_{n=1}^{\infty} n[pR_1(x+n+1)+(1-p)R_2(x+n+1)]$$

$$= p\,R_1(x+1)M_1(x) + (1-p)\,R_2(x+1)M_2(x) \qquad (2.14)$$

As before eliminating R_1 and R_2 from (2.2), (2.5) and (2.14),

$$M(x)[h_1(x+1) - h_2(x+1)] - h(x+1)[M_1(x) - M_2(x)]$$

$$= h_1(x+1)M_2(x) - h_2(x+1)M_1(x). \qquad (2.15)$$

2.3 Criteria for aging

This section deals with some approaches to the phenomenon of aging in the case of life times following a mixture distribution. Through the conditions presented here one can check whether the life length of the device is increasing or decreasing together with the manner in which these improvements or deterioration in the effectiveness of the device takes place with regard to its age.

From equation (2.5)

$$h(x+1) - h(x) = [p\,R_1(x+1)\,h_1(x+1)+(1-p)R_2(x+1)h_2(x+1)]/R(x+1)$$
$$- [p\,R_1(x)\,h_1(x) + (1-p)R_2(x)h_2(x)]/R(x).$$

and hence

$$R(x)R(x+1)[\,h(x+1) - h(x)] = [p\,R_1(x+1)h_1(x+1)+(1-p)R_2(x+1)h_2(x+1)]$$
$$[pR_1(x)+(1-p)R_2(x)]$$
$$- [p\,R_1(x)\,h_1(x) + (1-p)R_2(x)h_2(x)]$$
$$[p\,R_1(x+1)+(1-p)R_2(x+1)]$$

$$= p^2 R_1(x)R_1(x+1)[h_1(x+1)- h_1(x)]$$
$$+ (1- p)^2 R_2(x)R_2(x+1)[h_2(x+1)- h_2(x)]$$
$$+ p(1-p) R_1(x)R_2(x+1)[h_2(x+1)- h_1(x)]$$
$$+p(1-p)R_1(x+1)R_2(x)[h_1(x+1)- h_2(x)]$$

$$= R(x)[(h_1(x+1)- h_1(x))pR_1(x) +(h_2(x+1)$$
$$- h_2(x))(1-p)R_2(x)]+ p(1-p)(h_1(x)$$
$$- h_2(x))[R_1(x+1)R_2(x) - R_1(x)R_2(x+1)]$$

$$= R(x)[(h_1(x+1)- h_1(x))pR_1(x) +(h_2(x+1)$$
$$- h_2(x))(1-p)R_2(x)]$$
$$- p(1-p)[h_1(x)-h_2(x)]^2 R_1(x)R_2(x). \quad (2.16)$$

On using $\dfrac{R_i(x+1)}{R_i(x)} = 1-h_i(x)$. Equation (2.16) yields the following **theorem.**

Theorem 2.1

The mixture distribution (2.1) is IFR (DFR) if and only if

$$R(x)[(h_1(x+1)- h_1(x))pR_1(x) +(h_2(x+1)- h_2(x))(1-p)R_2(x)] \geq (\leq)$$
$$p(1-p)(h_1(x)- h_2(x))^2 R_1(x)R_2(x).$$

Corollary 2.1

If the component distributions are DFR then the mixture is always DFR.

In view of the similarity between (2.5) and (2.8), we have by similar computations

$$
\begin{aligned}
R(x+1)\,R(x+2)[r(x+1)-r(x)] = R(x+1)\,&[(r_1(x+1)-r_1(x))pR_1(x+1) \\
&+(r_2(x+1)-r_2(x))(1-p)R_2(x+1)] \\
&-p(1-p)[r_1(x)-r_2(x)]^2\,R_1(x+1)R_2(x+1). \quad (2.17)
\end{aligned}
$$

and the following.

Theorem 2.2

The mixture distribution (2.1) is IMRL (DMRL) if and only if

$$
\begin{aligned}
R(x+1)\,[(r_1(x+1)-r_1(x))pR_1(x+1) &+(r_2(x+1)-r_2(x))(1-p)R_2(x+1)] \\
&\geq (\leq)\; p(1-p)[r_1(x)-r_2(x)]^2\,R_1(x+1)R_2(x+1). \quad (2.18)
\end{aligned}
$$

Another condition can be obtained if we substitute

$$pR_1(x) = \frac{h(x)-h_2(x)}{h_1(x)-h_2(x)}R(x) \text{ and } (1-p)R_2(x) = \frac{h(x)-h_1(x)}{h_2(x)-h_1(x)}R(x)$$

obtained from equations (2.2) and (2.5). This gives

$$
\begin{aligned}
r(x+1) - r(x) = [h_1(x+1)-h_2(x+1)]^{-1}\,&[h(x+1)(r_1(x+1)-r_2(x+1)) \\
&+ h_1(x+1)r_2(x+1)-h_2(x+1)r_1(x+1)] \\
&- [h_1(x)-h_2(x)]^{-1}\,[h(x)(r_1(x)-r_2(x)) \\
&+ h_1(x)r_2(x)-h_2(x)r_1(x)]. \quad (2.19)
\end{aligned}
$$

Theorem 2.3

The mixture distribution (2.1) is IMRL (DMRL) if and only if

$$\frac{h(x)(r_1(x)-r_2(x))+h_1(x)r_2(x)-h_2(x)r_1(x)}{h_1(x)-h_2(x)} \tag{2.20}$$

is an increasing (decreasing) function in x.

Note

1. In Theorem 2.1 we have not made any specific assumption on the monotonicity of $h_1(x)$ and $h_2(x)$. Therefore the DFR nature of the mixture holds under the given conditions even if the component distributions are IFR. A similar comment is true of the IMRL nature of (2.1) even when the component distributions are DMRL.
2. The other aging properties like IFRA (increasing failure rate average), NBU (new better than used) NBUE (new better than used in expectation) HNBUE (harmonically new better than used in expectation) etc defined in terms of $h(x)$ and $r(x)$ can be obtained directly from the above expressions. They are therefore, not considered separately.

2.4 Equilibrium distributions

The definition and some reliability characteristics of equilibrium distributions were discussed in Section 1.4. The importance of such distributions arise from the fact that the corresponding random variable Y describes the residual life of a component in a system where a component of life length X is replaced upon failure by another having the same life time, so that the sequence of life lengths form a renewal process. Deshpande et al. (1986) have obtained certain new notions of aging based on the distribution of Y. They argue that life distribution of a device which ages more rapidly will come off worse in a comparison between the reliability functions of X and Y.

For the mixture model (2.1) the density function of Y from (1.31) is

$$g(y) = \mu^{-1} R(y+1), \quad y = 0, 1, 2, \ldots; \ \mu = \mathrm{E}(X) \tag{2.21}$$

and therefore the survival function becomes

$$S(y) = \mu^{-1} \left[p\sum_{y}^{\infty} R_1(x+1) + (1-p)\sum_{y}^{\infty} R_2(x+1) \right]$$

$$= \mu^{-1} \left[p\sum_{y+1}^{\infty} R_1(x) + (1-p)\sum_{y+1}^{\infty} R_2(x) \right]$$

$$= \mu^{-1} \left[pR_1(x+1)r_1(x) + (1-p)R_2(x+1)r_2(x) \right]$$

The failure rate of Y is accordingly

$$k(x) = \frac{\mu^{-1}[pR_1(x+1) + (1-p)R_2(x+1)]}{\mu^{-1}[pR_1(x+1)r_1(x) + (1-p)R_2(x+1)r_2(x)]}$$

$$= \frac{\dfrac{h(x+1)-h_1(x+1)}{h_2(x+1)-h_1(x+1)} + \dfrac{h(x+1)-h_2(x+1)}{h_1(x+1)-h_2(x+1)}}{r_2(x)\dfrac{h(x+1)-h_1(x+1)}{h_2(x+1)-h_1(x+1)} + r_1(x)\dfrac{h(x+1)-h_2(x+1)}{h_1(x+1)-h_2(x+1)}}$$

$$= \frac{h_1(x+1)-h_2(x+1)}{h(x+1)[r_1(x)-r_2(x)]+h_1(x+1)r_2(x)-h_2(x+1)r_1(x)}. \tag{2.22}$$

Equation (2.22) will be used in the sequel to extract mutual characterizations of the distributions of X and Y.

2.5 Mixture with a geometric component

The results in the preceding sections assume a much simplified structure when one of the components is geometric because of the constancy of the failure rate and mean residual life in that case. For example the necessary and sufficient condition for the mixture to be IFR(DFR) and DMRL (IMRL) respectively reduce to

$$h_2(x+1) - h_2(x) \geq (\leq) \frac{p(p_1 - h_2(x))^2 q_1^x}{R(x)} \tag{2.23}$$

and

$$r_2(x+1) - r_2(x) \geq (\leq) \frac{p(p_1^{-1} - r_2(x))^2 q_1^{x+1}}{R(x+1)}. \tag{2.24}$$

when the first component is geometric with parameter p_1. Further explanation of these results will be taken up in Chapter III when we consider mixtures of specific distributions.

CHAPTER III

SOME MIXTURE DISTRIBUTIONS USEFUL IN RELIABILITY MODELLING

3.1 Preliminaries

Some results relating to two-component mixtures of discrete distributions were presented in the previous Chapter. Also a survey of important distributions used in the context of reliability modelling and their properties in this connection was made in Chapter I. We now consider mixtures of these models and prove that several results in the case of single distributions extend to mixtures under specified conditions. Apart from the applications in reliability studies, many of the results are found to be useful in a more general situation and suppliments the general discussion available in literature on mixture models.

3.2 Geometric mixture

A two-component mixture of geometric distributions with parameters p_1 and p_2 has probability mass function

$$f(x) = p\, p_1\, q_1^x + (1-p)\, p_2\, q_2^x,\ x = 0, 1, 2, \ldots \tag{3.1}$$

$0<p_i<1$ and $q_i = 1 - p_i$, $i=1, 2$.

3.2.1 General properties.

The r^{th} factorial moment is

$$\mu_{(r)} = p\, r! \left(\frac{q_1}{p_1}\right)^r + (1-p)\, r! \left(\frac{q_2}{p_2}\right)^r. \tag{3.2}$$

so that mean and variance of X are

$$\mu = pq_1 p_1^{-1} + (1-p)q_2 p_2^{-1} \tag{3.3}$$

and

$$\sigma^2 = pq_1 p_1^{-2} + (1-p)q_2 p_2^{-2}. \tag{3.4}$$

The reliability function, failure rate and MRL are respectively

$$R(x) = p\, q_1^x + (1-p)\, q_2^x \tag{3.5}$$

$$h(x) = \frac{pp_1q_1^x + (1-p)p_2q_2^x}{pq_1^x + (1-p)q_2^x} \tag{3.6}$$

$$r(x) = \frac{pq_1^{x+1}p_1^{-1} + (1-p)q_2^{x+1}p_2^{-1}}{pq_1^{x+1} + (1-p)q_2^{x+1}}. \tag{3.7}$$

Further in the notations of the previous chapter,

$$h_i(x) = p_i,\ r_i(x) = p_i^{-1} \text{ and } M_i(x) = 2q_i p_i^{-2}.$$

Substituting these values in (2.11)

$$r(x) + p_1^{-1} p_2^{-1} h(x+1) = p_1^{-1} + p_2^{-1} \tag{3.8}$$

giving a compact relationship between the failure rate and MRL of the mixture independently of the reliability measures of the components. A similar relationship between $M(x)$ and $h(x)$ is obtained from equation (2.15)

$$M(x) - 2p_1^{-2} p_2^{-2}(p_1p_2-p_1-p_2)\, h(x+1) = 2p_1^{-2} p_2^{-2}[p_1^2+p_2^2+p_1p_2(1-p_1-p_2)] \tag{3.9}$$

The left hand expression of the condition in Theorem 2.1 is zero. Accordingly since

$$p(1-p)(h_1(x)-h_2(x))^2 R_1(x) R_2(x) = p(1-p)(p_1-p_2)^2 q_1^x q_2^x \geq 0$$

the mixture is DFR. Thus eventhough the component distribution possesses no aging property, their mixture has decreasing failure rate. Likewise from Theorem 2.2 the distribution is IMRL. From equation (2.22) the failure rate of the equilibrium distribution is

$$k(x) = \frac{p_1 - p_2}{h(x+1)[p_1^{-1} - p_2^{-1}] + p_1p_2^{-1} - p_2p_1^{-1}}$$

$$= \frac{p_1p_2}{p_1 + p_2 - h(x+1)}. \tag{3.10}$$

3.2.2 Characterizations

In this section we establish several characterizations of the mixture geometric law.

Theorem 3.1

Let X be a discrete random variable in the support of I^+ with $E(X)<\infty$. Then

$$r(x) = (p_1^{-1} + p_2^{-1}) - p_1^{-1} p_2^{-1} h(x+1)$$

for all x in I^+ if and only if X has distribution with probability mass function (3.1).

Proof: The 'if' part is proved in section 3.2.1. To prove the 'only if' part we write equation (3.8) as

$$\sum_{x+1}^{\infty} R(t) = (p_1^{-1} + p_2^{-1}) R(x+1) - p_1^{-1} p_2^{-1} f(x+1) \tag{3.11}$$

on using the definitions of $r(x)$ and $h(x)$. Now (3.11) is

$$\sum_{x+1}^{\infty} R(t) = (p_1^{-1} + p_2^{-1}) R(x+1) - p_1^{-1} p_2^{-1} (R(x+1) - R(x+2)). \tag{3.12}$$

Changing x to x-1

$$\sum_{x}^{\infty} R(t) = (p_1^{-1} + p_2^{-1}) R(x) - p_1^{-1} p_2^{-1} (R(x) - R(x+1)). \tag{3.13}$$

Subtracting (3.13) from (3.12),

$$R(x+2) + (p_1 + p_2 - 2)R(x+1) + (1 - p_1)(1 - p_2) R(x) = 0$$

This is a homogeneous difference equation of the second order, whose solution is of the form

$$R(x) = p\ m_1^x + q\ m_2^x, \text{ for some constants } p,\ q$$

where m_1 and m_2 are the roots of the auxillary equation

$$m^2 - (1-p_1+1-p_2)m + (1-p_1)(1-p_2) = 0. \tag{3.14}$$

Equation (3.14) has roots

$$m_1 = 1\text{-}p_1 = q_1 \text{ and } m_2 = 1\text{-}p_2 = q_2.$$

Thus

$$R(x) = p\ q_1^x + q\ q_2^x.$$

Since $R(0) = 1$, $p+q = 1$ and hence

$$R(x) = p\ q_1^x + (1\text{-}p)\ q_2^x$$

as claimed in the Theorem.

Corollary 3.1

X follows geometric law with

$$\mathrm{P}(X=x) = q^x p,\ x = 0,\ 1,\ 2,\ \ldots$$

if and only if

$$r(x) = 2p^{-1} - p^{-2}\ h(x+1). \tag{3.15}$$

The result follows by taking $p_1 = p_2 = p$ in Theorem 3.1. This is a particular case of the result for the Ord family proved in Nair and Sankaran (1991). Further if one notes that $h(x)$ is a constant p for the geometric distribution, we have Shanbhag's (1970) result from (3.15).

Theorem 3.2

If X be a discrete random variable in the support of I^+ with $E(X^2)<\infty$. Then the distribution of X is geometric mixture as in (3.1) if and only if

$$M(x) = Ah(x+1)+B$$

where

$$A = 2p_1^{-2}p_2^{-2}(p_1p_2-p_1-p_2) \tag{3.16}$$

and

$$B = 2p_1^{-2}p_2^{-2}[p_1^2+p_2^2+p_1p_2(1-p_1-p_2)] \tag{3.17}$$

Proof: When the distribution is mixture geometric it is shown in the previous section that (3.9) holds.

Conversely assume that,

$$M(x) = Ah(x+1)+B$$

That is,

$$2[R(x+1)]^{-1}\sum_{n=1}^{\infty} nR(x+n+1) = A[f(x+1)/R(x+1)] + B$$

or

$$\sum_{n=1}^{\infty} nR(x+n+1) = A\,f(x+1) + B\,R(x+1) \tag{3.18}$$

and

$$\sum_{n=1}^{\infty} nR(x+n) = A\,f(x) + B\,R(x) \tag{3.19}$$

Subtracting (3.18) from (3.19) gives,

$$\sum_{n=1}^{\infty} R(x+n) = A\,(f(x)-f(x+1)) + B\,(R(x)-R(x+1))$$

or

$$\sum_{n=1}^{\infty} R(x+n) = A\,(f(x)-f(x+1)) + Bf(x) \tag{3.20}$$

From (3.20),

$$\sum_{n=1}^{\infty} R(x+n+1) = A\,(f(x+1)-f(x+2)) + Bf(x+1) \tag{3.21}$$

and then from (3.20) and (3.21) gives,

$$R(x+1) = A(f(x)-2f(x+1)+f(x+2))+\mathrm{B}(f(x)-f(x+1)) \tag{3.22}$$

$$R(x+2) = A(f(x+1)-2f(x+2)+f(x+3)+B(f(x+1)-f(x+2)) \tag{3.23}$$

Lastly

$$\begin{aligned} f(x+1) &= R(x+1) - R(x+2) \\ &= A(f(x)-3f(x+1)+3f(x+2)-f(x+3))+B(f(x)-2f(x+1)+f(x+2)). \end{aligned} \tag{3.24}$$

Introducing the forward shift operator U defined by $Uf(x)=f(x+1)$, equation (3.24) becomes,

$$[AU^3-(3A+B)U^2+(3A+2B+1)U-(A+B)]f(x) = 0$$

whose solution is of the form

$$f(x) = C_1\ t_1^x + C_2\ t_2^x + C_3\ t_3^x,$$

where t_1, t_2 and t_3 are solutions of

$$At^3-(3A+B)t^2+(3A+2B+1)t-(A+B)=0.$$

Substituting for A and B and solving,

$$t_1=q_1,\ t_2 = q_2 \text{ and } t_3 = (p_2^2-p_1^2)/(p_2^2-p_1^2-p_1p_2(p_1+p_2)).$$

Thus

$$f(x) = \alpha p_1 q_1^x + \beta p_2 q_2^x + C_3\ [(p_2^2-p_1^2)/(p_2^2-p_1^2-p_1p_2(p_1+p_2))]^x$$

with $\alpha = \dfrac{C_1}{p_1}$ and $\beta = \dfrac{C_2}{p_2}$. It can be readily verified that $f(x)$ cannot be a probability mass function unless $C_3=0$ and $\beta=1-\alpha$, since otherwise the value of $(p_2^2-p_1^2)/(p_2^2-p_1^2-p_1p_2(p_1+p_2))$ would be greater than one.

Hence,

$$f(x) = \alpha\, p_1\, q_1^x + (1-\alpha)\, p_2\, q_2^x,$$

as required.

Corollary 3.2

X follows the geometric law with probability mass function

$$f(x)= q^x p,\ x = 0, 1, 2, \ldots, 0<p<1,\ q = 1-p,$$

if and only if for all x,

$$M(x) = 2p^{-3}\,(p\text{-}2)h(x+1)+2p^{-2}\,(3\text{-}2p) \qquad (3.25)$$

If one uses $h(x) = p$ in equation (3.25) that equation becomes

$$E[(X-x)(X-x-1)|X>x] = EX(X+1) \text{ for all } x$$

which is closely related to the result of Navarro et al. (1998).

Theorem 3.2 exhibit an identity connecting the failure rate with the second moment of residual life. Another direction by which the results in Theorem 3.1 and Theorem 3.2 can be generalised is by increasing the number of component densities in $f(x)$. Accordingly the next Theorem concerns three component geometric mixtures in which the second moment and mean of residual life and failure rate are involved. When the observations arise from a system which can be decomposed into three sub-populations Theorem 3.3 can help in identifying whether the underlying distribution is a geometric mixture or not. Apart from this the form of the identity established here will give us a hint to the general structure of the characteristic property in the case of any finite mixture.

Theorem 3.3

A discrete random variable in the support of I^+ with $E(X^2)<\infty$ will be a geometric mixture with probability mass function

$$f(x) = \sum_{i=1}^{3} \alpha_i p_i q_i^{x_i}, \quad \alpha_i > 0; \ \sum \alpha_i = 1 \tag{3.26}$$

if and only if

$$M(x) = A_1 r(x) + A_2 + A_3\, h(x+1) \tag{3.27}$$

where

$A_1 = 2 \sum p_i^{-1} - 2$; $A_2 = -2 \sum p_i^{-1} p_j^{-1}$, $i, j = 1, 2, 3$, $i<j$; $A_3 = 2\ p_1^{-1}\, p_2^{-1}\, p_3^{-1}$.

Proof: When the distribution is specified by (3.26) by direct calculations

$$R(x) = \alpha_1 q_1^x + \alpha_2\, q_2^x + \alpha_3 q_3^x$$

$$h(x) = [R(x)]^{-1}(\alpha_1 q_1^x p_1 + \alpha_2\, q_2^x p_2 + \alpha_3 q_3^x p_3)$$

$$r(x) = [R(x+1)]^{-1} \sum_{x+1}^{\infty} (\alpha_1 q_1^t + \alpha_2\, q_2^t + \alpha_3 q_3^t)$$

$$= [R(x+1)]^{-1} \sum_{i=1}^{3} \alpha_i q_i^{x+1} p_i^{-1}$$

and

$$M(x) = 2[R(x+1)]^{-1} \sum_{n=1}^{\infty} n(\alpha_1 q_1^{x+n+1} + \alpha_2\, q_2^{x+n+1} + \alpha_3 q_3^{x+n+1})$$

$$= 2[R(x+1)]^{-1} \sum_{i=1}^{3} \alpha_i q_i^{x+2} p_i^{-2}\,.$$

By direct substitution of these values in (3.27) we can verify that (3.27) holds.

Conversely assume that (3.27) is valid for some distribution in the support of I^+. Then by definition of $h(x)$, $r(x)$ and $M(x)$

$$2\sum_{n=1}^{\infty} nR(x+n+1) = A_1 \sum_{x+1}^{\infty} R(t) + A_2\, R(x+1) + A_3\, f(x+1) \tag{3.28}$$

Differencing with respect to x,

$$2\sum_{x+1}^{\infty} R(t) = A_1\, R(x) + A_2\,[R(x) - R(x+1)] + A_3[R(x) - 2R(x+1) + R(x+2)]$$

Further differencing successively leads to the equation,

$$A_3R(x+3) - (A_2+3\,A_3)R(x+2) + (2+A_1+2A_2+3A_3)\,R(x+1)$$
$$- (A_1+A_2+A_3)R(x) = 0 \tag{3.29}$$

Setting $R(x) = m^x$,

$$m^3 - \frac{A_2+3A_3}{A_3}m^2 + \frac{2+A_1+2A_2+3A_3}{A_3}m - \frac{A_1+A_2+A_3}{A_3} = 0 \tag{3.30}$$

Replacing A_1, A_2 and A_3 by the values in the Theorem,

$$m^3 - (3-p_1-p_2-p_3)\,m^2 + \left(\sum p_ip_j - 2\sum p_i + 3\right)m -$$
$$\left(1 - \sum p_i + \sum p_ip_j - p_1p_2p_3\right) = 0$$

$$[m-(1-p_1)]\;[m-(1-p_2)]\;[m-(1-p_3)] = 0$$

Now the solution of the equation (3.29) is of the form

$$R(x) = \alpha_1 m_1^x + \alpha_2\, m_2^x + \alpha_3 m_3^x$$

where m_1, m_2 and m_3 are the roots of the auxillary equation (3.30) which are q_1, q_2 and q_3. Thus

$$R(x) = \alpha_1 q_1^x + \alpha_2\, q_2^x + \alpha_3 q_3^x.$$

Since $R(0) = 1$, $\sum \alpha_i = 1$ and the Theorem is completely proved.

Now we are in a position to state the general result in the case of any finite mixture of geometric laws with n components, n=1, 2, 3, ...

Theorem 3.4

Let X be a discrete random variable in the support of I^+ with $E(X^n)<\infty$. Then if X has a probability mass function of the form

$$f(x) = \sum_{i=1}^{n} \alpha_i q_i^x p_i, \quad \alpha_i>0, \ \sum \alpha_i = 1,$$

then,

$$m_n(x) = A_{n-1}\, m_{n-1}(x) + \ldots + A_2\, m_1(x) + A_1 + A_0\, h(x) \qquad (3.31)$$

where

$$A_{n-1} = (S_{n-1} - n)$$

$$A_i = n(n-1)\ldots(n-\overline{i-1})(-1)^{n-i}\, S_i,\ i = 1, 2, \ldots, n-2.$$

$$A_0 = (-1)^n\, S_n.$$

where S_r is the product of $p_1^{-1}, \ldots, p_n^{-1}$ taken r at a time and

$$m_i = E[(X-x)(X-x-1)\ldots(X-x-\overline{i-1})|X>x].$$

Proof: Since the method of proof is identical to that of Theorem 3.4, except for the number of components, we give only an outline. We note that in the present case

$$R(x) = \sum_{i=1}^{n} \alpha_i q_i^x$$

$$h(x) = [R(x)]^{-1} \sum_{i=1}^{n} \alpha_i q_i^x p_i$$

$$m_n(x) = [R(x+1)]^{-1}\, n! \sum_{i=1}^{n} q_i^{x+n}\, p_i^{-n},\ n = 1, 2, 3, \ldots$$

Substituting these expressions in (3.31) and comparing coefficients of α_i, on both sides we get the values of A_i, i=0, 1, ..., n-1 stated in the Theorem.

3.2.3 Inference of parameters

Two basic problems that have to be settled while modelling life time data are (i) identification of the appropriate distribution and (ii) estimation of parameters to test model adequacy. The results obtained in the previous section can be employed to tackle both the problems.

(i) identification

From Theorem 3.1

$$r(x) = A + Bh(x+1)$$

where $A = (p_1^{-1} + p_2^{-1})$ and $B = -p_1^{-1}\, p_2^{-1}$, which show that the graph $(h(x+1), r(x))$ is a straight line with slope B and intercept A. Given a random sample of size n, say X_1, ..., X_n from the distribution (3.1), we can have the estimates of $h(x)$ and $r(x)$ viz.

$$\hat{h}(x) = \frac{n_x - n_{x+1}}{n_x} \qquad (3.32)$$

and

$$\hat{r}(x) = \frac{\sum_{x+1}^{\infty} n_t}{n_{x+1}} \qquad (3.33)$$

and plot the resulting estimates to produce the graph $(\hat{h}(x+1), \hat{r}(x))$. If the points fall approximately on a straight line then it could be concluded that mixture of geometric laws is a plausible model.

(ii) estimation of parameters.

Rough and ready estimates of A and B can be obtained from the graph $(\hat{h}(x+1), \hat{r}(x))$, by measuring the slope and the intercept of the line about which the points cluster. A more accurate method is to derive the least square estimates of A and B and then solve for p_1 and p_2 from the equations

$$\hat{A} = \hat{p}_1^{-1} + \hat{p}_2^{-1} \qquad \text{and} \qquad \hat{B} = -\hat{p}_1^{-1}\hat{p}_2^{-1}.$$

Since the mean of the geometric mixture is

$$\mathrm{E}(X) = pq_1 p_1^{-1} + (1-p)q_2 p_2^{-1}$$

replacing $\mathrm{E}(X)$ by the sample mean and the parameters by their estimates, the value of p can also be estimated.

Of the classical methods, one that is simple to use is the method of moments. From (3.2) the first three moments of (3.1) are

$$\mu_{(1)} = pq_1 p_1^{-1} + (1-p)q_2 p_2^{-1}$$

$$\mu_{(2)} = 2[p\, q_1^2\, p_1^{-2} + (1-p)\ q_2^2\, p_2^{-2}]$$

$$\mu_{(3)} = 6[p\, q_1^3\, p_1^{-3} + (1-p)\ q_2^3\, p_2^{-3}].$$

If the first three sample factorial moments are s_1, s_2 and s_3, we have the moment estimates as the solution of

$$pt + (1-p)s = a_1 \tag{3.34}$$

$$pt^2 + (1-p)s^2 = a_2 \tag{3.35}$$

$$pt^3 + (1-p)s^3 = a_3 \tag{3.36}$$

where $a_1 = s_1$, $a_2 = ½\, s_2$, $a_3 = s_3/6$, $t = \dfrac{q_1}{p_1}$, $s = \dfrac{q_2}{p_2}$.

From (3.34)

$$p(t-s) = a_1 - s$$

and from (3.35)

$$p(t^2-s^2) = a_2 - s^2$$

or

$$t+s = \frac{a_2 - s^2}{a_1 - s}. \tag{3.37}$$

Again equation (3.36) gives

$$p(t^3-s^3) = a_3 - s^3$$

or

$$t^2+s^2+ ts = \frac{a_3 - s^3}{a_1 - s}.$$

Eliminating t in the last equation by virtue of (3.37), we have a quadratic equation in s,

$$(a_2^2 - a_3a_1) + (a_3 - a_1a_2)s + (a_1^2 - a_2)\, s^2 = 0$$

which can be solved to give the values of s that provides the estimate of p_2 in the parameter space (0,1). Then t is obtained from (3.37), which will give $\hat{p}_1$ and finally p from (3.34) on substituting $\hat{p}_1$ and $\hat{p}_2$. Harris (1983) discusses other methods of estimation that makes use of sophisticated numerical techniques.

In order to verify the correctness of the methods of identification and estimation using the characterization theorem, we have simulated random samples of different sizes for chosen values of p, p_1 and p_2. The data corresponding to the samples are provided in the Appendix. The graph of $(\hat{h}(x+1), \hat{r}(x))$ for typical sample with $p = 0.7$, $p_1 = 0.35$ and $p_2 = 0.85$ producing 100 observations is shown in Figure 3.1. As expected, the sample points provided a reasonable straight line graph, confirming the model property. In order to study the behaviour of estimates random samples of sizes 100, 200 and 400 were generated for $p = 0.7$, p_1=0.35 and p_2=0.85.

Estimates derived by the method of least squares from the linear relationship between $r(x)$ and $h(x+1)$ are Shown in table 3.1.

Table 3.1: Estimation of the parameters of mixture geometric law

Sample No.	Sample Size	$\hat{p}_1$	$\hat{p}_2$	$\hat{p}$
1	100	0.42	0.82	0.70
2	100	0.40	0.89	0.60
3	200	0.31	0.76	0.60
4	200	0.39	0.88	0.73
5	400	0.32	0.83	0.66

Since the estimates $\hat{h}(x)$ and $\hat{r}(x)$ used in the least square procedure are ratios of random variables, the expressions for the standard errors of estimates become analytically intractable. That does not leave scope for a theoretical study of the efficiency or bias of the estimates. Hence we have simulated random samples of various sizes and computed the mean square errors. The values obtained were reasonable in all cases and in comparison the estimates of p showed more instability. For instance we quote below the mean square errors of estimates obtained when sample of size 200 with the above true parameter values were replicated ten times.

$$\text{MSE}(\hat{p}_1) = 0.00133 \ \text{MSE}(\hat{p}_2) = 0.009089, \ \text{MSE}(\hat{p}) = 0.01335.$$

The corresponding data is given in the Appendix.

Important aspects of the new estimation procedure suggested above can be summarised as (1) only pooled data is needed for the purpose. That is, observations need not be identifed as belonging to a particular component (2) no iteration as in the case of maximum likelihood nor solution of equations needed in the method of moments are required. The method provides quick estimates without much loss of efficiency.

3.2.4 Equilibrium distribution

From section 2.3, we note that the equilibrium distribution corresponding to a two-component mixture geometric distribution is found to be

$$g(x) = \mu^{-1}R(x+1),\ x = 0, 1, 2, \ldots,\ \mu = \mathrm{E}(X)$$

$$= \frac{p\sum_{x+1}^{\infty} q_1^t p_1 + (1-p)\sum_{x+1}^{\infty} q_2^t p_2}{p\left(\frac{q_1}{p_1}\right) + (1-p)\left(\frac{q_2}{p_2}\right)}$$

$$= \frac{pq_1^{x+1} + (1-p)q_2^{x+1}}{pp_1^{-1} + (1-p)p_2^{-1} - 1}. \tag{3.38}$$

Further the distribution (3.38) is characterized by the property

$$k(x) = [\,p_1^{-1} + p_2^{-1} - p_1^{-1}\, p_2^{-1} h(x+1)]^{-1} \tag{3.39}$$

which follows from equation (2.22).

Relation between failure rate and mean residual life for the geometric mixture

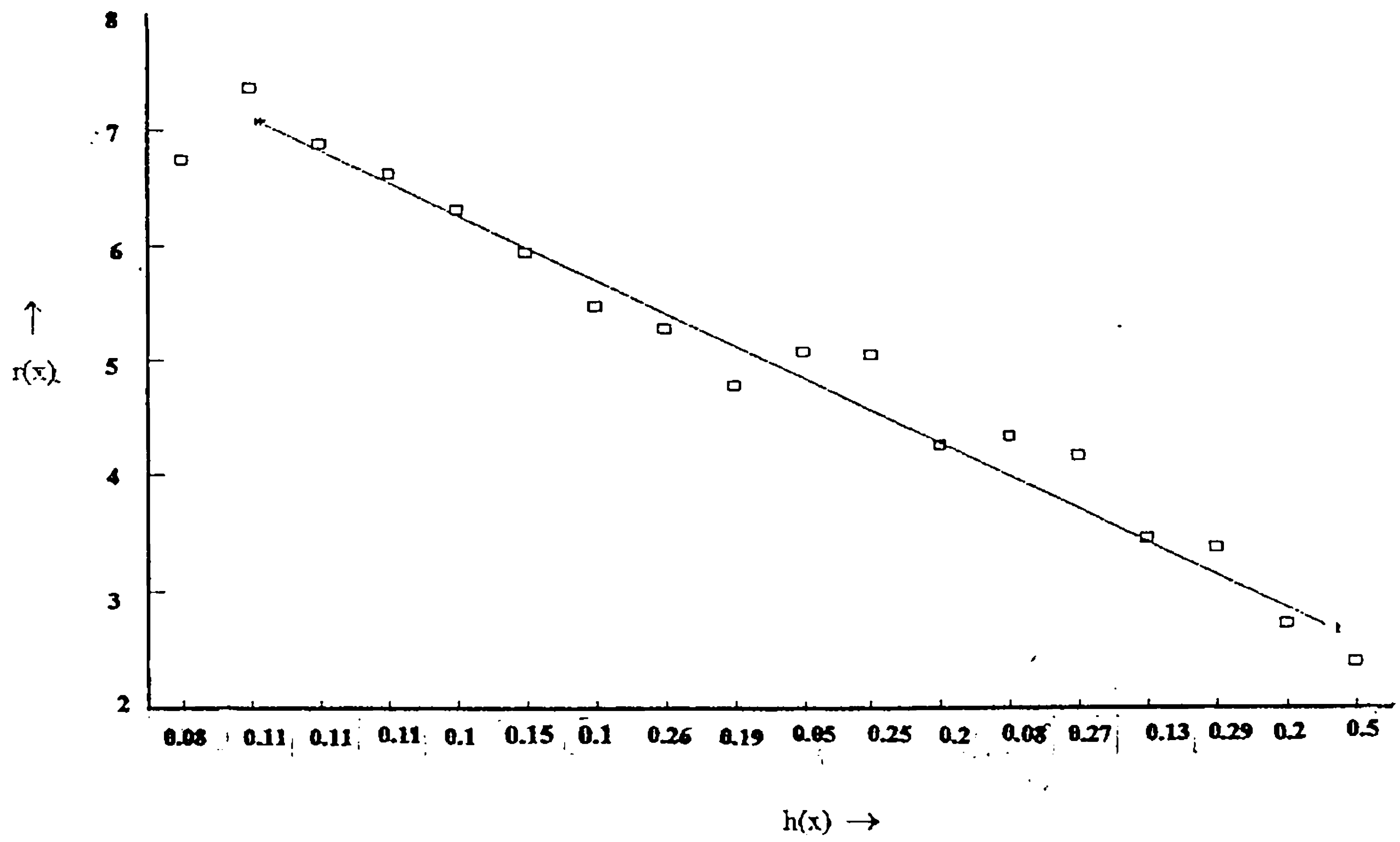

Figure 3.1

We now prove the following result concerning form of the equilibrium distribution

Theorem 3.5

A necessary and sufficient condition that the life time distribution is a geometric mixture is that its equilibrium distribution is also of the same form with the same component distributions.

Proof: when the parent distribution is a finite mixture of geometric distributions

$$f(x) = \sum_{i=1}^{n} \alpha_i q_i^x p_i\,, \quad \alpha_i > 0,\ \sum \alpha_i = 1$$

and hence

$$g(x) = \mu^{-1} \sum_{i=1}^{n} R_i(x+1) = \frac{\sum_{i=1}^{n} \alpha_i q_i^{x+1}}{\sum_{i=1}^{n} \alpha_i q_i\, p_i^{-1}}$$

$$= \sum_{i=1}^{n} \left(\frac{\alpha_i q_i\, p_i^{-1}}{\sum \alpha_i q_i\, p_i^{-1}} \right) p_i q_i^x$$

$$= \sum_{i=1}^{n} \beta_i q_i^x p_i\,,\ \beta_i > 0 \text{ and } \sum \beta_i = 1. \tag{3.40}$$

Conversely from the relationship (2.21), if $g(x)$ is a finite mixture of geometric laws with mixing weight β_i,

$$R(x+1) = \mu\sum_{i=1}^{n}\beta_i q_i^x p_i$$

and

$$f(x) = R(x) - R(x+1)$$

$$= \mu\sum_{i=1}^{n}\beta_i p_i q_i^{x-1} p_i$$

$$= \mu\sum_{i=1}^{n}\left(\frac{\beta_i p_i}{q_i}\right) q_i^x p_i$$

$$= \sum_{i=1}^{n}\left(\frac{\frac{\beta_i p_i}{q_i}}{\sum\frac{\beta_i p_i}{q_i}}\right) q_i^x p_i, \text{ since} \sum_{x=0}^{\infty} f(x) = 1$$

$$= \sum_{i=1}^{n}\alpha_i q_i^x p_i\,,\ \alpha_i > 0,\ \sum\alpha_i = 1.$$

Identically we note that the weights α_i and β_i of X and Y should be such that $\left[\sum\frac{\beta_i p_i}{q_i}\right]^{-1} = \sum\frac{\alpha_i q_i}{p_i} = \mu$. Further in view of the previous Theorem all the properties of geometric mixtures hold good for their equilibrium distribution too if we replace α_i by β_i. Since Theorem 3.1 through 3.3 are independent of the mixing weights, they continue to hold for the random variable Y, provided the failure rate and residual life of X are replaced by the corresponding quantities for Y.

3.3 Mixture of Waring distributions

The Waring distribution introduced by Irwin (1963) with probability mass function (1.6) was found to be of special interest as model for reliability in discrete time. Specifically, the distribution has linear MRL, reciprocal linear failure rate and DFR. Hitha (1990) uses the slope-ordinate ratio method to prove that a continuous approximation of the Waring distribution is the Lomax law and thereby justifies the similarity of the reliability characteristics of the two. The distribution has a longer tail than the geometric model and is J-shaped.

3.3.1 General properties

In this section we consider a two-component mixture of Waring distribution of the form

$$f(x) = p(a - b_1) \frac{(b_1)_x}{(a)_{x+1}} + (1-p)(a - b_2) \frac{(b_2)_x}{(a)_{x+1}} \tag{3.41}$$

$$x = 0, 1, 2, \ldots, a > b_1, b_2 > 0, b_1 > b_2.$$

with survival function

$$R(x) = p \frac{(b_1)_x}{(a)_x} + (1-p) \frac{(b_2)_x}{(a)_x}. \tag{3.42}$$

The expressions for the failure rate and MRL function of (3.41) are

$$h(x) = \frac{p(a+x)^{-1}(a-b_1)(b_1)_x + (1-p)(a+x)^{-1}(a-b_2)(b_2)_x}{p(b_1)_x + (1-p)(b_2)_x} \tag{3.43}$$

and

$$r(x) = \frac{p(a+x)(a-b_1-1)^{-1}(b_1)_{x+1} + (1-p)(a+x)(a-b_2-1)^{-1}(b_2)_{x+1}}{p(b_1)_{x+1} + (1-p)(b_2)_{x+1}}. \tag{3.44}$$

The r^{th} factorial moment can be written as

$$\mu_{(r)} = p\frac{(b_1)_r}{(a-b_1-1)^{(r)}} + (1\text{-}p)\ \frac{(b_2)_r}{(a-b_2-1)^{(r)}}. \tag{3.45}$$

In particular the mean is

$$\mu = \frac{pb_1}{a-b_1-1} + \frac{(1-p)b_2}{a-b_2-1}.$$

When $b_1 = 1 = b_2$ in (3.41)

$$\begin{aligned} f(x) &= p(a-1)\ \frac{x!}{(a)_{x+1}} + (1-p)(a-1)\ \frac{x!}{(a)_{x+1}} \\ &= (a-1)\ \frac{x!}{(a)_{x+1}} \end{aligned}$$

which is the Yule distribution. For the component densities in (3.41) we have the following

$$f_i(x) = p(a-b_i)\ \frac{(b_i)_x}{(a)_{x+1}}$$

$$h_i(x) = \frac{a-b_i}{a+x} \tag{3.46}$$

$$r_i(x) = \frac{a+x}{a-b_i-1}. \tag{3.47}$$

These derive from the expressions

$$R_i(x) = \frac{(b_i)_x}{(a)_x}$$

and

$$\sum_x^{\infty} \frac{(b_i)_t}{(a)_t} = \frac{(b_i)_x}{(a)_x} + \frac{(b_i)_{x+1}}{(a)_{x+1}} + \dots$$

$$= \frac{(b_i)_x}{(a)_x}\left[1+\frac{b_i+x}{a+x}+\frac{(b_i+x)(b_i+x+1)}{(a+x)(a+x+1)}+\dots\right]$$

$$= \frac{(b_i)_x}{(a)_x}\;\frac{a+x-1}{a-b_i-1}$$

on using the waring expansion

$$\frac{x}{x-a} = 1 + \frac{a}{x+1} + \frac{a(a+1)}{(x+1)(x+2)} + \dots$$

To evaluate $M_i(x)$ of $f_i(x)$, we note that

$$R_i(x+1)\; M_i(x) = 2\; \sum_{n=1}^{\infty} n \frac{(b_i)_{x+n+1}}{(a)_{x+n+1}}$$

$$= 2\; \left\{1.\frac{(b_i)_{x+2}}{(a)_{x+2}} + 2\frac{(b_i)_{x+3}}{(a)_{x+3}} + 3.\; \frac{(b_i)_{x+4}}{(a)_{x+4}} + \dots\right\}$$

$$= 2\; \left\{\sum_{j=2}^{\infty}\frac{(b_i)_{x+j}}{(a)_{x+j}} + \sum_{j=3}^{\infty}\frac{(b_i)_{x+j}}{(a)_{x+j}} + \dots\right\}$$

$$= 2\; \sum_{i=2}^{\infty}\sum_{j=i}^{\infty}\frac{(b_i)_{x+j}}{(a)_{x+j}}$$

$$= 2\ \sum_{j=2}^{\infty}\frac{(b_i)_{x+j}}{(a)_{x+j}}\left\{1+\frac{b_i+x+j}{a+x+j}+\frac{(b_i+x+j)(b_i+x+j+1)}{(a+x+j)(a+x+j+1)}+\ldots\right\}$$

$$= 2\ \sum_{j=2}^{\infty}\frac{(b_i)_{x+j}}{(a)_{x+j}}\ \frac{a+x+j-1}{a-b_i-1}$$

$$= 2(a-b_i-1)^{-1}\sum_{j=2}^{\infty}\frac{(b_i)_{x+j}}{(a)_{x+j}}$$

$$= 2\ \frac{(b_i)_{x+2}}{(a)_{x+2}}\ \frac{a+x}{(a-b_i-1)(a-b_i-2)}.$$

Hence

$$M_i(x) = \frac{2(b_i+x+1)(a+x)}{(a-b_i-1)(a-b_i-2)},\quad a>b+2. \tag{3.48}$$

Entering the expression for $r_i(x)$ and $h_i(x)$ in equation (2.11),

$$r(x)\left[\frac{a-b_1}{a+x+1}-\frac{a-b_2}{a+x+1}\right] - h(x+1)\left[\frac{a+x}{a-b_1-1}-\frac{a+x}{a-b_2-1}\right]$$

$$=\frac{a-b_1}{a+x+1}\ \frac{a+x}{a-b_2-1}-\frac{a-b_2}{a+x+1}\ \frac{a+x}{a-b_1-1}$$

which simplifies to

$$r(x) = \frac{(2a-b_1-b_2-1)}{(a-b_1-1)}\ \frac{(a+x)}{(a-b_2-1)} - \frac{(a+x)(a+x+1)}{(a-b_1-1)(a-b_2-1)}h(x+1). \tag{3.49}$$

Again from (2.15) we arrive at the identity connecting $M(x)$ and $h(x)$ as

$$\left[\frac{a-b_1}{a+x+1} - \frac{a-b_2}{a+x+1}\right] M(x)$$

$$= h(x+1) \left[\frac{2(a+x)(b_1+x+1)}{(a-b_1-1)(a-b_1-2)} - \frac{2(a+x)(b_2+x+1)}{(a-b_2-1)(a-b_2-2)}\right]$$

$$+ \left[\frac{2(a+x)(b_2+x+1)}{(a-b_2-1)(a-b_2-2)} \frac{a-b_1}{a+x+1} - \frac{a-b_2}{a+x+1} \frac{2(a+x)(b_1+x+1)}{(a-b_1-1)(a-b_1-2)}\right]$$

or

$$M(x) = \frac{2(a+x)}{b_2-b_1} [(a\text{-}b_1\text{-}1)(a\text{-}b_1\text{-}2)\ (a\text{-}b_2\text{-}1)\ (a\text{-}b_2\text{-}2)]^{-1}$$

$$[\{(a\text{-}b_2\text{-}1)(a\text{-}b_2\text{-}2)(b_1+x+1)\text{-}(a\text{-}b_1\text{-}1)(a\text{-}b_1\text{-}2)(b_2+x+1)\}(a+x+1)h(x+1)$$

$$(a\text{-}b_1)(a\text{-}b_1\text{-}1)(a\text{-}b_1\text{-}2)(b_2+x+1)\text{-}\ (a\text{-}b_2)(a\text{-}b_2\text{-}1)(a\text{-}b_2\text{-}2)(b_1+x+1)].$$

To infer the nature of the failure rate we need the quantities

$$R(x)\{[h_1(x+1)\text{-}\ h_1(x)]\ p\ R_1(x) + [h_2(x+1)\text{-}\ h_2(x)](1\text{-}\ p)\ R_2(x)\}$$

$$= R(x)\left\{\left[\frac{a-b_1}{a+x+1} - \frac{a-b_1}{a+x}\right] p\ \frac{(b_1)_x}{(a)_x} + \left[\frac{a-b_2}{a+x+1} - \frac{a-b_2}{a+x}\right] (1\text{-}\ p)\ \frac{(b_2)_x}{(a)_x}\right\}$$

$$= \frac{R(x)}{(a+x)(a+x+1)} \left[p \frac{(b_1)_x}{(a)_x} (b_1\text{-}a) + (1\text{-}\ p)\ \frac{(b_2)_x}{(a)_x} (b_2-a)\right] \qquad (3.50)$$

and

$$p(1\text{-}p)[h_1(x)\text{-}\ h_2(x)]^2 R_1(x) R_2(x) = \frac{p(1-p)(b_2-b_1)^2}{(a+x)^2} \frac{(b_1)_x}{(a)_x} \frac{(b_2)_x}{(a)_x}. \qquad (3.51)$$

Since $a>b_1$, (3.50) is always negative and (3.51) is always positive. Hence by Theorem 2.1 we conclude that the mixture of Waring distributions is always DFR. Compare this with the fact that the component distributions are also DFR.

Further

$$r_1(x+1) - r_1(x) = \frac{a+x+1}{a-b_1-1} - \frac{a+x}{a-b_1-1} = \frac{1}{a-b_1-1} > 0$$

since $a > b_1+1$ for the MRLF to exist. Similarly

$$r_2(x+1) - r_2(x) = \frac{1}{a-b_2-1} > 0.$$

Thus

$R(x+1)$ $\{[r_1(x+1) - r_1(x)]p\, R_1(x+1) - [r_2(x+1) - r_2(x)](1-p)\, R_2(x+1)\}$

$$= R(x+1)\left[\frac{p}{(a-b_1-1)}\frac{(b_1)_{x+1}}{(a)_{x+1}} + \frac{(1-p)}{(a-b_2-1)}\frac{(b_2)_{x+1}}{(a)_{x+1}}\right]. \qquad (3.52)$$

and

$p(1-p)[r_1(x)-r_2(x)]^2 R_1(x+1)R_2(x+1)$

$$= \frac{p(1-p)(b_2-b_1)^2(a+x)^2}{(a-b_1-1)(a-b_2-1)}\frac{(b_1)_{x+1}}{(a)_{x+1}}\frac{(b_2)_{x+1}}{(a)_{x+1}} \qquad (3.53)$$

The difference of (3.52) and (3.53) is calculated as

$$p^2(b_1)^2_{x+1}\, t^{-1} + p(1-p)(b_1)_{x+1}\,(b_2)_{x+1}\left\{t^{-1}+s^{-1}-\frac{(t-s)^2}{ts}\right\} + (1-p)^2(b_2)^2_{x+1}\, s^{-1}.$$

For the last expression to be non-negative we should have

$$\left\{\frac{1-s}{t}+\frac{1-t}{s}+2\right\}^2 \le \frac{4}{ts} \qquad (3.54)$$

with $t = (a-b_1-1)$ and $s = (a-b_2-1)$. Thus by Theorem 2.2 the condition that the mixture is IMRL is given by (3.54). If the inequality is reversed we have the condition for the distribution to be DMRL.

3.3.2 Characterizations

As in the case of the geometric distribution, we will make use of the relationship between MRL and failure rate to characterize the Waring mixture. The relevant result is stated in the following.

Theorem 3.6

Let X be a non-negative random variable in the support of I^+. Then the distribution of X is a mixture of Waring distributions with pmf (3.41) if and only if

$$r(x) = \frac{(2a-b_1-b_2-1)}{(a-b_1-1)}\frac{(a+x)}{(a-b_2-1)} - \frac{(a+x)(a+x+1)}{(a-b_1-1)(a-b_2-1)}h(x+1)$$

for all $x = 0, 1, 2, \ldots$

Proof: Equation (3.49) is already established in the previous section. It remains to prove the converse. Assuming (3.49), we can write as,

$$(a-b_1-1)\ (a-b_2-1)\ \sum_{x+1}^{\infty} R(t) = (a+x)(2a-b_1-b_2-1)\ R(x+1)$$

$$-\ (a+x+1)(a+x)\ (R(x+1)-R(x+2)). \quad (3.55)$$

Changing x to x-1

$$(a-b_1-1)\ (a-b_2-1)\ \sum_{x}^{\infty} R(t) = (a+x-1)(2a-b_1-b_2-1)\ R(x)$$

$$-\ (a+x-1)(a+x)(R(x)-R(x+1)). \quad (3.56)$$

Subtracting (3.56) from (3.55) and rearranging the terms,

$$(a+x)(a+x+1)R(x+2)-(a+x)(1+b_1+b_2+2x)R(x+1)$$

$$+(b_1+x)(b_2+x)R(x) = 0. \quad (3.57)$$

which is a second order difference equation in $R(x)$. Equation (3.57) can be written as,

$$\frac{(b_1)_{x+1}}{(b_1)_x}\frac{(b_2)_{x+1}}{(b_2)_x}R(x)-\frac{(a)_{x+1}}{(a)_x}\left[\frac{(b_1)_{x+1}}{(b_1)_x}+\frac{(b_2)_{x+2}}{(b_2)_{x+1}}\right]R(x+1)$$

$$+\frac{(a)_{x+1}}{(a)_x}\frac{(a)_{x+2}}{(a)_{x+1}}R(x+2)=0$$

or

$$\frac{(b_1)_{x+1}}{(b_1)_x}\left[\frac{(b_2)_{x+1}}{(b_2)_x}\ R(x)\ -\ \frac{(a)_{x+1}}{(a)_x}R(x+1)\right]$$

$$=\ \frac{(a)_{x+1}}{(a)_x}\left[\frac{(b_2)_{x+2}}{(b_2)_{x+1}}\ R(x+1)\ -\ \frac{(a)_{x+2}}{(a)_{x+1}}R(x+2)\right]$$

which is same as

$$\frac{(a)_x}{(b_1)_x}\left[\frac{(b_2)_{x+1}}{(b_2)_x}R(x) - \frac{(a)_{x+1}}{(a)_x}R(x+1)\right]$$

$$= \frac{(a)_{x+1}}{(b_1)_{x+1}}\left[\frac{(b_2)_{x+2}}{(b_2)_{x+1}}R(x+1) - \frac{(a)_{x+2}}{(a)_{x+1}}R(x+2)\right] \qquad (3.58)$$

Denoting the left side of equation (3.58) by v_x we can write (3.58) as a first order equation of the form

$$v_x = v_{x+1}$$

which has the unique solution $v_x = C$, where C is some constant to be evaluated from the boundary conditions. That is

$$\frac{(a)_x}{(b_1)_x}\left[\frac{(b_2)_{x+1}}{(b_2)_x}R(x) - \frac{(a)_{x+1}}{(a)_x}R(x+1)\right] = C,$$

Now setting $C = \alpha\,(b_1 - b_2)$,

$$R(x+1) = \frac{b_2+x}{a+x}R(x) + \alpha\,(b_1-b_2)\,\frac{(b_1)_x}{(a)_{x+1}}\ ,\ x = 0,1,2,\ldots \qquad (3.59)$$

Iterating (3.59) for $x=0,1,2,\ldots$ leads to

$$R(x) = \alpha\,\frac{(b_1)_x}{(a)_x} + (1-\alpha)\,\frac{(b_2)_x}{(a)_x}.$$

This completes the proof of the Theorem.

Corollary 3.3

X follows the Waring distribution with mass function,

$$f(x) = (a\text{-}b)\,\frac{(b)_x}{(a)_{x+1}}, \qquad x=0,1,2,\ldots,\ a>b,\ a,\ b>0.$$

if and only if

$$r(x) = \frac{(2a-2b-1)}{(a-b-1)^2}(a+x) - \frac{(a+x)(a+x+1)}{(a-b-1)^2}h(x+1). \qquad (3.60)$$

The corollary is proved by setting $b_1 = b_2$ in (3.49).

Corollary 3.4

X follows the Yule distribution with mass function,

$$f(x) = \frac{(a-1)x!}{(a)_{x+1}}, \qquad x=0,1,2,\ldots,\ a>0.$$

if and only if for all x,

$$r(x) = \frac{(2a-3)}{(a-2)^2}(a+x) - \frac{(a+x)(a+x+1)}{(a-2)^2}h(x+1). \qquad (3.61)$$

3.3.3 Equilibrium distribution

The equilibrium distribution corresponding to (3.41) is specified by the probability mass function

$$g(x) = \mu^{-1}[pR_1(x+1) + (1-p)\,R_2(x+1)],\ \mu = \mathrm{E}(X)$$

$$= \frac{p\{(b_1)_{x+1}/(a)_{x+1}\} + (1-p)(b_2)_{x+1}/(a)_{x+1}\}}{p[b_1/(a-b_1-1)] + (1-p)[b_2/(a-b_2-1)]}$$

$$= \left[\frac{pb_1}{a-b_1-1}\frac{(b_1+1)_x}{(a)_{x+1}} + \frac{(1-p)b_2}{a-b_2-1}\frac{(b_2+1)_x}{(a)_{x+1}}\right]\left[\frac{pb_1}{a-b_1-1} + \frac{(1-p)b_2}{a-b_2-1}\right]^{-1}$$

$$= \beta\frac{(b_1+1)_x}{(a)_{x+1}} + (1-\beta)\frac{(b_2+1)_x}{(a)_{x+1}},\ \beta = \frac{pb_1/(a-b_1-1)}{\left(\dfrac{pb_1}{a-b_1-1}\right)+\left(\dfrac{pb_2}{a-b_2-1}\right)}. \quad (3.62)$$

This leads us to the following theorem.

Theorem 3.7

A necessary and sufficient condition that the life time distribution is an n-component mixture of Waring distribution with parameters (a,b_i) and mixing constants α_i is that its equilibrium distribution is also of the same form with components as Waring with parameters (a,b_i+1) and mixing constants

$$\beta_i = \frac{\alpha_i b_i/(a-b_i-1)}{\sum \alpha_i b_i/(a-b_i-1)}.$$

From the deliberations it follows that the two-component mixture (3.62) is characterized by the property

$$k(x) = \left[\frac{2a-b_1-b_2-1}{(a-b_1-1)(a-b_2-1)}(a+x) - \frac{(a+x)(a+x+1)}{(a-b_1-1)(a-b_2-1)}h(x+1)\right]^{-1}$$

$$g(x) = \mu^{-1}[pR_1(x+1) + (1-p)\, R_2(x+1)],\ \mu = \mathrm{E}(X)$$

$$= \frac{p\{(b_1)_{x+1}/(a)_{x+1}\} + (1-p)(b_2)_{x+1}/(a)_{x+1}\}}{p[b_1/(a-b_1-1)] + (1-p)[b_2/(a-b_2-1)]}$$

$$= \left[\frac{pb_1}{a-b_1-1}\frac{(b_1+1)_x}{(a)_{x+1}} + \frac{(1-p)b_2}{a-b_2-1}\frac{(b_2+1)_x}{(a)_{x+1}}\right]\left[\frac{pb_1}{a-b_1-1} + \frac{(1-p)b_2}{a-b_2-1}\right]^{-1}$$

$$= \beta\frac{(b_1+1)_x}{(a)_{x+1}} + (1-\beta)\frac{(b_2+1)_x}{(a)_{x+1}},\ \beta = \frac{pb_1/(a-b_1-1)}{\left(\frac{pb_1}{a-b_1-1}\right) + \left(\frac{pb_2}{a-b_2-1}\right)}. \quad (3.62)$$

This leads us to the following theorem.

Theorem 3.7

A necessary and sufficient condition that the life time distribution is an n-component mixture of Waring distribution with parameters (a,b_i) and mixing constants α_i is that its equilibrium distribution is also of the same form with components as Waring with parameters (a,b_i+1) and mixing constants

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From the deliberations it follows that the two-component mixture (3.62) is characterized by the property

$$k(x) = \left[\frac{2a-b_1-b_2-1}{(a-b_1-1)(a-b_2-1)}(a+x) - \frac{(a+x)(a+x+1)}{(a-b_1-1)(a-b_2-1)}h(x+1)\right]^{-1}$$

It could be taken that in the above discussions it was assumed that the same parameter a was used in the component distributions. We can assume the mixture to be in a more general form with

$$f(x) = p\, \frac{(a_1 - b_1)(b_1)_x}{(a_1)_{x+1}} + (1-p)\, \frac{(a_2 - b_2)(b_2)_x}{(a_2)_{x+1}}.$$

Though this model may impart more flexibility with the addition of a new parameter, the resulting computation becomes too much involved and the estimation problem becomes very much complicated. Even for the four-parameter model discussed earlier application of the methods of moments, the simplest available for most mixture models, results in a system of non-linear equations which can be solved only on a computer numerically. To get an idea of the extend of complexity in the various results, we quote below the relationship connecting the MRL and failure rate in the five parameter mixture.

$$r(x) = \frac{(a_1 - b_1)(a_1 - b_1 - 1)(a_2 + x) - (a_2 - b_2)(a_2 - b_2 - 1)(a_1 + x)}{(a_1 - b_1 - 1)(a_2 - b_2 - 1)[(a_1 - b_1)(a_2 + x + 1) - (a_2 - b_2)(a_1 + x + 1)]}$$

$$- \frac{[(a_1 - b_1 - 1)(a_2 + x) - (a_2 - b_2 - 1)(a_1 + x)]}{(a_1 - b_1)(a_2 + x + 1) - (a_2 - b_2)(a_1 + x + 1)} (a_1+x+1)\ (a_2+x+1)\ h(x+1).$$

The particular case $b_1 = b_2$ does not hamper the utility of the model as none of the characteristics of the general model is entirely dependent on b_1 or b_2; in fact, the means, variances etc of the distribution (3.41) are different.

3.4 Mixture of negative hypergeometric distributions

The negative hypergeometric law is one of the oldest probability distributions derived as early as 1785. It has probability mass function of the form

$$P(X=x) = \binom{-\alpha}{x}\binom{-\beta}{n-x} \Big/ \binom{-\alpha-\beta}{n}, \; x = 0, 1, 2, \ldots n. \qquad (3.63)$$

For a historical perspective of the distribution and the different modes of its derivation we refer to Johnson, Kotz and Kemp (1992). Along with the geometric and Waring distribution, a particular case of (3.63) when $\alpha = 1$ form a class that is of special importance in reliability modelling. This is given particular emphasis in the review of literature made in Chapter I. In the present section we deal with lifetime X having two-component mixtures with mass function of the form

$$f(x) = p\,\frac{\binom{-1}{x}\binom{-\beta_1}{n_1-x}}{\binom{-1-\beta_1}{n_1}} + (1-p)\,\frac{\binom{-1}{x}\binom{-\beta_2}{n_2-x}}{\binom{-1-\beta_2}{n_2}} \qquad (3.64)$$

where $n_1 \geq n_2$ and $x=0, 1, 2, \ldots, n_1$, $\beta_i > 0$. Notice that unlike geometric and Waring mixtures the support of the components need not be the same. In the case of (3.64) the support of the first component is

(0, 1, 2, ..., n_1) and that of the second is (0, 1, 2, ..., n_2), where $n_1 \geq n_2$ is assumed without of loss of generality.

3.4.1 General properties

The survival function of X is

$$R(x) = p \sum_{x}^{n_1} \frac{\binom{\beta_1 + n_1 - t - 1}{n_1 - t}}{\binom{\beta_1 + n_1}{n_1}} + (1\text{-}p) \sum_{x}^{n_2} \frac{\binom{\beta_2 + n_2 - t - 1}{n_2 - t}}{\binom{\beta_2 + n_2}{n_2}} \tag{3.65}$$

on using

$$\binom{-u}{v} = (-1)^v \binom{u + v - 1}{v} \tag{3.66}$$

Equation (3.65) is the same as

$$R(x) = p \sum_{0}^{n_1 - x} \frac{\binom{\beta_1 + t - 1}{t}}{\binom{\beta_1 + n_1}{n_1}} + (1\text{-}p) \sum_{0}^{n_2 - x} \frac{\binom{\beta_2 + t - 1}{t}}{\binom{\beta_2 + n_2}{n_2}}$$

$$= p \frac{\binom{\beta_1 + n_1 - x}{n_1 - x}}{\binom{\beta_1 + n_1}{n_1}} + (1\text{-}p) \frac{\binom{\beta_2 + n_2 - x}{n_2 - x}}{\binom{\beta_2 + n_2}{n_2}}.$$

by virtue of the combinatorial identity

$$\sum_{x=0}^{n} \binom{u + n - x - 1}{n - x} = \binom{u + n}{n}.$$

Since $f(x)$ can also be written as

$$f(x)=p\,\frac{(-1)^x(-1)^{n_1-x}}{(-1)^{n_1}}\,\frac{\binom{\beta_1+n_1-x-1}{n_1-x}}{\binom{\beta_1+n_1}{n_1}}+(1\text{-}p)\,\frac{(-1)^x(-1)^{n_2-x}}{(-1)^{n_2}}\,\frac{\binom{\beta_2+n_2-x-1}{n_2-x}}{\binom{\beta_2+n_2}{n_2}}$$

$$=p\,\frac{\binom{\beta_1+n_1-x-1}{n_1-x}}{\binom{\beta_1+n_1}{n_1}}+(1\text{-}p)\,\frac{\binom{\beta_2+n_2-x-1}{n_2-x}}{\binom{\beta_2+n_2}{n_2}} \tag{3.67}$$

Hence

$$h(x)=\frac{p\dfrac{\binom{\beta_1+n_1-x-1}{n_1-x}}{\binom{\beta_1+n_1}{n_1}}+(1-p)\dfrac{\binom{\beta_2+n_2-x-1}{n_2-x}}{\binom{\beta_2+n_2}{n_2}}}{p\dfrac{\binom{\beta_1+n_1-x}{n_1-x}}{\binom{\beta_1+n_1}{n_1}}+(1-p)\dfrac{\binom{\beta_2+n_2-x}{n_2-x}}{\binom{\beta_2+n_2}{n_2}}} \tag{3.68}$$

which, though in closed form, is not of a nice functional form for a direct characterization. The MRL is

$$r(x)=\frac{p\dfrac{\binom{\beta_1+n_1-x}{n_1-x-1}}{\binom{\beta_1+n_1}{n_1}}+(1-p)\dfrac{\binom{\beta_2+n_2-x}{n_2-x-1}}{\binom{\beta_2+n_2}{n_2}}}{p\dfrac{\binom{\beta_1+n_1-x-1}{n_1-x-1}}{\binom{\beta_1+n_1}{n_1}}+(1-p)\dfrac{\binom{\beta_2+n_2-x-1}{n_2-x-1}}{\binom{\beta_2+n_2}{n_2}}} \tag{3.69}$$

For the components, the expressions for the failure rate and MRL are

$$h_i(x) = \beta_i/(\beta_i + n_i - x)$$

$$r_i(x) = (\beta_i + n_i - x)/(\beta_i + 1)$$

Substituting these values in (2.11) we find

$$r(x)\left[\frac{\beta_1}{\beta_1+n_1-x-1}-\frac{\beta_2}{\beta_2+n_2-x-1}\right] - h(x+1)\left[\frac{\beta_1+n_1-x}{\beta_1+1}-\frac{\beta_2+n_2-x}{\beta_2+1}\right]$$

$$= \frac{\beta_1}{\beta_1+n_1-x-1}\,\frac{\beta_2+n_2-x}{\beta_2+1} - \frac{\beta_2}{\beta_2+n_2-x-1}\,\frac{\beta_1+n_1-x}{\beta_1+1} \qquad (3.70)$$

which simplifies to

$r(x) =$

$$\frac{\beta_1(\beta_1+1)(\beta_2+n_2-x)(\beta_2+n_2-x-1)-\beta_2(\beta_2+1)(\beta_1+n_1-x)(\beta_1+n_1-x-1)}{(\beta_1+1)(\beta_2+1)[\beta_1(n_2-x-1)-\beta_2(n_1-x-1)]} -$$

$$\frac{[(\beta_2+1)(\beta_1+n_1-x)-(\beta_1+1)(\beta_2+n_2-x)](\beta_1+n_1-x-1)(\beta_2+n_2-x-1)}{\beta_1(n_2-x-1)-\beta_2(n_1-x-1)}$$

$$h(x+1). \qquad (3.71)$$

For reasons specified at the end of Section 3.3.3, in the present case we take $\beta_1 = \beta_2$ to achieve a significant reduction in the above expression. In that case

$$r(x)=\frac{2\beta+n_1+n_2-2x-1}{\beta+1}-\frac{h(x+1)}{\beta(\beta+1)}(\beta + n_1-x-1)(\beta + n_2-x-1). \qquad (3.72)$$

On the other hand if we assume $n_1 = n_2$, the support of the two components become identical and the identity (3.71) takes the simple form

$$r(x) = \frac{(1+\beta_1+\beta_2)(n-x)+2\beta_1\beta_2}{\beta_1\beta_2(1+\beta_1)(1+\beta_2)} - \frac{(\beta_1+n-x-1)(\beta_2+n-x-1)}{(1+\beta_1)(1+\beta_2)}\, h(x+1). \tag{3.73}$$

The nature of the failure rate derives from the fact that the mixture of two DFR distributions is DFR (Barlow and Proschan (1976)).

The second factorial moment of residual life of $f_i(x)$ with parameters (β, n_i),

$$R_i(x+1)\, M_i(x) = 2\sum t R(x+t+1)$$

$$\binom{\beta+n_i-x-1}{n_i-x-1} M_i(x) = 2\sum t \binom{\beta+n_i-x-t-1}{n_i-x-t-1}$$

$$= 2\sum_{j=2}^{n_i}\sum_{y=0}^{n_i-j}\binom{\beta+m-y-x}{m-y-x},\ m = n_i - j$$

$$= 2\sum_{j=2}^{n_i}\sum_{y=0}^{n_i-j}\binom{\beta+1+m-y-x-1}{m-y-x}$$

$$= 2\sum_{j=2}^{n_i}\sum_{y=0}^{n_i-j}\binom{\beta+1+n_i-j-y-x-1}{n_i-j-y-x}$$

$$= 2\sum_{j=2}^{n_i}\binom{\beta+1+n_i-j-x}{n_i-j-x}$$

$$= 2\sum_{y=0}^{n_i-2}\binom{\beta+1+n_i-y-2-x}{n_i-y-2-x}$$

$$= 2\binom{\beta+n_i-x}{n_i-x-2}.$$

Thus

$$M_i(x) = 2\,\frac{\binom{\beta+n_i-x}{n_i-x-2}}{\binom{\beta+n_i-x-1}{n_i-x-1}}$$

$$= 2\,\frac{(\beta+n_i-x)(n_i-x-1)}{(\beta+1)(\beta+2)},\ i=1,\ 2.$$

Hence from equation (2.15), we have

$$M(x)\left(\frac{\beta}{\beta+n_1-x-1}-\frac{\beta}{\beta+n_2-x-1}\right)$$

$$= 2h(x+1)\left[\frac{(\beta+n_1-x)(n_1-x-1)}{(\beta+1)(\beta+2)} - \frac{(\beta+n_2-x)(n_2-x-1)}{(\beta+1)(\beta+2)}\right]$$

$$+\ 2\frac{\beta}{\beta+n_1-x-1}\,\frac{(\beta+n_2-x)(n_2-x-1)}{(\beta+1)(\beta+2)}$$

$$-2\frac{\beta}{\beta+n_2-x-1}\,\frac{(\beta+n_1-x)(n_1-x-1)}{(\beta+1)(\beta+2)}.$$

This simplifies to

$$M(x) = \frac{2}{(n_2-n_1)\beta(\beta+1)(\beta+2)}\{(\beta+n_1-x-1)\ (\beta+n_2-x-1)$$

$$[(\beta+n_1-x)\ (n_1-x-1)-\ (\beta+n_2-x)\ (n_2-x-1)]\ h(x+1)$$

$$+\ [(\beta+n_2-x)\ (\beta+n_2-x-1)\ (n_2-x-1)$$

$$-\ (\beta+n_1-x)\ (\beta+n_1-x-1)\ (n_1-x-1)]\}$$

$$= \frac{2}{\beta(\beta+1)(\beta+2)}[\beta(\beta\text{-}3) + (2\beta - 1)\ (n_1+n_2-2x) + (n_2-x)^2$$

$$+(n_1-x)^2 - (n_1-x)(n_2-x)] - (\beta+n_1+n_2-2x-1)\ h(x+1). \quad (3.74)$$

The negative hypergeometric law with parameter (β, n) noted in (3.63) has survival function

$$\mathrm{P}(X \geq x) = \binom{\beta+n-x}{n-x} / \binom{\beta+n}{n}$$

and mean value

$$\mathrm{E}(X) = r(-1) = \frac{\beta+n+1}{\beta+1}.$$

Thus the equilibrium distribution is specified by the probability mass function

$$g(x) = \mu^{-1}\ [pR_1(x+1) + (1-p)R_2(x+1)]$$

$$= \mu^{-1} \left[\frac{p\binom{\beta_1+n_1-x-1}{n_1-x-1}}{\binom{\beta_1+n_1}{n_1}} + \frac{(1-p)\binom{\beta_2+n_2-x-1}{n_2-x-1}}{\binom{\beta_2+n_2}{n_2}} \right]. \quad (3.75)$$

where

$$\mu = p\,\frac{\beta_1+n_1+1}{\beta_1+1} + (1-p)\,\frac{\beta_2+n_2+1}{\beta_2+1}.$$

We can write (3.75) as

$$g(x) = p\,\frac{\beta_1+n_1+1}{\beta_1+1}\left[\binom{\beta_1+1+n_1-x}{n_1-x}\Big/\binom{\beta_1+n_1+1}{n_1}\right]$$

$$+\ (1\text{-}p)\,\frac{\beta_2+n_2+1}{\beta_2+1}\left[\binom{\beta_2+1+n_2-x}{n_2-x}\Big/\binom{\beta_2+n_2+1}{n_2}\right]$$

The above representation shows that $g(x)$ is a mixture of two negative hypergeometric mixtures with parameters (β_i+1, n_i). Further, the converse is also true since the calculations made above holds good for any number of components, we have the following theorem.

Theorem 3.8

A random variable in the support of Γ^+ is a finite mixture of negative hypergeometric distributions with parameters (β_i, n_i)

$$f(x) = \sum_{i=1}^{m}\alpha_i\left[\binom{\beta_i+n_i-x-1}{n_i-x}\Big/\binom{\beta_i+n_i}{n_i}\right],\ i=1, 2, \ldots, m,\ \sum\alpha_i=1$$

iff its equilibrium distribution is of the same form with parameters (β_i+1, n_i) and mixing constants

$$\beta_i = \frac{\alpha_i(\beta_i+n_i+1)}{(\beta_i+1)\mu}$$

where

$$\mu = \sum\alpha_i\,\frac{\beta_i+n_i+1}{\beta_i+1}.$$

In Chapter II we have presented some general results involving identities connecting various reliability concepts in the case of mixture distributions. This was followed in the present chapter by specialising the results for mixtures of discrete distributions that have closed forms for their failure rate, mean residual life etc. It may be noted that similar results are possible for any discrete mixture, but most of them do not admit convenient forms that are analytically tractable as in the above cases, to be of theoretical consideration for a characterization or inference. A special feature of the distributions considered here is that the identities we have proposed are independent of the failure rates or MRL's of the component populations from where they come. Part of the results in this Chapter has been published in Nair, Geetha and Priya (1999).

CHAPTER IV

PARTIAL MOMENTS AND THEIR PROPERTIES

4.1 Introduction

Many of the characteristics of probability distributions introduced earlier to the development of reliability theory as an independent discipline, came to be associated with life time later with meaningful interpretations of immense practical utility. The truncated mean was viewed as mean residual life and the reciprocal of the Mill's ratio as the failure rate. In the present chapter we look at the concept of partial moments associated with probability distributions and establish that like the earlier concepts, partial moments are also useful in reliability modelling. The review of literature initiated in Chapter I on partial moments reveals that an intensive study of this type of moments have not been carried out both from the theoretical and practical points of view. The absence of an indepth study of the topic in the discrete set up is more evident, although partial moments of discrete random variables have been touched upon in Gupta and Gupta (1983). The close analogy the partial moments have with truncated moments has motivated a deeper look at the former and to explore the possibilities of it being used in the context of reliability modelling. Accordingly the aim of the present chapter is two-fold. First we define and study the properties of partial moments in the case of integer valued random variables. Secondly we examine their role in modelling lifetime data, through characterization theorems and properties that describe the aging process. Some results in this chapter have appeared in Nair, Priya and Geetha (2000).

4.2 Definition and properties

Let X be a discrete random variable in the support of I^+ with probability mass function $f(x)$ and survival function $R(x)$ and finite moments of order r, $r = 1, 2, \dots$. Then the r^{th} descending factorial moment of X about a positive integer t is defined by

$$\alpha_{(r)}(t) = E[(X-t)^+]^{(r)},\ r = 1, 2, \dots, X>t+r-1.$$

$$= \sum_{t+r}^{\infty}(x-t)^{(r)}f(x) \tag{4.1}$$

where,

$$(X-t)^+ = \begin{cases} X-t & X>t \\ 0 & X \le t \end{cases}$$

and (4.2)

$$X^{(r)} = X(X-1)\dots(X-r+1).$$

The r^{th} raw partial moment of X at a point t is

$$\alpha_r(t) = \mathrm{E}[(X\text{-}t)^+]^r$$

which is more primary than factorial moments in describing the distribution. However, if $S(r, k)$ is the Stirling number of the second kind

$$[(X\text{-}t)^+]^r = \sum_{k=0}^{r}[(X-t)^+]^{(r)}\ S(r, k) \tag{4.3}$$

Equation (4.3) implies that α_r can be computed from $\alpha_{(r)}$ and vice-versa and hence our discussion is mostly confined to $\alpha_{(r)}$ only in view of the ease in computation of the latter for many distributions.

From the definition, it follows that

$$\alpha_1(t) = \alpha_{(1)}(t) = \sum_{t+1}^{\infty} R(x) \tag{4.4}$$

$$R(x+1) = \alpha_1(x) - \alpha_1(x+1) \tag{4.5}$$

$$f(x+1) = \alpha_1(x) - 2\alpha_1(x+1) + \alpha_1(x+2) \tag{4.6}$$

The expressions (4.4) through (4.6) enable the evaluation of the probability mass function and survival function in terms of the partial means. Further it is seen that the probability mass function is completely determined from the sequence of partial means $<\alpha_1(x)>$, $x = 0, 1, 2, \ldots$. We now establish some properties of partial moments.

Theorem 4.1

$$\alpha_{(r)}(t) = r \sum_{t+r}^{\infty}(x-t-1)^{(r-1)} \; R(x) \tag{4.7}$$

Proof: From equation (4.1),

$$\alpha_{(r)}(t) = \sum_{t+r}^{\infty}(x-t)^{(r)} \; [R(x) - R(x+1)]$$

$$= \sum_{t+r}^{\infty}(x-t)^{(r)} R(x) - \sum_{t+r}^{\infty}(x-t)^{(r)} R(x+1)$$

$$= \sum_{t+r}^{\infty}(x-t)^{(r)} R(x) - \sum_{t+r+1}^{\infty}(x-t-1)^{(r)} R(x)$$

$$= \sum_{t+r+1}^{\infty}[(x-t)^{(r)} - (x-t-1)^{(r)}]R(x) + r^{(r)} \; R(t+r)$$

$$= \sum_{t+r+1}^{\infty}[\{(x-t)(x\text{-}t\text{-}1)\ldots(x\text{-}t\text{-}r+1)\} - \{(x\text{-}t\text{-}1)(x\text{-}t\text{-}2)\ldots(x\text{-}t\text{-}r)\}]R(x)$$

$$+ \; r^{(r)} \; R(t+r)$$

$$= \sum_{t+r+1}^{\infty}(x-t-1)(x\text{-}t\text{-}2)\ldots(x\text{-}t\text{-}r+1)\} rR(x) + r^{(r)} \; R(t+r)$$

$$= r \sum_{t+r+1}^{\infty}(x-t-1)^{(r-1)} R(x) + r^{(r)} \; R(t+r)$$

$$= r \sum_{t+r}^{\infty}(x-t-1)^{(r-1)} R(x).$$

It could be observed that there are two constants associated with the sequence of partial moments relating to a distribution.

One is the order represented by r and other is the point t about which the moment is taken. The implication of these two constants is that, unlike the usual moments all the partial moments are not independent. We prove this fact in the next theorem.

Theorem 4.2

The partial moment satisfies the recurrence relation,

$$\alpha_{(r)}(t) - \alpha_{(r)}(t+1) = r\ \alpha_{(r-1)}(t+1) \tag{4.8}$$

for $r = 1, 2, 3, \ldots$ and $t = 0, 1, 2, \ldots$

Proof: From equation (4.7)

$$\alpha_{(r)}(t) - \alpha_{(r)}(t+1) = r\sum_{t+r}^{\infty}(x-t-1)^{(r-1)}R(x) - r\sum_{t+r+1}^{\infty}(x-t-2)^{(r-1)}R(x)$$

$$= r\sum_{t+r}^{\infty}(x-t-1)^{(r-1)}R(x) - r\sum_{t+r}^{\infty}(x-t-1)^{(r-1)}R(x+1)$$

$$= r\sum_{t+r}^{\infty}(x-t-1)^{(r-1)}[R(x) - R(x+1)]$$

$$= r\sum_{t+r}^{\infty}(x-t-1)^{(r-1)}f(x)$$

$$= r\alpha_{(r-1)}(t+1).$$

From the above recurrence relation (4.8), once $\alpha_{(r)}(t)$ is known for two consecutive values of t, the corresponding lower order moments can be determined. Furthur if the entire sequence $(\alpha_{(r)}(t))$ is known for all t and a specified r it is possible to compute all the lower moments of order r-1. More specifically, we have the following theorem on the subject.

Theorem 4.3

Let X be a non-negative integer valued random variable such that $E(X^r)<\infty$. Then for any one positive integer r, $\alpha_r(t)$ determines all the lower order factorial moments that exist.

Proof: From (4.8), we have

$$\alpha_{(r-1)}(t+1) = \frac{1}{r}\,[\alpha_{(r)}(t) - \alpha_{(r)}(t+1)],$$

which provides,

$$\alpha_{(r-1)}(t+1) = \frac{1}{r}(1-E)\ \alpha_{(r)}(t),$$

where E is the usual forward shift operator.

Similarly,

$$\begin{aligned}\alpha_{(r-2)}(t+1) &= \frac{1}{r-1}(1-E)\ \alpha_{(r-1)}(t)\\ &= \frac{1}{r-1}(1-E)\ \frac{1}{r}(1-E)\ \alpha_{(r)}(t-1)\\ &= \frac{1}{r(r-1)}(1-E)^2\,\alpha_{(r)}(t-1).\end{aligned}$$

Similar iteration on r gives,

$$\alpha_{(r-s)}(t+1) = \frac{1}{r(r-1)\ldots(r-s+1)}(1\text{-}E)^s\ \alpha_{(r)}(t\text{-}s+1),$$

$$= \frac{1}{r^{(s)}}\ (1\text{-}E)^s\ \alpha_{(r)}(t\text{-}s+1),\ s = 1,\ 2,\ \ldots,\ r\text{-}1,\ r>s. \quad (4.9)$$

we can verify theorem (4.3) for geometric distribution.

Example 4.1

For geometric distribution with probability mass function

$$f(x) = pq^x,\ x = 0,\ 1,\ 2,\ \ldots$$

we obtain by direct calculation

$$\alpha_{(2)}(t) = 2q^{t+2}\ p^{-2}.$$

From (4.9),

$$\alpha_{(1)}(t) = \tfrac{1}{2}\ (1\text{-}E)\alpha_{(2)}(t).$$

$$= \tfrac{1}{2}\ (1\text{-}E)\ 2q^{t+2}\ p^{-2}.$$

$$= \frac{1}{p^2}[q^{t+2} - q^{t+3}] = q^{t+2}\ p^{-1}.$$

4.3 Partial moments of modified power series distribution

In modelling statistical data families of distributions follow an important role. A recent emphasis in distribution theory is to develop methodology and techniques towards identifying the random mechanism that generates the data in question. We may often have to formulate hypothesis to ascertain the chance mechanism for the most appropriate explanation of the underlying phenomena. When physical characteristics of the system is not known, one has to be satisfied with determination of the distribution from the empirical information in the form of data. In such cases, the approach to modelling is to start with a family of distributions with enough members that can accommodate different shapes and characteristics and then choose that member from the family which resembles the data. An aid to such choices is the characteristic properties that are alluded to the family. A second advantage of discussing properties of family is that, a uniform analytic treatment can be made available to all members so that specific methods for individual members can be dispensed with. Some times, such considerations can also unearth new members in the family. Therefore it is worthwhile to consider properties of partial moments for families than for individual distributions. One important family of discrete distribution is the modified power series family. The family of modified power series distributions introduced by Gupta (1974) has probability mass function of the form

$$f(x) = \frac{a(x)b(\theta)^x}{A(\theta)}, \tag{4.10}$$

where the support of X is I^+, or a subset thereof, $a(x)\geq 0$ and $b(\theta)$ and $A(\theta)$ are finite, positive and differentiable functions of the parameter θ. When $b(\theta)$ is invertible it reduces to the generalised power series family. This class of distributions includes the Poisson, logarithmic series, generalised negative binomial and the lost games distributions. First we prove a recurrence relation, that characterizes the MPSD.

Theorem 4.4

A random variable X in the support of I^+ or a subset of it with $E(X^{r+1})< \infty$, $r = 0, 1, 2, \ldots$ has a MPSD if and only if for a finite, positive and differentiable function $b(\theta)$, its factorial partial moments satisfy the recurrence relation,

$$\alpha_{(r+1)}(t) = [\mu - (t+r)]\ \alpha_{(r)}(t) + \frac{b(\theta)}{b'(\theta)}\alpha^{'}_{(r)}(t) \tag{4.11}$$

Proof: Suppose that X has MPSD. Then from (4.10),

$$A(\theta)\alpha_{(r)}(t) = \sum_{t+r}^{\infty}(x-t)^{(r)}\, a(x)b(\theta)^x. \tag{4.12}$$

Differentiating (4.12) with respect to θ, we get,

$$A(\theta)\,\alpha'_{(r)}(t) + \alpha_{(r)}(t)\,A'(\theta) = \sum_{t+r}^{\infty}(x-t)^{(r)}\;x\;a(x)b(\theta)^{x-1}\;b'(\theta)$$

or

$$A(\theta)\,\alpha'_{(r)}(t) + \alpha_{(r)}(t)\,A'(\theta) = A(\theta)\sum_{t+r}^{\infty}x(x-t)^{(r)}\;a(x)\;\frac{b(\theta)^{x}}{A(\theta)}\,\frac{b'(\theta)}{b(\theta)}$$

which gives

$$\alpha'_{(r)}(t) + \alpha_{(r)}(t)\;\frac{A'(\theta)}{A(\theta)} = \frac{b'(\theta)}{b(\theta)}\sum_{t+r}^{\infty}x(x-t)^{(r)}\;a(x)\;\frac{b(\theta)^{x}}{A(\theta)}$$

or

$$\frac{b(\theta)}{b'(\theta)}\,\alpha'_{(r)}(t) + \alpha_{(r)}(t)\;\frac{A'(\theta)}{A(\theta)}\,\frac{b(\theta)}{b'(\theta)} = \sum_{t+r}^{\infty}x(x-t)^{(r)}\;a(x)\;\frac{b(\theta)^{x}}{A(\theta)}$$

The last equation is same as

$$\frac{b(\theta)}{b'(\theta)}\,\alpha'_{(r)}(t) + \alpha_{(r)}(t)\;\frac{A'(\theta)}{A(\theta)}\,\frac{b(\theta)}{b'(\theta)}$$

$$= \sum_{t+r}^{\infty}(x-t)(x\text{-}t\text{-}1)\ldots\;(x\text{-}t\text{-}r+1)\;(x\text{-}t\text{-}r+t+r)a(x)\;\frac{b(\theta)^{x}}{A(\theta)}.$$

or

$$\frac{b(\theta)}{b'(\theta)}\,\alpha'_{(r)}(t) + \mu\;\alpha_{(r)}(t) = (t+r)\;\alpha_{(r)}(t) + \alpha_{(r+1)}(t),$$

since $\mu = \dfrac{A'(\theta)}{A(\theta)}\,\dfrac{b(\theta)}{b'(\theta)}$,

$$\alpha_{(r+1)}(t) = [\mu - (t+r)]\;\alpha_{(r)}(t) + \frac{b(\theta)}{b'(\theta)}\,\alpha'_{(r)}(t).$$

which is (4.11).

Conversely, if the probability mass function $f(x)$ of X satisfies (4.11), we can write.

$$\frac{b(\theta)}{b'(\theta)}\sum_{t+r}^{\infty}(x-t)^{(r)}\ \frac{\partial f}{\partial\theta}= \sum_{t+r+1}^{\infty}(x-t)^{(r+1)}f(x) + (t+r-\mu)\sum_{t+r}^{\infty}(x-t)^{(r)}f(x)$$

$$= \sum_{t+r}^{\infty}(x-t)(x-t-1)\dots\ (x-t-r)f(x)$$

$$+ (t+r)\sum_{t+r}^{\infty}(x-t)(x-t-1)\dots\ (x-t-r+1)f(x)$$

$$- \mu\sum_{t+r}^{\infty}(x-t)(x-t-1)\dots\ (x-t-r+1)f(x).$$

$$= \sum_{t+r}^{\infty}x(x-t)(x-t-1)\dots\ (x-t-r+1)f(x)$$

$$- (t+r)\sum_{t+r+1}^{\infty}(x-t)(x-t-1)\dots\ (x-t-r+1)f(x)$$

$$+ (t+r)\sum_{t+r}^{\infty}(x-t)(x-t-1)\dots\ (x-t-r+1)f(x)$$

$$- \mu\sum_{t+r}^{\infty}(x-t)(x-t-1)\dots\ (x-t-r+1)f(x).$$

or

$$\sum_{t+r}^{\infty}(x-t)^{(r)}(x-\mu)\,f(x) = \frac{b(\theta)}{b'(\theta)}\sum_{t+r}^{\infty}(x-t)^{(r)}\ \frac{\partial f}{\partial\theta}. \qquad (4.13)$$

Changing t to $t+1$ in (4.13), we get

$$\sum_{t+r+1}^{\infty}(x-t)^{(r)}(x-\mu)\,f(x) = \frac{b(\theta)}{b'(\theta)}\sum_{t+r+1}^{\infty}(x-t)^{(r)}\ \frac{\partial f}{\partial\theta}. \qquad (4.14)$$

Subtracting (4.14) from (4.13), we get

$$\frac{b'(\theta)}{b(\theta)}\,(t+r-\mu) = \frac{1}{f(r+t)}\,\frac{\partial f(t+r)}{\partial\theta}. \qquad (4.15)$$

Since this equation is true for all t and r, we can write,

$$\frac{\partial \log f(x)}{\partial\theta} = \frac{b'(\theta)}{b(\theta)}\,(x-\mu),\ \text{for all } x = 0,\ 1,\ 2,\ \ldots$$

Integrating with respect to θ,

$$\log f(x) = (x-\mu)\log b(\theta) + \log a(x)$$

$$f(x) = a(x)[b(\theta)]^{(x-\mu)} \exp\int\frac{\partial\mu}{\partial\theta}\log b(\theta)\,d\theta$$

$$= \frac{a(x)b(\theta)^x}{A(\theta)}$$

where $A(\theta) = [b(\theta)]^{\mu}\ \exp -\int\frac{\partial\mu}{\partial\theta}\log b(\theta)\,d\theta.$

This completes the proof.

Corollary 4.1

The usual factorial moments of MPSD satisfy the recurrence relation,

$$\mu'_{(r+1)} = \frac{b(\theta)}{b'(\theta)}\,\frac{d\mu'_{(r)}}{d\theta} + (\mu-r)\,\mu'_{(r)}.$$

This result proved in Gupta (1974) is obtained by setting $t = 0$ in (4.11) since $\alpha_{(r)}(0) = \mu'_{(r)}$.

For the MPSD class and even for the subclass of generalized power series distributions, moments have not been obtained exclusively in literature. Only difference differential equations connecting the successive moments have been derived in literature which is difficult to solve in particular cases. The same arguments apply for partial moments also. However, we will illustrate the results for some specific distributions.

4.3.1 Generalized negative binomial distribution

A discrete random variable X has generalised negative binomial distribution if its probability mass function is

$$f(x) = \frac{n\Gamma(n+\beta x)}{x!\Gamma(n+\beta x-x+1)}\,\frac{[\theta(1-\theta)^{\beta-1}]^x}{(1-\theta)^{-n}},\ x = 0,\ 1,\ 2,\ \ldots,\ 0<\theta<1,\ |\theta\beta|<1.$$

In the notations given above

$$b(\theta) = \theta\,(1-\theta)^{\beta-1};\ A(\theta) = (1-\theta)^{-n} \text{ and } a(x) = \frac{n\Gamma(n+\beta x)}{x!\Gamma(n+\beta x-x+1)}$$

so that for this distribution

$$\mu = n\theta\,(1-\theta\beta)^{-1}.$$

and

$$\alpha_{(r+1)}(t) = (\mu-t-r)\ \alpha_{(r)}(t) + \frac{\theta(1-\theta)}{\theta(\beta-2)+1}\,\alpha'_{(r)}(t).$$

In particular, when β=0, X has binomial distribution with

$$\alpha_{(r+1)}(t) = (n\theta\text{-}t\text{-}r)\ \alpha_{(r)}(t) + \frac{\theta(1-\theta)}{1-\theta\beta}\,\alpha'_{(r)}(t) \tag{4.16}$$

and when β=1, we have negative binomial distribution for X in which case

$$\alpha_{(r+1)}(t) = \left(\frac{n\theta}{1-\theta} - t - r\right) \alpha_{(r)}(t) + \theta\ \alpha'_{(r)}(t). \tag{4.17}$$

Further when t = 0, (4.16) and (4.17) reduce to recurrence relation connecting factorial moments of the two distributions, viz

$$\mu'_{(r+1)} = \frac{\theta(1-\theta)}{1-\theta\beta}\ \frac{d\mu'_{(r)}}{d\theta} + (n\theta\text{-}r)\ \mu'_{(r)}$$

and

$$\mu'_{(r+1)} = \theta\ \frac{d\mu'_{(r)}}{d\theta} + \frac{n\theta}{1-\theta}\ \mu'_{(r)},$$

results which are not quoted often in literature.

4.3.2 Generalised Poisson distribution

The generalised Poisson distribution is given by

$$f(x) = \frac{\lambda_1(\lambda_1+\lambda_2 x)^{x-1}}{x!}\ \frac{[\theta e^{-\lambda_2\theta}]^x}{e^{\lambda_1\theta}},\ x = 0, 1, 2, \ldots,\ \theta\lambda_1 > 0,\ |\theta\lambda_2| < 1.$$

Here

$$b(\theta) = \theta e^{-\lambda_2\theta},\ A(\theta) = e^{\lambda_1\theta},\ \text{ and } a(x) = \frac{\lambda_1(\lambda_1+\lambda_2 x)^{x-1}}{x!}.$$

For this distribution,

$$\mu = \frac{\lambda_1\theta}{1-\lambda_2\theta}$$

and

$$\alpha_{(r+1)}(t) = (\mu\text{-}t\text{-}r)\ \alpha_{(r)}(t) + \frac{\theta}{(1-\lambda_2\theta)}\ \alpha'_{(r)}(t).$$

In particular when $\lambda_2 = 0$and $\theta = 1$, we have the Poisson distribution for X, in which case,

$$\alpha_{(r+1)}(t) = (\lambda_1\text{-}t\text{-}r)\ \alpha_{(r)}(t) + \alpha'_{(r)}(t). \tag{4.18}$$

Further when $t = 0$, (4.18) reduces to the recurrence relation connecting factorial moments, viz,

$$\mu'_{(r+1)} = \frac{d\mu'_{(r)}}{d\lambda_1} + (\lambda_1\text{-}r)\ \mu'_{(r)}.$$

4.3.3 Generalised logarithmic series distribution

This distribution has probability mass function

$$f(x) = \frac{\Gamma(x\beta)}{x\Gamma(x)\Gamma(x\beta-x+1)}\frac{\theta^x(1-\theta)^{\beta x-x}}{[-\log(1-\theta)]},\ x = 1,2,3,\ldots,\ 0<\theta<1,\ 0<\theta\beta<1, \beta\geq 1.$$

It is seen that

$$b(\theta) = \theta\,(1\text{-}\theta)^{\beta\text{-}1};\ A(\theta) = \text{-}\log\,(1\text{-}\theta) \text{ and } a(x) = \frac{\Gamma(x\beta)}{x\Gamma(x)\Gamma(x\beta-x+1)}$$

so that for this distribution

$$\mu = \frac{\theta}{(1-\beta\theta)\log(1-\theta)^{-1}}$$

and

$$\alpha_{(r+1)}(t) = (\mu\text{-}t\text{-}r)\ \alpha_{(r)}(t) + \frac{\theta(1-\theta)}{(1-\theta\beta)}\,\alpha'_{(r)}(t).$$

In particular when $\beta = 1$, we have the logarithmic distribution for X, in which case,

$$\alpha_{(r+1)}(t) = \left(\frac{\theta}{(1-\theta)\log(1-\theta)^{-1}} - t - r\right)\ \alpha_{(r)}(t) + \theta\alpha'_{(r)}(t). \qquad (4.19)$$

Further when t=0, (4.19) reduces to the recurrence relation connecting factorial moments, viz,

$$\mu'_{(r+1)} = \theta\,\frac{d\mu'_{(r)}}{d\theta} + \left(\frac{\theta}{(1-\theta)\log(1-\theta)^{-1}} - r\right)\mu'_{(r)}.$$

4.3.4 Lost-games distribution

The lost games distribution of Kemp and Kemp (1968) is specified by

$$f(x) = \frac{a\binom{2x-a}{x}}{2x-a}\,\frac{[\theta(1-\theta)]^x}{\theta^a},\ x = a,\ a+1,\ \ldots,\ a\geq 1.$$

By comparison with the general form of MPSD

$$b(\theta) = \theta\,(1\text{-}\theta),\ A(\theta) = \theta^a \text{ and } a(x) = \frac{a\binom{2x-a}{x}}{2x-a}.$$

For this distribution,

$$\mu = \frac{a(1-\theta)}{(1-2\theta)}$$

and

$$\alpha_{(r+1)}(t) = (\mu\text{-}t\text{-}r)\ \alpha_{(r)}(t) + \frac{\theta(1-\theta)}{(1-2\theta)} \alpha'_{(r)}(t).$$

Theorem 4.5

For the modified power series family

$$\alpha_{(r+1)}(t)=(\mu\text{-}t)\ \alpha_{(r)}(t+1) + \frac{b(\theta)}{b'(\theta)} \alpha'_{(r)}(t+1),\ r = 0,\ 1,\ 2,\ \ldots \quad (4.20)$$

Proof: Changing t to t+1 in (4.11) we get

$$\alpha_{(r+1)}(t) = [\mu\text{-}(t+r+1)]\ \alpha_{(r)}(t+1) + \frac{b(\theta)}{b'(\theta)} \alpha'_{(r)}(t+1)$$

$$= (\mu\text{-}t)\ \alpha_{(r)}(t+1)\text{-}(r+1)\ \alpha_{(r)}(t+1) + \frac{b(\theta)}{b'(\theta)} \alpha'_{(r)}(t+1)$$

$$\alpha_{(r+1)}(t+1) + (r+1)\ \alpha_{(r)}(t+1) = (\mu\text{-}t)\ \alpha_{(r)}(t+1) + \frac{b(\theta)}{b'(\theta)} \alpha'_{(r)}(t+1).$$

Using the recurrence relation in (4.8)

$$(r+1)\ \alpha_{(r)}(t\text{-}1) = \alpha_{(r+1)}(t) - \alpha_{(r+1)}(t+1),$$

we get

$$\alpha_{(r+1)}(t) = (\mu\text{-}t)\ \alpha_{(r)}(t+1) + \frac{b(\theta)}{b'(\theta)} \alpha'_{(r)}(t+1).$$

Remark

Equation (4.11) enables us to calculate the higher order moments for a given value of t, while (4.20) helps to get the moments for successive values of t.

Although there exists an identity between partial moments and partial factorial moments, it is not apparent that a recurrence relation for the former exists from the relationship (4.11). A direct derivation of the recurrence relation for partial moments that characterize the MPSD is proved in the next theorem.

Theorem 4.6

A random variable X in the support of I^+ or a subset of it with $E(X^{r+1})<\infty$, $r = 0, 1, 2, \ldots$ has a MPSD if and only if for a positive, finite and differentiable function $b(\theta)$, the partial moments satisfy the relationship,

$$\alpha_{r+1}(t) = (\mu - t)\ \alpha_r(t) + \frac{b(\theta)}{b'(\theta)}\ \alpha_r^{'}(t). \tag{4.21}$$

for all t in I^+ and $r = 0, 1, 2, \ldots$ and $\mu = E(X)$.

Proof: We have,

$$A(\theta)\ \alpha_r(t) = \sum_{t+1}^{\infty}(x-t)^{(r)}\, a(x)\ b(\theta)^x \tag{4.22}$$

Differentiating (4.22) with respect to θ, we get

$$A(\theta)\,\alpha_r'(t) + \alpha_r(t)\,A'(\theta) = \sum_{t+1}^{\infty}(x-t)^r\,a(x)x\,b(\theta)^{x-1}\,b'(\theta)$$

$$= A(\theta)\sum_{t+1}^{\infty}(x-t)^r\,a(x)\;x\;\frac{b(\theta)^x}{A(\theta)}\,\frac{b'(\theta)}{b(\theta)}.$$

This gives,

$$\alpha_r'(t) + \alpha_r(t)\,\frac{A'(\theta)}{A(\theta)} = \frac{b'(\theta)}{b(\theta)}\sum_{t+1}^{\infty}(x-t)^r\,a(x)\;x\;\frac{b(\theta)^x}{A(\theta)}.$$

or,

$$\alpha_r'(t)\,\frac{b(\theta)}{b'(\theta)} + \alpha_r(t)\,\frac{A'(\theta)}{A(\theta)}\,\frac{b(\theta)}{b'(\theta)} = \sum_{t+1}^{\infty}(x-t)^r\,(x\text{-}t+t)a(x)\,\frac{b(\theta)^x}{A(\theta)}.$$

$$= \sum_{t+1}^{\infty}(x-t)^{r+1}\,a(x)\,\frac{b(\theta)^x}{A(\theta)} + t\,\alpha_r(t)$$

$$\alpha_r'(t)\,\frac{b(\theta)}{b'(\theta)} + \alpha_r(t)\,\mu = \alpha_{r+1}(t) + t\,\alpha_r(t)$$

where

$$\mu = \mathrm{E}(X) = \frac{A'(\theta)}{A(\theta)}\,\frac{b(\theta)}{b'(\theta)}$$

or

$$\alpha_{r+1}(t) = (\mu\text{-}t)\,\alpha_r(t) + \frac{b(\theta)}{b'(\theta)}\,\alpha_r'(t)$$

which is (4.21).

Conversely, if the probability mass function $f(x)$ satisfies (4.21), we can write,

$$\frac{b(\theta)}{b'(\theta)} \sum_{t+1}^{\infty}(x-t)^r \frac{\partial f}{\partial \theta} = \sum_{t+1}^{\infty}(x-t)^{r+1} f(x) + (t - \mu)\sum_{t+1}^{\infty}(x-t)^r f(x) \quad (4.23)$$

changing t to $t+1$ in (4.23) we get

$$\frac{b(\theta)}{b'(\theta)} \sum_{t+2}^{\infty}(x-t)^r \frac{\partial f}{\partial \theta} = \sum_{t+2}^{\infty}(x-t)^{r+1} f(x) + (t - \mu)\sum_{t+2}^{\infty}(x-t)^r f(x) \quad (4.24)$$

Subtracting (4.24) from (4.23), we get,

$$\frac{b(\theta)}{b'(\theta)} \frac{\partial f(t+1)}{\partial \theta} = f(t+1) + (t - \mu) f(t+1)$$

$$\frac{b(\theta)}{b'(\theta)} \frac{\partial f(t+1)}{\partial \theta} = (t + 1 - \mu) f(t+1). \quad (4.25)$$

The equation is valid for all t, we can write,

$$\frac{\partial \log f(x)}{\partial \theta} = \frac{b'(\theta)}{b(\theta)}(x-\mu).$$

Integrating with respect to θ,

$$f(x) = a(x)\ b(\theta)^{x-\mu} \exp\left[\int \frac{d\mu}{d\theta} \log b(\theta) d\theta\right]$$

which is of the form (4.10) and our proof is complete.

Corollary 4.2

The raw moments $\mu_r' = E(X^r)$ of MPSD satisfies the recurrence relation,

$$\mu'_{r+1} = \mu\ \mu'_r + \frac{d\mu'_r}{d\theta}\frac{b(\theta)}{b'(\theta)}.$$

The result is obtained by setting t=0 in (4.21).

Special Cases

4.3.1.1 Generalised negative binomial distribution

The recurrence relation for the partial moments is,

$$\alpha_{r+1}(t) = (\mu\text{-}t)\ \alpha_r(t) + \frac{\theta(1-\theta)}{1-\theta\beta}\alpha'_r(t)$$

where $\mu = n\theta\,(1\text{-}\beta\theta)^{-1}$.

In particular, when β=0, X has binomial distribution with

$$\alpha_{r+1}(t) = (n\theta\text{-}t)\ \alpha_r(t) + \theta\,(1\text{-}\theta)\,\alpha'_r(t) \tag{4.26}$$

and when β=1, we have negative binomial distribution for X in which case

$$\alpha_{r+1}(t) = \left(\frac{n\theta}{1-\theta}-t\right)\alpha_r(t) + \theta\ \alpha'_r(t). \tag{4.27}$$

Further when t = 0, (4.26) and (4.27) reduce to recurrence relation connecting raw moments of the two distributions, viz

$$\mu'_{r+1} = \theta\,(1\text{-}\theta)\,\frac{d\mu'_r}{d\theta}+n\theta\ \mu'_r$$

and

$$\mu'_{r+1} = \theta\,\frac{d\mu'_r}{d\theta}+n\ \theta\,(1\text{-}\theta)^{-1}\,\mu'_r.$$

4.3.1.2 Generalised Poisson distribution

The recurrence relation is,

$$\alpha_{r+1}(t) = (\mu\text{-}t)\ \alpha_r(t) + \frac{\theta}{(1-\lambda_2\theta)}\,\alpha_r^{'}(t).$$

where

$$\mu = \frac{\lambda_1\theta}{1-\lambda_2\theta}$$

In particular when λ_2 = 0and $\theta = 1$, we have the Poisson distribution for X, is which case,

$$\alpha_{r+1}(t) = (\lambda_1\text{-}t)\ \alpha_{(r)}(t) + \alpha_r^{'}(t). \tag{4.28}$$

Further when t = 0, (4.28) reduces to,

$$\mu_{r+1}^{'} = \frac{d\mu_r^{'}}{d\lambda_1} + \lambda_1\ \mu_r^{'}.$$

4.3.3 Generalised logarithmic series distribution

The recurrence relation is,

$$\alpha_{r+1}(t) = (\mu\text{-}t)\ \alpha_r(t) + \frac{\theta(1-\theta)}{(1-\theta\beta)}\,\alpha_r^{'}(t).$$

where

$$\mu = \frac{\theta}{(1-\beta\theta)\log(1-\theta)^{-1}}$$

When $\beta = 1$, we have the logarithmic distribution for X, in which case,

$$\alpha_{r+1}(t) = \left(\frac{\theta}{(1-\theta)\log(1-\theta)^{-1}} - t\right) \alpha_r(t) + \theta\alpha_r'(t). \tag{4.29}$$

Further when t=0, (4.29) reduces to,

$$\mu_{r+1}' = \theta \frac{d\mu_r'}{d\theta} + \frac{\theta}{(1-\theta)\log(1-\theta)^{-1}} \mu_r'.$$

4.3.4 Lost-games distribution

The recurrence relation is specified by

$$\alpha_{r+1}(t) = (\mu\text{-}t)\ \alpha_r(t) + \frac{\theta(1-\theta)}{(1-2\theta)} \alpha_r'(t). \tag{4.30}$$

where

$$\mu = \frac{a(1-\theta)}{(1-2\theta)}.$$

When $t = 0$, (4.30) reduces to

$$\mu_{r+1}' = \frac{\theta(1-\theta)}{(1-2\theta)} \frac{d\mu_r'}{d\theta} + \frac{a(1-\theta)}{(1-2\theta)} \mu_r'.$$

Another important family of discrete distributions will be considered in this section, which is the discrete version of the Pearson family in the continuous case.

4.4 Partial moments of Ord family

The Ord (1967) family comprises of all distributions that satisfy,

$$f(x) - f(x\text{-}1) = \frac{(a-x)f(x-1)}{b_0 + b_1 x + b_2 x(x-1)}$$

where $f(x)$ is the probability mass function and X has support some subset of integers. Some important members of the family are binomial, Poisson, negative binomial, hypergeometric, beta-binomial, beta-Pascal and the discrete student's t distributions.

Theorem 4.7

If a random variable X in the support of I^+ or a subset thereof with $E(X^{r+1})<\infty$ has a distribution belonging to the Ord family on I^+, then

$$-r[a+b_0+(t+r)(b_1+b_2(3t+3r+1)-1)]\,\alpha_{(r-1)}(t)$$

$$-[a+(r+1)(b_1-1)+(t+r)(2b_2-1)]\,\alpha_{(r)}(t) - [(r+1)b_2-1]\,\alpha_{(r+1)}(t)=0 \qquad (4.31)$$

Proof: When the distribution belongs to Ord's family,

$[b_0+(b_1 - b_2)x+b_2\, x^2]\, f(x) = [a+b_0+(b_1 - b_2-1)x+b_2\, x^2]\, f(x-1)$

Multiplying by $(x-t)^{(r)}$.

$[b_0+(b_1 - b_2)x+b_2\, x^2]\ (x-t)^{(r)}f(x)$

$$= [a+b_0+(b_1 - b_2-1)x+b_2\, x^2]\ (x-t)^{(r)}f(x-1) \qquad (4.32)$$

Using,

$x = x\text{-}t\text{-}r+t+r$

$x^2 = (x\text{-}t\text{-}r)(x\text{-}t\text{-}r\text{-}1) + (2t+2r+1)(x\text{-}t\text{-}r) + (t+r)(3t+3r+2)$

(4.32) becomes,

$$[b_0+(b_1 - b_2)(x\text{-}t\text{-}r+t+r)+b_2 \{(x\text{-}t\text{-}r)(x\text{-}t\text{-}r\text{-}1) + (2t+2r+1)(x\text{-}t\text{-}r)$$
$$+(t+r)(3t+3r+2)\}](x\text{-}t)^{(r)}f(x) = [a+b_0+(b_1 - b_2\text{-}1)(x\text{-}t\text{-}r+t+r)$$
$$+b_2 \{(x\text{-}t\text{-}r)(x\text{-}t\text{-}r\text{-}1)+(2t+2r+1)(x\text{-}t\text{-}r)$$
$$+ (t+r)(3t+3r+2)\}] (x\text{-}t)^{(r)}f(x\text{-}1)$$

which gives,

$$b_0(x\text{-}t)^{(r)}f(x) + (b_1 - b_2)(x\text{-}t\text{-}r)(x\text{-}t)^{(r)}f(x) +(b_1 - b_2)(t+r)(x\text{-}t)^{(r)}f(x)$$
$$+b_2(x\text{-}t\text{-}r)(x\text{-}t\text{-}r\text{-}1)(x\text{-}t)^{(r)}f(x) + b_2 (2t+2r+1)(x\text{-}t\text{-}r)(x\text{-}t)^{(r)}f(x)$$
$$+(t+r)(3t+3r+2)(x\text{-}t)^{(r)}f(x)$$
$$= (a+b_0)(x\text{-}t)^{(r)}f(x\text{-}1) + (b_1\text{-}b_2\text{-}1)(x\text{-}t\text{-}r)(x\text{-}t)^{(r)}f(x\text{-}1)$$
$$+ (b_1\text{-}b_2\text{-}1)(t+\mathrm{r})(x\text{-}t)^{(r)}f(x\text{-}1) +b_2 (x\text{-}t\text{-}r)(x\text{-}t\text{-}r\text{-}1)(x\text{-}t)^{(r)}f(x\text{-}1)$$
$$+b_2(2t+2r+1)(x\text{-}t\text{-}r)(x\text{-}t)^{(r)}f(x\text{-}1)+(t+r)(3t+3r+2)(x\text{-}t)^{(r)}f(x\text{-}1).$$

Taking summation from $(t+r)$

$$b_0 \sum_{t+r}^{\infty} (x\text{-}t)^{(r)}f(x) + (b_1\text{-}b_2)\sum_{t+r}^{\infty} (x\text{-}t\text{-}r)(x\text{-}t)^{(r)}f(x)$$

$$+(b_1 - b_2)(t+r) \sum_{t+r}^{\infty} (x\text{-}t)^{(r)}f(x) +b_2\sum_{t+r}^{\infty} (x\text{-}t\text{-}r)(x\text{-}t\text{-}r\text{-}1)(x\text{-}t)^{(r)}f(x)$$

$$+ b_2(2t+2r+1)\sum_{t+r}^{\infty} (x\text{-}t\text{-}r)(x\text{-}t)^{(r)}f(x) +(t+r)(3t+3r+2) \sum_{t+r}^{\infty} (x\text{-}t)^{(r)}f(x)$$

$$= (a+b_0) \sum_{t+r}^{\infty} (x-t)^{(r)} f(x-1) + (b_1-b_2-1) \sum_{t+r}^{\infty} (x-t-r)\,(x-t)^{(r)} f(x-1)$$

$$+(b_1-b_2-1)(t+\mathrm{r})\sum_{t+r}^{\infty} (x-t)^{(r)} f(x-1)$$

$$+b_2 \sum_{t+r}^{\infty} (x-t-r)(x-t-r-1)(x-t)^{(r)} f(x-1)$$

$$+(2t+2r+1)\, b_2 \sum_{t+r}^{\infty} (x-t-r)\,(x-t)^{(r)} f(x-1)$$

$$+ (t+r)(3t+3r+2) \sum_{t+r}^{\infty} (x-t)^{(r)} f(x-1).$$

Rearranging the terms we get

$$[b_0+(t+r)\{b_1+(3t+3r+1)b_2\}]\,\alpha_{(r)}(t) + \{b_1+2(t+r)b_2\}\,\alpha_{(r+1)}(t)$$

$$+ b_2\, \alpha_{(r+2)}(t) = [a+b_0+(t+r)\{b_1 -1+(3t+3r+1)b_2\}]\ \alpha_{(r)}(t-1)$$

$$+ \{b_1 -1+2(t+r)b_2\}]\,\alpha_{(r+1)}(t-1) + b_2\, \alpha_{(r+2)}(t-1). \quad (4.33)$$

Using,

$$\alpha_{(r)}(t-1) = \alpha_{(r)}(t) + r\ \alpha_{(r-1)}(t).$$

(4.33) becomes,

$$b_0+(t+r)\{b_1+(3t+3r+1)b_2\}\,\alpha_{(r)}(t) + \{b_1+2(t+r)b_2\}\,\alpha_{(r+1)}(t)$$

$$+ b_2\, \alpha_{(r+2)}(t) = a+b_0+(t+r)\{b_1 -1+(3t+3r+1)b_2\}\{\alpha_{(r)}(t) + r\ \alpha_{(r-1)}(t)\}$$

$$+ \{b_1 -1+2(t+r)b_2\}\{\alpha_{(r+1)}(t) + (r+1)\alpha_{(r)}(t)\}$$

$$+ b_2\, \{\alpha_{(r+2)}(t) + (r+2)\alpha_{(r+1)}(t)\}.$$

Rearranging the terms, we get the final relation

$-r\ [a+b_0+(t+r)\{b_1+(3t+3r+1)b_2-1\}]\alpha_{(r-1)}(t)$

$-\ [a+(1+r)(b_1-1)+(t+r)(2b_2-1)]\alpha_{(r)}(t)\ -\ [(r+2)b_2-1]\alpha_{(r+1)}(t) = 0.$

We will illustrate the result for some specific distributions.

4.4.1 Particular cases

(i) For a discrete r.v. following Poisson distribution with probability mass function

$$f(x) = \frac{e^{-\lambda}\lambda^x}{x!},\quad x = 0, 1, 2, \ldots, \lambda>0. \tag{4.34}$$

On comparing (4.34) with the difference equation for the Ord's family, we get, $a=\lambda$, $b_0 = 0$, $b_1 = 1$, $b_2 = 0$. The recurrence relation (4.31) becomes

$$-r\,\lambda\,\alpha_{(r-1)}(t) - [\lambda-(t+r)]\ \alpha_{(r)}(t) + \alpha_{(r+1)}(t) = 0$$

or

$$\alpha_{(r+1)}(t) - (\lambda-t-r)\ \alpha_{(r)}(t) - r\,\lambda\ \alpha_{(r-1)}(t) = 0.$$

When $t = 0$, this becomes

$$\mu'_{(r+1)} - (\lambda-r)\ \mu'_{(r)} - \lambda r\,\mu'_{(r-1)} = 0$$

(ii) Let X has binomial distribution with probability mass function

$$f(x) = \binom{n}{x} p^x q^{n-x},\ x = 0, 1, 2, \ldots, n,\ 0<p<1,\ q=1-p.$$

Comparing with the difference equation for the Ord's family, we get, $a = (n+1)p$, $b_0 = 0$, $b_1 = q$, $b_2 = 0$, so that the recurrence relation becomes,

$$\alpha_{(r+1)}(t) - [(n-r)p-t-r]\ \alpha_{(r)}(t) - rp(n-t-r+1)\ \alpha_{(r-1)}(t) = 0.$$

When $t = 0$, we have

$$\mu'_{(r+1)} - [(n-r)p-r]\ \mu'_{(r)} - r\,p(n-r+1)\,\mu'_{(r-1)} = 0.$$

(iii) When X has the hypergeometric distribution with probability mass function,

$$f(x) = \frac{\binom{M}{x}\binom{N-M}{n-x}}{\binom{N}{n}},\ x = 0, 1, 2, \ldots \min(M,n),\ n-N+M \le x \le \min(M, n).$$

Comparing this with the difference equation for the Ord's family, we get,

$$a = \frac{Mn+M+n+1}{N+2},\ b_0 = 0,\ b_1 = \frac{N-M-n+1}{N+2},\ b_2 = (N+2)^{-1},$$

so that the recurrence relation becomes,

$$(N+1-r)\alpha_{(r+1)}(t) + [N(t+r)+(r-1)(M+n+1) - (M\,n+ M +n+1)]\alpha_{(r)}(t)$$

$$-\ r[M\,n+ M +n+1) +(t+r)(3t+3r- M -n)]\alpha_{(r-1)}(t) = 0.$$

Some comments seem to be in order. The MPSD and Ord's family overlap in terms of members of the families, e.g. binomial, Poisson etc. But the recurrence relations developed here are different in both the cases as could be seen from the particular cases derived in

Section 4.3 and 4.4. Further a separate consideration for each family is also essential because there are members in one family that do not belong to the other. For example, discrete t distribution in Ord's family and Legrangian distribution in MPSD.

4.5 Characterizations

The definition and some reliability characteristics of equilibrium distributions were discussed in the previous chapters. In this section we characterize the geometric, Waring and negative hypergeometric distributions in terms of $\alpha_{(r)}(t)$ and the factorial partial moments $\beta_{(r)}(t)$ of the equilibrium distributions.

The r^{th} factorial partial moment of the equilibrium distribution about t is obtained from (2.21)

$$\beta_{(r)}(t) = \mu^{-1} \sum_{t+r}^{\infty}(x-t)^{(r)} R(x+1). \tag{4.35}$$

From (4.7) and (4.35) we get,

$$\alpha_{(r+1)}(t) = \mu(r+1)\ \beta_{(r)}(t). \tag{4.36}$$

Theorem 4.8

Let X be a discrete random variable in the support of I^{+} such that $E(X^{r})<\infty$. Then X has

(a) geometric law with probability mass function (1.5) if and only if for every r

$$\alpha_{(r)}(t) = \beta_{(r)}(t) \tag{4.37}$$

(b) Waring distribution with probability mass function (1.6) if and only if for every r.

$$\alpha_{(r)}(t) = \frac{a-b-r-1}{b+t+r}\ \mu\ \beta_{(r)}(t),\ a>b+r+1 \tag{4.38}$$

(c) negative hypergeometric law with probability mass function (1.7) if and only if for every r

$$\alpha_{(r)}(t) = \frac{k+r+1}{n-t-r}\ \mu\ \beta_{(r)}(t) \tag{4.39}$$

Proof: (a) For the geometric distribution

$$\alpha_{(r)}(t) = r!\ \frac{q^{t+r}}{p^{r}}.$$

By direct calculation the distribution of equilibrium distribution is again geometric with parameter p and hence

$$\beta_{(r)}(t) = r!\ \frac{q^{t+r}}{p^{r}}.$$

giving relation (4.37).

To prove the converse, assume (4.37).

Substituting (4.37) in (4.36),

$$\alpha_{(r+1)}(t) = \mu\ (r+1)\ \alpha_{(r)}(t)$$

or

$$\alpha_{(r)}(t) = \mu\, r\, \alpha_{(r-1)}(t)$$

$$= \mu^2\, r(r-1)\, \alpha_{(r-2)}(t).$$

Similar iteration on r gives,

$$\alpha_{(r)}(t) = \mu^r\, r!\, \alpha_{(0)}(t).$$

$$= \mu^r\, r!\, R(t+1). \tag{4.40}$$

Substituting (4.40) in (4.8), we get

$$\mu^r\, r!\, R(t) - \mu^r\, r!\, R(t+1) = r\, \mu^{r-1}\, (r-1)!\, R(t+1).$$

or

$$\frac{R(t+1)}{R(t)} = \frac{\mu}{\mu+1}.$$

Iteration on t gives,

$$R(t) = \left(\frac{\mu}{\mu+1}\right)^t$$

and hence the distribution is geometric.

(b) For the Waring distribution,

$$\alpha_{(r)}(t) = r!\, \frac{(b+t)_r}{(a-b-1)^{(r)}}\, \frac{(b)_t}{(a)_t}.$$

By direct calculation, the distribution of the equilibrium distribution is again Waring with parameters a and $b+1$ and hence

$$\beta_{(r)}(t) = r!\, \frac{(b+t+1)_r}{(a-b-2)^{(r)}}\, \frac{(b+1)_t}{(a)_t}.$$

so that

$$\alpha_{(r)}(t) = \frac{a-b-r-1}{b+t+r}\mu\beta_{(r)}(t),$$

where

$$\mu = \frac{b}{a-b-1}.$$

To prove the converse assume (4.38).

Substituting (4.38) in (4.36) we get,

$$\alpha_{(r+1)}(t) = \frac{(b+t+r)(r+1)}{(a-b-r-1)}\alpha_{(r)}(t)$$

$$\alpha_{(r)}(t) = \frac{(b+t+r)r}{(a-b-r)}\alpha_{(r-1)}(t).$$

Similar iteration on r gives,

$$\alpha_{(r)}(t) = r!\,\frac{(b+t)_r}{(a-b-1)^{(r)}}R(t). \tag{4.41}$$

Substituting (4.41) in (4.8), we get,

$$r!\,\frac{(b+t)_r}{(a-b-r)^{(r)}}R(t) - r!\,\frac{(b+t+1)_r}{(a-b-1)^{(r)}}R(t+1) = r(r-1)!\,\frac{(b+t)_{r-1}}{(a-b-1)^{(r-1)}}R(t+1).$$

or

$$\frac{R(t+1)}{R(t)} = \frac{b+t}{a+t}.$$

Iteration on t gives,

$$R(t) = \frac{(b)_t}{(a)_t}$$

and hence the distribution is Waring.

(c) For the negative hypergeometric distribution,

$$\alpha_{(r)}(t) = \frac{r!}{\binom{k+n}{n}} \binom{k+n-t}{n-t-r}.$$

The equilibrium distribution also has the negative hypergeometric distribution with parameters $k+1$ and $n-1$.

$$\beta_{(r)}(t) = \frac{r!}{\binom{k+n}{n-1}} \binom{k+n-t}{n-t-r-1}$$

and hence

$$\alpha_{(r)}(t) = \frac{k+r+1}{n-t-r} \mu \beta_{(r)}(t),$$

where

$$\mu = \frac{n}{k+1}.$$

Conversely assume that (4.39) holds.

Substituting (4.39) in (4.36) we get,

$$\alpha_{(r+1)}(t) = \frac{n-t-r}{k+r+1} (r+1)\ \alpha_{(r)}(t)$$

$$\alpha_{(r)}(t) = \frac{n-t-r+1}{k+r}\ r\ \alpha_{(r-1)}(t)$$

Iteration on r gives,

$$\alpha_{(r)}(t) = r! \frac{(n-t-r+1)_r}{(k+r)^{(r)}} R(t) \tag{4.42}$$

Substituting (4.42) in (4.8) we get

$$r!\,\frac{(n-t-r+1)_r}{(k+r)^{(r)}}R(t) - r!\,\frac{(n-t-r)_r}{(k+r)^{(r)}}R(t+1) = r(r-1)!\,\frac{(n-t-r+1)_{r-1}}{(k+r-1)^{(r-1)}}R(t+1).$$

Proceeding as in the above cases, we get

$$R(t) = \frac{\binom{k+n-t}{n-t}}{\binom{k+n}{n}}$$

and the distribution is negative hypergeometric.

4.6 Application to reliability modelling

In the present section we point out the application of partial moments in modelling life time data.

We can see that from equations (1.2) and (1.8), giving the definitions of failure rate $h(x)$ and MRLF $r(x)$ that they can be expressed in terms of partial means in a straight forward manner, from equation (4.6) and (4.5),

$$1 - h(x+1) = \frac{\alpha_1(x+1) - \alpha_1(x+2)}{\alpha_1(x) - \alpha_1(x+1)} \tag{4.43}$$

and

$$r(x) = \frac{\alpha_1(x)}{\alpha_1(x) - \alpha_1(x+1)} \tag{4.44}$$

or

$$\frac{\alpha_1(x+1)}{\alpha_1(x)} = \frac{r(x)-1}{r(x)} \tag{4.45}$$

Equation (4.43) and (4.44) can be used for translating all characterization theorems given by properties of failure rate and mean residual life time in terms of partial means. As an illustration we prove the following theorem.

Theorem 4.9

The ratio of partial means of a discrete random variable in the support of I^+ with $E(X) < \infty$ is of the bilinear form

$$a(x) = \frac{\alpha_1(x+1)}{\alpha_1(x)} = \frac{(A-1)+Bx}{A+Bx}, \; A>1$$

if and only if the distribution is geometric for B=0, Waring for $B>0$ or negative hypergeometric for $B<0$.

Proof: The Theorem follows from the result of Nair and Hitha (1989). A discrete random variable with support I^+ has geometric ($B=0$) or Waring ($B>0$) or negative hypergeometric distribution ($B<0$) if and only if $r(x) = A + Bx$.

In a similar manner one can deduce the functional forms of $\alpha_1(x)$ corresponding to known functional forms of $h(x)$ and $r(x)$ that generate various well known distributions. Like the functions $h(x)$ and $r(x)$, the sequence of values of $\alpha_1(x)$ also determines the distribution of X uniquely. $\alpha_1(x)$ also enjoy most of the properties alike to those of $h(x)$ and $r(x)$. Thus for all practical purposes, $\alpha_1(x)$ can be used in the place of the other two functions in modelling lifetime data. Apart from this equivalence with the other two concepts for all practical purposes, partial means enjoy one additional advantage. The failure rate and the mean residual life when estimated from sample data, are in the form of ratios of random variables and therefore, it is very difficult for most populations to calculate their sampling variabilities in terms of standard errors or confidence intervals. On the other hand, the sample counterpart of partial mean is not a ratio and therefore do not suffer from the above handicap. This is an important reason that motivates the study of partial moments instead of truncated moments.

We now give an illustration to the fact that partial moments can be equally useful in describing aging properties.

In the next theorem we study the implications of the connection between ratio of partial means and MRL given in (4.44) towards describing aging behaviour.

Theorem 4.10

A necessary and sufficient condition that X has increasing (decreasing) mean residual life is that $\frac{\alpha_1(x+1)}{\alpha_1(x)}$ is non-decreasing (non-increasing) in x.

Proof: From (4.44) it follows that

$$\frac{1}{r(x)} - \frac{1}{r(x+1)} \geq 0 \ (\leq 0) \Leftrightarrow \frac{\alpha_1(x+2)}{\alpha_1(x+1)} - \frac{\alpha_1(x+1)}{\alpha_1(x)} \geq 0 \ (\leq 0)$$

or X has increasing (decreasing) mean residual life if and only if $\frac{\alpha_1(x+1)}{\alpha_1(x)}$ is non-decreasing (non-increasing) in x.

It is well known that if X has increasing failure rate then this implies that X has decreasing mean residual life. In the next theorem we investigate the additional property along with DMRL that ensures a distribution to be IFR.

Theorem 4.11

If X has DMRL and the ratio $\frac{r(x+1)}{r(x)}$ is decreasing then X has IFR.

Proof: From (4.43) and (4.45) we see that,

$$1 - h(x+1) = \frac{\alpha_1(x+1)}{\alpha_1(x)} \frac{r(x)}{r(x+1)}. \tag{4.46}$$

The result follows from (4.46) and Theorem 4.10.

The last two Theorems are intended only for illustration purpose and similar exposition of many well known result can be translated in terms of partial means.

CHAPTER V

ASCENDING FACTORIAL PARTIAL MOMENTS

5.1 Introduction

In the previous chapter we have discussed some important properties of partial moments and characterizations based on them relating to families and individual distributions. These were based on descending factorial expressions. A similar treatment with appropriate modifications can be attempted by replacing descending factorial by ascending factorials. Almost all the results we have encountered in chapter IV for descending factorial moments will work out for the ascending ones also. To illustrate the role of ascending factorial partial moments we define them and discuss some results which are not parallel to those in the previous chapter.

5.2 Definition and properties

Let X be a discrete random variable in the support of I^{+} with probability mass function $f(x)$ and survival function $R(x)$ and finite moments of order r, $r = 1, 2, \ldots$. Then the r^{th} ascending factorial moment of X about a positive integer t is defined as

$$m_{(r)}(t) = \mathrm{E}[(X-t)^+]^{(r)},\ r = 1, 2, \ldots, \tag{5.1}$$

$$= \sum_{t+1}^{\infty} (x-t)^{(r)} f(x)$$

where,

$$(X-t)^+ = \begin{cases} X-t & X > t \\ 0 & X \le t \end{cases}$$

and (5.2)

$$X^{(r)} = X(X+1)\ \ldots\ (X+r-1).$$

From the definition, it follows that

$$m_1(t) = m_{(1)}(t) = \sum_{t+1}^{\infty} R(x) \tag{5.3}$$

$$R(t+1) = m_1(t) - m_1(t+1) \tag{5.4}$$

$$f(t+1) = m_1(t) - 2\, m_1(t+1) + m_1(t+2) \tag{5.5}$$

The expression (5.3) through (5.5) enable the evaluation of the probability mass function and survival function in terms of the partial means. It is seen that the probability mass function is completely determined from the sequence of partial means $\langle m_1(x) \rangle$, x=0, 1, 2, ...

From the equation (5.3) and (5.4), we can see that $m_1(t)$ is related to MRL function $r(t)$ by,

$$r(t) = \frac{m_1(t)}{m_1(t) - m_1(t+1)}. \tag{5.6}$$

The relation between $m_1(t)$ and failure rate is,

$$1 - h(t+1) = \frac{m_1(t+1) - m_1(t+2)}{m_1(t) - m_1(t+1)}, \tag{5.7}$$

We now establish some properties of partial moments,

Theorem 5.1

The partial moments satisfies the recurrence relation,

$$m_{(r)}(t) - m_{(r)}(t+1) = r\, m_{(r-1)}(t), \qquad t \geq 0,\ r \geq 1. \tag{5.8}$$

Proof: From (5.1), we have

$$m_{(r)}(t) - m_{(r)}(t+1) = \sum_{t+1}^{\infty}(x-t)(x-t+1)\ldots(x-t+r-1)\, f(x)$$

$$- \sum_{t+2}^{\infty}(x-t-1)(x-t)\ldots(x-t+r-2)\, f(x)$$

$$= \sum_{t+1}^{\infty}(x-t)\ (x-t+1)\ldots(x-t+r-2)\, f(x)[(x-t+r-1)-(x-t-1)]$$

$$= r\sum_{t+1}^{\infty}(x-t)\ (x-t+1)\ldots(x-t+r-2)\, f(x)$$

$$= r\, m_{(r-1)}(t).$$

From (5.8) it is seen that, once $m_{(r)}(t)$ is known for two consecutive values of t, the corresponding lower order moments can be determined.

The next theorem gives a method of determining the distribution from the knowledge of partial moments.

Theorem 5.2

Let X be a non-negative integer valued random variable such that $E(X^r)<\infty$. Then for any positive integer r, $m_{(r)}(t)$ determines all the factorial moments that exist.

Proof: From the relationship (5.8), we have

$$m_{(r-1)}(t) = \frac{1}{r}[\, m_{(r)}(t) - m_{(r)}(t+1)]$$

which provides

$$m_{(r-1)}(t) = \frac{1}{r}(1-E)m_{(r)}(t)$$

where E is the usual forward shift operator.

Similarly,

$$m_{(r-2)}(t) = \frac{1}{r-1}(1-E)m_{(r-1)}(t)$$

$$= \frac{1}{r(r-1)}\,(1-E)^2\, m_{(r)}(t).$$

Similar iteration or r gives

$$m_{(r-s)}(t) = \frac{1}{r^{(s)}}(1-E)^s\, m_{(r)}(t),\ s = 1, 2, \ldots, (r-1),\ r>s. \tag{5.9}$$

On the other hand from (5.8),

$$m_{(r+1)}(t) = (r+1)\ (1\text{-}E)^{-1}\ m_{(r)}(t),$$

Similarly,

$$m_{(r+2)}(t) = (r+2)\ (1\text{-}E)^{-1}\ m_{(r+1)}(t),$$
$$= (r+1)(r+2)\ (1\text{-}E)^{-2}\ m_{(r)}(t),$$

Iteration on r gives,

$$m_{(r+s)}(t) = \frac{(r+s)!}{r!}\ (1\text{-}E)^{-s}\ m_{(r)}(t),\ s = 1, 2, \ldots \qquad (5.10)$$

We can verify Theorem 5.2 for geometric and Waring distributions.

Example 5.1

For geometric distribution with probability mass function (1.5), we obtain by direct calculation,

$$m_1(t) = q^{t+1}\ p^{-1}$$

and from (5.10), we have

$$m_{(2)}(t) = (1\text{-}E)^{-1}\ 2!\ m_1(t)$$
$$= (1\text{-}E)^{-1}\ 2!\ q^{t+1}\ p^{-1}$$
$$= 2!\ p^{-1}(1+E+E^2+\ldots)\ q^{t+1}$$
$$= 2!\ q^{t+1}\ p^{-2}.$$

Example 5.2

For the Waring distribution with probability mass function (1.6), we obtained by direct calculation

$$m_1(t) = \frac{(b+t)}{(a-b-1)}\frac{(b)_t}{(a)_t}.$$

Then from (5.10) we get

$$m_{(2)}(t) = 2!\ p(1+E+E^2+\dots)\frac{(b+t)}{(a-b-1)}\frac{(b)_t}{(a)_t}$$

$$= \frac{2!}{(a-b-1)}\left[(b+t)\frac{(b)_t}{(a)_t} + (b+t+1)\frac{(b)_{t+1}}{(a)_{t+1}} + (b+t+2)\frac{(b)_{t+2}}{(a)_{t+2}} + \dots\right]$$

$$= \frac{2!}{(a-b-1)}(b+t)\frac{(b)_t}{(a)_{t-1}}\left[\frac{1}{a+t-1} + \frac{b+t+1}{(a+t-1)(a+t)} + \dots\right]$$

$$= \frac{2!}{(a-b-1)}(b+t)\frac{(b)_t}{(a)_{t-1}}\ \frac{1}{(a-b-2)}$$

$$= \frac{2!(b+t)(a+t-1)(b)_t}{(a-b-1)(a-b-2)(a)_t}.$$

In the next theorem we prove that the r^{th} factorial partial moment $m_{(r)}(t)$ determines the distribution completely for any positive integer r.

Theorem 5.3

Assume that E(X) is finite. Then for any positive integer r, $m_{(r)}(t)$determines the distribution completely.

Proof: From the equation (5.9), taking s=r-1, we get

$$m_1(t) = \frac{(1-E)^{(r-1)}}{r!}\ m_{(r)}(t) \tag{5.11}$$

From the equation (5.4) we see that the sequence $\langle m_1(t)\rangle$, t=0,1,2,... determines the sequence $\langle f_1(t)\rangle$, t = 1, 2, ... and this along with the condition $R(0)=1$ gives $\langle f_1(t)\rangle$ for t = 0,1,2, ... Thus the distribution is completely determined by the sequence $\langle m_1(t)\rangle$.

The above theorem has wide applications in modelling statistical data. It is well known that MRL, $r(t)$ determines the distribution uniquely. However, partial moment $m_r(t)$ is superior to $r(t)$ in the sense that $m_r(t)$ accounts for the mean of $(X - t)^+$ only while $r(t)$ measures the average of $X\text{-}t \mid X>t$.

Remark

From the definition of $m_{(r)}(t)$ we have $m_{(r)}(t) \geq 0$, r = 1, 2, ... and thus the relationship (5.8) proves $m_{(r)}(t) \geq m_{(r)}(t+1)$, t = 0, 1, 2, ... This means that $m_{(r)}(t)$ is non-increasing in t. However, it is easy to verify that $m_{(r)}(t)$ is a non-decreasing function in r, for a fixed t.

5.3 Characterizations

In this section, we present some characterizations of some discrete models by properties of ratios of factorial partial moments.

Theorem 5.4

The relationship

$$\frac{m_{(r)}(t+1)}{m_{(r)}(t)} = c, \; 0<c<1 \tag{5.12}$$

where c is a constant is satisfied for every t = 0,1, 2, ... and r= 1, 2, ... if and only if the distribution of X is geometric.

Proof: For the geometric distribution, by direct calculation we get

$$m_{(r)}(t) = r!q^{t+r}\, p^{-r}$$

$$\frac{m_{(r)}(t+1)}{m_{(r)}(t)} = \frac{r!q^{r+t+1}p^{-(r+1)}}{r!q^{r+1}p^{-r}}$$

$$= q, \text{ a constant.}$$

Thus (5.12) is verified.

Conversely, assume that (5.12) holds.

Substituting (5.12) in (5.8) we get

$$m_{(r)}(t) - c\; m_{(r)}(t) = r\; m_{(r-1)}(t)$$

or

$$\frac{m_{(r-1)}(t)}{m_{(r)}(t)} = \frac{1-c}{r}$$

Iteration on r gives,

$$m_{(r)}(t) = \frac{r!}{(1-c)^{r-1}} \sum_{t+1}^{\infty} R(x). \tag{5.13}$$

Substituting (5.13) in (5.8), we get,

$$\frac{r!}{(1-c)^{r-1}} \sum_{t+1}^{\infty} R(x) - \frac{r!}{(1-c)^{r-1}} \sum_{t+2}^{\infty} R(x) = \frac{r(r-1)!}{(1-c)^{r-2}} \sum_{t+1}^{\infty} R(x).$$

or

$$R(t+1) = (1-c) \sum_{t+1}^{\infty} R(x)$$

or

$$r(t) = \frac{1}{1-c}. \tag{5.14}$$

The equation (5.14) characterizes the geometric distribution (1.5), with $p = 1-c$.

Theorem 5.5

The distribution of X is Waring with probability mass function (1.6), if and only if for $r = 1, 2, \ldots$ and $t = 0, 1, 2, \ldots$

$$\frac{m_{(r)}(t+1)}{m_{(r)}(t)} = \frac{A+t}{B-r+t} \text{ with } A = b+1,\ B = a+1. \tag{5.15}$$

Proof: For the Waring distribution with probability mass function (1.6), we get by direct calculation

$$m_{(r)}(t) = r!\ \frac{(b+t)(a+t-1)(a+t-2)\ldots(a+t-r+1)(b)_t}{(a-b-1)(a-b-2)\ldots(a-b-r)(a)_t}$$

So

$$\frac{m_{(r)}(t+1)}{m_{(r)}(t)} = \frac{b+t+1}{a+t-r+1} = \frac{A+t}{B-r+t}.$$

Thus (5.15) is verified.

Conversely, suppose that (5.15) holds. Substituting $m_{(r)}(t+1)$ in terms of $m_{(r)}(t)$ in the relationship (5.8), we get

$$m_{(r)}(t) - \frac{A+t}{B-r+t}\, m_{(r)}(t) = r\, m_{(r-1)}(t)$$

which provides

$$m_{(r)}(t) = r!\ \frac{B-r+t}{B-A-r}\ \frac{B-r+t-1}{B-A-r+1} \cdots \frac{B+t-1}{B-A-1} R(t+1). \tag{5.16}$$

Applying the identity (5.16) in (5.15), we get

$$R(t+2) = \frac{A+t}{B+t}\ R(t+1).$$

Iteration on t gives

$$R(t) = \frac{A+t-2}{B+t-2}\ \frac{A+t-3}{B+t-3} \cdots \frac{A-1}{B-1} R(0). \tag{5.17}$$

Since $R(0) = 1$, (5.17) gives,

$$R(t) = \frac{(A-1)_t}{(B-1)_t} = \frac{(b)_t}{(a)_t}.$$

Hence the proof is complete

In the next theorem we have a characterization for the generalized power series family of distributions using factorial partial moments.

Theorem 5.6

The distribution of X belongs to the generalized power series family with p.m.f

$$f(x) = \frac{a(x)\theta^x}{g(\theta)}, \quad x = 0,\ 1,\ 2,\ \dots \tag{5.18}$$

where $a(.)$ is a non-negative function of X and $g(\theta) = \sum_x a(x)\theta^x$ if and only if the factorial partial moments satisfy the relationship

$$\theta \frac{\partial m_{(r)}(t)}{\partial\theta} = m_{(r+1)}(t+1) + (t+1-\mu)\ m_{(r)}(t) \tag{5.19}$$

for every $t = 0,\ 1,\ 2\ \dots$ and $r = 1,\ 2,\ \dots$ where $\mu = \mathrm{E}(X)$.

Proof: When X has the distribution (5.18), then

$$m_{(r)}(t) = \sum_{t+1}^{\infty}(x-t)\dots(x-t+r-1)\ \frac{a(x)\theta^x}{g(\theta)} \tag{5.20}$$

Differentiating (5.20) with respect to θ, we get

$$\frac{\partial m_{(r)}(t)}{\partial\theta} = \sum_{t+1}^{\infty}(x-t)\dots(x-t+r-1)\frac{a(x)x\theta^{x-1}}{g(\theta)}$$

$$- \sum_{t+1}^{\infty}(x-t)\dots(x-t+r-1)\ \frac{g'(\theta)}{[g(\theta)]^2}a(x)\theta^x$$

$$= \theta^{-1}[(t+1-\mu)\ m_{(r)}(t) + m_{(r+1)}(t+1)]$$

since

$$\mu = \frac{\theta g'(\theta)}{g(\theta)}.$$

Thus

$$\theta \frac{\partial m_{(r)}(t)}{\partial \theta} = m_{(r+1)}(t+1) + (t+1-\mu)\, m_{(r)}(t).$$

Conversely, if (5.19) holds,

$$\theta \sum_{t+1}^{\infty} (x-t)\ldots(x-t+r-1)\ \frac{\partial f(x)}{\partial \theta}$$

$$= \sum_{t+1}^{\infty} (x-t-1)\ldots(x-t+r-1)\ f(x) + (t+1-\mu) \sum_{t+1}^{\infty} (x-t)\ldots(x-t+r-1) f(x)$$

which gives

$$\theta \sum_{t+1}^{\infty} (x-t)\ldots(x-t+r-1) \frac{\partial f(x)}{\partial \theta} = \sum_{t+1}^{\infty} x(x-t)\ldots(x-t+r-1)\ f(x)$$

$$- \mu \sum_{t+1}^{\infty} (x-t)\ldots(x-t+r-1) f(x). \qquad (5.21)$$

Changing t to $t+1$ in (5.21), we get

$$\theta \sum_{t+2}^{\infty} (x-t)\ldots(x-t+r-1) \frac{\partial f(x)}{\partial \theta} = \sum_{t+2}^{\infty} x(x-t)\ldots(x-t+r-1)\ f(x)$$

$$- \mu \sum_{t+2}^{\infty} (x-t)\ldots(x-t+r-1) f(x) \qquad (5.22)$$

Subtracting (5.22) from (5.21), we get,

$$\frac{\partial f(t+1)}{\partial \theta} = \theta^{-1}\ (t+1-\mu) f(t+1)$$

or

$$\frac{1}{f(t+1)} \frac{\partial f(t+1)}{\partial \theta} = \theta^{-1}\ (t+1-\mu).$$

Since this equation is true for all t, we can write

$$\frac{\partial \log f(t)}{\partial \theta} = \frac{t-\mu}{\theta}.$$

Integrating with respect to θ,

$$\log f(t) = (t-\mu) \log \theta + \int \frac{d\mu}{d\theta} \log \theta \, d\theta + \log a(t).$$

$$f(t) = a(t)\ \theta^{\,t-\mu} \exp\{ \int \frac{d\mu}{d\theta} \log \theta \, d\theta \}$$

$$= \frac{a(x)\theta^{x}}{g(\theta)}$$

where $g(\theta) = \theta^{\mu} \exp\{-\int \frac{d\mu}{d\theta} \log \theta \, d\theta\}$.

This completes the proof.

We will illustrate the result for some specific distributions. Table 5.1 provides the relationship (5.19) for some popular models belonging to the Generalized power series family.

In the next theorem we give a characterization for the Poisson distribution using a recurrence relation among factorial partial moments.

Theorem 5.7

The identity,

$$m_{(r+1)}(t+1) = (r-t-1)\ m_{(r)}(t+1) + \lambda\ m_{(r)}(t),\ \lambda > 0. \qquad (5.23)$$

Table 5.1

Model	Probability mass function	Recurrence relationship
Binomial	$\binom{n}{x} p^x q^x$, $x = 0, 1, 2, \ldots, n.$	$p\,(1\text{-}p)\,\frac{\partial m_{(r)}(t)}{\partial p} = m_{(r+1)}(t+1) + (t+1\text{-}np)\, m_{(r)}(t)$
Poisson	$\frac{e^{-\lambda}\lambda^x}{x!}$, $x = 0, 1, 2, \ldots, \lambda>0$	$\lambda\,\frac{\partial m_{(r)}(t)}{\partial \lambda} = m_{(r+1)}\,(t+1) + (t+1\text{-}\,\lambda)\, m_{(r)}(t)$
Negative Binomial (Geometric for $r=1$)	$\binom{x+r-1}{r-1} p^r q^x$, $x = 0, 1, 2, \ldots, n.$	$q\,\frac{\partial m_{(r)}(t)}{\partial q} = m_{(r+1)}(t+1) + \left(t+1-\frac{rq}{p}\right) m_{(r)}(t)$
Logarithmic	$\frac{a\theta^x}{x}$, x=0, 1, 2, …, $a = \frac{-1}{\log(1-\theta)}$	$\theta\,\frac{\partial m_{(r)}(t)}{\partial \theta} = m_{(r+1)}\,(t+1) + \left(t+1-\frac{a\theta}{1-\theta}\right) m_{(r)}(t)$

is satisfied for every t = 0, 1, 2, ... and r = 1, 2,... if and only if the distribution of X is Poisson with mean λ.

Proof: When X has Poisson distribution with mean λ, we have

$$m_{(r+1)}(t+1) = \sum_{t+2}^{\infty}(x-t-1)(x\text{-}t)\ldots(x\text{-}t+r\text{-}1)\ \frac{\lambda^{x}e^{-\lambda}}{x!}$$

$$= (r\text{-}t\text{-}1)\sum_{t+2}^{\infty}(x-t-1)(x\text{-}t)\ldots(x\text{-}t+r\text{-}2)\ \frac{\lambda^{x}e^{-\lambda}}{x!}$$

$$+\ \lambda\ \sum_{t+2}^{\infty}(x-t-1)(x\text{-}t)\ldots(x\text{-}t+r\text{-}2)\ \frac{\lambda^{x-1}e^{-\lambda}}{(x-1)!}$$

$$= (r\text{-}t\text{-}1)\ m_{(r)}(t+1) + \lambda\ m_{(r)}(t).$$

Hence (5.23) is verified.

Conversely suppose that (5.23) holds. Then we get

$$\sum_{t+2}^{\infty}(x-t-1)(x-t)\ldots(x-t+r-1)\ f(x)$$

$$= (r\text{-}t\text{-}1)\sum_{t+2}^{\infty}(x-t-1)(x-t)\ldots(x-t+r-2)\ f(x)$$

$$+\ \lambda\ \sum_{t+2}^{\infty}(x-t-1)(x-t)\ldots(x-t+r-2)\ f(x\text{-}1)$$

or

$$\sum_{t+2}^{\infty}x(x-t-1)(x-t)\ldots(x-t+r-2)\ f(x)$$

$$-\ \lambda\ \sum_{t+2}^{\infty}(x-t-1)(x-t)\ldots(x-t+r-2)\ f(x\text{-}1)=0. \qquad (5.24)$$

Changing t to t+1, we get

$$\sum_{t+3}^{\infty} x(x-t-1)(x\text{-}t)\ \ldots(x\text{-}t\text{+}r\text{-}2)\,f(x)$$

$$-\ \lambda \sum_{t+3}^{\infty}(x-t-1)(x\text{-}t)\ \ldots(x\text{-}t\text{+}r\text{-}2)\,f(x\text{-}1)=0 \qquad (5.25)$$

Subtracting (5.25) from (5.24), we get

$$(t\text{+}2)f(t\text{+}2)\ \text{-}\ \lambda\ f(t\text{+}1) = 0. \qquad (5.26)$$

The equation (5.26) is true for all t. So we can write,

$$t\,f(t)\ \text{-}\ \lambda\ f(t\text{-}1) = 0. \qquad (5.27)$$

The solution of (5.27) is given by

$$f(t) = \frac{\lambda^t}{t!} f(0). \qquad (5.28)$$

Since $\sum f(t)$=1, (5.28) provides $f(0) = e^{-\lambda}$ and thus the distribution of X is Poisson. Some of the result in this chapter is due to appear in Priya, Sankaran and Nair (2000).

5.4 Conclusion

The present study established some results that are useful in modelling and analysis of life time treated as a discrete random variable. Arising from the study of mixture distributions, it was illustrated in the geometric case that characterization connecting mean residual life and failure rate could be used in the identification of the model as well as in inferring the parameters.

However the idea concieved here, could not be extended to other mixture models. Attempts are being made to ensure that characterizations of this type are indeed useful as a general tool by finding more illustrations. Although some indications as to the applications of partial moments in reliability analysis was made, a more indepth study of its advantages vis-a-vis truncated moments which are currently employed seems worth investigating. Identification of specific members of families, through some indices formed out of parameters is also a problem under consideration. It is hoped that some results in these directions can be reported in a future work.

REFERENCES

1. Ahmed, A.N. (1991), Characterization of beta binomial and Poisson distributions, *IEEE Transactions on reliability*, 40, 290-295.

2. Barlow, R.E. and Proschan, F (1975), *Statistical theory of reliability and life testing, Probability Models*, Holt, Rinehart and Winston, Inc. New York.

3. Chong, K.M. (1977), On characterizations of exponential and geometric distribution by expectation, *Journal of the American Statistical Association*, 72, 160-161.

4. Cox, D.R. (1959), The analysis of exponentially distributed life time data with two types of failure, *Journal of Royal Statistical Society*, B 21, 411-421.

5. Cox. D.R. (1972), Regression models and life tables, *Journal of Royal Statistical Society*, B 34, 187-220.

6. Despande, J.V., Kochar, S.C. and Singh, H. (1986), Aspects of positive ageing, *Journal of Applied probability*, 23, 748-758.

7. Ebrahami, N. (1986), Classes of discrete decreasing and increasing mean residual life distributions, *IEEE Transactions on reliability*, 35, 403-405.

8. Everitt, B.S and Hand, D.J. (1981), *Finite mixture distributions*, Chapman and Hall, London.

9. Fowlkes, E.B. (1979), Some methods of studying the mixture of two normal (lognormal) distributions, *Journal of the American Mathematical Association*, 74, 561-575.

10. Guess, F.M. and Park, D.H. (1988), Modelling discrete bath tub and upside down bath tub mean residual life functions, *IEEE Transactions on reliability*, 35, 545-549

11. Gupta, R.C. (1974), Modified power series distributions and some of its applications, *Sankhya Series B*, 35, 288-298.

12.Gupta, R.C. (1979), On the characterization of survival distributions in reliability by properties of their renewal densities, *Communications in Statistics– Theory and Methods*, 8(7), 685-697.

13.Gupta,P.L. and Gupta, R.C.(1983), On the moments of residual life in reliability and some characterization results, *Communications in Statistics- Theory & Methods*, 12(4), 449-461.

14.Harris, C.M. (1983), On finite mixtures of geometric and negative binomial distributions, *Communications in Statistics- Theory & Methods*, 12, 987-1000.

15.Hitha, N. (1991), *Some characterizations of Pareto and related populations*, Unpublished Ph.D. Thesis, Cochin University of Science and Technology.

16.Irwin, J.O. (1963), The place of mathematics in medical and biological statistics, *Journal of Royal Statistical Society*, A 126, 1-44.

17.Johnson, N.L., Kotz, S and Kemp, A.W. (1992), *Univariate discrete distributions*, Second edition, John Wiley and Sons, Inc, New York.

18.Kalbfleisch, J.D. and Prentice, R.L. (1980), *The Statistical Analysis of Failure Time Data*, John wiley, New York.

19.Kao, J.H.K. (1959), A graphical estimation of mixed Weibull parameters in life testing of electronic tubes, *Technometrics*, 1, 389-407.

20.Kemp, C.D. and Kemp, A.W. (1968), On a distribution associated with certain stochastic processes, *Journal of Royal Statistical Society*, B 30, 401-410.

21.Kupka, J and Loo,S (1989), The hazard and vitality measures of ageing, *Journal of Applied Probability*, 26, 532-542.

22.Klefsjo, B. (1982), The HNBUE and HNWUE classes of life distributions, *Naval research Logistics Quarterly*, 29, 331-344.

23.Lawless, J.F. (1982), *Statistical models and methods for life time data*, John wiley, New York.

24.Mendenhall, W and Hader, R.J. (1958), Estimation of parameters of a mixed exponential failure time distribution from censored life test data, *Biometrika*, 45, 504-520.

25.Mendelbaum, J and Harris, C.M. (1982), Parameter estimation under progressive censoring conditions for a finite mixture of Weibull distributions, *TIMS/Studies in Management Sciences*, 19, 239-260. (In '*Statistical analysis of finite mixture distributions*' by Titterington, Smith, Markov (1985), John Wiley and Sons, Ltd.)

26.Mi, J. (1994), Discrete bath tub failure rate and upside down bath tub mean residual life, *Naval research Logistics Quarterly*, 40, 361-371.

27.Nair, N.U. (1983), A measure of memory for some discrete distributions, *Journal of the Indian Statistical Association*, 21, 141-147.

28.Nair, N.U. (1989), Characterization of failure models by properties of their residual life. *Proceedings of the Symposium on reliability and renewal theory*, 1-9.

29.Nair, N.U., Geetha, K.G. and Priya, P. (1999), Modelling life length data using mixture distributions, *Journal of Japanese Statistical Society*, 29(1), 89-98.

30.Nair,N.U. and Hitha, N. (1989), Characterization of discrete models by distribution based on their partial sums, *Statistics and Probability Letters*, 8, 335-337.

31.Nair,N.U., Priya, P and Geetha,K.G. (2000), On partial moments of discrete distributions and their applications, *Far East Journal of Theoretical Statistics*, 4(1), 153-164.

32.Nair,N.U. and Sankaran,P.G. (1991), Characterization of the Pearson family of distributions, *IEEE Transactions on Reliability*, 40, 75-77.

33.Navarro,J., Franco, M. and Ruiz, J.M. (1998), *Characterization through moments of the residual life and conditional spacings*, Sankhya, Series A 60, 36-48.

34.Nelson, W. (1982), *Applied life data analysis*, John wiley, New York.

35.Neweichi, E.E. and Govindarajulu, Z. (1979), Characterization of geometric distribution and discrete IFR(DFR) distributions using order statistics, *Journal of Statistical Planning and Inference*, 3, 85-90.

36.Ord, J.K. (1967), On a system of discrete distributions, *Biometrika*, 54, 649-656.

37.Osaki, S. and Li, X. (1988), Characterization of gamma and negative binomial distributions, *IEEE Transactions on Reliability*, 37, 379-382.

38.Priya, P., Sankaran, P.G. and Nair, N.U. (2000), Factorial partial moments and their properties. (communicated)

39.Ruiz, J.M. and Navarro, J. (1994), Characterization of distributions by relationships between failure rate and mean residual life, *IEEE Transactions on Reliability*, 43, 640-644.

40.Roy, D and Gupta, R.P. (1992), Classifications of discrete lives, *Micro electron reliability*, 31, 1459-1473.

41.Shanbhag, D.N. (1970), The Characterizations for exponential and geometric distributions, *Journal of the American Statistical Association*, 65, 1256-1258.

42.Salvia, A.A. and Bollinger, R.C. (1982), On discrete hazard functions, *IEEE Transactions on Reliability*, 31(5), 458-459.

43.Salvia, A.A. (1996), Some results on discrete mean residual life, *IEEE Transactions on Reliability*, 45(3), 359-361.

44.Xekalaki,E. (1983) Hazard functions and life distributions in discrete time, *Communications in Statistics- Theory & Methods,* 12 (21), 2503-2509.

APPENDIX

Sample Size 100				Sample Size 200				Sample Size 400	
Sample 1		Sample 2		Sample 1		Sample 2			
x	$f(x)$	x	$f(x)$	x	$f(x)$	x	$f(x)$	x	$f(x)$
0	63	0	68	0	124	0	130	0	261
1	16	1	11	1	33	1	28	1	58
2	9	2	7	2	8	2	15	2	25
3	2	3	1	3	5	3	8	3	13
5	2	4	4	4	3	4	2	4	7
7	2	5	2	5	2	5	6	5	9
9	3	6	3	6	4	6	4	6	5
10	1	7	2	7	4	7	2	7	5
11	1	8	1	8	6	8	1	8	5
12	1	9	1	9	3	12	3	9	5
				11	2	13	1	10	2
				12	2			12	1
				13	2			14	1
				17	2			17	1
								20	1
								24	1
$\hat{p}_1 = 0.4$		$\hat{p}_1 = 0.42$		$\hat{p}_1 = 0.31$		$\hat{p}_1 = 0.39$		$\hat{p}_1 = 0.32$	
$\hat{p}_2 = 0.89$		$\hat{p}_2 = 0.82$		$\hat{p}_2 = 0.76$		$\hat{p}_2 = 0.88$		$\hat{p}_2 = 0.83$	
$\hat{p} = 0.6$		$\hat{p} = 0.7$		$\hat{p} = 0.6$		$\hat{p} = 0.73$		$\hat{p} = 0.66$	

Sample No.	$\hat{p}_1$	$\hat{p}_2$	$\hat{p}$	$(p_1 - \hat{p}_1)^2$	$(p_2 - \hat{p}_2)^2$	$(p - \hat{p})^2$
1	0.38	0.94	0.55	0.0009	0.0081	0.0225
2	0.37	0.92	0.68	0.0004	0.0049	0.0004
3	0.31	0.76	0.60	0.0016	0.0081	0.0100
4	0.34	1.01				
5	0.35	0.9	0.52	0	0.0025	0.0324
6	0.42	0.95	0.71	0.0049	0.0100	0.0001
7	0.39	0.94	0.57	0.0016	0.0081	0.0169
8	0.36	0.99	0.53	0.0001	0.0196	0.0289
9	0.39	0.88	0.73	0.0016	0.0009	0.0009
10	0.38	0.99	0.61	0.0009	0.0196	0.0081
MSE($\hat{p}_1$) = .00133, MSE($\hat{p}_2$) = 0.009089, MSE($\hat{p}$) = 0.01335						