STATISTICAL INFERENCE

BAYESIAN INFERENCE IN EXPONENTIAL AND PARETO POPULATIONS IN THE PRESENCE OF OUTLIERS

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By
JEEVANAND E. S.

DEPARTMENT OF MATHEMATICS AND STATISTICS
COCHIN UNIVERSITY OF SCIENCE AND TECHNOLOGY
COCHIN - 682 022

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CERTIFICATE

Certified that the thesis entitled "Bayesian Inference in Exponential and Pareto populations in the presence of Outliers" is a bonafide record of work done by Sri.E.S.Jeevanand under my guidance in the Department of Mathematics and Statistics, Cochin University of Science and Technology and that no part of it has been included anywhere previously for the award of any degree or title.

N. Unnikrishnan

Dr.N.Unnikrishnan Nair
Professor of Statistics

Cochin University of Science
and Technology
December 20,'93 :

DECLARATION

This thesis contains no material which has been accepted for the award of any other Degree or Diploma in any University and to the best of my knowldge and belief, it contains no material previously published by any other person, except where due references are made in the text of the thesis.

(JEEVANAND. E. S)

Cochin 22
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(JEEVANAND.E.S)

CONTENTS

CHAPTER I

PRELIMINARIES

1.1 Introduction

The origin of the concept of outliers in statistical data can be traced to the concern manifested by analysts in seemingly unrepresentative observations in a collection and the problems such observations created in the understanding of the real world phenomenon, the data was supposed to provide. An outlying observaton or outlier is one that appears to deviate markedly from the other members of the sample in which it occurs (Grubbs, (1969)). Thus the reliability of the observation is reflected by its relationship with the other members of the sample and as such, a decision on whether an observation is an outlier or not is essentially subjective. The literature on outliers is voluminous on its own and moreover shares many results with other areas of statistics like robust procedures, mixture models, slippage problems and data analysis. A detailed review material covering various aspects of statistical analysis in the presence of outliers is available in Anscombe(1960), Grubbs (1969), Stigler

(1973,1980), Barnett (1978), Kale (1979), Hawkins (1980), Barnett and Lewis (1984) and Gather and Kale(1992). In view of this, in the present study, only the basic concepts with the general framework required to develop the results in the subsequent chapters will be outlined.

There are three basic reasons, for the emergence of outliers identified in literature as, global model weakness that often requires a change in the initially assumed model to a new one for the entire sample, local model weakness that applies only to the seemingly outlying observations paving way for the individual treatment of such observations and natural variability, in which case outliers will naturally originate as a characteristic of the inherent model.

Two broad methods of dealing with the possibility of outliers are identification and accommodation. Identification procedures essentially lead to rejection of outlying observations if they are present or, to its incorporation into the analysis through a revision of the basic model or method of estimation or to the realisation that rogue observations were the result of defective mechanism that calls for renewed experimentation. On the other hand, accommodation approaches as the nomenclature suggests, advocate and practice preservation of the possible outliers via appropriate revision of models or methods of analysis or both. The methods of accommodation largely depend on the information at the disposal of the anlayst about the process generating outliers or they are so designed as to be unaffected by outlying observations. In any case the available apriori information, the philosophy towards approaching the problem and the specific goals one has set are important elements in shaping the appropriate procedure.

1.2 Data generating models

The null or working model adopted by the analyst in any practical problem is that $X_1, X_2, \ldots, X_n$ are independent and identically distributed (iid) observations from some target population specified by the distribution function $F(x,\theta) \in \mathcal{F} = \left\{ F(x,\theta) \,|\, \theta \in \Theta \right\}$ whose functional form is known except for the paramaters. To facilitate a theoritical framework for the treatment of the outliers it is necessary to evolve an alternative model.

The earliest of alternative models proposed in literature are the mixture models due to Newcomb (1886). If $x_1, x_2, \ldots, x_n$ are realisations of $X_1, X_2, \ldots, X_n$ the joint probability density function (pdf) of the X_i's in a mixture model has the form

$$L(\underline{x} \mid f, g, p) = \sum_{i=1}^{n} \left\{ (1-p)f(x_i) + pg(x_i) \right\} , \qquad (1.1)$$

where $\underline{x} = (x_1, x_2, \ldots, x_n)$, f and g are density functions and $0<p<1$. It is easy to notice that (1.1) represents the pdf of iid random variables each of which has distribution function

$$H(x) = (1 - p)\ F(x) + pG(x),$$

where $F \in \mathcal{F}$ and $G \in \mathcal{G}$ are distribution functions. However Tukey (1960), while disagreeing with the adequacy of (1.1) vis-a-vis its capability to explain the occurence of outliers, proposed contaminating models of the form

$$h(x_i) = (1-\alpha)f(x_i) + \alpha g(x_i) \ ,$$

in which f is the density of the target population and g is the density of the contaminating factor.

In contemporary literature much attention has been received to what are known as k-outlier models. To describe such models we assume that $x_1, x_2, \ldots, x_n$ result from independent random variables $X_1, X_2, \ldots, X_n$, $(n-k)$ of which are distributed as $F \in \mathcal{F}$ while the remaining k are distributed as $G=G(F)$ depending on F and belonging to the class $\mathcal{G}$. Let f and g denote the pdfs corresponding to F and G respectively and s, the subset of indices that form the observations belonging to G. Thus in the k outlier model,

$$S = \Big\{ s \,|\, s = (i_1, i_2, \ldots, i_k), \text{ the permutation of } k \text{ integers out of } (1,2,\ldots,n) \Big\},$$

contains all $\binom{n}{k}$ subsets formed by choosing k integers out of n. The likelihood of the sample is then

$$L(\underline{x} \mid f,g) = \prod_{i \notin s} f(x_i) \prod_{i \in s} g(x_i), \quad s \in S. \qquad (1.2)$$

If $T(x_1, x_2, \ldots, x_n)$ is a symmetric statistic, then its distribution does not depend on s and further there does not exist any non-trivial sufficient statistics in this case. It is shown in Kale(1976) that the k largest order statistics $X_{(n-k+1)}, \ldots, X_{(n)}$ are most likely to be observations distributed as G.

Another significant contribution in outlier generating models is that of the exchangeable class introduced by Kale(1969). The idea here is that any set of k random variables out of $(X_1, X_2, \ldots, X_n)$ has an equal probability of being distributed according to G while the other random variables have distribution specified by F. In this case the likelihood takes the form

$$L(\underline{x}|f,g) = \binom{n}{k}^{-1} \sum_{s\in S} \prod_{i\notin s} f(x_i) \prod_{i\in s} g(x_i). \tag{1.3}$$

Since the random variables are exchangeable, unlike in (1.2), in (1.3) the order statistics are sufficient, which renders inference based on it desirable. In a variant approach Barnett and Lewis (1984) considered the notion of labelled model which specifies the model in terms of the distribution of the order statistics assuming that the largest(smallest) k observations arise from G and the rest belongs to F. Thus $(X_{(1)}, \ldots, X_{(n)})$ is distributed as $(Y_{(1)}, \ldots, Y_{(n-k)}, Z_{(1)}, \ldots, Z_{(k)})$ where Y's following F and Z's following G, such that $\max_{1\le i\le n-k} Y_i \le \min_{1\le j\le k} Z_j$, $i=1,2,\ldots,n-k$. Hence the likelihood is

$$L(\underline{x}|f,g) = \frac{(n-k)!k!}{\psi_k(F,G)} \prod f(x_i) \prod g(x_j),$$

where $\psi_k(F,G)$ is the probability that $\max Y_i \le \min Z_j$.

Occasionally, it is of interest (see Veale (1975)) to treat the elements of the subset s as known, giving rise to a pooled sample of n comprising of (n-k) from F and k from G. The likelihood takes the simple form

$$L(\underline{x}|f,g) = \prod_{i \notin s} f(x_i) \prod_{j \in s} g(x_j), \qquad (1.4)$$

for a given s.

1.3 Bayes inference

Among the various researchers who have used the Bayesian approach, many look upon the problem of estimation rather than identification of outliers. The present study also makes use of the Bayesian approach to estimation in specific distributions assuming the existence of joint probability measures on $\Theta \times \mathcal{X}$, where $\Theta \subset R_k$ is the parametric space corresponding to a vector of parameters $\theta=(\theta_1,\theta_2,\ldots,\theta_k)$ and $\mathcal{X}$ is the sample space. This joint measure is determined through a prior measure on Θ and the conditional measure on $\mathcal{X}$ for a given θ in Θ, which in turn provides the posterior measure on Θ for a specified x in $\mathcal{X}$ along with a marginal measure on $\mathcal{X}$. In this formulation the posterior density function of θ can be obtained through

Bayes Theorem as (Raiffa and Schlaifer (1961))

$$f(\theta|\underline{x}) = \phi(\theta)\ell(\underline{x}|\theta)C(\underline{x}), \tag{1.5}$$

where $\phi(\theta)$ is the prior density and $C(\underline{x})$ is a normalising constant independent of θ given by

$$\int_{\Theta} f(\theta|\underline{x})\, d\theta = C(\underline{x}) \int_{\Theta} \phi(\theta)\ell(\underline{x}|\theta)d\theta = 1 . \tag{1.6}$$

Throughout the sequel we denote by C with or without suffixes, such normalising constants attached to the posterior density. In finding point estimates of θ we employ either the mode of (1.5) or make use of the quadratic loss function

$$L(\hat{\theta}(\underline{x}),\theta) = (\hat{\theta}(\underline{x})-\theta)^2 , \tag{1.7}$$

to prescribe the estimate as one that minimises

$$E\, L(\hat{\theta}(\underline{x}),\theta) = \int_{\Theta} (\hat{\theta}(\underline{x})-\theta)^2 f(\theta|\underline{x})d\theta, \tag{1.8}$$

or

$$\hat{\theta}(\underline{x}) = E(\theta|\underline{x}). \tag{1.9}$$

The expected loss, resulting from the use of (1.9) as the estimator of θ, is the posterior variance of θ. Since (1.9) is calculated for a specific sample point $\underline{x}$, some times it is of advantage to look at the Bayes risk.

$$R(\hat{\theta},\theta) = \int_{\Theta} \int_{\mathcal{X}} L(\hat{\theta},\theta)\ell(\underline{x}|\theta)\ \phi(\theta)\ d\underline{x}\ d\theta. \tag{1.10}$$

On the other hand, interval estimates for θ of the form $(\theta_L(\underline{x}),\theta_U(\underline{x}))$, that is to find two values θ_L and θ_U such that the interval (θ_L,θ_U) has a significant posterior probability for θ, are obtained as solution of the equation

$$\int_{\theta_L}^{\theta_U} f(\theta|\underline{x})d\theta = 1 - \alpha. \qquad (1.11)$$

Since there can be more than one set of (θ_L,θ_U) that satisfy (1.11), inorder to render the estimate unique, often the conditions

$$\int_{-\infty}^{\theta_L} f(\theta|\underline{x})d\theta = \alpha/2 = \int_{\theta_U}^{\infty} f(\theta|\underline{x})d\theta \qquad (1.12)$$

are also imposed.

What has been outlined so far in the present section is the general Bayesian framework applicable to all inference procedures, including the situation when the sample contains outlying observations. The Bayesian approach postulates the existence of prior distributions for the elements s in S as well as for the parameters in F and G involved in a k-outlier model represented by equation (1.2). Thus the mixture and exchangeable models provide examples of assigning distribution to s and they are amenable to a fuller Bayesian analysis when the parameters are also assumed to have appropriate prior distributions. Among the various researchers who have used this approach, many look upon the problem of estimation rather than identification of outliers. Restricting our attention only to specific probability models we review the important developments in this context. Of these, Box and Tiao (1968) presented an extensive systematic analysis for the normal case. Analysis of data from normal population containing outliers is also discussed in Guttman(1973), Guttman and Khatri (1975), O'Hagan (1979) and Goldstein (1982).

It seems that Kale (1969), was the first author to discuss the Bayesian methods for analysing outliers in the exponential samples. He obtained a semi-Bayesian estimator of θ, with $F(x;\theta) = \exp(\theta)$ and $G(x;\theta\lambda) = \exp(\theta\lambda)$, $\lambda \geq 1$ in the presence of an outlier using a beta prior for λ leaving θ without being assigned any prior distribution. Under the same exchangeable model with F having an exponential distribution with mean θ and G having an exponential distribution with mean θ/c, $0<c<1$, Sinha(1972) obtained the Bayes estimate of the survival function with a beta prior for c and no prior attached to θ. In a later paper

Sinha(1973a) extended the work by suggesting the same prior for c along with three possible prior families for θ in order to estimate these parameters and the survival function.

Lingappaiah (1976) investigated the estimation problem in the presence of outliers for a more general family that included the gamma, Weibull and exponential models as particular cases. The basic model has the pdf

$$f(x,\alpha,b,\beta) = \frac{b\ x^{\alpha-1}}{\Gamma(\alpha/b)}\ \beta^{\alpha/b}\ \exp[-\beta x^{b}], \qquad x>0. \qquad (1.13)$$

He considered the situation where among n-observations n-k are distributed as (1.13) and k of them following $f(x,\alpha,b,\theta_r\beta)$, $r=1,2,\ldots,k$, $0<\theta_r<1$. With an exchangeable model for outliers, he obtained the Bayes estimate of θ_r and β, using exponential prior for β and beta prior distributions for θ_r, for fixed k.

Dixit (1991) obtained the Bayes estimates of the parameters and also the power of the scale parameter for the gamma distribution, under various priors in the presence of k known outliers. He assumed that the random sample $x_1,x_2,\ldots,x_n$ of size n are such that k of them are distributed as

$$f(x;\sigma/\alpha) = \frac{(\alpha/\sigma)^p}{\Gamma(p)} x^{p-1} \exp[-(\alpha x/\sigma)], \qquad (1.14)$$

where $x>0$, $\sigma>0$, $\alpha\neq 1$ and p is known and the remaining 'n-k' random variables are distributed as $f(x;\sigma)$. With a beta prior for α and inverted gamma and quasi-priors for σ the Bayes estimate of σ^r under the loss function

$$L(g^r,\sigma^r) = (\sigma^r)^b \left\{(g^r)^s - (\sigma^r)^{\delta}\right\}^2 \qquad (1.15)$$

was derived.

It appears that the latest work in this category concerning the exponential model is that of Kale and Kale (1992). They assume that $X_1, X_2, \ldots, X_n$ are such that 'n-k' of these are independent identically distributed as exponential with mean θ having pdf.

$$f(x:\theta) = 1/\theta \exp[-(x/\theta)], \quad x\geq 0, \ \theta\geq 0, \qquad (1.16)$$

while the remaining k observations $X_{s_1}, X_{s_2}, \ldots, X_{s_k}$ are iid exponential with mean θ/α, where $0<\alpha\leq 1$. The indexing set of observations $s = (s_1, \ldots, s_k)$ is treated as a parameter, over S, the subset of k integers out of n. With uniform prior over S for s and three other priors for θ and α viz. inverted gamma x beta, quasi-prior x beta and Jeffrey's prior, they obtained the Bayes estimators of α and θ. They also gave two methods of which the first, with the aid of the predictive distribution of $X_{(k)}$ given $x_{(1)}, x_{(2)}, \ldots, x_{(k-1)}$, explains how to determine the unkown number k of outliers that label $(X_{(n-k+1)}, \ldots, X_{(n)})$ as outliers. The other method depends on the posterior distribution of the indexing set $s = (s_1, \ldots, s_k)$ in the determination of the number of outliers. The two methods have also been illustrated in the case of a real data situation available in Nelson (1982).

Another problem of interest in the area of outlier analysis is the prediction of a future observation using a random sample in which one observation is an outlier. The idea behind such a prediction, as described in Dunsmore (1974) is to provide either a point or an interval estimate for a future observation. Lingappaiah (1989a) used this idea to construct prediction intervals for the maxima and minima of future observations when the samples are from an exponential distribution which contain an outlier. In a later paper Lingappaiah (1989b, 1990) obtained the one-sided Bayes prediction interval for the r^{th} ordered future observation in the presence of an outlier when the sample are from gamma and Weibull distributions respectively.

1.4 The present work

As already discussed, a familiar topic in the vast amount of literature available on outliers is the problem of estimating parameters of specific probability models like the normal, gamma, Weibull, exponential etc when the data is known to contain one or more spurious observations. Inspite of the popularity of the Pareto law in analysing data on income, city population sizes, occurence of natural resources, stock prices fluctuations, insurance risks, business failures, reliability etc, the problem of estimating its parameters in the presence of outliers does not appear to have been considered in literature. Further, the model belong to the class of long-tailed distributions and as such, the appearance of extreme observations in the sample is quite common and their identification as outliers or not becomes important. Acoordingly the main theme of the present thesis is focussed on various estimation problems using the Bayesian appraoch, falling under the general category of accommodation procedures for analysing Pareto data containing outliers. We also derive some results that are pertaining to the exponential population that have relevance to life testing and reliability.

Now we present a chapterwise summary of the discussions included in the remaining chapters.

In Chapter II, the problem of estimation of parameters in the classical Pareto distribution specified by the density function,

$$f(x:\alpha,\sigma) = \alpha\sigma^{\alpha}\, x^{-(\alpha+1)} \quad , \qquad x\geq\sigma>0,\ \alpha>0, \qquad (1.17)$$

under the k-outlier exchangable model is presented. Thus of the n observation (n-k) are distributed as (1.17) while the remaining k follows the same type of distribution with density function,

$$g(x:\alpha,b,\sigma) = \alpha b\sigma^{\alpha b}\, x^{-(\alpha b+1)} \quad , \qquad x\geq\sigma>0, \alpha, b>0, \qquad (1.18)$$

where b is assumed to be known. Notice that when b<1 the discordant observation is a lower outlier, while b>1 indicates an upper outlier. With the above assumption we obtain the Bayes estimates of α and σ under quadratic loss, in the two situations when the scale parameter σ is known assuming a gamma prior for α and when σ is unknown, with a joint gamma-power family prior. It is also shown that our results reduce to those of Arnold and Press (1983) once we take b=1. A comparative study of the estimates is provided with the aid of simulated samples.

In Chapter III, the estimation problem is concieved in a more general and realistic situation in which the shape parameter of the contaminating distribution is also not known. Under the above model assumptions and prior distributions for α and σ and non-informative prior for b we obtain the Bayes estimates of α,b and σ in the two cases when σ is known and unknown.

Since the Pareto distribution is extensively used as a realistic model for personal incomes that exceed a specified level of income, the estimation of the survival function

$$R(x) = P[X>x] = (x/\sigma)^{-\alpha} , \qquad (1.19)$$

is often an important objective. Equation (1.19) also represents the reliability function in the context of life testing, where the Pareto model characterizes life times that have failure rate of the form, αx^{-1} which is ever increasing. In Chapter IV, we discuss the estimation of (1.19) when the sample contain a known number of outliers under three different data generating mechanisms, viz. the exchangeable model, the identifiable model and the censored model that utilises only the first (n-k) order statistics for estimation after identifying the last k as outliers. In this investigation we assume that b>1 and that the scale parameter σ which is the same for all the variables is known. The behaviour of the point and interval estimates obtained in all the three cases are also studied by varying the sample size and the hyper-parameters of the prior distributions.

As a natural continuation of the Bayesian frame work proposed earlier, we consider in Chapter V the prediction of a future observation based on a random sample that contains one contaminant. The object of the inference is the r^{th} prospective order statistic from the Pareto population (1.17). We present a $100(1-\beta)\%$ predictive interval for order statistics in both the cases where the shape parameter of contaminating distribution is known and unknown.

Chapter VI is devoted to the study of estimation problems concerning the exponential parameters under a k-outlier model. Assuming the exchangeable model for the outliers, Bayes point and interval estimates are obtained for the parameters and the survival function. We also suggest a method to determine the number of outliers present in a sample of size n using the predictive density.

The problem of obtaining a $100(1-\beta)\%$ predictive interval (two sided) for future order statistics from the exponential population in the presence of outliers is investigated in Chapter VII.

In The last chapter (Chapter VIII) we consider the estimation of R = P[X>Y] when X and Y are independent exponential random variables and data on each of them contain a discordant observation. The problem has relevance in the context of analysing the reliability of a component with strength X, which is subjected to a stress Y, where X and Y are exponentially distributed and stress is independent of strength. The component fails whenever Y>X so that R is a measure of component reliability. The estimates of R are derived under the exchangable, identifiable and the consored models.

CHAPTER II

ESTIMATION OF PARETO PARAMETERS

2.1 Introduction

In this chapter we discuss the problem of estimating the parameters of the classical Pareto distribution specified by the density function,

$$f(x;\sigma,\alpha) = \alpha\sigma^{\alpha} x^{-(\alpha+1)}, \qquad x\geq\sigma>0, \ \alpha>0, \qquad (2.1)$$

in the presence of k outlying observations using the Bayesian approach.

The use of the Pareto distribution as a model for various soci-economic phenomena dates back to the late ninteenth century when Pareto observed that the number of persons whose incomes exceed x can be approximated as $cx^{-\alpha}$, $\alpha>0$. Arnold (1983) gives an extensive historical survey of its use in the context of income analysis and also the various properties of the distribution. Though initially the Pareto distribution was used as a model for personal incomes and influenced the development of measures of income inequalities, later it has acquired prominence in

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The result in this chapter is due to appear in Jeevanand and Nair (1992a).

theoretical studies as a long tailed distribution as well as in several other areas of scientific activity some of which were mentioned in Section 1.4.

Studies on Bayesian inference procedures for the Pareto distribution when the sample is homogeneous have been discussed in Muniruzzaman (1968), Malik (1970) Zellner (1971), Rao Tummala (1977) and Sinha and Howlader (1980), where they take the scale parameter σ as known. Lwin (1972,1974) developed estimates of the both shape and scale parameters using a joint natural conjugate prior distribution for α and σ. Attributing Lewin's prior to be unnaturally restrictive, Arnold and Press (1983) suggested a gamma-power prior distribution for (σ,α) which also provide a posterior distribution belonging to the same family. Later the same authors (Arnold and Press (1986, 1989)) extended these results for grouped and censored data. Inspite of the wide applicability of the model (2.1), it seems that the problem of inferring the parameters of the Pareto population (2.1) in the presence of outliers has not yet been considered in literature. When some of the observations are infact contaminants, special inference procedures are required and this motivates the discussion in the present Chapter.

2.2 The Model

We assume that $\underline{x}=(x_1, x_2, \ldots, x_n)$ is a random sample from (2.1) containing k outliers (k known) but which of them are outliers is not known. Thus of the n observations (n-k) are distributed as (2.1) while the remaining k follow the same kind of distribution with density function

$$g(x:\alpha, b, \sigma) = \alpha b \ \sigma^{\alpha b} \ x^{-(\alpha b+1)}, \qquad x \geq \sigma > 0, b > 0, \qquad (2.2)$$

where b is assumed known. In this exchangeable model, the likelihood can be written according to (1.2) as

$$\mathcal{L}(\underline{x}|\alpha, b, \sigma) = \binom{n}{k}^{-1} \alpha^n \ b^k \ \sigma^{[n+(b-1)k]\alpha} \ (\prod_{i=1}^{n} x_i^{-(\alpha+1)})$$

$$\sum\nolimits_* (\prod_{j=1}^{k} x_{A_j}^{-\alpha(b-1)}), \qquad (2.3)$$

where

$$\sum\nolimits_* = \sum_{A_1=1}^{n-k-1} \cdots \sum_{A_k=A_{k-1}+1}^{n} .$$

When b=1, the product over j in (2.3) reduces to 1 so that the multiple sum is the number of ways of filling k-tuple

$(A_1, A_2, \ldots, A_k)$ with integers from 1 to n for which $A_1 < A_2, \ldots, < A_k$, which is $\binom{n}{k}$.

Customarily, the estimation problem is discussed by distinguising three cases; when one of the parameters is known and when both are objects of inference. However, the case when α is known and σ has to be estimated rarely occurs in practice and hence it is omitted from the present discussion.

2.3 Estimation with known scale parameter

Since σ is known, the form of the likelihood (2.3) gives the kernal as

$$k(\alpha|\underline{x}) = \alpha^n e^{-t\alpha} ,$$

so that the prior belongs to the gamma family. Thus we choose the prior density as

$$\phi(\alpha) = \frac{(t')^r}{\Gamma(r)} \alpha^{r-1} e^{-\alpha t'}, \qquad r, t', \alpha > 0, \qquad (2.4)$$

and the pasterior density from (2.4) and (2.3) turns out to be

$$f(\alpha|\underline{x}) = \ell(\underline{x}|\alpha)\phi(\alpha),$$

$$= C_1^{-1} \alpha^n e^{-t\alpha} \left[\sum\nolimits_* e^{-\alpha(b-1)t_A}\right] \alpha^{r-1} e^{-t'\alpha} ,$$

$$= C_1^{-1} \sum_{*} \alpha^{n+r-1} \exp\{-\alpha[t+t'+(b-1)t_A]\},$$

$$= [C_1(m,T)]^{-1} \sum_{*} \alpha^{m-1} \exp\{-\alpha[T+(b-1)t_A]\}, \quad \alpha>0, \quad (2.5)$$

where C with various suffixes denote the normalising constants and

$$T = t + t', \quad m = n+r, \quad t_A = \sum_{i=1}^{k} \log(x_{A_i}/\sigma),$$

and $$t = \sum_{i=1}^{n} \log(x_i/\sigma).$$

Now, to obtain $C_1(m,T)$ we have $\int_0^\infty f(\alpha|\underline{x})d\alpha = 1$ so that

$$C_1(m,T) = \int_0^\infty \sum_{*} \alpha^{m-1} \exp\{-\alpha[T+(b-1)t_A]\}\, d\alpha,$$

$$= \Gamma(m) \sum_{*} [T+(b-1)t_A]^{-m}. \quad (2.6)$$

One can have the estimator for α by specifying appropriate loss functions and using (2.5). Under quadratic loss, the Bayes estimator of α according to (1.9) is the mean of the posterior distribution (2.5). Thus the Bayes estimate $\hat{\alpha}_1$ is

$$\hat{\alpha}_1 = E(\alpha|\underline{x}) = [C_1(m,T)]^{-1} \int_0^\infty \sum_{*} \alpha^{m} \exp\{-\alpha[T+(b-1)t_A]\}d\alpha,$$

$$= [C_1(m,T)]^{-1}\Gamma(m+1)\sum_{*} [T+(b-1)t_A]^{-(m+1)} ,$$

$$= C_1(m+1,T)/C_1(m,T). \qquad (2.7)$$

The loss incurred when $\hat{\alpha}_1$ is used as the estimation of α is

$$V(\alpha_1 | \underline{x}) = E(\alpha - \hat{\alpha}_1)^2 ,$$

$$= E(\alpha^2 | \underline{x}) - \hat{\alpha}_1^2 ,$$

$$= [C_1(m+2,T)/C_1(m,T)] - \hat{\alpha}_1^{\,2}. \qquad (2.8)$$

Deductions

1 In (2.7) as t' and r tend to zero we have

$$\hat{\alpha}_1 = C_1(n+1,t)/C_1(n,t), \qquad (2.9)$$

which is the estimate corresponding to non-informative improper prior of Jeffrey (1961).

2 When b=1 in (2.7), the resulting estimate

$$\hat{\alpha}_1 = m/(t+t'),$$

is based on sample from (2.1) without contaminants and is the expression obtained in Arnold and Press (1983). In this case if t' and r tend to zero, one has the expression identical to that of the usual maximum likelihodd estimator of α.

2.4 Estimation with unknown scale parameter

We can now look at a more general data situation when both the scale and shape parameters remain unknown. The kernel of the likelihood suggests the following form for the joint prior density for α and σ

$$\phi(\alpha,\sigma) = C_2\ \alpha^r\ \sigma^{u\alpha-1}\ e^{-z'\alpha}, \quad \alpha>0, 0<\sigma\le\sigma_0,\ \sigma_0, z', u>0. \quad (2.10)$$

The corresponding posterior distribution is

$$f(\alpha,\sigma|\underline{x})=C_3\ \Sigma_{*}\alpha^n\ \sigma^{[n+(b-1)k]\alpha}\ \alpha^r\ \sigma^{u\alpha-1}\ e^{-z'\alpha}\ e^{-\alpha[z+(b-1)z_A]},$$

$$=C_3\ \Sigma_{*}\ \alpha^{n+r}\ \sigma^{[n+(b-1)k\ +\ u]\alpha}\ \exp(-\alpha\ [z+z'+(b-1)z_A]),$$

$$=C_3\ \Sigma_{*}\alpha^m\ \sigma^{U\alpha-1}\ \exp(-\alpha[S+(b-1)z_A]), \quad (2.11)$$

where

$$S=z+z',\ U= n+u+(b-1)k\ ,\ z_A = \sum_{i=1}^{k} \log(x_{A_i}),$$

$$z = \sum_{i=1}^{n} \log(x_i) \quad \text{and} \quad \lambda = \min(\ x_{(1)}, \sigma_0).$$

2.4.1 Estimation of α

From equation (2.11), the marginal posterior density of α is

$$f(\alpha|\underline{x}) = C_3 \sum_* \int_0^{\lambda} \alpha^m \sigma^{U\alpha-1} \exp(-\alpha[S+(b-1)z_A])d\sigma,$$

$$= C_3 \sum_* \alpha^m (\lambda^{U\alpha}/U\alpha) \exp(-\alpha[S+(b-1)z_A]),$$

$$= C_4^{-1} \sum_* \alpha^{m-1} \exp(-\alpha[S+(b-1)z_A - U \log\lambda]),$$

$$=[C_4(m,S_1)]^{-1} \sum_* \alpha^{m-1} \exp(-\alpha[S_1+(b-1)z_A]), \qquad (2.12)$$

where

$$S_1 = S - U \log\lambda \quad \text{and} \quad C_4(m,S_1) = \Gamma(m) \sum_* \{S_1+(b-1)z_A\}^{-m}.$$

The Bayes estimates for α under quadratic loss is

$$\hat{\alpha}_2 = E(\alpha|\underline{x}) = C_4(m+1,S_1)/C_4(m,S_1) \qquad (2.13)$$

with expected loss

$$V(\alpha_2|\underline{x}) = [C_4(m+2,S_1)/C_4(m,S_1)] - \hat{\alpha}_2^{\,2} . \qquad (2.14)$$

Deductions

1. The estimator corresponding to Jeffrey's prior is

obtained by allowing t',r and u to tend to zero. Thus the resulting estimator is got as

$$\hat{\alpha}_2 = C_4[n+1, z-(n+(b-1)k)\log x_{(1)}]/C_4[n, z-(n+(b-1)k)\log x_{(1)}].$$

(2.15)

2. Setting b=1 in (2.13) the Bayes estimator based on the uncontaminated Pareto sample in Arnold & Press (1983)

$$\hat{\alpha}_2 = m/(S-(n+u)\log\lambda), \qquad (2.16)$$

is obtained.

2.4.2 Estimation of σ

The marginal density of σ is

$$f(\sigma|\underline{x}) = C_3 \sum_{*} \int_0^\infty \alpha^m \sigma^{U\alpha-1} \exp(-\alpha[S+(b-1)z_A])d\alpha,$$

$$= C_3 \sum_{*} \sigma^{-1} \Gamma(m+1) [S+(b-1)z_A-U\log\sigma]^{-(m+1)},$$

$$= C_5^{-1} \sum_{*} [Q_A-\log\sigma]^{-(m+1)} \sigma^{-1}, \quad 0<\sigma\leq\lambda, \quad (2.17)$$

where

$$Q_A = [S+(b-1)z_A]/U, \; C_5 = \sum_{*} W_{\mu}(0,m+1)$$

and $\qquad \mu = Q_A - \log\lambda,$

To obtain the value of C_5 we consider the integral

$$I(\sigma^c) = \Sigma_* \int_0^\lambda \sigma^c [Q_A - \log\sigma]^{-(m+1)} \sigma^{-1} \, d\sigma.$$

Setting $y = Q_A - \log\sigma$, we have

$$I(\sigma^c) = \Sigma_* \int_0^{Q_A - \log\lambda} y^{-(m+1)} \exp(c(Q_A - y)) \, dy,$$

$$= \Sigma_* \exp[cQ_A] \; W_\mu(c, m+1).$$

so that

$$C_5 = I(\sigma^0) = \Sigma_* W_\mu(0, m+1).$$

The W(.) function given above is related to the well known exponential integral $E_m(.)$ (Abramowitz and Stegun (1972)) as $W_c(b,m) = c^{1-m} E_m(bc)$. The value of W(.) can be read from the tabulated value of $E_m(.)$ given by them for the integer values of m and by interpolation for non-integer values. The estimator of σ under the squared error loss is

$$\hat{\sigma}_1 = E(\sigma|\underline{x}) = \frac{\Sigma_* \exp[Q_A] \; W_\mu(1, m+1)}{\Sigma_* W_\mu(0, m+1)} \tag{2.18}$$

with expected loss

$$V(\sigma_1|\underline{x}) = \frac{\Sigma_* \exp[2Q_A] \; W_\mu(2, m+1)}{\Sigma_* W_\mu(0, m+1)} - \hat{\sigma}_1^2. \tag{2.19}$$

For the non-informative prior situation, we let t',u and r to tend to zero to give

$$\hat{\sigma}_1 = \Sigma_* W_{Q'_A - \log x_{(1)}}(1,n+1) / \Sigma_* W_{Q'_A - \log x_{(1)}}(0,n+1). \qquad (2.20)$$

with $Q'_A = [z+(b-1)z_A]/[n+(b-1)k]$

In the absence of outliers (2.19) reduces to

$$\hat{\sigma}_1 = e^{\theta} W_{\theta}(1,m+1)/W_{\theta}(0,m+1) \;;\; \theta = [S/(n+u)]-\log\lambda. \qquad (2.21)$$

2.5 Discussion

In order to assess how the various estimates behave in a specific situation a random sample of size 19 of which 18 comes from population with pdf (2.1) with parameters α=2.5, σ=150 and single observation with parameters σ=150 and αb=15 (i.e. b=6) was simulated producing the following observations:

152.2618	173.4242	183.9583	167.7641	177.0248
211.8959	157.9960	173.8940	202.2048	594.9689
192.2253	227.4415	165.8673	170.4876	183.9489
175.9670	198.2829	253.2669	203.0875	

The estimates of the parameters derived in Sections 2.2 through 2.4 based on the above sample are exhibited in Tables 2.1,2.2 and 2.3. The losses corresponding to the estimates given in each cell are shown in braces below each entry. It is to be noted that $\tilde{\alpha}_1, \tilde{\alpha}_2, \tilde{\sigma}_1$ are the Bayes estimates discussed in Arnold and Press (1983). Further, to learn the sampling behaviour of the estimates, samples of sizes 10,30 and 50 were also generated for the above parameter values and the bias and expected losses were calculated. The results obtained are given in Table 2.4 to 2.7. In all cases, the hyper-parameters of the prior were chosen as u= 0.1,0.001, r=1,2,3 and t'=1,2,2.5,3. The computation of the Bayes estimators and the corresponding risk are done on the mainframe computer using Fortran 77. The evaluation of the exponential integral is done using the fortran subroutine avaliable in mathematical library of IMSL and those programs are given in Appendix.

It can be observed that the bias and expected loss associated with the estimates (2.7),(2.8) and (2.18) in the present work are considerably less than those of Arnold and Press (1983) in almost all cases. Thus the procedure outlined provides improved estimates, justifying the choice of G in the accomodation approach. For moderate values of r', the bias tends to increase with the values assigned to t', while the expected losses become smaller under the same condition. However as the sample size increases, the prior parameters have lesser influence on both the bias and the expected loss, and the estimates become closer to the true parameter value. An interesting feature of the proposed estimates is that even for very moderate sample sizes, our approach substantially improves upon the estimates of Arnold and Press (1983), irrespective of whether σ is held known or unknown.

Table 2.1

Estimates of α when σ is known
for samples from Pareto distribution
with α = 2.5, σ=150, b = 6

r	t'	$\tilde{\alpha}_1$ (Arnold & Press)	$\hat{\alpha}_1$ (Present Study)
1	1	3.212 (0.516)	3.016 (0.479)
1	2	2.768 (0.383)	2.604 (0.355)
1	2.5	2.589 (0.335)	2.438 (0.310)
1	3	2.431 (0.296)	2.292 (0.273)
2	1	3.373 (0.542)	3.175 (0.506)
2	2	2.906 (0.402)	2.74 (0.374)
2	2.5	2.718 (0.352)	2.565 (0.327)
2	3	2.553 (0.310)	2.412 (0.288)

Cont...

r	t'	$\tilde{\alpha}_1$ (Arnold & Press)	$\hat{\alpha}_1$ (Present Study)
3	1	3.534 (0.568)	3.334 (0.533)
3	2	3.045 (0.421)	2.876 (0.394)
3	2.5	2.848 (0.369)	2.692 (0.344)
3	3	2.674 (0.325)	2.531 (0.303)
Non-informative prior		3.636 (0.698)	3.407 (0.642)

Table 2.2

Estimates when σ is unknown and $\sigma_0 < x_{(1)}$ (σ_0 = 150)

		u=0.001			
r	z'	$\tilde{\alpha}_2$	$\hat{\alpha}_2$	$\tilde{\sigma}_1$	$\hat{\sigma}_1$
1	1	3.215 (0.517)	3.018 (0.480)	146.894 (2.158×10^{-4})	147.860 (2.156×10^{-4})
1	2	2.770 (0.384)	2.605 (0.355)	146.871 (2.156×10^{-4})	147.694 (2.151×10^{-4})
1	2.5	2.590 (0.366)	2.439 (0.310)	146.124 (2.161×10^{-4})	147.018 (2.135×10^{-4})
1	3	2.443 (0.296)	2.293 (0.274)	146.857 (2.167×10^{-4})	146.881 (2.161×10^{-4})
2	1	3.376 (0.543)	3.177 (0.507)	146.891 (2.156×10^{-4})	147.861 (2.156×10^{-4})
2	2	2.908 (0.403)	2.742 (0.375)	146.874 (2.156×10^{-4})	147.694 (2.151×10^{-4})
2	2.5	2.720 (0.352)	2.567 (0.327)	146.120 (2.160×10^{-4})	147.019 (2.151×10^{-4})
2	3	2.554 (0.311)	2.413 (0.288)	146.856 (2.166×10^{-4})	146.882 (2.163×10^{-4})

cont...

r	z'	$\tilde{\alpha}_2$	$\hat{\alpha}_2$	$\tilde{\sigma}_1$	$\hat{\sigma}_1$
3	1	3.536	3.337	146.894	147.860
		(0.568)	(0.533)	(2.157×10^{-4})	(2.157×10^{-4})
3	2	3.047	2.878	146.872	147.693
		(0.422)	(0.394)	(2.155×10^{-4})	(2.150×10^{-4})
3	2.5	2.849	2.694	146.121	147.017
		(0.369)	(0.344)	(2.164×10^{-4})	(2.13×10^{-4})
3	3	2.676	2.553	146.855	146.881
		(0.326)	(0.303)	(2.167×10^{-4})	(2.161×10^{-4})
			u=0.1		
1	1	3.494	3.227	147.501	147.877
		(0.610)	(0.569)	(2.187×10^{-4})	(2.176×10^{-4})
1	2	2.974	2.795	147.455	147.645
		(0.442)	(0.410)	(2.180×10^{-4})	(2.174×10^{-4})
1	2.5	2.768	2.604	146.13	145.59
		(0.383)	(0.355)	(2.178×10^{-4})	(2.156×10^{-4})
1	3	2.589	2.438	147.384	147.461
		(0.335)	(0.310)	(2.174×10^{-4})	(2.15×10^{-4})
2	1	3.668	3.451	147.591	147.887
		(0.641)	(0.600)	(2.187×10^{-4})	(2.177×10^{-4})

cont...

r	z'	$\tilde{\alpha}_2$	$\hat{\alpha}_2$	$\tilde{\sigma}_1$	$\hat{\sigma}_1$
2	2	3.123 (0.464)	2.941 (0.433)	147.465 (2.181×10^{-4})	147.646 (2.176×10^{-4})
2	2.5	2.907 (0.402)	2.740 (0.374)	146.623 (2.178×10^{-4})	147.61 (2.171×10^{-4})
2	3	2.719 (0.352)	2.565 (0.327)	147.384 (2.174×10^{-4})	147.460 (2.153×10^{-4})
3	1	3.843 (0.671)	3.089 (0.632)	147.506 (2.189×10^{-4})	147.889 (2.176×10^{-4})
3	2	3.271 (0.486)	3.089 (0.455)	147.464 (2.182×10^{-4})	147.691 (2.174×10^{-4})
3	2.5	3.045 (0.421)	2.877 (0.394)	146.622 (2.173×10^{-4})	147.651 (2.178×10^{-4})
3	3	2.848 (0.369)	2.693 (0.344)	147.382 (2.17×10^{-4})	147.62 (2.151×10^{-4})
Jeffery's prior		3.494 (0.355)	2.768 (0.311)	146.871 (2.157×10^{-4})	147.124 (2.135×10^{-4})

Table 2.3

Estimates when σ is unknown and $\sigma_0 > x_{(1)}$

		u = 0.001			
r	z'	$\tilde{\alpha}_2$	$\hat{\alpha}_2$	$\tilde{\sigma}_1$	$\hat{\sigma}_1$
1	1	3.369 (0.568)	3.202 (0.542)	150.018 (2.251×10^{-4})	149.993 (2.223×10^{-4})
1	2	2.883 (0.416)	2.74 (0.394)	148.455 (2.234×10^{-4})	149.456 (2.204×10^{-4})
1	2.5	2.689 (0.362)	2.556 (0.342)	149.204 (2.236×10^{-4})	149.39 (2.232×10^{-4})
1	3	2.520 (0.318)	2.397 (0.299)	148.438 (2.231×10^{-4})	149.039 (2.221×10^{-4})
2	1	3.537 (0.596)	3.371 (0.572)	150.019 (2.252×10^{-4})	149.994 (2.223×10^{-4})
2	2	3.027 (0.436)	2.883 (0.415)	148.456 (2.233×10^{-4})	149.455 (2.204×10^{-4})
2	2.5	2.824 (0.380)	2.690 (0.360)	149.202 (2.238×10^{-4})	149.39 (2.232×10^{-4})
2	3	2.646 (0.333)	2.521 (0.361)	148.429 (2.228×10^{-4})	149.101 (2.219×10^{-4})

cont...

r	z'	$\tilde{\alpha}_2$	$\hat{\alpha}_2$	$\tilde{\sigma}_1$	$\hat{\sigma}_1$
3	1	3.706	3.540	150.017	149.991
		(0.624)	(0.602)	(2.250×10^{-4})	(2.221×10^{-4})
3	2	3.172	3.027	148.456	149.458
		(0.457)	(0.437)	(2.231×10^{-4})	(2.208×10^{-4})
3	2.5	2.958	2.824	149.204	149.29
		(0.398)	(0.379)	(2.238×10^{-4})	(2.226×10^{-4})
3	3	2.772	2.647	148.429	149.020
		(0.349)	(0.332)	(2.238×10^{-4})	(2.017×10^{-4})
			u = 0.1		
1	1	3.677	3.497	149.039	150.078
		(0.676)	(0.649)	(2.221×10^{-4})	(2.22×10^{-4})
1	2	3.106	2.251	148.985	149.422
		(0.482)	(0.458)	(2.223×10^{-4})	(2.221×10^{-4})
1	2.5	2.882	2.749	149.775	149.285
		(0.415)	(0.393)	(2.229×10^{-4})	(2.226×10^{-4})
1	3	2.689	2.556	148.921	149.225
		(0.361)	(0.341)	(2.218×10^{-4})	(2.217×10^{-4})
2	1	3.861	3.682	149.041	150.068
		(0.710)	(0.685)	(2.22×10^{-4})	(2.221×10^{-4})

cont...

r	z'	$\tilde{\alpha}_2$	$\hat{\alpha}_2$	$\tilde{\sigma}_1$	$\hat{\sigma}_1$
2	2	3.261 (0.507)	3.106 (0.484)	148.988 (2.223×10^{-4})	149.426 (2.221×10^{-4})
2	2.5	3.026 (0.436)	2.882 (0.415)	149.701 (2.228×10^{-4})	149.288 (2.226×10^{-4})
2	3	2.823 (0.379)	2.689 (0.360)	148.924 (2.217×10^{-4})	149.225 (2.217×10^{-4})
3	1	4.045 (0.744)	3.868 (0.721)	149.021 (2.223×10^{-4})	150.041 (2.22×10^{-4})
3	2	3.417 (0.531)	3.262 (0.509)	148.789 (2.221×10^{-4})	149.041 (2.221×10^{-4})
3	2.5	3.17 (0.457)	3.026 (0.437)	149.761 (2.218×10^{-4})	149.128 (2.216×10^{-4})
3	3	2.957 (0.398)	2.823 (0.379)	148.914 (2.213×10^{-4})	149.225 (2.207×10^{-4})
Jeffery's prior		3.845 (0.778)	3.648 (0.748)	150.996 (2.258×10^{-4})	150.028 (2.253×10^{-4})

Table 2.4

Absolute bias and expected loss of α when σ is known for different sample sizes.

		n=10		n=30		n=50	
r	t*	$\tilde{\alpha}_1$	$\hat{\alpha}_1$	$\tilde{\alpha}_1$	$\hat{\alpha}_1$	$\tilde{\alpha}_1$	$\hat{\alpha}_1$
1	1	0.507	0.229	0.462	0.225	0.365	0.204
		(0.492)	(0.361)	(0.266)	(0.229)	(0.110)	(0.089)
1	2	0.812	0.628	0.386	0.231	0.309	0.186
		(0.329)	(0.259)	(0.222)	(0.194)	(0.100)	(0.082)
1	3	1.037	0.906	0.530	0.404	0.228	0.117
		(0.236)	(0.185)	(0.187)	(0.167)	(0.091)	(0.076)
2	1	0.325	0.248	0.323	0.156	0.248	0.109
		(0.542)	(0.394)	(0.275)	(0.236)	(0.112)	(0.091)
2	2	0.659	0.452	0.410	0.263	0.301	0.172
		(0.362)	(0.282)	(0.229)	(0.200)	(0.102)	(0.084)
2	3	0.904	0.452	0.491	0.361	0.155	0.039
		(0.259)	(0.212)	(0.194)	(0.172)	(0.093)	(0.078)
3	1	0.281	0.158	0.209	0.134	0.144	0.108
		(0.593)	(0.427)	(0.285)	(0.243)	(0.115)	(0.093)

cont...

		n=10		n=30		n=50	
r	t'	$\tilde{\alpha}_1$	$\hat{\alpha}_1$	$\tilde{\alpha}_1$	$\hat{\alpha}_1$	$\tilde{\alpha}_1$	$\hat{\alpha}_1$
3	2	0.506	0.320	0.370	0.275	0.258	0.110
		(0.395)	(0.306)	(0.237)	(0.206)	(0.104)	(0.086)
3	3	0.771	0.609	0.453	0.317	0.181	0.040
		(0.283)	(0.230)	(0.200)	(0.177)	(0.095)	(0.079)
Jeff-ery's prior		0.315	0.139	0.287	0.130	0.152	0.119
		(0.736)	(0.490)	(0.315)	(0.265)	(0.119)	(0.094)

Table 2.5

Absolute bias and expected loss of α when σ is unknown and $\sigma_0 < x_{(1)}$ ($\sigma_0 = 150$)

u = 0.001							
		n=10		n=30		n=50	
r	z'	$\tilde{\alpha}_1$	$\hat{\alpha}_1$	$\tilde{\alpha}_1$	$\hat{\alpha}_1$	$\tilde{\alpha}_1$	$\hat{\alpha}_1$
1	1	1.452	0.331	0.739	0.233	0.635	0.204
		(0.493)	(0.100)	(0.266)	(0.112)	(0.110)	(0.061)
1	2	1.543	0.626	0.792	0.309	0.741	0.088
		(0.33)	(0.083)	(0.222)	(0.100)	(0.100)	(0.057)
1	3	1.620	0.905	0.848	0.404	0.836	0.166
		(0.237)	(0.070)	(0.188)	(0.089)	(0.091)	(0.054)
2	1	1.067	0.425	0.699	0.206	0.575	0.156
		(0.543)	(0.109)	(0.276)	(0.116)	(0.113)	(0.062)
2	2	1.456	0.450	0.759	0.263	0.684	0.173
		(0.363)	(0.091)	(0.229)	(0.103)	(0.102)	(0.058)
2	3	1.540	0.757	0.815	0.361	0.782	0.038
		(0.260)	(0.077)	(0.194)	(0.092)	(0.093)	(0.055)
3	1	1.262	0.519	0.644	0.212	0.515	0.108
		(0.594)	(0.118)	(0.285)	(0.119)	(0.155)	(0.064)

cont...

Table 2.5

Absolute bias and expected loss of α when σ is unknown and $\sigma_0 < x_{(1)}$ ($\sigma_0 = 150$)

		u = 0.001					
		n=10		n=30		n=50	
r	z'	$\tilde{\alpha}_1$	$\hat{\alpha}_1$	$\tilde{\alpha}_1$	$\hat{\alpha}_1$	$\tilde{\alpha}_1$	$\hat{\alpha}_1$
1	1	1.452	0.331	0.739	0.233	0.635	0:204
		(0.493)	(0.100)	(0.266)	(0.112)	(0.110)	(0.061)
1	2	1.543	0.626	0.792	0.309	0.741	0.088
		(0.33)	(0.083)	(0.222)	(0.100)	(0.100)	(0.057)
1	3	1.620	0.905	0.848	0.404	0.836	0.166
		(0.237)	(0.070)	(0.188)	(0.089)	(0.091)	(0.054)
2	1	1.067	0.425	0.699	0.206	0.575	0.156
		(0.543)	(0.109)	(0.276)	(0.116)	(0.113)	(0.062)
2	2	1.456	0.450	0.759	0.263	0.684	0.173
		(0.363)	(0.091)	(0.229)	(0.103)	(0.102)	(0.058)
2	3	1.540	0.757	0.815	0.361	0.782	0.038
		(0.260)	(0.077)	(0.194)	(0.092)	(0.093)	(0.055)
3	1	1.262	0.519	0.644	0.212	0.515	0.108
		(0.594)	(0.118)	(0.285)	(0.119)	(0.155)	(0.064)

cont...

		n=10		n=30		n=50	
r	z'	$\tilde{\alpha}_1$	$\hat{\alpha}_1$	$\tilde{\alpha}_1$	$\hat{\alpha}_1$	$\tilde{\alpha}_1$	$\hat{\alpha}_1$
3	2	1.190	0.273	0.725	0.259	0.628	0.217
		(0.396)	(0.098)	(0.237)	(0.106)	(0.104)	(0.059)
3	3	1.460	0.608	0.783	0.317	0.728	0.041
		(0.283)	(0.083)	(0.200)	(0.095)	(0.095)	(0.056)
				u = 0.1			
1	1	1.400	0.469	0.702	0.148	0.578	0.053
		(0.625)	(0.110)	(0.294)	(0.119)	(0.116)	(0.063)
1	2	1.500	0.446	0.764	0.258	0.690	0.203
		(0.398)	(0.091)	(0.242)	(0.106)	(0.105)	(0.059)
1	3	1.584	0.778	0.821	0.358	0.790	0.019
		(0.277)	(0.076)	(0.203)	(0.094)	(0.095)	(0.055)
2	1	1.300	0.568	0.667	0.297	0.516	0.098
		(0.690)	(0.130)	(0.304)	(0.127)	(0.118)	(0.065)
2	2	1.409	0.292	0.732	0.252	0.632	0.211
		(0.438)	(0.107)	(0.251)	(0.122)	(0.107)	(0.060)
2	3	1.500	0.617	0.788	0.313	0.735	0.063
		(0.304)	(0.090)	(0.210)	(0.100)	(0.098)	(0.056)

cont...

		n=10		n=30		n=50	
r	z'	$\tilde{\alpha}_1$	$\hat{\alpha}_1$	$\tilde{\alpha}_1$	$\hat{\alpha}_1$	$\tilde{\alpha}_1$	$\hat{\alpha}_1$
3	1	1.200 (0.755)	0.668 (0.120)	0.632 (0.314)	0.544 (0.123)	0.454 (0.121)	0.049 (0.066)
3	2	1.318 (0.479)	0.382 (0.990)	0.695 (0.259)	0.164 (0.109)	0.573 (0.109)	0.057 (0.061)
3	3	1.417 (0.332)	0.456 (0.083)	0.755 (0.217)	0.269 (0.097)	0.680 (0.100)	0.145 (0.057)
Non inf. prior		1.448 (0.736)	0.520 (0.111)	0.706 (0.315)	0.139 (0.123)	0.581 (0.119)	0.130 (0.064)

Table 2.6

Absolute bias and expected loss of α when $\sigma_0 > x_{(1)}$ and σ unknown

u = 0.001							
		n=10		n=30		n=50	
r	z'	$\tilde{\alpha}_1$	$\hat{\alpha}_1$	$\tilde{\alpha}_1$	$\hat{\alpha}_1$	$\tilde{\alpha}_1$	$\hat{\alpha}_1$
1	1	1.410	0.460	0.682	0.111	0.582	0.010
		(0.598)	(0.108)	(0.292)	(0.119)	(0.119)	(0.065)
1	2	1.508	0.481	0.745	0.224	0.694	0.194
		(0.365)	(0.089)	(0.241)	(0.105)	(0.108)	(0.060)
1	3	1.590	0.802	0.803	0.328	0.793	0.026
		(0.269)	(0.075)	(0.202)	(0.094)	(0.098)	(0.056)
2	1	1.311	0.558	0.647	0.238	0.520	0.061
		(0.660)	(0.118)	(0.302)	(0.123)	(0.122)	(0.066)
2	2	1.418	0.290	0.711	0.284	0.635	0.177
		(0.423)	(0.098)	(0.249)	(0.109)	(0.110)	(0.062)
2	3	1.508	0.644	0.770	0.282	0.738	0.055
		(0.296)	(0.082)	(0.209)	(0.097)	(0.100)	(0.058)
3	1	1.212	0.671	0.611	0.479	0.458	0.011
		(0.722)	(0.128)	(0.312)	(0.126)	(0.125)	(0.067)

cont...

		n=10		n=30		n=50	
r	z'	$\tilde{\alpha}_1$	$\hat{\alpha}_1$	$\tilde{\alpha}_1$	$\hat{\alpha}_1$	$\tilde{\alpha}_1$	$\hat{\alpha}_1$
3	2	1.328	0.373	0.676	0.129	0.577	0.099
		(0.462)	(0.106)	(0.258)	(0.112)	(0.133)	(0.063)
3	3	1.425	0.484	0.737	0.237	0.683	0.137
		(0.322)	(0.089)	(0.216)	(0.100)	(0.103)	(0.059)

u = 0.1

		n=10		n=30		n=50	
r	z'	$\tilde{\alpha}_1$	$\hat{\alpha}_1$	$\tilde{\alpha}_1$	$\hat{\alpha}_1$	$\tilde{\alpha}_1$	$\hat{\alpha}_1$
1	1	1.353	0.612	0.755	0.341	0.521	0.277
		(0.781)	(0.120)	(0.324)	(0.126)	(0.103)	(0.058)
1	2	1.461	0.320	0.714	0.267	0.640	0.169
		(0.474)	(0.098)	(0.264)	(0.112)	(0.113)	(0.063)
1	3	1.551	0.656	0.755	0.277	0.745	0.079
		(0.319)	(0.082)	(0.220)	(0.099)	(0.103)	(0.058)
2	1	1.249	0.716	0.613	0.616	0.457	0.101
		(0.863)	(0.130)	(0.335)	(0.130)	(0.128)	(0.068)
2	2	1.367	0.414	0.679	0.121	0.580	0.055
		(0.522)	(0.107)	(0.274)	(0.115)	(0.116)	(0.064)

cont..

		n=10		n=30		n=50	
r	z'	$\tilde{\alpha}_1$	$\hat{\alpha}_1$	$\tilde{\alpha}_1$	$\hat{\alpha}_1$	$\tilde{\alpha}_1$	$\hat{\alpha}_1$
2	3	1.465	0.483	0.741	0.231	0.688	0.164
		(0.357)	(0.089)	(0.228)	(0.103)	(0.105)	(0.060)
3	1	1.144	0.892	0.577	0.321	0.393	0.053
		(0.946)	(0.141)	(0.346)	(0.134)	(0.131)	(0.070)
3	2	1.272	0.508	0.644	0.158	0.520	0.072
		(0.570)	(0.116)	(0.283)	(0.119)	(0.118)	(0.065)
3	3	1.378	0.309	0.707	0.250	0.632	0.185
		(0.383)	(0.097)	(0.235)	(0.106)	(0.107)	(0.061)
Non-Inf. prior		1.401	0.673	0.653	0.471	0.522	0.032
		(0.949)	(0.121)	(0.348)	(0.130)	(0.130)	(0.068)

Table 2.7

Absolute bias and expected loss of σ when $\sigma_0 < x_{(1)}$

		u = 0.001					
		n=10		n=30		n=50	
r	z'	$\tilde{\sigma}_1$	$\hat{\sigma}_1$	$\tilde{\sigma}_1$	$\hat{\sigma}_1$	$\tilde{\sigma}_1$	$\hat{\sigma}_1$
1	1	3.876	2.499	3.145	2.981	3.219	2.140
		(2.198)	(2.176)	(2.164)	(2.156)	(2.179)	(2.156)
1	2	3.879	2.997	3.149	2.984	3.143	2.1571
		(2.181)	(2.176)	(2.164)	(2.130)	(2.129)	(2.1228)
1	3	3.88	2.855	3.143	2.833	3.129	2.1571
		(2.189)	(2.166)	(2.186)	(2.146)	(2.174)	(2.1226)
2	1	3.876	2.897	3.355	2.521	3.105	2.139
		(2.188)	(2.161)	(2.176)	(2.159)	(2.169)	(2.156)
2	2	3.618	2.866	3.249	2.824	3.109	2.335
		(2.183)	(2.151)	(2.161)	(2.141)	(2.156)	(2.116)
2	3	3.517	2.895	3.139	2.647	3.128	2.306
		(2.193)	(2.161)	(2.187)	(2.156)	(2.167)	(2.140)
3	1	3.355	2.771	3.241	2.690	3.12	2.139
		(2.176)	(2.131)	(2.163)	(2.117)	(2.159)	(2.101)

cont...

		n=10		n=30		n=50	
r	z*	$\tilde{\sigma}_1$	$\hat{\sigma}_1$	$\tilde{\sigma}_1$	$\hat{\sigma}_1$	$\tilde{\sigma}_1$	$\hat{\sigma}_1$
3	2	3.499	2.897	3.144	2.772	3.109	2.307
		(2.171)	(2.161)	(2.164)	(2.159)	(2.156)	(2.141)
3	3	3.918	2.984	3.161	2.824	3.105	2.217
		(2.183)	(2.13)	(2.169)	(2.107)	(2.159)	(2.121)

u=0.1

		n=10		n=30		n=50	
r	z*	$\tilde{\sigma}_1$	$\hat{\sigma}_1$	$\tilde{\sigma}_1$	$\hat{\sigma}_1$	$\tilde{\sigma}_1$	$\hat{\sigma}_1$
1	1	3.199	1.544	2.213	0.899	2.123	0.029
		(2.236)	(2.228)	(2.217)	(2.217)	(2.214)	(2.122)
1	2	3.016	1.571	2.14	0.544	2.106	0.026
		(2.24)	(2.237)	(2.286)	(2.128)	(2.219)	(2.1223)
1	3	3.387	1.562	2.219	0.542	2.113	0.009
		(2.248)	(2.236)	(2.2458)	(2.126)	(2.217)	(2.101)
2	1	3.129	1.545	2.111	0.825	2.007	0.019
		(2.235)	(2.234)	(2.215)	(2.211)	(2.211)	(2.164)
2	2	3.126	1.015	2.539	0.815	2.111	0.016
		(2.260)	(2.223)	(2.2176)	(2.207)	(2.216)	(2.117)

cont...

		n=10		n=30		n=50	
r	z'	$\tilde{\sigma}_1$	$\hat{\sigma}_1$	$\tilde{\sigma}_1$	$\hat{\sigma}_1$	$\tilde{\sigma}_1$	$\hat{\sigma}_1$
2	3	3.145	1.079	2.39	0.798	2.123	0.007
		(2.238)	(2.218)	(2.2171)	(2.2038)	(2.214)	(1.996)
3	1	3.119	1.076	2.139	0.799	2.109	0.018
		(2.32)	(2.217)	(2.2156)	(2.138)	(2.2147)	(2.1251)
3	2	3.106	1.086	2.123	0.610	2.106	0.010
		(2.234)	(2.213)	(2.2187)	(2.2021)	(2.216)	(2.061)
3	3	3.337	1.221	2.139	0.541	1.921	0.006
		(2.61)	(2.221)	(2.2156)	(2.107)	(2.219)	(1.978)
Non-		3.129	1.758	2.982	1.123	2.544	0.98
Inf.		(2.257)	(2.2077	(2.135)	(2.204)	(2.123)	(2.017)
prior							

where (a) = $a \times 10^{-4}$ is the expected loss

CHAPTER III

ESTIMATION WHEN b IS UNKNOWN

3.1 Introduction

In the previous chapter we have dealt with the estimation of the parameters under the assumption that the outlying observations also have Pareto distribution with shape parameter αb, where b is known. The discussion in the succeeding sections relax this assumption by treating b as an unknown quantity and possessing a non-informative prior.

We derive the Bayes estimates of the parameters α,b and σ under various alternatives such as the scale parameter may be known or unknown and by taking $b<1$ and $b>1$.

3.2 Estimation when $0<b<1$

3.2.1 Estimation with known σ

When the scale parameter σ is known the joint prior density for α and b is calculated on the assumption that they are independently distributed with α following a gamma

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Some of the result in this chapter are to be published in Jeevanand and Nair (1992a).

distribution and b having a uniform distribution over (0,1). That is,

$$\phi(\alpha,b) = C_1\ \alpha^{r-1}\ e^{-t'\alpha},\quad r,t'>0. \tag{3.1}$$

Now using (2.3) and (3.1) the joint posterior density of α and b is obtained as

$$f(\alpha,b|\underline{x}) = C_2\ \Sigma_*\ \alpha^n\ b^k\ e^{-[t+(b-1)t_A]\alpha}\ \alpha^{r-1}\ e^{-t'\alpha},$$

$$= C_2\ \Sigma_*\ \alpha^{n+r-1}\ b^k\ \exp\{-\alpha[t+t'+(b-1)t_A]\},$$

$$= C_2\ \Sigma_*\ \alpha^{m-1}\ b^k\ \exp\{-\alpha[T_A+bt_A]\},\quad \alpha>0, 0<b\le 1, \tag{3.2}$$

where

$$m = n+r,\quad T_A = t + t' - t_A,$$

$$t_A = \sum_{i=1}^{n} \log(x_{A_i}/\sigma) \text{ and } t = \sum_{i=1}^{n} \log(x_i/\sigma).$$

From (3.2) we can obtain the marginal density of α as

$$f(\alpha|\underline{x}) = C_2\ \Sigma_*\ \alpha^{m-1}\ e^{-\alpha T_A} \int_0^1 b^k\ e^{-bt_A\alpha}\ db.$$

Now from Erdelyi (1954) we have

$$\int_0^b t^n\ e^{-pt}\ dt = \frac{n!}{p^{n+1}} - e^{-bp} \sum_{m=0}^{n} (n!/m!)\ \frac{b^m}{p^{n-m+1}}. \tag{3.3}$$

Hence,

$$f(\alpha|\underline{x})=C_2 \sum_{*} \alpha^{m-1} e^{-\alpha T_A} \left\{ \frac{k!}{(\alpha t_A)^{k+1}} - e^{-\alpha t_A} \sum_{j=0}^{k} \frac{k!}{j!(\alpha t_A)^{k-j+1}} \right\},$$

$$= C_2 \sum_{*} \alpha^{m-1} \frac{e^{-\alpha T_A} k!}{(\alpha t_A)^{k+1}} \left\{ 1 - e^{-\alpha t_A} \sum_{j=0}^{k} \frac{(\alpha t_A)^j}{\Gamma(j+1)} \right\},$$

$$= [C_3(m,k,T_A,t_A)]^{-1} \sum_{*} \alpha^{m-k-2} t_A^{-(k+1)} \exp[-\alpha T_A]$$

$$\left\{1 - \sum_{j=0}^{k} \frac{(\alpha t_A)^j}{\Gamma(j+1)} \exp[-\alpha t_A]\right\}, \quad \alpha>0, \qquad (3.4)$$

where the normalising constant

$$C_3(m,k,T_A,t_A) = \sum_{*} t_A^{-(k+1)} \int_0^{\infty} \alpha^{m-k-2} e^{-\alpha T_A}$$

$$\left\{1 - \sum_{j=0}^{k} \frac{(\alpha t_A)^j e^{-\alpha t_A}}{\Gamma(j+1)} \right\} d\alpha,$$

$$= \sum_{*} t_A^{-(k+1)} \left\{ \Gamma(m-k-1) T_A^{-(m-k-1)} \right.$$

$$\left. - \sum_{j=0}^{k} \frac{\Gamma(m-k+j-1) t_A^j}{\Gamma(j+1)\ (T_A+t_A)^{m-k+j-1}} \right\},$$

$$= \sum_{*} t_A^{-(k+1)} \left\{ \Gamma(m-k-1) T_A^{-(m-k-1)} \right.$$

$$\left. - \sum_{j=0}^{k} t_A^{j} \frac{\Gamma(m-k+j-1)}{\Gamma(j+1)} (t+t')^{-(m-k+j-1)} \right\}.$$

(3.5)

The estimator of α under quadratic loss is the posterior mean,

$$\hat{\alpha}_1 = C_3(m+1,k,T_A,t_A)/C_3(m,k,T_A,t_A). \tag{3.6}$$

The expected loss associated with (3.6) is

$$V(\alpha_1|\underline{x}) = [C_3(m+2,k,T_A,t_A)/C_3(m,k,T_A,t_A)] - \hat{\alpha}_1^2. \tag{3.7}$$

In the non-informative prior situation as t' and r tend to zero our estimate have the form

$$\hat{\alpha}_1 = C_3(n+1,k,t-t_A,t_A)/C_3(n,k,t-t_A,t_A). \tag{3.8}$$

It is to be noted that estimates are expressed in terms of gamma functions whose values are readily available from the tables in Abramowitz and Stegun(1972). In finding the Bayes estimator of b we first compute the marginal density of b. This is

$$f(b|\underline{x}) = C_2 \sum_* b^{k-1} \int_0^\infty \alpha^{m-1} e^{-\alpha[T_A + bt_A]} d\alpha,$$

$$= [C_4(m,k,T_A,t_A)]^{-1} \sum_* b^k [T_A+bt_A]^{-m}, \quad 0<b\le 1, \tag{3.9}$$

where

$$C_4(m,k,T_A,t_A) = \sum_* \int_0^1 b^k [T_A+bt_A]^{-m} db,$$

$$= \Sigma_* \; T_A^{-m} \int_0^1 b^k \; [1+b(t_A/T_A)]^{-m} \, db,$$

$$= \Sigma_* \; T_A^{-m} \; (k+1)^{-1} \; {}_2F_1(m, k+1, k+2, (-t_A/T_A)) \qquad (3.10)$$

and $F(\alpha, \beta, y, z)$ is the usual hypergeometric function. In evaluating the integral leading to (3.10) we have used the result from Erdelyi (1954),

$$\int_0^b (1+ax)^{-v} \, x^{s-1} \, dx = s^{-1} \, b^s \; {}_2F_1(v, s, s+1; -ab). \qquad (3.11)$$

Also the estimator under squared error loss is

$$\hat{b}_1 = C_4(m, k+1, T_A, t_A)/C_4(m, k, T_A, t_A) \qquad (3.12)$$

with risk

$$V(b_1 | \underline{x}) = [C_4(m, k+2, T_A, t_A)/C_4(m, k, T_A, t_A)] - \hat{b}_1^{\,2}. \qquad (3.13)$$

Specialising (3.13) to the non-informative case,

$$\hat{b}_1 = C_4(n, k+1, t-t_A, t_A)/C_4(n, k, t-t_A, t_A). \qquad (3.14)$$

The evaluation of the estimators and the associated risks are accomplished either directly through the tables of the hypergeometric function or converting ${}_2F_1$ into gamma functions and then using Fortan subroutine available in IMSL.

3.2.2 Estimation with unknown scale parameter

In this section we deal with the estimation problem in the most general situation when the three parameters α, b, σ are objects of inference. The Bayes procedure involves the likelihood (2.3) in combination with the prior

$$\phi(\alpha,b,\sigma) = C_5\ \alpha^r\ \sigma^{u\alpha-1}\ e^{-z'\alpha}\ ,\quad r,u,z'>0, \tag{3.15}$$

leading to the posterior density

$$f(\alpha,b,\sigma|\underline{x}) = C_6\ \Sigma_*\ \alpha^n\ b^k\ \sigma^{[n+(b-1)k]\alpha}\ e^{-[z+(b-1)z_A]\alpha}\ \alpha^r\ \sigma^{u\alpha-1}\ e^{z'\alpha},$$

$$= C_6\ \Sigma_*\alpha^{n+r}\ b^k\sigma^{[n+u+(b-1)k]\alpha}\ \exp\{-\alpha[z+z'+(b-1)z_A]\},$$

$$= C_6\ \Sigma_*\ \alpha^m\ b^k\ \sigma^{(v+kb)\alpha-1}\ \exp\{-\alpha[R_A+bz_A]\}$$

$$\alpha>0,\ 0<b\le 1,\ 0<\sigma\le\lambda, \tag{3.16}$$

where

$$v = n+u-k,\qquad R_A = z+z'-z_A,\qquad \lambda = \min(x_{(1)},\sigma_0)$$

$$z = \sum_{i=1}^{n} \log x_i \qquad \text{and} \qquad z_A = \sum_{i=1}^{k} \log x_{A_i}\ .$$

It is convenient to work with the marginal densities of (3.16), $f(\alpha|\underline{x})$, $f(b|\underline{x})$ and $f(\sigma|\underline{x})$ in search of the estimators of the parameters. Thus

$$f(\alpha|\underline{x})=C_6\ \Sigma_* \ \alpha^m\ e^{-\alpha R_A} \int_0^1 \int_0^\lambda b^k\ \sigma^{(v+bk)\alpha-1}\ e^{-\alpha b z_A}\, d\sigma\ db,$$

$$=C_6\ \Sigma_* \ \alpha^m\ e^{-\alpha R_A} \int_0^1 b^k\ \frac{\lambda^{(v+bk)\alpha}}{(v+bk)\alpha}\ e^{-\alpha b z_A}\, db,$$

$$=C_6\ \Sigma_* \ \alpha^{m-1}\ e^{-\alpha R'_A} \int_0^1 (b^k/v)[1+(k/v)b]^{-1}\ e^{-\alpha b Q_A}\, db,$$

$$=C_6\ \Sigma_* \ \frac{\alpha^{m-1}}{v}\ e^{-\alpha R'_A} \int_0^1 b^k\ [1+(k/v)b]^{-1}\ (1-b)^{1-1}\ e^{-\alpha b Q_A}\, db,$$

But from Erdelyi (1954),

$$\int_0^1 t^{\alpha-1}\ (1-t)^{\beta-1}\ (1-\sigma t)^{-\nu}\ e^{-pt}\, dt = B(\alpha,\beta)\phi_1(\alpha,\nu,\alpha+\beta,\sigma,-p). \tag{3.17}$$

Hence,

$$f(\alpha|\underline{x})=C_6\ \Sigma_*\alpha^{m-1}\ e^{-\alpha R'_A}\ \nu^{-1}B(k+1,1)\phi_1(k+1,1,k+2,(-k/v),-\alpha Q_A),$$

$$= C_7\ \Sigma_*\ \alpha^{m-1}\ e^{-\alpha R'_A}\ \phi_1(k+1,1,k+2,(-k/v),-\alpha Q_A), \qquad \alpha>0, \tag{3.18}$$

where

$R'_A = R_A - v\log\lambda, \qquad Q_A = z_A - k\log\lambda$

and

$$\phi_1(\alpha,\beta,\gamma;x,y) = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} \frac{(\alpha)_{n+m}\,(\beta)_{n+m}}{(\gamma)_{n+m}\, n!\; m!}\; x^m\, y^n$$

is the confluent hypergeometric series of two variables.

Similarly,

$$f(b|\underline{x}) = C_6 \Sigma_* b^k \int_0^{\infty}\int_0^{\lambda} \alpha^m\, \sigma^{(\nu+kb)\alpha-1}\, e^{-\alpha[R_A+bz_A]}\, d\sigma\, d\alpha,$$

$$= C_6 \Sigma_* b^k \int_0^{\infty} \alpha^m \frac{\lambda^{(\nu+kb)\alpha}}{(\nu+kb)\alpha}\, e^{-\alpha[R_A+bz_A]}\, d\alpha,$$

$$= C_6 \Sigma_* b^k (\nu+bk)^{-1} \int_0^{\infty} \alpha^{m-1}\, e^{-\alpha[R_A'+bQ_A]}\, d\alpha,$$

$$= C_6\, \Gamma(m)\, \Sigma_* b^k (\nu+bk)^{-1} [R_A'+bQ_A]^{-m},$$

$$= C_8 \Sigma_* b^k (\nu+bk)^{-1} [R_A'+bQ_A]^{-m}, \qquad 0<b<1, \qquad (3.19)$$

and

$$f(\sigma|\underline{x}) = C_6 \Sigma_* \sigma^{-1} \int_0^{\infty}\int_0^{1} \alpha^m\, b^k\, \sigma^{(\nu+kb)\alpha-1}\, e^{-\alpha[R_A+bz_A]}\, db\, d\alpha,$$

$$= C_6 \Sigma_* \sigma^{-1} \int_0^{1} b^k\, [(R_A-\nu\log\sigma) + b(z_A-k\log\sigma)]^{-(m+1)} db,$$

$$= C_6 \Sigma_* \sigma^{-1} [(R_A-\nu\log\sigma)]^{-(m+1)} \int_0^{1} b^k [1+bP_A(\sigma)]^{-(m+1)} db.$$

Now using (3.11) we have

$$f(\sigma|\underline{x}) = C_6 \sum_* \sigma^{-1} [(R_A-\nu\log\sigma)]^{-(m+1)} (k+1)^{-1}$$

$$ {}_2F_1(m+1, k+1, k+2, -P_A(\sigma)),$$

$$=C_9 \sum_* \sigma^{-1} [(R_A-\nu\log\sigma)]^{-(m+1)} {}_2F_1(m+1, k+1, k+2, -P_A(\sigma)),$$

$$0<\sigma\leq\lambda, \quad (3.20)$$

where

$$P_A(\sigma) = (z_A-k\log\sigma)/(R_A - \nu\log\sigma).$$

The expression for these densities as evidenced from equations (3.18) through (3.20) are in terms of various special functions. Calculation of the means and variances of these distributions to generate the estimators and associated risks, requires integration of these special functions for which exact analytical methods appear to be lacking. Therefore we resort to numerical techniques which will be explained in Section 3.4

3.3 Estimation when b>1

Under the model assumptions we have made so far (n-k) observations are distributed as Pareto with parameters(α,σ) while k observations follow the same distribution with parameters$(\sigma,\alpha b)$. By virtue of the result in Kale(1976) for b>1 (b<1) the largest k (smallest) observations in the sample have the maximum posterior probability for being the outliers. Thus under the assumption b<1, the outlying observations are most likely to turn up at the left tail which is not likely to arouse much interest in Paretian data analysis, as the distribution provides a realistic model only for values of the variable that exceeds a specific threshold. For the same reason, the contaminating observations for the case b>1 that may possibly occur at the right tail call for special attention. In the succeeding subsections the estimators that are meaningful in such situations will be developed.

3.3.1. Estimation with known scale parameter

In the case of known σ, the likelihood (2.3) along with the gamma prior

$$\phi(\alpha,b) = C_{10}\ \alpha^{r-1}\ e^{-t'\alpha}, \qquad \alpha,r,t'>0,$$

gives the joint posterior for α and b as

$$f(\alpha,b|\underline{x}) = C_{11}\ \Sigma_{*}\ \alpha^{m-1}\ b^{k}\ \exp\{-\alpha[T_A+bt_A]\},\ \alpha>0,b>1. \qquad (3.21)$$

From (3.21) it is seen that α has density

$$f(\alpha|\underline{x}) = C_{11}\Sigma_* \alpha^{m-1} e^{-\alpha T_A} \int_1^\infty b^k e^{-bt_A\alpha} db.$$

But from Erdelyi (1954) we have,

$$\int_b^\infty t^n e^{-pt} dt = e^{-bp} \sum_{m=0}^{n} (n!/m!)\, b^n\, p^{-(n-m+1)}. \qquad (3.22)$$

Hence,

$$f(\alpha|\underline{x}) = C_{11}\, \Sigma_* \alpha^{m-1}\, e^{-\alpha T_A} \left\{ e^{-\alpha T_A} \sum_{j=0}^{k} \frac{k!}{\Gamma(j+1)} (\alpha t_A)^{-(k-j+1)} \right\},$$

$$= [C_{12}(m,k,T_A,t_A)]^{-1}\, \Sigma_* \sum_{j=0}^{k} \left\{ \frac{(t_A)^{-(k-j+1)}}{\Gamma(j+1)} \alpha^{m-k+j-2} e^{-\alpha[T_A+t_A]} \right\},$$

$$=[C_{12}(m,k,T,t_A)]^{-1}\, \Sigma_* \sum_{j=0}^{k} \frac{(t_A)^{-(k-j+1)}}{\Gamma(j+1)} \alpha^{m-k+j-2} e^{-\alpha T},$$

$$\alpha>0, \qquad (3.23)$$

where

$T = t+t'$ and

$$C_{12}(m,k,T,t_A) = \Sigma_* \sum_{j=0}^{k} \frac{(t_A)^{-(k-j+1)}}{\Gamma(j+1)} \int_0^\infty \alpha^{m-k+j-2} e^{-\alpha T} d\alpha.$$

$$= \Sigma_* \sum_{j=0}^{k} \frac{(t_A)^{-(k-j+1)}}{\Gamma(j+1)} \Gamma(m-k+j-1)\ T^{-(m-k+j-1)}.$$

The Bayes estimator of α under squared error loss is

$$\hat{\alpha}_3 = C_{12}(m+1,k,T,t_A)/C_{12}(m,k,T,t_A) \tag{3.24}$$

with risk

$$V(\alpha_3|\underline{x}) = [C_{12}(m+2,k,T,t_A)/C_{12}(m,k,T,t_A)] - \hat{\alpha}_3^2. \tag{3.25}$$

In the non-informative case (3.24) takes the form

$$\hat{\alpha}_3 = C_{12}(n+1,k,t,t_A)/C_{12}(n,k,t,t_A). \tag{3.26}$$

The marginal density of b is

$$f(b|\underline{x}) = [C_{13}(m,k,T_A,t_A)]^{-1} \Sigma_* b^k [T_A+bt_A]^{-m},\ b\geq 1, \tag{3.27}$$

where

$$C_{13}(m,k,T_A,t_A) = \Sigma_* \int_1^\infty b^k [T_A+bt_A]^{-m}\, db,$$

$$= \Sigma_* T_A^{-m} \int_1^\infty b^k [1+b(t_A/T_A)]^{-m}\, db,$$

$$= \Sigma_* \frac{t_A^{-m}}{(m-k-1)}\ {}_2F_1(m, m-k-1, m-k, -(T_A/t_A)).$$

since from Erdelyi (1954), we have

$$\int_1^\infty (1+ax)^{-v}\, x^{s-1}\, dx = \frac{a^{-v}}{(v-s)}\, b^{s-v}\, {}_2F_1(v, v-s, v-s+1, -(1/ab)).$$

(3.28)

Also the estimator under squared error loss, which is the mean of the posterior density, is

$$\hat{b}_3 = C_{13}(m, k+1, T_A, t_A)/C_{13}(m, k, T_A, t_A) \qquad (3.29)$$

with risk

$$V(b_3|\underline{x}) = C_{13}(m, k+2, T_A, t_A)/C_{13}(m, k, T_A, t_A) - \hat{b}_3^{\,2}. \qquad (3.30)$$

In the limiting case as r and t' tends to zero

$$\hat{b}_3 = C_{13}(n, k+1, t-t_A, t_A)/C_{13}(n, k, t-t_A, t_A), \qquad (3.31)$$

corresponds to a non-informative prior situation for the decision maker.

3.3.2. Estimation with unknown scale parameter

In this case when all the parameters are unknown the likelihood (2.3) along with the prior

$$\phi(\alpha, b, \sigma) = C_{14}\, \alpha^{r}\, \sigma^{u\alpha-1}\, e^{-z'\alpha} \qquad z', r, u > 0, \qquad (3.32)$$

lead to the posterior density

$$f(\alpha,b,\sigma|\underline{x}) = C_{15}\, \Sigma_{*}\, \alpha^{m}\, b^{k}\, \sigma^{[\nu+bk]\alpha-1}\, e^{-\alpha[R_A+bz_A]},$$

$$\alpha>0,\ b>1,\ 0<\sigma\le\lambda. \qquad (3.33)$$

We now calculate the marginal densities of α, b and σ.

$$f(\alpha|\underline{x}) = C_{15}\, \Sigma_{*}\, \alpha^{m}\, e^{-\alpha R_A} \int_0^{\lambda}\int_1^{\infty} b^{k}\, \sigma^{(\nu+bk)\alpha-1}\, e^{-b\alpha z_A}\, db\, d\sigma,$$

$$= C_{15}\, \Sigma_{*}\, \alpha^{m}\, e^{-\alpha R_A} \int_1^{\infty} b^{k}\, \frac{\lambda^{(\nu+bk)\alpha}}{(\nu+bk)\alpha}\, e^{-bz_A\alpha}\, db,$$

$$= C_{15}\, \Sigma_{*}\, \alpha^{m-1}\, e^{-\alpha R_A'} \int_1^{\infty} b^{k}\, (\nu+bk)^{-1}\, e^{-b\alpha Q_A}\, db,$$

$$= C_{15}\, \Sigma_{*}\, \alpha^{m-1}\, e^{-\alpha R_A'} \left\{ \int_0^{\infty} (b^{k}/k)\, [(\nu/k)+b]^{-1}\, e^{-\alpha b Q_A}\, db \right.$$

$$\left. - \int_0^{1} [b^{k}/\nu]\, (b(k/\nu)+1)^{-1}\, e^{-\alpha b Q_A}\, db \right\},$$

$$= C_{16}\, \Sigma_{*}\, \alpha^{m-1}\, e^{-\alpha R_A'} \left\{ \frac{\nu^{k}\, \Gamma(k+1)}{k^{k+1}}\, e^{\alpha Q_A(\nu/k)}\, \Gamma(-k,(\alpha\nu/k)Q_A) \right.$$

$$\left. - \frac{B(k+1,1)}{\nu}\, \Phi_1(k+1,1,k+2,(-k/\nu),-\alpha Q_A) \right\},$$

$$\alpha>0. \qquad (3.34)$$

Also,

$$f(b|\underline{x}) = C_{15}\, \Sigma_{*}\, b^{k} \int_0^{\infty}\int_0^{\lambda} \alpha^{m}\, \sigma^{(\nu+bk)\alpha-1}\, e^{-\alpha[R_A+bz_A]}\, db\, d\alpha,$$

$$= C_{15}\ \Sigma_*\ b^k \int_0^\infty \alpha^m \frac{\lambda^{(\nu+bk)\alpha}}{(\nu+bk)\alpha}\ e^{-\alpha[R_A+bz_A]}\ d\alpha,$$

$$= C_{15}\ \Sigma_*\ b^k \int_0^\infty \alpha^{m-1}\ (\nu+bk)^{-1}\ e^{-\alpha[R_A'+bQ_A]}\ d\alpha,$$

$$= C_{15}\ \Sigma_*\ b^k\ \Gamma(m)\ (\nu+bk)^{-1}\ [R_A' + bQ_A]^{-m},$$

$$= C_{17}\ \Sigma_*\ b^k\ (\nu+bk)^{-1}\ [R_A' + bQ_A]^{-m}, \qquad b>1. \qquad (3.35)$$

Finally,

$$f(\sigma|\underline{x}) = C_{15}\ \Sigma_* \int_0^\infty \int_1^\infty \alpha^m\ b^k\ \sigma^{(\nu+bk)\alpha-1}\ e^{-\alpha[R_A+bz_A]}\ db\ d\alpha,$$

$$= C_{15}\ \Sigma_*\ \sigma^{-1} \int_1^\infty b^k\ [R_A-\nu\log\sigma + b(z_A-k\log\sigma)]^{-(m+1)}\Gamma(m+1)db,$$

$$= C_{15}\ \Sigma_*\ \Gamma(m+1)\ \sigma^{-1}\ [R_A-\nu\log\sigma]^{-(m+1)} \int_1^\infty b^k\ [1+bP_A(\sigma)]^{-m}\ db.$$

Using (3.28) we have,

$$f(\sigma|\underline{x})=C_{15}\ \Sigma_*\ \Gamma(m+1)\ \sigma^{-1}\ [R_A-\nu\log\sigma]^{-(m+1)}[P_A(\sigma)]^{-m}\ (m-k-1)^{-1}$$

$$_2F_1(m+1, m-k, m-k+1, [-1/(P_A(\sigma)]),$$

$$=C_{18}\ \Sigma_*\sigma^{-1}\ [Q_A-\nu\log\sigma]^{-(m+1)}\ {}_2F_1(m+1, m-k, m-k+1, [-1/(P_A(\sigma)]),$$

$$0<\sigma\le\lambda=\min(\sigma_0, x_{(1)}), \qquad (3.36)$$

where

$$\Gamma(a,x) = \int_x^{\infty} e^{-t}\, t^{a-1}\, dt$$

is the incomplete gamma function.

In this case also we seek numerical methods to evaluate the estimates $\hat{\alpha}_4, \hat{b}_4$ and $\hat{\sigma}_2$ and their corresponding risks.

3.4 Discussion

In order to assess how the various estimates behave in a specific sampling situation samples of various sizes and with different parameter values were generated and the estimates of the parameters and their corresponding expected losses were calculated. One such typical example for the data in Section 2.5, the values of the estimates and their corresponding losses are presented in Tables 3.1,3.2 and 3.3. It is observed that the bias and the expected loss in all cases are less than that of the estimates calculated by treating the data as iid observation as in Arnold and Press (1983). Also the bias and the expected loss tend to dicrease along with increasing values of α. However, they are not much sensitive to σ values. With increasing values of n, the bias and the expected loss get decreased. Further the same behaviour appears when t' increases for a fixed r while when r increases for fixed t' no such phenomenon is guaranteed.

To obtain the estimates in the specific case when all the three parameters are unknown, we have used the Fortran 77 subroutine for numerical integral available in mathematical library of the IMSL, which is based on the Gauss-Kronard rule (see the IMSL reference manual, Math/Library, Vol.2,pp 561-606.) The computer program for the calculation of all the estimates and their expected loss is given in Appendix.

Some noteworthy features of the procedure proposed in this chapter are

1) it is not necessary to know which of the observation in the sample is the outlier.

2) the data need not be ordered to arrive at the estimates as in the case of the earlier methods (e.g. Kale and Kale(1992), which results in considerable saving of time, computational ease and lesser space for storage of data in the computer.

3) our estimates are not unduly affected by treating a suspected observation as an outlier. This means that

one can go ahead with the estimation of parameters without waiting for any special test of discordancy.

4) in the light of the computation carried out so far it appears that a sample of moderate size would give a reasonably good estimate of the parameters.

Table 3.1

Estimates when σ is known

r	t'	$\hat{\alpha}_1$	$\hat{b}_1$
1	1	2.905 (0.469)	5.185 (2.903×10^{-4})
1	2	2.501 (0.347)	6.019 (2.153×10^{-4})
1	2.5	2.339 (0.304)	6.437 (1.883×10^{-4})
1	3	2.196 (0.268)	6.854 (1.661×10^{-4})
2	1	3.066 (0.495)	5.225 (2.620×10^{-4})
2	2	2.64 (0.367)	6.067 (1.944×10^{-4})
2	2.5	2.469 (0.321)	6.487 (1.700×10^{-4})
2	3	2.318 (0.283)	6.908 (1.499×10^{-4})
3	1	3.227 (0.521)	5.263 (2.380×10^{-4})

cont...

r	t^*	$\hat{\alpha}_1$	$\hat{b}_1$
3	2	2.779 (0.386)	6.11 (1.760×10^{-4})
3	2.5	2.599 (0.388)	6.534 (1.540×10^{-4})
3	3	2.440 (0.298)	6.958 (1.360×10^{-4})
Non-informative case		3.271 (0.629)	5.594 (2.903×10^{-4})

Table 3.2

Estimates when σ is unknown and $\sigma_0 < x_{(1)}$ (σ_0=150)

u=0.001				
r	z'	$\hat{a}_4$	$\hat{b}_4$	$\hat{\sigma}_2$
1	1	3.125	5.186	147.501
		(0.501)	(2.903×10^{-4})	(2.176×10^{-4})
1	2	2.870	6.110	147.645
		(0.418)	(2.016×10^{-4})	(2.180×10^{-4})
1	2.5	2.490	6.310	147.166
		(0.334)	(1.940×10^{-4})	(2.166×10^{-4})
1	3	2.423	6.958	146.871
		(0.296)	(1.360×10^{-4})	(2.157×10^{-4})
2	1	3.218	5.305	147.877
		(0.541)	(2.63×10^{-4})	(2.187×10^{-4})
2	2	2.809	6.018	148.242
		(0.403)	(2.15×10^{-4})	(2.198×10^{-4})
2	2.5	2.69	6.37	149.388
		(0.375)	(1.67×10^{-4})	(2.178×10^{-4})
2	3	2.514	6.908	147.694
		(0.331)	(1.499×10^{-4})	(2.151×10^{-4})

cont...

r	z'	$\hat{a}_4$	$\hat{b}_4$	$\hat{\sigma}_2$
3	1	3.494	5.315	146.871
		(0.610)	(2.73×10^{-4})	(2.157×10^{-4})
3	2	2.947	6.07	146.857
		(0.542)	(1.901×10^{-4})	(2.156×10^{-4})
3	2.5	2.768	6.31	146.894
		(0.355)	(1.883×10^{-4})	(2.158×10^{-4})
3	3	2.589	6.917	147.166
		(0.355)	(1.47×10^{-4})	(2.166×10^{-4})

u=0.1

r	z'	$\hat{a}_4$	$\hat{b}_4$	$\hat{\sigma}_2$
1	1	3.272	5.19	147.86
		(0.569)	(2.91×10^{-4})	(2.166×10^{-4})
1	2	2.849	6.019	147.69
		(0.369)	(2.153×10^{-4})	(2.15×10^{-4})
1	2.5	2.676	6.36	147.124
		(0.326)	(2.014×10^{-4})	(2.135×10^{-4})
1	3	2.554	6.891	146.881
		(0.311)	(1.967×10^{-4})	(2.177×10^{-4})
2	1	3.536	5.325	148.079
		(0.568)	(2.62×10^{-4})	(2.193×10^{-4})

cont...

r	z'	$\hat{a}_4$	$\hat{b}_4$	$\hat{\sigma}_2$
2	2	2.877 (0.514)	6.11 (2.016×10^{-4})	147.501 (2.176×10^{-4})
2	2.5	2.77 (0.384)	6.487 (1.7×10^{-4})	148.877 (2.187×10^{-4})
2	3	2.59 (0.336)	6.991 (2.59×10^{-4})	147.645 (2.18×10^{-4})
3	1	3.829 (0.671)	5.415 (2.67×10^{-4})	146.984 (2.458×10^{-4})
3	2	3.018 (0.61)	6.17 (1.977×10^{-4})	146.613 (2.39×10^{-4})
3	2.5	2.941 (0.433)	6.39 (1.87×10^{-4})	147.891 (2.187×10^{-4})
3	3	2.72 (0.352)	6.861 (1.46×10^{-4})	148.079 (2.193×10^{-4})

Table 3.3

Estimates when σ unknown and $\sigma_0 \geq x_{(1)}$

u = 0.001				
r	z'	$\hat{a}_4$	$\hat{b}_4$	$\hat{\sigma}_2$
1	1	3.212 (0.546)	5.285 (2.913×10^{-4})	149.039 (2.221×10^{-4})
1	2	2.824 (0.416)	6.119 (2.153×10^{-4})	149.49 (2.32×10^{-4})
1	2.5	2.646 (0.333)	6.357 (1.99×10^{-4})	149.775 (2.22×10^{-4})
1	3	2.491 (0.296)	6.904 (1.661×10^{-4})	148.838 (2.251×10^{-4})
2	1	3.497 (0.549)	5.497 (2.64×10^{-4})	150.018 (2.251×10^{-4})
2	2	2.958 (0.428)	6.107 (2.013×10^{-4})	149.456 (2.234×10^{-4})
2	2.5	2.772 (0.349)	6.547 (1.883×10^{-4})	148.455 (2.226×10^{-4})
2	3	2.556 (0.342)	7.08 (1.819×10^{-4})	148.438 (2.203×10^{-4})
3	1	3.682 (0.685)	5.671 (1.99×10^{-4})	150.078 (2.22×10^{-4})

cont...

r	z'	$\hat{a}_4$	$\hat{b}_4$	$\hat{\sigma}_2$
3	2	3.107	6.31	149.438
		(0.483)	(1.763×10^{-4})	(2.213×10^{-4})
3	2.5	2.958	6.11	149.65
		(0.398)	(1.542×10^{-4})	(2.24×10^{-4})
3	3	2.772	6.95	149.532
		(0.349)	(1.96×10^{-4})	(2.236×10^{-4})

u = 0.10

r	z'	$\hat{a}_4$	$\hat{b}_4$	$\hat{\sigma}_2$
1	1	3.537	5.325	149.65
		(0.595)	(2.62×10^{-4})	(2.24×10^{-4})
1	2	2.958	6.207	149.455
		(0.398)	(2.16×10^{-4})	(2.204)
1	2.5	2.772	6.487	149.704
		(0.350)	(1.7×10^{-4})	(2.20)
1	3	2.646	6.067	149.921
		(0.333)	(1.44×10^{-4})	(2.231)
2	1	3.706	5.54	149.793
		(0.624)	(2.65×10^{-4})	(2.24)
2	2	3.172	6.271	149.204
		(0.458)	(2.213×10^{-4})	(2.226)

cont...

r	z'	$\hat{a}_4$	$\hat{b}_4$	$\hat{\sigma}_2$
2	2.5	2.958 (0.398)	6.646 (1.763×10^{-4})	148.704 (2.211)
2	3	2.772 (0.349)	6.994 (1.499×10^{-4})	148.021 (2.88×10^{-4})
3	1	3.861 (0.708)	5.791 (2.21×10^{-4})	149.65 (2.24×10^{-4})
3	2	3.369 (0.521)	6.41 (1.83×10^{-4})	149.351 (2.231×10^{-4})
3	2.5	3.066 (0.496)	6.41 (1.549×10^{-4})	149.351 (2.223×10^{-4})
3	3	2.883 (0.416)	7.04 (1.421×10^{-4})	149.285 (2.229×10^{-4})
Non-informative case		3.417 (0.669)	5.801 (2.19×10^{-4})	150.832 (2.273×10^{-4})

CHAPTER IV
ESTIMATION OF PARETO SURVIVAL FUNCTION

4.1 Introduction

In the present chapter the problem of estimating the survival function

$$R = R(x) = P[X>x] = \left(\frac{x}{\sigma}\right)^{-\alpha}, \qquad (4.1)$$

(or equivalently the distribution function) of the Pareto distribution when the sample contains discordant observations, is discussed. Bayes point estimates and credible intervals are obtained by assuming three different outlier generating models. These arise as a consequence of the assumptions (i)a subset of k observations has the same probability to be discordant as any other such subset (ii) the k larget members of the sample are outliers and (iii) after identifying the outliers as in (ii) only the first (n-k) order statistics are utilized for estimation. With a view to compare the results arising from the three models, the behaviour of the point and interval estimates are studied through simulation by varying the sample size and the hyper-parameters of the prior distribution. In the deliberations that follow the scale parameter σ which is same for all the models is taken as known. (Jeevanand and Nair (1993b))

4.2 The Exchangeable model

Following the discussions in Section 3.3.1 relating to the exchangeable model, by choosing a gammma (r,t') prior for α and a vague uniform prior for b, the joint posterior density for (α,b) is

$$f(\alpha,b|\underline{x}) = C_1 \Sigma_* \alpha^{m-1} b^k \exp(-\alpha[T_A+bt_A]\alpha), \quad \alpha>0, b>1, \quad (4.2)$$

where m, T_A, t_A and t are as defined in equation (3.21).

Since R is a parametric function, its posterior distribution is readily obtained by transforming (α,b) into (R,b). This leaves

$$f(R,b|\underline{x})=C_1 \Sigma_* \left(\frac{-1}{z}\log R\right)^{m-1} b^k \left(\frac{1}{Rz}\right)\exp\left\{-(T_A+bt_A)\left(\frac{-1}{z}\log R\right)\right\},$$

$$=C_2 \Sigma_*(-\log R)^{m-1} b^k R^{(T_A+bt_A)z^{-1}-1}, \quad 0<R\leq 1, b>1, \quad (4.3)$$

where as usual, $z = \log(x/\sigma)$.

From (4.3) integrating out b, the marginal distribution of R becomes

$$f_1(R|\underline{x}) = C_2 \sum_* (-\log R)^{m-1} R^{T_A z^{-1}-1} I(b), \qquad (4.4)$$

where

$$I(b) = \int_1^\infty b^k R^{bt_A z^{-1}} db,$$

$$= \int_1^\infty b^k \exp\{-bt_A z^{-1} \log R\} db,$$

$$= \exp\left\{t_A z^{-1} \log R\right\} \sum_{j=0}^{k} \frac{\Gamma(k+1)}{\Gamma(j+1)} \left[\frac{t_A}{z}\right]^{-(k-j+1)}$$

Using (3.22),

$$I(b) = R^{t_A z^{-1}} \sum_{j=0}^{k} \frac{\Gamma(k+1)}{\Gamma(j+1)} \left[\frac{t_A}{z}\right]^{-(k-j+1)} \qquad (4.5)$$

Substituting the last result in (4.4) and simplifying, the posterior distribution of R turns out to be

$$f_1(R|\underline{x}) = [C_3(m,k,0)]^{-1} \sum_* \sum_{j=0}^{k} \frac{(t_A/z)^j}{\Gamma(j+1)} R^{(t+t')z^{-1}-1}$$

$$(-\log R)^{m-k+j-2}, \quad 0<R\leq 1. \quad (4.6)$$

For a complete specification of (4.6), we need the value of the normalising constant, which is

$$C_3(m,k,d)=\Sigma_* \sum_{j=0}^{k} \frac{(t_A/z)^j}{\Gamma(j+1)} \int_0^1 (-\log R)^{m-k+j-2}\, R^d\, R^{(t+t')z^{-1}-1} dR.$$

Setting u = -logR in the integral on the right side,

$$\int_0^1 (-\log R)^{m-k+j-2}\ R^{(t+t')z^{-1}+d-1} dR$$

$$= \int_0^\infty u^{m-k+j-2}\ \exp\{-u[(t+t')z^{-1}+d]\}\ du,$$

$$= \Gamma(m-k+j-1)\ \left[(t+t')z^{-1}+d\right]^{-(m-k+j-1)},$$

$$=\Gamma(m-k+j-1)\ z^{(m-k+j-1)}(t+t'+dz)^{-(m-k+j-1)},$$

so that

$$C_3(m,k,d) = \Sigma_* \sum_{j=0}^{k} \frac{\Gamma(m-k+j-1)}{\Gamma(j+1)} \frac{t_A^{\ j}\ z^{m-k-1}}{(t+t'+dz)^{m-k+j-1}} . \qquad (4.7)$$

Under the quadratic loss, the Bayes estimator of R is

$$\hat{R}_1 = E(R|\underline{x}) = C_3(m,k,1)/C_3(m,k,0) \qquad (4.8)$$

with expected loss resulting from (4.8),

$$V(R_1|\underline{x}) = [C_3(m,k,2)/C_3(m,k,0)] - \hat{R}_1^{\ 2}. \qquad (4.9)$$

On the other hand, if the interest lies in providing an interval estimate of R we can employ the equal tail credible interval of the form (L_1, U_1) with coverage probability $(1-\beta)$, $0<\beta<1$. This is, however, the solution of the equations,

$$\int_0^{L_1} f_1(R|\underset{\sim}{x})\, dR = \beta/2 = \int_{U_1}^{1} f_1(R|\underset{\sim}{x})\, dR. \qquad (4.10)$$

As before, setting $u = -\log R$ we can deduce from (3.22) that

$$\int_0^{L_1} (-\log R)^{m-k+j-2}\, R^{(t+t')z^{-1}-1} dR$$

$$= \int_{\log L_1^{-1}}^{\infty} u^{m-k+j-2}\, e^{-(t+t')z^{-1}\, u}\, du,$$

$$= \sum_{\ell=0}^{m-k+j-2} L_1^{(t+t')z^{-1}} \frac{(-\log L_1)^{\ell}\, \Gamma(m-k+j-2)}{\Gamma(\ell+1)\, [(t+t')z^{-1}]^{m-k+j-1-\ell}},$$

$$= \sum_{\ell=0}^{m-k+j-2} \frac{\Gamma(m-k+j-2) L_1^{(t+t')z^{-1}} z^{m-k-\ell-1+j} (-\log L_1)^{\ell}}{\Gamma(\ell+1)\ (t+t')^{m-k-\ell+j-1}}.$$

Since $\int_{U_1}^{1} f_1(R|\underset{\sim}{x})\, dR = \beta/2$ implies $\int_0^{U_1} f_1(R|\underset{\sim}{x}) dR = 1 - \beta/2$,

we have the following equations to be solved for L_1 and U_1,

$$C_3^{-1} \Sigma_* \sum_{j=0}^{k} \sum_{\ell=0}^{m-k+j-2} \frac{(t_A)^j \, \Gamma(m-k+j-2) \, (-\log L_1)^{\ell} \, z^{m-k-\ell-1}}{\Gamma(j+1) \, \Gamma(\ell+1) \, (t+t')^{m-k+j-\ell-1}}$$

$$L_1^{(t+t')z^{-1}} = \beta/2, \qquad (4.11)$$

and

$$C_3^{-1} \Sigma_* \sum_{j=0}^{k} \sum_{\ell=0}^{m-k+j-2} \frac{(t_A)^j \, \Gamma(m-k+j-2) \, (-\log U_1)^{\ell} \, z^{m-k-\ell-1}}{\Gamma(j+1) \, \Gamma(\ell+1) \, (t+t')^{m-k+j-\ell-1}}$$

$$U_1^{(t+t')z^{-1}} = 1-\beta/2. \qquad (4.12)$$

4.3 The Identifiable model

Under the identified outlier model (see Section 1.2), the subset which forms the outliers is known. When uniform prior is assumed over the subsets of k integers out of n, Kale (1976) has shown that for b>1, the last k components of the order statistics of the sample have the largest probability of being outliers. Accordingly we can write the likelihood as

$$\ell(\underline{x}|\alpha, b) = C_4 \, \alpha^n \, b^k \exp[-(Q+bS)\alpha], \qquad (4.13)$$

with

$$S = \sum_{i=1}^{k} \log(x_{(n-i+1)}/\sigma) \qquad \text{and} \qquad Q = t - S.$$

By proceeding on lines exactly similar to those in the previous section, we have the joint posterior density of (α,b) as

$$f(\alpha,b|\underline{x}) = C_5\ \alpha^{m-1}\ b^{k}\ \exp[-\alpha(Q+t'+bS)],\quad \alpha>0, b>1, \qquad (4.14)$$

and the posterior density of R is

$$f_2(R|\underline{x})=[C_6(m,k,0)]^{-1}\sum_{j=0}^{k}\frac{(S/z)^{j}}{\Gamma(j+1)}\ R^{(t+t')z^{-1}-1}\ (-\log R)^{m-k+j-2},$$

$$0<R\le 1, \qquad (4.15)$$

where

$$C_6(m,k,d) = \sum_{j=0}^{k}\frac{(S/z)^{j}}{\Gamma(j+1)}\int_0^1 R^{(t+t')z^{-1}-1}\ R^{d}\ (-\log R)^{m-k+j-2}\,dR,$$

$$= \sum_{j=0}^{k}\frac{(S/z)^{j}}{\Gamma(j+1)}\int_0^\infty u^{m-k+j-2}\ \exp\{-u[(t+t')z^{-1}+d]\}du,$$

$$= \sum_{j=0}^{k}\frac{(S/z)^{j}}{\Gamma(j+1)}\ \Gamma(m-k+j-1)\ \left[(t+t')z^{-1}+d\right]^{-(m-k+j-1)}.$$

$$= \sum_{j=0}^{k}\frac{(S)^{j}}{\Gamma(j+1)}\ \frac{\Gamma(m-k+j-1)\ z^{m-k-1}}{(t+t'+dz)^{m-k+j-1}}. \qquad (4.16)$$

Hence the Bayees estimate of R is

$$\hat{R}_2 = C_6(m,k,1)/C_6(m,k,0) \qquad (4.17)$$

and the corresponding expected loss becomes

$$V(R_2|\underline{x}) = [C_6(m,k,2)/C_6(m,k,0)] - \hat{R}_2^{\,2}. \qquad (4.18)$$

In the case of interval estimation a $100(1-\beta)\%$ credible interval(L_2,U_2) for R is the solution of the equations,

$$C_6^{-1}\ L_2^{\,(t+t')z^{-1}} \sum_{j=0}^{k} \sum_{\ell=0}^{m-k+j-2} \frac{S^j\ \Gamma(m-k+j-2)(-\log L_2)^{\ell}\ z^{m-k-\ell-1}}{\Gamma(j+1)\ \Gamma(\ell+1)\ (t+t')^{m-k+j-\ell-1}} = \beta/2, \qquad (4.19)$$

and

$$C_6^{-1}\ U_2^{\,(t+t')z^{-1}} \sum_{j=0}^{k} \sum_{\ell=0}^{m-k+j-2} \frac{S^j\ \Gamma(m-k+j-2)(-\log U_2)^{\ell}\ z^{m-k-\ell-1}}{\Gamma(j+1)\ \Gamma(\ell+1)\ (t+t')^{m-k+j-\ell-1}} = 1-\beta/2. \qquad (4.20)$$

where $C_6 = C_6(m,k,0)$ is as given in equation (4.16)

4.4 The Censored model

When the sample observations are ordered and the largest k members therin are deemed to be outliers, one can carry out inference by discarding the outlying observations. In this connection,we make use of the trimmed estimator

$$y=\hat{\alpha} = (n-m')\left[\sum_{i=1}^{m'-1} \log(x_{(i)}/\sigma) + (n-m')\log(x_{(m')}/\sigma)\right]^{-1}, \quad (4.21)$$

based on the first $m'=(n-k)$ order statistics proposed by Arnold (1983). The choice of (4.21) is motivated by the fact that it is robust against the incidence of outlying observations in the samples, as estabilished in Kale and Sinha (1971). Further such trimmed estimates have been used in the context of the exponential distribution by Sinha (1973a) and Lingappaiah(1989a) and for the gamma distribution by Lingappaiah (1989b). Now, the distribution of y is specified by the density

$$f(y|\alpha) = \frac{(m'\alpha)^{m'}}{\Gamma(m')} y^{-(m'+1)} \exp(-m'\alpha/y), \quad y>\sigma, \alpha>0. \quad (4.22)$$

To arrive at (4.22) we have used the fact that y^{-1} is distributed as a gamma variate with parameters m' and $m'\alpha$. When the prior distribution for α is gamma with parameters r and t', the posterior density of α is [Arnold and Press (1989)]

$$f(\alpha|y) = \frac{\omega^{p}}{\Gamma(p)} \alpha^{p-1} \exp[-\omega\, \alpha], \quad \alpha>0, \quad (4.23)$$

where $p = m'+r$ and $\omega = t' + m'y^{-1}$.

It is now easy to see that the posterior density of R has the form

$$f_3(R|y) = \frac{\omega^p}{\Gamma(p)} \left[\frac{-1}{z} \log R\right]^{p-1} \exp\left(-\omega \left[\frac{-1}{z} \log R\right]\right) \frac{1}{2R} ,$$

$$= \frac{\omega^p}{z^p \Gamma(p)} (-\log R)^{p-1} R^{(\omega/z)-1}, \quad 0<R\le 1. \qquad (4.24)$$

From (4.24), the Bayes estimator under squared error loss is

$$\hat{R}_3 = (1+z/\omega)^{-p} \qquad (4.25)$$

with expected loss

$$V(R_3|\underline{x}) = [1+2z/\omega]^{-p} - [1+(z/\omega)]^{-2p}. \qquad (4.26)$$

Further, $100(1-\beta)\%$ credible interval (L_3, U_3) of R under this model is solved from

$$\int_0^{L_3} \frac{\omega^p}{z^p \Gamma(p)} (-\log R)^{p-1} R^{(w/z)-1} \, dR = \beta/2$$

and

$$\int_{U_3}^{1} \frac{\omega^p}{z^p \Gamma(p)} (-\log R)^{p-1} R^{(w/z)-1} \, dR = \beta/2.$$

Setting $u = -\log R$ in the first of the above equations and using the formula (3.22) we get

$$\beta/2 = \int_{\log L_3^{-1}}^{\infty} \frac{\omega^p}{z^p \Gamma(p)} u^{p-1} e^{(-\omega/z)u} \, du.$$

$$= \frac{(\omega/z)^p}{\Gamma(p)} \exp\{(\omega/z)\log L_3\} \sum_{j=0}^{p-1} \frac{\Gamma(p)\ (-\log L_3)^j}{\Gamma(j+1)(\omega/z)^{p-j}} ,$$

$$= \frac{(\omega/z)^p}{\Gamma(p)} (L_3)^{\omega/z} \sum_{j=0}^{p-1} \frac{\Gamma(p)}{\Gamma(j+1)} (-\log L_3)^j (\omega/z)^{-(p-j)},$$

which reduces to,

$$L_3^{\omega/2} \sum_{j=0}^{p-1} \frac{\left[\frac{-z}{\omega} \log L_3\right]^j}{\Gamma(j+1)} = \beta/2. \qquad (4.27)$$

Similarly U_3 is yielded as the solution of the equation

$$U_3^{\omega/2} \sum_{j=0}^{p-1} \frac{\left[\frac{-z}{\omega} \log U_3\right]^j}{\Gamma(j+1)} = 1-\beta/2. \qquad (4.28)$$

4.5 Discussion

The Bayes estimates derived above, except that in equation (4.25), do not have simple closed forms and therefore, it is difficult to compare their performance algebraically. Hence an empirical validation is attempted. This is done by generating samples of different sizes for specified values of the population parameters and comparing the losses and bias of various estimators. One such comparison is offered in Figure 4.1 where the bias in

$\hat{R}_1, \hat{R}_2, \hat{R}_3$ in that order is represented by bars (whose lengths are proportional to the absolute bias) at points that correspond to selected values of the hyper-parameters of the prior distribution. The bar diagram at the left hand corner represents the non-informative prior situation obtained when $r, t' \to 0$. In general, the absolute bias $|\hat{R}_i - R_i|$, $i=1,2,3$ is smallest for $\hat{R}_1$, the estimates under model I. As expected, when the samples become large both the bias and the expected loss tend to be smaller, irrespective of the value of R and the prior parameters. Table 4.1 presents the actual values of the estimates for a sample of 10 observations: 152.1978, 152.9322, 167.6085, 183.1596, 183.4318, 196.6562, 211.3890, 236.3444, 275.9214, 354.2501 when $\alpha=2.5$, $\sigma=150$, $k=1$ and $x=180$. Notice that the values inside the braces in the table provide the expected losses. The interval estimations of $\hat{R}_i$ when $r=t'=2$ are also presented in Table 4.1 for the sake of illustration. Also Figures 4.2 to 4.4 represent the posterior plot for $R_i (i = 1,2,3)$ when $r=t'=2$. Further, we have presented in Tables 4.2 and 4.3 the bias and expected losses of $R_i (i=1,2,3)$ for different values of x and n. The program to calculate the estimates are given in Appendix .

Table 4.1

Bayes estimates and risks of R(x=180)

r	t*	$\hat{R}_1$	$\hat{R}_2$	$\hat{R}_3$
1	1	0.596	0.573	0.537
		(0.011)	(0.012)	(0.012)
1	2	0.663	0.644	0.614
		(0.009)	(0.009)	(0.01)
1	3	0.711	0.696	0.67
		(0.007)	(0.007)	(0.008)
2	1	0.558	0.536	0.501
		(0.011)	(0.011)	(0.011)
2	2	0.629	0.611	0.582
		(0.009)	(0.009)	(0.010)
2	3	0.681	0.666	0.641
		(0.007)	(0.007)	(0.008)
3	1	0.523	0.501	0.467
		(0.011)	(0.011)	(0.011)
3	2	0.629	0.58	0.551
		(0.009)	(0.009)	(0.009)

cont...

r	t'	$\hat{R}_1$	$\hat{R}_2$	$\hat{R}_3$
3	3	0.653	0.637	0.613
		(0.008)	(0.008)	(0.008)
Non-informative		0.542	0.513	0.465
prior		(0.015)	(0.015)	(0.015)
Interval Estimates		(0.40,0.81)	(0.42,0.72)	(0.43,0.78)
r = t' = 2				

Table 4.2

Absolute bias and expected loss of R (x = 180)

		n=10			n=30		
r	t'	R_1	R_2	R_3	R_1	R_2	R_3
1	1	0.038 (0.011)	0.061 (0.012)	0.097 (0.012)	0.058 (0.004)	0.065 (0.004)	0.110 (0.004)
1	2	0.029 (0.009)	0.010 (0.009)	0.020 (0.010)	0.033 (0.003)	0.041 (0.003)	0.081 (0.004)
1	3	0.077 (0.007)	0.062 (0.007)	0.036 (0.008)	0.012 (0.003)	0.020 (0.003)	0.055 (0.004)
2	1	0.076 (0.011)	0.098 (0.011)	0.133 (0.011)	0.067 (0.004)	0.076 (0.004)	0.123 (0.004)
2	2	0.005 (0.009)	0.023 (0.009)	0.052 (0.010)	0.043 (0.003)	0.052 (0.003)	0.093 (0.004)
2	3	0.047 (0.007)	0.032 (0.007)	0.007 (0.008)	0.022 (0.003)	0.030 (0.003)	0.066 (0.004)
3	1	0.111 (0.011)	0.133 (0.011)	0.167 (0.011)	0.078 (0.004)	0.087 (0.004)	0.135 (0.004)

cont..

		n=10			n=30		
r	t'	R_1	R_2	R_3	R_1	R_2	R_3
3	2	0.005 (0.009)	0.054 (0.009)	0.092 (0.009)	0.054 (0.003)	0.063 (0.003)	0.104 (0.004)
3	3	0.019 (0.008)	0.003 (0.008)	0.021 (0.008)	0.032 (0.003)	0.040 (0.003)	0.077 (0.003)
Jeefry's prior		0.092 (0.015)	0.121 (0.015)	0.169 (0.015)	0.071 (0.004)	0.080 (0.004)	0.132 (0.004)

		n=50		
r	t'	R_1	R_2	R_3
1	1	0.009 (0.002)	0.015 (0.002)	0.030 (0.002)
1	2	0.004 (0.002)	0.002 (0.002)	0.016 (0.002)
1	3	0.017 (0.002)	0.011 (0.002)	0.002 (0.002)
2	1	0.016 (0.002)	0.022 (0.002)	0.037 (0.002)

cont...

n=50				
r	t'	R_1	R_2	R_3
2	2	0.003 (0.002)	0.009 (0.002)	0.023 (0.002)
2	3	0.010 (0.002)	0.004 (0.002)	0.009 (0.002)
3	1	0.023 (0.002)	0.028 (0.002)	0.044 (0.002)
3	2	0.010 (0.002)	0.015 (0.002)	0.030 (0.002)
3	3	0.003 (0.002)	0.003 (0.002)	0.016 (0.002)
Jeffery's prior		0.016 (0.002)	0.021 (0.002)	0.037 (0.002)

Table 4.3

Absolute bias and expected loss of R (x = 260)

		n=10			n=30		
r	t'	R_1	R_2	R_3	R_1	R_2	R_3
1	1	0.022	0.045	0.080	0.054	0.063	0.104
		(0.014)	(0.012)	(0.010)	(0.004)	(0.003)	(0.003)
1	2	0.056	0.032	0.004	0.030	0.039	0.078
		(0.016)	(0.015)	(0.013)	(0.004)	(0.004)	(0.003)
1	3	0.121	0.099	0.063	0.006	0.016	0.053
		(0.016)	(0.015)	(0.014)	(0.004)	(0.004)	(0.004)
2	1	0.060	0.081	0.110	0.065	0.074	0.113
		(0.011)	(0.010)	(0.008)	(0.004)	(0.003)	(0.003)
2	2	0.013	0.008	0.040	0.041	0.050	0.089
		(0.013)	(0.012)	(0.010)	(0.004)	(0.004)	(0.003)
2	3	0.078	0.057	0.019	0.018	0.027	0.064
		(0.016)	(0.014)	(0.012)	(0.004)	(0.004)	(0.003)
3	1	0.092	0.110	0.135	0.076	0.084	0.123
		(0.008)	(0.007)	(0.006)	(0.003)	(0.003)	(0.002)

cont...

		n=10			n=30		
r	t'	R_1	R_2	R_3	R_1	R_2	R_3
3	2	0.023 (0.011)	0.042 (0.010)	0.070 (0.009)	0.052 (0.003)	0.061 (0.003)	0.098 (0.003)
3	3	0.040 (0.012)	0.020 (0.012)	0.008 (0.010)	0.030 (0.004)	0.038 (0.004)	0.075 (0.003)
Jeffer-y's prior		0.070 (0.013)	0.095 (0.011)	0.031 (0.008)	0.068 (0.004)	0.077 (0.003)	0.120 (0.003)

		n=50		
r	t'	R_1	R_2	R_3
1	1	0.012 (0.002)	0.017 (0.002)	0.031 (0.002)
1	2	0.001 (0.003)	0.006 (0.003)	0.013 (0.002)
1	3	0.010 (0.003)	0.005 (0.003)	0.006 (0.003)

cont...

n=50				
r	t'	R_1	R_2	R_3
2	1	0.018	0.023	0.037
		(0.002)	(0.002)	(0.002)
2	2	0.007	0.012	0.024
		(0.002)	(0.002)	(0.002)
2	3	0.001	0.004	0.012
		(0.002)	(0.002)	(0.002)
3	1	0.024	0.029	0.043
		(0.002)	(0.002)	(0.002)
3	2	0.013	0.017	0.030
		(0.002)	(0.002)	(0.002)
3	3	0.002	0.006	0.018
		(0.002)	(0.002)	(0.002)
Jeffery's		0.018	0.023	0.037
prior		(0.002)	(0.002)	(0.002)

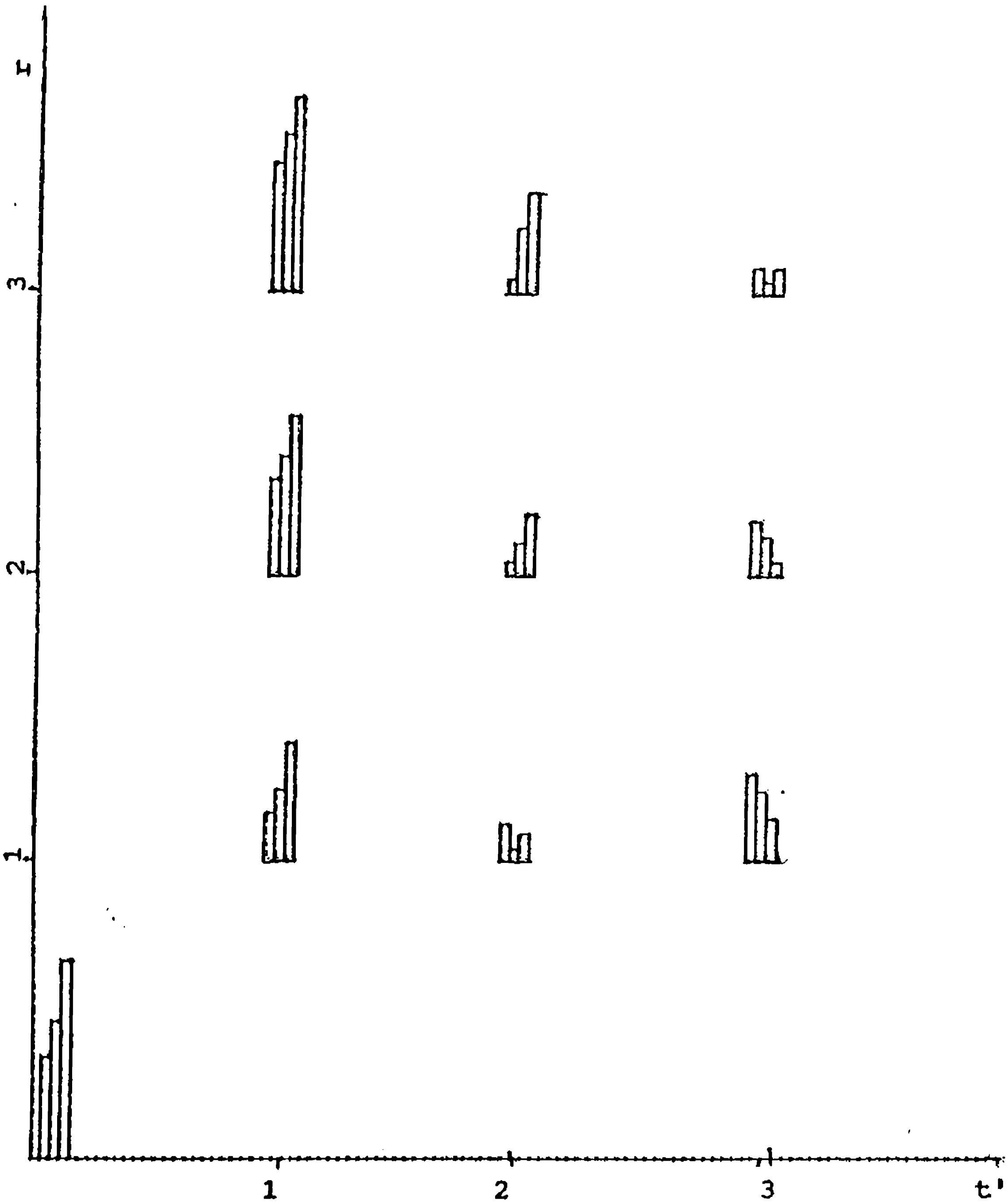

Fig. 4.1(a)
Bias in R_1, R_2, R_3 for selected values of prior parameters $r, t' = 0(1)3$, $n = 10$

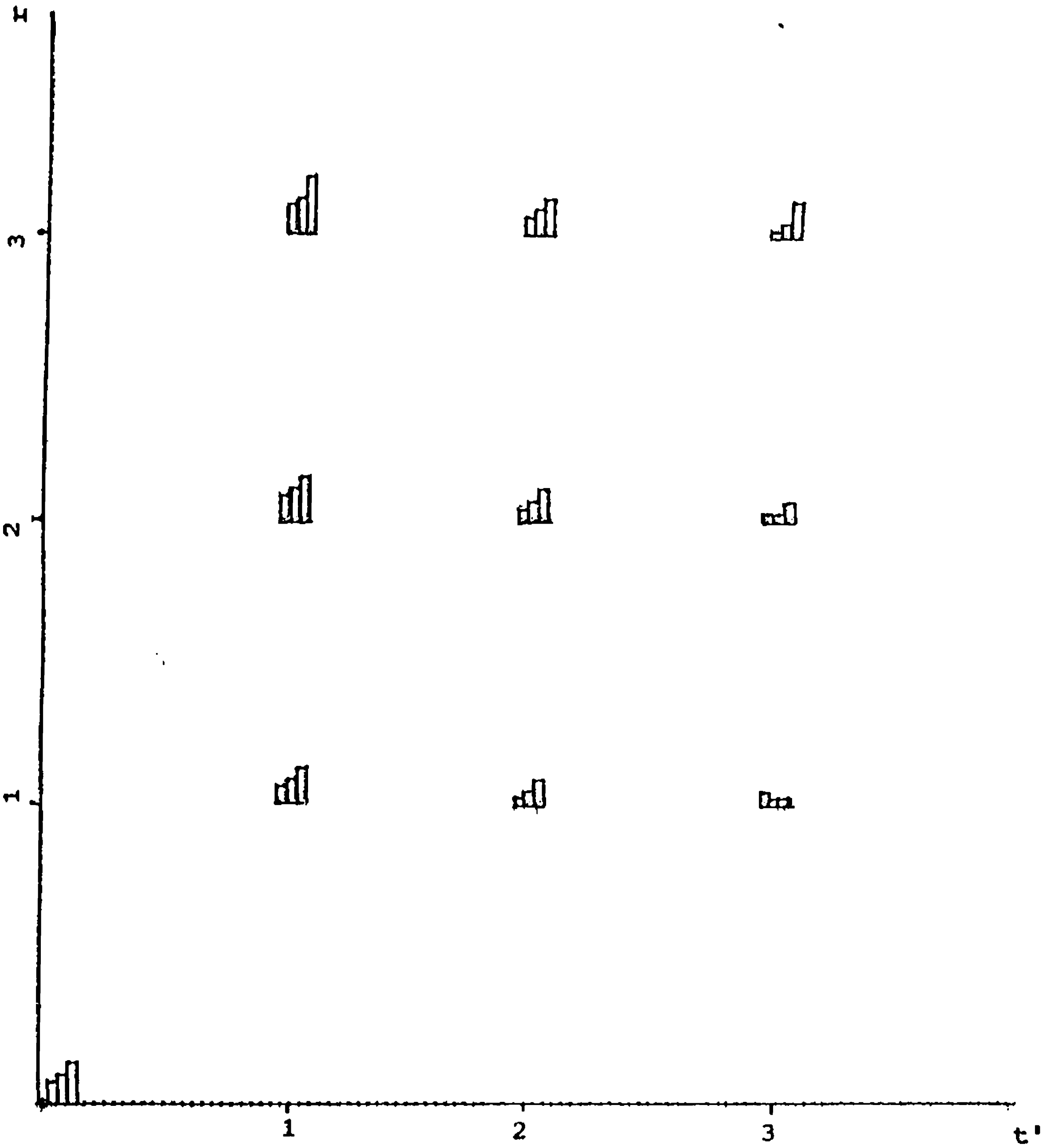

Fig. 4.1(b)

Bias in $\hat{R}_1, \hat{R}_2, \hat{R}_3$ for selected values of prior parameters
r, t' = 0(1)3,   n = 50.

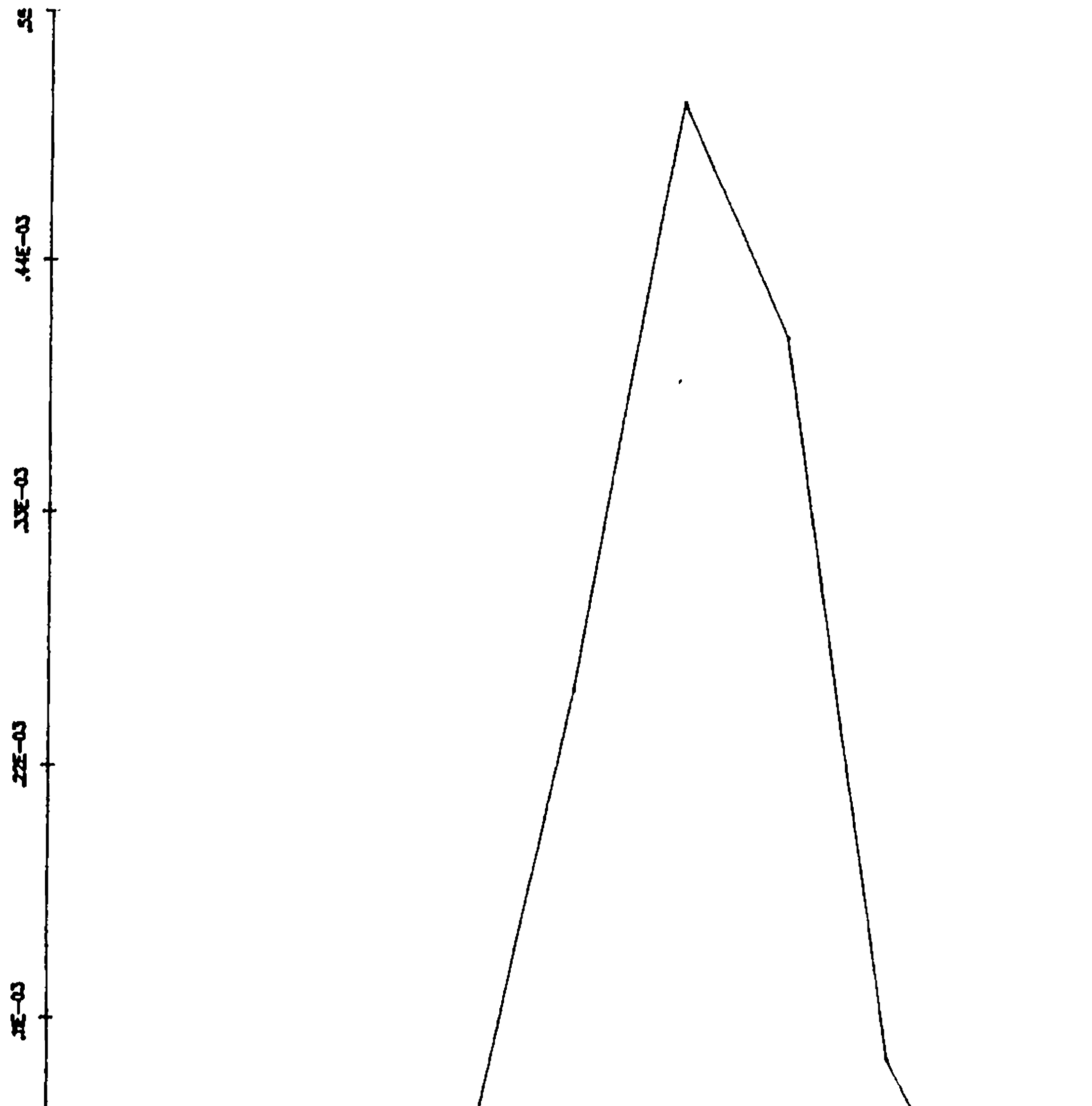

Fig. 4.2 Posterior plot R_1

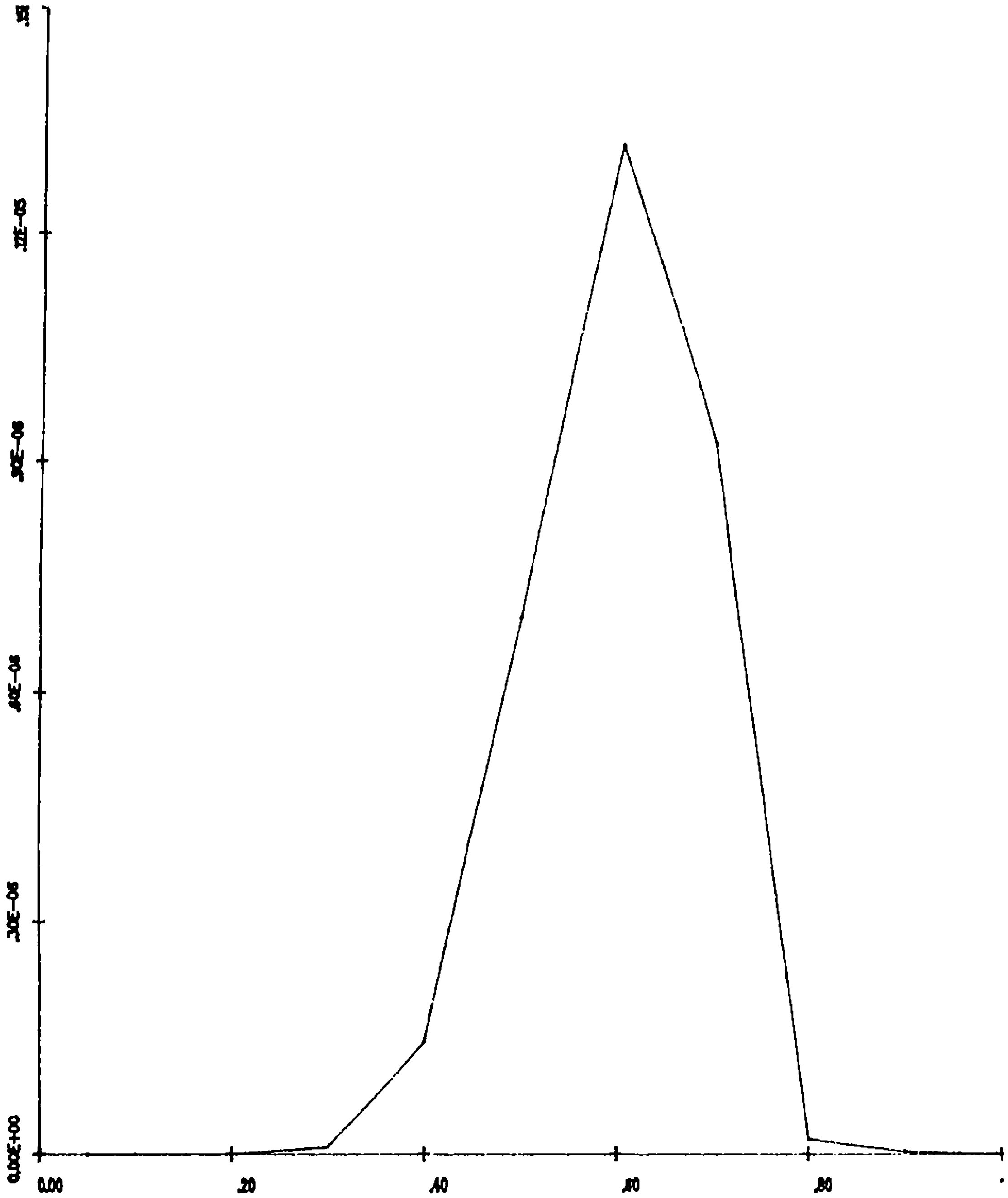

Fig.4.3 Posterior plot R_2

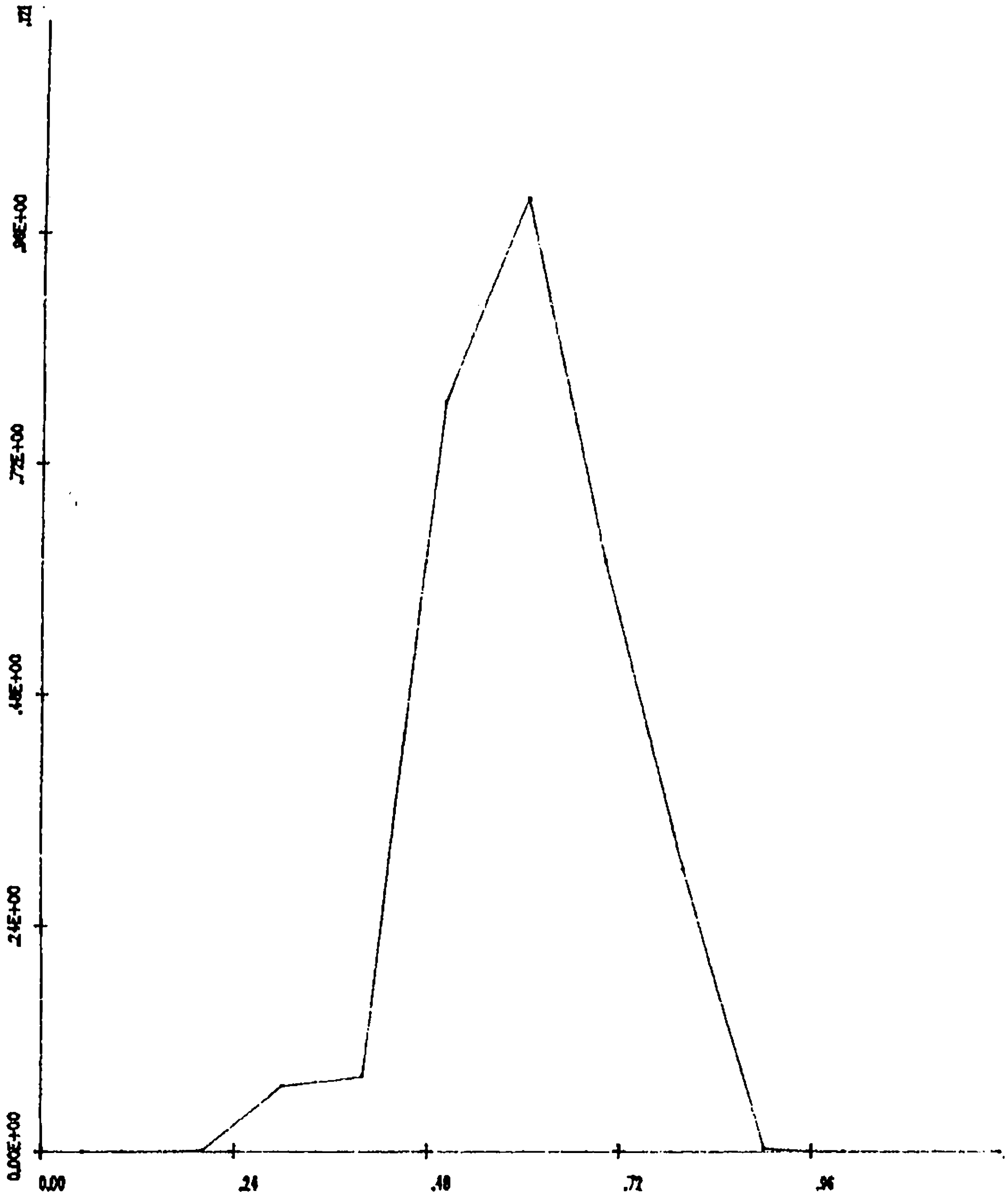

Fig.4.4 Posterior plot for R_3

CHAPTER V

PREDICTION BOUNDS FOR THE PARETO ORDER STATISTICS

5.1 Introduction

The problem of prediction of the r^{th} order statistics of future samples based on pre-specified values of the independent variables from a population has been considered by many researchers. Of these, the works of Geisser (1984,1985), Nigm and Hamdy (1987) and Arnold and Press (1989) concern the Pareto model while, Dunsmore (1974) and Lingappaiah (1973,1979b) deal with exponential model. The basic assumption in all these discussions is that the sample observations are independently and identically distributed. The possibility of the occurence of outliers in the sample has not been investigated by them. Lingappaiah (1989a) seems to be the first author to deal with the problem of predicting order statistics in future samples from the exponential population when the sample data contains outlying observation. Later he [Lingappaiah (1989b,1990)] extended the method and obtained similar results when the underlying distributions were *gamma* and Weibull.

The deliberations in the chapter centre around the necessary steps that enable the prediction of order statistics from the Pareto population on the basis of an observed sample using the Bayesian approach.

The predictive density of a future observation Y based on independent and identically distributed observations $X_1, X_2, \ldots, X_n$ from a population with p.d.f $f(x|\theta)$, $\theta \in \Theta$ is given by Dunsmore (1974) as

$$h(y|\underline{x}) = \int_{\Theta} f(y|\theta)\, f(\theta|\underline{x})\, d\theta, \tag{5.1}$$

where as usual, $\underline{x}=(x_1, x_2, \ldots, x_n)$ and $f(\theta|\underline{x})$ is the posterior density of θ. Then a Bayes prediction interval for Y of cover β can be specified as A with

$$P(A|\underline{x}) = \int_{A} h(y|\underline{x})dy = \beta. \tag{5.2}$$

In order to render the above interval unique, we take $A = \{y|p(y|\underline{x}) \geq \lambda\}$ where λ is determined such that $p(A|\underline{x}) = \beta$.

5.2 The Model

As described already, basic to the question of analysis of data containing outliers is the assumption about the model that generates such observations. In the present case, we assume that the sample consists of n observations of which 'n-1' belong to the population (2.1) and the remaining one observation is distributed as Pareto form with density

$$g(x;\alpha,b) = \alpha b\, \sigma^{\alpha b}\, x^{-(\alpha b+1)}, \qquad x\geq\sigma>0,\ \alpha,b>0. \tag{5.3}$$

Further we assume that the scale parameter σ which is same for all the observations is known. Notice that when $b<1$ the discordant observation is a lower outlier, while $b>1$ indicates an upper outlier. Under this assumption the joint density of $\underline{x}$ can be written as [see Chapter II, Section 2.2]

$$f(\underline{x}|\alpha,b,\sigma) = \frac{1}{n}\, \alpha^{n}\, b\, \sigma^{(n+b-1)\alpha} \left[\prod_{i=1}^{n} x_i^{-(\alpha+1)}\right]\left(\sum_{i=1}^{n} x_i^{-\alpha(b-1)}\right). \tag{5.4}$$

The distribution of the r^{th} order statistic in a sample of size n in the presence of an outlier is given in Balakrishnan (1987) as

$$h(y)= \binom{n-1}{r-1} [(r-1)F^{r-2}(1-F)^{n-r}\, Gf + F^{r-1}(1-F)^{n-r} g$$
$$+(n-r)F^{r-1}(1-F)^{n-r-1}(1-G)f], \tag{5.5}$$

where $f=f(y)$ and $F=F(y)$ are the density and distribution functions of the observations which are not outliers, while $g = g(y)$ and $G = G(y)$ are the corresponding functions of the outlier. In our context using (2.1) and (5.3)

$$h(y|\alpha,b) = \binom{n-1}{r-1}\left[\frac{\sigma}{y}\right]^{(n-r-1)\alpha}\left[1-\left(\frac{\sigma}{y}\right)^{\alpha}\right]^{r-2}$$

$$\left\{(r-1)\left[1-\left(\frac{\sigma}{y}\right)^{\alpha b}\right]\left(\frac{\sigma}{y}\right)^{\alpha}\frac{\alpha}{\sigma}\left(\frac{\sigma}{y}\right)^{\alpha+1}\right.$$

$$+\left[1-\left(\frac{\sigma}{y}\right)^{\alpha}\right]\left(\frac{\sigma}{y}\right)^{\alpha}\frac{\alpha b}{\sigma}\left(\frac{\sigma}{y}\right)^{\alpha b+1}$$

$$\left.+(n-r)\left[1-\left(\frac{\sigma}{y}\right)^{\alpha}\right]\left(\frac{\sigma}{y}\right)^{\alpha b}\frac{\alpha}{\sigma}\left(\frac{\sigma}{y}\right)^{\alpha+1}\right\}.$$

$$= \binom{n-1}{r-1}\left[\frac{\sigma}{y}\right]^{(n-r-1)\alpha}\left[1-\left(\frac{\sigma}{y}\right)^{\alpha}\right]^{r-2}\frac{\alpha}{\sigma}\left(\frac{\sigma}{y}\right)^{\alpha+1}$$

$$\left\{(r-1)\left[1-\left(\frac{\sigma}{y}\right)^{\alpha b}\right]\left(\frac{\sigma}{y}\right)^{\alpha}+b\left(\frac{\sigma}{y}\right)^{\alpha b}\left[1-\left(\frac{\sigma}{y}\right)^{\alpha}\right]\right.$$

$$\left.+(n-r)\left[1-\left(\frac{\sigma}{y}\right)^{\alpha}\right]\left(\frac{\sigma}{y}\right)^{\alpha b}\right\}.$$

$$= \binom{n-1}{r-1}\left[\frac{\sigma}{y}\right]^{(n-r)\alpha+1}\left[1-\left(\frac{\sigma}{y}\right)^{\alpha}\right]^{r-2}\frac{\alpha}{\sigma}\left\{(r-1)\left(\frac{\sigma}{y}\right)^{\alpha}\right.$$

$$\left.-(n+b-1)\left(\frac{\sigma}{y}\right)^{\alpha+\alpha b}+(n+b-r)\left(\frac{\sigma}{y}\right)^{\alpha b}\right\}.$$

$$= \binom{n-1}{r-1}\left[1-\left(\frac{\sigma}{y}\right)^{\alpha}\right]^{r-2}\left[\frac{\sigma}{y}\right]^{(n-r+1)\alpha+1}\left[\frac{\alpha}{\sigma}\right]$$

$$\left\{(r-1)-(n+b-1)\left(\frac{\sigma}{y}\right)^{\alpha b}+(n+b-r)\left(\frac{\sigma}{y}\right)^{\alpha(b-1)}\right\}. \tag{5.6}$$

Specialising for the first order statistic $Y_{(1)}$, (5.6) has the simplest form

$$h(y|\alpha,b) = (n+b-1)\ \alpha\sigma^{(n+b-1)\alpha}\ y^{-(n+b-1)\alpha-1}\ . \tag{5.7}$$

In the following sections we derive the Bayes prediction interval for the future observation under the two situations when b is known and unknown.

5.3 Prediction interval with known b *

In this case the likelihood takes the form

$$\ell(\underline{x}|\alpha) = C_1 \sum_{i=1}^{n} \alpha^n \exp(-\alpha[t+(b-1)t_i]), \tag{5.8}$$

where

$$t_i = \log\left(\frac{x_i}{\sigma}\right) \quad \text{and} \quad t = \sum_{i=1}^{n} t_i .$$

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* The main result in this Section is due to appear in Jeevanand and Nair (1992b).

$$= \binom{n-1}{r-1}\left[1-\left(\frac{\sigma}{y}\right)^{\alpha}\right]^{r-2}\left[\frac{\sigma}{y}\right]^{(n-r+1)\alpha+1}\left[\frac{\alpha}{\sigma}\right]$$

$$\left\{(r-1)-(n+b-1)\left(\frac{\sigma}{y}\right)^{\alpha b}+(n+b-r)\left(\frac{\sigma}{y}\right)^{\alpha(b-1)}\right\}. \tag{5.6}$$

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An informative prior for α can be prescribed as

$$\phi(\alpha) = C_2\, \alpha^{p-1}\, e^{-t'\alpha}, \qquad p, t', \alpha > 0. \tag{5.9}$$

The posterior distribution of α resulting from (5.8) and (5.9) is

$$f(\alpha|\underline{x}) = C_3 \sum_{i=1}^{n} \alpha^{m-1}\, e^{-T_i\alpha}, \qquad \alpha > 0. \tag{5.10}$$

where

$$C_3^{-1} = T(\Gamma(m)), \tag{5.11}$$

$$T = \sum T_i^{-m}, \quad T_i = t + t' + (b-1)t_i, \text{ and } \quad m = n+p.$$

In the general case, when r can be any integer from 1 to n the predictive density turns out to be

$$h(y|\underline{x}) = C_4 (1/y) \sum \int_0^{\infty} \left\{ \left[1 - \left[\frac{\sigma}{y}\right]^{\alpha} \right]^{r-2} \left[\frac{\sigma}{y}\right]^{(n-r+1)\alpha} \alpha^{m}\, e^{-T_i\alpha} \left[(r-1) - (n+b-1)\left[\frac{\sigma}{y}\right]^{\alpha b} + (n+b-r)\left[\frac{\sigma}{y}\right]^{\alpha(b-1)} \right] \right\} d\alpha. \tag{5.12}$$

In particular, the predictive density of $Y_{(1)}$ is

$$h(y|\underline{x}) = \frac{C_3}{y} \sum_{i=1}^{n} \int_0^{\infty} \alpha(n+b-1) \left[\frac{\sigma}{y}\right]^{(n+b-1)\alpha} \alpha^{m-1}\, e^{-T_i\alpha}\, d\alpha,$$

$$= \frac{C_3}{y}(n+b-1)\sum_{i=1}^{n}\int_0^{\infty}\alpha^m \exp\left\{-\alpha(T_i+(n+b-1)\log\left[\frac{y}{\sigma}\right]\right\} d\alpha,$$

$$= \frac{C_3}{y}(n+b-1)\sum_{i=1}^{n}\Gamma(m+1)\left(T_i+(n+b-1)\log\left[\frac{y}{\sigma}\right]\right)^{-(m+1)},$$

$$= \frac{(n+b-1)m\sigma}{yT}\sum_{i=1}^{n}\left(T_i+(n+b-1)\log\left[\frac{y}{\sigma}\right]\right)^{-(m+1)}. \qquad y\geq\sigma. \tag{5.13}$$

The prediction interval for the first order statistic is now obtained as (σ, u) where

$$\int_{\sigma}^{u} h(y|\underline{x})\, dy = \beta.$$

The last equation is equivalent to

$$\frac{m(n+b-1)}{T}\sum_{i=1}^{n}\int_{\sigma}^{u}\left(T_i+(n+b-1)\log\left[\frac{y}{\sigma}\right]\right)^{-(m+1)}\frac{dy}{y} = \beta,$$

which under the transformation

$$z = \left(T_i+(n+b-1)\log\left[\frac{y}{\sigma}\right]\right)^{-(m+1)}$$

yields

$$\frac{m}{T}\sum\int_{T_i}^{T_i + (n+b-1)\log(u/\sigma)} z^{-(m+1)}\, dz = \beta.$$

Simplifying

$$\beta = \frac{1}{T}\sum\left\{T_i^{-m} - \left(T_i+(n+b-1)\log\left[\frac{u}{\sigma}\right]\right)^{-m}\right\}.$$

$$= 1 - \frac{1}{T} \Sigma \left\{T_i + (n+b-1)\log\left[\frac{u}{\sigma}\right]\right\}^{-m},$$

and

$$\Sigma \left\{T_i + (n+b-1)\log\left[\frac{u}{\sigma}\right]\right\}^{-m} = (1-\beta)T. \qquad (5.14)$$

For $r \geq 2$ the predictive density is,

$$h(y|\underline{x}) = \frac{C_3}{y} \binom{n-1}{r-1} \sum_{j=0}^{r-2} (-1)^j \binom{r-2}{j} \sum_{i=1}^{n} \left\{ \int_0^\infty \left[\frac{\sigma}{y}\right]^{(n-r+j+1)\alpha} \right.$$

$$\left. \left\{(r-1) - (n+b-1)\left[\frac{\sigma}{y}\right]^{\alpha b} + (n+b-r)\left[\frac{\sigma}{y}\right]^{\alpha(b-1)}\right\} \alpha^m e^{-T_i \alpha} d\alpha \right\},$$

$$= \frac{m}{T} (\sigma/y) \binom{n-1}{r-1} \sum_{j=0}^{r-2} (-1)^j \binom{r-2}{j} \sum_{i=1}^{n} \left\{(r-1)z_j \right.$$

$$\left. -(n+b-1)z_{j+b} + (n+b-r)z_{j+b-1} \right\}, \qquad y \geq \sigma,$$

$$(5.15)$$

where

$$z_j = \int_0^\infty \left[\frac{\sigma}{y}\right]^{(n-r+j+1)\alpha} (\alpha^m / \Gamma(m+1))\, e^{-T_i \alpha} d\alpha,$$

$$= \left[T_i + (n-r+j+1) \log(y/\sigma)\right]^{-(m+1)}.$$

The predictive interal for the r^{th} order statistic $Y_{(r)}$ is then (L,U) satisfying

$$\int_L^U h(y|\underline{x})dy = \beta ,$$

is

$$\binom{n-1}{r-1} \frac{m\sigma}{T} \sum_{j=0}^{r-2} (-1)^j \binom{r-2}{j} \sum_{i=1}^{n} \Big\{(r-1)Q_j - (n+b-1)Q_{j+b} + (n+b-r)Q_{j+b-1}\Big\} = \beta, \quad (5.16)$$

where

$$Q_j = \int_L^U \Big[T_i + (n-r+j+1) \log(y/\sigma)\Big]^{-(m+1)} y^{-1} \sigma \, dy,$$

$$= (n-r+j+1)^{-1} \Big\{ \Big[T_i + (n-r+j+1) \log(L/\sigma)\Big]^{-m} - \Big[T_i + (n-r+j+1) \log(U/\sigma)\Big]^{-m} \Big\}.$$

Since an interval of the above form requires only that the probability content of (L,U) should be β, it may not be unique. This difficulty can be overcome if we insist that the probabilities are equally distributed at the two tails, that is

$$\int_\sigma^L h(y|\underline{x}) \, dy = \int_U^\infty h(y|\underline{x})dy = (1-\beta)/2.$$

5.4 Prediction interval with unknown b

5.4.1 The case when 0<b<1.

In this case the joint prior density for α and b is chosen assuming that α and b are independently distributed with α following a gamma distribution and b having uniform distribution over (0,1). Thus

$$\phi(\alpha,b) = C_5\ \alpha^{p-1}\ e^{-t'\alpha}\ , \qquad \alpha,p,t'>0. \tag{5.17}$$

Now using (5.4) and (5.17) the posterior density is seen to have the form

$$f(\alpha,b|\underline{x}) = C_6\ \Sigma\ \alpha^{m-1}\ b\ e^{-\alpha[A_i+bt_i]}\ , \qquad \alpha>0, 0<b<1, \tag{5.18}$$

where

$$A_i = t + t' - t_i \text{ and}$$

$$C_6^{-1} = \Sigma \int_0^\infty \int_0^1 \alpha^{m-1}\ b\ e^{-\alpha[A_i+bt_i]}\ db\ d\alpha,$$

$$= \Sigma \int_0^1 \Gamma(m) b\ [A_i + bt_i]^{-m}\ db,$$

$$= \Gamma(m)\ \Sigma\ A_i^{-m} \int_0^1 b\left(1 + b\left[\frac{t_i}{A_i}\right]\right)^{-m} db.$$

Now using (3.11) we have

$$C_6^{-1} = \Gamma(m) \sum_{i=1}^{n} (A_i^{-m}/2)\ {}_2F_1(m,2,3,(-t_i/A_i)),$$

$$= \Gamma(m)\ A, \quad \text{with } A = \sum_{i=1}^{n} (A_i^{-m}/2)\ {}_2F_1(m,2,3,(-t_i/A_i)).$$

Direct calculations from (5.18) and (5.7) lead to

$$h(y|\underline{x}) = C_6 \int_0^{\infty}\int_0^{1} \left\{ (\alpha/y)(n+b-1)\left[\frac{\sigma}{y}\right]^{(n+b-1)\alpha} \Sigma\, \alpha^{m-1}\, b \right.$$

$$\left. e^{-\alpha[A_i+bt_i]} \right\} d\alpha\, db,$$

$$=\frac{C_6}{y} \Sigma \int_0^1 (n-1+b)b\Gamma(m+1)[A_i+bt_i+(n+b-1)\log(y/\sigma)]^{-(m+1)}db,$$

$$= \Gamma(m+1)\ (C_6/y)\ \Sigma\ [A_i+(n-1)\log(y/\sigma)]^{-(m+1)}$$

$$\left\{\int_0^1 [1+bq_i]^{-(m+1)}((n-1)+b)\ db\right\},$$

$$= (C_7/y)\ \Sigma\ [A_i+(n-1)\log(y/\sigma)]^{-(m+1)}$$

$$\left\{ \frac{(n-1)}{2}\ {}_2F_1(m+1,2,3,-q_i)+(1/3)\ {}_2F_1(m+1,3,4,-q_i)\right\},$$

$$y \ge \sigma. \qquad (5.19)$$

where

$$C_7 = (m/A)$$

and

$$q_i = \frac{t_i + \log(y/\sigma)}{A_i + (n-1)\log(y/\sigma)} .$$

The predictive interval for the first order statistic is now obtained as (σ, u), where

$$\int_{\sigma}^{u} h(y|\underline{x})\, dy = \beta.$$

That is,

$$\beta = \int_{\sigma}^{u} (C_7/y) \Sigma \left[A_i + (n-1)\log(y/\sigma) \right]^{-(m+1)} \left\{ \frac{(n-1)}{2} \; {}_2F_1(m+1, 2, 3, -q_i) \right.$$

$$\left. + (1/3)\; F(m+1, 3, 4, -q_i) \right\} dy,$$

which reduces to

$$\sum_{i=1}^{n} \sum_{\ell=0}^{\infty} (-1)^{\ell}\, m\, (n-1)^{-(m+\ell+1)} \; (t + \log(u/\sigma))^{-m}$$

$${}_2F_1(m+\ell+1, m, m+1, -u_i) \left\{ \frac{(n-1)}{2} \; \frac{\Gamma(3)\; \Gamma(m+\ell+1)\; \Gamma(2+\ell)}{\Gamma(2)\; \Gamma(m+1)\; \Gamma(3+\ell)\; \Gamma(\ell+1)} \right.$$

$$\left. - \frac{\Gamma(4)\; \Gamma(m+\ell+1)\; \Gamma(\ell+3)}{3\Gamma(3)\; \Gamma(m+1)\; \Gamma(4+\ell)\; \Gamma(\ell+1)} \right\} = A\sigma(1-\beta).$$

$$\sum_{i=1}^{n} \sum_{\ell=0}^{\infty} (-1)^{\ell} \, m \, (n-1)^{-(m+\ell+1)} m \, (t + \log(u/\sigma))^{-m}$$

$$ {}_2F_1(m+\ell+1, m, m+1, -u_i) \, \frac{\Gamma(3)\, \Gamma(m+\ell+1)\, \Gamma(2+\ell)}{\Gamma(2)\, \Gamma(m+1)\, \Gamma(3+\ell)\, \Gamma(\ell+1)}$$

$$\left\{ \frac{(n-1)}{2} + \frac{(\ell+2)}{2(\ell+3)} \right\} = (1-\beta) A\sigma.$$

$$\sum_{i=1}^{n} \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell} (n-1)^{-(m+\ell+1)} \Gamma(3)\, \Gamma(m+\ell+1)}{\Gamma(2)\, \Gamma(m+1)\, \Gamma(\ell+3)\, \Gamma(\ell+1)} \, m \, (t+\log(u/\sigma))^{-m}$$

$$ {}_2F_1(m+\ell+1, m, m+1, -u_i) \, (n\ell + 3n - 1) = (1-\beta) A\sigma.$$

$$\sum_{i=1}^{n} \sum_{\ell=0}^{\infty} \frac{(-1)^{\ell} (n-1)^{-(m+1+1)} (n\ell+3n-1)}{\ell(\ell+2)(\ell+3)\, B(\ell, m+1)} \, (t_i + \log(u/\sigma))^{-m}$$

$$ {}_2F_1(m+\ell+1, m, m+1, -u_i) = (1-\beta) A\sigma, \qquad (5.20)$$

where

$$u_i = [(A_i/(n-1)) - t_i] \, (t_i + \log(u/\sigma))^{-1},$$

since

$$\int_u^{\infty} [A_i + (n-1)\log(y/\sigma)]^{-(m+1)} F(m+1, a, b, -q_i)(dy/y)$$

$$= \int_u^{\infty} [A_i + (n-1)\log(y/\sigma)]^{-(m+1)}$$

$$\left\{ \sum_{\ell=0}^{\infty} \frac{\Gamma(b)\, \Gamma(m+\ell+1)\, \Gamma(a+\ell)}{\Gamma(a)\, \Gamma(b)\, \Gamma(b+\ell)\, \Gamma(\ell+1)} (-q_i)^{\ell} \right\} (dy/y),$$

$$= \sum_{\ell=0}^{\infty} (-1)^{\ell} \frac{\Gamma(b)\, \Gamma(m+\ell+1)\, \Gamma(a+\ell)}{\Gamma(m+1)\, \Gamma(a)\, \Gamma(b+\ell)}$$

$$\int_u^{\infty} [A_i + (n-1)\log(y/\sigma)]^{-(m+\ell+1)} (t_i + \log(y/\sigma))^{-\ell} (dy/y),$$

$$= \sum_{\ell=0}^{\infty} (-1)^{\ell} C_{\ell} \int_{t_i+\log(u/\sigma)}^{\infty} z^{\ell} [1+a_i z]^{-(m+\ell+1)} dz \ , \ z=t_i+\log(y/\sigma),$$

$$= \sum_{\ell=1}^{\infty} (-1)^{\ell} C_{\ell} a_i^{-(m+\ell+1)} [t+\log(u/\sigma)]^{-m} F(m+\ell+1, m, m+1, u_i),$$

$$(a_i = [A_i - (n-1)t_i])$$

$$= \sum_{\ell=0}^{\infty} (-1)^{\ell} \left\{ \frac{\Gamma(b)\ \Gamma(m+\ell+1)\ \Gamma(a+\ell)}{\Gamma(m+1)\ \Gamma(a)\ \Gamma(b+\ell)} \sigma [A_i - (n-1)t_i]^{-(m+\ell+1)} \left[\frac{(n-1)}{A_i-(n-1)t_i}\right]^{-(m+\ell+1)} (t_i+\log(u/\sigma))^{-m} F(m+\ell+1, m, m+1, -u_i) \right\}.$$

$$= \sum_{\ell=0}^{\infty} (-1)^{\ell} \sigma \frac{\Gamma(b)\ \Gamma(m+\ell+1)\ \Gamma(a+\ell)}{\Gamma(m+1)\ \Gamma(a)\ \Gamma(b+\ell)} (n-1)^{-(m+\ell+1)} (t_i+\log(u/\sigma))^{-(m+\ell+1)} {}_2F_1(m+\ell+1, m, m+1, -u_i).$$

For $r \geq 2$, we have,

$$h(y|\underline{x}) = C_8(1/y) \sum_{j=0}^{r-2} (-1)^j \binom{r-2}{j} \sum_{i=1}^{n} \left\{ \frac{(r-1)}{2} B_j - \frac{(n-1)}{2} B_j'(2,3) - (1/3)B_j'(3,4) + \frac{(n-r)}{2} B_{j-1}'(2,3) + (1/3)B_{j-1}'(3,4) \right\},$$

$$y \geq \sigma, \quad (5.21)$$

where

$$B_j = \frac{1}{\Gamma(m+1)} \int_0^{\infty}\int_0^{1} (\sigma/y)^{(n-r+j+1)\alpha}\, b\, \alpha^m\, e^{-[A_i+bt_i]\alpha}\, db\, d\alpha,$$

$$= \frac{1}{\Gamma(m+1)} \int_0^{1} b\, \Gamma(m+1)\, [A_i+(n-r+j+1)\log(y/\sigma)+bt_i]^{-(m+1)}\, db,$$

$$= [A_i+(n-r+j+1)\log(y/\sigma)]^{-(m+1)}$$

$$\int_0^{1} b\left[1 + b\, t_i [A_i+(n-r+j+1)\log(y/\sigma)]^{-1}\right]^{-(m+1)} db,$$

$$= [A_i+(n-r+j+1)\log(y/\sigma)]^{-(m+1)}$$

$${}_2F_1(m+1, 2, 3, -t_i [A_i+(n-r+j+1)\log(y/\sigma)]^{-1}).$$

Similarly,

$$B_j^*(a,b) = [A_i+(n-r+j+1)\log(y/\sigma)]^{-(m+1)}\ {}_2F_1(m+1, a, b, -v_j),$$

with

$$v_j = [t_i + \log(y/\sigma)]\ [A_i+(n-r+j+1)\log(y/\sigma)]^{-1}$$

and

$$C_8 = \binom{n-1}{r-1} (m/A).$$

The predictive interval for the r^{th} order statistics $(r \geq 2)$ (L,U) is obtained from

$$\int_{\sigma}^{L} h(y|\underline{x})\, dy = \int_{U}^{\infty} h(y|\underline{x})\, dy = (1-\beta)/2. \qquad (5.22)$$

By direct integration the equation to be satisfied by L and U can be derived. These are

$$(C_8/\sigma) \sum_{j=0}^{r-2} (-1)^j \binom{r-2}{j} \sum_{i=1}^{n} \sum_{\ell=0}^{\infty} (-1)^{\ell} \left\{ \frac{(r-1)}{(\ell+2)\ (m+\ell)} D_j \right.$$

$$\left. - \frac{(n\ell + 3n - 1)}{(\ell+2)\ (\ell+3)} D'_j + \frac{[(n-r+1)\ell + 3n - 3r+2]}{(\ell+2)\ (\ell+3)} D'_{j-1} \right\} = \beta'. \qquad (5.23)$$

where

$$D_j = t_i^{\ell} \ [T_i + (n-r+j+1)\log(a/\sigma)]^{-(m+\ell)},$$

$$D'_j = \frac{(n-r+j+1)^{-(m+\ell+1)}}{m\ell B(\ell, m+1)} [t_i + \log(a/\sigma)]^{-m} F(m+\ell+1, m, m+1, -a_{j\ell}),$$

$$a_{j\ell} = \left[\frac{T_i}{(n-r+j+1)} - t_i \right] [t_i + \log(a/\sigma)]^{-1},$$

$a = L, U$

and

$$\beta' = \begin{cases} (1+\beta)/2 & \text{for } a=L \\ (1-\beta)/2 & \text{for } a=U. \end{cases}$$

5.4.2 The case when b>1

In this case, an improper uniform prior for b and a gamma prior for α along with assumption of independence of

α and b are employed to write the prior density of (α, b) as (see also Jeevanand and Nair (1993e))

$$\phi(\alpha, b) = C_9 \, \alpha^{p-1} \, e^{-t'\alpha}, \qquad \alpha, p, t' > 0, \; b > 1. \qquad (5.24)$$

This leads to the posterior density

$$f(\alpha, b|\underline{x}) = C_{10} \sum \alpha^{m-1} \, b \, e^{-\alpha[A_i + bt_i]}, \quad \alpha > 0, b > 1, \qquad (5.25)$$

where

$$C_{10}^{-1} = \int_0^\infty \int_1^\infty \sum \alpha^{m-1} \, b \, e^{-\alpha[A_i + bt_i]} \, db \, d\alpha,$$

$$= \sum \int_1^\infty \Gamma(m) b [A_i + bt_i]^{-m} \, db,$$

$$= \sum \Gamma(m) \; A_i^{-m} \int_1^\infty b[1 + b(t_i/A_i)]^{-m} \, db,$$

$$= \Gamma(m) \sum A_i^{-m} \, (t_i/A_i)^{-m} \, (m-2)^{-1} \; {}_2F_1(m, m-2, m-1, -(A_i/t_i)),$$

$$= \Gamma(m) \sum \frac{t_i^{-m}}{(m-2)} \; {}_2F_1(m, m-2, m-1, -(A_i/t_i)) = s\Gamma(m),$$

with $s = \sum \frac{t_i^{-m}}{(m-2)} \; {}_2F_1(m, m-2, m-1, -(A_i/t_i))$.

From (5.25) and (5.7) the density of $Y_{(1)}$ is

$$h(y|\underline{x}) = \frac{C_{10}}{y} \sum \int_0^\infty \int_1^\infty (n+b-1)(\alpha/y)^{(n+b-1)\alpha} \, \alpha^m \, b \, e^{-\alpha[A_i + bt_i]} \, d\alpha \, db,$$

$$=\frac{C_{10}}{y}\ \Sigma \int_1^{\infty} (n+b-1)\Gamma(m-1)[\,A_i+(n+b-1)\log(y/\sigma)+bt_i\,]^{-(m+1)}db,$$

$$= (C_{10}/y)\ \Sigma\ \Gamma(m+1)[\,A_i+(n-1)\log(y/\sigma)]^{-(m+1)}$$

$$\int_1^{\infty} b[\,1+z_i b]^{-(m+1)}(n-1+b)db,$$

$$= (C_{11}/y)\ \Sigma\ [\,A_i+(n-1)\log(y/\sigma)]^{-(m+1)}$$

$$\left\{z_i^{-(m+1)}\ \frac{(n-1)}{(m-1)}\ {}_2F_1(m+1,m-1,m,-z_i)\right.$$

$$\left.+\ \frac{z_i^{-(m+1)}}{(m-2)}\ \ {}_2F_1(m+1,m-2,m-1,-z_i)\right\},$$

$$= \frac{C_{11}}{y}\ \Sigma\ [\,t_i+\log(y/\sigma)]^{-(m+1)}\left\{\frac{(n-1)}{(m-1)}\ \ {}_2F_1(m+1,m-1,m,-z_i)\right.$$

$$\left.+(m-2)^{-1}\ \ {}_2F_1(m+1,m-2,m-1,-z_i)\right\},\quad y\geq\sigma, \quad (5.26)$$

where

$$C_{11} = m/s$$

and

$$z_i = \frac{A_i + (n-1)\log(y/\sigma)}{t_i + \log(y/\sigma)}\ .$$

The Bayes predictive interval for $Y_{(1)}$, (σ,u), is calculated exactly as in the preceeding sections. The relevant equation in the present case that derives u is

$$\sum_{i=1}^{n} \sum_{\ell=0}^{\infty} (-1)^{\ell} \frac{(n-1)^{m+\ell}}{\ell\, B(\ell, m+1)} \frac{[n(m+\ell-2)+1]}{(m+\ell-1)\ (m+\ell-1)} [T_i + (n-1)\log(u/\sigma)]^{-m}$$

$$_2F_1(m+\ell+1, m, m+1, -(n-1)[t_i - [T_i/(n-1)]][T_i + (n-1)\log(u/\sigma)]^{-1})$$

$$= (1-\beta)s\sigma. \qquad (5.27)$$

Also the predictive density of $Y_{(r)}$ for $r \geq 2$ has the form

$$h(y|\underline{x}) = (C_{12}/y) \sum_{j=0}^{r-2} (-1)^j \binom{r-2}{j} \sum_{i=1}^{n} \left\{ \frac{(r-1)}{(m-1)} w_j \right.$$

$$- \frac{(n-1)}{(m-1)} w_j^*(m-1, m) - \frac{w_j^*(m-2, m-1)}{(m-2)}$$

$$\left. + \frac{(n-r)}{(m-1)} w_{j-1}^*(m-1, m) + \frac{w_{j-1}^*(m-2, m-1)}{(m-2)} \right\},$$

$$y \geq \sigma, \qquad (5.28)$$

where

$$w_j = \frac{(m-1)}{\Gamma(m+1)} \int_0^{\infty}\int_1^{\infty} (\sigma/y)^{(n-r+j+1)\alpha}\ b\ \alpha^m\ e^{-[A_i + t_i b_i]}\ db\ d\alpha,$$

$$= (m-1) \int_1^{\infty} b[A_i + (n-r+j+1)\log(y/\sigma) + bt_i]^{-(m+1)}\ db,$$

$$= (m-1)[A_i + (n-r+j+1)\log(y/\sigma)]^{-(m+1)} \int_1^{\infty} b[1+\mu_i b]^{-(m+1)} db.$$

$$(\text{with } \mu_i = [A_i + (n-r+j+1)\log(y/\sigma)]\, t_i^{-1})$$

$$= (m-1)\ [A_i + (n-r+j+1)\log(y/\sigma)]^{-(m+1)}$$

$$\frac{t_i^{-(m+1)}}{(m-1)[A_i + (n-r+j+1)\log(y/\sigma)]^{-(m+1)}}$$

$${}_2F_1(m+1, m-1, m, -[A_i + (n-r+j+1)\log(y/\sigma)]\, t_i^{-1}),$$

$$w_j = t_i^{-(m+1)}\ {}_2F_1(m+1, m-1, m, -t_i^{-1}[A_i + (n-r+j+1)\log(y/\sigma)]).$$

Similarly,

$$w_j^{*}(a,b) = [t_i + \log(y/\sigma)]^{-(m+1)}\ {}_2F_1(m+1, a, b, -\varphi_{ij}),$$

$$\varphi_{ij} = [A_i + (n-r+j+1)\log(y/\sigma)][t_i + \log(y/\sigma)]^{-1}$$

and

$$C_{12} = \binom{n-1}{r-1}(m/s).$$

The corresponding predictive interval (L,U) is obtained from the equation

$$C_{12} \sum_{j=0}^{r-2} (-1)^j \sum_{i=1}^{n} \sum_{\ell=0}^{\infty} (-1)^{\ell} \left\{ \frac{(r-1)}{(m-1)} R_j - \left[\frac{(n-1)}{m+\ell+1} + \frac{1}{m+\ell-2}\right] R_j^{*} + \left[\frac{(n-r)}{(m+\ell-1)} + \frac{1}{m+\ell-2}\right] R_{j-1}^{*} \right\} = \beta', \quad (5.29)$$

where

$$R_j = \frac{t_i^{-1}}{n-r+j+1} \; [A_i + t_i + (n-r+j+1)\log(a/\sigma)]^{-(m-\ell)},$$

$$R_j' = \frac{(n-r+j+1)^{m+\ell}}{\ell \, B(\ell, m+1)} \; F(m+\ell+1, m, m+1, -a_{ij}')$$

and

$$a_{ij}' = (n-r+j+1) \left[t_i - \frac{T_i}{(n-r+j+1)} \right] [A_i + (n-r+j+1)\log(a/\sigma)]^{-1}.$$

There are several practical situations where a non-informative prior becomes realistic. Our discussion includes this case also, since the Jeffrey's (1961) prior results as a particular case as t' and p tend to zero .

5.5 Discussion

From the nature of the expressions given in the previous section it is clear that they are analytically intractable to render theoretical analysis of the general properties of the prediction intervals provided by them. Therefore we have applied the results to simulated random samples from the population with different values for the parameters of the model and hyper-parameters of the prior. One such example with α=5, σ=50, b=2.5, n=10 giving the observations, 50.298, 50.6898, 54.4666, 53.0231, 55.2714, 55.4298, 57.004, 58.3917, 65.8433 and 74.1340 produced prediction intervals as shown in Table 5.1 and 5.2. For the calculation of L and U values the Newton-Ralphson method with initial values $X_{(2)}, X_{(r-1)}, X_{(r+1)}$ for $r \neq n$ and $X_{(n)}$ for $r=n$ were used. The commputer programe for the numerical evaluation of the intervals in both the case when bis known and unknown are given in Appendix.

The role of the size of the sample in determining the prediction intervals was studied through simulation studies by varying the sample size. The general features of the procedure found from the investigations we have made so far, can be summarised in the following

(1) the interval becomes shorter as sample size increases or when σ becomes larger, and

(2) the increase in the hyper-parameter t' generally contributes to wider intervals.

Table 5.1

Prediction interval for the r^{th} order statistics

	Non-informative prior		P=1,t'=0.5		P=1,t'=1	
r	L	U	L	U	L	U
1	50.000	52.8584	50.0000	52.022	50.0000	51.6024
2	51.3469	55.895	50.432	53.7201	50.2114	52.4953
3	52.1742	57.060	51.3487	54.4033	50.6649	53.1201
4	52.4693	59.2830	52.1742	56.1200	52.3348	54.2174
5	53.546	59.7346	52.8433	57.394	52.6387	55.4033
6	54.0244	62.720	53.1096	60.726	53.0210	56.2174
7	55.356	67.0134	54.4693	62.4491	54.6934	58.2960
8	56.9843	68.241	55.2930	64.7024	55.4298	60.7346
9	58.126	68.9161	58.9843	66.983	58.8433	67.0134
10	60.231	70.7346	59.0134	71.045	59.9161	68.7346

Table 5.2

Prediction interval for the r^{th} order statistics from Pareto Population when b is unknown

	Non ifornative prior		P=1,t'=0.5		P=1,t'=1	
r	L	U	L	U	L	U
1	50.0000	53.584	50.0000	52.1953	50.000	52.465
2	51.3690	55.895	50.0244	53.7201	50.432	53.022
3	51.9070	56.406	51.3487	54.4033	50.830	53.789
4	52.9020	57.080	52.1742	55.7346	51.294	56.150
5	53.5460	59.283	52.8433	56.2974	53.184	57.391
6	53.9100	60.093	53.1096	57.2960	53.869	60.720
7	54.7290	62.120	54.4693	58.7346	56.277	62.419
8	57.3560	66.772	55.2930	60.2174	56.762	64.702
9	58.126	68.241	58.9843	67.0134	58.838	66.783
10	60.731	70.289	59.0134	68.9161	61.232	71.045

CHAPTER VI

ESTIMATION OF EXPONENTIAL PARAMETERS IN THE PRESENCE OF k-OUTLIERS

6.1 Introduction

Of the vast amount of literature available on the analysis of data containing outliers, those dealing with accommodation of outliers using the classical approach to inference outweigh quantitaively the discussions from a Bayesian position. Among them Kale and Sinha (1971), Joshi (1972,1988), Sinha (1973b,1973C),Kale (1975), Veale (1975), Lingappaiah (1979a), Chikkagoudar and Kunchur (1980), apply classical inference techniques. In a variant approach, Kale (1969) considered the case of n independent observations in which (n-1) are independently exponentially distributed with mean θ^{-1}, while one of the x_i where i is equally likely to be one of the integers from 1 to n, is distributed with mean $(\lambda\theta)^{-1}$, $\lambda>0$. He estiamted, the parameters by assuming a beta distribution of second kind for λ. Sinha (1972) estimated the survival function under the same framework. In a later paper Sinha (1973a) applied a fuller Bayesian treatment by suggesting the same prior for λ along with three possible prior families for θ in order to estimate these parameters and survival function. In what can be considered as an extension of these attempts Lingappaiah (1976) investigated the estimation problem in the presence of outliers for the generalized gamma family, that included the gamma, Weibull and exponential models as particular cases. The result in the exponential case was further extended by Kale and Kale(1992) to cover the case of arbitrary number of outliers under the identifiable model.

In the rest of the current chapter, we discuss the estimation of the parameter and the survival function of the exponential distribution in a k-outlier model by assuming that outliers are generated by an exchangable model. The Bayes interval end point estimates of the parameters along with the estimates of the survival function are derived under a general family of prior distributions*

6.2 The Model

Let X be the random variable representing the life time of a device with probability density function

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The results have been published in Jeevanand and Nair(1993a)

$$f(x:\theta) = \theta\, e^{-\theta x}, \qquad x>0,\ \theta>0, \qquad (6.1)$$

The inference situation considered here involves a random sample $x_1, x_2, \ldots, x_n$, in which $(n-k)$ of them are distributed as $f(x:\theta)$ while the remaining k observations $x_{\nu_1}, x_{\nu_2}, \ldots, x_{\nu_k}$ are distrbuted as $f(x:\theta b)$ where $b>0$ and $s=(\nu_1, \nu_2, \ldots, \nu_k)$ is the indexing set. It is assumed that there is no way to identify the set of possible k spurious observations and that every subset of k observations out of n is equally likely to be spurious. Thus the joint p.d.f of $\underline{x}=(x_1, x_2, \ldots x_n)$ is

$$\ell(\underline{x}|\theta,b) = \left\{\prod_{\nu_j\in\nu} f(x_{\nu_j};\theta b)\right\}\left\{\prod_{\nu_j\notin\nu} f(x_{\nu_j};\theta)\right\}\binom{n}{k}^{-1},$$

$$= \binom{n}{k}^{-1}\theta^n b^k\left[\prod_{i=1}^{n} e^{-\theta x_i}\right]\left[\Sigma_* \prod_{j=1}^{k} e^{-\theta(b-1)x_i}\right],$$

$$= \binom{n}{k}^{-1}\Sigma_* \prod_{j=1}^{k}\theta^n b^k \exp\left\{-\theta[n\bar{x}+(b-1)x_{A_j}]\right\}, \qquad (6.2)$$

where

$$\Sigma_* = \sum_{A_1=1}^{n-k+1}\cdots\sum_{A_k=A_{k-1}+1}^{n}.$$

6.3 Estimation when $0<b<1$

In this case the joint prior density for θ and b is chosen by assuming that θ and b are independently distributed with θ following gamma distribution and b having a uniform distribution over $(0,1)$. That is

$$\phi(\theta,b) = C_1\theta^{p-1}\ e^{-u\theta}, \qquad \theta,p,u>0,\ 0<b<1. \qquad (6.3)$$

Using (6.2) and (6.3) the posterior density of θ and b reduces to

$$f(\theta,b|\underline{x}) = C_2\ \Sigma_{*}\ b^k\ \theta^{m-1}\ \exp\{-\theta[Q+(b-1)x_A]\}, \quad \theta>0, 0<b<1, \qquad (6.4)$$

where

$$m = n+p,\ Q = n\bar{x} + u \text{ and } x_A = \sum_{i=1}^{k} x_{A_i}.$$

6.3.1 Estimation of θ

From equation (6.4) the marginal posterior density of θ is

$$f(\theta|\underline{x}) = C_2\ \Sigma_{*}\ \theta^{m-1}\ e^{-\theta T} \int_0^1 b^k\ e^{-\theta b x_A}\, db,$$

$$= C_2 \Sigma_* \theta^{m-1} e^{-\theta T} \left\{ \frac{\Gamma(k+1)}{(\theta x_A)^{k+1}} - e^{-\theta x_A} \sum_{j=0}^{k} \frac{\Gamma(k+1)}{\Gamma(j+1)} (\theta x_A)^{-(k-j+1)} \right\},$$

$$= [C_3(m,k)]^{-1} \Sigma_* \theta^{m-k-2} e^{-T\theta} x_A^{-(k+1)} \left\{ 1 - \sum_{j=0}^{k} \frac{(\theta x_A)^j}{\Gamma(j+1)} e^{-\theta x_A} \right\}, \theta > 0, \quad (6.5)$$

where

$$T = Q - x_A$$

and

$$C_3(m,k) = \Sigma_* \int_0^{\infty} \theta^{m-k-2} e^{-T\theta} (x_A)^{-(k+1)} \left\{ 1 - \sum_{j=0}^{k} \frac{(\theta x_A)^j}{\Gamma(j+1)} e^{-\theta x_A} \right\} d\theta,$$

$$= \Sigma_* (x_A)^{-(k+1)} \left\{ \Gamma(m-k-1)\ T^{-(m-k-1)} - \sum_{j=0}^{k} \frac{(x_A)^j}{\Gamma(j+1)} \Gamma(m-k+j-1)\ Q^{-(m-k+j-1)} \right\},$$

$$= \Sigma_* (x_A)^{-(k+1)} \left\{ \Gamma(m-k-1)\ T^{-(m-k-1)} - \sum_{j=0}^{k} \frac{(x_A)^j \Gamma(m-k+j-1)}{\Gamma(j+1)\ Q^{m-k+j-1}} \right\}. \quad (6.6)$$

Equations (6.5) and (6.6) enable us to make inferential statements on θ. Thus under the usual quadratic loss function, the Bayes estimator of θ is

$$\hat{\theta}_1 = E(\theta|\underline{x}) = C_3(m+1,k)/C_3(m,k) \qquad (6.7)$$

with expected loss, when $\hat{\theta}_1$ is used as estiamte of θ is given by

$$V(\theta_1|\underline{x}) = [C_3(m+2,k)/C_3(m,k)] - \hat{\theta}_1^{\,2}. \qquad (6.8)$$

Further a $100(1-\alpha)\%$ credible interval for θ, say (L_θ, U_θ) is determined as the solution of the equations

$$\int_0^{L_\theta} f(\theta|\underline{x})\, d\theta = \int_{U_\theta}^{\infty} f(\theta|\underline{x})\, d\theta = \alpha/2. \qquad (6.9)$$

The equation (6.9) can be simplified to

$$C_3^{-1} \sum_{*} x_A^{-(k+1)} \left\{ e^{-TL_\theta} \sum_{\ell=0}^{m-k-2} \frac{(m-k-2)!\; L_\theta^{\ell}}{\ell!\quad T^{m-k-\ell-1}} \right.$$

$$\left. - e^{-QL_\theta} \sum_{j=0}^{k} \frac{(x_A)^j}{\Gamma(j+1)} \sum_{r=0}^{m-k+j-2} \frac{(m-k+j-2)L_\theta^{r}}{r!\; Q^{m-k+j-r-1}} \right\} = 1-\alpha/2,$$

(6.10)

and

$$C_3^{-1} \Sigma_* \, x_A^{-(k+1)} \left\{ e^{-TU_\theta} \sum_{\ell=0}^{m-k-2} \frac{(m-k-2)! \; U_\theta^\ell}{\ell! \quad T^{m-k-\ell-1}} \right.$$

$$\left. - \, e^{-QU_\theta} \sum_{j=0}^{k} \frac{(x_A)^j}{\Gamma(j+1)} \sum_{r=0}^{m-k+j-2} \frac{(m-k+j-2)U_\theta^r}{r! \;\; Q^{m-k+j-r-1}} \right\} = \alpha/2. \tag{6.11}$$

From

$$\int_{U_\theta}^{\infty} f(\theta|\underline{x})d\theta = C_3^{-1} \Sigma_* \int_{U_\theta}^{\infty} \theta^{m-k-2} \, e^{-T\theta} \, (x_A)^{-(k+1)} \left\{ 1 - \sum_{j=0}^{k} \frac{(\theta x_A)^j}{\Gamma(j+1)} \, e^{-\theta x_A} \right\} d\theta,$$

$$= C_3^{-1} \Sigma_* \, x_A^{-(k+1)} \left\{ \int_{U_\theta}^{\infty} \theta^{m-k-2} \, e^{-T\theta} \, d\theta - \sum_{j=0}^{k} \int_{U_\theta}^{\infty} \frac{(x_A)^j}{\Gamma(j+1)} \, e^{-Q\theta} \, \theta^{m-k+j-1} \, d\theta \right\},$$

using (3.22) we have

$$\int_{U_\theta}^{\infty} \theta^{m-k-2} \, e^{-T\theta} \, d\theta = e^{-TU_\theta} \sum_{\ell=0}^{m-k-2} \frac{(m-k-2)! \;\; U_\theta^\ell}{\ell! \;\; T^{m-k-\ell-1}},$$

and

$$\int_{U_\theta}^{\infty} \theta^{m-k+j-2}\, e^{-Q\theta}\, d\theta = e^{-QU_\theta} \sum_{r=0}^{m-k+j-2} \frac{(m-k+j-2)!\; U_\theta^{r}}{r!\; Q^{m-k+j-r-1}} \;.$$

We thus find

$$\int_{U_\theta}^{\infty} f(\theta|\underline{x})d\theta = C_3^{-1} \;\Sigma_{*}\; x_A^{-(k+1)} \left\{ e^{-TU_\theta} \sum_{\ell=0}^{m-k-2} \frac{(m-k-2)!\; U_\theta^{\ell}}{\ell!\; T^{m-k-\ell-1}} \right.$$

$$\left. - \sum_{j=0}^{k} \frac{(x_A)^{j}}{\Gamma(j+1)}\; e^{-QU_\theta} \sum_{r=0}^{m-k+j-2} \frac{(m-k+j-2)!\; U_\theta^{r}}{r!\; Q^{m-k+j-r-1}} \right\}.$$

The shortest credible interval (A_θ, B_θ) arises from the equations

$$f(A_\theta|\underline{x}) = f(B_\theta|\underline{x}) \qquad (6.12)$$

and

$$\int_{A_\theta}^{B_\theta} f(\theta|\underline{x})d\theta = 1 - \alpha. \qquad (6.13)$$

This shortest credible interval becomes the HPD interval of Box and Tiao (1972) once the posterior density is unimodal.

6.3.2 Estimation of b

The marginal posterior density of b is

$$f(b|\underline{x}) = C_2 \sum_* b^k \int_0^\infty \theta^{m-1} \exp\{-\theta[Q+(b-1)x_A]\}d\theta,$$

$$= C_2 \sum_* b^k \Gamma(m) [Q+(b-1)x_A]^{-m},$$

$$= [C_4(m,k)]^{-1} \sum_* b^k [T+bx_A]^{-m}, \quad 0<b<1, \quad (6.14)$$

where

$$C_4(m,k) = \sum_* \int_0^1 b^k [T+bx_A]^{-m} db,$$

$$= \sum_* \int_0^1 T^{-m} b^k [1+b(x_A/T)]^{-m} db,$$

$$= \sum_* \frac{T^{-m}}{(k+1)} {}_2F_1(m,k+1,k+2,(-x_A/T)). \quad (6.15)$$

Proceeding exactly as in the previous section, the Bayes estimator is

$$\hat{b}_1 = C_4(m,k+1)/C_4(m,k) \quad (6.16)$$

and expected loss is

$$V(b_1|\underline{x}) = [C_4(m,k+2)/C_4(m,k)] - \hat{b}_1^{\,2}. \quad (6.17)$$

For 100(1-α)% credible interval (L_b, U_b) we seek the solution of

$$C_4^{-1} \sum_* \frac{T^{-m}}{(k+1)} L_b^{k+1} {}_2F_1\left(m,k+1,k+2,\frac{-x_A L_b}{T}\right) = \alpha/2. \quad (6.18)$$

and

$$C_4^{-1} \sum_* \frac{T^{-m}}{(k+1)} U_b^{k+1} {}_2F_1\left(m,k+1,k+2,\frac{-x_A U_b}{T}\right) = 1-\alpha/2. \quad (6.19)$$

where

$$[C_4(m,k)]^{-1} \Sigma_* \int_0^{L_b} b^k \, [T+bx_A]^{-m} \, db$$

$$= C_4^{-1} \Sigma_* T^{-m} \int_0^{L_b} b^k \, [T + (bx_A/T)]^{-m} \, db,$$

$$= C_4^{-1} \Sigma_* \frac{T^{-m}}{(k+1)} L_b^{(k+1)} {}_2F_1 \left[m, k+2, k+3, (-x_A L_b/T)\right].$$

6.3.3 Estimation of the survival function

Since the exponential distribution is extensively used as a lifetime model, it is important to estimate the survival function

$$R = R(t) = P[x>t] = \exp[-\theta t]. \qquad (6.20)$$

The posterior density $f(R|\underline{x})$ of the paramatric function R is arrived by transforming (θ,b) to (R,b) where R is as in (6.20). This lead to

$$f(R,b|\underline{x}) = C_4 \, \Sigma_* \, b^k \, ((1/t)\log R^{-1})^{m-1} \, (1/tR)$$

$$\exp\{(1/t)\log R[T+bx_A]\},$$

$$= C_5 \Sigma_* b^k (\log R^{-1})^{m-1} R^{[T+bx_A]t^{-1}-1}, \qquad 0<R,b<1.$$

(6.21)

From (6.21) we have,

$$f(R|\underline{x}) = C_5 \Sigma_* (\log R^{-1})^{m-1} R^{(T/t)-1} \int_0^1 b^k \exp\{(-1/t)\log R^{-1} b x_A\}\, db,$$

$$= C_5 \Sigma_* (\log R^{-1})^{m-1} R^{(T/t)-1}\Big\{(k+1)!\ [(x_A/t)\log R^{-1}]^{-(k+1)}$$

$$- R^{(x_A/t)} \sum_{j=0}^{k} \frac{[(x_A/t)\log R^{-1}]^{-(k-j+1)}\ (k+1)!}{\Gamma(j+1)}\Big\},$$

(By using (3.3))

$$= [C_6(m,k,0)]^{-1} \Sigma_* (x_A/t)^{-(k+1)} (\log R^{-1})^{m-k-2} R^{(T/t)-1}$$

$$\Big\{1 - R^{(x_A/t)} \sum_{j=0}^{k} \frac{[(x_A/t)\log R^{-1}]^j}{\Gamma(j+1)}\Big\}, \qquad 0<R\le 1,$$

(6.22)

where

$$C_6(m,k,d) = \Sigma_* (x_A/t)^{-(k+1)} \int_0^1 \Big\{(\log R^{-1})^{m-k-2} R^{[(T/t)+d]-1}$$

$$\Big[1 - R^{(x_A/t)} \sum_{j=0}^{k} \frac{[(x_A/t)\log R^{-1}]^j}{\Gamma(j+1)}\Big]\Big\}\, dR,$$

$$= \Sigma_{*} (x_A/t)^{-(k+1)} \int_0^{\infty} \Big\{ u^{m-k-2} \exp[-u((T/t)+d)]$$

$$\Big[1-e^{-(x_A/t)u} \sum_{j=0}^{k} \frac{[(x_A/t)^j u^j}{\Gamma(j+1)} \Big]\Big\} du,$$

(By letting u=-logR)

$$= \Sigma_{*} (x_A/t)^{-(k+1)} \Big\{ \frac{\Gamma(m-k-1)}{[(T+dt)t^{-1}]^{m-k-1}}$$

$$- \sum_{j=0}^{k} \frac{(x_A/t)^j}{\Gamma(j+1)} \frac{\Gamma(m-k+j-1)}{[(Q+dt)t^{-1}]^{m-k-j-1}} \Big\},$$

$$= t^m \Sigma_{*} (x_A)^{-k+1} \Big\{ \frac{\Gamma(m-k-1)}{(T+dt)^{m-k-1}}$$

$$- \sum_{j=0}^{k} \frac{\Gamma(m-k+j-1)\; x_A^{\,j}}{\Gamma(j+1)\; (Q+dt)^{m-k+j-1}} \Big\}. \qquad (6.23)$$

The Bayes estimate of R under quadratic loss is

$$\hat{R}_1 = C_6(m,k,1)/C_6(m,k,0) \qquad (6.24)$$

with expected loss

$$V(R_1|\underline{x}) = [C_6(m,k,2)/C_6(m,k,0)] - \hat{R}_1^{\,2}. \qquad (6.25)$$

Again a 100(1-α)% credible interval for R say (L_R,U_R) is determined as the solution of the equations

$$c_6^{-1} \Sigma_{*} (x_A)^{-(k+1)} t^m \left\{ L_R^{(T/t)} \sum_{\ell=0}^{m-k-2} \frac{(m-k-2)! \; (t\log L_R^{-1})^{\ell}}{\ell! \quad T^{m-k-\ell-1}} \right.$$

$$\left. - L_R^{(Q/t)} \sum_{j=0}^{k} \frac{(x_A)^j}{\Gamma(j+1)} \sum_{r=0}^{m-k+j-2} \frac{(m-k+j-2)! \; (t\log L_R^{-1})^{r}}{r! \quad Q^{m-k+j-r-1}} \right\} = \alpha/2.$$

(6.26)

and

$$c_6^{-1} \Sigma_{*} (x_A)^{-(k+1)} t^m \left\{ U_R^{(T/t)} \sum_{\ell=0}^{m-k-2} \frac{(m-k-2)! \; (t\log U_R^{-1})^{\ell}}{\ell! \quad T^{m-k-\ell-1}} \right.$$

$$\left. - U_R^{(Q/t)} \sum_{j=0}^{k} \frac{(x_A)^j}{\Gamma(j+1)} \sum_{r=0}^{m-k+j-2} \frac{(m-k+j-2)! \; (t\log U_R^{-1})^{r}}{r! \quad Q^{m-k+j-r-1}} \right\} = 1-\alpha/2.$$

(6.27)

Since,

$$\int_0^{L_R} (\log R^{-1})^M \, R^{(A/t)-1} \, dR$$

$$= \int_{\log L_R^{-1}}^{\infty} u^m e^{-(A/t)u} \, du, \quad \text{(By taking } u = -\log R \text{)}$$

$$= \exp\left\{- (A/t) \; (\log L_R^{-1})\right\} \sum_{\ell=0}^{M} \frac{M! \; (\log L_R^{-1})^{\ell}}{\ell! \; (A/t)^{m-\ell+1}},$$

$$= L_R^{(A/t)} \sum_{\ell=0}^{M} \frac{M!\ (\log L_R^{-1})^{\ell}}{\ell!\ (A/t)^{m-\ell+1}} \ ,$$

$$= L_R^{(A/t)} \sum_{\ell=0}^{M} \frac{M!\ (\log L_R^{-1})^{\ell}\ t^{m-\ell+1}}{\ell!\ A^{m-\ell+1}} \ , \qquad (6.28)$$

one can write,

$$\int_0^{L_R} f(R|\underset{\sim}{x})dR = C_5^{-1}\ \Sigma_*\ (x_A/t)^{-(k+1)}$$

$$\left\{ L_R^{(T/t)} \sum_{\ell=0}^{m-k-2} \frac{(m-k-2)!\ (\ \log L_R^{-1})^{\ell}}{\ell!\ \ \ (T/t)^{m-k-\ell-1}} \right.$$

$$\left. -L_R^{(Q/t)} \sum_{j=0}^{k} \frac{(x_A/t)^j}{\Gamma(j+1)} \sum_{r=0}^{m-k+j-2} \frac{(m-k+j-2)!\ (\ \log L_R^{-1})^r}{r!\,(Q/t)^{m-k+j-r-1}} \right\},$$

$$= C_6^{-1}\ \Sigma_*\ (x_A)^{-(k+1)} t^m$$

$$\left\{ L_R^{(T/t)} \sum_{\ell=0}^{m-k-2} \frac{(m-k-2)!\ (t\log L_R^{-1})^{\ell}}{\ell!\ \ \ T^{m-k-\ell-1}} \right.$$

$$\left. -\ L_R^{(Q/t)} \sum_{j=0}^{k} \frac{(x_A)^j}{\Gamma(j+1)} \sum_{r=0}^{m-k+j-2} \frac{(m-k+j-2)!\ (t\log L_R^{-1})^r}{r!\ \ \ Q^{m-k+j-r-1}} \right\}.$$

6.4 Estimation when b>1

In this case, an improper uniform prior for b and gamma prior for θ and the assumption of independence of θ and b give the joint prior for θ and b as

$$\phi(\theta, b) = C_7\ \theta^{p-1}\ e^{-u\theta}, \qquad \theta, u, p>0,\ b>1, \qquad (6.29)$$

and therefore the posterior density of θ and b as

$$f(\theta, b|\underline{x}) = C_8\ \Sigma_*\ b^k\ \theta^{n+p-1}\ \exp\{-\theta[\,n\bar{x} + u + (b-1)x_A]\},$$

$$= C_8\ \Sigma_*\ b^k\ \theta^{m-1}\ \exp\{-\theta[\,Q+(b-1)x_A]\},\ \theta>0, b>1.$$

(6.30)

6.4.1 Estimation of θ

From (6.30) we derive the marginal density of θ as follows.

$$f(\theta|\underline{x}) = C_8\ \Sigma_*\ \ \theta^{m-1}\ e^{-\theta T} \int_1^{\infty} b^k\ e^{-bx_A\theta}\ d\theta,$$

$$= C_8\ \Sigma_*\ \ \theta^{m-1}\ e^{-\theta T} \left\{ e^{-\theta x_A} \sum_{j=0}^{k} \frac{(k+1)!}{\Gamma(j+1)} (\theta x_A)^{-(k-j+1)} \right\}.$$

[By using (3.22)]

$$= [C_9(m,k)]^{-1} \Sigma_* \sum_{j=0}^{k} \theta^{m-k+j-2} e^{-Q\theta} \frac{(x_A)^{-(k-j+1)}}{\Gamma(j+1)},$$

$$= [C_9(m,k)]^{-1} \sum_{j=0}^{k} \frac{\theta^{m-k+j-2}}{\Gamma(j+1)} e^{-Q\theta} \left[\Sigma_* (x_A)^{-(k-j+1)}\right], \theta > 0, \qquad (6.31)$$

where

$$C_9(m,k) = \sum_{j=0}^{k} \frac{[\Sigma_* x_A^{-(k-j+1)}]}{\Gamma(j+1)} \int_0^{\infty} \theta^{m-k+j-2} e^{-Q\theta} d\theta,$$

$$= \sum_{j=0}^{k} \frac{[\Sigma_* x_A^{-(k-j+1)}] \Gamma(m-k+j-1)}{\Gamma(j+1) Q^{m-k+j-1}}. \qquad (6.32)$$

The Bayes estimate of θ is

$$\hat{\theta}_2 = C_9(m+1,k)/C_9(m,k) \qquad (6.33)$$

with expected loss

$$V(\theta_2|\underline{x}) = [C_9(m+2,k)/C_9(m,k)] - \hat{\theta}_2^2. \qquad (6.34)$$

When the interest is in a $100(1-\alpha)\%$ credible interval for θ of the form (L_θ, U_θ) we work out L_θ and U_θ from

$$\int_{L_\theta}^{\infty} f(\theta|\underline{x})d\theta = 1-\alpha/2 \text{ and } \int_{U_\theta}^{\infty} f(\theta|\underline{x})d\theta = \alpha/2.$$

Substituting (6.31) in the last two equations we get, after simplifications, the following equations which can be solved for U_θ and L_θ.

$$C_9^{-1} \sum_{j=0}^{k} \sum_{\ell=0}^{m-k+j-2} \frac{(m-k+j-2)!}{\ell!\ \Gamma(j+1)\ Q^{m-k+j-\ell-1}} \left[\Sigma_* \ x_A^{-(k-j-1)}\right] L_\theta^{\ell} = (1-\alpha/2) e^{QL_\theta} \quad (6.35)$$

and

$$C_9^{-1} \sum_{j=0}^{k} \sum_{\ell=0}^{m-k+j-2} \frac{(m-k+j-2)!}{\ell!\ \Gamma(j+1)\ Q^{m-k+j-\ell-1}} \left[\Sigma_* \ x_A^{-(k-j-1)}\right] U_\theta^{\ell} = (\alpha/2)\ e^{QU_\theta}. \quad (6.36)$$

With the help of equation (3.22) we have,

$$C_9^{-1} \int_{L_\theta}^{\infty} \sum_{j=0}^{k} \frac{[\Sigma_* \ x_A^{-(k-j+1)}]}{\Gamma(j+1)} e^{-Q\theta} \theta^{m-k+j-2}\, d\theta$$

$$= C_9^{-1} \sum_{j=0}^{k} \frac{\left[\Sigma_* \ x_A^{-(k-j+1)}\right]}{\Gamma(j+1)} \sum_{\ell=0}^{m-k+j-2} \frac{(m-k+j-2)!\ e^{-QL_\theta}\ L_\theta^{\ell}}{\ell!\ \ Q^{m-k+j-\ell-1}},$$

$$= e^{-QL_\theta}\ C_9^{-1} \sum_{j=0}^{k} \sum_{\ell=0}^{m-k+j-2} \frac{\left[\Sigma_* \ x_A^{-(k-j+1)}\right] (m-k+j-2)!\ \ L_\theta^{\ell}}{\ell!\ \Gamma(j+1)\ \ Q^{m-k+j-\ell-1}}.$$

6.4.2 Estimation of b

The marginal posterior density of b is

$$f(b|\underline{x}) = C_8 \sum_* b^k \int_0^\infty \theta^{m-1} \exp\{-(T+bx_A)\theta\} d\theta,$$

$$= C_8 \sum_* b^k \Gamma(m) [T+bx_A]^{-m},$$

$$= [C_{10}(m,k)]^{-1} \sum_* b^k [T+bx_A]^{-m}, \quad b>1, \qquad (6.37)$$

where

$$C_{10}(m,k) = \sum_* \int_1^\infty b^k [T+bx_A]^{-m} db,$$

$$= \sum_* \int_1^\infty T^{-m} b^k [1+b(x_A/T)]^{-m} db,$$

$$= \sum_* \frac{T^{-m}}{(m-k-1)} (x_A/T)^{-m} {}_2F_1(m, m-k-1, m-k, -[T_A/x_A]),$$

(By using (3.28))

$$= \sum_* (x_A)^{-m} (m-k-1)^{-1} {}_2F_1(m, m-k-1, m-k, -[T/x_A]), \qquad (6.38)$$

so that the Bayes estimate becomes

$$\hat{b}_2 = C_{10}(m,k+1)/C_{10}(m,k) \qquad (6.39)$$

with expected loss

$$V(b_2|\underline{x}) = [C_{10}(m,k+2)/C_{10}(m,k)] - \hat{b}_2^{2}. \qquad (6.40)$$

In order to calculate the interval estimate of b we repeat the procedure adopted in the case of θ. This gives the $100(1-\alpha)\%$ credible interval for b say (L_b, U_b) as the solution of the equations

$$\int_1^{L_b} f(b|\underline{x})db = \alpha/2 = \int_{U_b}^{\infty} f(b|\underline{x})\, db.$$

With the aid of (6.37) and (3.28) the last equation reduces to

$$C_{10}^{-1} \sum_{*} x_A^{-m} (m-k-1)^{-1} \; {}_2F_1(m, m-k-1, m-k, [-T/(x_A L_b)])$$

$$= (1-\alpha/2) L_b^{m-k-1} \qquad (6.41)$$

and

$$C_{10}^{-1} \sum_{*} x_A^{-m} (m-k-1)^{-1} \; {}_2F_1(m, m-k-1, m-k, [-T/(x_A U_b)])$$

$$= (\alpha/2) U_b^{m-k-1}. \qquad (6.42)$$

It may be noted that in arriving at the above calculation we have made use of the result

$$\int_{U_b}^{\infty} b^k \, [T+bx_A]^{-m} \, db = T^{-m} \int_{U_b}^{\infty} b^k \, [1+b(x_A/T)]^{-m} \, db,$$

$$= T^{-m} (x_A/T)^{-m} \frac{(U_b)^{-(m-k-1)}}{(m-k-1)} \; {}_2F_1(m, m-k-1, m-k, [-T/(x_A U_b)]),$$

$$= \frac{(x_A)^{-m}}{(m-k-1)} \quad (U_b)^{-(m-k-1)} \quad {}_2F_1(m, m-k-1, m-k, [-T/(x_A U_b)]).$$

6.4.3 Estimation of the survival function

In this case also, as in the earlier case, we transform (θ, b) to (R, b) to arrive the joint density of (R, b) where R is as in (6.20) as

$$f(R, b|\underline{x}) = C_8 \sum_{*} b^k \, [(1/t)\log R^{-1}]^{m-1} \, (1/tR)$$

$$\exp\{(1/t)\log R[T + bx_A]\},$$

$$= C_{11} \sum_{*} (\log R^{-1})^{m-1} \, b^k \, R^{[T+bx_A]t^{-1}-1}, \qquad 0 < R \le 1, b > 1.$$

(6.43)

so that the marginal density of R is,

$$f(R|\underline{x}) = C_{11} \sum_{*} (\log R^{-1})^{m-1} R^{(T/t)-1} \int_1^{\infty} b^k \exp\{(-1/t) b x_A \log R^{-1}\} db,$$

$$= C_{11} \sum_{*} (\log R^{-1})^{m-1} R^{(T/t)-1} R^{(x_A/t)}$$

$$\left\{ \sum_{j=0}^{k} \frac{[(x_A/t)\log R^{-1}]^{-(k-j+1)} (k+1)!}{\Gamma(j+1)} \right\},$$

$$= [C_{12}(m,k,0)]^{-1} \Sigma_* \sum_{j=0}^{k} \frac{[\log R^{-1}]^{m-k+j-2}}{\Gamma(j+1)} \left\{ (x_A/t)^{-(k-j+1)} R^{(Q/t)-1} \right\}, \quad 0<R\le 1.$$

(6.44)

with

$$C_{12}(m,k,d) = \Sigma_* \sum_{j=0}^{k} \frac{(x_A/t)^{-(k-j+1)}}{\Gamma(j+1)} \int_0^1 (\log R^{-1})^{m-k+j-2} R^{[(Q/t)+d]-1} \, dR,$$

$$= \Sigma_* \sum_{j=0}^{k} \frac{(x_A/t)^{-(k-j+1)}}{\Gamma(j+1)} \int_0^\infty z^{m-k+j-2} e^{-z[(Q/t)+d]} \, dz,$$

where $z = \log R^{-1}$. Hence

$$C_{12}(m,k,d) = \Sigma_* \sum_{j=0}^{k} \frac{(x_A/t)^{-(k-j+1)}}{\Gamma(j+1)} \frac{\Gamma(m-k+j-1)}{[(Q+dt)/t]^{m-k+j-1}} .$$

$$= t^m \Sigma_* \sum_{j=0}^{k} \frac{\Gamma(m-k+j-1)(x_A)^{-(k-j+1)}}{\Gamma(j+1)\ (Q+dt)^{m-k+j-1}} . \qquad (6.45)$$

The Bayes estimate of R under quadratic loss is

$$\hat{R}_2 = C_{12}(m,k,1)/C_{12}(m,k,0) \qquad (6.46)$$

with expected loss

$$V(R_2|\underline{x}) = [C_{12}(m,k,2)/C_{12}(m,k,0)] - \hat{R}_2^{\,2}. \qquad (6.47)$$

To obtain the 100(1-α)% credible interval for R say (L_R, U_R) we solve the equations

$$\int_0^{L_R} f(R|\underline{x})dR = \alpha/2 = \int_{U_R}^{1} f(R|\underline{x})dR. \qquad (6.48)$$

Introducing (6.44) in (6.48) and using (6.28) we have

$$\alpha/2 = \int_0^{L_R} [C_{12}(m,k,0)]^{-1} \Sigma_* \sum_{j=0}^{k} \frac{[\log R^{-1}]^{m-k+j-2}}{\Gamma(j+1)} \left\{(x_A/t)^{-(k-j+1)} R^{(Q/t)-1}\right\} dR,$$

$$= [C_{12}(m,k,0)]^{-1} \Sigma_* \sum_{j=0}^{k} \frac{(x_A/t)^{-(k-j+1)}}{\Gamma(j+1)} \left\{L_R^{(Q/t)} \sum_{\ell=0}^{m-k+j-2} \frac{(m-k+j-2)!\;(\log L_R^{-1})^{\ell}}{\ell!\;\;Q^{m-k+j-\ell-1}}\, t^{m-k+j-\ell-1}\right\},$$

which leads to

$$[C_{12}(m,k,0)]^{-1}\ t^{m}\ \Sigma_{*} \sum_{j=0}^{k} \frac{(x_A)^{-(k-j+1)}}{\Gamma(j+1)}\ L_R^{(Q/t)}$$

$$\sum_{\ell=0}^{m-k+j-2} \frac{(m-k+j-2)!}{\ell!} \frac{(t\log L_R^{-1})^{\ell}}{Q^{m-k+j-\ell-1}} = \alpha/2. \qquad (6.49)$$

Similarly the equation to be solved to U_R is of the form

$$[C_{12}(m,k,0)]^{-1}\ t^{m}\ \Sigma_{*} \sum_{j=0}^{k} \frac{(x_A)^{-(k-j+1)}}{\Gamma(j+1)}\ U_R^{(Q/t)}$$

$$\sum_{\ell=0}^{m-k+j-2} \frac{(m-k+j-2)!}{\ell!} \frac{(t\log U_R^{-1})^{\ell}}{Q^{m-k+j-\ell-1}} = 1-\alpha/2. \qquad (6.50)$$

6.5 Determination of the number of outliers

One of the major problems in the analysis of statistical data in which some of the observations appear to be discordant, is to determine the number of outliers present in the data and to identify the outlying observations. Kale and Kale (1992) consider the problem of determining the number of outliers k, present in a sample of size n for the exponential distribution with mean θ, using the predictive density of the r^{th} order statistics $X_{(r)}$ given $x_{(1)},\dots,x_{(r-1)}$ and a gamma prior for θ.

This method is applicable only when the prior information on the parameter can be represented through a proper density and does not take into consideration the influence of the observation to arrive at the desired result. Obviously this deprives the investigator of the prospect of revising his prior beliefs on the basis of available sample information. Accordingly in the present section we modify their procedure, using the predictive interval approach in the light of the posterior density, which renders the method applicable to any type of prior irrespective of whether it is proper or improper. The result is obtained here for the case of b>1. (See also Jeevanand and Nair (1993d))

6.5.1 Detection of outliers

When X_i, $i=1,2,\dots,n$ are independent and indentically distributed random variables as (6.1), let $X_{(1)},\dots,X_{(n)}$ denote the corresponding order statistics. The conditional distribution of $X_{(r)}$ given $x_{(1)},\dots,x_{(r-1)}$ is of the exponetial form with density

$$f(x_{(r)}|x_{(1)},\dots,_{(r-1)},\theta)= \theta(n-r+1)$$
$$\exp\left[-(n-r+1)\theta\,(x_{(r)}-x_{(r-1)})\right],$$
$$r = 2,3,\dots,n. \quad (6.51)$$

Following Kale (1976) and since b>1, the outliers are more probable to be the largest order statistics, we now obtain the posterior density of θ, using the trimmed estimate of θ based on the order statistics, proposed by Kale and Sinha (1971), viz.

$$y = \sum_{i=1}^{m'} x_{(i)} + (n-m)\, x_{(m')}. \qquad (6.52)$$

In (6.52) the value of m' can be determined from the table given by Joshi(1972). Thus if the prior distribution of θ is gamma as in (6.3), the posterior density of θ can be written as (seeLingappaiah (1989a))

$$f(\theta|y) = \frac{\omega^{N}}{\Gamma(N)}\,\theta^{N-1}\exp[-\theta\omega], \qquad \theta>0, \qquad (6.53)$$

where $N=m'+p$ and $\omega = u + y$.

Using (6.53) and (5.1) we have,

$$h(x_{(r)}|x_{(1)},\dots,x_{(r-1)})=\int_0^{\infty}\left\{\frac{(n-r+1)\omega^{N}}{\Gamma(N)}\,\theta^{N}\right.$$
$$\left.\exp[-\theta[(n-r+1)(x_{(r)}-x_{(r-1)})+\omega]]\right\}d\theta,$$

$$= \frac{(n-r+1)\Gamma(N+1)\ \omega^{N}}{\Gamma(N)[(n-r+1)(x_{(r)} - x_{(r-1)})+\omega\ ]^{N+1}},$$

$$= \frac{(n-r+1)N\ \omega^{N}}{[(n-r+1)(x_{(r)} - x_{(r-1)})+\omega\ ]^{N+1}}. \tag{6.54}$$

From (6.54), it follows that $X_{(r)}$ has distribution function

$$F_{X_{(r)}}(t) = P\ [X_{(r)} \le t\,|\,x_{(1)},\ldots,x_{(r-1)}]$$

$$= \begin{cases} 0 & \text{for } t < x_{(r-1)} \\ 1-[1+(n-r+1)(t-x_{(r-1)})/\omega]^{-N} & \text{for } t \ge x_{(r-1)}. \end{cases} \tag{6.55}$$

Now a one sided predictive interval for $X_{(r)}$ $[x_{(r-1)}, t_r]$ of cover α is obtained from (6.55) as

$$F_{X_{(r)}}(t_r) = \alpha. \tag{6.56}$$

Any observed value of $X_{(r)}$ that falls outside this interval can be treated as an outlier. The procedure starts with $r=n$ and terminates at that r_0 for which $x_{(r_0)}$ lies inside the predictive interval $[x_{(r_0-1)}, t_{r_0}]$ so that $k=n-r_0$ will be the number of outliers required. It is easy to see that the results in Kale and Kale (1992) derives as a special case when m* and y tends to zero.

6.6 Discussion

In order to learn the performance of the various estimates derived above, we generated samples of various sizes and parameter values and the Bayes estimates along with the losses were calculated. For a sample of size 20 with θ=0.5, b=5 having observations

0.40896	0.79785	4.4456	5.00769
0.4489	0.50784	2.47136	2.52012
3.49993	0.79667	0.92975	2.08597
0.54979	0.70342	2.06798	3.39681
1.00885	1.45273	2.97957	12.97798

The estimates and the expected losses of the parameters and the survival functions are presented in Table 6.1 and 6.2. We have also calculated the estimates and their corresponding losses for wide ranging values of θ,b and n, and the common features observed can be summarised as follows.

Our estimates had consistently less bias than the Bayes estimate when the sample is homogeneous. The bias and the expected losses for the estimates were seen to reduce gradually as the sample size increased. As the value of the parameter θ, is increased, the sample sizes required to produce a good estimate also increased. On the other hand, in estimating R the sample size was not so much influenced by the proximity of the estimate to the true value. Finally the bias and the expected loss get decreased when u increase for a fixed p or when p increase for fixed u.

In order to illustrate the procedure in the Section 6.5, we apply it to the data sets from Nelson (1982) (which was also used by Kale and Kale (1992)) representing the time to breakedown of an insulating fluid in a test as random sample from the entire production of units containing the fluid between electrodes recorded at two different voltages. After arranging the observations in increasing order of magnitude the 95% predictive interval for $X_{(r)}$ given $x_{(1)},\ldots,x_{(r-1)}$ and the number of outliers using the hyper-parameter (p,u), P=0,1,2 and u=0,1,2 for the two data sets are exhibited in Table 6.3 and 6.4.

A comparison of the methods proposed in Kale and Kale (1992) and in the present section reveals that (i) the predictive interval in some cases are wider when the sample information is incorprated into the analysis, possibly due to the disparity between the prior belief and the sample evidence (ii) because of the larger predictive intervals, the number of outliers in some cases are less than those derived when the prior information alone is used. This is to be expected as a result of the influence of sample observations in revising the prior opinion. The programs to calculate the estimates and the predictive intervals are given in Appendix.

Table 6.1

Estimate and expected loss of θ and b from the samples from the exponential distribution with θ=0.5, b =5

P	u	$\hat{\theta}$	$\tilde{\theta}$	$\hat{b}$
1	1	0.479568 (0.007584)	0.419548 (8.382×10^{-3})	5.274 (0.853)
1	2	0.472134 (0.007290)	0.4113314 (8.0568×10^{-3})	5.244 (0.852)
2	1	0.499548 (0.007983)	0.439528 (8.781×10^{-3})	5.291 (0.853)
2	2	0.491722 (0.0076773)	0.430919 (8.4405×10^{-3})	5.287 (0.853)
Jeffery's prior		0.466918 (0.00748)	0.407717 (8.3×10^{-3})	5.349 (0.855)

Table 6.2

Estimate with expected losses of the survival function for the samples from the exponential sample with $\theta=0, b=5$

P	u	$\hat{R}(t=0.05)$	$\tilde{R}(t=2)$
1	1	0.980829	0.467986
		(0.000019)	(0.006486)
1	2	0.980229	0.456751
		(0.000019)	(0.006256)
2	1	0.980612	0.4643681
		(0.000018)	(0.006203)
2	2	0.979653	0.439202
		(0.000019)	(0.006079)
Non-informative case		.979251	0.446202
		(0.000020)	(0.006036)

Table 6.3

Predictive intervals and number of outlying observations in data set I in Nelson(1982)

p=0, u=0

r	P.I for $x_{(r)}$	Observed Value of $x_{(r)}$	Value of k
19	[36.71,81.231]	72.89	0

p=1, u=1

r	P.I for $x_{(r)}$	Observed Value of $x_{(r)}$	Value of k
19	[36.71,54.919]	72.89	1
18	[33.91,43.410]	36.71	

p=1, u=2

r	P.I for $x_{(r)}$	Observed Value of $x_{(r)}$	Value of k
19	[36.71,98.755]	72.89	0

p=2, u=1

r	P.I for $x_{(r)}$	Observed Value of $x_{(r)}$	Value of k
19	[36.71,53.724]	72.89	1
28	[33.91,45.729]	36.71	

cont...

p=2, u=2

r	P.I for $x_{(r)}$	Observed Value of $x_{(r)}$	Value of k
19	[36.71,95.226]	72.89	0

Table 6.4

Predictive intervals and number of outlying observations in data set II in Nelson(1982)

p=0, u=0

r	P.I for $x_{(r)}$	Observed Value of $x_{(r)}$	Value of k
15	[13.77,24.271]	25.59	
14	[5.35,10.60]	13.77	2
13	[3.99,7.49]	5.35	

p=1, u=1

r	P.I for $x_{(r)}$	Observed Value of $x_{(r)}$	Value of k
15	[13.77,23.67]	25.59	
14	[5.35,10.303]	13.77	2
13	[3.99,7.292]	5.35	

p=1, u=2

r	P.I for $x_{(r)}$	Observed Value of $x_{(r)}$	Value of k
15	[13.77,23.915]	25.59	
14	[5.35,10.423]	13.77	2
13	[3.99,7.372]	5.35	

cont...

p=2, u=1

r	P.I for $x_{(r)}$	Observed Value of $x_{(r)}$	Value of k
15	[13.77,23.948]	25.59	
14	[5.35,9.939]	13.77	2
13	[3.99,7.049]	5.35	

p=2, u=2

r	P.I for $x_{(r)}$	Observed Value of $x_{(r)}$	Value of k
15	[13.77,23.169]	25.59	
14	[5.35,10.053]	13.77	2
13	[3.99,7.123]	5.35	

CHAPTER VII

PREDICTION INTERVAL FOR ORDER STATISTICS IN EXPONENTIAL SAMPLES

7.1 Introduction

The evaluation of prediction intervals for order statistics in future samples from a population is of recent origin and induced potential research. In view of the predominant role the exponential model has enjoyed in reliability and life testing, prediction analysis concerning this distribution assumes special significance. Researchers like Hewitt (1968), Guttman (1970), Lawless (1971), Faulkenberry (1973), and Kaminsky and Nelson (1975) used the classical inference procedure to arrive at tolerence regions for the extreme order statistics in a sample. On the other hand Kalbfleisch (1971) employed the likelihood approach for prediction of future order statistics. A variant method which differs in philosophy to these attempts is to use the Bayes approach as is done in Dunsmore (1974) and Lingappaiah (1973,1979b). In this setup the predictive density of some statistic Y based on independent and identically distributed observations $x_1, x_2, \ldots, x_n$ from a population is given by equation (5.1). Then the Bayesian prediction interval of cover α can be specified as A that satisfies equation (5.2).

Taking $f(x,\theta) = \theta e^{-\theta x}$, $\theta, x>0$ and using a gamma prior for θ Dunsmore(1974) obtained the predictive interval for the future order statistic. He also derived the predictive interval for the future order statistics when $f(x,\theta)$ is translated exponential based on both the complete and censored samples.

In the present chapter we investigate a more general problem than in Dunsmore (1974) by aiming to predict order statistics in future samples in terms of earlier samples that are non-homogeneous. (Jeevanand and Nair (1992c)) This type of problem was first considered by Lingappaiah (1989a) in which he obtained the prediction intervals for the maxima and minima of the future sample from exponential population, when the observed sample contains an outlier assuming censored model as the outlier generating mechanism.

7.2 The Model

Let X be a random variable representing the life time of a device with probability density function (6.1). The inference situation considered here involves a random sample

$x_1, x_2, \ldots, x_n$ in which (n-1) of them are distributed as (6.1) while the remaining one follows the distribution

$$g(x:\theta,b) = \theta\ b\ \exp[-\theta bx], \qquad \theta,b>0. \tag{7.1}$$

In this case, under the exchangeable model the joint density of $\underline{x} = (x_1, x_2, \ldots, x_n)$ is

$$f(\underline{x}|\theta,b) = (1/n)\sum_{i=1}^{n} \theta^n\ b\ \exp\{-\theta[n\bar{x} + (b-1)x_i]\}, \tag{7.2}$$

derived as a special case of (6.2) with k=1.

Now, the distribution of the r^{th} order statistics in a sample of size n in the presence of an outlier is given by (5.5)

Thus using (6.1) and (7.1) we have

$$h(y|\theta,b) = \binom{n-1}{r-1} [1-e^{-\theta y}]^{r-2} e^{-\theta(n-r-1)y}$$

$$\left\{(r-1)[1-e^{-\theta by}]\theta e^{-\theta y}e^{-\theta y} + e^{-\theta y}[1-e^{-\theta y}]\theta be^{-\theta by}\right.$$

$$\left. + (n-r)[1-e^{-\theta by}]\theta e^{-\theta y}e^{-\theta by}\right\},$$

$$= \binom{n-1}{r-1} [1-e^{-\theta y}]^{r-2} e^{-\theta(n-r)y}\ \theta\left\{(r-1)[1-e^{-\theta by}]e^{-\theta y}\right.$$

$$\left. +[1-e^{-\theta by}]be^{-\theta by} + (n-r)[1-e^{-\theta by}]e^{-\theta by}\right\},$$

$$= \binom{n-1}{r-1} [1-e^{-\theta y}]^{r-2} e^{-\theta(n-r)y} \theta e^{-\theta y}\Big\{(r-1)$$

$$- (n+b-1) e^{-\theta by} + (n+b-r)e^{-\theta(b-1)y}\Big\},$$

$$= \binom{n-1}{r-1} [1-e^{-\theta y}]^{r-2} \theta e^{-(n-r+1)\theta y} \Big\{(r-1)$$

$$- (n+b-1) e^{-\theta by} + (n+b-r)e^{-\theta(b-1)y}\Big\}. \qquad (7.3)$$

When $r = 1$, (7.6) assumes the simple form

$$h(y|\theta,b) = (n+b-1)\theta e^{-(n+b-1)\theta y}, \qquad \theta,b,y>0, \qquad (7.4)$$

as the distribution of the first order statistic $y_{(1)}$. In the following sections we distinguish between the two inference situations when b is known and unknown and derive the corresponding prediction interval for future observations.

7.3 Prediction interval with known b

With the choice of an informative prior for θ prescribed by

$$\phi(\theta) = c_2 \theta^{p-1} e^{-\theta u}, \qquad \theta,p,u>0. \qquad (7.5)$$

The posterior density of θ arrived at from (7.2) and (7.5) is

$$f(\theta|\underline{x}) = [A\Gamma(m)]^{-1} \sum_{i=1}^{n} \theta^{m-1} \exp\{A_i\theta\}, \qquad \theta>0, \qquad (7.6)$$

where

$$A_i = n\bar{x} + u + (b-1)x_i, \quad m = n+p \quad \text{and} \quad A = \sum_{i=1}^{n} A_i^{-m}.$$

For r ranging from 1 to n the predictive density derives from (7.6) and (7.3) as

$$h(y|\underline{x}) = C_3 \sum_{i=1}^{n} \int_0^{\infty} [1-e^{\theta y}]^{r-2} \theta^m e^{-[A_i+(m-r+1)y]\theta} \Big\{(r-1) -(n+b-1)e^{-\theta by} + (n+b-r)e^{-\theta(b-1)y}\Big\} d\theta. \quad (7.7)$$

In particular, the predictive density of $Y_{(1)}$ is

$$h(y|\underline{x}) = [A\Gamma(m)]^{-1} \sum_{i=1}^{n} \int_0^{\infty} (n+b-1)\ \theta^m \exp\{-[A_i+(n+b-1)y]\theta\}d\theta,$$

$$= (m/A)\ (n+b-1) \sum_{i=1}^{n} [A_i+(n+b-1)y]^{-(m+1)}, \quad y>0. \qquad (7.8)$$

The prediction interval for the first order statistics is now obtained as $(0,U)$ where

$$\int_0^U h(y|\underline{x})dy = \alpha,$$

which simplifies to

$$\sum_{i=1}^{n} \int_{0}^{U} [A_i + (n+b-1)y]^{-(m+1)} (n+b-1)(m/A)dy = \alpha.$$

or $$(1/A) \left\{ \sum_{i=1}^{n} A_i^{-m} - \sum_{i=1}^{n} [A_i + (n+b-1)U]^{-m} \right\} = \alpha.$$

or $$\sum_{i=1}^{n} [A_i + (n+b-1)U]^{-m} = (1-\alpha)A. \qquad (7.9)$$

For $r \geq 2$ (7.7) reduces to

$$h(y|\underline{x}) = \binom{n-1}{r-1} [\Gamma(m)A]^{-1} \sum_{j=0}^{r-2} (-1)^j \binom{r-2}{j} \sum_{i=1}^{n} \left\{ \int_0^{\infty} \left\{ (r-1) - (n+b-1)e^{-\theta by} + (n+b-r)e^{-\theta(b-1)y} \right\} \theta^m \exp\{-\theta[A_i + (n-r+j+1)y]\} \right\} d\theta,$$

$$= \binom{n-1}{r-1} (m/A) \sum_{j=0}^{r-2} (-1)^j \binom{r-2}{j} \sum_{i=1}^{n} \left\{ (r-1) [A_i + (n-r+j+1)y]^{-(m+1)} - (n+b-1)[A_i + (n-r+j+1)y]^{-(m+1)} + (n+b-r)[A_i + (n-r+j+b)y]^{-(m+1)} \right\}, \quad y \geq 0. \qquad (7.10)$$

The predictive interval for the r^{th} order statistics $(r \geq 2)$ is then (L, U) where

$$\int_L^U f(y|\underline{x})dy = \alpha.$$

By direct integration of (7.10) we find that

$$\binom{n-1}{r-1} \sum_{j=0}^{r-2} (-1)^j \binom{r-2}{j} \sum_{i=1}^{n} \left\{ (r-1)\ B_{j+1} - (n+b-1)\ B_{b+j+1} - (n+b-r)\ B_{j+b} \right\} = A\alpha, \quad (7.11)$$

where

$$B_j = m \int_L^U [\,A_i + (n-r+j)y\,]^{-(m+1)} dy,$$

$$= (n-r+j)^{-1} \left\{ [\,A_i + (n-r+j)U\,]^{-m} - [\,A_i + (n-r+j)L\,]^{-m} \right\}.$$

An interval defined above need not be unique as the only requirement there is that the probability content should be α. In order to avoid the multiplicity of intervals arising from (7.11) we choose L and U so as to satisfy the conditions

$$\int_0^L h(y|\underline{x})dy = \int_U^\infty h(y|\underline{x})dy = (1-\alpha)/2. \quad (7.12)$$

Notice that on taking b=1, in (7.8) and (7.10) we find that for the first order statistics the distribution is

$$h(y|\underline{x}) = nm\,(n\bar{x}+U)^{m}\,[n\bar{x}+U+ny]^{-(m+1)}, \quad y>0, \qquad (7.13)$$

and in general,

$$h(y|\underline{x}) = \binom{n-1}{r-1} m\,[n\bar{x}+U]^{m} \sum_{j=0}^{r-2} (-1)^{j} \binom{r-2}{j}$$

$$\left\{[n\bar{x}+U+(n-r+j+1)y]^{-(m+1)} - [n\bar{x}+U+(n-r+j+2)y]^{-(m+1)}\right\},$$

$$y>0. \qquad (7.14)$$

Equations (7.13) and (7.14) are the predictive densities based on samples from (6.1) without contaminants and they are in agreement with the expressions obtained in Dunsmore (1974).

7.4 Prediction interval with unknown b

In this section we attend to the problem of predicting the order statistics when both θ and b are unknown. In infering the parameter b we again discuss seperately the two specific cases, when b<1 and b>1. Such a distinction is necessitated by the fact that when b<1 the discordant observation is in the lower tail while $b>1$ indicates an upper outlier.

7.4.1 The case when $0<b<1$.

The joint prior density for θ and b is chosen by assuming that θ and b are independently distributed with θ following a gamma distribution and b having a proper uniform distribution over $(0,1)$. Hence

$$\phi(\theta,b) = C_4 \, \theta^{p-1} \, e^{-u\theta}, \qquad \theta,p,u>0, \; 0<b<1, \qquad (7.15)$$

so that the joint posterior density of θ and b becomes

$$f(\theta,b|\underset{\sim}{x}) = C_5 \sum_{i=1}^{n} \theta^{m-1} \; b \; e^{-\theta[T_i+bx_i]}, \quad \theta>0, 0<b<1, \quad (7.16)$$

where

$$T_i = n\bar{x} + u - x_i$$

and

$$C_5^{-1} = \sum_{i=1}^{n} \int_0^{\infty} \int_0^{1} \theta^{m-1} \; b \; e^{-\theta[T_i+bx_i]} \, db \, d\theta,$$

$$= \sum_{i=1}^{n} \int_0^{1} b\Gamma(m)[T_i+bx_i]^{-m} \, db,$$

$$= \sum_{i=1}^{n} \Gamma(m)\ T_i^{-m} \int_0^1 b\ [1+b(x_i/T_i)]^{-m}\ db,$$

$$= \Gamma(m) \sum_{i=n}^{n} [T_i^{-m}]/2\ {}_2F_1(m,2,3,-[x_i/T_i]) = T\ \Gamma(m),$$

with

$$T = \sum_{i=n}^{n} [T_i^{-m}]/2\ {}_2F_1(m,2,3,-[x_i/T_i]).$$

From (7.16) and (7.4) it is seen that $Y_{(1)}$ has density

$$h(y|\underline{x}) = C_5 \sum_{i=1}^{n} \int_0^{\infty} \int_0^1 (n+b-1)\theta^m\ b\ \exp\{-\theta[T_i+bx_i+(n+b-1)y]\,db\,d\theta,$$

$$= C_5\ \Gamma(m+1) \sum_{i=1}^{n} \int_0^1 (n+b-1)b\ [T_i+(n-1)y + b(x_i+y)]^{-(m+1)}db,$$

$$= (m/T) \sum_{i=1}^{n} [T_i+(n-1)y]^{-(m+1)} \int_0^1 b(n+b-1)[bq_i+1]^{-(m+1)}db,$$

$$= (m/T) \sum_{i=1}^{n} [T_i+(n-1)y]^{-(m+1)} \Big\{(n-1)/2\ {}_2F_1(m+1,2,3,-q_i)$$

$$+ (1/3)\ {}_2F_1(m+1,3,4,-q_i)\Big\},\quad y\geq 0, \qquad (7.17)$$

where

$$q_i = \frac{x_i+y}{T_i+(n-1)y}\ .$$

The predictive interval for the first order statistics is now obtained as $(0,U)$, where

$$\int_0^U f(y|\underline{x})dy = \alpha.$$

By direct integration we find that the equation to be satisfied by U is

$$(m/T)\sum_{i=1}^{n}\sum_{\ell=0}^{\infty}(-1)^{\ell}\frac{(n-1)^{-(m+\ell+1)}}{m(x_i+y)^m}F(m+\ell+1,m,m+1,-q_{i\ell})$$

$$\left\{(n-1)/2\,\frac{\Gamma(3)\,\Gamma(m+\ell+1)\,\Gamma(\ell+2)}{\Gamma(m+1)\,\Gamma(2)\,\Gamma(\ell+3)\,\Gamma(\ell+1)}\right.$$

$$\left.-(1/3)\,\frac{\Gamma(4)\,\Gamma(m+\ell+1)\,\Gamma(\ell+3)}{\Gamma(m+1)\,\Gamma(3)\,\Gamma(\ell+4)\,\Gamma(\ell+1)}\right\} - 1-\alpha.$$

or

$$\sum_{i=1}^{n}\sum_{\ell=0}^{\infty}(-1)^{\ell}(n-1)^{-(m+\ell+1)}(x_i+y)^{-m}F(m+\ell+1,m,m+1,-q_{i\ell})$$

$$\frac{\Gamma(3)\,\Gamma(m+\ell+1)\,\Gamma(\ell+2)}{\Gamma(m+1)\,\Gamma(2)\,\Gamma(\ell+3)\,\Gamma(\ell+1)}\left\{(n-1)/2+\frac{(\ell+2)}{2(\ell+3)}\right\} = (1-\alpha)T.$$

$$\sum_{i=1}^{n}\sum_{\ell=0}^{\infty}\frac{(-1)^{\ell}(n-1)^{-(m+\ell+1)}(n\ell+3n-1)}{(\ell+2)(\ell+3)\ell\,B(\ell,m+1)}(x_i+y)^{-m}$$

$$_2F_1(m+\ell+1,m,m+1,-q_{i\ell}) = (1-\alpha)T, \qquad (7.18)$$

with

$$q_{i\ell} = \left\{\frac{T_i}{n-1} - x_i\right\}(x_i+y)^{-1}.$$

In deriving equation (7.18) we have made use of the result

$$\int_U^\infty [T_i+Ny]^{-(m+1)} F(a,b,c,-Q_i)$$

$$=\sum_{\ell=0}^{\infty} \frac{\Gamma(c)\ (-1)^\ell}{\Gamma(a)\ \Gamma(b)} \frac{\Gamma(a+\ell)\ \Gamma(b+\ell)}{\Gamma(c+\ell)\ \Gamma(\ell+1)} \int_U^\infty [T_i+Ny]^{-(m+1)}\ Q_i^\ell\ dy,$$

(with $Q_i = (x_i+y)/(T_i+Ny)$)

But $\int_U^\infty [T_i+Ny]^{-(m+1)}\ (x_i+y)^\ell\ dy$

$$= \int_{x_i+U}^\infty [T_i+N(z-x_i)]^{-(m+\ell+1)}\ z^\ell\ dz\ , \quad (\text{with } z=x_i+y)$$

$$= (N)^{-(m+\ell-1)}\ (x_i+U)^{-m}(1/m)\ F(m+\ell+1,m,m+1,-\mu_\ell),$$

with $\mu_\ell = [(T/N)-x_i](x_i+U)^{-1}$. Hence

$$\int_U^\infty [T_i+Ny]^{-(m+1)} F(a,b,c,-Q_i)=\sum_{\ell=0}^{\infty} \frac{\Gamma(c)\ (-1)^\ell}{\Gamma(a)\ \Gamma(b)} \frac{\Gamma(a+\ell)\ \Gamma(b+\ell)}{\Gamma(c+\ell)\ \Gamma(\ell+1)}$$

$$\frac{N^{-(m+\ell-1)}}{m(x_i+U)^m}\ F(m+\ell+1,m,m+1,-\mu_\ell).$$

For higher order statistics $Y_{(r)}$, $r \geq 2$, we have

$$h(y|\underline{x}) = C_6 \sum_{j=0}^{r-2} (-1)^j \binom{r-2}{j} \sum_{i=1}^{n} \Big\{ \frac{(r-1)}{2} Q_j - \frac{(n-1)}{2} Q_j^{*}(2,3) - (1/3) Q_j^{*}(3,4) + \frac{(n-r)}{2} Q_{j-1}^{*}(2,3) + (1/3) Q_{j-1}^{*}(3,4) \Big\},$$

$$y \geq 0, \qquad (7.19)$$

where

$$Q_j = [T_i + (n-r+j+1)y)]^{-(m+1)} \; {}_2F_1(m+1, 2, 3, -x_i [T_i + (n-r+j+1)y]^{-1}),$$

$$Q_j^{*}(a,b) = [T_i + (n-r+j+1)y]^{-(m+1)} \; {}_2F_1(m+1, a, b, -v_j),$$

with

$$v_j = (x_i + y) \, [T_i + (n-r+j+1)y]^{-1}$$

and

$$C_6 = \binom{n-1}{r-1} (m/T).$$

The predictive interval for the r^{th} order statistics $(r \geq 2)$ is then (L,U), which satisfies the condition

$$\int_0^L f(y|\underline{x}) \, dy = \int_U^\infty f(y|\underline{x}) dy = (1-\alpha)/2. \qquad (7.20)$$

Introducing (7.19) into (7.20) and performing integration,

$$C_6 \sum_{j=0}^{r-2} (-1)^j \binom{r-2}{j} \sum_{i=1}^{n} \sum_{\ell=0}^{\infty} (-1)^\ell \Big\{ \frac{(r-1)}{(\ell+2)} R_j - \frac{(n\ell + 3n - 1)}{(\ell+2)(\ell+3)} R_j^{*} + \frac{[(n-r+1)\ell + 3n - 3r+2]}{(\ell+2)(\ell+3)} R_{j-1}^{*} \Big\} = \alpha^{*}.$$

$$(7.21)$$

where

$$R_j = x_i^{\ell} \, [\, T_i + (n-r+j+1)a\,]^{-(m+\ell)} \, [\,(m+\ell)\ell(n-r+j+1)B(\ell, m+1)\,]^{-1},$$

$$R'_j = \frac{(n-r+j+1)^{-(m+\ell+1)}}{m\ell B(\ell, m+1)} \, [\, x_i + a\,]^{-m} \, F(m+\ell+1, m, m+1, -a_{ij}),$$

$$a_{ij} = \left[\frac{T_i}{(n-r+j+1)} - x_i \right] [\, x_i + a\,]^{-1},$$

$a = L, U$

and

$$\alpha' = \begin{cases} (1+\alpha)/2 & \text{for } a=L \\ (1-\alpha)/2 & \text{for } a=U. \end{cases}$$

7.4.2 The case when b>1

In this case, an improper uniform prior for b and a gamma prior for θ and the assumption of independence of θ and b give the joint prior for θ and b as

$$\phi(\theta, b) = C_7 \, \theta^{p-1} \, e^{-\theta u}, \qquad \theta, p, u > 0, \; b > 1. \qquad (7.22)$$

Working with (7.22) and (7.2) the posterior distribution has density

$$f(\theta, b \mid \underline{x}) = C_8 \, \Sigma \, \theta^{m-1} \, b \, e^{-\theta[\, T_i + b x_i \,]}, \qquad \theta > 0, b > 1, \qquad (7.23)$$

in which

$$C_8^{-1} = \int_{0}^{\infty} \int_{1}^{\infty} \Sigma\, \theta^{m-1}\, b\, e^{-\theta[T_i + bx_i]}\, db\, d\theta,$$

$$= \Sigma \int_{1}^{\infty} \Gamma(m) b[T_i + bx_i]^{-m}\, db,$$

$$= \Sigma\, \Gamma(m)\, T_i^{-m} \int_{1}^{\infty} b[1 + b(x_i/T_i)]^{-m}\, db,$$

$$= \Gamma(m)\, \Sigma\, T_i^{-m} (x_i/T_i)^{-m} (m-2)^{-1}\; {}_2F_1(m, m-2, m-1, -(T_i/x_i)),$$

$$= \Gamma(m)\, \Sigma \frac{x_i^{-m}}{(m-2)}\; {}_2F_1(m, m-2, m-1, -(T_i/x_i)) = s\Gamma(m),$$

with $s = \Sigma \frac{x_i^{-m}}{(m-2)}\; {}_2F_1(m, m-2, m-1, -(T_i/x_i))$.

It now follows from (7.4) that the predictive density for $Y_{(1)}$ is

$$h(y|\underline{x}) = C_8\, \Sigma \int_{0}^{\infty} \int_{1}^{\infty} (n+b-1)\theta^{m}\, b\, \exp\{-\theta[T_i + bx_i + (n+b-1)y]\}\, d\theta\, db,$$

$$= C_8\, \Sigma \int_{1}^{\infty} b\, (n+b-1)\Gamma(m+1)[T_i + (n+b-1)y + bx_i]^{-(m+1)} db,$$

$$= C_8'\, \Sigma \int_{1}^{\infty} b\, \frac{(n-1+b)}{[T_i + (n-1)y]^{m+1}} \left[1 + b\, \frac{(x_i + y)}{T_i + (n-1)y}\right]^{-(m+1)} db,$$

$$=(m/s)\ \Sigma\ (x_i+y)^{-(m+1)}\left\{\frac{(n-1)}{(m-1)}\ {}_2F_1(m+1,m-1,m,-z_i)\right.$$

$$\left.+(m-2)^{-1}\ {}_2F_1(m+1,m-2,m-1,-z_i)\right\},\quad y\geq 0,\quad (7.24)$$

where

$$z_i = \frac{[\ T_i + (n-1)y]}{[x_i + y\ ]}\ .$$

Since the calculation of the predictive interval uses the same techinique as in Section 7.3, we omit the details of calculation and the predictive interval for $Y_{(1)}$ is $(0,u)$, where u satisfies the equation,

$$\sum_{i=1}^{n}\ \sum_{\ell=0}^{\infty}(-1)^{\ell}\ \frac{(n-1)^{m+\ell}\ [n(m+\ell-2)+1]}{\ell\ B(\ell,m+1)\ (m+\ell-1)\ (m+\ell-2)}\ [\ T_i+(n-1)u]^{-m}$$

$${}_2F_1(m+\ell+1,m,m+1,-(n-1)[\ x_i-\ [T_i/(n-1)]]\ [\ T_i+(n-1)u]^{-1})$$

$$= (1-\alpha)s.\quad (7.25)$$

Similarly the predictive density of $Y_{(r)}$, for $r\geq 2$, has the form

$$f(y|\underline{x})=C_9 \sum_{j=0}^{r-2} (-1)^j \binom{r-2}{j} \sum_{i=1}^{n} \left\{ \frac{(r-1)}{(m-1)} w_j - \frac{(n-1)}{(m-1)} w'_j(m-1,m) - \frac{w'_j(m-2,m-1)}{(m-2)} + \frac{(n-r)}{(m-1)} w'_{j-1}(m-1,m) + \frac{w'_{j-1}(m-2,m-1)}{(m-2)} \right\}, \quad y\geq 0, \quad (7.28)$$

where

$$w_j = \frac{(m-1)}{\Gamma(m+1)} \int_0^\infty \int_1^\infty e^{-\theta(n-r+j+1)y}\, b\, \theta^m\, e^{-\theta[T_i+x_i b_i]}\, db\, d\theta,$$

$$= (m-1) \int_1^\infty b[T_i+(n-r+j+1)y+x_i b]^{-(m+1)}\, db,$$

$$=(m-1)[T_i+(n-r+j+1)y]^{-(m+1)} \int_1^\infty b[1+x_i(T_i+(n-r+j+1)y)^{-1} b]^{-(m+1)} db,$$

$$= (m-1)\ [T_i+(n-r+j+1)y]^{-(m+1)} \frac{x_i^{-(m+1)}}{(m-1)[T_i+(n-r+j+1)y]^{-(m+1)}}\ {}_2F_1(m+1,m-1,m,-[T_i+(n-r+j+1)y]\,x_i^{-1}),$$

$$= x_i^{-(m+1)}\ {}_2F_1(m+1,m-1,m,-x_i^{-1}[T_i+(n-r+j+1)y]).$$

Similarly,

$$w'_j(a,b)= (x_i+y)^{-(m+1)}\ {}_2F_1(m+1,a,b,,-[T_i+(n-r+j+1)y](x_i+y)^{-1})$$

and

$$C_9 = \binom{n-1}{r-1}(m/s).$$

The corresponding predictive interval (L,U) obtained from the equation

$$C_9 \sum_{j=0}^{r-2} \sum_{i=1}^{n} \sum_{\ell=0}^{\infty} (-1)^{\ell+j} \left\{ \frac{(r-1)}{(m-1)} D_j - \left[\frac{(n-1)}{m+\ell+1} + \frac{1}{m+\ell-2} \right] D'_j \right.$$

$$\left. + \left[\frac{(n-r)}{(m+\ell-1)} + \frac{1}{m+\ell-2} \right] D'_{j-1} \right\} = \alpha'. \qquad (7.27)$$

where

$$D_j = \frac{(m-1)x_i^{-(m+\ell+1)}}{(n-r+j+1)} [T_i + (n-r+j+1)a]^{\ell-1} [\ell(\ell-1)(m+\ell-1)B(m+1,\ell)]^{-1},$$

$$D_j' = \frac{(n-r+j+1)^{m+\ell}}{\ell\, B(\ell, m+1)} {}_2F_1(m+\ell+1, m, m+1, -a'_{ij})$$

and

$$a'_{ij} = (n-r+j+1) \left[x_i - \frac{T_i}{(n-r+j+1)} \right] [T_i + (n-r+j+1)a]^{-1}.$$

7.5 Discussion

The analytical expression obtained in the previous section are too elaborate and analytically intractable to assess the nature of the prediction intervals provided by them. We have therefore made an empirical study of the performance of the results by simulated samples from the population with different values of parameters of the model and the hyper-parameters of the prior. One such example with θ=0.005, b=2.5, and n=10, giving the observations 22.6487, 50.9598, 134.4387, 150.0568, 158.5446, 183.3987, 310.8717, 326.2717, 347.4034, 1040.9046 produces prediction interval of cover α=0.95 as shown in Tables 7.1 and 7.2. For calculating the interval the usual Newtons-Raphson method with intial values $x_{(r-1)}$ and $x_{(r+1)}$ for $r \neq n$ and $x_{(n)}$ for r=n are used. The computer program is avliable in Appendix.

The prediction intervals were computed for wide ranging values of θ,b and n and the common features observed in our investigations are summerised as follows.

1) The intervals are not very much sensitive to b values for large samples.

2) With increasing values of n, the intervals become narrower.

3) The width of the interval increases along with the value of the parameter θ.

4) The length of the interval is seen to be smaller when b is known, than when b is unknown. The accommodation procedure suggested here performs adequately in the

sense that the interval for the homogeneous data using the formula given by Dunsmore (1974), is much wider than intervals in the two cases when b is known and unknown.

Table 7.1

Prediction interval for the r^{th} order statistics in exponential sample when b is known

	Non-informative prior		p=1 u=1	
r	L	U	L	U
1	0.0000	88.9341	0.0000	87.9413
2	22.0459	107.8813	21.9715	106.3375
3	49.1478	155.7990	48.9526	154.9922
4	83.9715	166.9256	82.9526	169.3050
5	105.7019	203.1400	113.5487	209.8614
6	129.9298	247.2935	124.5098	238.0499
7	159.1246	304.5322	163.5446	305.3884
8	192.4858	386.2923	210.0717	386.858
9	237.0387	525.8045	254.1977	565.4045
10	300.6783	887.1709	302.1709	875.0492

Table 7.2

Prediction interval for the r^{th} order statistics in exponential sample when b is known

	Non-informative prior		p=1 u=1	
r	L	U	L	U
1	0.0000	98.02646	0.0000	96.0265
2	21.0785	116.33755	19.9417	110.743
3	48.1628	173.3932	48.2760	166.8167
4	102.7019	218.4938	88.0658	209.8167
5	124.9298	248.1235	109.8614	237.9978
6	184.2200	276.2246	174.1246	264.0738
7	226.7665	330.4987	214.1364	315.3101
8	300.8717	507.005	287.7665	486.7536
9	324.3987	620.4716	302.8223	590.7357
10	337.4858	935.8032	335.4667	936.1526

CHAPTER VIII

ESTIMATION OF P[X>Y] FROM EXPONENTIAL SAMPLES

8.1 Introduction

In this chapter we discuss the problem of estimating P[X>Y] when X and Y are independent exponential random variables and the samples from each population contain one spurious observation. The estimates are derived for exchangeable, identifiable and censored models and their performances are evaluated with the aid of simulated samples.

The problem investigated here has relevance in the context of analysing the reliability of a component whose strength is represented by the random variable X which is subjected to a stress Y, where X and Y are exponentialy distributed and the stress, to which the component is subjected to, is independent of its strength. The component fails at the instant the stress applied to it exceeds the strength. If there is no other cause of failure in the sense that the component will function satisfactorily whenever X>Y, R = P[X>Y] is a measure of the component reliability.

Estimation of $R = P[X>Y]$ when X and Y are random varibles following a specified distribution has been discussed extensively in literature. The minimum variance unbaised estimate and the maximum likelihood estimate of R when X and Y are exponential random varibles, are discussed in the papers of Tong(1974,1975), Johnson(1975), Kelly et. al.(1976) and Bai and Hong(1992). All these papers assume that the observations are independently and identically distributed. The possibility of discordant observations in a sample from any of the populations while estimating R does not appear to have been discussed in literature.

Accordingly in the present chapter we investigate this problem and propose Bayes estimate of R when X and Y are independent exponential random varibles and data on each of them contains a discordant observation. The question of a sample derived from one of the populations contains a discordant observation while the sample from the other is homogeneous arises as a special case of this discussion as will be demonstrated below.(Jeevanand and Nair (1993c)).

8.2 The model

Let X and Y have probability density functions

$$f(x:\theta) = \theta \exp[-\theta x], \qquad x,\theta>0. \qquad (8.1)$$

and

$$f(y:\alpha) = \alpha \exp[-\alpha y], \qquad \alpha,y>0. \qquad (8.2)$$

Then

$$R = P[X>Y] = \theta/(\theta+\alpha). \qquad (8.3)$$

In the following sections we obtain the Bayes estimate of R based on a random samples from (8.1) and (8.2) under different data generating mechanisms that produce one spurious observation in a sample of size n. Let $\underline{x}=(x_1,\ldots,x_n)$ and $\underline{y}=(y_1,\ldots,y_n)$ be n independent observations on X and Y respectively, with (n-1) of $\underline{x}$ distributed as (8.1) and (n-1) of $\underline{y}$ distributed as (8.2) while one observation in $\underline{x}$ is distributed with density $f(x:b\theta)$ and one observation in $\underline{y}$ has density $f(y:d\alpha)$, $(d,b\geq 1)$.

8.3 The Exchangeable Model

It is assumed here that discordant observation is not identifiable and that any observation in the sample is as likely to be discordant as any other Accordingly the joint distribution of $\underline{x}=(x_1,\ldots,x_n)$ in the light of (1.3) is

$$\ell(\underline{x}|b,\theta) = (1/n)\sum_{i=1}^{n} b\ \theta^{n} \exp(\ -\theta[n\bar{x} + (b-1)x_i]). \qquad (8.4)$$

In the Bayesian framework proposed here θ and b are assumed to have the joint prior density (6.3). This gives the joint posterior density of θ and b as

$$f(\theta,b|\underline{x}) = C_1 \sum_{i=1}^{n} b\ \ \theta^{n} \exp(\ -\theta[n\bar{x} +(b-1)x_i])\ \ \theta^{\,p-1}\ e^{-\theta u}\ ,$$

$$= C_1 \sum_{i=1}^{n} b\ \ \theta^{n+p-1} \exp(\ -\theta[n\bar{x} + u + (b-1)x_i]),$$

$$= C_1 \sum_{i=1}^{n} b\ \ \theta^{m-1} \exp(\ -\theta[Q+(b-1)x_i]),\ \theta>0, b\geq 1,\ (8.5)$$

where

$$Q = n\bar{x}+u \quad \text{and} \quad m = n+p.$$

In a similar manner, it is easy to see that the posterior density of α and d has the form

$$f(\alpha,d|\underline{y}) = C_2 \sum_{j=1}^{n} d\ \alpha^{s-1} \exp(\ -\alpha[D+(d-1)y_j])\quad \alpha>0, d\geq 1,\quad (8.6)$$

with $D = n\bar{y}+v$ and $s = n+q$.

Thus the joint distribution of θ,b,α and d is specified by the density

$$f(\theta,\alpha,b,d|\underline{x},\underline{y})=C_3\left\{\sum_{i=1}^{n} b\ \theta^{m-1}\ \exp(\ -\theta[Q+(b-1)x_i])\right\}$$

$$\left\{\sum_{j=1}^{n} d\ \ \alpha^{s-1}\ \exp(\ -\alpha[D+(d-1)y_j])\right\},$$

$$=C_3\sum_{i=1}^{n}\ \sum_{j=1}^{n} b\ \theta^{m-1}\ d\ \alpha^{s-1}$$

$$\exp\{-\theta[Q+(b-1)x_i]-\alpha[D+(d-1)y_j]\}. \quad (8.7)$$

To obtain the posterior density $f_1(R|\underline{x},\underline{y})$ of the parametric function R, we transform (θ,b,α,d) to (R,b,α,d), where is as in (8.3). This leads to

$$f(R,\alpha,b,d|\underline{x},\underline{y})=C_4\sum_{i=1}^{n}\ \sum_{j=1}^{n} b\ [R\alpha/(1-R)]^{m-1}\ d\ \alpha^{s-1}\ \alpha\ (1-R)^{-2}$$

$$\exp\{-\frac{R\alpha}{(1-R)}\ [Q+(b-1)x_i]-\alpha[D+(d-1)y_j]\},$$

$$=C_4\sum_{i=1}^{n}\ \sum_{j=1}^{n} b\ d\ \alpha^{m+s-1}\ R^{m-1}\ (1-R)^{-(m+1)}$$

$$\exp\{-\frac{R\alpha}{(1-R)}\ [Q+(b-1)x_i]-\alpha[D+(d-1)y_j]\},$$

$$0<R\le 1,\alpha>0,b,d\ge 1. \quad (8.8)$$

The marginal density of R becomes,

$$f_1(R|\underline{x},\underline{y}) = C_4 \sum_{i=1}^{n} \sum_{j=1}^{n} R^{m-1} (1-R)^{-(m+1)} \left\{ \int_0^\infty \int_1^\infty \int_1^\infty b\ d\ \alpha^{m+s-1} \right.$$

$$\left. \exp\{-\frac{R\alpha}{(1-R)} [Q+(b-1)x_i] - \alpha [D+(d-1)y_j]\} d\alpha\ db\ dd \right\},$$

$$= C_4 \sum_{i=1}^{n} \sum_{j=1}^{n} R^{m-1} (1-R)^{-(m+1)} \left\{ \int_0^\infty \alpha^{m+s-1}\ I_1(\alpha) I_2(\alpha) \right.$$

$$\left. \exp\{-\frac{R\alpha}{(1-R)} [Q-x_i] - \alpha[D-y_j]\} d\alpha \right\},$$

where

$$I_1(\alpha) = \int_1^\infty d \exp(-d\alpha y_j)\ dd = \frac{d\ e^{-d\alpha y_j}}{-\alpha y_j} \Big]_1^\infty + \int_1^\infty (\alpha y_j)^{-1} \exp(-d\alpha y_j) dd,$$

$$= (\alpha y_j)^{-1} \exp(-\alpha y_j) - \Big[(\alpha y_j)^{-2} \exp(-\alpha d y_j)\Big]_1^\infty .$$

$$= (1+\alpha y_j)(\alpha y_j)^{-2} \exp(-\alpha y_j).$$

and

$$I_2(\alpha) = \int_1^\infty b \exp(-[R(1-R)^{-1} b\alpha x_i])\ db\ ,$$

$$= [1+[R(1-R)^{-1} x_i \alpha]] [R(1-R)^{-1} \alpha x_i]^{-2} \exp(-[R(1-R)^{-1} \alpha x_i]).$$

Hence,

$$f_1(R|\underline{x},\underline{y})=C_4 \sum_{i=1}^{n} \sum_{j=1}^{n} R^{m-1} (1-R)^{-(m+1)} (1-R)^2 (Rx_i y_j)^{-2}$$

$$\left\{ \int_0^\infty \alpha^{m+s-1} \left(1+ \frac{R\alpha}{1-R} x_i\right) \alpha^{-4} [1+\alpha y_j]\right.$$

$$\left.\exp\{-\frac{R\alpha}{(1-R)} [x_i+(Q-x_i)] -\alpha[D-y_j]+y_j\}d\alpha\right\},$$

$$=C_4 \sum_{i=1}^{n} \sum_{j=1}^{n} R^{m-3} (1-R)^{-(m-1)} (x_i y_j)^{-2} \left\{ \int_0^\infty \alpha^{m+s-5}\right.$$

$$\left.\left(1+ \frac{R\alpha}{1-R} x_i + \frac{R\alpha^2}{1-R} x_i y_j + \alpha y_j\right) \exp\{-\alpha[\frac{R}{(1-R)} Q+D] \}d\alpha\right\},$$

$$=C_4 \sum_{i=1}^{n} \sum_{j=1}^{n} R^{m-3}(1-R)^{-(m-1)}(x_i y_j)^{-2}$$

$$\left\{\Gamma(m+s-4) \left[\frac{R}{1-R}Q+D\right]^{-(m+s-4)} +y_j\Gamma(m+s-3) \left[\frac{R}{1-R}Q+D\right]^{-(m+s-3)}\right.$$

$$+R(1-R)^{-1} x_i \Gamma(m+s-3) \left[\frac{R}{1-R}Q+D\right]^{-(m+s-3)}$$

$$\left.+Rx_i y_j(1-R)^{-1}\Gamma(m+s-2) \left[\frac{R}{1-R}Q+D\right]^{-(m+s-2)} \right\},$$

$$=C_4' \sum_{i=1}^{n} \sum_{j=1}^{n} R^{m-3}(1-R)^{-(m-1)}(x_i y_j)^{-2}\left[\frac{R}{1-R}Q+D\right]^{-(m+s-2)}$$

$$\left\{\left[\frac{R}{1-R}Q+D\right]^{2} +(m+s-4)\left[\frac{R}{1-R}Q+D\right]\left[\frac{R}{1-R}x_i+y_j\right]\right.$$

$$\left.+\frac{R}{1-R}x_i y_j(m+s-3)(m+s-4)\right\},$$

$$= C_5 \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i y_j)^{-2} R^{m-3}(1-R)^{s-3}(1+AR)^{-(m+s-2)}$$

$$\left\{D^2(1+AR)^2 +\left(\frac{R}{1-R}x_i+y_j\right)(m+s-4)D(1+AR)(1-R)\right.$$

$$\left.+(1-R)R(m+s-4)(m+s-3)x_i y_j\right\},\quad 0<R\le 1, \qquad (8.9)$$

where

$$A = (Q/D)-1$$

and

$$C_5^{-1} = \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i y_j)^{-2} B(s-2,m-3)(D/Q)^{m-1}[D+(s-2)y_j][Q+(m-2)x_i]. \qquad (8.10)$$

To obtain C_5 we consider the integral,

$$L(m,s,t) = \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i y_j)^{-2} \int_0^1 R^{m+t-1}(1-R)^{s-3}(1+AR)^{-(m+s-2)}$$

$$\left\{D^2(1+AR)^2 +\left(\frac{R}{1-R}x_i+y_j\right)(m+s-4)D(1+AR)(1-R)\right.$$

$$\left.+(1-R)R(m+s-4)(m+s-3)x_i y_j\right\}\, dR,$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i y_j)^{-2} \left\{ D^2 \int_0^1 \frac{R^{m+t-3} (1-R)^{s-3}}{(1+AR)^{(m+s-4)}} dR \right.$$

$$+ (m+s-4) D x_i \int_0^1 \frac{R^{m+t-2} (1-R)^{s-3}}{(1+AR)^{(m+s-3)}} dR$$

$$+ (m+s-4) D y_j \int_0^1 \frac{R^{m+t-3} (1-R)^{s-2}}{(1+AR)^{(m+s-3)}} dR$$

$$\left. + (m+s-4)(m+s-3) x_i y_j \int_0^1 \frac{R^{m+t-2} (1-R)^{s-2}}{(1+AR)^{(m+s-2)}} dR \right\}.$$

Now using the result (3.11),

$$L(m,s,t) = \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i y_j)^{-2}$$

$$\Big\{ D^2 B(s-2, m+t-2)\, {}_2F_1(m+s-4, m+t-2, m+s+t-4, -A)$$

$$+ D x_i (m+s-4) B(s-2, m+t-1)\, {}_2F_1(m+s-3, m+t-1, m+s+t-3, -A)$$

$$+ D y_j (m+s-4) B(s-1, m+t-2)\, {}_2F_1(m+s-3, m+t-2, m+s+t-3, -A)$$

$$+ x_i\ y_j\ (m+s-4)(m+s-3) B(s-1, m+t-1)$$

$${}_2F_1(m+s-2, m+t-1, m+s+t-2, -A) \Big\} . \qquad (8.11)$$

The above calculation provides

$$C_5^{-1} = L(m,s,0)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i y_j)^{-2} \Big\{ D^2 B(s-2,m-2)\,{}_2F_1(m+s-4,m-2,m+s-4,-A)$$

$$+Dx_i\ (m+s-4)B(s-2,m-1)\,{}_2F_1(m+s-3,m-1,m+s-3,-A)$$

$$+Dy_j\ (m+s-4)B(s-1,m-2)\,{}_2F_1(m+s-3,m-2,m+s-3,-A)$$

$$+x_i\ y_j\ (m+s-4)(m+s-3)B(s-1,m-1)$$

$${}_2F_1(m+s-2,m-1,m+s-2,-A)\Big\}.$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i y_j)^{-2} \Big\{ D^2 \frac{\Gamma(s-2)\Gamma(m-2)}{\Gamma(m+s-4)} (1+A)^{-(m-2)}$$

$$+ Dx_i(m+s-4) \frac{\Gamma(s-2)\Gamma(m-1)}{\Gamma(m+s-3)} (1+A)^{-(m-1)}$$

$$+ Dy_j(m+s-4) \frac{\Gamma(s-1)\Gamma(m-2)}{\Gamma(m+s-3)} (1+A)^{-(m-2)}$$

$$+(m+s-4)(m+s-3)x_i y_j \frac{\Gamma(s-1)\Gamma(m-1)}{\Gamma(m+s-2)} (1+A)^{-(m-1)}\Big\}.$$

Since ${}_2F_1(a,b,a,z) = 1+bz+[b(b+1)/(2!)]z^2+\ldots$

$$= (1-z)^{-b},$$

$$C_5^{-1} = \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i y_j)^{-2} \frac{\Gamma(s-2)\Gamma(m-2)}{\Gamma(m+s-4)} (1+A)^{-(m-1)} \Big\{ D^2 (1+A)$$

$$+ (m-2)Dx_i + (1+A)(s-2)Dy_j + (s-2)(m-2)x_i y_j\Big\}.$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i y_j)^{-2} B(s-2, m-2) (D/Q)^{(m-1)} \Big\{ D\ Q + (m-2)Dx_i$$

$$+ (s-2)Qy_j + (s-2)(m-2)x_i y_j \Big\},$$

since $1+A = (Q/D)$. Hence,

$$C_5^{-1} = \sum_{i=1}^{n} \sum_{j=1}^{n} (x_i y_j)^{-2} B(s-2, m-3)(D/Q)^{m-1} [D+(s-2)y_j][Q+(m-2)x_i].$$

Under qudratic loss the Bayes estimator of R is

$$\hat{R}_1 = C_5\ L(m, s, 1) \tag{8.12}$$

with expected loss resulting from (8.12) is

$$V(R_1 | \underline{x}) = [C_5\ L(m, s, 2)] - \hat{R}_1^{\,2}. \tag{8.13}$$

Note that b=1, represent the situation when the data on X does not contain any contaminant and d=1, when the Y data is homogenous. For example, when d=1 the estimate is

$$\hat{R}_1 = L'(m, s, 1)/L'(m, s, 0),$$

where

$$L'(m, s, t) = \sum_{i=1}^{n} x_i^{-2} \Big\{ D\ B(s, m+t-2)\,{}_2F_1(m+s-2, m+t-2, m+s+t-2, -A)$$

$$+ x_i (m+s-2) B(s, m+t-1)\,{}_2F_1(m+s-1, m+t-1, m+s+t-1, -A) \Big\}.$$

8.4 The Identifiable Model

In this section we employ the identified outlier model, which stipulates that the subset which forms the outliers is known. Since b>1, it follows from Kale and Kale(1992) that the largest order statistics in the sample has the largest posterior probability of being outlier. Hence treating $X_{(n)}$ and $Y_{(n)}$ as outliers, the likelihood of $\underline{x}$ takes the form

$$\ell(\underline{x}|b,\theta) = C_6\ b\ \theta^{n} \exp\{\ -\theta[n\bar{x} + (b-1)x_{(n)}]\}. \qquad (8.14)$$

With the same prior for θ and b as given in (6.3) we obtain the joint posterior density of (θ,b) and (α,d) as

$$f(\theta,b|\underline{x}) = C_7\ b\ \theta^{m-1} \exp\{\ -\theta[Q+(b-1)x_{(n)}]\},\ \theta>0,b\geq 1, \qquad (8.15)$$

and

$$f(\alpha,d|\underline{y}) = C_8\ d\ \alpha^{s-1} \exp\{\ -\alpha[D+(d-1)y_{(n)}]\},\ \alpha>0,d\geq 1, \qquad (8.16)$$

By proceeding on lines exactly similar to those in the previous section, we have the joint posterior density of R,α,b,d as

$$f(R,\alpha,b,d\,|\,\underline{x},\underline{y}) = C_9\; b\; d\; \alpha^{m+s-1}\; R^{m-1}\; (1-R)^{-(m+1)}$$

$$\exp\{-\frac{R\alpha}{(1-R)}\,[Q+(b-1)x_{(n)}]-\alpha[D+(d-1)y_{(n)}]\},$$

$$0<R\le 1,\ \alpha>0,\ b,d\ge 1.$$

and the marginal density of R becomes

$$f_2(R\,|\,\underline{x},\underline{y}) = C_{10}\; R^{m-3}(1-R)^{s-3}(1+AR)^{-(m+s-2)}\;\Big\{D^2(1+AR)^2$$

$$+\Big(\frac{R}{1-R}\,x_{(n)}+y_{(n)}\Big)(m+s-4)D(1+AR)(1-R)$$

$$+(1-R)R(m+s-4)(m+s-3)x_{(n)}y_{(n)}\Big\},\ 0<R\le 1,\qquad (8.17)$$

where

$$C_{10}^{-1} = B(s-2,m-2)(D/Q)^{m-1}\;[D+(s-2)y_{(n)}]\;[Q+(m-2)x_{(n)}].$$

The Bayes estimate of R is

$$\hat{R}_2 = C_{10}\; L_1(m,s,1) \qquad (8.18)$$

with expected loss

$$V(R_2\,|\,\underline{x},\underline{y}) \;=\; [C_{10}\; L_1(m,s,2)] - \hat{R}_2^{\,2}. \qquad (8.19)$$

where

$$L_1(m,s,t) = \Big\{ D^2 B(s-2,m+t-2)\,{}_2F_1(m+s-4,m+t-2,m+s+t-4,-A)$$

$$+Dx_{(n)}\,(m+s-4)B(s-2,m+t-1)$$

$${}_2F_1(m+s-3,m+t-1,m+s+t-3,-A)$$

$$+Dy_{(n)}\,(m+s-4)B(s-1,m+t-2)$$

$${}_2F_1(m+s-3,m+t-2,m+s+t-3,-A)$$

$$+x_{(n)}\,y_{(n)}\,(m+s-4)(m+s-3)B(s-1,m+t-1)$$

$${}_2F_1(m+s-2,m+t-1,m+s+t-2,-A)\Big\} \,. \quad (8.20)$$

8.5 The Censored Model

In this model the trimmed estimate proposed by Kale and Sinha (1971) for θ when the sample contains one outlier is utilised to obtain the posterior distribution of θ and α . Taking

$$\hat{\theta} = \sum_{i=1}^{r} x_{(i)} + (n-r)x_{(r)},$$

and using the gamma prior for θ as in (6.3), Lingappaiah (1989a) obtain the posterior density of θ as

$$f(\theta|\hat{\theta}) = C_{11}\ \theta^{G-1}\ \exp[-\theta H], \qquad \theta>0, \qquad (8.21)$$

where

$$G = p+r \text{ and } H = u+\hat{\theta}.$$

Similarly the posterior density of α is

$$f(\alpha|\hat{\alpha}) = C_{12}\ \alpha^{k-1}\exp[-\alpha w], \qquad \alpha>0, \qquad (8.22)$$

with

$$k = q+r', \ w = v+\hat{\alpha} \text{ and } \hat{\alpha} = \sum_{i=1}^{r'} y_{(i)} + (n-r')y_{(r')}.$$

So the joint density of (α,θ) can be writen as

$$f(\alpha,\theta|\hat{\alpha},\hat{\theta}) = C_{13}\ \alpha^{k-1}\ \theta^{G-1}\ \exp(-[\theta H+\alpha w]), \quad \alpha,\theta>0.$$

Now using the transformation (α,θ) to (R,α), where R is as in (8.3) one can obtain the posterir density of (α,R) as

$$f(R,\alpha|\hat{\alpha},\hat{\theta}) = C_{14}\ \alpha^{k-1}\ [R\alpha/(1-R)]^{G-1}\ \exp(-\alpha[\frac{R}{1-R} H+w])\alpha(1-R)^{-2},$$

$$= C_{14}\ \alpha^{k+G-1}\ R^{G-1}\ (1-R)^{-(G+1)}\ \exp(-\alpha\ [\frac{R}{1-R} H+ w]).$$

(8.23)

The density of R obtained from (8.23) is

$$f_3(R|\hat{\theta},\hat{\alpha}) = C_{14}\ R^{G-1}(1-R)^{-(G+1)} \int_0^\infty \alpha^{G+k-1}\ \exp(-\alpha\ [\frac{R}{1-R} H+w])d\alpha,$$

$$= C_{14}\ R^{G-1}(1-R)^{-(G+1)}\Gamma(G+k)\ \left[\frac{R}{1-R} H+ w\right]^{-(G+K)},$$

$$=C_{14}\Gamma(G+k)R^{G-1}(1-R)^{-(G+1)}\left[\frac{w}{(1-R)}\right]^{-(G+k)}[1+ZR]^{-(G+k)},$$

$$= C_{15}\, R^{G-1}(1-R)^{k-1}(1+RZ)^{-(G+k)}, \quad 0<R\le 1, \qquad (8.24)$$

where

$$C_{15}^{-1} = B(k+G)\ (1+Z)^{-G} \quad \text{and } Z = (H/w)-1.$$

The Bayes estimate of R underqudratic loss is

$$\hat{R}_3 = (1+Z)^G\ G\ (G+k)^{-1}\ {}_2F_1(G+K, G+1, G+k+1, -Z) \qquad (8.25)$$

with expected loss

$$V(R_3|\hat{\alpha},\hat{\theta}) = \frac{G(G+1)(1+Z)^G}{(G+k)(G+k+1)}\ {}_2F_1(G+K, G+2, G+k+2, -Z) - \hat{R}_3^{\,2}. \quad (8.26)$$

8.6 Discussion

In this section we present some numerical results for some selected values of the hyper-parameters of the prior distributions and compare the bias and the expected loss of $\hat{R}$ for different sample. Samples of sizes 10,30 and 50 were generated with θ = 19 and α = 0.5. The estimates were calculated for repated samples and the results obtained are given in Table 8.1. It is observed that both $\hat{R}_1$ and $\hat{R}_2$ gives almost the same bias and the risk (expcted loss) for all values of n. In general, the absolute bais $|\hat{R}_i - R_i|$, $i=1,2,3$ is smaller for $\hat{R}_1$ than $\hat{R}_3$, the estimator under models I and III. As expected, when the samples become large both the bais and risk tend to be smaller, irrespective of the value of R and the prior parameters. Notice that the values inside the braces in the table provide the expected losses. The computer programe to calculate the estimates are given at the end of Appendix.

Table 8.1

Absolute bais and expected loss of R=P[X>Y] for different sample sizes for the Exponential model

p u q v	n=10		n=30		n=50	
	$R_1 = R_2$	R_3	$R_1 = R_1$	R_3	$R_1 = R_2$	R_3
1 1 1 1	0.0464	0.0584	0.0076	0.0504	0.0012	0.0245
	(6.08)	(9.428)	(5.23)	(5.929)	(2.93)	(4.937)
1 1 1 2	0.0464	0.0584	0.0077	0.0504	0.0012	0.0245
	(6.08)	(9.430)	(5.23)	(5.937)	(2.93)	(4.841)
1 1 2 1	0.0543	0.0634	0.0072	0.0584	0.0009	0.0146
	(6.18)	(9.323)	(5.37)	(6.431)	(2.98)	(5.23)
2 2 1 1	0.0453	0.0564	0.0078	0.0504	0.0014	0.0246
	(6.00)	(9.323)	(5.13)	(5.833)	(1.95)	(4.831)
2 2 1 2	0.0454	0.0574	0.0078	0.0504	0.0014	0.0255
	(6.00)	(9.329)	(5.13)	(5.841)	(1.95)	(4.947)
2 2 2 2	0.0543	0.0624	0.0073	0.0574	0.0010	0.0125
	(6.08)	(9.467)	(5.27)	(6.331)	(2.9)	(5.27)
Non-informative case	0.0389	0.0555	0.0081	0.0544	0.0015	0.0256
	(5.26)	(9.382)	(6.18)	(5.490)	(2.96)	(5.33)

*where (a)= a × 10^{-4} is the expected loss.

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APPENDIX

COMPUTER PROGRAMS

Program 2.1

```
C      This program calculates the estimate and the expected loss of
C      alpha when both the parameters b and sigma are known  and k=1

       real x(100),y(100),sum,Q0,t0,su1,su2,su3,a1,va,b,sig
       integer n,m,r
       character *10 dataf
       print*,'Programe to caculate the Estimate of Parmeters of the'
       print*,'Pareto population when b and sigma are known'
       print*,'Data file Name'
       print*,'Data file should contain No and value of sigma and b'
       print*,'then the values of observations in order'
       read(*,6)dataf
6      format(A10)
       open(unit=1,file=dataf,status='unknown')
       read(1,*)n,sig,b
       read(1,*)(x(I),i=1,n)
       print*,'Value of the prior parameter r'
       read(*,*)r
       print*,'Value of the prior parameter tdash'
       read(*,*)t0
       sum=0
       do 10 i=1,n
10     sum=sum+log(x(i)/sig)
       Q0=sum+t0
       m=n+r
       do 20 i=1,n
20     y(i)=log(x(I)/sig)
       su1=0
       su2=0
       su3=0
       do 30 i=1,n
       su1=su1+((Q0+((b-1)*y(i)))**(-m))
       su2=su2+((Q0+((b-1)*y(i)))**(-(m+1)))
       su3=su3+((Q0+((b-1)*y(i)))**(-(m+2)))
30     continue
       a1=m*su2/su1
       va=(m*(m+1)*su3/su1)-(a1*a1)
       write(*,106)a1,va
106    format(5x,'alpha1=',f14.6,5x,'V(a1/x) =',f14.6)
       stop
       end
```

Program 2.2

```
C      Thisprogram calculates the estimate and the expected loss of the
C      alpha and sigma when the parameters b is known and k=1'

       real x(100),y(100),sum,Q0,t0,su1,su2,su3,a1,va,b,sig
       real sig0,sg1,sg2,sg3,lam,u,u0,s,Q(100),vsig,Q1(100)
       integer n,m,r
       character *10 dataf
       print*,'Programe to caculate the Estimate of Parmeters of the'
       print*,'Pareto population when b and sigma are known'
       print*,'Data file Name'
       print*,'Data file should contain No and value of b'
       print*,'then the values of observations in order'
       read(*,6)dataf
6      format(A10)
       open(unit=1,file=dataf,status='unknown')
       read(1,*)n,b
       read(1,*)(x(I),i=1,n)
       print*,'Value of the prior parameter r'
       read(*,*)r
       print*,'Value of the prior parameter zdash'
       read(*,*)t0
       print*,'Value of the prior parameter u0'
       read(*,*)u0
       print*,'Value of the prior parameter sigma0'
       read(*,*)sig0

       sum=0
       do 10 i=1,n
10     sum=sum+log(x(i))
       s=sum+t0
       m=n+r
       lam=amin1(sig0,x(1))
       u=u0+n+b-1
       Q0=s-(u*lam)
       do 20 i=1,n
20     y(i)=log(x(i))
       su1=0
       su2=0
       su3=0
       do 30 i=1,n
       su1=su1+((Q0+((b-1)*y(i)))**(-m))
       su2=su2+((Q0+((b-1)*y(i)))**(-(m+1)))
       su3=su3+((Q0+((b-1)*y(i)))**(-(m+2)))
30     continue
       a1=m*su2/su1
       va=(m*(m+1)*su3/su1)-(a1*a1)
       Q(I)=s+((b-1)*y(i)/u)
```

```
      do 40 i=1,n
      Q1(i)=Q(I)-log(lam)
      sg1=sg1+We(Q1(i),1,m+1)
      sg2=sg2+(exp(Q(I))*We(Q1(i),1,m+1))
      sg3=sg3+(exp(2*Q(I))*We(Q1(i),1,m+1))
40    continue
      sig1=su2/su1
      vsig=(sg3/sg1)-(sig1*sig1)
      write(*,106)a1,va
106   format(5x,'alpha1=',f14.6,5x,'V(a1/x) =',f14.6)
      write(*,108)sig1,vsig
108   format(5x,'sigma1=',f14.6,5x,'V(s1/x) =',f14.6)
      stop
      end
      function we(c,b,n)
      external ene
      real x,f,c,b
      integer n
      x=c*b
      call ene(x,n,f)
      we=f*(c**(1-n))
      return
      end
```

Program 3.1

```
C     Thisprogram calculates the estimate and the expected loss of the
C     parameters alpha and b for Pareto population  when b>1 and k=1

      real x(100), u,sum,t,r1,vr1,var1,d,sig,r2,r3,vr2,var2
      real y(100),z,t0,vr3,var3,Q(100),Q0,sum1,Q1(100)
      integer n,m,r
      character *10 dataf
      print*,'Programe to caculate the Estimate alpha and b'
      print*,'Data file Name'
      print*,'Data file should contain No and value of sigma and then'
      print*,'the values of observations in order'
      read(*,6)dataf
6     format(A10)
      open(unit=1,file=dataf,status='unknown')
      read(1,*)n,sig
      read(1,*)(x(I),i=1,n)
      print*,'Value of the prior parameter r'
      read(*,*)r
      print*,'Value of the prior parameter tdash'
      read(*,*)t0
      sum=0
```

```
      do 10 i=1,n
10    sum=sum+log(x(i)/sig)
      Q0=sum+t0
      m=n+r
      d=0
      do 20 i=1,n
      y(i)=log(x(I)/sig)
      Q(I)=Q0-y(I)
20    continue

      r1=cr(m+1,n,Q,y,Q0)/cr(m,n,Q,y,Q0)
      vr=cr(m+2,n,Q,y,Q0)/cr(m,n,Q,y,Q0)
      var=vr-(r1*r1)

      r2=crto(m,d+1,n,Q,y,Q0)/crto(m,d,n,Q,y,Q0)
      vr2=crto(m,d+2,n,Q,y,Q0)/crto(m,d,n,Q,y,Q0)
      var2=vr2-(r2*r2)

      write(*,102)r1,var
102   format(5x,'alpha =',f14.6,5x,'V(a1/x) =',f14.6)
      write(*,104)r2,var2
104   format(5x,'b1=',f14.6,5x,'V(b1/x) =',f14.6)
      stop
      end

      function cr(m,n,Q,y,Q0)
      real Q0,sig1,y(100),Q(100),z
      integer n,m
      sig1=0
      do 20 j=1,2
      k=j-1
      gc=fact(m+k-3)*(Q0**(m-2+k))/fact(k)
      do 10 i=1,n
10    sig1=sig1+(gc*(y(i)**(k-2)))
20    continue
      cr=sig1
      return
      end

      function crto(m,d,n,Q,y,Q0)
      real Q0,sig2,y(100),Q(100),Q1(100)
      integer n,m
      sig2=0
      do 20 i=1,n
      Q1(i)=Q(i)/y(i)
      sig2 = sig2+((y(i)**(-m))*hyper(m,m-d-2,m-d-1,Q1(i))/(m-d-2))
20    continue
      crto=sig2
      return
      end
```

```
      function fact(n)
      fact=1
      if(n.eq.0)then
      fact=1
      else
      do 10 i=1,n
10    fact=fact*i
      endif
      return
      end

      function hyper(n,m,l,z)
      external gamma
      real z,sum
      integer n,m,l,k
      do 10 i=1,100
10    sum=sum+((gamma(n+i)*gamma(m+i)*(z**i))/(fact(i)*gamma(l+i)))
      d=gamma(l)/(gamma(n)*gamma(m))
      hyper=d*sum
      return
      end
```

Program 3.2

```
C  Thisprogram calculates the estimate and the expected loss
C  of aplha when all the three parameters are unknown and k=1

     real r1(100),r2(100),p1
      real x1(100),y(100),sum,Q0,t0,su1,su2,su3,a1,va,b,sig
      real sig0,sg1,sg2,sg3,lam,u,u0,s,Q(100),vsig,Q1(100),vs
      integer n,m,r
      character *10 dataf
      print*,'Programe to caculate the Estimate of alpha for Pareto'
      print*,'population when all the three parametrs are unknown'
      print*,'Data file Name'
      print*,'Data file should contain No of observations'
      print*,'then the values of observations in order'
      read(*,6)dataf
6     format(A10)
      open(unit=1,file=dataf,status='unknown')
      read(1,*)n
      read(1,*)(x1(I),i=1,n)
      print*,'Value of the prior parameter r'
      read(*,*)r
      print*,'Value of the prior parameter zdash'
      read(*,*)t0
      print*,'Value of the prior parameter u0'
      read(*,*)u0
```

```
      print*,'Value of the prior parameter sigma0'
      read(*,*)sig0

      sum=0
      do 10 i=1,n
10    sum=sum+log(x1(i))
      s=sum+t0
      m=n+r
      lam=amin1(sig0,x1(1))
      u=u0+n-1
      do 20 i=1,n
20    y(i)=log(x1(i))
      do 30 i=1,n
      Q(i)=y(i)-alog(lam)
      R1(i)=s-y(i)
      R2(i)=s-y(I)-(u*log(lam))
30    continue


      b1=ctwo(m+1,R1,R2,u,Q,y,lam,n)/ctwo(m,R1,R2,u,Q,y,lam,n)
      va=ctwo(m+2,R1,R2,u,Q,y,lam,n)/ctwo(m,R1,R2,u,Q,y,lam,n)
      vb=va-(b1*b1)
      write(*,107)b1,vb
107   format(5x,'a2    =',f14.6,5x,'V(a2/x) =',f14.6)
      stop
      end

      function ctwo(m,R1,R2,u,Q,y,lam,n)
      real r1(100),R2(100),y(100),q(100),lam,u
      real p1,q1,su,d
      integer interv, nout,m,n
       integer irule
      real a,b,errabs,errest,errrel,f,g,h,result
      external f,g,h, twodq, umach

      call umach(2,nout)

      a=0.0
      interv=1
      b=interv

      errabs=0.0
      errrel=0.01

      irule=6
      do 10 i=1,n
      su=su+result
      q1=q(i)
      y1=y(I)
      p1= r2(i)
```

```
      call twodq(f,a,b,g,h, errabs, errrel, irule, result, errest)

10    continue
      end
      real function f(x,y)
      real x,y
      real exp
      intrinsic exp
      f=(x**(m-1))*y*(exp(-x*p1))*(exp(-y*x*q1))/(v+y)
      return
      end

      real function g(x)
      real x
      g=1.0
      return
      end
      real function h(x)
      real x
      h=100000
      return
      end
      ctwo=su
      return
      end
```

program 3.3

```
C  This program calculates the estimate and the expected loss of
C  b when all the three parameters are unknown and k=1

      real r1(100),r2(100),p1
      real x1(100),y(100),sum,Q0,t0,su1,su2,su3,a1,va,b,sig
      real sig0,sg1,sg2,sg3,lam,u,u0,s,Q(100),vsig,Q1(100),vs
      integer n,m,r
      character *10 dataf
      print*,'Programe to caculate the Estimate of b for the Pareto'
      print*,'population when all the three parametrs are unknown'
      print*,'Data file Name'
      print*,'Data file should contain No of observations'
      print*,'then the values of observations in order'
      read(*,6)dataf
6     format(A10)
      open(unit=1,file=dataf,status='unknown')
      read(1,*)n
      read(1,*)(x1(I),i=1,n)
      print*,'Value of the prior parameter r'
```

```
      read(*,*)r
      print*,'Value of the prior parameter zdash'
      read(*,*)t0
      print*,'Value of the prior parameter u0'
      read(*,*)u0
      print*,'Value of the prior parameter sigma0'
      read(*,*)sig0

      sum=0
      do 10 i=1,n
10    sum=sum+log(x1(i))
      s=sum+t0
      m=n+r
      lam=amin1(sig0,x1(1))
      u=u0+n-1
      do 20 i=1,n
20    y(i)=log(x1(i))
      do 30 i=1,n
      Q(i)=y(i)-alog(lam)
      R1(i)=s-y(i)
      R2(i)=s-y(I)-(u*log(lam))
30    continue


      b1=ctwo(m,2,R1,R2,u,Q,y,lam,n)/ctwo(m,1,R1,R2,u,Q,y,lam,n)
      va=ctwo(m,3,R1,R2,u,Q,y,lam,n)/ctwo(m,1,R1,R2,u,Q,y,lam,n)
      vb=va-(b1*b1)
      write(*,107)b1,vb
107   format(5x,'b2    =',f14.6,5x,'V(b2/x) =',f14.6)
      stop
      end
      function ctwo(m,d,R1,R2,u,Q,y,lam,n)
      real r1(100),R2(100),y(100),q(100),lam,u
      real p1,q1,su,d
      integer interv, nout,m,n
      real abs, alog,atan, bound, errabs, errest, error
      real errrel, exact,f,pi, result, const
      intrinsic abs,alog
      external f,qdagi, umach, const

      call umach (2,nout)
      bound=1
      interv=1

      errabs=0.0
      errrel=0.001

      do 10 i=1,n
      su=su+result
      q1=q(i)
      y1=y(I)
```

```
      p1= r2(i)

      call Qdagi(f,bound, interv, errabs, errrel, result, errest)

      pi=const('pi')
      exact=-pi*alog(10.)/20.
      error=abs(result-exact)
10    continue
      end
      real function f(x)
      real x,q1,v,p1,d
      integer n,m
      intrinsic alog
      f(x) = (x**d)*((p1+(x*q1))**(-m))/(v+x)
      return
      end
      ctwo=su
      return
      end
```

program 3.4

```
C   This program calculates the estimate and the expected loss of
C   the sigma when all the three parameters are unknown and k=1

      real r1(100),r2(100),p1
      real x1(100),y(100),sum,Q0,t0,su1,su2,su3,a1,va,b,sig
      real sig0,sg1,sg2,sg3,lam,u,u0,s,Q(100),vsig,Q1(100),vs
      integer n,m,r
      character *10 dataf
      print*,'Programe to caculate the Estimate of sigma for the'
      print*,'Pareto population when all the three parameters are'
      print*,'unknown'
      print*,'Data file Name'
      print*,'Data file should contain No of observations'
      print*,'then the values of observations in order'
      read(*,6)dataf
6     format(A10)
      open(unit=1,file=dataf,status='unknown')
      read(1,*)n
      read(1,*)(x1(I),i=1,n)
      print*,'Value of the prior parameter r'
      read(*,*)r
      print*,'Value of the prior parameter zdash'
      read(*,*)t0
      print*,'Value of the prior parameter u0'
      read(*,*)u0
      print*,'Value of the prior parameter sigma0'
      read(*,*)sig0
```

```
      sum=0
      do 10 i=1,n
10    sum=sum+log(x1(i))
      s=sum+t0
      m=n+r
      lam=amin1(sig0,x1(1))
      u=u0+n-1
      do 20 i=1,n
20    y(i)=log(x1(i))
      do 30 i=1,n
      Q(i)=y(i)-alog(lam)
      R1(i)=s-y(i)
      R2(i)=s-y(I)-(u*log(lam))
30    continue

      sig1=cthe(m,1,R1,R2,u,Q,y,lam,n)/cthe(m,0,R1,R2,u,Q,y,lam,n)
      vs=(cthe(m,2,R1,R2,u,Q,y,lam,n)/cthe(m,0,R1,R2,u,Q,y,lam,n))
      vsig=vs-(a1*a1)
      write(*,108)sig1,vsig
108   format(5x,'sigma2=',f14.6,5x,'V(s2/x) =',f14.6)
      stop
      end

      function hyper(n,m,l,z)
      real z,su
      integer n,m,l
      external gamma
      su=0           ,fac
      do 10 i=1,5
10    su=su+((gamma(n+i)*gamma(m+i)*(z**i))/(gamma(l+i)*fac(i)))
      c=gamma(l)/(gamma(m)*gamma(n))
      hyper=c*su
      return
      end

      function cthe(m,d,R1,R2,u,Q,y,lam,n)
      integer irule,nout,m,n,d
      real a,abs,b,errabs,errest,error,errrel, exact,exp
      real f,result,R1(100),R2(100),y(100),q(100),lam,u
      real p1
      intrinsic abs, exp
      external f,qdag, umach
      su=0
      call umach(2,nout)

      a=0.0
      b=lam

      errabs=0.0
      errrel=0.001
```

```
      irule=2
      do 10 i=1,n
      su=su+result
      q1=q(i)
      y1=y(I)
      p1= r1(i)
      call qdag(f,a,b,errabs,errrel,irule, result, errest)

      exact=1.0+exp(2.0)
      error=abs(result-exact)
10    continue
      end

      real function f(x)
      real d,x,q,v,p,p1
      integer m
      intrinsic exp
      p=(p1-(v*lgo(x)))/(y1-log(x))
      f=(x**d)*((q-v*alog(x))**(-m))**hyper(m+1,m-1,m,-p)
      return
      end

      cthe=su
      return
      end
```

Program 4.1

```
C  This program calculates the estimate and the expected loss of
C  the Survival function for the Pareto population  when b>1 and
C  the under the three outlier generating mechanisms

      real x(100), u,sum,t,r1,vr1,var1,d,sig,r2,r3,vr2,var2
      real y(100),z,t0,vr3,var3,Q(100),Q0,w,alp,sum1,w0,w10
      integer n,m,r,m0,p
      character *10 dataf
      print*,'Programe to caculate the Estimate of R(t)'
      print*,'Data file Name'
      print*,'Data file should contain No and value of sigma and then'
      print*,'the values of observations in order'
      read(*,6)dataf
6     format(A10)
      open(unit=1,file=dataf,status='unknown')

      read(1,*)n,sig
      read(1,*)(x(I),i=1,n)
      print*,'Value of the prior parameter r'
      read(*,*)r
```

```
      print*,'Value of the prior parameter tdash'
      read(*,*)t0
      print*,'Value of m-dash obtained from the table'
      read(*,*)m0
      print*,'Value of the t'
      read(*,*)t

      sum=0
      do 10 i=1,n
10    sum=sum+log(x(i)/sig)
      Q0=sum+t0
      m=n+r
      d=0
      s=log(x(n)/sig)
      do 20 i=1,n
      y(i)=log(x(I)/sig)
      Q(I)=Q0-y(I)
20    continue
      z=log(t/sig)

      sum1=0
      do 30 i=1,m0-1
30    sum1=sum1+log(x(i)/sig)
      w0=sum1+((n-m0)*log(x(m0)/sig))
      w10=(n-m0)/w0
      w=t0+(m0/w10)
      p=m0+r

      r1=cr(m,d+1,n,Q,y,Q0,z)/cr(m,d,n,Q,y,Q0,z)
      vr=cr(m,d+2,n,Q,y,Q0,z)/cr(m,d,n,Q,y,Q0,z)
      var=vr-(r1*r1)
      r2=crto(m,d+1,n,s,y,Q0,z)/crto(m,d,n,s,y,Q0,z)
      vr2=crto(m,d+2,n,s,y,Q0,z)/crto(m,d,n,s,y,Q0,z)
      var2=vr2-(r2*r2)
      r3=((1+(z/w))**(-p))
      var3=((1+(2*z/w))**(-p))-((1+(z/w))**(-2*p))

      write(*,102)r1,var
102   format(5x,'R1=',f14.6,5x,'V(R1/x) =',f14.6)
      write(*,104)r2,var2
104   format(5x,'R2=',f14.6,5x,'V(R2/x) =',f14.6)
      write(*,106)r3,var3
106   format(5x,'R3=',f14.6,5x,'V(R3/x) =',f14.6)

      stop
      end

      function cr(m,d,n,Q,y,Q0,z)
      real Q0,sig1,y(100),Q(100),z
      integer n,m
```

```
      sig1=0
      do 20 j=1,2
      k=j-1
      gc=fact(m+k-3)*(z**(m-2))/fact(k)
      do 10 i=1,n
10    sig1=sig1+(gc*(y(i)**k)*((Q0+(d*z))**(2-m-j)))
20    continue
      cr=sig1
      return
      end

      function crto(m,d,n,s,y,Q0,z)
      real Q0,sig2,y(100),z,s
      integer n,m
      sig2=0
      do 20 j=1,2
      k=j-1
      gc=fact(m+k-4)*(z**(m-2))/fact(k)
      sig2=sig2+(gc*(s**k)*((Q0+(d*z))**(2-m-j)))
20    continue
      crto=sig2
      return
      end

      function fact(n)
      fact=1
      if(n.eq.0)then
      fact=1
      else
      do 10 i=1,n
10    fact=fact*i
      endif
      return
      end
```

Program 5.1

```
C   This program calculates the lower limit L for the future
C   order ststiatics from a Pareto sample when b is known

      Real x(100),l,a(100),b,alp,alpha,sum,d,a1,y(100)
      Integer m,n,p
      character *10 dataf
      print*,'Data file Name'
      print*,'Data file should contain No.the value of b
      print*,'the value of sigma and the value of data'
      read(*,6)dataf
6     format(a10)
```

```
      open(unit=1,file=dataf,status='unknown')
      Read(1,*)n,b,sig
      Read(1,*)(y(i),i=1,n)

      Print *,'Value of prior parameter p'
      Read(*,*)p
      Print *,'Value of prior parameter t'
      Read(*,*)u

      do 11 i=1,n
11    x(i)= log(y(i)/sig)

      sum=0
      do 10 i=1,n
10    sum=sum+x(i)

      m=n+p
      do 20 i=1,n
20    a(i)=u+x(i)*(b-1)+sum
      a1=0
      do 25 i=1,n
25    a1=a1+(a(i)**(-m))

      alpha=0.95
      alp=(1-alp)/2
      do 30 r=2,n
      l=x(r-1)
      Call Newton(l,n,b,alp,c,r,u)
      write(*,101)
101   format(/,5x,'r                L',/)
      write(*,102)r,l
102   format(5x,I2,8x,F10.4)
30    continue
      stop
      end

      Function getd(l,c,a1,r,b,n,m,a)
      Real l,c,a1,b,a(100)
      integer pwr,n,m,r
      pwr=-(m+1)
      sum=0
      do 20 j=1,r-1
      k=j-1
      t1=(fact(r-2)*((-1)**k))/(fact(k)*fact(r-2-k))
      do 10 i=1,n
      t2=(r-1)*(a(i)+((n-r+k+1)*log(l/sig))**pwr)
      t3=(n+b-r)*(a(i)+((n-r+k+b)*log(l/sig))**pwr)
      t4=(n+b-1)*(a(i)+((n-r+b+k+1)*log(l/sig))**pwr)
      sum=t1*(t2+t3-t4)
10    continue
20    continue
```

```
      getd=sum*c
      return
      end

      Function getg(l,c,a1,r,b,alp,n,m)
      real l,a1,a(100),alp,b
      integer r,n,m
      pr=-m
      sum=0
      do 20 j=1,r-1
      k=j-1
      t0=(fact(r-2)*((-1)**k))/(fact(k)*fact(r-2-k))
      do 10 i=1,n
      t11=(r-1)/(n-r+k+1)
      t1=(a(i)**pr)-(a(i)+((n-r+k+1)*log(l/sig))**pr)
      t21=(n+b-1)/(n-r+b+k+1)
      t2=(a(i)**pr)-(a(i)+((n-r+b+k+1)*log(l/sig))**pr)
      t31=(n+b-r)/(n-r+k+b)
      t3=(a(i)**pr)-(a(i)+((n-r+k+b)*log(l/sig))**pr)
      sum=sum+t0*(t11*t1-t21*t2+t31*t3)
10    continue
20    continue
      getg=(c*sum)-alp
      return
      end

      Subroutine Newton(l,n,b,alp,c,r,u)
      real l
      integer n,r
      eps=0.001
      c=getc(r,a1,n)
      do 10 i=1,50
      g=getg(l,c,a1,r,b,alp,n,m)
      gdash=getd(l,c,a1,r,b,n,m,a)
      l=l-g/gdash
      if(abs(g).lt.eps)return
10    continue
      return
      end

      Function getc(r,a1,n)
      real a1
      integer r,n
      sm=0
      sm=sm+(fact(n-1)/(fact(r-1)*fact(n-r)))
      getc=sm/a1
      return
      end
```

```
      Function fact(n)
      fact=1
      if(n.eq.0)then
      fact=1
      else
      do 10 i=1,n
10    fact=fact*i
      endif
      return
      end
```

Program 5.2

```
C   This program calculates the lower limit L for the future
C   order ststiatics from a Pareto sample when b is unknown

      Real x(100),l,a(100),b,alp,alpha,sum,d,a1,y(100)
      Integer m,n,p
      character *10 dataf
      print*,'Data file Name'
      print*,'Data file should contain number of observation'
      print*,'the value of sigma and the value of data'
      read(*,6)dataf
6     format(a10)
      open(unit=1,file=dataf,status='unknown')
      Read(1,*)n,sig
      Read(1,*)(x(i),i=1,n)

      Print *,'Value of prior parameter p'
      Read(*,*)p
      Print *,'Value of prior parameter u'
      Read(*,*)u

      do 11 i=1,n
11    x(i)=log(y(i)/sig)

      sum=0
      do 10 i=1,n
10    sum=sum+x(i)

      m=n+p
      do 20 i=1,n
20    a(i)=u+x(i)*(b-1)+sum
      a1=0
      do 25 i=1,n
25    a1=a1+(a(i)**(-m))
```

```
      alpha=0.95
      alp=(1+alp)/2
      do 30 r=2,n
      l=x(r-1)
      Call Newton(l,n,b,alp,c,r,u)
      write(*,101)
101   format(/,5x,'r                L',/)
      write(*,102)r,l
102   format(5x,I2,8x,F10.4)
30    continue
      stop
      end

      Function getd(l,c,a1,r,b,n,m,a)
      Real l,c,a1,b,a(100)
      integer pwr,n,m,r,j1,i1,l1
      external gamma
      pwr=-(m+1)
      sum=0
      do 20 j=1,r-1
      k=j-1
      do 30 l1= 0,100000
      do 10 i=1,n
      t0=(-1)**(l+j)
      t11=((r-1)/(l1*(m+l1-1)*beta(m+1,l1)*(n-r+k-+1))
      t10=(x(i)**(-(m+l1-1)))*(a(i)+((n-r+k+1)*log(l/sig))**(l1-2))
      t1=t11*t10
      t20=0
      t30=0
      do 40 i1=1,10000
      t21=(((n-1)/(m+l1-1))+(1/(m+l1-2)))*((n-r+k+1)**
     1(m+l))/(l1*beta(m+1,l))
      d=gamma(m+1)/(gamma(m+l1+1)*gamma(m))
      t20=t20+((gamma(m+1+l1+i1)*gamma(m+i1)/(fact(i1)*gamma(m+1+i1)))*
     1((((n-r+k+1)*x(i))-a(i))**i1)*((n-r+k+1)**2)/(a(i)+(n-r+k+1)*
     1log(l/sig))**2
      t2=t21*t20*d

      t31=(((n-r)/(m+l1-1))+(1/(m+l1-2)))*((n-r+k)**
     1(m+l))/(l1*beta(m+1,l))
      t30=t30+((gamma(m+1+l1+i1)*gamma(m+i1)/(fact(i1)*gamma(m+1+i1)))*
     1((((n-r+k)*x(i))-a(i))**i1)*((n-r+k)**2)/(a(i)+(n-r+k+1)*
     1log(l/sig))**2
      t3=t31*t30*d

      sum=t0*(t1+t3-t2)
10    continue
30    continue
20    continue
```

```
      getd=sum*c
      return
      end

      Function getg(l,c,a1,r,b,alp,n,m)
      real l,a1,a(100),alp,b
      integer r,n,m
      sum=0
      do 20 j=1,r-1
      k=j-1
      do 30 l1=0,10000
      t0=(-1)**(k+l1)
      do 10 i=1,n
      t11=((r-1)/(l1*(l1-1)*(m+l1-1)*beta(m+1,l1)*(n-r+k-+1))
      t1=(x(i)**(-(m+l1-1)))*(a(i)+((n-r+k+1)*log(l/sig))**(l1-1))
      t21=(((n-1)/(m+l1-1))+(1/(m+l1-2)))*((n-r+k+1)**
     1(m+l1))/(l1*beta(m+1,l1))
      t2=hyper(m+l1+1,m,m+1,((((n-r+k+1)*x(i))-a(i))/(a(i)+((n-r+k+1)*
     1log(l/sig))))
      t21=(((n-r)/(m+l1-1))+(1/(m+l1-2)))*((n-r+k)**
     1(m+l1))/(l*beta(m+1,l1))
      t2=hyper(m+l1+1,m,m+1,((((n-r+k)*x(i))-a(i))/(a(i)+((n-r+k)*
     1log(l/sig))))
      sum=sum+t0*(t11*t1-t21*t2+t31*t3)
10    continue
20    continue
30    continue
      getg=(c*sum)-alp
      return
      end

      Subroutine Newton(l,n,b,alp,c,r,u)
      real l
      integer n,r
      eps=0.001
      c=getc(r,a1,n)
      do 10 i=1,50
      g=getg(l,c,a1,r,b,alp,n,m)
      gdash=getd(l,c,a1,r,b,n,m,a)
      l=l-g/gdash
      if(abs(g).lt.eps)return
10    continue
      return
      end

      Function getc(r,a1,n)
      real a1
      integer r,n
      sm=0
      sm=sm+(fact(n-1)/(fact(r-1)*fact(n-r)))
```

```
      getc=sm/a1
      return
      end

      Function fact(n)
      fact=1
      if(n.eq.0)then
      fact=1
      else
      do 10 i=1,n
10    fact=fact*i
      endif
      return
      end

      function hyper(l,m,n,z)
      external gamma
      real z,sum
      integer l,m,n
      sum=0
      do 10 i=1,10000
10    sum=sum+((gamma(l+i)*gamma(m+i)**(z**i))/(fact(i)*gamma(n+i)))
      d=gamma(n)/(gamma(l)*gamma(m))
      hyper=sums*d
      return
      end

      function beta(n,m)
      external gamma
      integer n,m
      beta=gamma(n)*gamma(m)/gamma(m+n)
      return
      end
```

Note: To obtain the upper limits we can use the same programs with the only difference being that we must change the intial values and the alpha value.

Program 6.1

```
C  program to calculate estimate of the parameter in the
C  exponential case when b>1 and k=1

      Real x(100),u,sum,Q,T(100),theata1,b1,varth,varb
      Real th,th1,th2
      integr m,n,p,k
```

```
      Character *10 dataf
      print *,'Programe to calculate the Estimates of the Parameters'
      print *,'for the Exponential distribution'
      print *,'Data file Name'
      print *,'Data   file should contain No. and Values of data'
      read(*,6)dataf
6     format(a10)
      open(unit=1,file=dataf,status='unknown')
      read(1,*)n
      read(1,*)(x(i),i=1,n)

      print*,'Value of the prior parameter p'
      Read(*,*)p
      print*,'Value of the prior parameter u'
      Read(*,*)u

      sum=0
      do 10 i=1,n
10    sum=sum+x(i)
      Q=sum+u
      m=n+p
      do 20 i=1,n
20    T(i)=Q-x(i)
      k=1

      theta1= cone(m+1,n,Q,x,sum,t)/cone(m,n,Q,x,sum,t)
      th= cone(m+2,n,Q,x,sum,t)/cone(m,n,Q,x,sum,t)
      varth = th-(theta1**2)

      b1= ctwo(m,k+1,n,Q,x,sum,t)/cone(m,k,n,Q,x,sum,t)
      th1= ctwo(m,k+2,n,Q,x,sum,t)/cone(m,k,n,Q,x,sum,t)
      varb = th1-(b1**2)

      write(*,101)Theta1,varth
101   format(5x,'Theta = ',f14.6,5x,'Expected loss = ',f14.6)

      write(*,101)b1,varb
101   format(5x,'B = ',f14.6,5x,'Expected loss = ',f14.6)

      stop
      end

      Function fact(n)
      if(n.eq.0)then
      fact=1
      else
      do 10 i=1,n
10    fact=fact*i
      endif
      return
      end
```

```
      Function hyper(n,m,k,z)
      real z,sum
      external gamma
      integer n,m,k
      sum=0
      do 10 i=1,1000000
10    sum=sum+((gamma(n+i)*gamma(m+i)*(z**i))/(fact(i)*gamma(k+i)))
      gc=gamma(k)/(gamma(n)*gamma(m))
      hyper=sum*gc
      return
      end

      Function cone(m,n,Q,x,sum,T)
      real Q,T(100),x(100),sig,sum
      intger n,m,m1
      sig=0
      do 20 l=1,2
      j=l-1
      m1=-(m-2+j)
      co=fact(m-3+j)*(Q**m1)/fact(j)
      do 10 i=1,n
10    sig=sig+(co*(x(i)**(j-2)))
20    continue
      cone=sig
      return
      end

      Function ctwo(m,k,n,Q,x,sum,T)
      Real Q,T(100),x(100),sig,sum,a(100)
      integer n,m
      sig = 0
      do 10 i=1,n
      a(i)=-(T(i)/x(i))
      sig=sig+(hyper(m,m-k-1,m-k,a
      sig=sig+(hyper(m,m-k-1,m-k,a(i))/((x(i)**m)*(m-k-1)))
10    continue
      ctwo=sig
      return
      end
```

Program 6.2

```
C  This program calculates the estimate and the expected loss of
C  the Survival function for the exponential sample when b>1 & k=1

      real x(100), u,sum,Q,t,r1,vr,var,d
      integer n,m,p
      character *10 dataf
```

```
      print*,'Programe to caculate the Estimate of R(t)'
      print*,'Data file Name'
      print*,'Data file should contain No and value of data'
      read(*,6)dataf
6     format(A10)
      open(unit=1,file=dataf,status='unknown')

      read(1,*)n
      read(1,*)(x(I),i=1,n)
      print*,'Value of the prior parameter p'
      read(*,*)p
      print*,'Value of the prior parameter u'
      read(*,*)u
      print*,'Value of the t'
      read(*,*)t

      sum=0
      do 10 i=1,n
10    sum=sum+x(i)
      Q=sum+u
      m=n+p
      d=0

      r1=cr(m,d+1,n,Q,x,sum,t)/cr(m,d,n,Q,x,sum,t)
      vr=cr(m,d+2,n,Q,x,sum,t)/cr(m,d,n,Q,x,sum,t)
      var=vr-(r1*r1)
      write(*,102)r1,var

102   format(5x,'R=',f14.6,5x,'V(R/x) =',f14.6)
      stop
      end
      function cr(m,d,n,Q,x,sum,t)
      real Q,x(100),sig
      integer n,m
      sig=0
      do 20 j=1,2
      k=j-1
      gc=fact(m-k-3)*(t**m)/fact(k)
      do 10 i=1,n
10    sig=sig+(gc*(x(i)**(k-2))*((Q+(d*t))**(2-m-j)))
20    continue
      cr=sig
      return
      end

      function fact(n)
      fact=1
      if(n.eq.0)then
      fact=1
      else
```

```
      do 10 i=1,n
10    fact=fact*i
      endif
      return
      end
```

Program 6.3

```
C   This program calculates the predictive interval for the rth
C   order statistics  for the exponential sample and therefore to
C   determine the number of outliers present inn the sample

      real x(100), u,sum,Q,t0,r1,vr,d
      integer n,m,p,n1,m0,r
      character *10 dataf
      print*,'Programe to caculate the interval of x(r) for a given r'
      print*,'Data file Name'
      print*,'Data file should contain No and orderd value of data'
      read(*,6)dataf
6     format(A10)
      open(unit=1,file=dataf,status='unknown')

      read(1,*)n
      read(1,*)(x(I),i=1,n)
      print*,'Value of the prior parameter p'
      read(*,*)p
      print*,'Value of the prior parameter u'
      read(*,*)u
      print*,'Value of the m obtained from the table'
      read(*,*)m0
      print*,'Value of the r'
      read(*,*)r

      sum=0
      do 10 i=1,m0
10    sum=sum+x(i)
      Q=sum+u+((n-m0)*x(m0))
      m=m0+p
      d=0.95
      n1=1/m
      r1=((1-d)**n1)
      vr=((Q/r1)-1)/(n-r+1)
      t0=vr+x(r-1)
      write(*,102)x(r-1),t0

102   format(5x,'( ',f14.6,5x,',',f14.6,2x,')')
      stop
      end
```

Program 7.1

```
C    This program calculates the lower limit L for the future
C    order ststiatics from an exponential sample when b is known

       Real x(100),l,a(100),b,alp,alpha,sum,d,a1
       Integer m,n,p
       character *10 dataf
       print*,'Data file Name'
       print*,'Data file should contain number of observation
       pritn*,'the value of b and the value of data

       read(*,6)dataf
6      format(a10)
       open(unit=1,file=dataf,status='unknown')
       Read(1,*)n,b
       Read(1,*)(x(i),i=1,n)

       Print *,'Value of prior parameter p'
       Read(*,*)p
       Print *,'Value of prior parameter u'
       Read(*,*)u

       sum=0
       do 10 i=1,n
10     sum=sum+x(i)

       m=n+p
       do 20 i=1,n
20     a(i)=u+x(i)*(b-1)+sum
       a1=0
       do 25 i=1,n
25     a1=a1+(a(i)**(-m))

       alpha=0.95
       alp=(1-alp)/2
       do 30 r=2,n
       l=x(r-1)
       Call Newton(l,n,b,alp,c,r,u)
       write(*,101)
101    format(/,5x,'r              L',/)
       write(*,102)r,l
102    format(5x,I2,8x,F10.4)
30     continue
       stop
       end
```

```
      Function getd(l,c,a1,r,b,n,m,a)
      Real l,c,a1,b,a(100)
      integer pwr,n,m,r
      pwr=-(m+1)
      sum=0
      do 20 j=1,r-1
      k=j-1
      t1=(fact(r-2)*((-1)**k))/(fact(k)*fact(r-2-k))
      do 10 i=1,n
      t2=(r-1)*(a(i)+((n-r+k+1)*l)**pwr)
      t3=(n+b-r)*(a(i)+((n-r+k+b)*l)**pwr)
      t4=(n+b-1)*(a(i)+((n-r+b+k+1)*l)**pwr)
      sum=t1*(t2+t3-t4)
10    continue
20    continue
      getd=sum*c
      return
      end

      Function getg(l,c,a1,r,b,alp,n,m)
      real l,a1,a(100),alp,b
      integer r,n,m
      pr=-m
      sum=0
      do 20 j=1,r-1
      k=j-1
      t0=(fact(r-2)*((-1)**k))/(fact(k)*fact(r-2-k))
      do 10 i=1,n
      t11=(r-1)/(n-r+k+1)
      t1=(a(i)**pr)-(a(i)+((n-r+k+1)*l)**pr)
      t21=(n+b-1)/(n-r+b+k+1)
      t2=(a(i)**pr)-(a(i)+((n-r+b+k+1)*l)**pr)
      t31=(n+b-r)/(n-r+k+b)
      t3=(a(i)**pr)-(a(i)+((n-r+k+b)*l)**pr)
      sum=sum+t0*(t11*t1-t21*t2+t31*t3)
10    continue
20    continue
      getg=(c*sum)-alp
      return
      end

      Subroutine Newton(l,n,b,alp,c,r,u)
      real l
      integer n,r
      eps=0.001
      c=getc(r,a1,n)
      do 10 i=1,50
      g=getg(l,c,a1,r,b,alp,n,m)
      gdash=getd(l,c,a1,r,b,n,m,a)
      l=l-g/gdash
```

```
      if(abs(g).lt.eps)return
10    continue
      return
      end

      Function getc(r,a1,n)
      real a1
      integer r,n
      sm=0
      sm=sm+(fact(n-1)/(fact(r-1)*fact(n-r)))
      getc=sm/a1
      return
      end

      Function fact(n)
      fact=1
      if(n.eq.0)then
      fact=1
      else
      do 10 i=1,n
10    fact=fact*i
      endif
      return
      end
```

Program 7.2

```
C  This program calculates the lower limit L for the future
C  order ststiatics from an exponential sample when b is unknown

      Real x(100),l,a(100),b,alp,alpha,sum,d,a1
      Integer m,n,p
      character *10 dataf
      print*,'Data file Name'
      print*,'Data file should contain No. and the value of data'
      read(*,6)dataf
6     format(a10)
      open(unit=1,file=dataf,status='unknown')
      Read(1,*)n
      Read(1,*)(x(i),i=1,n)

      Print *,'Value of prior parameter p'
      Read(*,*)p
      Print *,'Value of prior parameter u'
      Read(*,*)u

      sum=0
      do 10 i=1,n
10    sum=sum+x(i)
```

```
      m=n+p
      do 20 i=1,n
20    a(i)=u+x(i)*(b-1)+sum
      a1=0
      do 25 i=1,n
25    a1=a1+(a(i)**(-m))

      alpha=0.95
      alp=(1+alp)/2
      do 30 r=2,n
      l=x(r-1)
      Call Newton(l,n,b,alp,c,r,u)
      write(*,101)
101   format(/,5x,'r                L',/)
      write(*,102)r,l
102   format(5x,I2,8x,F10.4)
30    continue
      stop
      end

      Function getd(l,c,a1,r,b,n,m,a)
      Real l,c,a1,b,a(100)
      integer pwr,n,m,r,j1,i1,l1
      external gamma
      pwr=-(m+1)
      sum=0
      do 20 j=1,r-1
      k=j-1
      do 30 l1= 0,100000
      do 10 i=1,n
      t0=(-1)**(l+j)
      t11=((r-1)/(l1*(m+l1-1)*beta(m+1,l1)*(n-r+k-+1))
      t10=(x(i)**(-(m+l1-1)))*(a(i)+((n-r+k+1)*l)**(l1-2))
      t1=t11*t10
      t20=0
      t30=0
      do 40 i1=1,10000
      t21=(((n-1)/(m+l1-1))+(1/(m+l1-2)))*((n-r+k+1)**
     1(m+l))/(l1*beta(m+1,l))
      d=gamma(m+1)/(gamma(m+l1+1)*gamma(m))
      t20=t20+((gamma(m+1+l1+i1)*gamma(m+i1)/(fact(i1)*gamma(m+1+i1)))*
     1((((n-r+k+1)*x(i))-a(i))**i1)*((n-r+k+1)**2)/(a(i)+(n-r+k+1)*l)**2
      t2=t21*t20*d
      t31=(((n-r)/(m+l1-1))+(1/(m+l1-2)))*((n-r+k)**
     1(m+l))/(l1*beta(m+1,l))
      t30=t30+((gamma(m+1+l1+i1)*gamma(m+i1)/(fact(i1)*gamma(m+1+i1)))*
     1((((n-r+k)*x(i))-a(i))**i1)*((n-r+k)**2)/(a(i)+(n-r+k+1)*l)**2
      t3=t31*t30*d
      sum=t0*(t1+t3-t2)
10    continue
```

```
30     continue
20     continue
       getd=sum*c
       return
       end

       Function getg(l,c,a1,r,b,alp,n,m)
       real l,a1,a(100),alp,b
       integer r,n,m
       sum=0
       do 20 j=1,r-1
       k=j-1
       do 30 l1=0,10000
       t0=(-1)**(k+l1)
       do 10 i=1,n
       t11=((r-1)/(l1*(l1-1)*(m+l1-1)*beta(m+1,l1)*(n-r+k-+1))
       t1=(x(i)**(-(m+l1-1)))*(a(i)+((n-r+k+1)*l)**(l1-1))
       t21=(((n-1)/(m+l1-1))+(1/(m+l1-2)))*((n-r+k+1)**
      1(m+l1))/(l1*beta(m+1,l1))
       t2=hyper(m+l1+1,m,m+1,((((n-r+k+1)*x(i))-a(i))/(a(i)+((n-r+k+1)*
      1a)))
       t21=(((n-r)/(m+l1-1))+(1/(m+l1-2)))*((n-r+k)**
      1(m+l1))/(l*beta(m+1,l1))
       t2=hyper(m+l1+1,m,m+1,((((n-r+k)*x(i))-a(i))/(a(i)+((n-r+k)*l)))
       sum=sum+t0*(t11*t1-t21*t2+t31*t3)
10     continue
20     continue
30     continue
       getg=(c*sum)-alp
       return
       end

       Subroutine Newton(l,n,b,alp,c,r,u)
       real l
       integer n,r
       eps=0.001
       c=getc(r,a1,n)
       do 10 i=1,50
       g=getg(l,c,a1,r,b,alp,n,m)
       gdash=getd(l,c,a1,r,b,n,m,a)
       l=l-g/gdash
       if(abs(g).lt.eps)return
10     continue
       return
       end

       Function getc(r,a1,n)
       real a1
       integer r,n
       sm=0
```

```
      sm=sm+(fact(n-1)/(fact(r-1)*fact(n-r)))
      getc=sm/a1
      return
      end

      Function fact(n)
      fact=1
      if(n.eq.0)then
      fact=1
      else
      do 10 i=1,n
10    fact=fact*i
      endif
      return
      end

      function hyper(l,m,n,z)
      external gamma
      real z,sum
      integer l,m,n
      sum=0
      do 10 i=1,10000
10    sum=sum+((gamma(l+i)*gamma(m+i)**(z**i))/(fact(i)*gamma(n+i)))
      d=gamma(n)/(gamma(l)*gamma(m))
      hyper=sums*d
      return
      end

      function beta(n,m)
      external gamma
      integer n,m
      beta=gamma(n)*gamma(m)/gamma(m+n)
      return
      end
```

Note: To obtain the upper limits we can use the same programs with the only difference being that we must change the intial values and the alpha value.

Program 8.1

```
C     Thisprograme calculate the estimate and the expected loss of the
C     P[X>Y]  in the case of the exponential distribution when
C     both the sample on x and y contain an outlier observation

      real x(100),y(100),sum,sum1,sum2,sum3,u,v,QO,D,A,sum4
```

```
      real w,h,r1,r2,r3,vr1,vr2,vr3,var1,var2,var3
      integer n,m,q,m0,p,s,k,r0
      character *10 dataf
      print*,'Programe to caculate the Estimate of R(t)'
      print*,'Data file Name'
      print*,'Data file should contain No and value of sigma and then'
      print*,'the values of observations in order'
      read(*,6)dataf
6     format(A10)
      open(unit=1,file=dataf,status='unknown')

      read(1,*)n
      read(1,*)(x(I),y(i),i=1,n)
      print*,'Value of the prior parameter p'
      read(*,*)p
      print*,'Value of the prior parameter u'
      read(*,*)u
      print*,'Value of the prior parameter q'
      read(*,*)q
      print*,'Value of the prior parameter v'
      read(*,*)v
      print*,'Value of r obtained from the table'
      read(*,*)m0
      print*,'Value of r-dash obtained from the table'
      read(*,*)r0

      sum=0
      do 10 i=1,n
10    sum=sum+x(i)
      G0=sum+u
      m=n+p

      sum2=0
      do 20 i=1,n
20    sum2=sum2+y(i)
      D=sum+v
      s=n+q
      A=(G0/D)-1

      sum3=0
      do 30 i=1,m0-1
30    sum3=sum3+x(i)
      h=sum3+((n-m0)*x(m0))+u
      r=m0+p

      sum4=0
      do 40 i=1,r0-1
40    sum4=sum4+y(i)
      w=sum4+((n-r0)*y(r0))+v
      k=r0+q
      z=(h/w)-1
```

```
      r1=lone(m,x,y,s,A,Q0,D,n,t+1)
      vr1=lone(m,x,y,s,A,Q0,D,n,t+2)
      var1=vr1-(r1 *r1)
      r2=ltwo(m,x,y,s,A,Q0,D,n,t+1)
      vr2=ltwo(m,x,y,s,A,Q0,D,n,t+2)
      var2=vr2-(r2 *r2)
      r3=((1+z)**(-r))*(r/(r+k))*hyper(r+k,r+1,r+k+1,-z)
      vr3=r*(r+1)*((1+z)**r)*hyper(r+k,r+2,r+k+2,-z)/((r+k)*(r+k+1))
      var3=vr3-(r3*r3)

      write(*,102)r1,var1
102   format(5x,'R1=',f14.6,5x,'V(R1/x) =',f14.6)
      write(*,104)r2,var2
104   format(5x,'R2=',f14.6,5x,'V(R2/x) =',f14.6)
      write(*,106)r3,var3
106   format(5x,'R3=',f14.6,5x,'V(R3/x) =',f14.6)
      stop
      end

      function lone(m,x,y,s,A,Q0,D,n,t)
      real x(100),y(100),A,D,Q0
      integer m,n,s
      su1=0
      su2=0
      do 10 i=1,n
      do 20 j=1,n
      su1=su1+(((x(i)*y(j))**(-2))*((D+((s-2)*y(j))))*(Q0+((m-2)*X(i))))
      su2=su2+((D*D*beta(s-2,m+t-2)*hyper(m+s-4,m+t-2,m+s-4+t,-A))
     1+(D*x(i)*(m+s-4)*beta(s-2,m+t-1)*hyper(m+s-3,m+t-1,m+s-3+t,-A))+
     1(D*y(j)*(m+s-4)*beta(s-1,m+t-2)*hyper(m+s-3,m+t-2,m+s+t-3,-A))+
     1(x(i)*y(j)*(m+s-3)*(m+s-4)*beta(s-1,m+t-1)*
     1hyper(m+s-2,m+t-1,m+s+t-3,-A)))
20    continue
10    continue
      lone=su1*su2
      return
      end

      function ltwo(m,x,y,s,A,Q0,D,n,t)
      real x(100),y(100),A,D,Q0
      integer m,n,s
      su1=0
      su2=0
      su1=su1+(((x(n)*y(n))**(-2))*((D+((s-2)*y(n))))*(Q0+((m-2)*X(n))))
      su2=su2+((D*D*beta(s-2,m+t-2)*hyper(m+s-4,m+t-2,m+s-4+t,-A))
     1+(D*x(n)*(m+s-4)*beta(s-2,m+t-1)*hyper(m+s-3,m+t-1,m+s-3+t,-A))+
     1(D*y(n)*(m+s-4)*beta(s-1,m+t-2)*hyper(m+s-3,m+t-2,m+s+t-3,-A))+
     1(x(n)*y(n)*(m+s-3)*(m+s-4)*beta(s-1,m+t-1)*
     1hyper(m+s-2,m+t-1,m+s+t-3,-A)))
```

```
ltwo=su1*su2
return
end

function fact(n)
fact=1
if(n.eq.0)then
fact=1
else
do 10 i=1,n
fact=fact*i
endif
return
end

function hyper(l,m,n,z)
external gamma
real z,sum
integer l,m,n
sum=0
do 10 i=1,10000
sum=sum+((gamma(l+i)*gamma(m+i)**(z**i))/(fact(i)*gamma(n+i)))
d=gamma(n)/(gamma(l)*gamma(m))
hyper=sums*d
return
end

function beta(n,m)
external gamma
integer n,m
beta=gamma(n)*gamma(m)/gamma(m+n)
return
end
```