FLUID MECHANICS

TOPOLOGICAL INVARIANTS IN HYDRODYNAMICS AND HYDROMAGNETICS

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Subin P. Joseph

Certificate

This is to certify that this thesis entitled **'Topological Invariants in Hydrodynamics and Hydromagnetics'** is a bonafide record of the research work carried out by Mr. Subin P. Joseph under my supervision in the Department of Mathematics, Cochin University of Science and Technology. The result embodied in the thesis have not been included in any other thesis submitted previously for the award of any degree or diploma.

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Declaration

I hereby declare that the work presented in this thesis entitled **'Topological Invariants in Hydrodynamics and Hydromagnetics'** is based on the original work done by me under the supervision of Dr. M. Jathavedan in the Department of Mathematics, Cochin University of Science and Technology, Cochin-22, Kerala; and no part thereof has been presented for the award of any other degree or diploma.

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Chapter 1

Introduction

1.1 Topological fluid mechanics

Hermann Helmholtz's 1858 paper on vortex motion made it possible to apply topological ideas to fluid mechanics [30]. Lord Kelvin was so impressed by Helmholtz's laws that he believed in the eternal existence of vortex atoms as fundamental constituents of nature. In this theory atoms are thought to be tiny vortex filaments in the fluid *ether*. The different chemical compounds are given birth by topological combinations of linked and knotted vortices [38]. In fact, this use of topological ideas in physics and fluid mechanics in particular dates back to the very origin of topology as an independent science in the days of Carl Gauss [27], Johan Benedict Listing [51] et.al. But this topological approach was pushed to the back by later developments using differential and integral calculus. Recent years have witnessed a revived interest in the topological studies of fluid flows [15, 62, 63, 72, 73]. This has resulted in the birth of a new branch of research - 'Topological Fluid Mechanics'. According to Moffatt [60]: 'Topological, rather than analytical, techniques and language provide the natural frame work for many aspects of fluid mechanical research that are now attracting intensive study'.

It was Maxwell, more than any other, who truly saw the physical implications of topology. The whole preface of his treatise on electricity and magnetism is permeated by topological ideas [56]. While Kelvin's dream of explaining atoms as knotted vortex rings in a fluid ether never came to fruition, his work was seminal in the development of topological approach to fluid flow analysis. Other works followed soon. The work of J.J. Thomson on vortex links [78] and studies of fluid flows in multiply connected regions (see [48]) are notable works. Leon Lichtenstein dedicated two of the eleven chapters in his book on hydrodynamics to topological ideas [50]. But difficulty of an immediate application and testing of these ideas limited for many years the use of these concepts. In recent years the application of modern results from topology and knot theory and greater access to direct numerical simulation of fluid flows have led to new developments in the qualitative study of fluid mechanics.

In simple terms Helmholtz's laws of vortex motion says that in an ideal fluid flow vortex structures live for ever. It is well known that Kelvin's circulation theorem, Helmholtz's vorticity theorems and Euler's equation of motion are equivalent. But Kelvin's theorem is an integral theorem and requires the knowledge of detailed evolution of material surfaces in the fluid. The vorticity equation, though deals directly with the vector character of vorticity, is more a description of how vorticity change than a usual constraint on that change. Conservation of potential vorticity due to Ertel [19] provides the way of translating the informations in the Kelvin's circulation theorem into invariants of a local kinematic quantity. It can be seen that invariance of potential vorticity is associated with the invariance of the scalar field quantity, the entropy, that is conserved along fluid particles. Since potential vorticity itself is such a conserved quantity we can generate an infinite number of invariants using it again and again. This property is associated with the integrability of Euler's equation, though not all these invariants are of physical significance. Thus the theorems of Helmholtz, Kelvin and Ertel give rise to three types of invariants. Other invariants also have been identified in hydrodynamics and magnetohydrodynamics.

Invariants play a crucial role in physical systems and contain essential features of their motion. The knowledge of invariants often leads to an elegant, qualitative picture of the behavior of the system and to a simplification of the search for exact solutions.

In fluid mechanics the fundamental assumption is that the fluid is a continuum. In mathematical language this says that, at least locally, the fluid looks like usual two or three dimensional space. A fluid particle is identified with a mathematical point in the two or three dimensional space. We see that the theories and technologies based on continuum models have been widely successful. The basic model of fluid mechanics is the set consisting of the fluid body and the various additional structures that allow one to discuss such properties as continuity, volume, velocity and deformation. The flow itself is a transformation of the fluid body to itself and a basic object of study is the transport of structures under the flow.

The complete evolution of the fluid is described by a family of maps, ϕ_t, parametrised by time t. The particle at position X at time $t = 0$ is at the position $\phi_t(X)$ after time t. If we fix X and vary t, the positions $\phi_t(X)$ sweep out the trajectory or path of the particle. The family of fluid maps ϕ_t is best described as simple map

$$\phi : D \times \mathbb{R} \rightarrow D,$$

with $\phi(X,t) = \phi_t(X)$, where D is the domain of the fluid. Given a fluid motion ϕ_t, its velocity field $\mathbf{u}$ at the point X at a time t is the instantaneous velocity of the fluid particle that occupies that point at that time. The velocity field $\mathbf{u}$ is given by

$$\mathbf{u}\left(\phi_t\left(X\right), t\right) = \frac{\partial \phi}{\partial t}\left(X, t\right)$$

Given a velocity field $\mathbf{u}$ its fluid motion ϕ_t is obtained by solving the above differential equation.

The more deeply one penetrates the general character of fluid motions, the more apparent it becomes that the dynamical properties of fluids in the main are but names, interpretations, and methods of measuring purely kinematical quantities, and that in general the flow of fluid, whether perfect or viscous may be defined by purely kinematical conditions. It is no accident that the greatest contributions to practical fluid dynamics were preceded by kinematical analysis which in themselves belong to pure mathematics rather than to mechanics or physics.

Topological fluid mechanics has a wide range of applications. Topological factors control the behavior of magnetized turbulent plasmas through constraints on the magnetic field and through the topology of the mechanical and magentic boundaries.

Fluid mechanics provides a very natural setting for the consideration of knotted or linked structures. In an ideal barotropic fluid flow under conservative body forces the vortex lines are frozen in fluid. This has the important consequence that any kind of linkage and knottedness of vortex lines are conserved. The measure of net linkage and knottedness of the field inside a Lagrangian surface S on which $\mathbf{n}\cdot\boldsymbol{\omega} = 0$ is the helicity $H = \int_V \mathbf{u}\cdot\boldsymbol{\omega} dV$, where V is the volume bounded by S.

For a simple interpretation of helicity, we may assume that the vorticity vanishes everywhere except in two linked vortex filaments of vanishingly small cross sections and with circulation K_1 and K_2 respectively. Then it can be easily shown that

$$H = \pm 2NK_1K_2,$$

where N is the Gauss linking number of the axes of the tubes, provided each vortex tube is unknotted and the vorticity field has no internal twist within each tube [57, 58, 74]. The sign is positive or negative according as the orientation of the linkage is right handed or left handed. In ideal magnetohydrodynamic flow, the magnetic helicity is defined similarly by $H_M = \int_V \mathbf{A}\cdot\mathbf{B}dV$, where $\mathbf{A}$ is the magnetic potential and V is the volume bounded by a closed surface S.

Suppose $\mathbf{P}$ and $\mathbf{Q}$ are two vector fields where $\mathbf{P} = \nabla\times\mathbf{M}$, and $\mathbf{Q} = \nabla\times\mathbf{N}$, satisfying the conditions

$$\frac{\partial \mathbf{P}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{P}) \quad \text{and} \quad \frac{\partial \mathbf{Q}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{Q})$$

Then there is a generalisation of helicity to such fields [59]. Let S be a closed material surface with unit normal $\mathbf{n}$, on which $\mathbf{n} \cdot \mathbf{Q} = 0$. Then it can be shown that $H_{\mathbf{PQ}} = \int_V \mathbf{M} \cdot \mathbf{Q} dV$ is an invariant of fluid motion. When $\mathbf{P} = \mathbf{Q} = \boldsymbol{\omega}$ we get the helicity invariant and when $\mathbf{P} = \mathbf{Q} = \mathbf{B}$ we get the magnetic helicity invariant. When $\mathbf{P} = \boldsymbol{\omega}$ and $\mathbf{Q} = \mathbf{B}$, we get the so called cross helicity found by Woltjer [83].

A detailed discussion on the invariance and topological interpretation of helicity and its significance in the dynamo theory of celestial magnetic fields and in turbulent flows with and without magnetic fields can be found in [58, 61] . Arnold has found that helicity integrals can be described mathematically in terms of topological objects such as Hopf invariant and the Gauss linkage integral [7].

The study of helicity is important in the context of turbulence. Here helicity as an inviscid invariant implies some degree of constraint on the energy cascade process. The magnetic helicity gives a lower bound for magnetic energy through the inequality [61]

$$\int_V B^2 dV \geq q_0 \left| \int_V \mathbf{A} \cdot \mathbf{B} dV \right|,$$

where q_0 is a constant depending on the geometry and size of the fluid flow. Even though global magnetic helicity is vanishing, the magnetic energy is still bounded away from zero as shown by Freedman [25] as long as the topology of the field is nontrivial. Such is a situation when the linkage of magnetic filaments are in the form of Borromean rings topology.

The helicity is quadratic in the magnetic fluxes and hence describes a second order linking. The subtle intertwining of the Borromean rings may be described by a third order linking integral as shown in [11, 12] . An extension to the n^{th} order linking integral is also possible. The discussions on higher order linking integral are simplified by the use of differential forms and Massey product [20, 21, 46, 52, 53]. Gunnar Horning and

Christoph Mayer [33] have obtained a more general third order topological invariant for magnetic fields using Chern-Simons three-form.

The topology of vortex lines is important in attempts to understand, describe and control flows in various applications. Changes in this topology may affect mixing in flows and may be significant for the dynamics of turbulence. Under reconnection of vortex rings helicity changes. Topological changes occur when dissipative effects become predominant. The changes occur through the formation and disappearance of reconnections in the fluid pattern. Reconnections take place when the vector field lines cross each other. One of the simplest and most fundamental experiments on vortex reconnection is the interaction of two colliding circular vortex rings [23, 67, 75, 68]. In the case of magnetic reconnection, the simple situation of two dimensional stationary reconnection was considered in the first models by Sweet [77] and Parker [69]. The analytical and numerical aspects of vortex reconnection and magnetic reconnection have initiated a rich branch of research [13, 14, 32, 35, 34, 41, 71]. Helicity and topological estimates together with detailed knowledge of reconnections, can prove to be very useful for the characterization and classification of the fundamental fluid flows.

A detailed discussion on circulation preserving motion and convection and diffusion of vorticity can be seen in the classic work by Truesdell [79]. In that work he showed that a flow of fluid of uniform density and viscosity, subjected to conservative body forces is circulation preserving if and only if it admits a flexion potential, where the curl of vorticity field is called flexion field. The important Couette and Poiseulle flows are included in such flows.

There exist many other vector fields which are frozen-in fields in ideal flows. The general problem is that it is required to find conditions under which a given dynamical system admits a direction field frozen in the phase flow. By rectification theorem for trajectories, a whole family of frozen direction fields always exist locally [44]. The existence of nontrivial frozen-in direction fields defined in the whole phase space of a viscous fluid is closely related to the well known problem of small denominators and it is presented in [44].

Many important achievements in the field of hydrodynamics are based on profound mathematical theories rather than on experiments. The existence problems for the smooth solutions of hydrodynamic equations of a three dimensional fluid and a rigorous mathematical theory for explaining the phenomenon of turbulence are still challenging problems open for mathematicians.

Arnold [3, 4, 5] has used variational principle to study the stability of stationary flows of an ideal incompressible fluid. He has shown that it is possible to construct variational principles for stationary flows using special combination of two integrals of motion, namely, integral of energy conservation and integral of vorticity conservation.

There is a group theoretic approach to hydrodynamics considering it to be the differential geometry of infinite dimensional group of diffeomorphisms that preserves the volume element of the domain of a fluid flow [6, 9, 8]. The principle of least action implies that the motion of a fluid is described by the geodesics on the group in the right invariant Riemannian metric given by the kinetic energy. The equation of an inviscid incompressible fluid are Hamiltonian ones on the orbits of the group of volume preserving diffeomorphisms. Khesin and Chekanov [39] have discussed the invariants of the Euler equation for ideal barotropic fluid in an arbitrary dimensional manifold M. They have shown that the Euler's equation in an m-dimensional manifold has an infinite series of integrals if m is even, called the generalised enstrophies and at least one integral if m is odd, called generalised helicity.

1.2 Use of space-time manifold

Kiehn [42] suggested that the quantization of flux, charge and angular momentum can be interpreted as a set of independent natural concepts which physically exhibit certain topological properties of the fields on a space- time manifold. The topology of fields built on a four dimensional space-time manifold was studied by him in terms of a set of fundamental differential forms. He stated that physical theories of matter can be put into correspondence with various gauge theories relating or defining the one, two and three-dimensional periods (integrals on closed manifolds) supported by space time. Processes without dissipation leave the topological periods invariant. He has also suggested two distinct methods for extracting topological information from a hydrodynamic flow, first by utilising Cartan's calculus based on a one form of action and second method considers the null sets of the six invariant scalar functions associated with the Jacobian of the unit speed tangent vector field [43]. Gumral [29] considered three-dimensional unsteady flow of fluids in the Lagrangian description as an autonomous dynamical system in four dimensions. He also constructed a scheme of generating symmetries and invariants. Fluid motions in a four dimensional space-time manifold rather than in three-dimensional space were also discussed by Kuvshinov and Schep [47] and Peradzynski [70] using differential forms.

Topological evolutions of magnetic or vortex flux tubes in the limit of small diffusivity within a time such that the helicity is conserved can be formalised as a bordism, which is an orientable surface without self intersections in four-dimensional space-time manifold bounded by the set of flux tubes initially and at any subsequent time [2]. The births and deaths of flux tubes correspond to the maxima and minima of the bordism and the reconnections are the saddle points of the bordism. In the trivial case of a one circular flux tube at rest, the bordism is a surface of the cylinder with axis directed along the time axis.

The concept of conservation laws plays a key role in the analysis of basic properties of the solutions of systems of differential equations. The general principle relating symmetry groups and conservation laws was established by Noether [65]. Drobot and Rybarski [18] have formulated a variational principle for barotropic flows by introducing hydromechanical variations of the fields in a four dimensional Euclidean space and made use of Noether's theorem to obtain conservation laws. Based on this Mathew and Vedan [54, 55] have developed the variational principle for non barotropic flows. The advantage of their method is that it avoids such conditions like Lin's constraints and provides systematic approach using Lie group theory leading to conservation laws.

1.3 Calculus of Differential forms

The theory of differential forms and its applications to mathematical physics can be found in many books (Eg:[1, 6, 22, 24, 66, 76, 82]). In this section we will review the materials needed for a mathematical description of invariants associated with a flow. Geometry is one of the most important branches of mathematics which is having direct application to many dynamical system. We can distinguish three branches of geometry, namely, Riemannian geometry, affine geometry and differential geometry. Riemannian geometry deals with spaces that are charecterised by some metric tensor g_{ik}. Affine geometry deals with spaces where a rule for parallel transport of vectors is given. This rule is conventionally introduced through the connection coeffients Γ^i_{jk}, which are the Christoffel symbols. Any metric g_{ik} determines uniquely a symmetric ($\Gamma^i_{jk} = \Gamma^i_{kj}$) connection. This means that Riemannian geometry is a particular case of affine geometry. Differential geometry incorporates both Riemannian and affine geometries. It considers the general case of spaces that have no additional structures, called manifolds.

A set of points M is said to be an n-dimensional manifold if each point of M has an open neighborhood which has a continuous one-one map onto an open set in $\mathbb{R}^n$. In simple words a manifold is a set that locally has a topology of n-dimensional Euclidean space $\mathbb{R}^n$. More rigorous definition of smooth manifolds can be found in the references given above. In our discussion a manifold means a smooth manifold.

The tangent space to a manifold M at $\mathbf{x} \in M$, written as $TM|_{\mathbf{x}}$, is the real vector space consisting of all tangent vectors to M at $\mathbf{x}$. If $(x^1, \cdots, x^n)$ is a local coordinate system for the manifold, then the n vectors $\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}$ are defined to form a basis of this n-dimensional vector space. A vector field $\mathbf{v}$ on M assigns a tangent vector $\mathbf{v}|_{\mathbf{x}} \in TM|_{\mathbf{x}}$ to each point $\mathbf{x} \in M$, with $\mathbf{v}|_{\mathbf{x}}$ varying smoothly from point to point. In local coordinates $(x^1, \cdots, x^n)$, a vector field has the form

$$\mathbf{v}|_{\mathbf{x}} = \eta^1(\mathbf{x})\frac{\partial}{\partial x^1} + \cdots + \eta^n(\mathbf{x})\frac{\partial}{\partial x^n} \equiv \eta^1(\mathbf{x})\partial_{x^1} + \cdots + \eta^n(\mathbf{x})\partial_{x^n},$$

where each $\eta^i(\mathbf{x})$ is a smooth function of $\mathbf{x}$ and $\partial_{x^i} = \frac{\partial}{\partial x^i}$.

Exterior forms

An exterior form ω of degree k is a function of k vectors in $\mathbb{R}^n$ which is k-linear and antisymmetric. That is

$$\omega : \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n \to \mathbb{R}$$

satisfies

$$\omega\left(a\boldsymbol{\xi}_1' + b\boldsymbol{\xi}_1'', \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_k\right) = a\omega\left(\boldsymbol{\xi}_1', \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_k\right) + b\omega\left(\boldsymbol{\xi}_1'', \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_k\right) \quad \text{and}$$
$$\omega\left(\boldsymbol{\xi}_{i_1}, \cdots, \boldsymbol{\xi}_{i_k}\right) = (-1)^{\nu}\omega\left(\boldsymbol{\xi}_1, \cdots, \boldsymbol{\xi}_k\right),$$

where

$$\nu = \begin{cases} 0 & \text{if the permutation } i_1, \cdots, i_k \text{ is even;} \\ 1 & \text{if the permutation } i_1, \cdots, i_k \text{ is odd.} \end{cases},$$

$a, b \in \mathbb{R}$ and $\boldsymbol{\xi}_1', \boldsymbol{\xi}_1'', \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_k \in \mathbb{R}^n$. The set of all exterior k-forms in $\mathbb{R}^n$ form a real vector space if we introduce operations of addition and multiplication by scalars as follows:

$$(\omega_1 + \omega_2)(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_k) = \omega_1(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_k) + \omega_2(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_k) \quad \text{and}$$

$$(a\omega)(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_k) = a\omega(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_k),$$

where $a \in \mathbb{R}$ and $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_k \in \mathbb{R}^n$.

Exterior multiplication

The exterior multiplication of an arbitrary exterior k-form ω^k on $\mathbb{R}^n$ by an arbitrary exterior l-form ω^l on $\mathbb{R}^n$ is defined to be an exterior $(k+l)$-form whose value on the $k+l$ vectors

$$\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_k, \boldsymbol{\xi}_{k+1}, \cdots, \boldsymbol{\xi}_{k+l} \in \mathbb{R}^n$$

is equal to

$$(\omega^k \wedge \omega^l)(\boldsymbol{\xi}_1, \cdots, \boldsymbol{\xi}_{k+l}) = \sum (-1)^\nu \omega^k(\boldsymbol{\xi}_{i_1}, \cdots, \boldsymbol{\xi}_{i_k})\omega^l(\boldsymbol{\xi}_{j_1}, \cdots, \boldsymbol{\xi}_{j_l}),$$

where $i_1 < \cdots < i_k$ and $j_1 < \cdots < j_l$; $(i_1, \cdots, i_k, j_1 \cdots j_l)$ is a permutation of the numbers $(1, 2, \cdots, k+l)$ and

$$\nu = \begin{cases} 0 & \text{if the permutation is even;} \\ 1 & \text{if the permutation is odd.} \end{cases}$$

Every partition of the above $k+l$ vectors into two sets of vectors, one containing k vectors and the other containing l vectors, gives one term in the above some. The operation of exterior multiplication has the following properties:

1. $\omega^k \wedge \omega^l = (-1)^{kl} \omega^l \wedge \omega^k$ (skew-commutative),

2. $(a\omega_1^k + b\omega_2^k) \wedge \omega^l = a\omega_1^k \wedge \omega^l + b\omega_2^k \wedge \omega^l$ (distributive) and

3. $(\omega^k \wedge \omega^l) \wedge \omega^m = \omega^k \wedge (\omega^l \wedge \omega^m)$ (associative).

For example, let ω_1 and ω_2 be two exterior one forms (that is $k = l = 1$). Then the above definition of exterior multiplication gives the exterior two form $\omega_1 \wedge \omega_2$ whose value on any pair of vectors $\boldsymbol{\xi}_1$ and $\boldsymbol{\xi}_2$ in $\mathbb{R}^n$ is given by

$$(\omega_1 \wedge \omega_2)(\boldsymbol{\xi}_1, \boldsymbol{\xi}_2) = \omega_1(\boldsymbol{\xi}_1)\omega_2(\boldsymbol{\xi}_2) - \omega_1(\boldsymbol{\xi}_2)\omega_2(\boldsymbol{\xi}_1)$$

From this it is clear that the exterior square of an exterior one form is zero. In general $\omega^k \wedge \omega^k = 0$ if k is an odd integer.

Differential forms

Let M be a smooth manifold. Then a differential k-form $\omega|_{\mathbf{x}}$ at a point $\mathbf{x}$ of the manifold M is an exterior k-form on the tangent space $TM_{\mathbf{x}}$ to M at $\mathbf{x}$, that is, a k-linear skew-symmetric function of k vectors $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_k$ tangent to M at $\mathbf{x}$.

A smooth differential k-form ω on M (or k-form for short) is a collection of smoothly varying k-linear skew-symmetric maps $\omega|_{\mathbf{x}}$, for each $\mathbf{x} \in M$. Here we require that for all smooth vector fields $\mathbf{v}_1, \cdots, \mathbf{v}_k$ on M

$$\omega(\mathbf{v}_1, \cdots, \mathbf{v}_k)(\mathbf{x}) \equiv \omega(\mathbf{v}_1|_{\mathbf{x}}, \cdots, \mathbf{v}_k|_{\mathbf{x}})$$

is a smooth real valued function of $\mathbf{x}$. In particular, a 0-form is just a smooth real valued function on M. It can be shown that the set of all k-forms forms an infinite-dimensional vector space, if k does not exceed the dimension of M. Also the set of k-forms has a natural structure as a module over the ring of infinitely differentiable real functions on M.

The space of all one forms at a point $\mathbf{x} \in M$ is called the cotangent space $T^*M|_{\mathbf{x}}$, which is the space of linear functionals on $TM|_{\mathbf{x}}$. If $(x^1, \cdots, x^n)$ is a local coordinate system for M, then $TM|_{\mathbf{x}}$ has a basis $\{\frac{\partial}{\partial x^1}, \cdots, \frac{\partial}{\partial x^n}\}$. Then the cotangent space has the dual basis denoted traditionally by $\{dx^1, \cdots, dx^n\}$. Note that

$$dx^i\left(\frac{\partial}{\partial x^j}\right) = \begin{cases} 1; & \text{if } i = j \\ 0; & \text{if } i \neq j. \end{cases}$$

So any one form ω has a local coordinate expression

$$\omega = a_1(\mathbf{x})dx^1 + \cdots + a_n(\mathbf{x})dx^n,$$

where each $a_i(\mathbf{x})$ is a smooth function. It can be shown that in local coordinates any differential k-form at a point $\mathbf{x}$ of the manifold M is spanned by

$$dx^I = dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

where I ranges over all strictly increasing multi-indices $1 \leq i_1 < \cdots < i_k \leq n$. So the space of all differential k-forms at a point $\mathbf{x}$ has dimension $\binom{n}{k}$. In particular when $k > n$, the space of all differential k-forms at $\mathbf{x}$ is a null space.

Any smooth differential k-form on M has the local coordinate expression

$$\omega = \sum_I a_I(\mathbf{x})dx^I$$

where for each strictly increasing multi-indices I, the coefficient a_I is a smooth real valued function. In particular, every differential k-form on the space $\mathbb{R}^n$ with a given coordinate system $(x^1, \cdots, x^n)$ can be written uniquely as above.

For a differential k-form ω, its domain is the set of all vectors in $TM|_{\mathbf{x}}$ (that is, the domain is the product space $TM|_{\mathbf{x}} \times \cdots \times TM|_{\mathbf{x}}$, of n copies of $TM|_{\mathbf{x}}$). The restriction of ω to a subspace V of $TM|_{\mathbf{x}}$ is the same k-form ω whose domain is now restricted to vectors in V. We denote it by $\omega|_V$:

$$\omega|_V(\boldsymbol{\xi}_1, \cdots, \boldsymbol{\xi}_k) = \omega(\boldsymbol{\xi}_1, \cdots, \boldsymbol{\xi}_k)$$

where all of $\boldsymbol{\xi}_1, \cdots, \boldsymbol{\xi}_k$ are in V. If the dimension of V is less than k, then $\omega|_V$ is necessarily zero. It is to be noted that a differential k-form ω is said to be annulled by a vector subspace if its restriction to it vanishes. A k-form is said to be annulled by a submanifold N of a manifold M when the tangent vectors of N annuls ω (that is restriction of ω to N vanishes).

Exterior derivative

The exterior derivative or differential of a k-form

$$\omega = \sum_I a_I(\mathbf{x}) dx^I$$

on the space $\mathbb{R}^n$ is the $(k+1)$-form

$$d\omega = \sum_I da_I \wedge dx^I = \sum_{I,j} \frac{\partial a_I}{\partial x^j} dx^j \wedge dx^I$$

This exterior derivative has the following properties:

1. $d(a\omega + a'\omega') = ad\omega + a'd\omega'$, where $a, a' \in \mathbb{R}$ (Linearity),

2. $d(\omega^k \wedge \omega^l) = d\omega^k \wedge \omega^l + (-1)^k \omega^k \wedge d\omega^l$, where ω^k is a k-from and ω^l is an l-form (Anti-derivation) and

3. $d(d\omega) = 0$ (Closure).

For example if $n = 3$, then the differential of a one form in $\mathbb{R}^3$ is

$$d\,(Adx + Bdy + Cdz) = (C_y - B_z)dy \wedge dz + (A_z - c_x)dz \wedge dx + (B_x - A_y)dx \wedge dy$$

which can be identified as taking curl of the vector field $\mathbf{A} = (A, B, C)$ in $\mathbb{R}^3$. Similarly differential of a two form can be identified with the divergence of the corresponding vector field. So the concept of exterior derivative generalizes the classical curl and divergence operators.

Interior product

If ω is a differential k-form and $\mathbf{v}$ is a smooth vector field, then we can define a $(k-1)$-form, $i_{\mathbf{v}}\omega$, called the interior product of $\mathbf{v}$ with ω as

$$(i_{\mathbf{v}}\omega)(\mathbf{v}_1, \cdots, \mathbf{v}_{k-1}) = \omega(\mathbf{v}, \mathbf{v}_1, \cdots, \mathbf{v}_{k-1})$$

for every set of vector fields $\mathbf{v}_1, \cdots, \mathbf{v}_{k-1}$. Interior product is bilinear in both its arguments. For basis elements the interior product is given by

$$i_{\frac{\partial}{\partial x^i}}\left(dx^{j_1} \wedge \cdots \wedge dx^{j_k}\right) = \begin{cases} (-1)^{\nu-1} dx^{j_1} \wedge \cdots \wedge dx^{j_{\nu-1}} \wedge dx^{j_{\nu+1}} \wedge \cdots \wedge dx^{j_k}; & \text{if } i = j_\nu, \\ 0; \text{if } i \neq j_\nu \text{ for all } \nu. \end{cases}$$

Interior product has the following properties:

If α and β are any two differential forms and $\mathbf{v}$ and $\mathbf{w}$ are any two vector fields, then

1. $i_{\mathbf{v}+\mathbf{w}}\alpha = i_{\mathbf{v}}\alpha + i_{\mathbf{w}}\alpha$,
2. $i_{f\mathbf{v}}\alpha = f i_{\mathbf{v}}\alpha$, where f is a scalar function,
3. $i_{\mathbf{v}} i_{\mathbf{v}}\alpha = 0$ and
4. $i_{\mathbf{v}}(\alpha \wedge \beta) = (i_{\mathbf{v}}\alpha) \wedge \beta + (-1)^k \alpha \wedge (i_{\mathbf{v}}\beta)$

Comparison between vector fields and differential forms in $\mathbb{R}^3$

Let $\Omega = dx \wedge dy \wedge dz$ be the elementary volume form in the three dimensional Euclidean space $\mathbb{R}^3$. For any one form in $\mathbb{R}^3$ there exist an associated vector field in $\mathbb{R}^3$ such that the coefficient functions of the form and vector field are the same. So There is a one-one correspondence between one forms and vector fields in the Euclidean space $\mathbb{R}^3$. In a general Riemannian manifold such a one-one correspondence is trough the corresponding metric tensor g_{ij}. Also there is one-one correspondence between differential two forms and vector fields in $\mathbb{R}^3$ through the relation $\omega^2 = i_{\mathbf{w}}\Omega$, where ω^2 is a two form and $\mathbf{w}$ is the associated vector field. A zero form is a function on $\mathbb{R}^3$. Let α^1 and β^1 are two one forms with associated vector fields $\mathbf{A}$ and $\mathbf{B}$ respectively and γ^2 is a two form with associated vector field $\mathbf{C}$. Then we can make the following symbolic identifications

$$\alpha^1 \wedge \beta^1 \longleftrightarrow \mathbf{A} \times \mathbf{B} \qquad \alpha^1 \wedge \gamma^2 \longleftrightarrow \mathbf{A} \cdot \mathbf{C}$$

$$i_{\mathbf{B}}\alpha^1 \longleftrightarrow \mathbf{B} \cdot \mathbf{A} \qquad i_{\mathbf{B}}\gamma^2 \longleftrightarrow -\mathbf{B} \times \mathbf{C}$$

$$df \longleftrightarrow \nabla f \qquad d\alpha^1 \longleftrightarrow \nabla \times \mathbf{A}$$

$$d\gamma^2 \longleftrightarrow \nabla \cdot \mathbf{C} \qquad i_{\mathbf{B}} i_{\mathbf{A}}\Omega \longleftrightarrow \mathbf{A} \times \mathbf{B}$$

1.4 Invariance of geometrical objects

In this section we will briefly discuss how to find out the invariants associated with vector fields and differential forms using Lie derivative.

Lie derivative

Consider a vector field $\mathbf{v}$ on a manifold M. We are often interested in how certain geometrical objects on M, such as functions, other vector fields and differential forms vary under the flow induced by $\mathbf{v}$. The Lie derivative of such an object gives its infinitesimal change when advected by the flow. The vanishing of the Lie derivative with respect to a flow field of a geometrical object associated with a physical quantity represents the conservation of that physical quantity. The properties of Lie derivative and the operations of differential geometry lead to general rules for the construction of invariant fields.

By definition, the Lie derivative of a scalar function with respect to a vector field $\mathbf{v}$ is the directional derivative $df(\mathbf{v})$. The Lie derivative of a vector field $\mathbf{w}$ with respect to the vector field $\mathbf{v}$ is given by [76]

$$L_{\mathbf{v}}\mathbf{w} = [\mathbf{v}, \mathbf{w}]$$

where [,] is the Lie bracket of the vector fields. On a coordinate basis the above Lie derivative becomes

$$L_{\mathbf{v}}\mathbf{w} = \left(\eta^i \frac{\partial}{\partial x^i} \zeta^j - \zeta^i \frac{\partial}{\partial x^i} \eta^j \right) \frac{\partial}{\partial x^j}$$

where the summation convention is assumed and

$$\mathbf{v} = \eta^1 \frac{\partial}{\partial x^1} + \cdots + \eta^n \frac{\partial}{\partial x^n} \text{ and } \mathbf{w} = \zeta^1 \frac{\partial}{\partial x^1} + \cdots + \zeta^n \frac{\partial}{\partial x^n}$$

. If $L_{\mathbf{v}}\mathbf{w} = 0$, then we say that the field lines of the vector field $\mathbf{w}$ is invariantly transported with the flow field $\mathbf{v}$, that is, the field lines of $\mathbf{w}$ are preserved by the flow.

The Lie derivative of a differential form ω with respect to a vector field $\mathbf{v}$ is given by the Cartan's formula

$$L_{\mathbf{v}}\omega = di_{\mathbf{v}}\omega + i_{\mathbf{v}}d\omega.$$

If $L_{\mathbf{v}}\omega = 0$, then the differential form ω is said to be invariantly transported under the flow field $\mathbf{v}$.

Following are the properties of Lie derivative:

Let α and β are any two differential forms and $\mathbf{v}$ and $\mathbf{w}$ are any two vector fields, then

1. $L_{\mathbf{v}} d\alpha = dL_{\mathbf{v}}\alpha$,

2. $L_{\mathbf{v}} i_{\mathbf{v}}\alpha = i_{\mathbf{v}} L_{\mathbf{v}}\alpha$,

3. $[L_{\mathbf{v}}, L_{\mathbf{w}}]\,\alpha = L_{[\mathbf{v},\mathbf{w}]}\alpha$,

4. $L_{\mathbf{v}+\mathbf{w}}\alpha = L_{\mathbf{v}}\alpha + L_{\mathbf{w}}\alpha$,

5. $L_{f\mathbf{v}}\alpha = fL_{\mathbf{v}}\alpha + df \wedge i_{\mathbf{v}}\alpha$, where f is a scalar function,

6. $[L_{\mathbf{v}}, i_{\mathbf{w}}] = i_{[\mathbf{v},\mathbf{w}]}$,

7. $i_{\mathbf{v}} df = L_{\mathbf{v}} f$ and

8. $L_{\mathbf{v}}(\alpha \wedge \beta) = (L_{\mathbf{v}}\alpha) \wedge \beta + \alpha \wedge L_{\mathbf{v}}\beta$

Frobenius theorem

Frobenius theorem gives the necessary and sufficient condition under which collections of vector fields or one forms determine families of submanifolds, also called hypersurfaces. In terms of vector fields the Frobenius theorem states that, the integral curves of r vector fields $\mathbf{v}_1, \cdots, \mathbf{v}_r$ form a family of hypersurfaces if and only if their Lie brackets in pairs are linear combinations of these r vector fields,

$$[\mathbf{v}_i, \mathbf{v}_j] = c_{ij}^k \mathbf{v}_k$$

In terms of one forms Frobenius theorem states that, there exist a family of $(n-r)$-dimensional hypersurfaces whose tangent vectors annihilate each of the r forms $\omega_1, \cdots, \omega_r$ if and only if $d\omega_i \wedge \omega_1 \wedge \cdots \wedge \omega_r = 0$ for all i. Such a family of hypersurfaces is also called the family of integral surfaces for the r one forms $\omega_1, \cdots, \omega_r$.

Let ω be a one form in $\mathbb{R}^3$ and $\mathbf{A}$ be the associated vector field. According to Frobenius theorem $\omega \wedge d\omega = 0$ if and only if there is a family of integral surfaces for the one form ω. In terms of the associated vector field $\mathbf{A}$ this is restated as: There is a family of orthogonal surfaces for the vector field $\mathbf{A}$ if and only if $\mathbf{A} \cdot \nabla \times \mathbf{A} = 0$.

Integral invariants

Consider a flow of a continuous media with a flow field $\mathbf{v}$. We can construct integral invariants by integrating k-forms over orientable k-dimensional surfaces. Let C be a k-dimensional surface moving with velocity $\mathbf{v}$, that is a k-dimensional comoving surface or simply a k-surface. The time derivative of the integral of a k-form ω over a k-surface C is equal to [1, 24]

$$\frac{d}{dt}\int_C \omega = \int_C L_{\mathbf{v}}\omega$$

So it follows that $\int_C \omega$ is an invariant if $L_{\mathbf{v}}\omega = 0$. Hence every invariant k-form generates an integral invariant. In non-autonomous case (time dependent case) we have the following equivalent formulation:

The time derivative of the integral of a k-form ω over a k-surface C is equal to

$$\frac{d}{dt}\int_C \omega = \int_C \frac{\partial \omega}{\partial t} + L_{\mathbf{v}}\omega$$

So it follows that $\int_C \omega$ is an invariant if

$$\frac{\partial \omega}{\partial t} + L_{\mathbf{v}}\omega = 0.$$

Invariance condition in $\mathbb{R}^3$

The invariance condition using Lie derivative for different kinds of forms and vector fields in $\mathbb{R}^3$ under the flow field $\mathbf{u}$ and there corresponding expression in vector notation are given below.

1. Consider the zero form f, which is a scalar function in $\mathbb{R}^3$. Then

$$\partial_t f + L_{\mathbf{u}} f = 0 \Leftrightarrow \partial_t f + (\mathbf{u} \cdot \nabla) f = 0 \tag{1.1}$$

2. Consider the one form $\omega^1 = A_1 dx^1 + A_2 dx^2 + A_3 dx^3$

   Then

$$\partial_t \omega^1 + L_{\mathbf{u}} \omega^1 = 0 \Leftrightarrow \partial_t \mathbf{A} + (\mathbf{u} \cdot \nabla)\mathbf{A} + (\mathbf{A} \cdot \nabla)\mathbf{u} + \mathbf{A} \times \nabla \times \mathbf{u} = 0 \tag{1.2}$$

   where $\mathbf{A} = (A_1, A_2, A_3)$.

3. Consider the two form $\omega^2 = H_1 dx^2 dx^3 + H_2 dx^3 dx^1 + H_3 dx^1 dx^2$. Then

$$\partial_t \omega^2 + L_{\mathbf{u}} \omega^2 = 0 \Leftrightarrow \partial_t \mathbf{H} + (\mathbf{u} \cdot \nabla)\mathbf{H} - (\mathbf{H} \cdot \nabla)\mathbf{u} + (\nabla \cdot \mathbf{u})\mathbf{H} = 0 \tag{1.3}$$

   where $\mathbf{H} = (H_1, H_2, H_3)$.

4. Consider the three form $\omega^3 = g dx^1 dx^2 dx^3$. Then

$$\partial_t \omega^3 + L_{\mathbf{u}} \omega^3 = 0 \Leftrightarrow \partial_t g + \nabla \cdot (g\mathbf{u}) = 0 \tag{1.4}$$

5. Consider the vector filed $\mathbf{J} = (J_1, J_2, J_3) = J_1 \partial_{x^1} + J_2 \partial_{x^2} + J_3 \partial_{x^3}$. Then

$$\partial_t \mathbf{J} + L_{\mathbf{u}} \mathbf{J} = 0 \Leftrightarrow \partial_t \mathbf{J} + [\mathbf{u}, \mathbf{J}] = 0 \Leftrightarrow \partial_t \mathbf{J} + (\mathbf{u} \cdot \nabla)\mathbf{J} - (\mathbf{J} \cdot \nabla)\mathbf{u} = 0 \tag{1.5}$$

   where [,] is the Lie bracket of vector fields.

1.5 Summary of the thesis

Fluid flow occurs in three dimensional space. The topology of field lines of a flow are related to the differential geometry of the three dimensional space. Using the concepts of differential forms, it has been shown that there exist four kinds of invariants associated with such a flow. But it is found useful to consider a four dimensional manifold in the analysis of fluid flows. Chapter 2 deals with the use of a four dimensional space-time manifold E^4 for obtaining vorticity invariants of hydrodynamic flows. A criterion using Lie derivative is applied to obtain the invariants.

The analogy between vorticity equation of a barotropic flow and the induction equation of a magnetohydrodynamics is well known. Vortex lines have properties similar to magnetic lines of force. Thus topological considerations apply to barotropic flows as well as magnetohydrodynamic flows. Though this has been mentioned in many magnetohydrodynamic studies, particular attention has not been given to it. There are vector fields other than vorticity and magnetic field which are frozen-in fields in ideal fluid flows. In chapter 3 we generalise the vorticity invariants to the case of any general divergence-free frozen-in vector field. Here we define a closed differential two form and its potential one form. Using this a three form and a four form are defined and the associated invariants are discussed. The use of a four dimensional manifold instead of three dimensional Euclidean space give rise to an additional invariant associated with four forms. This also is discussed in this chapter.

In chapter 4 we classify the possible invariants in a continuous media which generalises the concept of invariants in a dissipationless continuous media. We consider a p-form ω associated with a physical quantity which evolves under a flow field $\mathbf{u}$. If the Lie derivative of ω with respect to $\mathbf{u}$ vanishes, then for all comoving surfaces C^p the integral, $\int_{C^p} \omega$, will be a constant. But in all cases the Lie derivative need not be vanishing. Even then we can find some p-dimensional comoving surfaces C^p over which $\int_{C^p} \omega$ is a constant of motion. Such invariant surfaces can be used for the qualitative study of the flow of a continuous media in the absence of invariants of all p-dimensional comoving surfaces. In this chapter we investigate the sufficient conditions for the existence of such invariant surfaces. Some illustrative examples are also given for hydrodynamic and magnetohydrodynamic flows.

In chapter 5 we conclude the thesis with a general discussion of the results obtained.

Chapter 2

Vorticity Related Invariants in Hydrodynamics

2.1 Introduction

A vector field $\mathbf{u}$ on a manifold assigns a tangent vector at each point on it. A good physical example of a vector field is the velocity field of a steady fluid flow in some open subset $M \subset \mathbb{R}^3$. At each point of M the vector field $\mathbf{u}$ is the velocity of fluid particle passing through that point.

The maximal integral curve of $\mathbf{u}$ passing through a point $\mathbf{x} \in M$ is called the flow generated by $\mathbf{u}$. Thus a vector field defines a flow on M. We are often interested in how certain geometric objects on M such as functions, differential forms and other vector fields vary under the flow induced by $\mathbf{u}$. The Lie derivative of the object with respect to $\mathbf{u}$ gives the infinitesimal change resulted by the flow.

This geometrical interpretation of Lie derivative has been used by Tur and Yanovsky [80] to study the invariants in dissipationless hydrodynamic media. The Lie derivative with respect to the velocity vector $\mathbf{u}$ corresponds to the convective derivative of objects.

[0]Some of the results in this chapter has appeared in our paper entitled 'Vorticty invariants in hydrodynamics' *Zeitschrift fur Angewandte Mathematik und Physik* (ZAMP), Volume 55 (2), 2004

Thus the object is an invariant of the flow if this derivative is compensated by the local derivative. This leads to the invariance criterion of an object Φ : $\partial_t\Phi + L_{\mathbf{u}}\Phi = 0$, where $L_{\mathbf{u}}\Phi$ denotes the Lie derivative of Φ with respect to $\mathbf{u}$.

Fluid flow occurs in three dimensional space. The above considerations lead to the investigation of how the fluid flow is related to the geometry of three dimensional space. This is the problem investigated by Tur and Yanovsky using differential forms. Corresponding to the four kinds of forms in a three dimensional manifold they have obtained four types of invariants.

Drobot and Rybarski [18], Mathew and Vedan [54, 55] and Geetha, Thomas Joseph and Vedan [28] have used a four dimensional space-time manifold to study inviscid flows. A four vector $\mathcal{P}$ represents a flow in this manifold. In this case the above invariance condition becomes vanishing of the Lie derivative with respect to this flow vector $\mathcal{P}$. We are using this invariance condition to obtain the vorticity related invariants in hydrodynamics. We consider a four dimensional Euclidean space E^4 and a four vector field $\mathcal{P}$.

2.2 Invariants in inviscid flows

As mentioned in the introductory chapter, we can find numerous examples for different types of invariants in ideal fluid flows. It has been shown that in hydrodynamics there exists only four types of invariants which are associated with the four types of differential forms that exist in $\mathbb{R}^3$ [80]. These can be classified as follows.

1. Lagrangian invariants: These are scalar functions I satisfying the differential equation

$$\partial_t I + (\mathbf{u} \cdot \nabla) I = 0 \tag{2.1}$$

A well known example for such an invariant is the Ertel invariant $(\boldsymbol{\omega} \cdot \nabla S)/\rho$, where S and ρ are the entropy density and the density of the fluid respectively and $\boldsymbol{\omega}$ is the vorticity field.

2. Surface invariants: Surface invariants are those vector fields which are satisfying the equation

$$\partial_t \mathbf{S} + (\mathbf{u} \cdot \nabla)\mathbf{S} + (\mathbf{S} \cdot \nabla)\mathbf{u} + \mathbf{S} \times \nabla \times \mathbf{u} = 0 \tag{2.2}$$

3. Frozen in invariants: Frozen in invariants are those vector fields $\mathbf{Q}$ which satisfy the equation

$$\partial_t \mathbf{Q} + (\mathbf{u} \cdot \nabla)\mathbf{Q} - (\mathbf{Q} \cdot \nabla)\mathbf{u} + (\nabla \cdot \mathbf{u})\mathbf{Q} = 0 \tag{2.3}$$

   Such fields are clearly flux conserving fields.

4. Density invariants: A scalar function ϕ is said to be a density invariant in a fluid flow if it satisfies the equation

$$\partial_t \phi + (\mathbf{u} \cdot \nabla)\phi + (\nabla \cdot \mathbf{u})\phi = 0 \tag{2.4}$$

Euler's equation of motion

Consider the case of barotropic inviscid fluid flow. The Euler's equation of motion under conservative body force is

$$\partial_t \mathbf{u} + \frac{1}{2}\nabla \mathbf{u}^2 - \mathbf{u} \times \boldsymbol{\omega} = \nabla \psi. \tag{2.5}$$

Here $\boldsymbol{\omega}$ is the vorticity. ψ is defined as follows. Let φ be the potential of the body forces and $\nabla W = \nabla P/\rho$, where P is the pressure and ρ is the density of the fluid. Then $\psi = \varphi - W$. Taking $\mathbb{E} = (\partial_t \mathbf{u} + \frac{1}{2}\nabla \mathbf{u}^2)$, the above equation can be written as

$$\mathbb{E} + \boldsymbol{\omega} \times \mathbf{u} = \nabla \psi. \tag{2.6}$$

This can be compared with the Ohm's law in plasma flows for which the right hand side is a gradient(Here we introduced this notation deliberately for its use in coming sections).

Equation (2.6) leads to the vorticity equation

$$\partial_t \boldsymbol{\omega} + (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} + (\nabla \cdot \mathbf{u})\boldsymbol{\omega} = 0. \tag{2.7}$$

From this it is clear that the evolution of the vorticity field is flux conserving. This gives the well known result, which is known form the dates of Helmholtz, that in the case of a barotropic inviscid fluid flow with conservative body forces the vorticity field lines are frozen in the fluid.

Integral conservation

For a barotropic inviscid flow we have the equation of motion (2.6). Now for any closed surface $\mathbf{S}$ on which $\boldsymbol{\omega} \cdot d\mathbf{S} = 0$, we have

$$\int_V \mathbb{E} \cdot \boldsymbol{\omega} dV = \int_S \psi \boldsymbol{\omega} \cdot d\mathbf{S} = 0, \tag{2.8}$$

where V is the volume bounded by the closed surface $\mathbf{S}$. So, $\mathbb{E} \cdot \boldsymbol{\omega}$ is an invariant of the flow inside this closed surface. Let us assume that the vorticity field is zero except inside a closed vortex tube bounding a volume V. Then the above integral is vanishing throughout the flow. This invariant is much similar to the invariant $\mathbf{E} \cdot \mathbf{B}$ in the ideal magnetohydrodynamic (MHD) flows. But, in general $\boldsymbol{\omega} \cdot d\mathbf{S}$ need not be vanishing on all surfaces. In such cases, let $I = \int_V \mathbb{E} \cdot \boldsymbol{\omega}$.

Then

$$\begin{aligned} \frac{dI}{dt} &= \frac{d}{dt} \int_V \mathbb{E} \cdot \boldsymbol{\omega} dV = \frac{d}{dt} \int_S \psi \boldsymbol{\omega} \cdot d\mathbf{S} \\ &= \int_S \psi [\partial_t \boldsymbol{\omega} + \nabla \times (\boldsymbol{\omega} \times \mathbf{u})] \cdot d\mathbf{S} + \int_S \frac{d\psi}{dt} \boldsymbol{\omega} \cdot d\mathbf{S} \end{aligned} \tag{2.9}$$

Thus for any volume V, I is an invariant (or equivalently, $\mathbb{E}\cdot\boldsymbol{\omega}$ is an integral invariant) for an inviscid barotropic flow if $d\psi/dt = 0$ (or $\boldsymbol{\omega}\cdot d\mathbf{S} = 0$ on S).

As an example we can consider an isentropic flow for which $dP/dt = 0$, where P is the pressure (here the equation of motion becomes $\mathbb{E}+\boldsymbol{\omega}\times\mathbf{u} = \nabla P$). In this case it is clear from the above discussions that $\mathbb{E}\cdot\boldsymbol{\omega}$ is an integral invariant for any volume V.

Let us consider a viscous barotropic flow with the equation of motion

$$\mathbb{E}+\boldsymbol{\omega}\times\mathbf{u} = \nabla\psi - \nu\nabla\times\boldsymbol{\omega}. \tag{2.10}$$

Now, for a viscous flow inside a closed surface S enclosing a volume V we can assume that $\mathbf{u} = 0$ on the boundary so that $\boldsymbol{\omega}\cdot d\mathbf{S} = 0$ on the boundary, assuming that the kinematic viscosity is small [57]. Then

$$\begin{aligned}\int_V \mathbb{E}\cdot\boldsymbol{\omega}dV &= -\nu\int_V \boldsymbol{\omega}\cdot\nabla\times\boldsymbol{\omega}dV \quad (\text{since}\,\boldsymbol{\omega}\cdot d\mathbf{S} = 0\,\text{on the boundary})\\ &= \frac{1}{2}\frac{dh}{dt}\end{aligned} \tag{2.11}$$

where $h = \int_V \mathbf{u}\cdot\boldsymbol{\omega}dV$ is the total helicity.

For a non-barotropic inviscid flow, for any volume bounded by a closed surface over which $\boldsymbol{\omega}\cdot d\mathbf{S} = 0$, the volume Integral $\int_V \rho(\mathbb{E}\cdot\boldsymbol{\omega})$ vanishes, since

$$\mathbb{E}+\boldsymbol{\omega}\times\mathbf{u} = \frac{\nabla P}{\rho} \quad \Rightarrow \quad \int_V \rho(\mathbb{E}\cdot\boldsymbol{\omega}) = \int_V \boldsymbol{\omega}\cdot\nabla P = \int_S P\boldsymbol{\omega}\cdot d\mathbf{S} = 0. \tag{2.12}$$

2.3 Invariance of geometrical objects in E^4

The geometrical objects we consider here are those associated with the differential forms and vector fields. An object is invariantly transported or Lie transported by a vector field $\mathcal{P}$ if its Lie derivative with respect to $\mathcal{P}$ is zero. In E^4 there exists five kinds of differential forms ω^n for $n = 0, 1, 2, 3$ and 4. We will discuss the invariance criterion for each of these forms and also for vector fields. Since we are familiar with vector field notations in three dimensional space, we will represent differential forms in vector notations also. The flow field defined by the four vector field $\mathcal{P}$ will be taken to be $\mathcal{P} = (p_0, \mathbf{p})$, where p_0 is the coefficient of $\mathcal{P}$ corresponding to the time coordinate and $\mathbf{p}$, which is a vector in $\mathbb{R}^3$, corresponds to space coordinates.

Invariance of differential forms

In this section we will give the invariant condition for differential forms ω^n for $n = 0, 1, 2, 3$ and 4 under the flow generated by the four vector field $\mathcal{P}$ in E^4.

Zero forms

Consider a zero form, *i.e.*, a function f in E^4. This is an invariant if it satisfies $L_{\mathcal{P}} f = 0$, which in vector notation takes the form

$$p_0 \partial_t f + (\mathbf{p} \cdot \nabla) f = 0, \tag{2.13}$$

where ∇ is the gradient operator with respect to spatial coordinates. Clearly this is the definition of a Lagrangian invariant. An example for such an invariant is the Ertel invariant given by $(\boldsymbol{\omega} \cdot \nabla S)/\rho$, where S and ρ are the entropy density and the density of the fluid respectively and $\boldsymbol{\omega}$ is the vorticity.

One forms

Usually a one form is given by

$$\omega^1 = \sum A_i dx^i = A_0 dx^0 + A_1 dx^1 + A_2 dx^2 + A_3 dx^3$$

and can be represented in vector notation as $\mathcal{A} = (A_0, \mathbf{A})$, where $\mathbf{A} = (A_1, A_2, A_3)$, the coefficients corresponding to spatial coordinate differentials, and A_0 corresponding to time differential. The one form is invariant if $L_{\mathcal{P}}\omega^1 = 0$. In vector notation this gives two simultaneous equations

$$\partial_t(p_0 A_0) + (\mathbf{p}\cdot\nabla)A_0 + \mathbf{A}\cdot\partial_t\mathbf{p} = 0 \tag{2.14a}$$

and

$$p_0\partial_t\mathbf{A} + (\mathbf{p}\cdot\nabla)\mathbf{A} + (\mathbf{A}\cdot\nabla)\mathbf{p} + \mathbf{A}\times\nabla\times\mathbf{p} + A_0\nabla p_0 = 0. \tag{2.14b}$$

Two forms

Consider a two form

$$\omega^2 = dx^0\wedge(P_1dx^1 + P_2dx^2 + P_3dx^3) + Q_1dx^2\wedge dx^3 + Q_2dx^3\wedge dx^1 + Q_3dx^1\wedge dx^2$$

We will denote this two form in vector notation as $(\mathbf{P},\mathbf{Q})$, where $\mathbf{P} = (P_1, P_2, P_3)$ and $\mathbf{Q} = (Q_1, Q_2, Q_3)$. Then $L_{\mathcal{P}}\omega^2 = 0$ is equivalent to the equations

$$\partial_t(p_0\mathbf{P}) + (\mathbf{p}\cdot\nabla)\mathbf{P} + (\mathbf{P}\cdot\nabla)\mathbf{p} + \mathbf{P}\times(\nabla\times\mathbf{p}) + \mathbf{Q}\times\partial_t\mathbf{p} = 0 \tag{2.15a}$$

and

$$p_0\partial_t\mathbf{Q} + (\nabla\cdot\mathbf{Q})\mathbf{p} + \nabla\times(\mathbf{Q}\times\mathbf{p}) + \nabla p_0\times\mathbf{p} = 0. \tag{2.15b}$$

Three form

Generally a three form in E^4 is of the form

$$\omega^3 = dx^0\wedge(R_1dx^2\wedge dx^3 + R_2dx^3\wedge dx^1 + R_3dx^1\wedge dx^2) + R_0dx^1\wedge dx^2\wedge dx^3$$

In vector notation it is $(\mathbf{R}, R_0)$, where $\mathbf{R} = (R_1, R_2, R_3)$. Then $L_{\mathcal{P}}\omega^3 = 0$ can be identified with the equations

$$\partial_t(p_0\mathbf{R}) + (\mathbf{p}\cdot\nabla)\mathbf{R} + \mathbf{R}(\nabla\cdot\mathbf{p}) - (\mathbf{R}\cdot\nabla)\mathbf{p} + R_0\partial_t\mathbf{p} = 0 \tag{2.16a}$$

and

$$p_0\partial_t R_0 + \nabla\cdot(R_0\mathbf{p}) + \mathbf{R}\cdot\nabla p_0 = 0. \tag{2.16b}$$

Four forms

Any four form in E^4 is given by $\omega^4 = T dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$. Then $L_{\mathcal{P}}\omega^4 = 0$ gives the equation

$$\partial_t(p_0 T) + \nabla\cdot(T\mathbf{p}) = 0. \tag{2.17}$$

Invariance of vector fields

Suppose $J = (J_0, \mathbf{J}) = J_0\partial_{x^0} + J_1\partial_{x^1} + J_2\partial_{x^2} + J_3\partial_{x^3}$ be any general vector field defined in E^4. Then $L_{\mathcal{P}}J = [\mathcal{P}, J]$, where $[\ ,\ ]$ is the commutator of the vector fields. Thus the invariant transport of J is given by

$$p_0\partial_t J_0 + (\mathbf{p}\cdot\nabla)J_0 - (\mathbf{J}\cdot\nabla)p_0 - J_0\partial_t p_0 = 0 \tag{2.18a}$$

and

$$p_0\partial_t\mathbf{J} + (\mathbf{p}\cdot\nabla)\mathbf{J} - (\mathbf{J}\cdot\nabla)\mathbf{p} - J_0\partial_t\mathbf{p} = 0. \tag{2.18b}$$

Now we discuss some useful relations between different kinds of forms and vector fields

1. If $\omega^n = d\omega^{n-1}$, then $L_{\mathcal{P}}\omega^{n-1} = 0 \Rightarrow L_{\mathcal{P}}\omega^n = 0$, where ω^n is an n form and ω^{n-1} is an $(n-1)$ form.

2. If ω^n and ω^m are Lie transported, then $\omega^{n+m} = \omega^n \wedge \omega^m$ is also Lie transported.

3. Consider a Lie transported three form $(\mathbf{R}, R_0)$. Now define a vector field $J = (J_0, \mathbf{J}) = (-R_0/\rho, \mathbf{R}/\rho)$. Then from (2.16) and (2.18) it follows that J is an invariant vector field.

4. We have the identity $L_Y L_Z - L_Z L_Y = L_{[Y,Z]}$, where Y and Z are any two vector fields. Now suppose that J is an invariant vector field, then if ω^n is also an invariant n-form, then $L_{\mathfrak{p}} L_J \omega^n - L_J L_{\mathfrak{p}} \omega^n = L_{[\mathfrak{p},J]} \omega^n$. This implies that $L_J \omega^n$ is an invariant n-form. Let $L_J^m = L_J \circ L_J \circ \ldots \circ L_J$ (up to m-terms). Then, if ω^n and J are invariant, $L_J^m \omega^n$ is also an invariant n-form for any $m = 1, 2, 3, \cdots$.

5. We know that $L_Y i_Z - i_Z L_Y = i_{[Y,Z]}$. Now suppose that J is an invariant vector field, then $L_{\mathfrak{p}} i_J - i_J L_{\mathfrak{p}} = i_{[\mathfrak{p},J]}$. Hence $i_J \omega^n$ is an invariant n-form if ω^n is an invariant form. In particular $i_{\mathfrak{p}} \omega^n$ is an invariant for any invariant n-form ω^n. From this result also it is true that $L_J \omega^n$ is an invariant for invariant J and ω^n, using the Cartan's formula $L_J \omega^n = i_J d\omega^n + d i_J \omega^n$.

Thus from known invariants we can construct many invariants, each of which belongs to the class of invariant differential forms and vector fields. Also, for any n, the Lie transport of ω^n implies the integral conservation

$$\int_{C^n} \omega^n = \text{const.}$$

where C^n is any comoving n dimensional surface.

Remark 1. *Even though we have defined above the Lie derivative with respect to a general vector field in E^4, we are interested only in the case where the four dimensional flow field is $p = (1, \mathbf{u}) = dX/dt$, where $X = (x^0, x^1, x^2, x^3)$, $x^0 = t$ and $\mathbf{u}$ is the velocity field of fluid motion in $\mathbb{R}^3$.*

2.4 Vorticity two form in E^4

Let us define a one form ω^1 as

$$\omega^1 = A_0 dx^0 + A_1 dx^1 + A_2 dx^2 + A_3 dx^3. \tag{2.19}$$

We denote it as a four vector $A_i = (A_0, \mathbf{A})$. Here $\mathbf{A} = \mathbf{u}$, the velocity vector and $A_0 = -\frac{1}{2}\mathbf{u}^2$.

Taking the exterior derivative of ω^1 we will get a two form

$$\omega^2 = d\omega^1 = \omega_1 dx^2 \wedge dx^3 + \omega_2 dx^3 \wedge dx^1 + \omega_3 dx^1 \wedge dx^2 + \\ dx^0 \wedge (\mathbb{E}_1 dx^1 + \mathbb{E}_2 dx^2 + \mathbb{E}_3 dx^3), \tag{2.20}$$

where $(\omega_1, \omega_2, \omega_3) = \boldsymbol{\omega}$ is the usual vorticity vector and $(\mathbb{E}_1, \mathbb{E}_2, \mathbb{E}_3) = \mathbb{E}$ with $\mathbb{E} = \partial_t \mathbf{u} - \nabla A_0$. It is convenient to represent ω^2 as $(\mathbb{E}, \boldsymbol{\omega})$, in vector form. Also we have $\nabla \times \mathbb{E} = \partial_t \boldsymbol{\omega}$ and $\nabla \cdot \boldsymbol{\omega} = 0$. By our definition of $\mathbb{E}$, $\mathbb{E} + \boldsymbol{\omega} \times \mathbf{u} = \nabla \xi$ can be identified as the Euler's equation of motion for an inviscid barotropic flow under conservative body forces.

In tensor notation ω^2 is $\mathcal{F}_{\alpha\beta} dx^\alpha \wedge dx^\beta$, where

$$\mathcal{F}_{\alpha\beta} = \frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta}. \tag{2.21}$$

We call the tensor $\mathcal{F}_{\alpha\beta}$ the vorticity field tensor in E^4. Clearly it is a skew-symmetric tensor of rank two. This corresponds to the electromagnetic field tensor in magneto-hydrodynamics. The corresponding matrix representation for vorticity field tensor is given by

$$\mathcal{F}_{\alpha\beta} = \begin{bmatrix} 0 & \mathbb{E}_1 & \mathbb{E}_2 & \mathbb{E}_3 \\ -\mathbb{E}_1 & 0 & \omega_3 & -\omega_2 \\ -\mathbb{E}_2 & -\omega_3 & 0 & \omega_1 \\ -\mathbb{E}_3 & \omega_2 & -\omega_1 & 0 \end{bmatrix}$$

In what follows we will consider only the differential form representation ω^2 of $\mathcal{F}_{\alpha\beta}$. Using the above definition of vorticity two form we will investigate the invariants of hydrodynamic flows in E^4.

2.5 Invariant transport of vorticity related fields

In this section we discuss the invariant transport of vorticity related fields which are obtained from vorticity two form.

Invariance of vorticity

We are looking for a condition for the invariance of vorticity two form ω^2 in E^4 under a flow induced by the four vector field $p = (1, \mathbf{u})$. This is given by the vanishing of the Lie derivative of ω^2 with respect to p, i.e., $L_p\omega^2 = 0$. This is equivalent to the vector equations

$$\left.\begin{aligned} \partial_t(\mathbb{E} + \boldsymbol{\omega} \times \mathbf{p}) + \nabla(\mathbb{E} \cdot \mathbf{p}) = 0 \\ \nabla \times (\mathbb{E} + \boldsymbol{\omega} \times \mathbf{p}) = 0 \end{aligned}\right\} \tag{2.22}$$

Here we have used the relation $\nabla \times \mathbb{E} = \partial_t \boldsymbol{\omega}$.

We have,

$$L_p\omega^2 = i_p d\omega^2 + d i_p \omega^2. \tag{2.23}$$

Now, ω^2 is a closed two form so that $i_p d\omega^2 = 0$. Thus $L_p\omega^2 = 0$ if and only if $d i_p \omega^2 = 0$.

This implies that $i_p\omega^2 = d\phi$ for some ϕ. Again, in vector notation this is equivalent to

$$\left.\begin{aligned} \partial_t\phi + (\mathbf{u}\cdot\nabla)\phi = 0 \\ \mathbb{E} + \boldsymbol{\omega}\times\mathbf{u} = \nabla\phi \end{aligned}\right\} \tag{2.24}$$

The second of equations. (2.24) is similar to the Euler's equation of motion in barotropic flows and the first one determines the evolution of the potential ϕ. Taking the curl of the second equation we get the vorticity equation which is the condition for the invariance of vorticity in $\mathbb{R}^3$.

We have

$$L_p\omega^2 = 0 \Rightarrow \int_{C^2}\omega^2 = \text{const.}$$
$$\Rightarrow \int_{C^2}\boldsymbol{\omega}\cdot d\mathbf{a} + \int_{C^2}\mathbb{E}\cdot d\mathbf{a}_0 = \text{const.}$$

Here $d\mathbf{a}$ and $d\mathbf{a}_0$ are surface elements with the convention that $d\mathbf{a} = (dx^2\wedge dx^3, dx^3\wedge dx^1, dx^1\wedge dx^2)$ and $d\mathbf{a}_0 = dx^0\wedge(dx^1, dx^2, dx^3) = dx^0\wedge d\mathbf{l}$, and the integration is over a two dimensional surface C^2.

If the surface under consideration is space like (*i.e.*, t = constant), then we have

$$\int \boldsymbol{\omega}\cdot d\mathbf{a} = \text{const.}$$

On the other hand if we start with a plane spanned by a vortex line and the time axis, the first term vanishes so that

$$\int \mathbb{E}\cdot d\mathbf{a}_0 = \text{const.} \quad \Rightarrow \quad \int \mathbb{E}\cdot d\mathbf{l}dx^0 = \text{const.}$$

Conservation of $\mathbb{E}$ as a one form in $\mathbb{R}^3$

Suppose the fluid flow is irrotational. Then clearly $\mathbb{E} = \nabla\phi$ for some ϕ and vorticity two form becomes $(\mathbb{E}, \boldsymbol{\omega}) = (\nabla\phi, 0)$. Now from (2.24) the conservation of vorticity two form implies

$$\left.\begin{aligned} \partial_t \phi + (\mathbf{u} \cdot \nabla)\phi = 0 \\ \mathbb{E} = \nabla \phi \end{aligned}\right\}$$

Taking the gradient of first equation will lead to

$$\left.\begin{aligned} \partial_t \mathbb{E} + (\mathbf{u} \cdot \nabla)\mathbb{E} + (\mathbb{E} \cdot \nabla)\mathbf{u} = 0 \\ \mathbb{E} = \nabla \phi \end{aligned}\right\} \tag{2.25}$$

But in $\mathbb{R}^3$ any field $\mathbf{S}$ satisfying the equation

$$\partial_t \mathbf{S} + (\mathbf{u} \cdot \nabla)\mathbf{S} + (\mathbf{S} \cdot \nabla)\mathbf{u} + \mathbf{S} \times (\nabla \times \mathbf{u}) = 0$$

is called an invariant one form or S -invariant [80]. Now from (2.25) it follows that $\mathbb{E}$ satisfies an equation of this form. *i.e.*, the field $\mathbb{E}$ is an invariant one form, which implies

$$\int \mathbb{E} \cdot d\mathbf{l} = \text{const.}$$

The invariance of $\mathbb{E}$ is not restricted to the case of irrotational flows only. For example consider the case of a Beltrami flow. Then also from (2.24) we have $\mathbb{E} = \nabla\phi$ and (2.25) takes the form

$$\left.\begin{aligned} \partial_t \mathbb{E} + (\mathbf{u} \cdot \nabla)\mathbb{E} + (\mathbb{E} \cdot \nabla)\mathbf{u} + \mathbb{E} \times (\nabla \times \mathbf{u}) = 0 \\ \mathbb{E} = \nabla \phi \end{aligned}\right\}$$

In this case also $\mathbb{E}$ is an invariant one from in $\mathbb{R}^3$.

Invariant transport of one form

We have for any n-form ω^n, $dL_p\omega^n = L_p d\omega^n$. Thus if the one form $\omega^1 = A_0 dx^0 + A_1 dx^1 + A_2 dx^2 + A_3 dx^3$ is invariantly transported (or Lie transported, *i.e.*, Lie derivative of ω^1

with respect to the vector field p is zero), then $\omega^2 = d\omega^1$ is also Lie transported. The converse is not true always. We will find the condition for the converse to be true.

Now, $L_p\omega^1 = 0$ implies

$$\left.\begin{aligned} \partial_t A_0 + (\mathbf{u}\cdot\nabla)A_0 + \mathbf{u}\cdot\partial_t\mathbf{u} = 0 \\ \partial_t\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \mathbf{u}\times(\nabla\times\mathbf{u}) = 0 \end{aligned}\right\} \tag{2.26}$$

using $\mathbf{A} = \mathbf{u}$. Now $L_p\omega^2 = 0 \quad \Rightarrow \quad i_p\omega^2 = d\phi$.

Then

$$\begin{aligned} L_p\omega^1 &= i_p d\omega^1 + di_p\omega^1 \\ &= i_p\omega^2 + di_p\omega^1 \\ &= d\phi + di_p\omega^1 \\ &= d\psi. \end{aligned}$$

Therefore, for $L_p\omega^1 = 0$, we should have $di_p\omega^1 = -d\phi$, *i.e.*, $i_p\omega^1 = -\phi + c$, where c is a constant. When $p = (1, \mathbf{u})$ this becomes $A_0 + \mathbf{u}^2 = -\phi + c$, *i.e.*, $\phi = A_0 + c$.

Hence, $L_p\omega^2 = 0$ implies $L_p\omega^1 = 0$ only if $i_p\omega^2 = dA_0$.

But $L_p\omega^1 = 0$ implies

$$\int_{C^1} \omega^1 = \text{const.}$$

$$\Rightarrow \qquad \int_{C^1} A_0 dx^0 + \int_{C^1} \mathbf{A}\cdot d\mathbf{l} = \text{const.}$$

Now, if the curve C^1 is a space curve then this becomes

$$\int_{C^1} \mathbf{A}\cdot d\mathbf{l} = \text{const.}$$

Lie transport of helicity three form

We have discussed in the previous sections invariant transport of the one form ω^1 and the vorticity two form ω^2 in E^4. Now we consider the three form $\omega^3 = \omega^1 \wedge \omega^2$ in E^4. In vector notation this can be written as

$$\begin{aligned}(H_0, \mathbf{H}) &= (\mathbf{A} \cdot \boldsymbol{\omega}, A_0\boldsymbol{\omega} + \mathbb{E} \times \mathbf{A}) \\ &= (\mathbf{u} \cdot \boldsymbol{\omega}, A_0\boldsymbol{\omega} + \mathbb{E} \times \mathbf{u}).\end{aligned} \tag{2.27}$$

We see that the component corresponding to $dx^1 \wedge dx^2 \wedge dx^3$ is the usual helicity in $\mathbb{R}^3$. We call ω^3 as the four-helicity corresponding to the vorticity two form ω^2.

Now $L_p\omega^3 = 0$ is equivalent to

$$\left.\begin{aligned}\partial_t H_0 + \nabla \cdot (H_0\mathbf{u}) = 0 \\ \partial_t \mathbf{H} - \nabla \times (\mathbf{u} \times \mathbf{H}) + (\nabla \cdot \mathbf{H})\mathbf{u} + H_0 \partial_t \mathbf{u} = 0\end{aligned}\right\} \tag{2.28}$$

Also $L_p\omega^3 = 0$ implies

$$\int_{C^3} \omega^3 = \text{const.}$$

$$\Rightarrow \qquad \int_{C^3} H_0 dV + \int_{C^3} \mathbf{H} \cdot d\mathbf{V}_0 = \text{ const.}, \tag{2.29}$$

where $dV = dx^1 \wedge dx^2 \wedge dx^3$ is the three dimensional space volume element and $d\mathbf{V}_0 = dx^0 \wedge (dx^2 \wedge dx^3 + dx^3 \wedge dx^1 + dx^1 \wedge dx^2) = dx^0 \wedge d\mathbf{a}$. If C^3 is independent of time coordinates, *i.e.*, C^3 is a space volume, then (2.29) becomes the usual conservation of helicity density $H_0 = \mathbf{u} \cdot \boldsymbol{\omega}$.

$$i.e., \qquad \int_{C^3} \mathbf{u} \cdot \boldsymbol{\omega} dV = \text{const.}$$

Now suppose the helicity density $H_0 = \mathbf{u} \cdot \boldsymbol{\omega} = 0$ for a given flow. Then from (2.28) and (2.29) we get the invariance condition for the vector $\mathbf{H} = A_0\boldsymbol{\omega} + \mathbb{E} \times \mathbf{U}$ as

$$\int_{C^3} \mathbf{H} \cdot da dx^0 = \text{const.}$$

Also in this case (2.28) becomes the single equation

$$\partial_t \mathbf{H} - \nabla \times (\mathbf{u} \times \mathbf{H}) + (\nabla \cdot \mathbf{H})\mathbf{u} = 0 \tag{2.30}$$

This is the condition for the invariance of a two form in $\mathbb{R}^3$. Thus if we take $\mathbf{H}$ as a two form,

$$H_1 dx^2 \wedge dx^3 + H_2 dx^3 \wedge dx^1 + H_3 dx^1 \wedge dx^2,$$

in $\mathbb{R}^3$, then (2.30) implies that $\mathbf{H}$ is Lie transported in $\mathbb{R}^3$. Then, following Tur and Yanovsky [80], we see that $\mathbf{J} = \mathbf{H}/\rho$ is a frozen-in field. Also using the definition of $\mathbf{H}$ as given in (2.27)

$$\mathbf{u} \cdot \mathbf{H} = A_0 H_0 \qquad \Rightarrow \qquad H_0 = \mathbf{u} \cdot \mathbf{H}/A_0$$

i.e., $\mathbf{u} \cdot \mathbf{H}/A_0$ is the usual helicity three form in $\mathbb{R}^3$.

Now, if ω^1 and ω^2 are Lie transported forms then their wedge product ω^3 is also Lie transported, for,

$$\begin{aligned} L_p \omega^3 &= L_p(\omega^1 \wedge \omega^2) \\ &= L_p\omega^1 \wedge \omega^2 + \omega^1 \wedge L_p \omega^2 \\ &= 0. \end{aligned}$$

Thus if ω^2 is Lie transported, then for flows for which $i_p\omega^2 = dA_0$ the four helicity is also Lie transported.

Transport of $\mathbb{E}\cdot\boldsymbol{\omega}$

Let us consider the four form

$$\begin{aligned}\omega^4 &= \omega^2 \wedge \omega^2 \\ &= 2\mathbb{E}\cdot\boldsymbol{\omega} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3.\end{aligned}$$

The invariance condition of this form, $L_p\omega^4 = 0$, implies

$$\int_{C^4} \omega^4 = \text{const.}$$

$$\Rightarrow \qquad \int_{C^4} \mathbb{E}\cdot\boldsymbol{\omega} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 = \text{const.}$$

Also, $L_p\omega^4 = 0$ is equivalent to the vector equation

$$\partial_t(\mathbb{E}\cdot\boldsymbol{\omega}) + \nabla\cdot(\mathbb{E}\cdot\boldsymbol{\omega}\mathbf{u}) = 0. \tag{2.31}$$

This can be considered as the conservation equation for $\mathbb{E}\cdot\boldsymbol{\omega}$ in $\mathbb{R}^3$. Hence by considering a three dimensional space volume C^3 we have

$$\int_{C^3} \mathbb{E}\cdot\boldsymbol{\omega} dx^1 \wedge dx^2 \wedge dx^3 = \text{const.}$$

Thus, by considering the invariance of the four form we obtain an invariant quantity $\mathbb{E}\cdot\boldsymbol{\omega}$.

Also, since $L_p\omega^4 = 2L_p\omega^2 \wedge \omega^2$, we have

$$L_p\omega^2 = 0 \Rightarrow L_p\omega^4 = 0,$$

i.e., when ω^2 is Lie transported, ω^4 is also Lie transported. In fact from (2.22) we can deduce (2.31). But the converse is not true always, *i.e.*, ω^4 can be Lie transported without ω^2 being Lie transported.

In the case of an inviscid incompressible isentropic flow the governing equations are [10]

$$\frac{dP}{dt} = \partial_t P + (\mathbf{u} \cdot \nabla) P = 0$$

and

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \partial_t \mathbf{u} + \nabla(\frac{1}{2}\mathbf{u}^2) + \boldsymbol{\omega} \times \mathbf{u} = -\nabla P$$

where P is the pressure. These two equations can be written as

$$\partial_t \psi + \mathbf{u} \cdot \nabla \psi = 0$$

and

$$\mathbb{E} + \boldsymbol{\omega} \times \mathbf{u} = \nabla \psi,$$

where $\psi = -P$ and $\mathbb{E} = \partial_t \mathbf{u} + \nabla(\frac{1}{2}\mathbf{u}^2)$. Thus, comparing these equations with (2.24) it is clear that the vorticity two form is Lie transported and hence $\mathbb{E} \cdot \boldsymbol{\omega}$ is also Lie transported.

2.6 Discussion

In this chapter we have discussed the topological invariants associated with the vorticity field. We see that the Euler's equation of motion for inviscid barotropic flows can be compared to the ideal form of Ohm's law. We have also discussed about the integral conservation of $\mathbb{E} \cdot \boldsymbol{\omega}$. An isentropic flow gives a good example in which this quantity is an integral invariant.

The geometry of three dimensional space has been used by Tur and Yanovsky to study the invariants in hydrodynamics. There are differential forms of order $0, 1, 2$ and 3 defined on a three dimensional manifold. An equation involving Lie derivative of these forms is taken as the condition for invariants of differential forms. This condition for the invariance has been used in [31] to make it covariant in discussing the invariants of magnetohydrodynamic flows. The covariant form is the Lie derivative in Minkowski space applied to different objects transported by a four velocity.

It has been shown by Mathew and Vedan that a four dimensional Euclidean space representing the space-time continuum can be used in the study of hydrodynamics. This points to the use of a four dimensional manifold and associated differential forms instead of the three dimensional manifold used by Tur and Yanovsky. There are differential forms of order four also associated with this manifold. This concept is used to obtain additional invariants of hydrodynamics.

We start with a one form; its exterior derivative is a closed two form. The invariance condition for this two form leads to the invariance of vorticity field. Our choice of A_0 and $\mathbf{A}$ in the one form leads to the vorticity field tensor corresponding to the electromagnetic field tensor. Further, comparing the equation

$$\rho\frac{d\mathbf{u}}{dt} = \rho(\partial_t\mathbf{u} + \frac{1}{2}\nabla\mathbf{u}^2 - \mathbf{u}\times\boldsymbol{\omega})$$

with the equation of motion of a particle in an electromagnetic field, $\mathbb{E} = \partial_t\mathbf{u} + \frac{1}{2}\nabla\mathbf{u}^2$ corresponds to the electric field intensity [49]. The vorticity field tensor $\mathcal{F}_{\alpha\beta}$ is similar to the electromagnetic field tensor. This again confirms the strong analogy between vorticity field and magnetic field in ideal flows. The hydrodynamics equations are non-relativistic and Galilean invariant. Thus we use only four-dimensional Euclidean space instead of the Minkowski space used by Hornig.

In irrotational or Beltrami flows $\mathbb{E}$ is invariant as a 1-form in $\mathbb{R}^3$. This corresponds to the S-invariants defined by Tur and Yanovsky. Similarly, when helicity is zero, we get a Lie transported two form $\mathbf{H}$ in $\mathbb{R}^3$. Associated with the four form we obtain an invariant quantity $\mathbb{E}\cdot\boldsymbol{\omega}$. This invariant is similar to the invariant $\mathbf{E}\cdot\mathbf{B}$ in magnetohydrodynamic flows. Also we have shown that in the case of an incompressible isentropic flow the vorticity two form is Lie transported in E^4.

Chapter 3

Topological invariants in hydrodynamics

3.1 Introduction

The analogy between vorticity equation of a barotropic flow and the induction equation of a magnetohydrodynamic flow is well known [58]. Vortex lines have properties similar to magnetic lines of force. Thus topological considerations apply to barotropic flows as well as to magnetohydrodynamic flows. Though this has been mentioned in almost all magnetohydrodynamic studies (e.g.[58, 17]), particular attention has not been given to it. In this case it is to be noted that there exists a significant difference between magnetic induction equation and vorticity equation. Vorticity equation is only a particular case of the induction equation, where vorticity is related to the fluid velocity. Thus, corresponding to the results for induction equation we can get results for vorticity also, but there may have results for vorticity that do not have counterparts in the case of magnetic fields [58]. Considering this analogy we can say that the geometrical properties of both flows share same properties, but dynamical properties may differ. Another difference is in the comparative significance of dissipations of magnetic field and vorticity field in non-ideal cases[17].

As mentioned in the introduction, the study of fluid flows in $\mathbb{R}^3$ can be carried out by considering a four dimensional Euclidean space-time manifold E^4. In the previous chapter we have considered the vorticity related invariants using such a manifold. In this chapter we continue our study obtaining topological invariants of hydrodynamic flows by generalising the concept of vorticity related invariants to any general frozen-in vector field.

In the next section we consider some invariants associated with vector fields. The geometry of invariant vector fields in $\mathbb{R}^3$ is discussed in the third section. Then we define a closed differential two form and its potential one form corresponding to a solenoidal vector field which is having an invariance nature. The invariance conditions for different kinds of forms, which are obtained from the closed two form, are given in the subsequent sections. After that some examples are discussed.

3.2 Invariants associated with vector fields

We will discuss in this section conservation of vector field lines and flux conservation of arbitrary vector fields. We give a generalisation of steady Euler flows which admits a family of integral surfaces. An identity involving two frozen-in vector fields is also given.

Topology conservation of vector fields

A necessary and sufficient condition that the flux of an arbitrary vector field $\mathbf{Q}$ through an arbitrary material surface is constant as the motion proceeds is (see [79, 64])

$$\partial_t \mathbf{Q} + (\mathbf{w} \cdot \nabla)\mathbf{Q} - (\mathbf{Q} \cdot \nabla)\ \mathbf{w} + \mathbf{Q}(\nabla \cdot \mathbf{w}) = 0, \tag{3.1}$$

and a necessary and sufficient condition for the vector tubes of $\mathbf{Q}$ to be material tubes is

$$\mathbf{Q} \times [\partial_t \mathbf{Q} + (\mathbf{w} \cdot \nabla)\mathbf{Q} - (\mathbf{Q} \cdot \nabla)\mathbf{w}] = 0. \tag{3.2}$$

In equations (3.1) and (3.2) $\mathbf{w}$ is the generating vector field. The subscript means partial derivative with respect to time. Thus if any vector field $\mathbf{Q}$ satisfies equation (3.1), we say that flux of the field $\mathbf{Q}$ is conserved. From (3.1) and (3.2) it is clear that if a field is flux conserving, then its vector tubes are material tubes.

The evolution of a smooth vector field $\mathbf{Q}$ conserves the topology if a generating vector field $\mathbf{w}$ and a scalar function λ exist so that

$$\partial_t \mathbf{Q} + (\mathbf{w} \cdot \nabla)\mathbf{Q} - (\mathbf{Q} \cdot \nabla)\mathbf{w} = \lambda \mathbf{Q}. \tag{3.3}$$

Topological conservation according to this definition requires the preservation of null points of the field and also the orientation of field lines, but (3.2) need not preserve such properties [36].

In the case of magnetohydrodynamic flows ideal form of Ohm's law

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = 0 \tag{3.4}$$

leads to the induction equation

$$\partial_t \mathbf{B} + (\mathbf{u} \cdot \nabla)\mathbf{B} - (\mathbf{B} \cdot \nabla)\mathbf{u} + (\nabla \cdot \mathbf{u})\mathbf{B} = 0. \tag{3.5}$$

Here $\mathbf{E}$ is the electric field, $\mathbf{B}$ the magnetic field and $\mathbf{u}$ is the flow velocity. This is an equation of the form (3.3), where $\mathbf{w}$ is replaced by $\mathbf{u}$ and $\lambda = -\nabla \cdot \mathbf{u}$. Hence it is clear that the evolution of magnetic field in ideal magnetohydrodynamic flows conserves the flux across any material surface.

The more general form of Ohm's law that satisfies the topology conservation according to (3.3) is given by

$$\mathbf{E} + \mathbf{u} \ \times \mathbf{B} = \mathbf{C}, \tag{3.6}$$

with

$$\nabla \times \mathbf{C} \ = \mu \mathbf{B} = -(\lambda + \nabla \mathbf{u}) \cdot \mathbf{B}.$$

Here we restrict our field $\mathbf{B}$ to be divergence free [36], so that $\nabla \cdot (\lambda + \nabla.\mathbf{u})\mathbf{B} = 0$.

Also there are many other vector fields in ideal fluid flows which satisfy the flux conserving equation (3.1), as noted in [80]. We will discuss some of them in coming sections.

We have already discussed the conservation of vorticity in the case of inviscid barotropic flows, which satisfies the equation (2.3)

$$\partial_t \boldsymbol{\omega} + (\mathbf{u} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\mathbf{u} + (\nabla \cdot \mathbf{u})\boldsymbol{\omega} = 0 \tag{3.7}$$

Taking cross product of equation (3.7) with $\mathbf{u}$ gives the equation

$$\partial_t(\boldsymbol{\omega} \times \mathbf{u}) + (\mathbf{u} \cdot \nabla)(\boldsymbol{\omega} \times \mathbf{u}) + (\boldsymbol{\omega} \times \mathbf{u}) \cdot \nabla \mathbf{u} + (\boldsymbol{\omega} \times \mathbf{u}) \times (\nabla \times \mathbf{u}) + \boldsymbol{\omega} \times \partial_t \mathbf{u} = 0, \tag{3.8}$$

using the identity

$$(\mathbf{A} \times \mathbf{B}) \cdot \nabla \mathbf{C} = \mathbf{B} \times (\mathbf{A} \cdot \nabla)\mathbf{C} - \mathbf{A} \times (\mathbf{B} \cdot \nabla)\mathbf{C} - (\mathbf{A} \times \mathbf{B}) \times \nabla \times \mathbf{C} + (\nabla \cdot \mathbf{C})(\mathbf{A} \times \mathbf{B}).$$

From equations (2.2) and (3.8) it follows that $\boldsymbol{\omega} \times \mathbf{u}$ is a surface invariant if and only if $\boldsymbol{\omega} \times \partial_t \mathbf{u} = 0$. Here instead of vorticity field if we have any other frozen in field, such as magnetic field $\mathbf{B}$ in MHD, this result is true. So, in an inviscid flow if the time derivative of velocity field is parallel to the vorticity field (or to any other frozen in field $\mathbf{Q}$), then the field $\mathbf{u} \times \boldsymbol{\omega}$ (or $\mathbf{u} \times \mathbf{Q}$) is a surface invariant.

Now, if we start with a surface invariant $\mathbf{S}$, then we can show that $\mathbf{S}.\mathbf{u}$ is a Lagrangian invariant if and only if $\mathbf{S} \cdot \partial_t \mathbf{u} = 0$, by taking the dot product of equation(2.2) with $\mathbf{u}$ and using the identity

$$\mathbf{A} \cdot (\mathbf{B} \cdot \nabla)\mathbf{C} = \mathbf{B}[(\mathbf{A} \cdot \nabla)\mathbf{C} + \mathbf{A} \times \nabla \times \mathbf{C}].$$

Similarly it can be shown that, in the case of an invariant density ϕ, $\phi\mathbf{u}$ is a flux conserving filed if and only if the flow is steady.

Integral Surfaces

Consider an incompressible flow. Suppose $\mathbf{Q}$ be an invariant vector field satisfying the equation

$$\partial_t \mathbf{Q} + (\mathbf{u} \cdot \nabla)\mathbf{Q} - (\mathbf{Q} \cdot \nabla)\mathbf{u} = 0 \tag{3.9}$$

Then $\mathbf{Q} \times \mathbf{u}$ is a surface invariant if and only if $\mathbf{Q} \times \partial_t \mathbf{u} = 0$ (from (3.8), replacing $\boldsymbol{\omega}$ by $\mathbf{Q}$). Now the vector $\mathbf{P} = \mathbf{Q} \times \mathbf{u} (\neq 0)$ is both perpendicular to $\mathbf{Q}$ and $\mathbf{u}$.

Now suppose at some instant, say at $t = 0$, we have $\mathbf{P}.\nabla \times \mathbf{P} = 0$ which is the Frobenius integrability condition. Then from the invariance of $\mathbf{P}$ it can be shown that $\mathbf{P} \cdot \nabla \times \mathbf{P} \equiv 0$ for all time. Hence there exist a family of integral surfaces for the field $\mathbf{P}$ at any time. The vectors $\mathbf{Q}$ and $\mathbf{u}$ are on this surface and if they are linearly independent, *i.e.*, $\mathbf{Q} \times \mathbf{u} \neq 0$, then these vectors generates the surface. But, a family of integral surface for the vectors $\mathbf{Q}$ and $\mathbf{u}$ exist if and only if the Lie bracket $[\mathbf{Q}, \mathbf{u}] = \nabla \times (\mathbf{Q} \times \mathbf{u})$ is a linear combination of $\mathbf{Q}$ and $\mathbf{u}$. That is

$$\nabla \times \mathbf{P} = [\mathbf{Q}, \mathbf{u}] = \alpha \mathbf{Q} + \beta \mathbf{u} \tag{3.10}$$

Even though $\mathbf{P} \cdot \nabla \times \mathbf{P} \equiv 0$, the integral surface at different times may be distinct. But the invariance of $\mathbf{P}$ implies the invariance of the integral surfaces also, as the surface orthogonal to $\mathbf{P}$ is frozen([80]) into the medium.

Here the family of integral surfaces $\mathcal{S}$ on which the velocity vector always lies moves invariantly with the flow. Let us take the invariant vector field $\mathbf{Q}$ as the vorticity vector $\boldsymbol{\omega}$. Then from the above discussion we can conclude the following: In an unsteady inviscid barotropic flow if the partial derivative of velocity field with respect to time is parallel to the vorticity field and there exist a family of integral surfaces for $\mathbf{u}$ and $\boldsymbol{\omega}$ initially, then this family of integral surfaces will remain spanned by the field lines of $\mathbf{u}$ and $\boldsymbol{\omega}$. It is known that ([9]) in a steady Euler flow $\mathbf{u} \times \boldsymbol{\omega} = \nabla h$, the vector lines of $\mathbf{u}$ and $\boldsymbol{\omega}$ are lying on the integral surfaces h =const. Thus our result can be considered as a generalisation of steady Euler flows which admits a family of integral surfaces.

An identity involving two frozen-in vector fields

Consider an incompressible fluid flow. Let $\mathbf{P}$ and $\mathbf{Q}$ be two divergence free flux conserving vector fields in $\mathbb{R}^3$ satisfying the equations

$$\partial_t \mathbf{P} + (\mathbf{u} \cdot \nabla)\mathbf{P} - (\mathbf{P} \cdot \nabla)\mathbf{u} = 0 \tag{3.11a}$$

and

$$\partial_t \mathbf{Q} + (\mathbf{u} \cdot \nabla)\mathbf{Q} - (\mathbf{Q} \cdot \nabla)\mathbf{u} = 0 \tag{3.11b}$$

where $\nabla \cdot \mathbf{P} = \nabla \cdot \mathbf{Q} = \nabla \cdot \mathbf{u} = 0$. Now from the above equations it follows that $\mathbf{P}$ and $\mathbf{Q}$ are also frozen-in vector fields. Hence their cross product $\mathbf{S} = \mathbf{P} \times \mathbf{Q}$ corresponds to an invariant one form in $\mathbb{R}^3$ [80]. Thus $\nabla \times \mathbf{S}$ corresponds to an invariant two form, which means that $\nabla \times \mathbf{S}$ satisfies the equation

$$\partial_t(\nabla \times \mathbf{S}) + (\mathbf{u} \cdot \nabla)(\nabla \times \mathbf{S}) - ((\nabla \times \mathbf{S}) \cdot \nabla)\mathbf{u} = 0. \tag{3.12}$$

This equation gives

$$\partial_t \mathbf{S} + (\nabla \times \mathbf{S}) \times \mathbf{u} = \nabla \phi, \tag{3.13}$$

for some scalar function ϕ, which is the equation governing the evolution of the vector field $\mathbf{S}$. Now, taking cross product of (3.11a) with $\mathbf{Q}$ and (3.11b) with $\mathbf{P}$ and adding we get

$$\partial_t(\mathbf{P}\times\mathbf{Q})+\mathbf{P}\times(\mathbf{u}\cdot\nabla)\mathbf{Q}+\mathbf{Q}\times(\mathbf{P}\cdot\nabla)\mathbf{u}-\mathbf{P}\times(\mathbf{Q}\cdot\nabla)\mathbf{u}-\mathbf{Q}\times(\mathbf{u}\cdot\nabla)\mathbf{P}=0. \quad (3.14)$$

Using the identity $(\mathbf{A}\cdot\nabla)(\mathbf{B}\times\mathbf{C})=\mathbf{B}\times(\mathbf{A}\cdot\nabla)\mathbf{C}-\mathbf{C}\times(\mathbf{A}\cdot\nabla)\mathbf{B}$, this becomes

$$\partial_t(\mathbf{P}\times\mathbf{Q})+[\nabla\times(\mathbf{P}\times\mathbf{Q})]\times\mathbf{u}+(\mathbf{u}\cdot\nabla)(\mathbf{P}\times\mathbf{Q})+(\mathbf{P}\cdot\nabla)(\mathbf{Q}\times\mathbf{u})+(\mathbf{Q}\cdot\nabla)(\mathbf{u}\times\mathbf{P})=0;$$

$$\textit{i.e.},\ \partial_t\mathbf{S}+(\nabla\times\mathbf{S})\times\mathbf{u}+(\mathbf{u}\cdot\nabla)(\mathbf{P}\times\mathbf{Q})+(\mathbf{P}\cdot\nabla)(\mathbf{Q}\times\mathbf{u})+(\mathbf{Q}\cdot\nabla)(\mathbf{u}\times\mathbf{P})=0. \quad (3.15)$$

Thus $\mathbf{S}$ satisfies both the equations (3.13) and (3.15). Then it follows that

$$(\mathbf{u}\cdot\nabla)(\mathbf{P}\times\mathbf{Q})+\mathbf{P}\cdot\nabla(\mathbf{Q}\times\mathbf{u})+(\mathbf{Q}\cdot\nabla)(\mathbf{u}\times\mathbf{P})=\nabla\zeta \quad (3.16)$$

for some scalar function ζ.

Suppose $\mathbf{S}=\mathbf{P}\times\mathbf{Q}\neq 0$, then $\mathbf{P}$ and $\mathbf{Q}$ are linearly independent. Assume that $\mathbf{S}\cdot\nabla\times\mathbf{S}\equiv 0$, which means that the helicity density of $\mathbf{S}$ is identically zero for the flow. *i.e.*, $(\mathbf{P}\times\mathbf{Q})\cdot[\mathbf{P},\mathbf{Q}]\equiv 0$, where $[\ ,\ ]$ is the commutator of $\mathbf{P}$ and $\mathbf{Q}$. It follows that $[\mathbf{P},\mathbf{Q}]$ is a linear combination of $\mathbf{P}$ and $\mathbf{Q}$. Two linearly independent vector fields $\mathbf{P}$ and $\mathbf{Q}$ will generate a family of integral surfaces if and only if the commutator product $[\mathbf{P},\mathbf{Q}]$ is a linear combination of $\mathbf{P}$ and $\mathbf{Q}$ [64]. Thus it follows that in such cases $\mathbf{P}$ and $\mathbf{Q}$ are having a family of integral surfaces, which is orthogonal to $\mathbf{P}\times\mathbf{Q}$ and on which $\nabla\times(\mathbf{P}\times\mathbf{Q})=[\mathbf{P},\mathbf{Q}]$ lies. On such a surface

$$\text{const.}=\int_S\nabla\times(\mathbf{P}\times\mathbf{Q})\cdot d\mathbf{S}=\int_S[\mathbf{P},\mathbf{Q}]\cdot d\mathbf{S}=0$$

Let α, β and γ be three Lagrangian invariants in an inviscid incompressible fluid flow. Then it follows that $\nabla\alpha$ and $\nabla\beta$ are surface invariants [80] and hence $\mathbf{A} = \nabla\alpha \times \nabla\beta$ is a divergence-free frozen-in vector field. Since vorticity field is also a frozen-in field, it follows that $\mathbf{A} \times \boldsymbol{\omega}$ satisfies the identity (3.16).

3.3 Geometry of flux conserving fields

Consider the vector equation

$$\partial_t \mathbf{Q} + (\mathbf{u} \cdot \nabla)\mathbf{Q} - (\mathbf{Q} \cdot \nabla)\mathbf{u} + (\nabla \cdot \mathbf{u})\mathbf{Q} = 0 \tag{3.17}$$

where $\mathbf{u}$ is the fluid velocity and $\mathbf{Q}$ is any vector field. Any vector field $\mathbf{Q}$ satisfying this equation conserves the flux across any material surface. For an inviscid fluid there exist a number of vector fields satisfying this condition. As said earlier, most familiar examples are that of vorticity vector $\boldsymbol{\omega}$ and magnetic field $\mathbf{B}$ in the case of ideal flows.

Let S be the entropy density of a compressible adiabatic flow, which is a Lagrangian invariant. Then $\mathbf{A} = S\boldsymbol{\omega}$ is also a flux conserving field. Also it may be noted that here $\nabla \cdot \mathbf{A} = \nabla S \cdot \boldsymbol{\omega}$, which is generally non zero and $\nabla S \cdot \boldsymbol{\omega}/\rho$ is the Ertel invariant. A vector field which satisfies (3.17) can be identified as an invariant two form in three dimensional Euclidean space $\mathbb{R}^3$, as described by Tur and Yanovsky [80]. If our field $\mathbf{Q}$ is a solenoidal field then (3.17) can be written as

$$\partial_t \mathbf{Q} + \nabla \times (\mathbf{Q} \times \mathbf{u}) = 0. \tag{3.18}$$

Now we recall the following result due to Appel [79]: 'Given any family of lines furnished with continuously turning tangents, which in a given flow $\mathbf{u}$ are material lines, there exists a continuously differentiable solenoidal vector field $\mathbf{K}$ satisfying (3.17) and whose vector lines coincide with the given material lines. Since $\mathbf{K}$ is solenoidal there exist a continuously differentiable vector field $\mathbf{w}$ such that $\mathbf{K} = \nabla \times \mathbf{w}$. Also the circulation of $\mathbf{w}$ about any material curve is constant and the vector lines of $\mathbf{K}$ are material lines'.

Now, any smooth vector field $\mathbf{Q}$ which satisfies the equation

$$\partial_t \mathbf{Q} + (\mathbf{u} \cdot \nabla)\mathbf{Q} - (\mathbf{Q} \cdot \nabla)\mathbf{u} = \lambda \mathbf{Q} \tag{3.19}$$

where λ is any scalar field, is said to have field lines which are material lines [79]. Any smooth vector field $\mathbf{Q}$ which satisfies equation (3.17), or more generally equation (3.19), is having material field lines with continuously turning tangents. So by the Appel's result, there exists a continuously differentiable vector field $\mathbf{K}$ having field lines as that of $\mathbf{Q}$, satisfying the condition (3.17), and $\mathbf{K} = \nabla \times \mathbf{w}$, where $\mathbf{w}$ is also a continuously differentiable vector field. We call $\mathbf{K}$ an associated solenoidal vector field of $\mathbf{Q}$ (if $\mathbf{Q}$ itself is solenoidal, then $\mathbf{K} = \mathbf{Q}$). In particular if the field $\mathbf{Q}$ is a frozen-in field (*i.e.*, taking $\lambda = 0$ in (3.19)), then also there exists a flux conserving solenoidal vector field $\mathbf{K}$ satisfying (3.17) and having the field lines as that of given field.

In the above discussion, since the field lines of a given vector field $\mathbf{Q}$ and its associated solenoidal vector field $\mathbf{K}$ are the same, there exists a scalar field f such that $\mathbf{K} = f\mathbf{Q}$. The non vanishing of divergence of $\mathbf{Q}$ implies that the net flux across a closed surface is nonzero and thus there is a source of $\mathbf{Q}$ in that domain. This does not affect the geometry of the field lines, as pointed out by Kida and Takaoka [41] in the discussion of helicity invariants. Thus to study the geometric properties of the above discussed invariant vector fields, or more generally to study the geometric properties of vector fields whose vector lines are material lines, we can restrict ourselves to the associated solenoidal vector field.

Now, if we consider any invariant vector field $\mathbf{Q}$ (which need not be solenoidal), satisfying (3.17), then by Appel's result we can find a solenoidal vector field $\mathbf{K}$, satisfying (3.17), which is again an invariant vector field. Thus now onwards we consider only solenoidal vector fields $\mathbf{Q} = \nabla \times \mathbf{w}$ satisfying the equation (3.18). That is

$$\nabla \times [\partial_t \mathbf{w} + (\mathbf{Q} \times \mathbf{u})] = 0 \tag{3.20}$$

which implies

$$\partial_t \mathbf{w} + \nabla\phi + \mathbf{Q} \times \mathbf{u} = \nabla\psi \tag{3.21}$$

for some scalar functions ϕ and ψ.

If we take $\mathbb{E} = \partial_t \mathbf{w} + \nabla\phi$, then this equation becomes

$$\mathbb{E} + \mathbf{Q} \times \mathbf{u} = \nabla\psi. \tag{3.22}$$

This is the equation governing the invariant evolution of the field $\mathbf{Q}$ under the flow generated by $\mathbf{u}$. Thus for the invariant transport of $\mathbf{Q}$, $\nabla \times (\mathbb{E} + \mathbf{Q} \times \mathbf{u}) = 0$. But in a general flow the equation governing the evolution of $\mathbf{Q}$ (or $\mathbf{w} = \text{curl}^{-1}(\mathbf{Q})$) may not always satisfy (3.18) (or (3.22)); instead the equation may be of the form

$$\nabla \times (\mathbb{E} + \mathbf{Q} \times \mathbf{u}) = \mathbf{D} \tag{3.23}$$

for some vector field $\mathbf{D}$. This term may arise from the physical nature of the flow (for example from the non conservative body forces, dissipative forces etc.). If $\mathbf{D}$ happens to be zero, the condition for the invariance of $\mathbf{Q}$ is obtained. Now if we uncurl this equation, we get the general equation governing the evolution of the field $\mathbf{Q}$ (or the field $\mathbf{w}$) as

$$\mathbb{E} + \mathbf{Q} \times \mathbf{u} = \mathbf{F} \tag{3.24}$$

where $\mathbf{F} = \text{curl}^{-1}(\mathbf{D})$, or equivalently

$$\partial_t \mathbf{w} + \nabla\phi + (\nabla \times \mathbf{w}) \times \mathbf{u} = \mathbf{F}. \tag{3.25}$$

In the next section we shall discuss the geometric properties of such solenoidal vector fields using the language of differential forms.

3.4 Closed two forms and related invariants in E^4

We wish to study the local invariants of differential forms and associated geometrical objects in the four dimensional space-time manifold E^4. Our starting point is an invariant vector field $\mathbf{Q}$ in $\mathbb{R}^3$ satisfying an equation of the form (3.22).

Let us define a one form ω^1 in E^4 as

$$\omega^1 = \varphi dx^0 + w_1 dx^1 + w_2 dx^2 + w_3 dx^3 \tag{3.26}$$

which is represented conveniently as $(\varphi, \mathbf{w})$. Here $\varphi = -\phi$ and $\mathbf{w} = (w_1, w_2, w_3)$ are supposed to satisfy (3.25). Now taking the exterior derivative of ω^1 we get a closed two form ω^2,

$$\omega^2 = dx^0 \wedge (\mathbb{E}_1 dx^1 + \mathbb{E}_2 dx^2 + \mathbb{E}_3 dx^3) + Q_1 dx^2 \wedge dx^3 + Q_2 dx^3 \wedge dx^1 + Q_3 dx^1 \wedge dx^2 \tag{3.27}$$

where $(\mathbb{E}, \mathbf{Q}) = (\partial_t \mathbf{w} - \nabla\varphi, \nabla \times \mathbf{w})$. We have $\nabla \times \mathbb{E} = \partial_t \mathbf{Q}$ and $\nabla \cdot \mathbf{Q} = 0$, which is the exact consequences of closedness of the two form ω^2. In tensor notation ω^2 is $\mathfrak{F}_{\alpha\beta} dx^\alpha \wedge dx^\beta$, where $\mathfrak{F}_{\alpha\beta} = \partial_{x^\alpha} w_\beta - \partial_{x^\beta} w_\alpha$ is a skew symmetric tensor of rank two (here $\alpha, \beta = 0, 1, 2, 3$ and $w_0 = \phi$).

Using this definition of two form ω^2 we will investigate the invariants in hydrodynamic flows.

Invariant transport of two form

The invariance condition for the two form in E^4 is $L_p\omega^2 = 0$, where $p = (1, \mathbf{u})$. The equivalent expression for this equation in vector notation is

$$\partial_t(\mathbb{E} + \mathbf{Q} \times \mathbf{u}) + \nabla(\mathbb{E} \cdot \mathbf{u}) = 0 \tag{3.28a}$$

and

$$\nabla \times (\mathbb{E} + \mathbf{Q} \times \mathbf{u}) = 0. \tag{3.28b}$$

But as ω^2 is a closed two form and $L_p\omega^2 = di_p\omega^2 + i_p d\omega^2$, we have $i_p\omega^2 = d\psi$ for the invariance of ω^2. This is equivalent to

$$\partial_t\psi + (\mathbf{u} \cdot \nabla)\psi = 0 \tag{3.29a}$$

and

$$\mathbb{E} + \mathbf{Q} \times \mathbf{u} = \nabla\psi. \tag{3.29b}$$

From (3.29a) it is clear that ψ is a Lie transported zero form. Equation (3.29b) is the equation governing the invariant evolution of the field $\mathbf{Q}$ under the flow $\mathbf{u}$ (also see (3.22)). We can consider (3.29) as the set of equations which governs the invariant transport of ω^2 in E^4, under the flow generated by $(1, \mathbf{u})$. Taking gradient of (3.29a) and using (3.29b) we obtain (3.28a) and taking curl of (3.29b) gives (3.28b). The study of invariance of the two form ω^2 in E^4 also includes the notion of invariance of $\mathbb{E}$ and $\mathbf{Q}$ simultaneously.

Now $L_p\omega^2 = 0$

$$\Rightarrow \int_{C^2} \omega^2 = \text{const.} \quad \Leftrightarrow \quad \int_{C^2} \mathbf{Q} \cdot d\mathbf{a} - \int_{C^2} \mathbb{E} \cdot d\mathbf{l}dt = \text{const.},$$

where $d\mathbf{a} = (dx^2 \wedge dx^3, dx^3 \wedge dx^1, dx^1 \wedge dx^2)$, $d\mathbf{l} = (dx^1, dx^2, dx^3)$ and the integration is over any comoving two dimensional surface C^2 in E^4. If the surface under consideration is space like then we have $\int \mathbf{Q} \cdot d\mathbf{a} = \text{const}$. On the other hand if we start with a plane spanned by a $\mathbf{Q}$ line and the time axis, the first term vanishes so that $\int \mathbb{E} \cdot dldt = \text{const}$.

Conservation of $\mathbb{E}$ as a one form in $\mathbb{R}^3$

In a flow, it may happen that the vector field $\mathbf{w}$ is a potential field. Then $\mathbf{Q} = \nabla \times \mathbf{w} = 0$ and $\mathbb{E} = \nabla\psi$ and (3.29) becomes

$$\partial_t \mathbb{E} + (\mathbf{u} \cdot \nabla)\mathbb{E} + (\mathbb{E} \cdot \nabla)\mathbf{u} + \mathbb{E} \times (\nabla \times \mathbf{u}) = 0 \tag{3.30a}$$

and

$$\mathbb{E} = \nabla\psi. \tag{3.30b}$$

So it follows that $\mathbb{E}$ satisfies an equation of the form (2.2). That is the field $\mathbb{E}$ can be considered as an invariant one form

$$\omega^1_{\mathbb{E}} = \mathbb{E}_1 dx^1 + \mathbb{E}_2 dx^2 + \mathbb{E}_3 dx^3$$

in $\mathbb{R}^3$. Thus (3.30) implies

$$\int \mathbb{E} \cdot d\mathbf{l} = \text{const.},$$

Here note that the fluid flow need not be irrotational, in contrast to the corresponding result in the case of vorticity two form.

It may also happen that $\mathbf{w} \times \mathbf{Q} \equiv 0$ without being $\mathbf{Q} = 0$. Then also from (3.29b) it follows that $\mathbb{E}$ satisfies an equation of the form (2.2). So here also $\omega^1_{\mathbb{E}}$ is an invariant form.

Invariance of ω^1

Now we consider the one form $\omega^1 = \varphi dx^0 + w_1 dx^1 + w_2 dx^2 + w_3 dx^3$. Here, since $\omega^2 = d\omega^1$, Lie transport of ω^1 implies that of ω^2. Now $L_p\omega^1 = 0$ implies

$$\partial_t\varphi + (\mathbf{u}\cdot\nabla)\varphi + \mathbf{w}\cdot\partial_t\mathbf{u} = 0 \tag{3.31a}$$

and

$$\partial_t\mathbf{w} + (\mathbf{u}\cdot\nabla)\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{u} + \mathbf{w}\times(\nabla\times\mathbf{u}) = 0. \tag{3.31b}$$

In terms of integrals this means

$$\int_{C^1}\omega^1 = \text{const.} \Leftrightarrow \int_{C^1}\mathbf{w}\cdot d\mathbf{l} + \int_{C^1}\varphi dt = \text{const.}$$

If the curve C^1 is a space curve , then this becomes $\int_{C^1}\mathbf{w}\cdot d\mathbf{l} = \text{const.}$

Since $d\omega^1 = \omega^2$, if ω^2 is an integral invariant then ω^1 is said to be relative integral invariant. Now the question is that when does the invariance of ω^2 imply the invariance of ω^1?

We have $L_p\omega^2 = 0 \Rightarrow i_p\omega^2 = d\psi$ for some scalar function ψ. This implies $L_p\omega^1 = d\chi$, for some scalar function $\chi = \psi + i_p\omega^1$. Thus for the Lie transport of ω^1 we should have

$$di_p\omega^1 = -d\psi \Rightarrow \psi = c - \varphi - \mathbf{u}\cdot\mathbf{w} \tag{3.32}$$

where c is some arbitrary constant. But for arbitrary choices of ψ this is not satisfied. Thus generally it is not always possible to find a Lie transported one form corresponding to a Lie transported two form (see section 3.6).

But in our actual situation, if we are giving more importance to the two form than the one form ω^1 in such a way that the transformation $\omega^1 \to \omega^1 + d\gamma$ does not affect the physical model in a significant way, then it may be possible to make a choice of the scalar function γ so that the condition $\psi = c - i_p(\omega^1 + d\gamma)$ is satisfied for an arbitrary

ψ. Such a transformation is considered as a gauge transformation. Also in some cases it may be possible to choose φ $(= -\phi)$ satisfying (3.25) as an arbitrary scalar function as we wish (in all cases this is not possible, which is discussed in section 3.6). In such cases a choice of φ as $\varphi = \psi + \mathbf{u}\cdot\mathbf{w} - c$ is possible so that ω^1 is an invariant one form. Thus in such cases the Lie transport of ω^2 implies the Lie transport of the corresponding one form.

Invariance of three form in E^4

Using the one form ω^1 and the two form ω^2 defined earlier we can define a three form $\omega^3 = \omega^1 \wedge \omega^2$. If both ω^1 and ω^2 are Lie transported then ω^3 is also Lie transported. Thus if ω^2 is given to be Lie transported, then in some special cases where the invariance of ω^2 implies the invariance of the corresponding one form, ω^3 is also Lie transported.

Here ω^3 can be represented in vector notation as $\mathcal{H} = (\mathbf{H}, H_\mathbf{Q})$, where $H_\mathbf{Q} = \mathbf{w}\cdot\mathbf{Q}$ (which corresponds to the coefficient of spacial volume element) and $\mathbf{H} = \varphi\mathbf{Q} + \mathbb{E}\times\mathbf{w}$ (corresponding to remaining terms). The component $H_\mathbf{Q}$ corresponding to $dx^1 \wedge dx^2 \wedge dx^3$ is the helicity density of the vector field $\mathbf{w}$ in $\mathbb{R}^3$. Hence $\mathcal{H}$ can be considered to be the generalized helicity density in E^4.

Now $L_p\omega^3 = 0$ is equivalent to the vector equations

$$\partial_t\mathbf{H} + \nabla\times(\mathbf{H}\times\mathbf{u}) + (\nabla\cdot\mathbf{H})\mathbf{u} + H_\mathbf{Q}\partial_t\mathbf{u} = 0 \tag{3.33a}$$

and

$$\partial_t H_\mathbf{Q} + \nabla\cdot(H_\mathbf{Q}\mathbf{u}) = 0. \tag{3.33b}$$

Equation (3.33b) gives the conservation law for the helicity density $\mathbf{w}\cdot\mathbf{Q}$ in $\mathbb{R}^3$.

The Lie transport of three form ω^3 gives the integral invariant

$$\int_{C^3} \omega^3 = \text{const.}$$

where C^3 is any three dimensional volume. If the volume C^3 is a space volume, then this means that

$$\int_{C^3} H_{\mathbf{Q}} dv = \text{const.} \Rightarrow \int \mathbf{w} \cdot \mathbf{Q} = \text{const.}$$

where $dv = dx^1 \wedge dx^2 \wedge dx^3$. From the definition of $\mathbf{H}$, the helicity density can be represented in terms of $\mathbf{H}$ as $H_{\mathbf{Q}} = \mathbf{w} \cdot \mathbf{H}/\varphi$.

Now if $H_{\mathbf{Q}}$ is identically zero for a certain flow then the invariance of ω^3 gives

$$\partial_t \mathbf{H} + \nabla \times (\mathbf{H} \times \mathbf{u}) + (\nabla \cdot \mathbf{H})\mathbf{u} = 0.$$

But this is the condition for the invariance of a two form in $\mathbb{R}^3$. Thus if we take $\mathbf{H}$ as a two form $H_1 dx^2 \wedge dx^3 + H_2 dx^3 \wedge dx^1 + H_3 dx^1 \wedge dx^2$ in $\mathbb{R}^3$, the above equation implies the invariance of $\mathbf{H}$ as a two form in $\mathbb{R}^3$. Then it follows that $\mathbf{H}/\rho$ is a frozen in field.

Invariant transport of four form

Now from the two form ω^2 we can define a four form $\omega^4 = \omega^2 \wedge \omega^2$. That is

$$\omega^4 = 2\mathbb{E} \cdot \mathbf{Q} dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3.$$

If ω^2 is an invariant two form, then ω^4 is clearly invariant but not conversely. $L_p \omega^4 = 0$ is equivalent to

$$\partial_t(\mathbb{E} \cdot \mathbf{Q}) + \nabla \cdot (\mathbb{E} \cdot \mathbf{Q}\mathbf{u}) = 0. \qquad (3.34)$$

This is the equation of continuity for the scalar field $\mathbb{E} \cdot \mathbf{Q}$ in E^4. This same equation can be identified to be the continuity equation for the three form $\mathbb{E} \cdot \mathbf{Q} dx^1 \wedge dx^2 \wedge dx^3$ in $\mathbb{R}^3$. The invariant transport of ω^4 gives the the integral invariant

$$\int_{C^4} \omega^4 = \text{const.}$$

Since $\mathbb{E} \cdot \mathbf{Q}$ satisfies (3.34), it is also clear that

$$\int \mathbb{E} \cdot \mathbf{Q} = \text{const.}$$

over any three dimensional space volume.

So far we have discussed the local invariance of differential forms in E^4 associated with a vector field $\mathbf{Q}$ in $\mathbb{R}^3$ and its vector potential $\mathbf{w}$. One of the most important example is that of vorticity two form, which we have discussed in detail in the previous chapter. Now we will see some other important examples which illustrate the concepts discussed in previous sections.

3.5 Examples

Magnetohydrodynamics

In a magnetohydrodynamic flow the induction equation is given by

$$\partial_t \mathbf{B} + \nabla \times (\mathbf{B} \times \mathbf{u}) = \eta \nabla^2 \mathbf{B} \tag{3.35}$$

which corresponds to (3.23). This equation is obtained by taking the curl of Ohm's law

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \eta \mathbf{J} \tag{3.36}$$

where $\mathbf{B}$ is the magnetic field, $\mathbf{J}$ is the electric current density, $\mathbf{E}$ is the electric field and η is the magnetic diffusivity of the fluid. (Note that we have written (3.24) in analogy with the Ohm's law. Also note that according to our definition of $\mathbb{E}$, the electric field $\mathbf{E}$ and $\mathbb{E}$ are having opposite sign). In the case of ideal Ohm's law or in the slightly non-ideal case $\mathbf{E} + \mathbf{u} \times \mathbf{B} = \nabla \psi$, the induction equation is given by

$$\partial_t \mathbf{B} + \nabla \times (\mathbf{B} \times \mathbf{u}) = 0$$

which conserves the magnetic flux across any material surface. Now in terms of vector potential $\mathbf{A}$ of $\mathbf{B}$ we can write the Ohm's law as

$$\partial_t \mathbf{A} - \nabla\varphi + [(\nabla \times \mathbf{A}) \times \mathbf{u}] = \mathbf{F}$$

where $\mathbf{F} = \eta\nabla \times (\nabla \times \mathbf{A}) = \eta\mathbf{J}$.

Then we can define a one form, the electromagnetic potential, as

$$\omega^1 = \varphi dx^0 + A_1 dx^1 + A_2 dx^2 + A_3 dx^3 \tag{3.37}$$

which gives on exterior differentiation the electromagnetic two form

$$\omega^2 = dx^0 \wedge (E_1 dx^1 + E_2 dx^2 + E_3 dx^3) + B_1 dx^2 \wedge dx^3 + B_2 dx^3 \wedge dx^1 + B_3 dx^1 \wedge dx^2 \tag{3.38}$$

Here we are free to choose the potential ω^1 ($\mathcal{A} = (\varphi, \mathbf{A})$) up to a gauge transformation $\omega^1 \rightarrow \omega^1 + d\gamma$, since the physical quantities we are interested in are the magnetic field and electric field and not their potential $\mathcal{A}$, which is not a measurable quantity. Thus the guage transformation does not affect the physical model. Hence in this case if $L_p\omega^2 = 0$, then we can choose a gauge potential satisfying some initial conditions so that the one form ω^1 is also Lie transported [31]. In magnetohydrodynamic case (3.29b) corresponds to the Ohm's law which may be slightly non-ideal, but conserves magnetic flux.

Since the Lie transport of ω^2 implies that of ω^1 with a suitable gauge, it also implies the Lie transport of generalized helicity three form $\omega^3 = \omega^1 \wedge \omega^2$. This also includes the conservation of magnetic helicity $H_0 = \mathbf{A} \cdot \mathbf{B}$. Corresponding to the four form we obtain the classical invariant of electromagnetic field $\mathbf{E} \cdot \mathbf{B}$. Even though our approach is non-relativistic this can be easily extended to the case of relativistic MHD as well [31].

Non-barotropic flows

Here we will discuss the invariance of the generalised vorticity in a non-barotropic perfect fluid flow [37]. For a non-barotropic perfect fluid flow, there comes into picture a function λ of entropy and temperature satisfying $\mathrm{d}\lambda/\mathrm{d}t = 0$. Assuming the existence of such a function $\lambda = \lambda(S, T)$ we introduce a function I such that $dI = Td\lambda + VdP$. This amounts to replacing the state variable entropy S in terms of λ and T assuming that

$$\left(\frac{\partial T}{\partial P}\right)_\lambda = \left(\frac{\partial V}{\partial \lambda}\right)_P,$$

where T the absolute temperature, V the specific volume and P the pressure. We are also having the following thermodynamic relation to describe the non-barotropic flow $\nabla P/\rho = \nabla I - T\nabla\lambda$, where $\lambda = \lambda(S, T)$ with $d\lambda/dt = 0$. Also we have the relation

$$\nabla\lambda = \frac{\partial\lambda}{\partial S}\nabla S + \frac{\partial\lambda}{\partial T}\nabla T.$$

The conservation of momentum now takes the form

$$\frac{\mathrm{D}\mathbf{u}}{\mathrm{D}t} + \nabla(I + \delta) - T\nabla\lambda = 0 \tag{3.39}$$

where $\mathbf{u}$ is the velocity field and δ is the potential of conservative body forces. Let η be the thermasy, which is the time integral of temperature defined by $\mathrm{d}\eta/\mathrm{d}t = T$, $\eta = 0$ at $t = 0$. We can now define a barotropic flow as the one in which $\nabla\eta \times \nabla\lambda = 0$.

Now it can be shown that [37] for a non-barotropic flow $\mathbf{Q} = \boldsymbol{\omega} - \nabla\eta \times \nabla\lambda$ satisfies the equation

$$\frac{d\mathbf{Q}}{dt} = (\mathbf{Q} \cdot \nabla)\mathbf{u} - \mathbf{Q}(\nabla \cdot \mathbf{u}) \tag{3.40}$$

Here we take $\mathbf{w} = (\mathbf{u} - \eta\nabla\lambda)$. Then from (3.40) we obtain the equation governing the motion of the field $\mathbf{w}$ as

$$\partial_t \mathbf{w} + \nabla\phi + (\mathbf{Q} \times \mathbf{u}) = \nabla\psi \tag{3.41}$$

for some scalar function ϕ and ψ. That is

$$\mathbb{E} + (\mathbf{Q} \times \mathbf{u}) = \nabla\psi$$

where $\mathbb{E} = \partial_t(\mathbf{u} - \eta\nabla\lambda) + \nabla\phi$. Now, from (3.39), after some known substitutions and rearrangements of the terms, we get

$$\partial_t(\mathbf{u} - \eta\nabla\lambda) + [\nabla \times (\mathbf{u} - \eta\nabla\lambda)] \times \mathbf{u} + \nabla(\frac{1}{2}\mathbf{u}^2 + I + \delta + \eta\partial_t\lambda) = 0 \tag{3.42}$$

From (3.41) and (3.42) we see that $\phi = \psi + \frac{1}{2}\mathbf{u}^2 + I + \delta + \eta\partial_t\lambda$. As for the general case here we can define one form, two form, three form and four form in E^4. Also the invariance of each forms can be obtained.

Two fluid model

Consider a two fluid model for the macroscopic dynamics of plasmas. The generating equations are [80, 84]

$$\partial_t \mathbf{v} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{e}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) - \frac{1}{mn}\nabla P_e \tag{3.43a}$$

and

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} = \frac{e}{M}(\mathbf{E} + \mathbf{u} \times \mathbf{B}) - \frac{1}{Mn}\nabla P_i \tag{3.43b}$$

where $\mathbf{v}$ and $\mathbf{u}$ are electron and ion velocities respectively. These equations can be written again as

$$\partial_t(\mathbf{v} - \frac{e}{m}\mathbf{A}) + \nabla(\frac{1}{2}\mathbf{v}^2 - \frac{e}{m}\phi) + (\nabla \times \mathbf{v} - \frac{e}{m}\mathbf{B}) \times \mathbf{v} = \nabla\psi_e \tag{3.44a}$$

and

$$\partial_t(\mathbf{u} + \frac{e}{M}\mathbf{A}) + \nabla(\frac{1}{2}\mathbf{u}^2 + \frac{e}{M}\phi) + (\nabla \times \mathbf{u} + \frac{e}{M}\mathbf{B}) \times \mathbf{u} = \nabla\psi_i \tag{3.44b}$$

where electric field $\mathbf{E} = \partial_t\mathbf{A} - \nabla\phi$, $\psi_e = P_e/mn$, $\psi_i = P_i/Mn$ and $n = const.$ Now, taking the curl of these equations, we obtain the conservation of

$$\mathbf{Q_e} = (\nabla \times \mathbf{v} - \frac{e}{m}\mathbf{B}) = \nabla \times \mathbf{w_e}, \quad \text{where} \;\; \mathbf{w_e} = \mathbf{v} - \frac{e}{m}\mathbf{A}$$

and

$$\mathbf{Q_i} = (\nabla \times \mathbf{u} + \frac{e}{M}\mathbf{B}) = \nabla \times \mathbf{w_i}, \quad \text{where} \;\; \mathbf{w_i} = \mathbf{u} + \frac{e}{M}\mathbf{A}.$$

Here we can define one forms corresponding to each of these equations as $\omega_e^1 = (-\frac{1}{2}\mathbf{v}^2 + \frac{e}{m}\phi, \mathbf{v} - \frac{e}{m}\mathbf{A})$ and $\omega_i^1 = (-\frac{1}{2}\mathbf{u}^2 - \frac{e}{M}\phi, \mathbf{u} + \frac{e}{M}\mathbf{A})$. From the gauge invariance of electromagnetic field it is possible to obtain the gauge invariance of the above one forms. Thus, in the case of two fluid plasmas the Lie transport of two forms implies the Lie transport of one forms in suitable guages. Thus the invariance of two forms leads to the invariance of one forms, three forms and four forms. Here the spatial components of four helicity density is $(\mathbf{v} - \frac{e}{m}\mathbf{A}) \cdot (\nabla \times \mathbf{v} - \frac{e}{m}\mathbf{B})$ for electron velocity and similarly for ion velocity. The invariant associated with the four form is given by $\mathbb{E}_e \cdot \mathbf{Q_e}$, for electron velocity, where $\mathbb{E}_e = \partial_t(\mathbf{v} - \frac{e}{m}\mathbf{A}) + \nabla(\frac{1}{2}\mathbf{v}^2 - \frac{e}{m}\phi)$ and similarly for ion velocity.

An example using Lagrangian invariants

So far we have considered the examples involving solenoidal vector fields only. Now we will construct an example of an invariant vector field which is not solenoidal and will find out its associated invariant solenoidal vector field. For this, consider three Lagrangian invariants ϕ, ψ and χ, where $\chi \neq 0$ in $\mathbb{R}^3$. Then $\nabla\phi$ and $\psi\nabla\phi$ are invariant one forms in $\mathbb{R}^3$. Also the vector field $\mathbf{Q} = \nabla\chi \times (\psi\nabla\phi)$ is an invariant two form. That is, it satisfies (3.17).

Now $\nabla \cdot \mathbf{Q} = \nabla\psi \cdot (\nabla\chi \times \nabla\phi) \neq 0$ in general. Put $\mathbf{C} = \mathbf{Q}/\psi$, then $\nabla \cdot \mathbf{C} = 0$ and $\mathbf{C}$ also satisfies (3.17) and hence (3.18)

Here, $\mathbf{C}$ is a solenoidal vector field having field lines as that of $\mathbf{Q}$. Thus corresponding to an invariant vector field $\mathbf{Q}$, which is not solenoidal, we are able to obtain an invariant solenoidal vector field $\mathbf{C} = \mathbf{Q}/\psi$. Uncurling the equation satisfied by $\mathbf{C}$ we obtain the evolution equation for the vector potential $\mathbf{w} = \chi\nabla\phi$ as

$$\partial_t \mathbf{w} + \nabla\zeta + [(\nabla \times \mathbf{w}) \times \mathbf{u}] = \nabla\eta.$$

As for the general case here we can define one form, two form, three form and four form in E^4 corresponding to the field $\mathbf{C}$. Also the invariance of each forms can be obtained.

Polarization

Again, in the case of MHD, we may introduce a dimensionless quantity called the polarization $\mathbf{P}$ which is defined by [16, 81]

$$\mathbf{J} = \partial_t \mathbf{P} + \nabla \times (\mathbf{P} \times \mathbf{u}), \nabla \cdot \mathbf{P} = 0$$

where $\mathbf{J}$ is the current density. Then the equation of motion for the barotropic inviscid flows becomes

$$\partial_t \mathbf{Q} + \nabla \times (\mathbf{Q} \times \mathbf{u}) = 0$$

where $\mathbf{Q} = \nabla \times (\mathbf{u} + (\mathbf{B} \times \mathbf{P})/\rho)$. Here also we can define different forms as in the case of hydrodynamics.

3.6 Comparison between hydrodynamics and MHD

In $\mathbb{R}^3$ the equation satisfied by vorticity and induction equation satisfied by magnetic field are having the same form [59, 58, 17]. We have defined a vorticity two form for a hydrodynamic flow, given by (2.20). This is a closed two form and corresponding one form is given by (2.19). Also we have

$$d\omega^2 = 0 \Rightarrow \nabla \times \mathbb{E} = \partial_t \boldsymbol{\omega} \text{ and } \nabla \cdot \boldsymbol{\omega} = 0.$$

Now consider the density four form $\omega^4 = \rho dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3$. Using this we define a three form $\omega^3_{HD} = i_p \omega^4 = (-\rho \mathbf{u}, \rho)$. Then the equation of continuity is given by $d\omega^3 = 0$. (This is clearly due to the fact that ω^4 is Lie transported gives the equation of continuity).

In MHD we are having the closed electromagnetic two form (3.38) and the potential one form (3.37). Then $d\omega^2 = 0$ gives half of the set of Maxwell's equations. Here also we are having a three form $\omega^3_{MHD} = (-\mathbf{J}, \rho')$, where $\mathbf{J}$ is the electric current density [22]. $d\omega^3_{MHD} = 0$ gives the equation of continuity of charge $\partial_t \rho + \nabla \cdot \mathbf{J} = 0$.

Thus in hydrodynamics and MHD we are having similar skew symmetric tensors of rank two. Using this two form and its potential one form we can define a three form $\omega^1 \wedge \omega^2$ and a four form $d(\omega^1 \wedge \omega^2)$. In $\mathbb{R}^3$ the comparison between hydrodynamics and MHD can be seen as

$$\mathbf{u} \to \mathbf{A}, \quad \boldsymbol{\omega} \to \mathbf{B}, \quad \mathcal{F}(= \nabla \times \boldsymbol{\omega}) \to \mathbf{J}(= \nabla \times \mathbf{B})$$

Thus the topological information in both type of flows have some similarities, which is useful for further investigation.

When we consider the electromagnetic field two form and the corresponding one form in the context of magnetohydrodynamics, the electromagnetic field two form is independent of gauge transformation. In the case of hydrodynamics the vorticity field tensor $\omega^2 = d\omega^1$ is also invariant under a gauge transformation. But here the components of the one form ω^1 is directly related to a physical quantity, the velocity of fluid flow. Hence a gauge transformation of the one form changes the flow field and hence there is a change in the equation of motion also. Thus unlike electromagnetic field and its potential, the Lie transport of vorticity field two form does not imply the Lie transport of its potential, the one form ω^1. Now consider the equation of motion given in the form (see second equation of (2.24))

$$\mathbb{E} + \boldsymbol{\omega} \times \mathbf{u} = \nabla\phi$$

and the Ohm's law corresponding to the invariant transport of electromagnetic field in the non relativistic case [31]

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \nabla\zeta$$

where $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field and ζ is some scalar function. These two are the fundamental equations governing the invariant evolution of the vorticity field and magnetic field respectively. Taking the curl of these equations we obtain the well known vorticity equation and induction equation respectively. Both of these equations have the same form with the only difference that the first equation is nonlinear for the evolution of $\boldsymbol{\omega}$ as it is the curl of $\mathbf{u}$, but the second one is linear in $\mathbf{B}$. The first equation involving $\mathbf{u}$ and $\boldsymbol{\omega}$ is not invariant under a gauge transformation, but the second equation is gauge invariant. Therefore, we cannot always find properties of $\boldsymbol{\omega}$ corresponding to the properties of magnetic field $\mathbf{B}$. Hence, even though we can find a Lie transported one form of potentials corresponding to a Lie transported electromagnetic two form [31], it is not physically possible always to find a Lie transported one form of potentials corresponding to a vorticity two form.

3.7 Discussion

In this paper we have discussed the invariants of fluid flows using a four dimensional space-time continuum E^4 and using the language of differential forms. The knowledge of invariants provides an effective way for studying the qualitative behavior of the system. We have used E^4 since it is more convenient and it simplifies the way in identifying different kinds of invariants and finding relation between them. From known invariants we can derive an infinite number of new invariants using the relationships among differential forms and vector fields.

Even though we have used a space-time manifold E^4 we have not assumed any specific metric structure defined on it. One can easily extend the discussion in the context of MHD to relativistic case also using Lorentz metric. If a metric structure is defined, then there is a one to one correspondence between one forms and vector fields. In [9], the Lie group formed by the volume preserving diffeomorphisms on a domain M has been used to study the geometric properties of the fluid flow in that domain. This group can be considered as the infinite dimensional configuration space of an incompressible inviscid fluid filling the domain on which the kinetic energy defines a Riemannian metric (see [9] and references there in). In contrast to this approach, we have considered the four dimensional space-time manifold for obtaining invariants of the fluid flow in $\mathbb{R}^3$.

The induction equation in magnetohydrodynamic flows and vorticity equation for incompressible barotropic flows are having the same form. Therefore when we discuss the topological properties of magnetic field and vorticity field, the study of topological properties of any one of these fields will give some insight into the topological properties of the other field.

We have shown that in the case of a frozen in invariant $\mathbf{Q}$, the vector field $\mathbf{Q} \times \mathbf{u}$ is a surface invariant if and only if the time derivative of velocity field is parallel to the vector field $\mathbf{Q}$. Similar results for other invariants are also given.

suppose for an ideal incompressible flow $\boldsymbol{\omega} \times \partial_t \mathbf{u} = 0$, then we are obtaining a family of integral surfaces on which the vector lines of $\mathbf{u}$ and $\boldsymbol{\omega}$ lie, given that at some instant $\mathbf{P} \cdot \nabla \times \mathbf{P} = 0$ where $\mathbf{P} = \boldsymbol{\omega} \times \mathbf{u}$. This result can be considered as a generalisation of steady Euler flows, which admits a family of integral surfaces, to the unsteady flows. We have also obtained an identity satisfied by the vector field $\mathbf{P} \times \mathbf{Q}$, where $\mathbf{P}$ and $\mathbf{Q}$ are two divergence free frozen-in vector fields satisfying (3.11) in an incompressible ideal flow.

The condition for constant flux in $\mathbb{R}^3$ for a vector field $\mathbf{Q}$ in a flow with velocity $\mathbf{u}$ is given by $\partial_t \omega^2 + L_{\mathbf{u}} \omega^2 = 0$, where $\omega^2 = Q_1 dx^2 \wedge dx^3 + Q_2 dx^3 \wedge dx^1 + Q_3 dx^1 \wedge dx^2$ is the two form associated with the vector field $\mathbf{Q}$. Corresponding to this we find another vector field $\mathbf{K}$ which is solenoidal and having the same field lines as that of $\mathbf{Q}$. Equation (3.22) becomes Ohm's law in the case of MHD (with a change of sign for the field $\mathbb{E}$). In hydrodynamics it is the Euler's equation of motion.

Using this $\mathbf{K}$ and its vector potential we define a closed two form in E^4. The solenoidal property of the field is necessary for the closedness of two form . For this two form in E^4 we have obtained an invariant criterion using Lie derivatives. From this closed two form we get a potential one form and we discuss its invariance also. Using this one form we define a sequence of forms

$$\left\{\omega^1,\ \omega^2 = d\omega^1,\ \omega^3 = \omega^1 \wedge d\omega^1,\ \omega^4 = d\omega^1 \wedge d\omega^1\right\}$$

Here ω^4 is the exterior derivative of ω^3. We have also discussed the invariance of ω^3 and ω^4. If the one form is Lie transported, then all other forms derived from this one are also Lie transported.

Now if we are having a given set of invariants we can find a larger collection of invariants by making wedge products, exterior derivatives or any other operations as described in section 2.3. Thus if $\alpha = (\mathbf{P}, \mathbf{Q})$ and $\beta = (\mathbf{P}', \mathbf{Q}')$ are invariant two forms, then $\omega^4 = \alpha \wedge \beta = (\mathbf{P} \cdot \mathbf{Q}' + \mathbf{P}' \cdot \mathbf{Q})$ is an invariant four form.

If the fluid is incompressible the invariance conditions for a zero form and a four form coincide and it is given by the equation

$$\partial_t I + (\mathbf{u} \cdot \nabla) I = 0$$

Hence in this case an invariant zero form can be used to define an invariant four form and vice versa. Thus we have the following cycle of invariant forms.

$$\boxed{\text{0-forms}} \rightarrow \boxed{\text{1-forms}} \rightarrow \boxed{\text{2-forms}} \rightarrow \boxed{\text{3-forms}} \rightarrow \boxed{\text{4-forms}} \rightarrow \boxed{\text{0-forms}}$$

Suppose we are given four Lagrangian invariants α, β, γ and δ. Then α is a zero form, $d\alpha$ is a Lie transported one form, $d\alpha \wedge d\beta$ is a Lie transported two form, $d\alpha \wedge d\beta \wedge d\gamma$ is a Lie transported three form, $d\alpha \wedge d\beta \wedge d\gamma \wedge d\delta$ is a Lie transported four form and again $(1/\rho) d\alpha \wedge d\beta \wedge d\gamma \wedge d\delta$ is a Lie transported zero form.

Using an invariant solenoidal vector field, we have defined a closed two form in section 3.4. In MHD it is the electromagnetic two form and in hydrodynamics it is the vorticity two form. In our definition of invariance, corresponding to zero form and four form, we get usual invariants which are related to a zero form and a three form in $\mathbb{R}^3$.

Invariance of vector fields in E^4 also involves the invariance of the time component J_0 as a zero form and if J_0 is zero then the space component of the vector field is said to be frozen-in. Corresponding to an invariant vector field in E^4 we obtain an invariant three form $i_J(\rho dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3)$.

Chapter 4

Surface invariants

4.1 Introduction

Consider a p-form ω associated with a physical quantity which evolves under a flow field $\mathbf{u}$. Then $\partial_t\omega + L_{\mathbf{u}}\omega = 0$ means that the surface integral $\int_{C^p}\omega = const.$, where C^p is a p-dimensional comoving surface. But in some cases $\partial_t\omega + L_{\mathbf{u}}\omega$ need not be vanishing. So in this case we cannot say that $\int_{C^p}\omega$ is conserved for all p-dimensional comoving surfaces. Even then there may exist some p-dimensional comoving surfaces over which $\int_{C^p}\omega$ is a constant of the motion. Such invariant surfaces may be used for the qualitative study of the flow of continuous media in the absence of invariance of all comoving p-dimensional surfaces. In this chapter we are investigating what are the sufficient conditions for the existence of such invariant surfaces in continuous media using the language of differential forms and vector fields. Some illustrative examples are also given from hydrodynamics and magnetohydrodynamics.

4.2 Differential forms, vector fields and their invariance

In our classical vector calculus the flux preservation and line preservation are related to a vector field in the Euclidean space $\mathbb{R}^3$. But when we use the language of differential forms and vector fields, the flux preservation is identified with the integral invariance of differential forms and the line preservation is attributed to the vector fields [6, 24, 76]. This identification is significant for a general n-dimensional manifold. In introduction we have seen that in the usual Euclidean space $\mathbb{R}^3$ there is a one to one correspondence between vector fields and differential forms of degreeone. Also a vector field $\mathbf{H}$ can be identified as a two form ω^2 by means of the interior product of $\mathbf{H}$ with the volume form $\Omega = dx^1 \wedge dx^2 \wedge dx^3$, *i.e.*, $\omega^2 = i_{\mathbf{H}}\Omega$. Let S be a surface such that $\mathbf{H} \cdot d\mathbf{S}$ vanishes at every point of the surface. Then if we consider the associated two form ω^2; this means that on the surface S the two form ω^2 is annihilated or S annuls ω^2. In the Euclidean space $\mathbb{R}^3$, vector fields and differential forms of degreeone or two may be used interchangeably. For a better analysis of topological properties of objects in the three dimensional space (or generally in an n-dimensional space) we should use the calculus of differential forms and vector fields. In the study of invariants in hydrodynamics this distinction gives rise to the classification of invariants as given in previous chapters and in [80].

We consider a general p-form ω^p and a flow field $\mathcal{P}$ in an n-dimensional manifold. Let C be a p-dimensional surface moving with velocity $\mathcal{P}$, that is a p-dimensional comoving surface or simply a p-surface, then as given in chapter 1

$$\frac{d}{dt}\int_C \omega^p = \int_C \partial_t \omega^p + L_{\mathcal{P}}\omega^p \tag{4.1}$$

So, if $\partial_t \omega^p + L_{\mathcal{P}}\omega^p = 0$ then $\int_C \omega^p = const.$ for any p-surface C.

4.3 Sequence of Lie derivatives

We consider a tensor field T and a vector field $\mathcal{P}$ in an n-dimensional manifold. We define a sequence of tensor fields of the same type as that of T recursively as follows:

$T^{(0)} = T$,

$T^{(1)} = \partial_t T^{(0)} + L_{\mathcal{P}} T^{(0)}$ and

$T^{(m+1)} = \partial_t T^{(m)} + L_{\mathcal{P}} T^{(m)}$; for all $m \geq 1$.

Here the operator $L_{\mathcal{P}}$ is the Lie derivative with respect to the vector field $\mathcal{P}$.

Let $\mathcal{T} = \{T^{(0)}, T^{(1)}, T^{(2)}, \ldots, T^{(m)}, \ldots\}$, which is called the family of derived fields. The tensor field T is said to be invariant under the flow if $T^{(1)} = 0$. The tensor fields which we are going to consider include differential forms and vector fields only.

Let ω be a p-form and consider the derived space

$$\mathcal{W} = \{\omega^{(0)}, \omega^{(1)}, \ldots \omega^{(m)} \ldots\}$$

Here if $\omega^{(1)} = 0$, then we can say that the p-form ω is invariantly transported under the flow field $\mathcal{P}$. In terms of integral it means that the surface integral $\int_C \omega = const.$, where C is a p-surface in the n-dimensional space. In general $\omega^{(1)}$ need not be vanishing. In some cases it may be of the form $\omega^{(1)} = d\alpha$ where α is a $(p-1)$-form. Then from the Stoke's theorem it follows that

$$\frac{d}{dt}\int_{\partial C} \omega = \frac{d}{dt}\int_C d\omega = \int_C \partial_t d\omega + L_{\mathcal{P}} d\omega = \int_C d\omega^{(1)} = 0$$

where C is a $(p+1)$-surface. This means that ω is a relative integral invariant. For the sequence of the derived space the relation (4.1) becomes

$$\frac{d}{dt}\int_C \omega^{(m)} = \int_C \omega^{(m+1)} \tag{4.2}$$

Now for the sequence of derived forms we will prove the following theorem by induction.

Theorem 1. *For a fixed integer $r \geq 1$ and for some $q \geq 0$, if $\omega^{(q+r)} = \sum_{i=0}^{r-1} f_i \omega^{(q+i)}$, where $f_0, f_1, \ldots, f_{r-1}$ are scalar functions, then $\omega^{(m)} = \sum_{i=0}^{r-1} g_i \omega^{(q+i)}$, for all $m \geq q+r$, where $g_0, g_1, \ldots, g_{r-1}$ are also scalar functions.*

Proof: The result is given to be true for $m = q+r$. So assume it is true for $m = k$, where $k \geq q+r$; *i.e.*, $\omega^{(k)} = \sum_{i=0}^{r-1} h_i \omega^{(q+i)} = h_0 \omega^{(q)} + h_1 \omega^{(q+1)} + \ldots + h_{r-1} \omega^{(q+r-1)}$, where h_i's are scalar functions.Then

$$\begin{aligned}
\omega^{(k+1)} &= \partial_t \omega^{(k)} + L_{\mathcal{P}} \omega^{(k)} \\
&= \partial_t \Big(\sum_{i=0}^{r-1} h_i \omega^{(q+i)}\Big) + L_{\mathcal{P}} \Big(\sum_{i=0}^{r-1} h_i \omega^{(q+i)}\Big), \text{by induction hypothesis} \\
&= \sum_{i=0}^{r-1} \left[\partial_t (h_i \omega^{(q+i)}) + L_{\mathcal{P}} (h_i \omega^{(q+i)})\right], \text{since Lie derivative is linear} \\
&= \sum_{i=0}^{r-1} \left[(\partial_t h_i + L_{\mathcal{P}} h_i)\, \omega^{(q+i)} + h_i \left(\partial_t \omega^{(q+i)} + L_{\mathcal{P}} \omega^{(q+i)}\right)\right] \\
&= \sum_{i=0}^{r-1} \left[l_i \omega^{(q+i)} + h_i \omega^{(q+i+1)}\right], \text{ where } l_i = \partial_t h_i + L_{\mathcal{P}} h_i \\
&\qquad\qquad \text{is a scalar function for all } i \\
&= \sum_{i=0}^{r-1} l_i \omega^{(q+i)} + \sum_{i=0}^{r-2} h_i \omega^{(q+i+1)} + h_{r-1} \omega^{(q+r)} \\
&= \sum_{i=0}^{r-1} l_i \omega^{(q+i)} + \sum_{i=0}^{r-2} h_i \omega^{(q+i+1)} + h_{r-1} \sum_{i=0}^{r-1} f_i \omega^{(q+i)}, \text{from the basis step} \\
&= \sum_{i=0}^{r-1} (l_i + t_i)\, \omega^{(q+i)} + \sum_{i=0}^{r-2} h_i \omega^{(q+i+1)}, \text{where } t_i = h_{r-1} f_i \\
&= \sum_{i=0}^{r-1} g_i \omega^{(q+i)}, \text{where } g_0 = l_0 + t_0 \text{ and } g_i = l_i + t_i + h_{i-1} \\
&\qquad\qquad \text{for } 1 \leq i \leq r-1
\end{aligned}$$

So the result is true for $m = k+1$ also. Hence by induction it follows that the result is true for all $m \geq q+r$. Here note that if all the f_i's are constants, then g_i's are also constant functions.

For $r = 1$, this result says that, if $\omega^{(q+1)} = f_0\omega^{(q)}$ for some q then $\omega^{(m)} = f_m\omega^{(q)}$, for all $m \geq q+1$, where f_m is a scalar function depending on m. Also assume $q = 0$, then this becomes: if $\omega^{(1)} = \alpha\omega$, then $\omega^{(m)} = \alpha_m\omega$ for all $m \geq 1$. When $r = 2$ and $q = 1$, this result is as follows: If $\omega^{(3)} = \alpha\omega^{(1)} + \beta\omega^{(2)}$, then $\omega^{(m)} = \alpha_m\omega^{(1)} + \beta_m\omega^{(2)}$, for all $m \geq 3$.

In continuous media we may assume that the surface integral $\int_C \omega$ is an analytic function of time, where ω is any p-form and C is any p-surface. Let

$$\omega^{(q+r)} = \sum_{i=0}^{r-1} g_i\omega^{(q+i)} \tag{4.3}$$

for a p-form ω . Also let over some p-surface C the forms $\omega^{(q)}, \omega^{(q+1)}, \ldots, \omega^{(q+r-1)}$ are annihilated initially. Then $\phi(t) = \int_C \omega^{(q-1)}$ will be a constant of flow. This follows from theorem 1, and the equation (4.2), using a Taylor's series expansion of $\phi(t)$. Also it follows that $\int_C \omega^{(q)} = 0$ throughout the flow. In general it follows that $\int_C \omega^{(m)} = 0$ throughout the flow for all $m \geq q$.

Again, let the condition (4.3) is satisfied for a p-form ω with all g_i's constant. Also, if $\int_C \omega^{(q+i)}$ $(0 \leq i \leq r-1)$ are vanishing initially for a p-surface C, then $\int_C \omega^{(q-1)}$ will be a constant throughout the flow. Here note that the condition $\omega^{(q+i)}$ $(0 \leq i \leq r-1)$ are annihilated initially need not be satisfied.

Let α is a $(p-1)$-form. Consider the derived fields $\omega^{(m)}$ of the p-form $\omega = d\alpha$ for all m. Then by induction it follows that $\omega^{(m)} = d\alpha^{(m)}$, for all $m \geq 1$. Then the following theorem is obvious:

Theorem 2. *Consider the derived fields of the p-form $\omega = d\alpha$, where α is a $(p-1)$-form. If $\alpha^{(q+r)} = \sum_{i=0}^{r-1} f_i\alpha^{(q+i)}$, then $\omega^{(q+r)} = \sum_{i=0}^{r-1} f_i\omega^{(q+i)}$, provided f_i's are constants.*

Let α and β are differential forms of degree$k$ and $p-k$ respectively, for some p.

Then $\omega = \alpha \wedge \beta$ is a p-form. Now we have the following theorem for the derived space $\mathcal{W} = \{\omega, \omega^{(1)}, \cdots, \omega^{(m)} \cdots\}$ of ω by induction.

Theorem 3. *If* $\omega = \alpha \wedge \beta$, *then* $\omega^{(m)} = \sum_{r=o}^{m} \binom{m}{r} \alpha^{(r)} \wedge \beta^{(m-r)}$

Proof: When $m = 1$,

$$\omega^{(1)} = \alpha \wedge \beta^{(1)} + \alpha^{(1)} \wedge \beta$$

So the result is true when $m = 1$. Let the result is true for $m = k$. That is $\omega^{(k)} = \sum_{r=0}^{k} \binom{k}{r} \alpha^{(r)} \wedge \beta^{(k-r)}$. Then

$$\begin{aligned}
\omega^{(k+1)} &= \partial_t \omega^{(k)} + L_{\mathcal{P}} \omega^{(k)} \\
&= \sum_{r=0}^{k} \binom{k}{r} \{\alpha^{(r)} \wedge \partial_t \beta^{(k-r)} + \partial_t \alpha^{(r)} \wedge \beta^{(k-r)} + \\
&\qquad\qquad \alpha^{(r)} \wedge L_{\mathcal{P}} \beta^{(k-r)} + L_{\mathcal{P}} \alpha^{(r)} \wedge \beta^{(k-r)}\} \\
&= \sum_{r=0}^{k} \binom{k}{r} \{\alpha^{(r)} \wedge \beta^{(k-r+1)} + \alpha^{(r+1)} \wedge \beta^{(k-r)}\} \\
&= \sum_{r=0}^{k} \binom{k}{r} \alpha^{(r)} \wedge \beta^{(k-r+1)} + \sum_{r=1}^{k+1} \binom{k}{r-1} \alpha^{(r)} \wedge \beta^{(k-r+1)} \\
&= \alpha \wedge \beta^{(k+1)} + \sum_{r=1}^{k} \left\{ \binom{k}{r} + \binom{k}{r-1} \right\} \alpha^{(r)} \wedge \beta^{(k-r+1)} + \alpha^{(k+1)} \wedge \beta \\
&= \alpha \wedge \beta^{(k+1)} + \sum_{r=1}^{k} \binom{k+1}{r} \alpha^{(r)} \wedge \beta^{(k+1-r)} + \alpha^{(k+1)} \wedge \beta \\
&= \sum_{r=0}^{k+1} \binom{k+1}{r} \alpha^{(r)} \wedge \beta^{(k+1-r)}
\end{aligned}$$

So the result is true for $m = k + 1$ also. Hence the theorem.

The following different cases of theorem 3 attract particular attention.

Case I: Let the derived fields of α and β satisfy the conditions $\alpha^{(q+r)} = \sum_{i=0}^{r-1} f_i \alpha^{(q+i)}$ and $\beta^{(1)} = 0$. then for all m, $\omega^{(m)} = \alpha^{(m)} \wedge \beta$ and $\omega^{(q+r)} = \sum_{0}^{r-1} f_i \omega^{(q+i)}$.

Case II: Let the form α satisfy the condition $\alpha^{(r)} = \sum_{i=0}^{r-1} f_i \alpha^{(i)}$. Then, from theorem 3, it is clear that the expansion of $\omega^{(m)}$ (for all m) contains $\alpha^{(i)}$ $(0 \leq i \leq r-1)$ as a factor of each of the wedge product. So if $\alpha^{(i)}$'s are annihilated initially on any p-surface, then this p-surface will be an integral invariant surface for ω and the integral is vanishing over all such surfaces. Here note that β can be any $(p-k)$-form.

Case III: Let α and β satisfy the conditions $\alpha^{(1)} = f\alpha$ and $\beta^{(1)} = g\beta$. Then $\omega^{(1)} = (f+g)\alpha \wedge \beta = h\omega$. If the k-form α is vanishing on any p-surface initially or the $(p-k)$-form β is vanishing on any p-surface initially, then the p-surface will be an integral invariant surface for ω. Also if ω is annihilated on any p-surface initially then such surfaces will be invariant surfaces for ω.

For an arbitrary vector field J in an n-dimensional space, consider the derived space $\mathfrak{J} = \{J^{(0)}, J^{(1)}, \ldots, j^{(m)}, \ldots\}$. Then the theorem 1 is true for this family of vector fields also, which can be stated as follows:

Theorem 4. *Let J be a vector field in an n-dimensional space. For $r \geq 1$ and $q \geq 0$, if $J^{(q+r)} = f_0 J^{(q)} + f_1 J^{(q+1)} + \ldots + f_{r-1} J^{(q+r-1)}$ then for all $m \geq q+r$, $J^{(m)} = g_0 J^{(q)} + g_1 J^{(q+1)} \ldots + g_{r-1} J^{(q+r-1)}$, where f_i's and g_i's are scalar functions.*

Let ω be a $(p+1)$-form. Then consider the p-form $\alpha = i_J\omega$, where J is a vector field. Then as theorem 3 we can prove the following theorem.

Theorem 5. *If $\alpha = i_J\omega$, where ω is a $(p+1)$-form and J is a vector field in an n-dimensional space, then $\alpha^{(m)} = \sum_{r=0}^{m} \binom{m}{r} i_{J^{(r)}} \omega^{(m-r)}$*

The following different cases of theorem 5 are interesting:

Case I: If $\omega^{(1)} = 0$, then $\alpha^{(m)} = i_{J^{(m)}}\omega$. Also let $J^{(q+r)} = \sum_{i=0}^{r-1} f_i J^{(q+i)}$. Then we get $\alpha^{(q+r)} = \sum_{i=0}^{r-1} f_i \alpha^{(q+i)}$.

Case II: When $J^{(1)} = 0$ and $\omega^{(q+r)} = \sum_{i=0}^{r-1} f_i \omega^{(q+i)}$, we get $\alpha^{(m)} = i_J\omega^{(m)}$ and $\alpha^{(q+r)} = \sum_{i=0}^{r-1} f_i \alpha^{(q+i)}$

Case III: Here if $\omega^{(1)} = f\omega$ and $J^{(1)} = gJ$, then we have $\alpha^{(1)} = h\alpha$, where $h = f+g$.

We will use the above results in the coming sections to charecterize the different types of surface invariants in three dimensional continuous media. We shall discuss line preservation and surface preservation of vector fields also.

4.4 Surface invariants related to one forms

Let ω be a one form associated with a geometrical object in a three dimensional flow with velocity vector $\mathbf{u}$. Consider the derived space of this one form. We will discuss the following three cases.

Case I: Here we consider the case where $\omega^{(1)} = 0$. Then it follows that the integral $\int_C \omega$ over any 1-surface C(that is, over a comoving curve C) is a constant of the flow.

Case II: If $\omega^{(1)}$ is not vanishing and $\omega^{(1)} = f\omega$, where f is a scalar function, then from theorem 1 it follows that $\omega^{(m)} = f_m\omega$ for all $m \geq 1$. So if there exists a curve over which ω is vanishing initially, then $\int_C \omega = 0$ throughout the flow. So these curves over which ω vanishes initially will constitute the invariant curves of the flow. Also it follows that the line integrals $\int_C \omega^{(m)}$ are vanishing throughout the flow for all $m \geq 1$.

Now consider the case where $\omega^{(2)} = f\omega^{(1)}$ for some scalar function f, then any curve C over which $\omega^{(1)}$ vanishes initially will be an integral invariant 1-surface for the one form ω. This line integral of ω over C need not be zero. Let S be a 2-dimensional surface over which $\omega^{(1)}$ vanishes initially. Then for any curve C on this surface, $\int_C \omega$ is a constant. But by Frobenius integrability condition a family of integral surfaces for the one form $\omega^{(1)}$ exists only if $\omega^{(1)} \wedge d\omega^{(1)} = 0$. So when this condition is satisfied initially, then for all curves C on this family of surfaces $\int_C \omega$ is a constant of the flow, provided $\omega^{(2)} = f\omega^{(1)}$. Even though the Frobenius integrability condition is not satisfied there may exist many curves over which the integral is a constant of the flow. For example, let us consider the vector field associated with the one form $\omega^{(1)}$ in the Euclidean space $\mathbb{R}^3$. Then consider the vector tubes of this field. If this tube has the topology of a cylinder initially, then for any curve of intersection of this tube with a plane perpendicular to the axis of the tube will be an integral invariant curve for the one form ω. Let $\mathbf{A}$ be the vector field associated with $\omega^{(1)}$ and $\mathbf{B}$ be an arbitrary vector field not collinear with $\mathbf{A}$. Then initially on the vector lines of the vector field $\mathbf{A} \times \mathbf{B}$, $\omega^{(1)}$ is annihilated. So all these vector lines form a family of invariant 1-surface for ω.

In general, if $\omega^{(q+1)} = f\omega^{(q)}$, then the curves C on which $\omega^{(q)}$ vanishes initially constitute the invariant curves for the one form $\omega^{(q-1)}$. If there exists a family of integral surfaces for the one form $\omega^{(q)}$ initially, then for all curves on this surface $\int_C \omega^{(q-1)}$ is an invariant. The condition for the existence of such a family of integral surface initially is that $\omega^{(q)} \wedge d\omega^{(q)}$ vanishes initially. Also note that for all such invariant curves C of $\omega^{(q-1)}$, the line integral $\int_C \omega^{(m)}$ is an integral invariant which vanishes, for all $m \geq q$.

Case III: Here we assume that $\omega^{(1)}$ and $\omega^{(2)}$ are independent forms, but $\omega^{(3)} = f\omega^{(1)} + g\omega^{(2)}$ where f and g are scalar functions. Then, if there exists a curve C over which both $\omega^{(1)}$ and $\omega^{(2)}$ are annihilated, then $\int_C \omega$ is a constant of the flow. If $\omega^{(2)} = f\omega + g\omega^{(1)}$, then$\int_C \omega = 0$ over the curves C on which ω and $\omega^{(1)}$ are annihilated. In general if $\omega^{(q+2)} = f\omega^{(q)} + g\omega^{(q+1)}$, and if there exists a curve on which $\omega^{(q)}$ and $\omega^{(q+1)}$ are annihilated, then $\int_C \omega^{(q-1)}$ is a constant. Also consider $\alpha = \omega^{(q)} \wedge \omega^{(q+1)}$ which is a two form . Then

$$
\begin{aligned}
(\omega^{(q)} \wedge \omega^{(q+1)})^{(1)} &= \partial_t(\omega^{(q)} \wedge \omega^{(q+1)}) + L_{\mathbf{u}}(\omega^{(q)} \wedge \omega^{(q+1)}) \\
&= \omega^{(q+1)} \wedge \omega^{(q+1)} + \omega^{(q)} \wedge \omega^{(q+2)} \\
&= \omega^{(q)} \wedge \omega^{(q+2)} \\
&= \omega^{(q)} \wedge (f\omega^{(q)} + g\omega^{(q+1)}) \\
&= g\omega^{(q)} \wedge \omega^{(q+1)}
\end{aligned}
$$

So $\alpha^{(1)} = g\alpha$ for the two form $\alpha = \omega^{(q)} \wedge \omega^{(q+1)}$ when $\omega^{(q+2)} = f\omega^{(q)} + g\omega^{(q+1)}$. Hence if initially $\omega^{(q)}$(or $\omega^{(q+1)}$) is a surface forming one form, *i.e.*, $\omega^{(q)} \wedge d\omega^{(q)} = 0$ (or $\omega^{(q+1)} \wedge d\omega^{(q+1)} = 0$), then over these family of surfaces the two form α is annihilated and hence these surfaces will be invariant surfaces of the two form α. That is, for all such 2-surfaces C, $\int_C \alpha$ is a constant, which is zero. For example when $q = 0$, $\omega^{(2)} = f\omega + g\omega^{(1)}$, then $\alpha = \omega \wedge \omega^{(1)}$ is having such invariant surfaces provided $\omega \wedge d\omega = 0$ or $\omega^{(1)} \wedge d\omega^{(1)} = 0$, or both.

Now we will discuss the three special cases of theorem 3 as given in section 4.3, when α and β are 1-forms. In the first case let $\alpha^{(2)} = f\alpha^{(1)}$ and $\beta^{(1)} = 0$. Then $\omega^{(2)} = f\omega^{(1)}$, where $\omega = \alpha \wedge \beta$. If initially $\alpha^{(1)} \wedge d\alpha^{(1)} = 0$, then on the integral surfaces of $\alpha^{(1)}$, $\omega^{(1)}$ is annihilated initially as well as $\alpha^{(1)}$. So $\int_C \omega$ will be a constant of motion for any 2-surface C on this family of integral surfaces.

In the second case, let $\alpha^{(1)} = f\alpha$ and β is any 1-form. Then $\omega = \alpha \wedge \beta$ is such that each term in $\omega^{(m)}$ contains α as a factor in the wedge product. So if there exist integral surfaces for α initially (that is, $\alpha \wedge d\alpha = 0$ initially), then any 2-surface on this family of integral surfaces will be an integral invariant surface for ω. This is also true when $\alpha^{(1)} = 0$.

In the last case, let α and β be 1-forms satisfying $\alpha^{(1)} = f\alpha$ and $\beta^{(1)} = g\beta$.Then $\omega^{(1)} = h\omega$. So if $\alpha \wedge d\alpha = 0$ (or, $\beta \wedge d\beta = 0$), then any 2-surface lying on the family of integral surfaces of α(or, β) will be an integral invariant surface for ω.

4.5 Line and surface preservation of vector fields

Let $\mathbf{J}$ be a vector field in the flow of a continuous media. Then consider the sequence of derived vector fields of same type. We will distinguish the following different cases:
Case I: If $\mathbf{J}^{(1)} = 0$, then the **J**-lines are said to be frozen into the medium. In other words, if this condition is satisfied, then the vector lines of $\mathbf{J}$ are said to be material lines. The vorticity field in the ideal incompressible hydrodynamic flow and magnetic field in the ideal incompressible MHD flows are well known examples of frozen in fields.
Case II: Let $\mathbf{J}^{(1)} = \lambda\mathbf{J}$ for some scalar function λ. In this case also the vector lines are said to be preserved, that is, the vector lines of $\mathbf{J}$ are material lines. Also from theorem 4 it follows that all the derived vector fields $\mathbf{J}^{(m)}$ satisfy the equation $\mathbf{J}^{(m)} = \lambda_m\mathbf{J}$. That is, the vector lines of each of these derived fields $\mathbf{J}^{(m)}$ coincide with that of $\mathbf{J}$. So the vector lines of the derived fields are also material lines.

Case III: Let

$$\mathbf{J}^{(2)} = f\mathbf{J} + g\mathbf{J}^{(1)} \tag{4.4}$$

where f and g are some scalar functions. Then for any $m \geq 2$; $\mathbf{J}^{(m)} = f_m\mathbf{J} + g_m\mathbf{J}^{(1)}$, from theorem 1. Also let there exist a family of integral surface for $\mathbf{J}$ and $\mathbf{J}^{(1)}$. The necessary and sufficient condition for the existence of such a family of integral surfaces for these vector fields is that their Lie bracket is a linear combination of themselves (Frobenius theorem). That is

$$\left[\mathbf{J}, \mathbf{J}^{(1)}\right] = f'\mathbf{J} + g'\mathbf{J}^{(1)} \tag{4.5}$$

where f' and g' are scalar functions. So when equations (4.4) and (4.5) are satisfied simultaneously, then on the integral surfaces of $\mathbf{J}$ and $\mathbf{J}^{(1)}$ lie all $\mathbf{J}^{(m)}$, for $m \geq 0$.

Now we are going to show that if the condition (4.5) is satisfied initially, that is there exist a family of integral surfaces for $\mathbf{J}$ and $\mathbf{J}^{(1)}$ initially, then these surfaces will remain as integral surfaces for these vector fields throughout the flow. Consider the one form $\omega = i_{\mathbf{J}^{(1)}} i_{\mathbf{J}}\Omega$, where $\Omega = dx^1 \wedge dx^2 \wedge dx^3$ is the volume element form. Clearly the vector field corresponding to this one form is $\mathbf{J} \times \mathbf{J}^{(1)}$. Then the integral surfaces of $\mathbf{J}$ and $\mathbf{J}^{(1)}$, if they exist, annuls the one form ω and they are the integral surfaces of ω also.

Remark: Here integral surfaces for ω exists if and only if $\omega \wedge d\omega = 0$. But $\omega \wedge d\omega = 0 \Leftrightarrow (\mathbf{J}\times\mathbf{J}^{(1)})\cdot\nabla\times(\mathbf{J}\times\mathbf{J}^{(1)}) = 0 \Leftrightarrow (\mathbf{J}\times\mathbf{J}^{(1)})\cdot\{[\mathbf{J},\mathbf{J}^{(1)}]-(\nabla\cdot\mathbf{J})\mathbf{J}^{(1)}+(\nabla\cdot\mathbf{J}^{(1)})\mathbf{J}\} = 0 \Leftrightarrow [\mathbf{J}, \mathbf{J}^{(1)}] = f'\mathbf{J}+g'\mathbf{J}^{(1)}$. So the integral surfaces for ω exist if and only if there exist integral surfaces for $\mathbf{J}$ and $\mathbf{J}^{(1)}$.

For the above one form ω,

$$\begin{aligned}
\omega^{(1)} &= \partial_t\omega + L_{\mathbf{u}}\omega \\
&= \partial_t(i_{\mathbf{J}^{(1)}} i_{\mathbf{J}}\Omega) + L_{\mathbf{u}}(i_{\mathbf{J}^{(1)}} i_{\mathbf{J}}\Omega)
\end{aligned}$$

$$= \left\{ i_{\partial_t \mathbf{J}^{(1)}} + i_{\mathbf{J}^{(1)}} \partial_t \right\} i_{\mathbf{J}} \Omega + \left\{ i_{[\mathbf{u}, \mathbf{J}^{(1)}]} + i_{\mathbf{J}^{(1)}} L_{\mathbf{u}} \right\} i_{\mathbf{J}} \Omega$$

(since $\partial_t i_{\mathbf{X}} - i_{\mathbf{X}} \partial_t = i_{\partial_t \mathbf{X}}$ and using properties of Lie derivative)

$$= \left\{ i_{\partial_t \mathbf{J}^{(1)} + [\mathbf{u}, \mathbf{J}^{(1)}]} + i_{\mathbf{J}^{(1)}} \partial_t + i_{\mathbf{J}^{(1)}} L_{\mathbf{u}} \right\} i_{\mathbf{J}} \Omega$$

$$= i_{\mathbf{J}^{(2)}} i_{\mathbf{J}} \Omega + i_{\mathbf{J}^{(1)}} \left\{ \partial_t i_{\mathbf{J}} + L_{\mathbf{u}} i_{\mathbf{J}} \right\} \Omega$$

$$= i_{\mathbf{J}^{(2)}} i_{\mathbf{J}} \Omega + i_{\mathbf{J}^{(1)}} \left\{ i_{\partial_t \mathbf{J}} + i_{\mathbf{J}} \partial_t + i_{[\mathbf{u}, \mathbf{J}]} + i_{\mathbf{J}} L_{\mathbf{u}} \right\} \Omega$$

$$= i_{\mathbf{J}^{(2)}} i_{\mathbf{J}} \Omega + i_{\mathbf{J}^{(1)}} \left[i_{\partial_t \mathbf{J} + [\mathbf{u}, \mathbf{J}]} + i_{\mathbf{J}} (\partial_t + L_{\mathbf{u}}) \right] \Omega$$

$$= i_{\mathbf{J}^{(2)}} i_{\mathbf{J}} \Omega + i_{\mathbf{J}^{(1)}} \left[i_{\mathbf{J}^{(1)}} + i_{\mathbf{J}} (\partial_t + L_{\mathbf{u}}) \right] \Omega$$

$$= i_{\mathbf{J}^{(2)}} i_{\mathbf{J}} \Omega + i_{\mathbf{J}^{(1)}} i_{\mathbf{J}} (\partial_t \Omega + L_{\mathbf{u}} \Omega), \text{ since } \; i_{\mathbf{J}^{(1)}} i_{\mathbf{J}^{(1)}} \Omega = 0$$

$$= i_{\mathbf{J}^{(2)}} i_{\mathbf{J}} \Omega + i_{\mathbf{J}^{(1)}} i_{\mathbf{J}} (\nabla \cdot \mathbf{u}) \Omega, \text{ since } \partial_t \Omega = 0 \; \text{ and } \; L_{\mathbf{u}} \Omega = (\nabla \cdot \mathbf{u}) \Omega$$

$$= i_{\mathbf{J}^{(2)}} i_{\mathbf{J}} \Omega + (\nabla \cdot \mathbf{u}) \omega, \text{ since } i_{\mathbf{X}} f \Omega = f i_{\mathbf{X}} \Omega \; \text{ and } \; i_{\mathbf{J}^{(1)}} i_{\mathbf{J}} \Omega = \omega. \tag{4.6}$$

So

$$\omega^{(1)} = h\omega \Leftrightarrow i_{\mathbf{J}^{(2)}} i_{\mathbf{J}} \Omega = f\omega, \; \text{ for the scalar function } f = h - \nabla \cdot \mathbf{u}$$

$$\Leftrightarrow i_{\mathbf{J}^{(2)}} i_{\mathbf{J}} \Omega - f i_{\mathbf{J}^{(1)}} i_{\mathbf{J}} \Omega = 0$$

$$\Leftrightarrow i_{\mathbf{J}^{(2)} - f\mathbf{J}^{(1)}} i_{\mathbf{J}} \Omega = 0, \; \text{ since } \; i_{f\mathbf{X}} = f i_{\mathbf{X}}$$

$$\Leftrightarrow \mathbf{J}^{(2)} - f\mathbf{J}^{(1)} = g\mathbf{J}, \text{ for some scalar function } g$$

(since $i_{\mathbf{X}} i_{\mathbf{Y}} \Omega = 0 \Leftrightarrow \mathbf{X} = h' \mathbf{Y}$ for some scalar function h')

$$\Leftrightarrow \mathbf{J}^{(2)} = f\mathbf{J} + g\mathbf{J}^{(1)} \; \text{ where } \; f \text{ and } g \text{ are scalar functions..}$$

So $\omega^{(1)} = h\omega \Leftrightarrow \mathbf{J}^{(2)} = f\mathbf{J} + g\mathbf{J}^{(1)}$ for some scalar functions h, f and g. Thus if any of these equivalent conditions is satisfied and if initially there exists a family of integral surfaces for ω, then on this family of integral surfaces, ω is annihilated initially. So from the previous section it follows that the integration of ω over an arbitrary material curve on this family of surfaces vanishes initially and hence identically. So this family of integral surfaces for ω will remain as a family of integral surfaces for ω. Then it follows that this family of integral surfaces which is a family of orthogonal surfaces for $\mathbf{J} \times \mathbf{J}^{(1)}$ will remain as a family of orthogonal surfaces for $\mathbf{J} \times \mathbf{J}^{(1)}$. Hence the vector fields $\mathbf{J}$ and $\mathbf{J}^{(1)}$ which span this family of material surfaces initially will remain spanning this family of material surfaces (also see the above remark). A vector field is said to preserve a material surface if initially this material surface is generated by the field lines of this vector field and this material surface remains generated by the field lines of the same vector field throughout the flow. That is, if the field lines initially lie on this comoving surface, then they will remain lying on this surface. Hence if there exist an integral surface for $\mathbf{J}$ and $\mathbf{J}^{(1)}$ initially, then (4.4) is the condition for the surface preservation of the vector field $\mathbf{J}$. Here a material line initially which is a $\mathbf{J}$-line lying on this invariant surface need not be a $\mathbf{J}$-line as the field evolves. When the field $\mathbf{J}^{(1)}$ vanishes identically, then as given in case I all the vector lines are material lines. Then clearly all the vector surfaces of the vector field $\mathbf{J}$ are preserved, that is they are material surfaces. If $\mathbf{J}^{(2)} = 0$, then from (4.6) $\omega^{(1)} = (\nabla \cdot \mathbf{u})\omega$. Then also the integral surfaces of $\mathbf{J}$ and $\mathbf{J}^{(1)}$ are preserved by the flow, if they exist initially.

4.6 Invariant surfaces related to two forms

Let ω be a two form associated with a physical quantity in a continuous media. Consider the derived space of two forms as given in section 3. We will consider three cases given below.

Case I: Here we will consider the case where $\omega^{(1)} = 0$ for a flow. Then the integral of ω over any 2-surface will be a constant of motion. In terms of associated vector field $\mathbf{H}$ it says that $\int_C \mathbf{H} \cdot d\mathbf{S}$ is a constant for all 2-surfaces.

Case II: In some cases it may happen that $\omega^{(2)} = \lambda\omega^{(1)}$, then for all $m \geq 2$, $\omega^{(m)} = \lambda_m \omega^{(1)}$ for some scalar functions λ_m. Then consider a surface over which $\omega^{(1)}$ vanishes initially. Any 2-surface in this surface will be an integral invariant surface for ω. That is, $\int_C \omega$ is a constant. Let $\mathbf{H}$ be the associated vector field of the two form $\omega^{(1)}$. Here initially consider the vector sheets of $\mathbf{H}$. Then clearly $\omega^{(1)}$ is annihilated on these vector sheets of $\mathbf{H}$. So all 2-surfaces lying on these vector sheets of $\mathbf{H}$ constitute a class of surfaces with constant surface integral of the two form ω.

Let ω be a two form for which $\omega^{(2)} = f\omega^{(1)}$ and $\mathbf{J}$ is an invariant vector field (that is $\mathbf{J}^{(1)} = 0$). If $\alpha = i_{\mathbf{J}}\omega$, then $\alpha^{(2)} = h\alpha^{(1)}$, from case 2 of theorem 5. Let $\mathbf{H}$ be the associated vector field of $\omega^{(1)}$. Then $\mathbf{H} \times \mathbf{J}$ will be the associated vector field of the one form $\alpha^{(1)}$ (since, $\alpha^{(1)} = i_{\mathbf{J}} i_{\mathbf{H}}\Omega$). Then on the vector lines of $\mathbf{H}$ and $\mathbf{J}$, $\alpha^{(1)}$ is annihilated and these vector lines are integral invariant families of curves for α. If the Frobenius integrability condition is satisfied initially for the vector fields $\mathbf{H}$ and $\mathbf{J}$, then any curve on these integral surfaces form an integral invariant 1-surface for α. A similar discussion is possible when $\omega^{(1)} = 0$ and $\mathbf{J}^{(2)} = f\mathbf{J}^{(1)}$, using case 1 of theorem 5. The third case of theorem 5 can also be discussed similarly.

If $\omega^{(1)} = \lambda\omega$, then the constant flux surfaces will include vector sheets of the field $\mathbf{K}$, where $\omega = i_{\mathbf{K}}\Omega$. Here the value of the constant flux across the surface is zero. Here we can also see that the associated vector field $\mathbf{K}$ is line preserving. We have

$$\begin{aligned}
\omega^{(1)} &= \partial_t(i_{\mathbf{K}}\Omega) + L_{\mathbf{u}}(i_{\mathbf{K}}\Omega) \\
&= (i_{\partial_t \mathbf{K}} + i_{\mathbf{K}}\partial_t)\Omega + (i_{[\mathbf{u},\mathbf{K}]} + i_{\mathbf{K}})L_{\mathbf{u}}\Omega, \quad \text{since } \partial_t i_{\mathbf{Y}} - i_{\mathbf{Y}}\partial_t = i_{\partial_t \mathbf{Y}} \\
&\qquad \text{and } L_{\mathbf{u}} i_{\mathbf{X}} - i_{\mathbf{X}} L_{\mathbf{u}} = i_{[\mathbf{u},\mathbf{X}]} \\
&= i_{\partial_t \mathbf{K} + [\mathbf{u},\mathbf{K}]}\Omega + i_{\mathbf{K}} L_{\mathbf{u}}\Omega, \quad \text{since } \partial_t\Omega = 0 \\
&= i_{\mathbf{K}^{(1)}}\Omega + (\nabla\cdot\mathbf{u})i_{\mathbf{K}}\Omega \\
&= i_{\mathbf{K}^{(1)} + (\nabla\cdot\mathbf{u})\mathbf{K}}\Omega.
\end{aligned} \tag{4.7}$$

So $\omega^{(1)} = \lambda\omega$ implies that $i_{\mathbf{K}^{(1)} + (\nabla\cdot\mathbf{u})\mathbf{K}}\Omega = \lambda i_{\mathbf{K}}\Omega = i_{\lambda\mathbf{K}}\Omega$.

Hence,

$$i_{\mathbf{K}^{(1)} + (\nabla\cdot\mathbf{u})\mathbf{K} - \lambda\mathbf{K}}\Omega = 0 \;\Rightarrow\; \mathbf{K}^{(1)} + (\nabla\cdot\mathbf{u})\mathbf{K} - \lambda\mathbf{K} = 0 \;\Rightarrow\; \mathbf{K}^{(1)} = \beta\mathbf{K}$$

where $\beta = \lambda - \nabla \cdot \mathbf{u}$. Hence it follows that the field lines of $\mathbf{K}$ are preserved.

Similarly, if $\omega^{(2)} = \lambda\omega^{(1)}$, then in addition to the flux preservation of the two form ω over all the 2-surfaces over which $\omega^{(1)}$ is annihilated, the field lines of the vector field $\mathbf{H}$ are preserved, where $\omega^{(1)} = i_{\mathbf{H}}\Omega$.

Case III: Here we consider the case where

$$\omega^{(3)} = f\omega^{(1)} + g\omega^{(2)} \tag{4.8}$$

for some scalar functions f and g. If there exists a surface C which annihilates both $\omega^{(1)}$ and $\omega^{(2)}$ initially, then $\int_C \omega$ is a constant. Let $\mathbf{J}$ and $\mathbf{K}$ be two vector fields satisfying $\omega^{(1)} = i_{\mathbf{J}}\Omega$ and $\omega^{(2)} = i_{\mathbf{K}}\Omega$. If $[\mathbf{J}, \mathbf{K}]$ is a linear combination of $\mathbf{J}$ and $\mathbf{K}$ initially, that is

$$[\mathbf{J}, \mathbf{K}] = \lambda\mathbf{J} + \beta\mathbf{K}, \tag{4.9}$$

then on the integral surfaces of $\mathbf{J}$ and $\mathbf{K}$, $\omega^{(1)}$ and $\omega^{(2)}$ are vanishing initially, so that the surface integral of ω over any of the 2-surface lying on this family of integral surfaces are constant of the motion.

Now we are going to show that the condition (4.8) is equivalent to the condition that the vector field $\mathbf{J}^{(2)}$ is a linear combination of the vector fields $\mathbf{J}$ and $\mathbf{J}^{(1)}$.

Clearly

$$\mathbf{K} = \mathbf{J}^{(1)} + (\nabla \cdot \mathbf{u})\mathbf{J} \tag{4.10}$$

since $\omega^{(2)} = \partial_t\omega^{(1)} + L_{\mathbf{u}}\omega^{(1)}$ and proceeding as in (4.7).

Then

$$\omega^{(3)} = f\omega^{(1)} + g\omega^{(2)} = fi_{\mathbf{J}}\Omega + gi_{\mathbf{K}}\Omega = i_{f\mathbf{J}}\Omega + i_{g\mathbf{K}}\Omega = i_{f\mathbf{J}+g\mathbf{K}}\Omega \tag{4.11}$$

So vector field corresponding to $\omega^{(3)}$ is $f\mathbf{J} + g\mathbf{K}$. But, from the definition of $\omega^{(3)}$ and proceeding as in (4.7), we have

$$\omega^{(3)} = i_{\mathbf{K}^{(1)}+(\nabla\cdot\mathbf{u})\mathbf{K}}\Omega. \tag{4.12}$$

So from (4.11) and (4.12)

$$\mathbf{K}^{(1)} + (\nabla \cdot \mathbf{u})\mathbf{K} = f\mathbf{J} + g\mathbf{K}. \tag{4.13}$$

Hence

$$\mathbf{K}^{(1)} = f\mathbf{J} + h\mathbf{K} \tag{4.14}$$

where $h = g - \nabla \cdot \mathbf{u}$.

Also from equation (4.10)

$$\mathbf{K}^{(1)} = (\partial_t + L_{\mathbf{u}})[\mathbf{J}^{(1)} + (\nabla \cdot \mathbf{u})\mathbf{J}] = \mathbf{J}^{(2)} + (\nabla \cdot \mathbf{u})\mathbf{J}^{(1)} + k\mathbf{J}, \tag{4.15}$$

for some scalar function k.

Also substituting the value of $\mathbf{K}$ from (4.10) in (4.14) we have

$$\mathbf{K}^{(1)} = t\mathbf{J} + h\mathbf{J}^{(1)} \tag{4.16}$$

where $t = f + \nabla \cdot \mathbf{u}$. Now from (4.15) and (4.16) we get

$$\mathbf{J}^{(2)} = f_1\mathbf{J} + f_2\mathbf{J}^{(1)} \tag{4.17}$$

for some scalar functions f_1 and f_2. Hence $\mathbf{J}^{(2)}$ is a linear combination of the vector fields $\mathbf{J}$ and $\mathbf{J}^{(1)}$.

If initially

$$[\mathbf{J}, \mathbf{J}^{(1)}] = g_1\mathbf{J} + g_2\mathbf{J}^{(1)} \tag{4.18}$$

for some scalar functions g_1 and g_2, then the integral surfaces for the fields $\mathbf{J}$ and $\mathbf{J}^{(1)}$ exist initially. Then from (4.17) it follows that these integral surfaces of $\mathbf{J}$-lines (and $\mathbf{J}^{(1)}$-lines) are preserved during the flow as discussed in previous section. Here note that

(4.9) and (4.18) are also equivalent (since, $[\mathbf{J},\mathbf{K}] = \lambda\mathbf{J} + \beta\mathbf{K} \Leftrightarrow [\mathbf{J},\mathbf{J}^{(1)} + (\nabla\cdot\mathbf{u})\mathbf{J}] = \lambda\mathbf{J} + \beta\mathbf{K} \Leftrightarrow [\mathbf{J},\mathbf{J}^{(1)}] + [\mathbf{J},(\nabla\cdot\mathbf{u})\mathbf{J}] = \lambda\mathbf{J} + \beta\mathbf{K} \Leftrightarrow [\mathbf{J},\mathbf{J}^{(1)}] = \lambda'\mathbf{J} + \beta\mathbf{K}$ $\{\text{since}[\mathbf{J},\gamma\mathbf{J}] = (\mathbf{J}\cdot\nabla\gamma)\mathbf{J}$, where $\gamma = \nabla\cdot\mathbf{u}$ and $\lambda' = \lambda - \mathbf{J}\cdot\nabla\gamma\}$ $= g_1\mathbf{J} + g_2\mathbf{J}^{(1)}$ where $g_1 = \lambda' + \beta(\nabla\cdot\mathbf{u})$ and $g_2 = \beta$, from (4.10))

4.7 Some illustrative examples

In this section we will discuss some examples in the Euclidean space $\mathbb{R}^3$ which illustrate some of the developments given in the previous sections. For convenience differential forms are represented in vector notations.

Example I

Consider the equation of motion of a homogeneous incompressible fluid in a potential field of external forces, taking into account viscous friction in Rayleigh's form [26, 44, 45].

$$\partial_t\mathbf{u} + (\nabla\times\mathbf{u})\times\mathbf{u} = -\nabla f - k\mathbf{u} \tag{4.19}$$

where k is the coefficient of viscous friction constant. This equation can be written as

$$\partial_t\Theta + L_{\mathbf{u}}\Theta = d(u^2 - f) - k\Theta, \quad \text{where } \Theta = u_1dx^1 + u_2dx^2 + u_3dx^3 \tag{4.20}$$

This equation can be again written as

$$\partial_t\omega + L_{\mathbf{u}}\omega = -k\omega \tag{4.21}$$

under some gauge transformation $\omega = \Theta + d\phi$, where the potential ϕ is to be so chosen that it satisfies $u^2 - f + d\phi/dt + k\phi = 0$. Here the one form ω satisfies $\omega^{(1)} = -k\omega$. So from section 4.4 , for any comoving curve C for which ω vanishes $\int_C\omega$ is a constant. If $\omega\wedge d\omega = 0$ initially, then there exists a family of integral surfaces for the one form

ω and all curves on this family of surfaces will be integral invariant curves. Here the one form ω corresponds to the velocity vector under some gauge potential. Similar potentials have been used in [31, 80].

Let $\mathbf{A}$ be the vector field corresponding to the one form ω. and let $\mathbf{B}$ be an arbitrary vector field not parallel to the field $\mathbf{A}$. Then on the vector lines of the field $\mathbf{A} \times \mathbf{B}$ the one form ω is annihilated. so these vector lines forms family of integral invariant curves for ω.

Let there exist a family of integral surface for the one form ω initially. Also let β be a one form associated with some physical quantity. Then, from case 2 of theorem 3, the two form $\omega \wedge \beta$ will have this family of integral surfaces for ω as integral invariant surfaces.

Example 2:

Here we discuss the integral invariant curves of a one form corresponding to the magnetic potential of a magnetic field using particle drift velocity. Let a uniform electric field $\mathbf{E}$ drive a current through an infinitely long straight wire. The magnetic field $\mathbf{B}$ consists of a uniform external field parallel to the wire, in addition to the field produced by the current in the wire [64]. Then the fields out side the wire may be written in cartesian coordinates as follows:

$$\mathbf{B} = \left(\frac{-y}{x^2+y^2},\ \frac{x}{x^2+y^2},\ 1 \right) \quad \text{and}$$

$$\mathbf{E} = (0,\ 0,\ 1)\,.$$

Then the particle drift velocity is given by

$$\mathbf{v} = \left(\frac{-x}{1+x^2+y^2},\ \frac{-y^2}{1+x^2+y^2},\ 0 \right).$$

Magnetic potential is taken to be

$$\mathbf{A} = \left(\frac{-y}{2}, \frac{x}{2}, -log\sqrt{x^2+y^2} \right).$$

Let $\omega_{\mathbf{A}}$ be the one form associated with the magnetic potential $\mathbf{A}$. In vector notation the first three derived fields of this one form are given below.

$$\begin{aligned} \omega_{\mathbf{A}}^{(1)} &\longleftrightarrow \frac{1}{1+x^2+y^2}(y, -x, 1), \\ \omega_{\mathbf{A}}^{(2)} &\longleftrightarrow \frac{2}{(1+x^2+y^2)^3}\left(-y, x, x^2+y^2\right) \quad \text{and} \\ \omega_{\mathbf{A}}^{(3)} &\longleftrightarrow \frac{4(-1+2x^2+2y^2)}{(1+x^2+y^2)^5}\left(-y, x, x^2+y^2\right) \end{aligned}$$

Clearly $\omega_{\mathbf{A}}^{(3)} = f\omega_{\mathbf{A}}^{(2)}$, where $f = 2(-1+2x^2+2y^2)/(1+x^2+y^2)^2$. So for all $m \geq 3$, $\omega_{\mathbf{A}}^{(m)} = f_m\omega_{\mathbf{A}}^{(2)}$ for some scalar function f_m. If initially $\omega_{\mathbf{A}}^{(1)}$ and $\omega_{\mathbf{A}}^{(2)}$ are annihilated on some curve C, then $\int_C \omega_{\mathbf{A}}$ is an invariant. This follows from section 4.4. Clearly vector lines of the vector field $\mathbf{A}_1 \times \mathbf{A}_2$ are such curves, where $\mathbf{A}_1$ and $\mathbf{A}_2$ are the associated vector fields of the one forms $\omega_{\mathbf{A}}^{(1)}$ and $\omega_{\mathbf{A}}^{(2)}$ respectively. Also we may consider the two form $\alpha = \omega_{\mathbf{A}}^{(1)} \wedge \omega_{\mathbf{A}}^{(2)}$. Then $\alpha^{(1)} = g\alpha$, where g is some scalar function. Here the one form $\omega_{\mathbf{A}}^{(2)}$ is a surface forming one form, since $\omega_{\mathbf{A}}^{(2)} \wedge d\omega_{\mathbf{A}}^{(2)} = 0$. So initially on this family of surfaces α is annihilated. Hence over these 2-surfaces the integrals of α are vanishing invariants.

Also form $\omega_{\mathbf{A}}^{(3)} = f\omega_{\mathbf{A}}^{(2)}$, it follows that for all curves C on which $\omega_{\mathbf{A}}^{(2)}$ is annihilated, $\int_C \omega_{\mathbf{A}}^{(1)}$ is a constant of motion. Since $\omega_{\mathbf{A}}^{(2)}$ is a surface forming one form, it is annihilated on all curves lying on these surfaces and hence all such curves will be integral invariant curves for $\int_C \omega_{\mathbf{A}}^{(1)}$.

Example 3:

Consider the magnetic field

$$\mathbf{B} = (cos\ z,\ sin\ z,\ 0)$$

produced by the current

$$\mathbf{J} = -\frac{1}{4\pi}\,(cos\ z,\ sin\ z, 0)$$

and the electric field

$$\mathbf{E} = (1,\ 0,\ 0)\,.$$

Then the particle drift velocity is given by

$$\mathbf{v} = (0,\ 0,\ sin\ z)$$

and the magnetic potential is

$$\mathbf{A} = (-cos\ z, -sin\ z, 0)\,.$$

Let $\omega_{\mathbf{A}}$ be the one form associated to the magnetic potential $\mathbf{A}$. Then the first two derived fields of $\omega_{\mathbf{A}}$ are given below:

$$\omega_{\mathbf{A}}^{(1)} \longleftrightarrow \left(sin^2\ z,\ -sin\ z\ cos\ z,\ 0\right) \text{ and}$$
$$\omega_{\mathbf{A}}^{(2)} \longleftrightarrow (sin\ z\ sin\ 2z,\ -sin\ z\ cos\ 2z,\ 0)\,.$$

Then $\omega_{\mathbf{A}}^{(2)} = f\omega_{\mathbf{A}} + g\omega_{\mathbf{A}}^{(1)}$, where $f = -sin^2\ z$ and $g = cos\ z$. So $\omega_{\mathbf{A}}^{(m)} = f_m\omega_{\mathbf{A}} + g_m\omega_{\mathbf{A}}^{(1)}$, for all $m \geq 2$. Hence, if $\omega_{\mathbf{A}}$ and $\omega_{\mathbf{A}}^{(1)}$ are annihilated initially on some curve C, then $\int_C \omega_{\mathbf{A}}$ is a vanishing invariant of the flow. Clearly $\omega_{\mathbf{A}}$ and $\omega_{\mathbf{A}}^{(1)}$ are annihilated on the lines parallel to the z-axis. So they are curves with constant line integral of $\omega_{\mathbf{A}}$. Here note that $\mathbf{B}$ is the magnetic field of a circularly polarized electromagnetic wave in free space.

Example 4:

Again consider the equation of motion of a fluid in Rayleigh's form given by (4.19). Taking the curl of this equation we get the vorticity equation

$$\partial_t \mathbf{w} + \nabla \times (\mathbf{w} \times \mathbf{u}) = -k\mathbf{w} \tag{4.22}$$

where $\mathbf{w} = \nabla \times \mathbf{u}$ is the vorticity. Then consider a two form ω defined by $\omega = i_{\mathbf{w}}\Omega$. Then the above equation is equivalent to

$$\partial_t \omega + L_{\mathbf{u}}\omega = -k\omega.$$

That is

$$\omega^{(1)} = -k\omega.$$

So if ω is annihilated on some 2-surface initially, then such 2-surfaces will be integral invariant surfaces for ω. Clearly such surfaces include vector sheets of $\mathbf{w}$.

Here the equation 4.22 can also be expressed as

$$\partial_t \mathbf{w} + L_{\mathbf{u}}\mathbf{w} = l\mathbf{w}$$

where $l = -k - \nabla \cdot \mathbf{u}$. This is equivalent to the equation $\mathbf{w}^{(1)} = l\mathbf{w}$. So from case 2 of section 4.5 it follows that the vector lines of $\mathbf{w}$ are preserved by the flow.

Example 5:

Consider the motion of the magnetic lines of force under drift velocity, as given in example 2. Let $\omega_{\mathbf{B}}$ is the two form corresponding to this magnetic field

$$\mathbf{B} = \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, 1 \right).$$

Then the corresponding derived fields of $\omega_{\mathbf{B}}$ are given by

$$\omega_{\mathbf{B}}^{(1)} \longleftrightarrow \frac{2}{(1+x^2+y^2)^2}\left(-y,\ x,\ -1\right) \quad \text{and}$$
$$\omega_{\mathbf{B}}^{(2)} \longleftrightarrow \frac{4(-1+2x^2+2y^2)}{(1+x^2+y^2)^4}\left(-y,\ x,\ -1\right).$$

So $\omega_{\mathbf{B}}^{(2)} = f\omega_{\mathbf{B}}^{(1)}$, where $f = 2(-1+2x^2+2y^2)/(1+x^2+y^2)^2$. Hence for all $m \geq 2$, $\omega_{\mathbf{B}}^{(m)} = f_m\omega_{\mathbf{B}}^{(1)}$ So any surface C on which $\omega_{\mathbf{B}}^{(1)}$ is initially annihilated will be an integral invariant 2-surface for the two form $\omega_{\mathbf{B}}$. Let $\mathbf{B}_1$ be the vector field associated with the two form $\omega_{\mathbf{B}}^{(1)}$. Then clearly on the vector sheets of the vector field $\mathbf{B}_1$ the two form $\omega_{\mathbf{B}}^{(1)}$ is annihilated initially and hence any 2-surface in this family of vector sheets will be integral invariant surfaces for $\omega_{\mathbf{B}}$. So these 2-surfaces are constant flux surfaces of the associated magnetic field $\mathbf{B}$.

Example 6:

Here we will consider a possible incompressible viscous flow under no external body forces and with vanishing pressure gradient. An exact solution to the corresponding Navier-Stokes equation is given by

$$\mathbf{u} = \left(e^{-(y+at)},\ -(a+\nu),\ b\right)$$

where a and b are constants and ν is the coefficient of kinematic viscosity. Then the vorticity field $\mathbf{w}$ is given by

$$\mathbf{w} = \left(0,\ 0,\ e^{-(at+y)}\right)$$

Let $\omega_{\mathbf{w}}$ is the two form associated to this vorticity vector field. Then

$$\omega_{\mathbf{w}}^{(1)} \longleftrightarrow \left(0,\ 0,\ \nu e^{-(at+y)}\right)$$
$$\omega_{\mathbf{w}}^{(2)} \longleftrightarrow \left(0,\ 0,\ \nu^2 e^{-(at+y)}\right)$$

So we have $\omega_{\mathbf{w}}^{(1)} = \nu\omega_{\mathbf{w}}$ and hence $\omega_{\mathbf{w}}^{(m)} = \nu^m\omega_{\mathbf{w}}$ for all $m \geq 1$. Hence for all 2-surfaces C on which the two form $\omega_{\mathbf{w}}$ vanishes initially, we have $\int_C \omega_{\mathbf{w}} = 0$ throughout the flow. The family of planes initially parallel to z-axis will constitute such a class of invariant surfaces.

Example 7:

Here we are giving an example which illustrates the surface preservation of vector fields. Consider the fields as given in example 3. Here $\mathbf{B}$ is a vector field in the three dimensional Euclidean space $\mathbb{R}^3$. Then consider the derived fields of the vector field $\mathbf{B}$.

$$\mathbf{B}^{(1)} = \left(-sin^2\, z,\ cos\ z\ sin\ z,\ 0\right)$$
$$\mathbf{B}^{(2)} = \left(-2\ cos\ z\ sin^2\, z,\ cos\ 2z\ sin\ z,\ 0\right)$$
$$\mathbf{B}^{(3)} = \left((-1 + 3\ cos\ 2z)\ sin^2\, z,\ -\frac{1}{2}(cos\ z - 3\ cos\ 3\ z)\ sin\ z,\ 0\right)$$

Then we have $\mathbf{B}^{(2)} = f\mathbf{B} + g\mathbf{B}^{(1)}$, where $f = -sin^2\, z$ and $g = cos\ z$. Here clearly $\left[\mathbf{B}\ ,\ \mathbf{B}^{(1)}\right] = 0$ and hence there exist a family of integral surfaces, which are planes perpendicular to the z-axis. Initially consider such a plane material surface. Then from case 3 of section 4.5, it follows that this material surface will remain as the integral surface for $\mathbf{B}$ and $\mathbf{B}^{(1)}$.

Here the one form $\omega = i_{\mathbf{B}^{(1)}} i_{\mathbf{B}}\Omega$ is given by

$$\omega \longleftrightarrow (0,\ 0,\ sin\ z)$$

Also $\omega^{(1)}$ is given by

$$\omega^{(1)} \longleftrightarrow (0,\ 0,\ sin\ 2\ z))$$

So $\omega^{(1)} = f\omega$, where $f = 2\ cos\ z$. Here $\omega \wedge d\omega = 0$ and the integral surfaces of ω are clearly that of **B** and $\mathbf{B}^{(1)}$, namely planes perpendicular to z-axis. For any curve on this family of surfaces, ω is a vanishing integral invariant. So this family of surfaces annuls ω initially and hence identically.

The material surface which is an integral surface of **B** and $\mathbf{B}^{(1)}$ initially, remain as an integral surface of **B** and $\mathbf{B}^{(1)}$. We conclude that the material planes perpendicular to the z-axis remains perpendicular to z-axis. Hence the material surface spanned by **B** and $\mathbf{B}^{(1)}$ remain spanned by **B** and $\mathbf{B}^{(1)}$. So the surface preservation of **B** (and $\mathbf{B}^{(1)}$) is obtained. Here note that the vector lines of **B** are not material lines. That is, a material line initially parallel to a **B**-line need not be parallel to the **B**-line during the flow.

4.8 Discussion

In this chapter we have obtained some sufficient conditions for the integral invariance of one forms and two forms under a flow field in $\mathbb{R}^3$. Also we have obtained some sufficient conditions for surface preservation of vector fields.

We have proved some general results which holds for any n-dimensional manifold. we have given sufficient conditions for the invariance of integral of a p-form ω over a p-surface C^p. We explained these results in the context of three dimensional flows in $\mathbb{R}^3$.

Let ω be a one form in the Euclidean space $\mathbb{R}^3$. Also let $\omega^{(2)} = f\omega^{(1)}$. Then it is possible to find out some particular curves for which the line integral of ω is a constant of motion. If **A** is the associated vector field of $\omega^{(1)}$ and **B** is any arbitrary smooth vector field not collinear with **A**, then the vector lines of $\mathbf{A} \times \mathbf{B}$ will constitute a family of curves over which the line integral of ω is a constant. Moreover, if Frobenius integrability condition is satisfied for $\omega^{(1)}$, then there exist a family of orthogonal surfaces for the vector field **A**. So any curve on this surfaces will be integral invariant curve.

We have shown that equation (4.4) is the sufficient condition for the surface preservation of vector fields provided that there exist an integral surfaces for $\mathbf{J}$ and $\mathbf{J}^{(1)}$. Also we have given conditions for the invariance of surface integrals of a two form in different cases.

At the end of the chapter we have given some examples which illustrate the concepts developed in the chapter. Now let us consider a particular example of viscous ABC flow, which is used in [40, 41] in the study of vortex reconnection. Here the velocity field and vorticity field given by

$$\mathbf{u} = \boldsymbol{\omega} = (A\ sin\ z + C\ cos\ y,\ B\ sin\ x + A\ cos\ z,\ C\ sin\ y + B\ cos\ x),$$

where $A = A_0 e^{-\nu t}, B = B_0 e^{-\nu t}, C = C_0 e^{-\nu t}$ and $A_0,\ B_0,\ C_0$ are constants, is an exact solution of the Navier-stokes equation. Here topology of vortex lines is invariant but helicity changes because the magnitude of vorticity changes. Also note that when there is a reconnection of vortex lines, then the topology of vortex lines changes. Let ω be the two form associated with this vorticity field. Then we have $\omega^{(1)} = -\nu\omega$. So $\omega^{(m)} = (-\nu)^m \omega$, for all $m \geq 1$. Then from section 4.6 it is clear that the vortex sheets of $\boldsymbol{\omega}$ are integral invariant 2-surfaces for this two form. So a material surface which is initially a vortex sheet will remain as a vortex sheet for this flow.

Chapter 5

Conclusion

Fluid mechanics is the source of many of the ideas and concepts that are central to modern mathematics. Mathematicians have abstracted and vastly generalised many of the fluid mechanical concepts and have a deep and powerful body of knowledge. But many of them are unfortunately now unknown to fluid mechanicians, while mathematicians themselves have lost all but a passing knowledge of physical origins of many of their basic notions. It will be surprising to a student of classical fluid mechanics to see that early mathematicians like C.F Gauss had thought of applying topological ideas in electricity and magnetism and had inspired Lord Kelvin to develop a theory of matter based on vortex knots and links. The past two decades have witnessed a revival of interest in these studies and has resulted in the origin of a new branch of study called topological fluid mechanics. The present thesis is the outcome of our investigation of some of these topological aspects of hydrodynamics and magnetohydrodynamics.

In this thesis we are studying possible invariants in hydrodynamics and hydromagnetics. The concept of flux preservation and line preservation of vector fields, especially vorticity vector fields, have been studied from the very beginning of the study of fluid mechanics by Helmholtz and others. In ideal magnetohydrodynamic flows the magnetic fields satisfy the same conservation laws as that of vorticity field in ideal hydrodynamic flows. Apart from these there are many other fields also in ideal hydrodynamic and magnetohydrodynamic flows which preserves flux across a surface or whose vector lines are preserved.

A general study using this analogy had not been made for a long time. Moreover there are other physical quantities which are also invariant under the flow, such as Ertel invariant. Using the calculus of differential forms Tur and Yanovsky classified the possible invariants in hydrodynamics. This mathematical abstraction of physical quantities to topological objects is needed for an elegant and complete analysis of invariants.

Many authors used a four dimensional space-time manifold for analysing fluid flows. We have also used such a space-time manifold in obtaining invariants in the usual three dimensional flows.

In chapter one we have discussed the invariants related to vorticity field using vorticity field two form ω^2 in E^4. Corresponding to the invariance of four form $\omega^2 \wedge \omega^2$ we have got the invariance of the quantity $\mathbb{E} \cdot \boldsymbol{\omega}$. We have shown that in an isentropic flow this quantity is an invariant over an arbitrary volume.

In chapter three we have extended this method to any divergence-free frozen-in field. In a four dimensional space-time manifold we have defined a closed differential two form and its potential one from corresponding to such a frozen-in field. Using this potential one form ω^1, it is possible to define the forms $d\omega^1$, $\omega^1 \wedge d\omega^1$ and $d\omega^1 \wedge d\omega^1$. Corresponding to the invariance of the four form we have got an additional invariant in the usual hydrodynamic flows, which can not be obtained by considering three dimensional space.

In chapter four we have classified the possible integral invariants associated with the physical quantities which can be expressed using one form or two form in a three dimensional flow. After deriving some general results which hold for an arbitrary dimensional manifold we have illustrated them in the context of flows in three dimensional Euclidean space $\mathbb{R}^3$. If the Lie derivative of a differential p-form ω is not vanishing, then the surface integral of ω over all p-surfaces need not be constant of flow. Even then there exist some special p-surfaces over which the integral is a constant of motion, if the Lie derivative of ω satisfies certain conditions. Such surfaces can be utilised for investigating the qualitative properties of a flow in the absence of invariance over all p-surfaces. We have also discussed the conditions for line preservation and surface preservation of vector fields. We see that the surface preservation need not imply the line preservation. We have given some examples which illustrate the above results.

The study given in this thesis is a continuation of that started by Vedan et.el. As mentioned earlier, they have used a four dimensional space-time manifold to obtain invariants of flow from variational formulation and application of Noether's theorem. This was from the point of view of hydrodynamic stability studies using Arnold's method.

The use of a four dimensional manifold has great significance in the study of knots and links. In the context of hydrodynamics, helicity is a measure of knottedness of vortex lines. We are interested in the use of differential forms in E^4 in the study of vortex knots and links. The knowledge of surface invariants given in chapter 4 may also be utilised for the analysis of vortex and magnetic reconnections.

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