Stochastic Modelling and Analysis

Impact of Self-generation of Priorities and Non-preemptive Service in Single/Multiserver Queues

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by

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Dedicated to the loving memory of

my Parents

Certificate

This is to certify that the work reported in the thesis entitled **'Impact of Self-generation of Priorities and Non-preemptive Service in Single/Multiserver Queues'** that is being submitted by Sri.S. Babu for the award of Doctor of Philosophy to Cochin University of Science and Technology is based on bonafide research work carried out by him under my supervision in the Department of Mathematics, Cochin University of Science and Technology. The results embodied in this thesis have not been included in any other thesis submitted previously for the award of any degree or diploma.

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Declaration

I, S.Babu here by declare that this thesis entitled **'Impact of Self-generation of Priorities and Non-preemptive Service in Single/Multiserver Queues'** contains no material which had been accepted for any other Degree or Diploma in any University and to the best of my knowledge and belief, it contains no material previously published by any person except where due reference is made in the text of the thesis.

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S.Babu

Contents

Chapter 1

Introduction

There are many situations in real life where customers have to wait in a line (queue) for getting service. This happens when there is more demand for service than there is facility available for service. Such situations arise due to shortage of servers, economical infeasibility in providing more service to avoid waiting, etc. For examples customers wait in a bank counter, patients wait in a doctors' clinic, airplanes wait to take off or landing etc.

A study on any queueing system is mainly based on the arrival of customers, type of service provided, the number of servers, the capacity of the service station and service discipline. Queueing theory has a wide range of applications in the field of telecommunication systems, computer networks, hospital management etc.

Here the queueing models are analysed by means of continuous time Markov chains in which we use the modelling tools such as Markovian Arrival/Service Process (MA/(S)P) and Phase type distributions (PH-distributions). Numerically tractable tools like these help us to model and analyse the structures so obtained in a general manner. Resulting quasi-birth-and-death processes are solved numerically by matrix analytic methods.

Now we list in brief, the definitions and terminology used in this thesis.

Continuous time PH-distribution

Let $\{X(t), t \geq 0\}$ be a continuous time Markov chain with finite state space $\{1, 2, ..., m+1\}$ and generator

$$Q = \begin{bmatrix} T & T_0 \\ 0 & 0 \end{bmatrix}$$

where the $m \times m$ matrix $T = (T_{ij})$ satisfies $T_{ii} < 0$ for $1 \leq i \leq m$ and $T_{ij} \geq 0$ for $i \neq j$. Also $T\underset{\sim}{e} + T_0 = 0$ where $\underset{\sim}{e}$ is a column vector of 1's of appropriate order. The initial probability vector of Q is given by (α, α_{m+1}) with $\alpha\underset{\sim}{e} + \alpha_{m+1} = 1$. Assume that $1, 2, \ldots, m$ are transient, so that absorption into the state $m+1$ from any initial state is certain. Let Z be the random variable representing the time until absorption *i.e.*, $Z = \inf\{t \geq 0 : X(t) = m+1\}$. Then the distribution of Z is called phase type distribution (PH- distribution) with representation (α, T). The dimension m of T is called the the order of the distribution. The distribution function of Z is

$$F(t) = \Pr(Z \leq t) = 1 - \alpha e^{Tt}\underset{\sim}{e}$$

and its probability density function is

$$f(t) = \alpha e^{Tt} T_0.$$

PH- renewal Process

A renewal process in which the renewal intervals follow PH-distribution is called a PH- renewal process. To construct a PH renewal process, consider a continuous time Markov Chain with state space $\{1, 2, ..., m+1\}$ having infinitesimal generator

$$Q = \begin{bmatrix} T & T_0 \\ 0 & 0 \end{bmatrix}.$$

The $m \times m$ matrix T is taken to be nonsingular so that absorption into the state $m+1$ is certain from any initial state. Let $(\alpha, 0)$ be the initial probability vector with $\alpha \underline{e} = 1$. When absorption occurs in the above chain we say that a renewal event, may be in the form of an arrival, has occurred and the process immediately starts anew in one of the states in $\{1, 2, ..., m\}$ according to the probability vector α, thereby continuing the process. If $0 = t_o < t_1 < t_2$......... are the time points at which the Markov process is preinitialized with α, this process forms a renewal process with inter renewal distribution PH(α, T). Here there is a rate matrix $T_1 = T_0 \cdot \alpha$ which gives transitions with arrivals. The transitions without arrivals are described by the matrix T. Now the matrix $D = T + T_1$ will be an infinitesimal generator of a Markov process $\{J(t) : t \geq 0\}$ on $\{1, 2, ..., m\}$, which is the 'phase process' associated with the PH-renewal process.

Let $N(t)$ denote the number of renewals in $(0, t)$. Then $\{(N(t), J(t)) : t \geq 0\}$ is a two dimensional Markov process with generator

$$Q = \begin{bmatrix} T & T_1 & 0 & 0 & \dots \\ 0 & T & T_1 & 0 & \dots \\ 0 & 0 & T & T_1 & \\ \vdots & & & & \ddots \end{bmatrix}.$$

Markovian Arrival Process

In PH renewal process, immediately after the occurrence of an event, the phase distribution is always α. Therefore a new phase after an arrival is chosen independently of the phase immediately before that arrival. If this restriction is relaxed we arrive at a new rate matrix T' (instead of T_1 in PH renewal process) corresponding to transitions with arrival. *i.e.*, $T' = \left(T'_{ij}\right)$ where $T'_{ij} = (T_0)_i\,(\alpha_i)_j \geq 0, (\alpha_i)_j$ is the probability that the process restart at phase j immediately after absorption at the phase i and $(T_0)_i$ is the i^{th} component of T_0. $\alpha_i = \left((\alpha_i)_1\,, (\alpha_i)_2\,, ..., (\alpha_i)_m\right)$ is a probability vector with $\alpha_i\underline{e} = 1$, for $1 \leq i \leq m$.

By choosing $D_0 = T$ and $D_1 = T'$, the matrix $D = D_0 + D_1$ is the generator of the Markov process $\{Y(t) : t \geq 0\}$ on the state space $\{1, 2, ..., m\}$. If $N(t)$ denotes the number of arrivals in $(0, t)$, then the two dimensional Markov process $\{(N(t), Y(t)) : t \geq 0\}$ with state space $\{(i,j)\,, i \geq 0, 1 \leq j \leq m\}$ has infinitesimal generator

$$Q = \begin{bmatrix} D_0 & D_1 & 0 & 0 & \cdots \\ 0 & D_0 & D_1 & 0 & \cdots \\ 0 & 0 & D_0 & D_1 & \\ \vdots & & & & \ddots \end{bmatrix}.$$

$D_1 = (\delta_{jk})$ and $D_0 = (\delta'_{jk})$ are $m \times m$ matrices where δ_{jk} denotes the transition rate from (i,j) to $(i+1,k)$, $i \geq 0; 1 \leq j,k \leq m$ and δ'_{jk} denotes the transition rate from (i,j) to (i,k), $i \geq 0; 1 \leq j,k \leq m; j \neq k$. A Markov process with such a generator Q is called Markovian Arrival Process (MAP). In BMAP (Batch Markovian Arrival Process) we have a sequence $\{D_k\}$ of matrices where entries of D_0 represents transition without an arrival and those of D_k represent transitions coupled with a batch arrival of size k (= 1,2,...)

If the process $\{Y(t) : t \geq 0\}$ is irreducible, then this Markov chain has a unique stationary distribution π such that $\pi D = 0$. The fundamental arrival rate of the MAP is given by $\lambda = \pi D_1\underline{e}$.

Quasi-birth-and-death process(QBD process)

Consider a Markov chain on a state space $\bigcup_{i\geq 0} l\,(i)$ where $l\,(i) = \{(i,j) : 1 \leq j \leq m\}$ for $i \geq 0$. The vector $l(i)$ is called the i^{th} level and j stands for the phase of the state (i,j).

The Markov chain is called a QBD process if the transitions from a state are restricted to the states in the same level or to the two adjacent levels. i.e., move in one step from (i,j) to (i',j') only if $i' = i, i+1$ for $i = 0$ and $i' = i, i+1, i-1$ for $i \geq 1$.

If the transition rates are level independent, the resulting QBD process is called level independent quasi-birth-and-death process (LIQBD). The infinitesimal generator of an LIQBD with state space defined above has the form

$$Q = \begin{bmatrix} B_0 & A_0 & 0 & 0 & \\ A_2 & A_1 & A_0 & 0 & \\ 0 & A_2 & A_1 & A_0 & \\ & & & & \\ \cdot\cdot & & & & \ddots \end{bmatrix}$$

where A_2, A_1 A_0 and B_0 are square matrices of order m. If Q is irreducible the following theorem holds [45].

Theorem 1.1. *The process Q is positive recurrent if and only if the minimal non negative solution R to the matrix quadratic equation*

$$R^2A_2 + RA_1 + A_0 = 0$$

has spectral radius less than 1 and the finite system of equations

$$x_0(B_0 + RA_2) = 0, \quad x_0(I - R)^{-1}e = 1$$

has a unique positive solution x_0.
If the matrix $A = A_0 + A_1 + A_2$ is irreducible, then $sp(R) < 1$ if and only if $\pi A_2 e > \pi A_0 e$, where π is the stationary probability vector of the generator matrix A and $sp(R)$ is the spectral radius of R.

The stationary probability vector $x = (x_0, x_1, ...)$ of Q is given by $x_i = x_0 R^i$ for $i \geq 0$.

A QBD process in which transition rates are level dependent is known as level dependent QBD process (LDQBD). The infinitesimal generator of an LDQBD on the state space $\bigcup_{i \geq 0} l\,(i)$ in which $l\,(i) = \{(i,j) : 1 \leq j \leq m_i\}$ has the form

$$Q^* = \begin{bmatrix} B_0 & A_0 & 0 & 0 & \\ C_1 & B_1 & A_1 & 0 & \\ 0 & C_2 & B_2 & A_2 & \\ \cdots & \cdots & \cdots & \cdots & \\ \cdots & \cdots & \cdots & \cdots & \end{bmatrix},$$

where $B_0\underline{e} + A_0\underline{e} = 0$ and $C_i\underline{e} + B_i\underline{e} + A_i\underline{e} = 0$ for $i \geq 1$.

Here all $B_i's$ are square matrices but the $A_i's$ and $C_i's$ in the boundary states are, in general, rectangular. Assume that the QBD is irreducible. Then we have the following theorem.

Theorem 1.2. *When Q^* is positive recurrent, its steady state distribution*

$x = (x_0, x_1, ...)$ *satisfies the relation*

$$x_{i+1} = x_i R_i \text{ for } i \geq 0$$

where the matrices R_i *are the minimal nonnegative solutions of the system of equations*

$$x_0 (B_0 + R_1 C_1) = 0$$
$$R_i R_{i+1} C_{i+1} + R_i B_i + A_{i-1} = 0, \text{for } i \geq 1$$

Regarding the positive recurrence of Q^* we have the following theorem.

Theorem 1.3. *Q^* is positive recurrent if and only if the system of equations*

$$x_0 (B_0 + R_1 C_1) = 0$$
$$x_o \underline{e} + \sum_{i=1}^{\infty} \left\{ x_0 \left(\prod_{1 \leq k \leq i-1} R_k \right) \underline{e} \right\} = 1$$

has a positive solution for x_0.

For more details one can refer [40, 45]

1.1 Review of related works

Recent application in health care systems (Brahimi[10], Taylor[54], Wang[57]), in queues with impatient customers arriving in telecommunication networks (Baccelli [4], Zhao [58], Zohar[59])and inventory systems with perishable goods (Graves[32], Perry[48]) resulted in a spurt of interest in prioritization of customers in queueing models.

Quite a large number of probabilistic models possessing a variety of properties have been discussed in the literature on priority queues (see books by Gross and Harris [33], Jaiswal[36], Takagi [52] and Stanford[49]). All these treat priority queues with exogeneous priority rules which means that the decision of selecting the next unit for service may depend only on the knowledge of the priority class to which the unit belongs. However in many applications this discipline may not be an accurate modelling approach. This is especially the case in several medical procedure - patients are treated according to the urgency of their requirements (seriousness of illness); aircraft landing and in several communication related problems. At the time of arrival a customer does not assume (i.e., not assigned)any priority; however while waiting in a clinic his condition may worsen resulting in the need urgent attention. Similarly an aircraft in queue for landing may develop problems (running out of fuel, for example)and so has to be given the next chance to land, irrespective of its position in the queue. We shall call such customers as priority generated customers(see Krishnamoorthy, Viswanath and Deepak[37], Gomez Corral, Krishnamoorthy and Viswanath[30]). Self generation of priorities by units in queue may be thought of as a consequence of their impatient behavior [58]. Classical queueing theory on impatient units [4, 5, 50, 51] usually concerns with models in which units wait for service for a limited time only and leave the system for ever if service has not begun within that time.

MAP and its extension allowing group arrival (BMAP) are wide generalization of the Poisson process and it encompasses a large class of numerically tractable point process as special case (for example Markov modulated Poisson process, Renewal process with phase type inter-renewal times, and superpositions of such processes). This class of versatile point process was introduced by Neuts[44] and further extended by Lucantoni [42]. It has the advantage of being almost as computationally tractable as the scalar Poisson process, while yielding a dense class within the Space of point processes on $[0, \infty)$. Because of its computational tractability, it is regarded as the simplest modelling choice when correlation aspect cannot be ignored. For results concerning MAPs, MAPs with marked transition and queues with exogenous priority rules, one may refer paper by He[35], Krieger [37], Leemans [41], Takine[53] and Wager [56]. Chakravarthy[14] provides an excellent account of BMAP. In all these the analysis is essentially based on Matrix Analytic Method, a thorough discussion of which can be found in Latouche and Ramaswamy [40] and Neuts [45].

Matrix analytic methods introduced by M.F. Neuts in late 1970's, establish a success story, illustrating the enrichment of science and applied probability. As a modelling tool one can use this to construct and analyze a wide class of stochastic models in an algorithmically tractable manner.

Retrial queues have been extensively studied by Fallin and Templeton [24]. An exhaustive survey of the recent developments in retrial and other queueing models tackled with matrix analytic method is given in Gomez-Corral [31].

1.2 Summary of the thesis

This thesis entitled 'Impact of Self-generation of Priorities and Non-preemptive Service in Single/Multiserver Queues' is divided into 5 chapters including this introductory one. In Chapters 2 to 5 the systems under study are always stable due to the phenomenon 'self-generation of priorities'. We employ the Bright and Taylor [13] procedure for obtaining a dominating process to arrive at a truncation level. Then the Neuts-Rao algorithm [46]is employed to obtain the steady state system state distribution.

Chapter 2 discusses a single server queueing system in which waiting customers generate priority at a constant rate. A customer in service will be completely served before this priority generated customer is taken for service. We call this service discipline as 'non-preemptive service'. Priority generated customer can wait in a waiting space of capacity 1 specially provided such class of customers. Only one priority generated customer can wait at a time and a customer generating into priority at that time will leave the system in search of emergency service. Arrival process is according to MAP and service process follow PH-distribution. Performance measures such as probability of n consecutive services of priority generated customers, that of ordinary customers, mean waiting time of a tagged customer are found by approximating them by the corresponding value in a truncated system.

In the third chapter we consider a c server queueing system in which waiting customers generate priority. Such a customer is immediately taken for service if at least one of the servers is free. Else it waits in a waiting space of capacity c exclusively for priority generated customers, provided there is space. As in Chapter 2, a customer in service will be completely served before the priority generated customer is taken for service. If there is no space and if all servers are busy, the priority generated customer will leave the system in search of urgent service elsewhere. Arrival of customers follow MAP and service times of ordinary and priority generated customers follow PH distribution. Several performance measures are evaluated and we attempt to compute the optimal number of servers to be employed to minimize the loss of customers due to priority generation.

In Chapter 4, a multi server retrial queue is considered. An arriving customer who finds the server busy, join an orbit of infinite capacity. Each customer in the orbit tries independently of others to access the server. Customers in the orbit generate priority at constant rate and such a customer is immediately taken for service if any of the server is free. Else they wait in a waiting space as described in Chapter 3 or leave the system if all servers are busy and if there is no waiting space. Arrival, service patterns and performance measures are also discussed.

Chapter 5 deals with a multi-priority retrial queue with a finite number of priority classes having finite waiting space and an orbit of infinite capacity for the least priority customers. The system has only one server. An arriving customer join the waiting line of the a priority class to which he belongs if there are vacant spaces. There is a super priority class of capacity 1 in which no arrival from outside the system takes place. The customers waiting in the lower priority classes generate priority and joins the higher priority class as dictated by the priority generated; provided there are vacant spaces. Else they leave the system in search of emergency service elsewhere. Customers of least priority join the orbit if at the time of arrival the server is busy and tries independently of each other to access the server at a constant rate. Priority generation of customers in the orbit is only to the super priority class. The service discipline is non-preemptive. Customers arrive to the system according to a marked Markovian arrival process and the service time distribution of each customer is phase type. System performance measures are provided with numerical illustrations.

Chapter 2

MAP/(PH,PH)/1 Queue

In this chapter we consider single server queueing system in which customers arrive according to a Markovian arrival process. Waiting customers generate priority at a constant rate. Such a customer waits in a waiting space of capacity 1,if the server is busy and if this waiting space is not already occupied by a priority generated customer. A customer in service will be completely served before the priority generated customer is taken for service (non - preemptive service discipline). Only one priority generated customer can wait at a time and a customer generating into priority at that time, will have to leave the system in search of emergency service elsewhere. The service times of ordinary and priority generated customers follow distinct PH-distributions. Matrix Analytic method is used to compute steady state distribution and performance evaluation. Performance measures such as probability of n consecutive services of priority generated customers, probability of the same for ordinary customers, mean waiting time of a tagged customer are found by approximating them by their corresponding values in a truncated system.

This chapter is arranged as follows. In section 2.1 the problem is mathematically formulated and analysed. In section 2.2 we see that the system under study is always stable. We employ Bright and Taylor procedure for obtaining a dominating process to arrive at a truncation level. Then the Neuts-Rao algorithm is employed to obtain the steady state system state distribution. These are done in section 2.3. In section 2.4 we provide various system performance measures of interest. Finally, numerical illustration are given in section 2.5.

2.1 Mathematical modelling

Customers arrive to a single server counter according to MAP with representation (D_0, D_1) of order m_1. At the time of arrival all customers are classified as 'ordinary'. If the server is busy the arriving customers join a queue. Waiting customers 'generate priority' at a constant rate γ in such a way that if there are n customers in the queue then the rate of priority generation is $n\gamma$. Such a customer waits in waiting space of capacity 1 (exclusively for priority generated customers) for service which begins on completion of the present service. A second priority generated customer during that time period (while the previously generated priority customer is waiting) will have to leave the system in search of emergency service elsewhere.

The service time of ordinary and priority generated customers follow PH-distribution with representation (α,T) and (β,S) respectively with T_0= -T $\underline{e}$ and S_0=-S $\underline{e}$ where $\underline{e}$ is a column vector of 1's of appropriate order. Let

$N_1(t) = \sharp$ of ordinary customers in the system at time t

$N_2(t) = \sharp$ of priority generated customers in service.

$N_3(t) = \sharp$ of priority generated customers waiting for service.

$M_1(t)$ = phase of arrival process at time t.

$M_2(t)$ = phase of service process of ordinary customers at time t.

$M_3(t)$ = phase of service process of priority generated customer at time t.

If $X(t) = \{N_1(t), N_2(t), N_3(t), M_1(t), M_2(t)/M_3(t)\}$,then $\{X(t), t \geq 0\}$ form a continuous time Markov chain $\mathcal{H}$ with state space

$\{(0,0,0,k_1); 1 \leq k_1 \leq m_1\} \bigcup \{(0,1,j_2,k_1,k_3); j_2 = 0,1; 1 \leq k_1 \leq m_1; 1 \leq k_3 \leq m_3\} \bigcup \{(i,0,j_2,k_1,k_2); i \geq 1; j_2 = 0,1; 1 \leq k_1 \leq m_1; 1 \leq k_2 \leq m_2\} \bigcup$
$\{(i,1,j_2,k_1,k_3); i \geq 1; j_2 = 0,1; 1 \leq k_1 \leq m_1; 1 \leq k_3 \leq m_3\}$

By partitioning the state space into levels with respect to the number of ordinary customers in the system, the generator of the above Markov chain is of the form

$$Q = \begin{bmatrix} B_0 & A_0 & & & \\ C_1 & B_1 & A_1 & & \\ & C_2 & B_2 & A_1 & \\ & & & & \end{bmatrix}, \text{ where}$$

$$A_0 = \begin{bmatrix} (D_1 \otimes \alpha)_{\nu_1 \times \nu_2} & 0_{\nu_1 \times \nu_2} & 0_{\nu_1 \times \nu_3} \\ 0_{\nu_3 \times \nu_2} & 0_{\nu_3 \times \nu_2} & \left(U_0^{(0)}\right)_{\nu_3 \times \nu_3} \end{bmatrix}, \quad \text{with}$$

$\nu_1 = m_1, \ \nu_2 = m_1 m_2$ and $\nu_3 = 2m_1 m_3$,

$$U_0^{(0)} = \begin{bmatrix} D_1 \otimes I_{m_3} & 0 \\ 0 & D_1 \otimes I_{m_3} \end{bmatrix}; \quad A_1 = \begin{bmatrix} U_1^{(0)} & 0 \\ 0 & U_0^{(0)} \end{bmatrix},$$

$$U_1^{(0)} = \begin{bmatrix} D_1 \otimes I_{m_2} & 0 \\ 0 & D_1 \otimes I_{m_2} \end{bmatrix};$$

$$B_0 = \begin{bmatrix} D_0 & 0 \\ V_0^{(0)} & V_0^{(1)} \end{bmatrix}, \quad V_0^{(0)} = \begin{bmatrix} I_{m_1} \otimes S_0 \\ 0_{m_1 m_3 \times m_1} \end{bmatrix},$$

$$V_0^{(1)} = \begin{bmatrix} D_0 \oplus S & 0 \\ I_{m_1.} \otimes S_0 \beta & D_0 \oplus S \end{bmatrix}; \quad \text{for k} \geq 1, \ B_k = \begin{bmatrix} V_k^{(1)} & 0 \\ V_k^{(2)} & V_k^{(3)} \end{bmatrix},$$

$$V_k^{(1)} = \begin{bmatrix} D_0 \oplus T - (k-1)\gamma I_{m_1m_2} & 0 \\ 0 & D_0 \oplus T - (k-1)\gamma I_{m_1m_2} \end{bmatrix},$$

$$V_k^{(2)} = \begin{bmatrix} I_{m_1} \otimes \alpha \otimes S_0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$V_k^{(3)} = \begin{bmatrix} D_0 \oplus S - k\gamma I_{m_1m_3} & 0 \\ I_{m_1} \otimes (S_0 \otimes \beta) & D_0 \oplus S - k\gamma I_{m_1m_3} \end{bmatrix};$$

$$C_1 = \begin{bmatrix} W_1^{(0)} & W_1^{(1)} \\ 0 & W_1^{(2)} \end{bmatrix}, \quad W_1^{(0)} = \begin{bmatrix} I_{m_1} \otimes T_0 \\ 0 \end{bmatrix},$$

$$W_1^{(1)} = \begin{bmatrix} 0 & 0 \\ I_{m_1} \otimes (T_0 \otimes \beta) & 0 \end{bmatrix}, \quad W_1^{(2)} = \begin{bmatrix} 0 & \gamma I_{m_1m_3} \\ 0 & \gamma I_{m_1m_3} \end{bmatrix};$$

$$\text{for } k \geq 2, \quad C_k = \begin{bmatrix} W_k^{(0)} & W_k^{(1)} \\ 0 & W_k^{(2)} \end{bmatrix}, \quad W_k^{(0)} = \begin{bmatrix} I_{m_1} \otimes (T_0\alpha) & (k-1)\gamma \mathrm{I}_{m_1m_2} \\ 0 & (k-1)\gamma \mathrm{I}_{m_1m_2} \end{bmatrix},$$

$$W_k^{(1)} = \begin{bmatrix} 0 & 0 \\ I_{m_1} \otimes (T_0\beta) & 0 \end{bmatrix}, \quad W_k^{(2)} = \begin{bmatrix} 0 & k\gamma \mathrm{I}_{m_1m_3} \\ 0 & k\gamma \mathrm{I}_{m_1m_3} \end{bmatrix}.$$

2.2 System stability

Theorem 2.1. *The system under discussion is always stable.*

Proof. Consider the Lyapunov test function defined by $\varphi(s) = i$, where 's' is a state in level i. Then for a state 's' in level i, the mean drift y_s is given by

$$y_s = \sum_{p \neq s} [\phi(p) - \phi(s)] q_{sp}$$

$$= \sum_{s'} [\phi(s') - \phi(s)]\, q_{ss'} + \sum_{s''} [\phi(s'') - \phi(s)]\, q_{ss''} + \sum_{s'''} [\phi(s''') - \phi(s)]\, q_{ss'''}$$

where s', s'' and s''' vary over the states belonging to levels $i-1$, i and $i+1$, respectively. Then $\varphi(s) = i,\ \varphi(s') = i-1,\ \varphi(s'') = i$ *and* $\varphi(s''') = i+1$

$$y_s = -\sum_{s'} q_{ss'} + \sum_{s'''} q_{ss'''}$$

$$= \begin{cases} -(i-1)\gamma - (e_{2m_1} \otimes T_0)_s + \sum\limits_{s'''} q_{ss'''}, \text{if the server is busy with ordinary customer} \\ -i\gamma + \sum\limits_{s'''} q_{ss'''} \text{ ,if the server is busy with priority generated customer} \end{cases}$$

where $(e_{2m_1} \otimes T_0)_s$ denotes the s^{th} entry of the vector $e_{2m_1} \otimes T_0$. Since the number of phase is finite, $\sum\limits_{s'''} q_{ss'''}$ is bounded by some fixed constant for any s in level $i \geq 1$. Hence we can find a positive real number K such that $\sum\limits_{s'''} q_{ss'''} < K$ for all s in level $i \geq 1$. Thus, for any $\varepsilon > 0$, we can find N large enough that $y_s < -\varepsilon$ for any s belonging to level $i \geq$ N. Hence the theorem follows from Tweedie [55]. □

2.3 Steady state distribution

Let $x = (x_0, x_1, ...)$ be the equilibrium distribution. For a positive recurrent $LDQBD$, x_i satisfies the relationship $x_{k+1} = x_k R_k$, $k \geq 0$, which gives $x_{k+1} = x_0 \prod\limits_{l=0}^{k} R_l$ where the family of matrices $\{R_k : k \geq 0\}$ are the minimal nonnegative solution of the system of equations

$$A_0 + R_0 B_1 + R_0 R_1 C_2 = 0, \tag{2.1}$$

$$A_1 + R_k B_{k+1} + R_k R_{k+1} C_{k+2} = 0 \text{ for } k \geq 1 \tag{2.2}$$

and x_0 is the solution of

$$x_0(B_0 + R_0 C_1) = 0 \tag{2.3}$$

subject to

$$x_0 \underline{e} + x_0 \left(\sum_{k=1}^{\infty} \prod_{l=0}^{k-1} R_l \right) \underline{e} = 1 \tag{2.4}$$

Before we pass on to the numerical computations we construct a dominating process. Here the process under discussion, $\{X(t), t \geq 0\}$, satisfies the condition that for all k$\geq$ 1 and for all i, there exists j such that $(C_k)_{ij} > 0$. Therefore, there exists a dominating process $\bar{X}(t)$ (see Bright and Taylor [13]) on the same state space as X(t) and with generator

$$\bar{Q} = \begin{bmatrix} B_0 & A_0 & & & & \\ 0 & \bar{B}_1 & \bar{A}_1 & & & \\ . & \bar{C}_2 & \bar{B}_2 & \bar{A}_1 & & \\ & & \bar{C}_3 & \bar{B}_3 & \bar{A}_1 & \\ & & & & & \ddots \end{bmatrix},$$

where, $(\bar{A}_1)_{i,j} = \frac{1}{\mu}\left((A_1 e)_{\max}\right)$,

$(\bar{C}_k)_{i,j} = \frac{1}{\mu}\left((C_{k-1}e)_{\min}\right), k \geq 2,$

$(\bar{B}_k)_{i,j} = (B_k)_{i,j}, i \neq j$ and $k \geq 1$

and $\mu = 2m_1(m_2 + m_3)$ is the dimension of the level $k \geq 1$, and $(A_1\underline{e})_{max}$ is the maximum element of the column vector $A_1\underline{e}$

Let $\{l_n, n \geq 0\}$ and $\{\bar{l}_n, n \geq 1\}$ be the marginal distributions of the levels of X(t) and $\bar{X}(t)$, respectively, in the long run as the system get stabilized. Let $\bar{z} = (z_1, z_2, ...)$ be an invariant measure for $\bar{X}(t)$. Define $\bar{L}_n = \bar{z}_n\underline{e}$ and $P_0^{-1} = \sum_{n=1}^{\infty} \bar{L}_n$. If $P_0^{-1} < \infty$, then an equilibrium distribution for $\bar{X}(t)$ exists and $\bar{l}_n = P_0\bar{L}_n$. But the structure of $\bar{X}(t)$ shows that $\{\bar{l}_n, n \geq 1\}$ can be considered as an equilibrium distribution of a standard birth-and-death process on state space $\{i \geq 1\}$ with transition rates $\bar{q}(i, j)$ given by

$$\bar{q}(0, 1) = 0,$$

$$\bar{q}(i, i+1) = (A_1\underline{e})_{max}, \; i \geq 1,$$

$$\bar{q}(1, 0) = 0,$$

$$\bar{q}(i, i-1) = (C_{i-1}\underline{e})_{min}, \; i \geq 2$$

So $\{\bar{l}_n, n \geq 1\}$ is given by

$$\bar{l}_n = P_0 \prod_{i=1}^{n-1} \frac{\bar{q}(i,i+1)}{\bar{q}(i+1,i)}, \; n \geq 1. \tag{2.5}$$

Equation (2.5) shows that a sufficient condition for $P_0^{-1} < \infty$ is that $\frac{\bar{q}(i,i+1)}{\bar{q}(i+1,i)} < r < 1$, $\forall$ i$\geq$ N for some N. Thus if $\{\bar{l}_n, n \geq 1\}$ exists, $\bar{x}$, the steady state distribution of $\bar{X}(t)$, must exist and therefore x must exist since $\bar{X}(t)$ stochastically dominate X(t). Now we fix the truncation level K^* such that $\sum_{n=K^*}^{\infty} l_n < \varepsilon$. Since $\bar{X}(t)$ dominates $X(t)$ we have $\sum_{n=K^*}^{\infty} l_n \leq \sum_{n=K^*}^{\infty} \bar{l}_n$, so it is sufficient to fix K^* such that $\sum_{n=K^*}^{\infty} \bar{l}_n < \varepsilon$.

We use the K^* obtained by the above method to fix the truncation level and employ Neuts-Rao procedure in numerical computations. Thus $x_k(K^*), 1 \leq k \leq K^*$, is given by $x_k(K^*) = x_0(K^*) \prod_{l=0}^{k-1} R_l$ where $x_0(K^*)$ satisfies $x_0(B_0 + R_0C_1) = 0$. The components of x above the level K^* are given by $x_{K^*+i} = x_{K^*} \prod_{j=1}^{i} R_{K^*+j}$ and eq. (2.4) becomes $x\underline{e} = x_{K^*+1}(I - R_{K^*})^{-1}\underline{e} + x_0(K^*)\underline{e} + x_0(K^*) \sum_{k=1}^{K^*} \prod_{l=0}^{k-1} R_l)\underline{e} = 1$. Note that $x_{K^*+1}(I - R_{K^*})^{-1}\underline{e} < \varepsilon$ for our choice of K^*.

2.4 System performance measures

For the evaluation of system performance measures we partition each x_i in the steady state probability vector $x = (x_0, x_1, x_2, ...)$ as follows

$x_0 = (y_0(0,0), y_0(1,0), y_0(1,1))$

$x_i = (y_i(0,0), y_i(0,1), y_i(1,0)), y_i(1,1))$ for $i \geq 1$

where $y_k(i,j)$ is a row vector corresponding to $N_2(t) = i$ and $N_3(t) = j$.

We concentrate on the following performance measures.

- Average number E_1, of ordinary customers in the system $= \sum_{i=0}^{\infty} i\ x_i \underline{e}$
- Average number E_2, of priority generated customers in the system

$= y_0(1,0)\underline{e} + 2y_0(1,1)\underline{e} + \sum_{k=1}^{\infty} (y_k(0,1) + y_k(1,0))\,\underline{e} + \sum_{k=1}^{\infty} 2y_k(1,1)\underline{e}$

- Average number E_3, of priority generated customers lost per unit time

$= \sum_{k=2}^{\infty}(k-1)\gamma y_k(0,1)\underline{e} + \sum_{k=1}^{\infty} k\gamma y_k(1,1)\underline{e})$

- Probability $\mathcal{P}_1$ that a priority generated customer is waiting for service

$= y_0(1,1)\underline{e} + \sum_{k=1}^{\infty} y_k(0,1)\underline{e} + \sum_{k=1}^{\infty} y_k(1,1)\underline{e}$

For convenience we partition the probability vector $y_i(0,0)$ as

$y_i\,(0,0) = (y_{i00}\,(1)\,,......, y_{i00}\,(m_1m_2))$

(here we make the substitution $j = (k_1-1)m_2 + k_2$ which maps $(k_1,k_2), 1 \le k_1 \le m_1, 1 \le k_2 \le m_2$ respectively, into $1,2,.....,m_1m_2$). By a similar argument we can write,

$y_i\,(1,0) = (y_{i10}\,(m_1m_2+1)\,......, y_{i10}\,(2m_1m_2)),$

$y_i\,(0,1) = (y_{i01}\,(2m_1m_2+1)\,,......, y_{i01}, (2m_1m_2+m_1m_3))$ and

$y_i\,(1,1) = (y_{i11}\,(2m_1m_2+m_1m_3+1)\,,......, y_{i11}\,(2m_1m_2+2m_1m_3))$. Then

- probability $\mathcal{P}_2$ that a priority generated customer is lost to the system

$= \sum_{i=1}^{\infty}\left\{\sum_{j=m_1m_2+1}^{2m_1m_2} ((i-1)\gamma/(-B_i(j,j))y_{i01}\,(j) + \sum_{j=2m_1m_2+m_1m_3+1}^{2m_1(m_2+m_3)} (i\gamma/(-B_i(j,j))y_{i11}\,(j)\right\}$

- Probability $\mathcal{P}_3$ that a priority generated customer is retained in the system

$= \sum_{i=1}^{\infty}\left\{\sum_{j=1}^{m_1m_2} ((i-1)\gamma/(-B_i(j,j))y_{i00}\,(j) + \sum_{j=2m_1m_2+1}^{2m_1m_2+m_1m_3} (i\gamma/(-B_i(j,j))y_{i10}\,(j)\right\}$

- Probability $\mathcal{P}_4$ that the server is idle $= y_0(0,0)$

Probability of n consecutive services for priority generated customers.

Here we obtain the probability $\mathcal{PP}_n$ that there are exactly n consecutive services for priority generated customers between the services of two ordinary customers. We note that for this event to happen there should be a priority generation during the service of the ordinary customer to be followed by at least one priority generation during the service to each of the $n-1$ priority generated customers and there should not be a priority generation during the service to the n^{th} priority customer.

We find the probability $\mathcal{PP}_n$ by approximating it using the probabilities $\mathcal{PP}_n^{(N)}$ as $N \longrightarrow \infty$. The probability $\mathcal{PP}_n^N$ is defined to be the probability for exactly n consecutive priority services following an ordinary service in the queueing system $\mathcal{H}_N$, which is obtained by truncating the original system $\mathcal{H}$, where the truncation is done such that no customer is allowed to join the system if the number of ordinary customers in the system (including the one in service) is equal to N with the waiting space for priority generated customer either empty or occupied. We note that $\mathcal{H}_N$will have the state space $\bigcup\limits_{i=0}^{N} l(i)$ where $l(i)$ is the same as that defined for $\mathcal{H}$

Now consider the case of a priority service starting in $\mathcal{H}_N$ following an ordinary service. i.e., $\mathcal{H}_N$ is in one of the states $(0,1,0),(1,1,0),.......,(N-1,1,0)$, (for convenience we use the first three coordinates of elements of the state space). From a state in sub level $(i,1,0)$, $1 \leq i \leq N-1$, the chain $\mathcal{H}_N$ can move to the states in the sub level $(i+1,1,0)$ due to an arrival or to state $(i,0,0)$ due to a service completion and to states $(i-1,1,1)$ due to a priority generation; from the state $(0,1,0)$ the chain can move either to $(1,1,0)$ due to arrival or to $(0,0,0)$ due to a service completion. From state $(N,1,0)$ the chain $\mathcal{H}_N$ can move to states $(N,0,0)$ due to a service completion and to state $(N-1,1,1)$ due to a priority generation. We want to find the probability that starting from one of the states in $\{(0,1,0),(1,1,0),....,(N-1,1,0)\}$ the chain reaches one of the states in $\{(0,1,1),(1,1,1),....,(N-1,1,1)\}$ before reaching any of the states in $\{(0,0,0),(1,0,0),....,(N,0,0)\}$. This probability can be found in the vector $z_1 = \bar{y}(I-P_N)^{-1}\tilde{P}_N$, where $\bar{y} = (y_{0,1,0}, y_{1,1,0}, ..., y_{N-1,1,0}, \underline{0})$ with $\underline{0}$ being a zero vector containing m_1m_3 elements.

$(P_N)_{i,j} = \frac{(W_{N_0})_{i,j}}{-(W_{N_0})_{i,i}}, i \neq j,\ 1 \leq i,j \leq (N+1)m_1m_3,\quad (P_N)_{i,i} = 0$ and

$(\tilde{P}_N)_{i,j} = \frac{(\tilde{W}_{N_0})_{i,j}}{-(W_{N_0})_{i,i}}, 1 \leq i,j \leq (N+1)m_1m_3;$

$$W_{N_0} = \begin{bmatrix} D_0 \oplus S & D_1 \otimes I_{m_3} & & & \\ & D_0 \oplus S - \Gamma_1 & D_1 \otimes I_{m_3} & & \\ & & \ddots & \ddots & \\ & & & D_0 \oplus S - \Gamma_{N-1} & D_1 \otimes I_{m_3} \\ & & & & D \oplus S - \Gamma_N \end{bmatrix}$$

with $\Gamma_i = i\gamma I_{m_1m_3}$ and $D = D_0 + D_1$;

$$\tilde{W}_{N_0} = \begin{array}{c} \\ (0,1,0) \\ (1,1,0) \\ (2,1,0) \\ \\ (N-1,1,0) \\ (N,1,0) \end{array} \begin{array}{c} \begin{array}{cccc} (0,1,1) & (1,1,1) & (N-1,1,1) & (N,1,1) \end{array} \\ \begin{bmatrix} 0 & & & \\ \gamma I_{m_1m_3} & & & \\ & 2\gamma I_{m_1m_3} & & \\ & & \ddots & \\ & & 0 & 0 \\ & & N\gamma I_{m_1m_3} & 0 \end{bmatrix} \end{array}.$$

Similarly the probability that starting from one of the states in $\{(0,1,0),(1,1,0),...,(N-1,1,0)\}$ the chain reaches one of the states in $\{(0,0,0),(1,0,0),....,(N,0,0)\}$ before reaching any of the states in $\{(0,1,1),(1,1,1),....,(N,1,1)\}$ is given by

$$\bar{z}_1 = \bar{y}(I - P_N)^{-1}\tilde{\tilde{P}}_N$$

$(\tilde{\tilde{P}}_N)_{i,j} = \frac{(\tilde{\tilde{W}}_{N_0})_{i,j}}{-(W_{N_0})_{i,i}}, 1 \leq i,j \leq (N+1)m_1m_3$

$$\tilde{\tilde{W}}_{N_0} = \begin{bmatrix} I_{m_1} \otimes S_0 & & & \\ & I_{m_1} \otimes \alpha \otimes S_0 & & \\ & & \ddots & \\ & & & I_{m_1} \otimes \alpha \otimes S_0 & \\ & & & & I_{m_1} \otimes \alpha \otimes S_0 \end{bmatrix}$$

We note that $z_1\underline{e}$ and $\bar{z}_1\underline{e}$ give respectively, the probability that there will be at least one transition due to priority generation and there will not be any transition due to priority generation before the service completion of a priority customer who is selected for service after an ordinary service completion.

Now starting from the states $\{(0,1,1),(1,1,1),....,(N-1,1,1),(N,1,1)\}$ with probabilities recorded in z_1,the probability that the chain reaches the states in $\{(0,1,0),(1,1,0),....,(N-1,1,0),(N,1,0)\}$ is given by the vector

$$z_2 = z_1(I - P_{N_1})^{-1}\tilde{P}_{N_1}.$$

$(P_{N_1})_{i,j} = \frac{(W_{N_1})_{i,j}}{-(W_{N_1})_{i,i}}, i \neq j, 1 \leq i,j \leq (N+1)m_1m_3, (P_{N_1})_{i,i} = 0$ and
$(\tilde{P}_{N_1})_{i,j} = \frac{(\tilde{W}_{N_1})_{i,j}}{-(W_{N_1})_{i,i}}, 1 \leq i,j \leq (N+1)m_1m_3.$ where

$$W_{N_1} = \begin{bmatrix} D_0 \oplus S & D_1 \otimes I_{m_3} & & & \\ \Gamma_1 & D_0 \oplus S - \Gamma_1 & D_1 \otimes I_{m_3} & & \\ & \ddots & \ddots & \ddots & \\ & & \Gamma_{N-1} & D_0 \oplus S - \Gamma_{N-1} & D_1 \otimes I_{m_3} \\ & & & \Gamma_N & D \oplus S - \Gamma_N \end{bmatrix},$$

with $\Gamma_i = i\gamma I_{m_1m_3}$ and $D = (D_0 + D_1)$;

$$\tilde{W}_{N_1} = \begin{bmatrix} I_{m_1} \otimes S_0 \otimes \beta & & & \\ & I_{m_1} \otimes S_0 \otimes \beta & & \\ & & \ddots & \\ & & & I_{m_1} \otimes S_0 \otimes \beta & \\ & & & & I_{m_1} \otimes S_0 \otimes \beta \end{bmatrix}.$$

Note that, since $(I - P_{N_1})^{-1}\tilde{P}_{N_1}\underline{e} = \underline{e}$,

$$\begin{aligned} z_2\ \underline{e} &= z_1(I - P_{N_1})^{-1}\tilde{P}_{N_1}\underline{e} \\ &= z_1\ \underline{e}. \end{aligned}$$

Again starting in $\{(0,1,0),(1,1,0),....,(N,1,0)\}$, according to z_2, the probability that before a transition due to service completion, there is at least one transition due to priority generation or no transition due to priority generation are given by

$$z_2(I - P_N)^{-1}\tilde{P}_N\underline{e} \text{ and } z_2(I - P_N)^{-1}\tilde{\tilde{P}}_N\underline{e} \text{ respectively.}$$

So the probability that there are exactly two consecutive priority services after an ordinary customer's service completion, followed by a priority customer's selection for service is

$$\mathcal{PP}_2^{(N)} = \bar{y}(I - P_N)^{-1}\tilde{P}_N(I - P_{N_1})^{-1}\tilde{P}_{N_1}(I - P_N)^{-1}\tilde{\tilde{P}}_N\underline{e}$$

Proceeding like this

$$\mathcal{PP}_n^{(N)} = \bar{y}[(I - P_N)^{-1}\tilde{P}_N(I - P_{N_1})^{-1}\tilde{P}_{N_1}]^{n-1}(I - P_N)^{-1}\tilde{\tilde{P}}_N\underline{e}$$

In table 3 we produce numerical evidence for convergence of $\mathcal{PP}_n^{(N)}$ to $\mathcal{PP}_n$

Probability of n consecutive services for ordinary customers.

Here we compute the probability $\mathcal{OP}_n$ that, starting with the srvice of an ordinary customer there are exactly n consecutive ordinary services, to be followed by the service of a priority generated customer. As in the previous section we approximate $\mathcal{OP}_n$ by using the probability $\mathcal{OP}_n^N$ as N $\longrightarrow \infty$, where $\mathcal{OP}_n^N$ is defined as the required probability in the system $\mathcal{H}_N$

After an ordinary service started in $\mathcal{H}_N$ in one of the states $\{(1,0,0),(2,0,0),..,(N,0,0)\}$ according to $\bar{y}_1 = (y_{1,0,0}, y_{2,0,0}, ..., y_{N,0,0})$, the probability that $\mathcal{H}_N$ reaches one of the states in $\{(0,0,0),(1,0,0),....,(N-1,0,0)\}$ due to a service completion and before any priority generation, is given by

$$z_3 = \bar{y}_1(I - P_{N_2})^{-1}\tilde{P}_{N_2} \ ,$$

where $(P_{N_2})_{i,j} = \frac{(W_{N_2})_{i,j}}{-(W_{N_2})_{i,i}}, i \neq j, 1 \leq i,j \leq Nm_1m_2, (P_{N_2})_{i,i} = 0,$

$(\tilde{P}_{N_2})_{i,j} = \frac{(\tilde{W}_{N_2})_{i,j}}{-(W_{N_2})_{i,i}}, 1 \leq i,j \leq Nm_1m_2.$

$$W_{N_2} = \begin{bmatrix} D_0 \oplus S & D_1 \otimes I_{m_2} & & & \\ & D_0 \oplus T - \Gamma_1' & D_1 \otimes I_{m_2} & & \\ & & \ddots & \ddots & \\ & & & D_0 \oplus T - \Gamma_{N-2}' & D_1 \otimes I_{m_2} \\ & & & & D \oplus T - \Gamma_{N-2}' \end{bmatrix},$$

where $D = (D_0 + D_1)$ and $\Gamma_i' = i\gamma I_{m_1m_2}$.

$$\tilde{W}_{N_2} = \begin{array}{c} \\ (1,0,0) \\ (2,0,0) \\ (3,0,0) \\ \\ (N,0,0) \end{array} \begin{array}{c} \begin{array}{cccc} (0,0,0) & (1,0,0) & (2,0,0) & \quad (N-1,0,0) \end{array} \\ \left[\begin{array}{cccc} I_{m_1} \otimes T_0 & & & \\ & I_{m_1} \otimes \alpha \otimes T_0 & & \\ & & I_{m_1} \otimes \alpha \otimes T_0 & \\ & & & \ddots \\ & & & I_{m_1} \otimes \alpha \otimes T_0 \end{array} \right] \end{array}.$$

Again starting in one of the states in $\{(0,0,0),(1,0,0),....,(N-1,0,0),(N,0,0)\}$ according to $z_4 = (z_3, \underline{0})$, where $\underline{0}$ is a zero row vector of order $m_1 m_2$, the probability that the chain reaches $\{(0,0,0),(1,0,0),....,(N-1,0,0),(N,0,0)\}$ due to a service completion and before any priority generation, is given by the vector

$$z_5 = z_4 (I - P_{N_3})^{-1} \tilde{P}_{N_3}$$

and the probability that there will be a priority generation before service completion is given by

$$\begin{aligned} z_6 \underline{e} &= z_4 (I - P_{N_3})^{-1} \tilde{\tilde{P}}_{N_3} \underline{e}, \text{ where} \\ (P_{N_3})_{i,j} &= \tfrac{(W_{N_3})_{i,j}}{-(W_{N_3})_{i,i}}, i \neq j, 1 \leq i,j \leq N m_1 m_2, \ (P_{N_3})_{i,i} = 0, \\ (\tilde{P}_{N_3})_{i,j} &= \tfrac{(\tilde{W}_{N_3})_{i,j}}{-(W_{N_3})_{i,i}}, 1 \leq i,j \leq N m_1 m_2 \text{ and} \\ (\tilde{\tilde{P}}_{N_3}) \underline{e} &= \tfrac{(\tilde{\tilde{W}}_{N_3}) \underline{e}}{-(W_{N_3})_{i,i}}. \end{aligned}$$

$$W_{N_3} = \left[\begin{array}{cccccc} D_0 & D_1 \otimes \alpha & & & & \\ & D_0 \oplus T & D_1 \otimes I_{m_2} & & & \\ & & D_0 \oplus T - \Gamma_1' & D_1 \otimes I_{m_2} & & \\ & & & \ddots & \ddots & \\ & & & & D_0 \oplus T - \Gamma_{N-1}' & D_1 \otimes I_{m_2} \\ & & & & & D \oplus T - \Gamma_{N-1}' \end{array} \right],$$

where $\Gamma_i' = i\gamma I_{m_1m_2}$ and $D = D_0 + D_1$,

$$\tilde{W}_{N_3} = \begin{array}{c} \\ (0,0,0) \\ (1,0,0) \\ (2,0,0) \\ \\ (N,0,0) \end{array} \begin{array}{c} \begin{array}{cccc} (0,0,0) & (1,0,0) & (N-1,0,0) & (N,0,0) \end{array} \\ \left[\begin{array}{ccccc} 0I_{m_1} & & & & \\ I_{m_1}\otimes T_0 & & & & \\ & I_{m_1}\otimes\alpha\otimes T_0 & & & \\ & & \ddots & & \\ & & & I_{m_1}\otimes\alpha\otimes T_0 & 0I_{m_1m_2} \end{array}\right] \end{array},$$

$$\tilde{\tilde{W}}_{N_3} = \left[\begin{array}{c} 0I_{m_1} \\ 0I_{m_1m_2} \\ \gamma I_{m_1m_2} \\ 2\gamma I_{m_1m_2} \\ \vdots \\ (N-1)\gamma I_{m_1m_2} \end{array}\right].$$

Thus the probability of exactly two consecutive services to ordinary customers is $\mathcal{OP}_2^N = z_6\underline{e}$. Proceeding like this

$$\mathcal{OP}_n^N = z_4[(I - P_{N_3})^{-1}\tilde{P}_{N_3}]^{n-2}(I - P_{N_3})^{-1}\tilde{\tilde{P}}_{N_3}\underline{e}\ , \quad n = 3, 4, 5, ...$$

In table 4 we produce numerical evidence for convergence of $\mathcal{OP}_n^{(N)}$ to $\mathcal{OP}_n$.

Expected waiting time of a tagged customer.

The waiting time W of a tagged customer is defined to be the amount of time a tagged customer waits in the system (either as ordinary customer or as a priority generated customer) until he is taken for service or leave the system by priority generation because the waiting space is already occupied by a priority generated customer. We find the expected value $E(W)$ by approximating it with the expected waiting time of a tagged customer in the truncated system $\mathcal{H}_N$ which we defined in the previous section.

Let us define $W^{(N)}$ as the time until absorption in the Markov Chain $\mathcal{H}_N^*$ defined as $\mathcal{H}_N^* = \{N_T^{(t)}, N_A^{(t)}, N_B^{(t)}, S(t), N_p(t), M_1(t), M_2(t)/M_3(t) | t \geq 0\}$, where

$$N_T(t) = \begin{cases} 0 & \text{if the tagged customer waits as an ordinary customer} \\ 1 & \text{if the tagged customer waits as an priority generated customer} \\ 2 & \begin{cases} \text{if the tagged customer has to leave the system} \\ \text{on priority generation or is selected for service} \end{cases} \end{cases}$$

$N_A(t) = \sharp$ of ordinary customers ahead of the tagged customer.

$N_B(t) = \sharp$ of ordinary customers behind the tagged customer.

$$S(t) = \text{server status}$$
$$= \begin{cases} 0 \text{ if the server is busy with the service of an ordinary customer} \\ 1 \text{ if the server is busy with a priority generated customer} \end{cases}$$

$N_p(t) = \sharp$ of priority generated customers waiting.

We note that when $N_T(t)$=0, $N(t) = N_A(t)+N_B(t)+1$. When $N_T(t)$=1, the tagged customer will be the next one to be served. Further in this case $N_p(t) = N_T(t)$. Therefore when $N_T(t) = 1$, we need only know the status of the server (i.e., $S(t)$). Finally when $N_T(t)$=2, the waiting time $W^{(N)}$ of the tagged customer ends (i.e., absorption takes place in the chain $\mathcal{H}_N^*$). The state space for $\mathcal{H}_N^*$ is

$$S^* = \left\{ \bigcup_{i=0}^{N-1} l_0^*(i) \right\} \bigcup l_1^* \bigcup \{\Delta\}$$

where $l_0^*(i), 0 \leq i \leq N-1$, consists of states for which $N_T(t)$=0, l_1^* consists of states for which $N_T(t)$=1 and Δ is the absorbing state that corresponds to $N_T(t)$=2.

$l_0^*(0) = \{(0,0,j,1,l,h_l,r_l) | 0 \leq j \leq N-1; l = 0,1, 1 \leq h_l \leq m_1; 1 \leq r_{2+l} \leq m_{2+l}$

and for $1 \leq i \leq N-1$,

$l_0^*(i) = \{(0,i,j,k,l,h_l,r_{2+l}) | 0 \leq j \leq N-1-i; k = 0,1; l = 0,1; 1 \leq h_l \leq m_1; 1 \leq r_{2+l} \leq m_{2+l}\}$, $l_1^* = \{(1,k,h_l,r_{2+l}) | k = 0,1; 1 \leq h_l \leq m_1; 1 \leq r_{2+l} \leq m_{2+l}\}$.

Arranging the state space lexicographically, we get the infinitesimal generator of

$\mathcal{H}_N^*$ as

$$\tilde{Q} = \begin{bmatrix} \tilde{\tilde{Q}}_N & -\tilde{\tilde{Q}}_N \underline{e} \\ 0 & 0 \end{bmatrix},$$

$$\tilde{\tilde{Q}}_N = \begin{array}{c} \\ l_0^*(0) \\ l_0^*(1) \\ l_0^*(2) \\ \\ l_0^*(N-1) \\ l_1^* \end{array} \begin{array}{c} \begin{array}{cccccc} l_0^*(0) & l_0^*(1) & l_0^*(2) & & l_0^*(N-1) & l_1^* \end{array} \\ \begin{bmatrix} A_{10}^* & & & & & A_{00}^* \\ A_{21}^* & A_{10}^* & & & & A_{01}^* \\ & A_{22}^* & A_{12}^* & & & A_{02}^* \\ & & & & & \vdots \\ & & & A_{2N-1}^* & A_{1N-1}^* & A_{0N-1}^* \\ & & & & 0 & A_{1N}^* \end{bmatrix} \end{array}$$

$$-\tilde{\tilde{Q}}_N \underline{e} = \text{transpose of } \begin{bmatrix} A_{30}^* & A_{31}^* & A_{32}^* & \dots & A_{3N-1}^* & A_{3N}^* \end{bmatrix}$$

$$A_{10}^* = \begin{bmatrix} A_{10}^{(1,1)} & A_{10}^{(0)} & & & & \\ A_{10}^{(2,1)} & A_{10}^{(1,2)} & A_{10}^{(0)} & & & \\ & A_{10}^{(2,2)} & A_{10}^{(1,3)} & & & \\ & & \ddots & \ddots & & \\ & & & A_{10}^{(2,N-2)} & A_{10}^{(1,N-1)} & A_{10}^{(0)} \\ & & & & A_{10}^{(2,N-1)} & A_{10}^{(1,N)} \end{bmatrix},$$

$$A_{10}^{(1,i)} = \begin{bmatrix} D_0 \oplus S - i\gamma I_{m_1 m_3} & 0 \\ I_{m_1} \otimes S_0 \beta & D_0 \oplus S - i\gamma I_{m_1 m_3} \end{bmatrix}, 1 \le i \le N-1,$$

$$A_{10}^{(1,N)} = \begin{bmatrix} (D_0 + D_1) \oplus S - N\gamma I_{m_1 m_3} & 0 \\ I_{m_1} \otimes S_0 \beta & (D_0 + D_1) \oplus S - N\gamma I_{m_1 m_3} \end{bmatrix},$$

$$A_{10}^{(0)} = \begin{bmatrix} D_1 \otimes I_{m_3} & 0 \\ 0 & D_1 \otimes I_{m_3} \end{bmatrix}, \; A_{10}^{(2,i)} = \begin{bmatrix} 0 & i\gamma I_{m_1 m_3} \\ 0 & i\gamma I_{m_1 m_3} \end{bmatrix}, \; 1 \le i \le N-1;$$

for $1 \leq i \leq N-2$,

$$A^{*}_{1i} = \begin{bmatrix} A_{11}^{(1,i)} & A_{11}^{(0)} & & & & \\ A_{11}^{(2,1)} & A_{11}^{(1,i+1)} & A_{11}^{(0)} & & & \\ & A_{11}^{(2,2)} & A_{11}^{(1,i+2)} & & & \\ & & & & & \\ & & & A_{11}^{(2,N-2-i)} & A_{11}^{(1,N-2)} & A_{11}^{(0)} \\ & & & & A_{11}^{(2,N-1-i)} & A_{11}^{(1,N-1)} \end{bmatrix},$$

$$A_{11}^{(0)} = \begin{bmatrix} D_1 \otimes I_{m_2} & & & \\ & D_1 \otimes I_{m_2} & & \\ & & D_1 \otimes I_{m_3} & \\ & & & D_1 \otimes I_{m_3} \end{bmatrix}, \text{ for} 1 \leq \mathrm{j} \leq \mathrm{N}-2,$$

$$A_{11}^{(1,j)} = \begin{bmatrix} D_0 \oplus T - \Gamma_j' & 0 & 0 & 0 \\ 0 & D_0 \oplus T - \Gamma_j' & 0 & 0 \\ I_{m_1} \otimes \alpha \otimes S_0 & 0 & D_0 \oplus S - \Gamma_{j+1} & 0 \\ 0 & 0 & I_{m_1} \otimes S_0\beta & D_0 \oplus S - \Gamma_{j+1} \end{bmatrix},$$

where $\Gamma_j' = j\gamma I_{m_1 m_2}$ and $\Gamma_j = j\gamma I_{m_1 m_3}$,

$$A_{11}^{(1,N-1)} = \begin{bmatrix} D \oplus T - \Gamma_{N-1}' & 0 & 0 & 0 \\ 0 & D \oplus T - \Gamma_{N-1}' & 0 & 0 \\ I_{m_1} \otimes \alpha \otimes S_0 & 0 & D \oplus S - \Gamma_N & 0 \\ 0 & 0 & I_{m_1} \otimes S_0\beta & D \oplus S - \Gamma_N \end{bmatrix},$$

$$A_{11}^{(2,j)} = \begin{bmatrix} 0 & j\gamma I_{m_1 m_2} & 0 & 0 \\ 0 & j\gamma I_{m_1 m_2} & 0 & 0 \\ 0 & 0 & 0 & j\gamma I_{m_1 m_3} \\ 0 & 0 & 0 & j\gamma I_{m_1 m_3} \end{bmatrix}, \ 1 \leq j \leq N-2;$$

$$A^*_{1N-1} = A_{11}^{(1,N-1)}, \quad A^*_{1N} = \begin{bmatrix} T & 0 \\ 0 & S \end{bmatrix};$$

$$A^*_{21} = \begin{bmatrix} A_{21}^{(0)} & & & 0 \\ & A_{21}^{(1)} & & 0 \\ & & & \\ & & A_{21}^{(N-2)} & A_{21}^{(N-1)} \end{bmatrix},$$

$A_{21}^{(N-1)}$ is a zero matrix of order same as that of $A_{21}^{(0)}$,

$$A_{21}^{(i)} = \begin{bmatrix} 0 & 0 \\ I_{m_1} \otimes T_0 \otimes \beta & 0 \\ & \gamma I_{m_1 m_3} \\ & \gamma I_{m_1 m_3} \end{bmatrix}, \ 0 \le i \le N-2;$$

for $2 \le i \le N-1$

$$A^*_{2i} = \begin{bmatrix} A_{2i}^{(0)} & & & 0 \\ & A_{2i}^{(1)} & & 0 \\ & & & \\ & & A_{2i}^{(N-i-1)} & A_{2i}^{(N-i)} \end{bmatrix},$$

$A_{2i}^{(N-i)}$ is a zero matrix of order same as that of $A_{2i}^{(0)}$,

$$A_{2i}^{(j)} = \begin{bmatrix} I_{m_1} \otimes \alpha \otimes T_0 & (i-1)\gamma I_{m_1 m_2} & 0 & 0 \\ 0 & (i-1)\gamma I_{m_1 m_2} & I_{m_1} \otimes T \otimes \beta & 0 \\ 0 & 0 & 0 & i\gamma I_{m_1 m_2} \\ 0 & 0 & 0 & i\gamma I_{m_1 m_2} \end{bmatrix},$$

$0 \le j \le N-i-1$;

$$A_{00}^{*} = \begin{bmatrix} A_{00}^{(0)} \\ A_{00}^{(1)} \\ \vdots \\ A_{00}^{(N-1)} \end{bmatrix},$$

for $0 \le j \le N-1$,

$$A_{00}^{(j)} = \begin{bmatrix} 0 & \gamma e_{m_1} \otimes I_{m_3} \\ 0 & 0 \end{bmatrix}_{2m_1m_3 \times (m_2+m_3)};$$

for $1 \le i \le N-1$,

$$A_{0i}^{*} = \begin{bmatrix} A_{0i}^{(0)} \\ A_{0i}^{(1)} \\ \\ A_{0i}^{(N-i-1)} \end{bmatrix}, \text{ where for all } i \text{ and } j$$

$$A_{0i}^{(j)} = \begin{bmatrix} \gamma e_{m_1} \otimes I_{m_2} & 0 \\ 0 & 0 \\ 0 & \gamma e_{m_1} \otimes I_{m_3} \\ 0 & 0 \end{bmatrix}_{2m_1(m_2+m_3); \times (m_2+m_3)};$$

$$A_{30}^{*} = \begin{bmatrix} A_{30}^{(0)} \\ A_{30}^{(1)} \\ \vdots \\ A_{30}^{(N-1)} \end{bmatrix}, \text{ for } 0 \le j \le N-1,$$

$$A_{30}^{(j)} = \begin{bmatrix} e_{m_1} \otimes S_0 \\ \gamma e_{m_1 m_3} \end{bmatrix}; \quad A_{31}^{*} = \begin{bmatrix} A_{31}^{(0)} \\ A_{31}^{(1)} \\ \\ A_{31}^{(N-2,)} \end{bmatrix},$$

$$A_{31}^{(j)} = \begin{bmatrix} e_{m_1} \otimes T_0 \\ \gamma e_{m_1 m_2} \\ 0 e_{m_1 m_2} \\ \gamma e_{m_1 m_3} \end{bmatrix}, \quad 0 \le j \le N-2;$$

for $2 \le i \le N-1$,

$$A_{3i}^{*} = \begin{bmatrix} A_{3i}^{(0)} \\ A_{3i}^{(1)} \\ \\ A_{3i}^{(N-i-1)} \end{bmatrix},$$

for all i and j,

$$A_{3i}^{(j)} = \begin{bmatrix} 0 e_{m_1 m_2} \\ \gamma e_{m_1 m_2} \\ 0 e_{m_1 m_2} \\ \gamma e_{m_1 m_3} \end{bmatrix} \quad \text{and} \quad A_{3N}^{*} = \begin{bmatrix} T_0 \\ S_0 \end{bmatrix}.$$

Since the matrix $\tilde{\tilde{Q}}_N$ is invertible, absorption occurs with probability 1 in the chain $\mathcal{H}_N^*$. Also $W^{(N)}$follows phase type distribution with representation $(\tilde{\xi_N}, \tilde{\tilde{Q}}_N)$ where the row vector $\tilde{\xi_N}$ is given by $\tilde{\xi_N} = (\xi_0, \xi_1, ..., \xi_{N-1}, \xi_N)$, in which $\xi_0 = (\xi_{00}, \xi_{01}, ..., \xi_{0N-1})$, where $\xi_{00} = (y_{010}, y_{011})$ and for $1 \le j \le (N-1)$, $\xi_{0j} = 0.\xi_{00}$ for $1 \le i \le (N-2)$, $\xi_i = (\xi_{i0}, \xi_{i1}, ..., \xi_{iN-i-1})$, where $\xi_{i0} = (y_{i00}, y_{i01}, y_{i10}, y_{i11})$, and $\xi_{ij} = 0.\xi_{i0}$, for $1 \le j \le N-i-1$; $\xi_{N-1} = (y_{N-1,0,0}, y_{N-1,0,1}, y_{N-1,1,0}, y_{N-1,1,1})$ and $\xi_N = 0$, a vector containing $m_2 + m_3$ entries. Thus $E(W^{(N)}) = -\tilde{\xi}_N(\tilde{\tilde{Q}}_N)^{-1}\underline{e}$. We approximate E(W) as $\lim_{n \longrightarrow \infty} E(W^{(N)})$. In table 5 we give numerical evidence for the convergence of the sequence $\{E(W^{(N)})\}$.

Computation of $(\tilde{\tilde{Q}}_N)^{-1}\underline{e}$

Let $(\tilde{\tilde{Q}}_N)^{-1}\underline{e} = \bar{a} =$ transpose of $\begin{bmatrix} a_0 & a_1 & & a_{N-1} & a_N \end{bmatrix}$ where a_0 is a column vector containing $2Nm_1m_3$ entries; for $1 \leq i \leq N-1$, a_i is a column vector of order $(N-1)2m_1(m_2+m_3)$ and a_N is a column vector of order m_2+m_3. Then $\tilde{\tilde{Q}}_N\bar{a} = \underline{e}$ and which gives rise to the equations

$$A^*_{10}a_0 + A^*_{00}a_N = \underline{e} \tag{2.6}$$

$$A^*_{2i}a_{i-1} + A^*_{1i}a_i + A^*_{0i}a_N = \underline{e}, 1 \leq i \leq N-1 \tag{2.7}$$

$$A^*_{1N}a_N = \underline{e} \tag{2.8}$$

From equation(2.8), we get

$$a_N = (A^*_{1N})^{-1}\underline{e} \tag{2.9}$$

From equation (2.6) ,

$$a_0 = (A^*_{10})^{-1}(\underline{e} - A^*_{00}a_N) \tag{2.10}$$

$$a_i = (A^*_{1i})^{-1}(\underline{e} - A^*_{2i}a_{i-1} - A^*_{0i}a_N), 1 \leq i \leq N-1. \tag{2.11}$$

Using equations (2.9),(2.10) and (2.11) we get $(\tilde{\tilde{Q}}_N)^{-1}\underline{e}$.

2.5 Numerical examples

Example 1.

Take $D_0 = \begin{bmatrix} -8.5 & 0.25 \\ 0.25 & -0.75 \end{bmatrix}$ and $D_1 = \begin{bmatrix} 8.0 & 0.25 \\ 0.25 & 0.25 \end{bmatrix}$.

Here fundamental arrival rate = 4.37500 and correlation = 0.12681

Let $S = \begin{bmatrix} -8.0 & 4.0 \\ 4.0 & -8.0 \end{bmatrix}$, $\quad S_0 = \begin{bmatrix} 4.0 \\ 4.0 \end{bmatrix}$

$T = \begin{bmatrix} -15.0 & 3.0 \\ 3.0 & -15.0 \end{bmatrix}$ and $T_0 = \begin{bmatrix} 12.0 \\ 12.0 \end{bmatrix}$

with $\alpha = \begin{bmatrix} 0.3 & 0.7 \end{bmatrix}$ and $\beta = \begin{bmatrix} 0.4 & 0.6 \end{bmatrix}$.

Table 1. γ versus performance measures.

γ	E_1	E_2	E_3	$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_3$	$\mathcal{P}_4$
5	0.60257	0.42363	1.09431	0.13112	0.04031	0.03423	0.56901
10	0.40416	0.52446	1.32334	0.11279	0.04091	0.03184	0.57048
15	0.32246	0.58466	1.42594	0.09815	0.03823	0.02891	0.57066
20	0.27733	0.62400	1.48538	0.08812	0.03518	0.02626	0.57049
30	0.22852	0.67206	1.55229	0.07578	0.02980	0.02201	0.56994
40	0.20242	0.70031	1.58934	0.06859	0.02564	0.01887	0.56943

Example 2.

Here we have $D_0 = \begin{bmatrix} -10.5 & 0.25 \\ 0.25 & -0.75 \end{bmatrix}$ and $D_1 = \begin{bmatrix} 10.0 & 0.25 \\ 0.25 & 0.25 \end{bmatrix}$.

Then the fundamental arrival rate = 5.37500 and correlation = 0.13398

S, S_0, T, T_0, α, and β are same as that of example 1.

Table 2. γ versus performance measures.

γ	E_1	E_2	E_3	$\mathcal{P}_1$	$\mathcal{P}_2$	$\mathcal{P}_3$	$\mathcal{P}_4$
5	0.81560	0.55416	1.90306	0.19170	0.06272	0.04115 0	.50259
10	0.50601	0.65172	2.16532	0.15953	0.06080	0.03677	0.51248
15	0.38730	0.71754	2.28062	0.13573	0.05625	0.03305	0.51610
20	0.32372	0.76240	2.34705	0.11948	0.05167	0.02995	0.51784
30	0.25648	0.81855	2.42165	0.09940	0.04385	0.02513	0.51937
40	0.22118	0.85206	2.46292	0.08765	0.03785	0.02160	0.51998

Example 3.

Next we compute the probability of consecutive services for ordinary and priority generated customers. Here we take

$$D_0 = \begin{bmatrix} -20.0 & 0.5 \\ 0.25 & -10.75 \end{bmatrix} \text{ and } D_1 = \begin{bmatrix} 1.0 & 18.5 \\ 0.25 & 10.25 \end{bmatrix}.$$

Then the fundamental arrival rate = 10.73077 and correlation = -0.00001

Let $S = \begin{bmatrix} -6.0 & 4.0 \\ 4.0 & -6.0 \end{bmatrix}$, $S_0 = \begin{bmatrix} 2.0 \\ 2.0 \end{bmatrix}$, $T = \begin{bmatrix} -5.0 & 3.0 \\ 3.0 & -5.0 \end{bmatrix}$ and $T_0 = \begin{bmatrix} 2.0 \\ 2.0 \end{bmatrix}$

with $\alpha = \begin{bmatrix} 0.3 & 0.7 \end{bmatrix}$ and $\beta = \begin{bmatrix} 0.4 & 0.6 \end{bmatrix}$.

Table 3. Probability of consecutive services of ordinary customers.

N	2 consec. services	3 consec. services	4 consec. services	5 consec. services
5	0.005743	0.001407	0.000380	0.000106
6	0.005772	0.001400	0.000377	0.000105
7	0.005780	0.001398	0.000376	0.000105
8	0.005782	0.001397	0.000376	
9	0.005783	0.001397		
10	0.005783			

Table 4. Probability of consecutive services of priority gen. customers.

N	2 consec. services	3 consec. services	4 consec. services	5 consec. services
5	0.017331	0.014565	0.012252	0.010307
6	0.017369	0.014611	0.012305	0.010366
7	0.017378	0.014623	0.012320	0.010382
8	0.017381	0.014626	0.012322	0.010386
9	0.017381	0.014627	0.012324	0.010387
10		0.014627	0.012325	0.010388
11			0.012325	0.010388

Example 4.

Here we calculate the expected waiting time of a tagged customer for different arrival rates, service rates and γ, with $\alpha = [0.3 \;\; 0.7]$ and $\beta = [0.4 \;\; 0.6]$.

(I) Take $D_0 = \begin{bmatrix} -8.5 & 0.25 \\ 0.25 & -0.75 \end{bmatrix}$ and $D_1 = \begin{bmatrix} 8.0 & 0.25 \\ 0.25 & .25 \end{bmatrix}$.

Fundamental arrival rate=4.37500 and Correlation =0.12681

$Let, S = \begin{bmatrix} -6.0 & 4.0 \\ 4.0 & -6.0 \end{bmatrix}$, $S_0 = \begin{bmatrix} 2.0 \\ 2.0 \end{bmatrix}$, $T = \begin{bmatrix} -5.0 & 3.0 \\ 3.0 & -5.0 \end{bmatrix}$ and $T_0 = \begin{bmatrix} 2.0 \\ 2.0 \end{bmatrix}$,
with $\gamma = 5$.

(II) $\gamma =$ 10 , all other parameters are same as in (I)

(III) Here $D_0 = \begin{bmatrix} -6.5 & 0.25 \\ 0.25 & -0.75 \end{bmatrix}$ and $D_1 = \begin{bmatrix} 6.0 & 0.25 \\ 0.25 & .25 \end{bmatrix}$

Fundamental arrival rate=3.37500 and correlation =0.11568,
all other parameters are same as in (I)

(IV) Here D_0, D_1, T, and T_0 are same as in (I) and γ=10,

$S = \begin{bmatrix} -10.0 & 5.0 \\ 5.0 & -10.0 \end{bmatrix}$ and $S_0 = \begin{bmatrix} 5.0 \\ 5.0 \end{bmatrix}$.

(V) Here D_0, D_1, $S$and S_0 are same as in (I) and γ=10,

with $T = \begin{bmatrix} -8.0 & 3.0 \\ 3.0 & -8.0 \end{bmatrix}$ and $T_0 = \begin{bmatrix} 5.0 \\ 5.0 \end{bmatrix}$.

Table 5. Expected waiting time of tagged customers

N	I	II	III	IV	V
5	0.233564	0.169384	0.176110	0.126818	0.142625
6	0.234276	0.169406	0.176116	0.126849	0.142639
7	0.234440	0.169409	0.176117	0.126853	0.142641
8	0.234474	0.169409	0.176117	0.126853	0.142641
9	0.234480	0.169409			
10	0.234481				
11	0.234481				

Tables 1 and 2 show that when γ increases average number of ordinary customers decreases, average number of priority generated customers and average number of customers lost per unit time increases. However the server idle probability shows only a slight fluctuation. Probability that priority generated customers are lost to the system and probability of priority generated customer retained in the system decreases.

In Table 3 the probability of n consecutive services of ordinary customers gradually decreases as n increases; also this table shows convergence of $\mathcal{OP}_n^{(N)}$ as N increases.

Table 4 gives the probability of n consecutive services of priority generated customers approximated by $\mathcal{PP}_n^{(N)}$ as N increases.

Columns I and II of Table 5 show that when γ increases expected waiting time of the tagged customer decreases. This can be attributed to the fact that as γ increases more priority generated customers will leave the system in search of emergency service, including the tagged customer himself. Columns II and III show that expected waiting time of tagged customer increases as the arrival rate increases, when the priority generation rate is fixed. This can also be considered as a characteristic of systems with self generation of priorities. Also, as can be expected, columns IV and V of table 5 show that when service rate increases, the expected waiting time of the tagged customer decreases.

Chapter 3

MAP/(PH,PH)/c Queue

This chapter deals with a multi-server system in which the input stream of customers form a Markovian arrival process and service requirements are of phase type. As in Chapter 2 waiting customers generate into priority at a constant. Such a customer is immediately taken for service if at least one of the servers is free. Else the customer waits at a waiting space of capacity c, exclusively for priority generated customers, provided there is space. A customer in service will be completely served before the priority generated customer is taken for service. Any waiting customer generating into priority at an epoch when all servers are busy and c priority generated customer are already in the wait, will leave the system in search of urgent service elsewhere. We provide a numerical procedure to compute the optimal number of servers to be employed to minimize the loss to the system. It is proved that the system is always stable. We compute the long run system state probabilities and performance measures.

This chapter is arranged as follows. In section 3.1 the problem is mathematically formulated and analysed. In section 3.2 we see that the system under study is always stable. We construct a dominating process to arrive at a truncation level.

Then the Neuts-Rao algorithm is employed to obtain the steady state system state distribution. These are done in section 3.3. In section 3.4 we provide a number of system performance measures of interest. Finally, in section 3.5 we investigate the optimal value of c numerically.

3.1 Mathematical modelling

Customers arrive to a c - server counter, according to a Markovian arrival process with representation (D_0, D_1) of order m_1. If all servers are busy, the arriving customers join a queue. At the time of arrival all customers are classified as 'ordinary'. Waiting customers 'generate priority' at a rate γ (i.e., if there are n customers in the queue then the rate of priority generation is $n\gamma$). Such a customer is immediately taken for service if at least one of the servers is free. Else it waits in a waiting space (specially for the priority generated customers) of capacity c, if the waiting space is not already filled by priority generated customers. If this waiting space is also full, the present priority generated customer leaves the system for ever in search of emergency service. A customer in service will be completely served before the next customer (priority generated / ordinary customer) is taken for service.

The service time of ordinary and priority generated customers follow PH-distribution with representation (α, T) and (β, S) respectively. Define $T_0 = -T\underline{e}$ and $S_0 = -S\underline{e}$ where $\underline{e}$ is a column vector of 1's of appropriate order.

We use the following definitions based on Kronecker product $\otimes$ and Kronecker sum $\oplus$.

Definition 1. *For a given square matrix A, define $A^{\oplus m}$ as the matrix*
$A^{\oplus m} = A \oplus A \oplus \ldots\ldots \oplus A$, *$m$ terms for $m \geq 1$ and*

$A^{\oplus 0} = 0$, *the scalar.*

Definition 2. *For a column vector B with n entries the matrix*

$B^{\oplus m} = B \otimes I_{n^{m-1}} + I_n \otimes B \otimes I_{n^{m-2}} + \ldots\ldots + I_{n^{m-1}} \otimes B$, *for* $m \geq 1$ *and*

$B^{\oplus 0} = 1$, *the scalar.*

Let $N_1(t) = \sharp$ of ordinary customers at time t in the system.

$N_2(t) = \sharp$ of priority generated customers in service at time t.

$N_3(t) = \sharp$ of priority generated customers waiting for service at time t.

$M(t) =$ phase of arrival process at time t.

$\mathcal{M}_1(t) =$ vector of phase of service process of ordinary customers .

$\mathcal{M}_2(t) =$ vector of phase of service process of priority generated customers.

If $X(t) = \{N_1(t), N_2(t), N_3(t), M(t), \mathcal{M}_1(t), \mathcal{M}_2(t)\}$, then $\{X(t), t \geq 0\}$ form a continuous time Markov chain with state space $S = \bigcup_{k=0}^{\infty} L(k)$

in which the k^{th} level

$$L(k) = \begin{cases} \left(\bigcup_{i=0}^{c-k-1} l\,(k,i,0)\right) \bigcup \left(\bigcup_{i=c-k}^{c} l'\,(k,i,j)\right), \text{for } k < c \text{ and } 0 \leq j \leq c; \\ \bigcup_{i=0}^{c} l''\,(k,i,j), \text{for } k \geq c \text{ and } 0 \leq j \leq c. \end{cases}$$

The subset $l(k,i,0)$ represents

$\{(k, i, 0, \nu, \mu_1, \ldots\ldots, \mu_k, \eta_1, \ldots\ldots, \eta_i)\,;\ 1 \leq \nu \leq m_1,$

$1 \leq \mu_1, \ldots\ldots, \mu_k \leq m_2, 1 \leq \eta_1, \ldots\ldots, \eta_i \leq m_3\}$

(here we consider the service phase only for busy servers). $l'(k,i,j)$ represents

$\{(k, i, j, \nu, \mu_1, \ldots, \mu_{c-i}, \eta_1, \ldots, \eta_i)\,;\ 0 \leq j \leq c, 1 \leq \nu \leq m_1,$

$1 \leq \mu_1, \ldots, \mu_{c-i} \leq m_2, 1 \leq \eta_1, \ldots, \eta_i \leq m_3\}$

and $l''(k,i,j)$ represents

$\{(k, i, j, \nu, \mu_1, \ldots, \mu_{c-i}, \eta_1, \ldots, \eta_i)\,;\ 0 \leq j \leq c, 1 \leq \nu \leq m_1,$

$1 \leq \mu_1, \ldots, \mu_{c-i} \leq m_2, 1 \leq \eta_1, \ldots, \eta_i \leq m_3\}$

The number of states in each

$$L(k) = \begin{cases} m_1 m_2^k \sum\limits_{i=0}^{c-k-1} m_3^i + (c+1)\, m_1 \sum\limits_{i=c-k}^{c} m_2^{c-i} m_3^i \ ; k < c \\ (c+1)\, m_1 \sum\limits_{i=0}^{c} m_2^{c-i} m_3^i \ ; k \geq c \end{cases}$$

If we partition the state space into levels based on the number of ordinary customers in the system, the generator of the Markov chain is

$$Q = \begin{bmatrix} B_0 & A_0 & & & & & & & \\ C_1 & B_1 & A_1 & & & & & & \\ & C_2 & B_2 & A_2 & & & & & \\ & & & & & & & & \\ & & & C_{c-1} & B_{c-1} & A_{c-1} & & & \\ & & & & C_c & B_c & A_c & & \\ & & & & & C_{c+1} & B_{c+1} & A_c & \\ & & & & & & & & \ddots \end{bmatrix},$$

where for $0 \leq k < c$,

$$A_k = \begin{bmatrix} U_k^{(0)} & 0_{\nu_1 \times \nu_3} & 0_{\nu_1 \times \nu_2} \\ 0_{\nu_2 \times m\nu_1} & 0_{\nu_2 \times \nu_3} & U_k^{(1)} \end{bmatrix},$$

with ν_1, ν_2, ν_3 functions of k and are given by

$\nu_1(k) = m_1 m_2^k (1 + m_3 + m_3^2 + \cdots\cdots + m_3^{c-k-1})$

$\nu_2(k) = (c+1) m_1 (m_2^k m_3^{c-k} + m_2^{k-1} m_3^{c-k+1} + \cdots\cdots + m_3^c)$

$\nu_3(k) = c m_1 m_2^{k+1} m_3^{c-k-1}$

the order of A_k for $1 \leq k < c$ is $(\nu_1(k) + \nu_2(k)) \times (\nu_1(k+1) + \nu_2(k+1))$,

$U_k^{(0)} = \text{diag}(D_1 \otimes \alpha \otimes I_{m_2^k}, D_1 \otimes \alpha \otimes I_{m_2^k m_3}, \ldots.., D_1 \otimes \alpha \otimes I_{m_2^k m_3^{c-k-1}})$

$U_k^{(1)} = \text{diag}(I_{c+1} \otimes D_1 \otimes I_{m_2^k m_3^{c-k}}, I_{c+1} \otimes D_1 \otimes I_{m_2^{k-1} m_3^{c-k+1}}, \ldots.., I_{c+1} \otimes D_1 \otimes I_{m_3^c})$

$A_c = \text{diag}(I_{c+1} \otimes D_1 \otimes I_{m_2^c}, I_{c+1} \otimes D_1 \otimes I_{m_2^{c-1} m_3}, \ldots., I_{c+1} \otimes D_1 \otimes I_{m_3^c})$;

for $0 \leq k < c$, $\quad B_k = \begin{bmatrix} V_k^{(0)} & 0 \\ V_k^{(1)} & V_k^{(2)} \end{bmatrix},$

$$V_k^{(0)} = \begin{bmatrix} \psi(k,0) & 0 & & 0 & 0 \\ I_{m_1 m_2^k} \otimes S_0 & \psi(k-1,1) & & 0 & 0 \\ & & & & \\ 0 & 0 & .. & I_{m_1 m_2^k} \otimes S_0^{\oplus(c-k-1)} & \psi(0,c-k-1) \end{bmatrix},$$

$\psi(i,j) = D_0 \oplus T^{\oplus i} \oplus S^{\oplus(j)}$,

$$V_k^{(1)} = \begin{bmatrix} v_1^{(0)} \\ 0 \\ \\ 0 \end{bmatrix}_{(k+1)\times 1},$$

each block in $V_k^{(1)}$ is of order $(c+1)\times(c-k)$ and the j^{th} entry of the i^{th} block is matrix of order $(m_1 m_2^{k-i-1} m_3^{c-k+i-1} \times m_1 m_2^k m_3^{j-1})$ for $i = 1, ..., (k+1)$, $j = 1, ..., (c-k)$ and

$$v_1^{(0)} = \begin{bmatrix} 0 & \cdots\cdots & I_{m_1 m_2^k} \otimes S_0^{\oplus(c-k)} \\ 0 & \cdots\cdots & 0 \\ 0 & \cdots\cdots & 0 \end{bmatrix};$$

$$V_k^{(2)} = \begin{bmatrix} v_k^{(1)} & 0 & 0 & & 0 & 0 \\ v_k^{(2)} & v_k^{(3)} & 0 & & 0 & 0 \\ 0 & v_k^{(4)} & v_k^{(5)} & & 0 & 0 \\ \cdots & & & & & \\ \cdots & & & \cdots & & \\ 0 & 0 & 0 & \cdots & v_k^{(2k)} & v_k^{(2k+1)} \end{bmatrix},$$

$$v_k^{(2q+1)} = \begin{bmatrix} \psi_1 & 0 & 0 & & 0 \\ \psi_2 & \psi_1 & 0 & & 0 \\ 0 & \psi_2 & \psi_1 & & 0 \\ \cdots & & & & \cdots \\ 0 & & & \psi_2 & \psi_1 \end{bmatrix}, \; q = 0, 1, ..., k,$$

$$\psi_1 = \left(D_0 \oplus S^{\oplus(c-k+q)}\right) - q\gamma I_{m_1 m_2^{k-q} m_3^{c-k+q}} \text{ and}$$

$$\psi_2 = I_{m_1 m_2^{k-q}} \otimes S_0^{\oplus(c-k+q)} \otimes \beta,$$

$$v_k^{(2q)} = \begin{bmatrix} I_{m_1 m_2^{k-q}} \otimes \alpha \otimes S_0{}^{\oplus c-k+q} & 0 & 0 \\ 0 & 0 & 0 \\ & \cdots & \cdots \\ 0 & 0 & 0 \end{bmatrix}, \; q = 1, 2, ..., k,$$

order of each zero matrix is same as that of $I_{m_1 m_2^{k-q}} \otimes \alpha \otimes S_0{}^{\oplus c-k+q}$;

$$\text{for k} \geq \text{c}, \quad \text{B}_\text{k} = \begin{bmatrix} v_k^{(1)} & 0 & 0 & 0 & 0 \\ v_k^{(2)} & v_k^{(3)} & 0 & 0 & 0 \\ 0 & v_k^{(4)} & v_k^{(5)} & & \\ & & & \cdots & \\ 0 & 0 & 0 & v_k^{(2c)} & v_k^{(2c+1)} \end{bmatrix},$$

$$v_k^{(2q+1)} = \begin{bmatrix} \varphi_1 & 0 & 0 & & 0 \\ \varphi_2 & \varphi_1 & 0 & & 0 \\ 0 & \varphi_2 & \varphi_1 & & 0 \\ \cdots & & & & \cdots \\ 0 & 0 & \cdots & \varphi_2 & \varphi_1 \end{bmatrix}_{(c+1)\times(c+1)} , \; q = 0, 1, ..., c$$

$$\varphi_1 = \left(D_0 \oplus T^{\oplus(c-q)} \oplus S^{\oplus q}\right) - (k-c+q)\gamma I_{m_1 m_2^{c-q} m_3^q},$$

$$\varphi_2 = I_{m_1 m_2^{c-q}} \otimes (S_0)^{\oplus q} \otimes \beta,$$

$$v_k^{(2q)} = \begin{bmatrix} I_{m_1 m_2^{c-q}} \otimes \alpha \otimes S_0{}^{\oplus q} & 0 & 0 \\ 0 & 0 & 0 \\ & & \cdots \\ 0 & 0 & 0 \end{bmatrix}_{(c+1)\times(c+1)}, \quad q = 1, 2, ..., c;$$

for $1 \le k < c$, $$C_k = \begin{bmatrix} W_k^{(0)} & 0_{\nu_1(k)\times\nu_5(k)} & 0_{\nu_1(k)\times\nu_2(k-1)} \\ 0_{\nu_2(k)\times\nu_4(k)} & W_k^{(1)} & W_k^{(2)} \end{bmatrix},$$

where ν_1 *and* ν_2, as functions of k are defined while describing A_k and, ν_4 and ν_5 as functions of k are given by

$\nu_4(k) = m_1 m_2^{k-1}(1 + m_3 + \cdots\cdots + m_3^{c-k-1}),$

$\nu_5(k) = m_1 m_2^{k-1} m_3^{c-k};$

$$W_k^{(0)} = \begin{bmatrix} I_{m_1} \otimes T_0^{\oplus k} & 0 & & 0 \\ 0 & I_{m_1} \otimes T_0^{\oplus k} \otimes I_{m_3} & \cdots\cdot & 0 \\ & & & \\ 0 & 0 & \cdots\cdot & I_{m_1} \otimes T_0^{\oplus k} \otimes I_{m_3^{c-k-1}} \end{bmatrix},$$

$$W_k^{(1)} = \begin{bmatrix} w_k^{(0)} \\ 0 \\ \\ 0 \end{bmatrix}_{(k+1)\times 1} \quad \text{in which} \quad w_k^{(0)} = \begin{bmatrix} I_{m_1} \otimes T_0^{\oplus k} \otimes I_{m_3^{c-k}} \\ 0 \\ \\ 0 \end{bmatrix}_{(c+1)\times 1},$$

$$W_k^{(2)} = \begin{bmatrix} w_k^{(1)} & 0 & & 0 \\ w_k^{(2)} & w_k^{(3)} & & 0 \\ \cdots & & & \cdots \\ 0 & 0 & \cdots & w_k^{(2k-1)} \\ 0 & 0 & & w_k^{(2k)} \end{bmatrix},$$

$$w_k^{(2q-1)} = \begin{bmatrix} 0 & 0 & & 0 & 0 \\ \varphi_k(q-1) & 0 & & 0 & 0 \\ 0 & \varphi_k(q-1) & & 0 & 0 \\ & & & & \cdots \\ o & o & \cdots & \varphi_k(q-1) & 0 \end{bmatrix}_{(c+1)\times(c+1)},$$

$$\varphi_k(q-1) = I_{m_1} \otimes T_0^{\oplus(k-q+1)} \otimes I_{m_3^{c-k+q-1}} \otimes \beta, \quad q = 1, 2, \ldots\ldots, k,$$

$$w_k^{(2q)} = \begin{bmatrix} I_{m_1} \otimes \alpha \otimes (T_0)^{\oplus(k-q)} \otimes I_{m_3^{c-k+q}} & \Gamma_q & 0 & 0 & 0 \\ 0 & 0 & \Gamma_q & 0 & 0 \\ & & & & \cdots \\ 0 & 0 & 0 & 0 & \Gamma_q \\ 0 & 0 & 0 & 0 & \Gamma_q \end{bmatrix},$$

$$\Gamma_q = q\gamma I_{m_1 m_2^{k-q} m_3^{c-k+q}}, \ q = 1, 2, \ldots, (k-1),$$

$$w_k^{(2k)} = \begin{bmatrix} 0 & k\gamma I_{m_1 m_3^k} & 0 & 0 & & 0 \\ 0 & 0 & k\gamma I_{m_1 m_3^k} & 0 & & 0 \\ \cdots & & & & & \\ 0 & 0 & 0 & 0 & \cdots & k\gamma I_{m_1 m_3^k} \\ 0 & 0 & 0 & 0 & \cdots & k\gamma I_{m_1 m_3^k} \end{bmatrix};$$

$$C_c = \begin{bmatrix} W_c^{(0)} & 0 \\ 0 & W_c^{(1)} \end{bmatrix}, \quad W_c^{(0)} = \left[I_{m_1} \otimes T_0^{\oplus c} \right],$$

$$W_c^{(1)} = \begin{bmatrix} w_c^{(1)} & 0 & & 0 \\ w_c^{(2)} & w_c^{(3)} & & 0 \\ .... & & & .... \\ 0 & 0 & .... & w_c^{(2c-1)} \\ 0 & 0 & & w_c^{(2c)} \end{bmatrix},$$

$$w_c^{(2q-1)} = \begin{bmatrix} 0 & 0 & & 0 & 0 \\ \varphi_c(q-1) & 0 & & 0 & 0 \\ 0 & \varphi_c(q-1) & & 0 & 0 \\ & & & & ... \\ o & o & .... & \varphi_c(q-1) & 0 \end{bmatrix}_{(c+1)\times(c+1)},$$

$$\varphi_c(q-1) = I_{m_1} \otimes T_0^{\oplus(c-q+1)} \otimes I_{m_3^{q-1}} \otimes \beta, \quad q = 1, 2, ..., c,$$

$$w_c^{(2q)} = \begin{bmatrix} I_{m_1} \otimes \alpha \otimes (T_0)^{\oplus(c-q)} \otimes I_{m_3^q} & \Gamma_q' & 0 & 0 & ... & 0 \\ 0 & 0 & \Gamma_q' & 0 & & 0 \\ & & & & & .... \\ 0 & 0 & 0 & 0 & ... & \Gamma_q' \\ 0 & 0 & 0 & 0 & ... & \Gamma_q' \end{bmatrix},$$

$$\Gamma_q' = q\gamma I_{m_1 m_2^{c-q} m_3^q} \text{ for } q = 1, ..., (c-1)$$

$$w_c^{(2c)} = \begin{bmatrix} 0 & c\gamma I_{m_1 m_3^k} & 0 & 0 & .... & 0 \\ 0 & 0 & c\gamma I_{m_1 m_3^k} & 0 & .... & 0 \\ ... & & & & & \\ 0 & 0 & 0 & 0 & .... & c\gamma I_{m_1 m_3^k} \\ 0 & 0 & 0 & 0 & & c\gamma I_{m_1 m_3^k} \end{bmatrix};$$

$$
\text{for } k > c, \ C_k = \begin{bmatrix} w_k^{(0)} & w_k^{(1)} & 0 & 0 & 0 \\ 0 & w_k^{(2)} & w_k^{(3)} & 0 & 0 \\ & & & & \cdots \\ 0 & 0 & 0 & w_k^{(2c-2)} & w_k^{(2c-1)} \\ 0 & 0 & 0 & 0 & w_k^{(2c)} \end{bmatrix},
$$

$$
w_k^{(2q)} = \begin{bmatrix} I_{m_1} \otimes \alpha \otimes (T_0)^{\oplus(c-q)} \otimes I_{m_3^q} & \Gamma'_{k-c+q} & 0 & & 0 \\ 0 & 0 & \Gamma'_{k-c+q} & & 0 \\ & & & & \\ 0 & 0 & 0 & \cdots & \Gamma'_{k-c+q} \\ 0 & 0 & 0 & \cdots & \Gamma'_{k-c+q} \end{bmatrix},
$$

$$
\Gamma'_{k-c+q} = (k-c+q)\gamma I_{m_1 m_2^{c-q} m_3^q}, \ q = 0, 1, \ldots\ldots, (c-1),
$$

$$
w_k^{(2c)} = \begin{bmatrix} 0 & k\gamma I_{m_1 m_3^c} & 0 & 0 & 0 \\ 0 & 0 & k\gamma I_{m_1 m_3^c} & 0 & 0 \\ \cdots & & & & \\ 0 & 0 & 0 & 0 & k\gamma I_{m_1 m_3^c} \\ 0 & 0 & 0 & 0 & k\gamma I_{m_1 m_3^c} \end{bmatrix},
$$

$$
w_k^{(2q+1)} = \begin{bmatrix} 0 & 0 & & 0 & 0 \\ \varphi_c(q) & 0 & & 0 & 0 \\ 0 & \varphi_c(q) & \cdots & 0 & 0 \\ & & & & \cdots \\ o & o & & \varphi_c(q) & 0 \end{bmatrix},
$$

$$
\varphi_c(q) = I_{m_1} \otimes T_0^{\oplus(c-q)} \otimes I_{m_3^q} \otimes \beta, \quad q = 0, 1, 2, \ldots\ldots, (c-1).
$$

3.2 System stability

Theorem 3.1. *The system under discussion is always stable.*

Proof. Consider the Lyapunov test function defined by $\varphi(s) = k$, where 's' is a state in level k. Then for a state s in level $k \geq c$, the mean drift y_s is given by

$$\begin{aligned} y_s &= \sum_{p \neq s} [\phi(p) - \phi(s)] q_{sp} \\ &= \sum_{s'} [\phi(s') - \phi(s)]\, q_{ss'} + \sum_{s''} [\phi(s'') - \phi(s)]\, q_{ss''} + \sum_{s'''} [\phi(s''') - \phi(s)]\, q_{ss'''} \end{aligned}$$

where s', s'' and s''' vary over the states belonging to levels $k-1$, k and $k+1$ respectively. Then $\varphi(s) = k$, $\varphi(s') = k-1$, $\varphi(s'') = k$ and $\varphi(s''') = k+1$

Thus $y_s = -\sum_{s'} q_{ss'} + \sum_{s'''} q_{ss'''}$

$$= \begin{cases} \sum_{s'''} q_{ss'''} - (k+j-c)\,\gamma - \left(\underline{e}_{(c+1)m_1} \otimes \left(T_0^{\oplus(c-j)}\right) \underline{e}\right)_s \text{ if (c - j) servers} \\ \text{are busy with ordinary customers, } j = 0, 1, \ldots\ldots, (c-1). \\ \sum_{s'''} q_{ss'''} - k\gamma\ , \\ \text{if all servers are busy with priority generated customers.} \end{cases}$$

where $\left(\underline{e}_{(c+1)m_1} \otimes \left(T_0^{\oplus(c-j)}\right) \underline{e}\right)_s$ denotes the s^{th} entry of the vector $\left(\underline{e}_{(c+1)m_1} \otimes \left(T_0^{\oplus(c-j)}\right) \underline{e}\right)$. Since the number of phase is finite, $\sum_{s'''} q_{ss'''}$ is bounded by some fixed constant for any s in level $k \geq c$. Hence we can find a positive real number K such that $\sum_{s'''} q_{ss'''} < K$ for all s in level $k \geq c$. Thus, for any $\varepsilon > 0$, we can find K^* large enough that $y_s < -\epsilon$ for any s belonging to level $k \geq K^*$.

Hence the theorem follows from Tweedie [55]. □

3.3 Steady state distribution

Let $x = (x_0, x_1, ...)$ be the equilibrium distribution. For a positive recurrent $LDQBD$, x_i satisfies the relationship $x_{k+1} = x_k R_k$, $k \geq 0$, which gives $x_{k+1} = x_0 \prod_{l=0}^{k} R_l$ where the family of matrices $\{R_k : k \geq 0\}$ is the minimal nonnegative solution of the system of equations

$$A_k + R_k B_{k+1} + R_k R_{k+1} C_{k+2} = 0 \text{ for } 0 \leq k < c \tag{3.1}$$

$$\text{and } A_k + R_k B_{k+1} + R_k R_{k+1} C_{k+2} = 0 \text{ for } k \geq c, \tag{3.2}$$

and x_0 is the solution of

$$x_0(B_0 + R_0 C_1) = 0 \tag{3.3}$$

subject to

$$x_0 \underline{e} + x_0 \sum_{k=1}^{\infty} \left(\left(\prod_{l=0}^{k-1} R_l \right) \underline{e} \right) = 1 \tag{3.4}$$

Here the process $\{X(t), t \geq 0\}$, under discussion, satisfies the condition ' for all k$\geq$ 1 and for all i, there exists j such that $(C_k)_{ij} > 0$ '. Therefore, there exist a dominating process $\bar{X}(t)$ (see Bright and Taylor [13]) on the same state space as $X(t)$ and with generator

$$\bar{Q} = \begin{bmatrix} B_0 & A_0 & & & & & & \\ 0 & \bar{B}_1 & \bar{A}_1 & & & & & \\ & \bar{C}_2 & \bar{B}_2 & \bar{A}_2 & & & & \\ & & & & & & & \\ & & & \bar{C}_{c-1} & \bar{B}_{c-1} & \bar{A}_{c-1} & & \\ & & & & \bar{C}_c & \bar{B}_c & \bar{A}_c & \\ & & & & & \bar{C}_{c+1} & \bar{B}_{c+1} & \bar{A}_c \\ & & & & & & & \ddots \end{bmatrix}$$

$\left(\overline{A}_0\right)_{i,j} = (A_0)_{i,j}$,
$\left(\overline{A}_k\right)_{i,j} = \left(\frac{1}{\nu_1(k+1)+\nu_2(k+1)}\right)\left((A_{k-1}\underline{e})_{\max}\right)$, $1 \leq k \leq c$,
$\left(\overline{A}_c\right)_{i,j} = \left(\frac{1}{N}\right)\left((A_c\underline{e})_{\max}\right)$ where $N = (c+1)\, m_1 \sum_{i=0}^{c}\left(m_2^{c-i} m_3^{i}\right)$, the dimension of the level for $k \geq c$ and $(A_c\underline{e})_{max}$ is the maximum element of the column vector $A_c\underline{e}$
$\left(\overline{C}_1\right)_{i,j} = 0$,
$\left(\overline{C}_k\right)_{i,j} = \left(\frac{1}{\nu_1(k-1)+\nu_2(k-1)}\right)\left((C_{k-1}\underline{e})_{\min}\right)$, $2 \leq k \leq c$,
$\left(\overline{C}_k\right)_{i,j} = \left(\frac{1}{N}\right)\left((C_k\underline{e})_{\min}\right)$, $k > c$,
$\left(\overline{B}_k\right)_{i,j} = (B_k)_{i,j}$, $j \neq i$ and $k \geq 0$.

Let $\{l_n, n \geq 0\}$ and $\{\bar{l}_n, n \geq 1\}$ be the marginal distributions of the levels of X(t) and $\bar{X}(t)$ respectively in the long run as the system gets stabilized. Let $\bar{z} = (z_1, z_2, ...)$ be an invariant measure for $\bar{X}(t)$. Define $\bar{L}_n = \bar{z}_n\underline{e}$ and $P_0^{-1} = \sum_{n=1}^{\infty} \bar{L}_n$. If $P_0^{-1} < \infty$, then an equilibrium distribution for $\bar{X}(t)$ exists and $\bar{l}_n = P_0\bar{L}_n$. But the structure of $\bar{X}(t)$ shows that $\{\bar{l}_n, n \geq 1\}$ can be considered as an equilibrium distribution of a standard birth-and-death process on state space $\{i \geq 1\}$ with transition rates $\bar{q}(i,j)$given by

$$\bar{q}(0,1) = 0$$
$$\bar{q}(i,i+1) = \begin{cases} (A_{i-1}\underline{e})_{\max}, & 1 \leq i \leq c \\ (A_c\underline{e})_{\max}, & i > c \end{cases}$$
$$\bar{q}(1,0) = 0$$
$$\bar{q}(i,i-1) = (C_{i-1}\underline{e})_{min}, \; i >= 2$$

$$So \;\; \{\bar{l}_n, n \geq 1\} \text{ is given by } \bar{l}_n = P_0 \prod_{i=1}^{n-1} \frac{\bar{q}(i,i+1)}{\bar{q}(i+1,i)}, \; n \geq 1 \tag{3.5}$$

Equation (3.5) shows that a sufficient condition for $P_0^{-1} < \infty$ is that $\frac{\bar{q}(i,i+1)}{\bar{q}(i+1,i)} < r < 1$, $\forall$ i$\geq$ N for some N. Thus if $\{\bar{l}_n, n \geq 1\}$ exists, $\bar{x}$ the steady state distribution of $\bar{X}(t)$, must exist and therefore x must exist since $\bar{X}(t)$ stochastically dominate

$X(t)$. Now we fix the truncation level K^* such that $\sum\limits_{n=K^*}^{\infty} l_n < \epsilon$. Since $\bar{X}(t)$ dominates X(t) we have $\sum\limits_{n=K^*}^{\infty} l_n \leq \sum\limits_{n=K^*}^{\infty} \bar{l_n}$, so it is sufficient to fix K^* such that $\sum\limits_{n=K^*}^{\infty} \bar{l_n} < \epsilon$.

We use the K^* obtained by the above method to fix the truncation level and employ Neuts-Rao procedure in numerical computations. Thus $x_k(K^*), 1 \leq k \leq K^*$, is given by $x_k(K^*) = x_0(K^*) \prod\limits_{l=0}^{k-1} R_l$, where $x_0(K^*)$ satisfies $x_0(B_0 + R_0 C_1) = 0$. The components of x above the level K^* are given by $x_{K^*+i} = x_{K^*} \prod\limits_{j=1}^{i} R_{K^*+j}$ and eq.(3.4) becomes $x\underline{e} = x_{K^*+1}(I - R_{K^*})^{-1}\underline{e} + x_0(K^*)\underline{e} + x_0(K^*) \sum\limits_{k=1}^{K^*} \left(\left(\prod\limits_{l=0}^{k-1} R_l \right) \underline{e} \right) = 1$. Note that $x_{K^*+1}(I - R_{K^*})^{-1}\underline{e} < \epsilon$ for our choice of K^*.

3.4 System performance measures

The steady state probability vector of $X(t)$ process is $x = (x_0, x_1, ...)$. Let us partition x_k as

$$x_k = \begin{cases} y_k(i,0) \text{ if } k < c \text{ and } 0 \leq i \leq c-k \\ y_k(i,j) \begin{cases} \text{if } k < c,\ c-k \leq i \leq c \text{ and } 0 \leq j \leq c \\ \text{or if } 0 \leq i,j \leq c \text{ and } k \geq c \end{cases} \end{cases}$$

where y_k(i,j) is a row vector corresponding to $N_2(t) = i$ and $N_3(t) = j$.

We concentrate on the following measures of interest.

- Average number E_1 of ordinary customers in the system $= \sum\limits_{i=0}^{\infty} i\ x_i \underline{e}$
- Average number E_2 of priority generated customers in the system
$$= \sum_{k=0}^{c-1} \left(\sum_{i=0}^{c-k-1} i y_k(i,0)\underline{e} + \sum_{i=c-k}^{c} \sum_{j=0}^{c} (i+j)\, y_k(i,j)\underline{e} \right) + \sum_{k=c}^{\infty} \left(\sum_{j=1}^{c} j y_k(0,j)\underline{e} + \sum_{i=1}^{c} \sum_{j=0}^{c} (i+j)\, y_k(i,j)\underline{e} \right)$$
- Average number E_3 of priority generated customers waiting
$$= \sum_{k=0}^{c-1} \left(\sum_{i=c-k}^{c} \sum_{j=1}^{c} j y_k(i,j)\underline{e} \right) + \sum_{k=c}^{\infty} \left(\sum_{i=0}^{c} \sum_{j=1}^{c} j y_k(i,j)\underline{e} \right)$$

- Average number E_4 of priority generated customers lost per unit time to the system

$$= \sum_{k=1}^{c-1}\sum_{i=0}^{k-1} (k-i)\,\gamma y_k\,(c-i,c)\,\underline{e} + \sum_{k=c}^{\infty}\sum_{i=0}^{c} (k-c+i)\,\gamma y_k\,(i,c)\underline{e}$$

- Average number E_5 of Idle servers $= \sum_{k=0}^{c-1}\sum_{i=1}^{c-k} i y_k\,(c-k-i,0)\,\underline{e}$

Next we construct a cost function for numerical computation.

Let C_1=Holding cost per unit of the ordinary customers in the system.

C_2 = Holding cost per unit of the priority generated customers in service.

C_3 = Holding cost per unit of the priority generated customers waiting.

C_4 = Cost per unit due to the loss of priority generated customers.

C_5 = Cost per unit of idle servers per server.

•The expected total cost ETC $= E_1C_1 + (E_2 - E_3)C_2 + E_3C_3 + E_4C_4 + E_5C_5$.

3.5 Numerical examples

We provide two illustrations.

Example 1.

Take $D_0 = \begin{bmatrix} -6.5 & 0.25 \\ 0.25 & -0.75 \end{bmatrix}$ and $D_1 = \begin{bmatrix} 6.0 & 0.25 \\ 0.25 & 0.25 \end{bmatrix}$.

Here fundamental arrival rate = 3.37500 and correlation = 0.11568

Let $S = \begin{bmatrix} -8.0 & 4.0 \\ 4.0 & -8.0 \end{bmatrix}$, $\qquad S_0 = \begin{bmatrix} 4.0 \\ 4.0 \end{bmatrix}$,

$T = \begin{bmatrix} -15.0 & 3.0 \\ 3.0 & -15.0 \end{bmatrix}$ and $T_0 = \begin{bmatrix} 12.0 \\ 12.0 \end{bmatrix}$

with $\alpha = \begin{bmatrix} 0.3 & 0.7 \end{bmatrix}$ and $\beta = \begin{bmatrix} 0.4 & 0.6 \end{bmatrix}$.

Further take $\gamma = 20, C_1 = 5, C_2 = 5, C_3 = 10, C_4 = 350, C_5 = 5$. We then have

Table 1. Number of servers versus expected total cost.

c	E_1	E_2	E_3	E_4	E_5	ETC
1	0.22851	0.44293	0.05720	0.76329	0.64589	274.02557
2	0.27314	0.07613	0.02081	0.02751	1.68045	19.90072
3	0.28055	0.00590	0.00087	0.00015	2.71551	15.06754
4	0.28119	0.00520	0.00005	0.00000	3.71823	20.00780
5	0.28124	0.00004	0.00000	0.00000	4.71837	25.00007

Example 2.

Here we have $D_0 = \begin{bmatrix} -12.0 & 0.25 \\ 0.25 & -3.25 \end{bmatrix}$ and $D_1 = \begin{bmatrix} 11.5 & 0.25 \\ 0.5 & 2.5 \end{bmatrix}$.

Then the fundamental arrival rate = 8.25000 and correlation = 0.18064

Further take $S = \begin{bmatrix} -8.0 & 4.0 \\ 4.0 & -8.0 \end{bmatrix}$, $S_0 = \begin{bmatrix} 4.0 \\ 4.0 \end{bmatrix}$,

$$T = \begin{bmatrix} -15.0 & 3.0 \\ 3.0 & -15.0 \end{bmatrix} \text{ and } T_0 = \begin{bmatrix} 12.0 \\ 12.0 \end{bmatrix}$$

with $\alpha = \begin{bmatrix} 0.3 & 0.7 \end{bmatrix}$ and $\beta = \begin{bmatrix} 0.4 & 0.6 \end{bmatrix}$.

Also we assume $\gamma = 10,\ C_1 = 5,\ C_2 = 5,\ C_3 = 10,\ C_4 = 350,\ C_5 = 5$.

Then we have

Table 2. Number of servers versus expected total cost.

c	E_1	E_2	E_3	E_4	E_5	ETC
1	0.81411	0.96927	0.27039	4.08202	0.29187	1440.43530
2	0.69839	0.51991	0.18472	0.60385	1.16249	224.17624
3	0.69303	0.07316	0.01723	0.02002	2.27852	22.31635
4	0.68795	0.00892	0.00102	0.00015	3.30635	20.07529
5	0.68757	0.00107	0.00009	0.00000	4.31188	25.00350

Note that in both tables 1 and 2 the Total expected cost first decreases with the increasing number of customers, reaches a minimum value and then starts increasing. Of course this has a bearing on the input parameters. In any case this expected cost, as function of number of servers, will be either strictly convex or monotone.

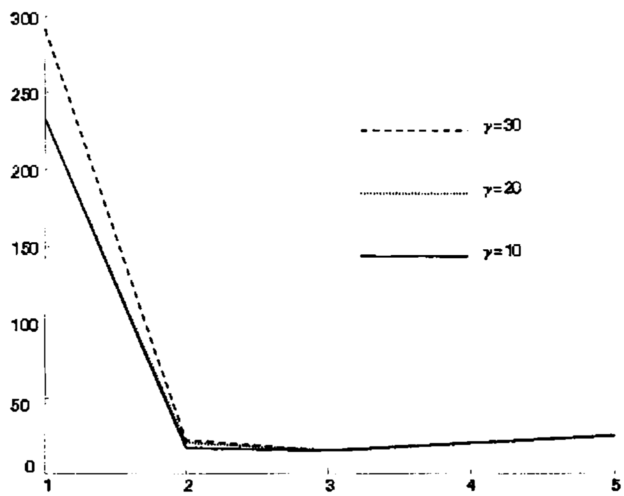

Figure 3.1: No. of servers versus expected total cost for different priority generation rates.(The performance measures corresponding to $\gamma = 20$ are given in Table 1.)

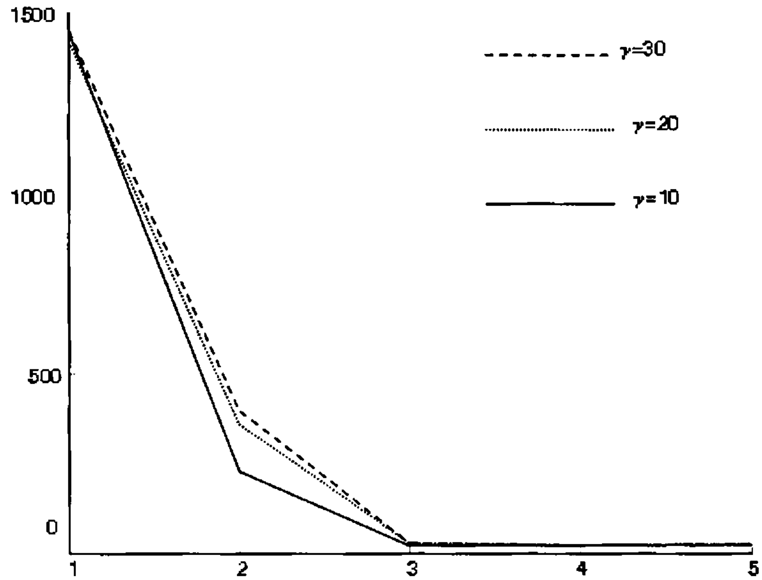

Figure 3.2: No. of servers versus expected total cost for different priority generation rates.(The performance measures corresponding to $\gamma = 10$ are given in Table 1.)

Chapter 4

MAP/(PH,PH)/c Retrial Queue

In this chapter we discuss about multi-server retrial queueing systems. Customers join the c server system according to a Markovian arrival process. If any of the servers is free, such a customer enters for service immediately. If all servers are busy the arriving customer enters an orbit of infinite capacity. Each customer in the orbit tries, independently of each other, to access the server at a constant rate θ. Each customer in the orbit, independently of others, generate into priority with inter occurrence time exponentially distributed with parameter γ. A priority generated customer is immediately taken for service if any of the server is free. Else it waits in a waiting space (specially for priority generated customers) of capacity c, if this waiting space is not full at that instant. If this waiting space is full the present priority generated customer leaves the system for ever. The service discipline is non-preemptive priority. The service times of ordinary and priority generated customers follow PH-distribution. We provide a numerical procedure to compute the optimal number of servers to be employed to minimize the loss of customers. It is proved that the system is always stable. Several performance measures are evaluated.

This chapter is arranged as follows. In section 4.1 the problem is mathematically formulated and analyzed. In section 4.2 we prove that the system under study is always stable. We construct a dominating process to arrive at a truncation level. From there we proceed to obtain the long run system state distribution. These are done in section 4.3. In section 4.4 we provide a number of system performance measures of interest. Finally, in section 4.5, we investigate the optimal value of c numerically.

4.1 Mathematical modelling

Here we consider a service system with c servers, to which customers arrive according to a Markovian arrival process with representation (D_0, D_1). An arriving customer enters service immediately if at least one server is free; on the other hand it enters an orbit of infinite capacity if all servers are busy. Each customer in the orbit tries independently of each other to access the server at a constant rate θ (i.e., if there are k customers in the orbit, the rate of retrial is $k\theta$). Each Customer in the orbit, independently of others, generate into priority with inter occurrence time exponentially distributed with parameter γ. A priority generated customer is immediately taken for service if at least one of the servers is free. Else it waits in a waiting space (specially for priority generated customers) of capacity c, if this waiting space is not full with priority generated customers at that instant. If this waiting space is full the present priority generated customer will leave the system for ever. A customer in service (priority generated or otherwise) will be completely served before the priority generated customer is taken for service.

The service time of ordinary and priority generated customers follow PH-distribution with representation (α,T) and (β,S) respectively. Write $T_0 = -T\underline{e}$

and $S_0 = -S\underline{e}$ where $\underline{e}$ is a column vector of 1's of appropriate order.

Let, $N_1(t)$ = number of customers in the orbit at time t,

$N_2(t)$ = number of busy servers,

$N_3(t)$ = number of priority generated customers in service,

$N_4(t)$ = number of priority generated customers waiting for service,

$M(t)$ = phase of Markovian arrival process,

$\mathcal{M}_1(t)$ = vector of phase of service process of ordinary customers and

$\mathcal{M}_2(t)$ = vector of phase of service process of priority generated customers. Write $X(t) = (N_1(t), N_2(t), N_3(t), N_4(t), M(t), \mathcal{M}_1(t), \mathcal{M}_2(t))$; then $\{X(t) : t \geq 0\}$ forms a continuous time Markov chain with state space $S = \bigcup_{k=0}^{\infty} L(k)$, in which the k^{th}level $L(k) = \bigcup_{i=0}^{c} l_k(i)$, where

$$l_k(i) = \begin{cases} l'(k,0,0,0) \text{ , if } i = 0 \\ \bigcup_{j=0}^{i} l''(k,i,j,0) \text{ ,if } 1 \leq i < c \\ \bigcup_{j,j_1=0}^{c} l'''(k,c,j,j_1) \end{cases}$$

The element $l'(k,0,0,0)$ represents $\{(k,0,0,0,\nu) : 1 \leq \nu \leq m_1\}$, which means all servers are idle. $l''(k,i,j,0)$ represents $\{(k,i,j,0,\nu,\mu_1,..,\mu_{i-j},\eta_1,..,\eta_j) : 1 \leq \nu \leq m_1, 1 \leq \mu_1,..,\mu_{i-j} \leq m_2, 1 \leq \eta_1,..,\eta_j \leq m_3\}$, here we consider the service phase only for busy servers. Finally $l'''(k,c,j,j_1)$ represents $\{(k,c,j,j_1,\nu,\mu_1,..,\mu_{c-j},\eta_1,..,\eta_j) : 1 \leq \nu \leq m_1, 1 \leq \mu_1,..,\mu_{c-j} \leq m_2, 1 \leq \eta_1,..,\eta_j \leq m_3\}$.

By partitioning the state space into levels based on the number of customers in the orbit, the generator of the above Markov chain has the block partitioned form

$$Q = \begin{bmatrix} B_0 & A_0 & & & \\ C_1 & B_1 & A_0 & & \\ & C_2 & B_2 & A_0 & \\ & & & \ddots & \ddots \end{bmatrix}$$

. The description of A_0, B_k and C_k are as follows

$$A_0 = \begin{bmatrix} 0_{\nu_1 \times \nu_1} & 0_{\nu_1 \times \nu_2} \\ 0_{\nu_2 \times \nu_1} & A \end{bmatrix}$$ in which

$\nu_1 = m_1 \sum_{i=0}^{c-1} \sum_{j=0}^{i} m_2^{i-j} m_3^j$, $\nu_2 = (c+1)\, m_1 \sum_{j=0}^{c} m_2^{c-j} m_3^j$ and

$$A = \begin{bmatrix} I_{c+1} \otimes D_1 \otimes I_{m_2^c} & 0 & & 0 \\ 0 & I_{c+1} \otimes D_1 \otimes I_{m_2^{c-1} m_3} & \cdots & 0 \\ & & & \\ 0 & 0 & \cdots & I_{c+1} \otimes D_1 \otimes I_{m_3^c} \end{bmatrix};$$

$$B_k = \begin{bmatrix} B_{1k0} & B_{0k0} & & & \\ B_{2k1} & B_{1k1} & B_{0k1} & & \\ & B_{2k2} & B_{1k2} & B_{0k2} & \\ & & & & \\ & & B_{2k,c-1} & B_{1k,c-1} & B_{0k,c-1} \\ & & & B_{2kc} & B_{1kc} \end{bmatrix},$$

for $i = 0, 1, \ldots\ldots, (c-2)$,

$$B_{0ki} = \begin{bmatrix} D' \otimes I_{m_2^i} & 0 & 0 & 0 & 0 \\ 0 & D' \otimes I_{m_2^{i-1} m_3} & 0 & 0 & 0 \\ 0 & 0 & D' \otimes I_{m_2^{i-2} m_3^2} & 0 & 0 \\ & & & & .. \\ 0 & 0 & 0 & \ldots \; D' \otimes I_{m_3^i} & 0 \end{bmatrix},$$

where $D' = D_1 \otimes \alpha$,

$$B_{0k,c-1} = \begin{bmatrix} V_0 & 0 & & 0 & 0 \\ 0 & V_1 & & 0 & 0 \\ \cdots & & & & \cdots \\ 0 & 0 & & V_{c-1} & 0 \end{bmatrix}, \text{ in which}$$

$$V_j = \begin{bmatrix} D_1 \otimes \alpha \otimes I_{m_2^{c-1-j} m_3^j} & 0 & \ldots & 0 \end{bmatrix}, \ j = 0, 1, \ldots, (c-1);$$

for $i = 0, 1, \ldots\ldots, (c-1)$,

$$B_{1ki} = \begin{bmatrix} \varphi_k(i,0) & & & \\ & \varphi_k(i-1,1) & & \\ & & \varphi_k(i-2,2) & \\ & & & \\ & & & \varphi_k(0,i) \end{bmatrix},$$

where $\varphi_k(i,j) = D_0 \oplus T^{\oplus i} \oplus S^{\oplus j} - k\theta I_{m_1 m_2^i m_3^j}$

$$B_{1kc} = \begin{bmatrix} V'_{k0} & V''_0 & & & \\ & V'_{k1} & V''_1 & & \\ & & & & \\ & & & & \\ & & & V'_{kc} & V''_c \end{bmatrix},$$

$$V''_j = \begin{bmatrix} 0 & 0 & 0 & 0 \\ I_{m_1} \oplus T_0^{(c-j)} \otimes I_{m_3^j} \otimes \beta & 0 & 0 & 0 \\ & & & \cdots \\ 0 & 0 & I_{m_1} \oplus T_0^{(c-j)} \otimes I_{m_3^j} \otimes \beta & 0 \end{bmatrix},$$

$$V'_{kj} = \begin{bmatrix} \varphi_1(j) & & & \\ \varphi_2(j) & \varphi_1(j) & & \\ & & & \\ & & \varphi_2(j) & \varphi_1(j) \end{bmatrix},$$

$$\varphi_1(j) = D_0 \oplus T^{\oplus(c-j)} \oplus S^{\oplus j} - k\gamma I_{m_1 m_2^{c-j} m_3^{j}},$$

$$\varphi_2(j) = I_{m_1 m_2^{c-j}} \otimes S_0^{\oplus j} \otimes \beta, \text{for j} = 0, 1, \ldots\ldots, \text{c};$$

for i $= 0, 1, \ldots\ldots, (c-1)$,

$$B_{2ki} = \begin{bmatrix} \xi_1(i,1) & 0 & 0 & \\ \xi_2(i,1) & \xi_1(i,2) & 0 & \\ 0 & \xi_2(i,2) & & 0 \\ & & & \\ 0 & 0 & .. & \xi_1(i,i) \\ 0 & 0 & & \xi_2(i,i) \end{bmatrix},$$

$$\xi_1(i,j) = I_{m_1} \otimes T_0^{\oplus(i-j+1)} \otimes I_{m_3^{j-1}}$$

$$\xi_2(i,j) = I_{m_1 m_2^{i-j}} \otimes S_0^{\oplus j}, j = 1, 2, \ldots., i;$$

$$B_{2kc} = \begin{bmatrix} V_0''' \\ V_1''' \\ \vdots \\ V_c''' \end{bmatrix},$$

$$V_0''' = \begin{bmatrix} I_{m_1} \otimes T_0^{\oplus c} & 0 & 0 \\ 0 & 0 & 0 \\ & \cdots & \cdots \\ 0 & 0 & 0 \end{bmatrix},$$

$$V_1''' = \begin{bmatrix} I_{m_1 m_2^{c-1}} \otimes S_0 & I_{m_1} \otimes T_0^{\oplus(c-1)} \otimes I_{m_3} & 0 & \dots & 0 \\ 0 & 0 & 0 & & 0 \\ & & \dots & \dots & \dots \\ 0 & 0 & 0 & & 0 \end{bmatrix},$$

$$V_2''' = \begin{bmatrix} 0 & I_{m_1 m_2^{c-2}} \otimes S_0^{\oplus 2} & I_{m_1} \otimes T_0^{\oplus(c-2)} \otimes I_{m_3^2} & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & & 0 \\ \dots & & & & & \dots \\ 0 & 0 & 0 & 0 & & 0 \end{bmatrix},$$

$$V_c''' = \begin{bmatrix} 0 & & 0 & I_{m_1} \otimes S_0^{\oplus c} \\ 0 & & 0 & 0 \\ \dots & \dots & \dots & \\ 0 & \dots & 0 & 0 \end{bmatrix}, V_i''' \text{ are block matrices of order } (c+1) \times c;$$

$$C_k = \begin{bmatrix} 0 & W_{k0} & 0 & 0 & 0 & 0 \\ 0 & 0 & W_{k1} & 0 & 0 & 0 \\ \dots & & & & & \\ 0 & 0 & 0 & 0 & W_{k,c-2} & 0 \\ 0 & 0 & 0 & 0 & 0 & W_{k,c-1} \\ 0 & 0 & 0 & 0 & 0 & W_{kc} \end{bmatrix},$$

$$W_{k0} = \begin{bmatrix} I_{m_1} \otimes (k\theta\alpha) & 0_{m_1 \times m_1 m_3} \end{bmatrix},$$

$$W_{k1} = \begin{bmatrix} I_{m_1 m_2} \otimes (k\theta\alpha) & 0 & 0_{m_1 m_2 \times m_1 m_3^2} \\ 0 & I_{m_1 m_2} \otimes (k\theta\alpha) & 0_{m_1 m_3 \times m_1 m_3^2} \end{bmatrix},$$

$$W_{kc-2} = \begin{bmatrix} I_{m_1 m_2^{c-2}} \otimes (k\theta\alpha) & 0 & 0 & & 0 & 0 \\ 0 & I_{m_1 m_2^{c-3} m_3} \otimes (k\theta\alpha) & 0 & & 0 & 0 \\ & & & & & \dots \\ 0 & 0 & 0 & \dots & I_{m_1 m_3^{c-2}} \otimes (k\theta\alpha) & 0 \end{bmatrix},$$

$$W_{kc-1} = \begin{bmatrix} w'_{k0} & 0 & & 0 & 0 \\ 0 & w'_{k1} & & 0 & 0 \\ & & & & \dots \\ 0 & 0 & \dots & w'_{kc-1} & 0 \end{bmatrix}, \text{ where}$$

$$w'_{ki} = \begin{bmatrix} I_{m_1 m_2^{c-i} m_3^{i-1}} \otimes (k\theta\alpha) & 0 & \dots & 0 \end{bmatrix}_{(c+1)\times 1},$$

the 0's in w'_{ki} are zero matrices of order $m_1 m_2^{c-i} m_3^{i-1} \times m_1 m_2^{c-i+1} m_3^{i-1}$

$$W_{kc} = \begin{bmatrix} w''_0 & 0 & & 0 \\ 0 & w''_1 & & 0 \\ \dots & & & \dots \\ 0 & 0 & & w''_c \end{bmatrix}, \text{ where}$$

$$w''_{ki} = \begin{bmatrix} 0 & k\gamma I_{m_1 m_2^{c-i+1} m_3^{i-1}} & 0 & 0 & & 0 \\ 0 & 0 & k\gamma I_{m_1 m_2^{c-i+1} m_3^{i-1}} & 0 & & 0 \\ \dots & & & & & \\ 0 & 0 & 0 & 0 & \dots & k\gamma I_{m_1 m_2^{c-i+1} m_3^{i-1}} \\ 0 & 0 & 0 & 0 & & k\gamma I_{m_1 m_2^{c-i+1} m_3^{i-1}} \end{bmatrix}$$

4.2 System stability

Theorem 4.1. *With $\gamma > 0$, the system under discussion is always stable.*

Proof. Consider the Lyapunov test function defined by $\varphi(s) = k$, where 's' is a state in level k. Then the mean drift y_s is given by

$$y_s = \sum_{p \neq s} [\phi(p) - \phi(s)] q_{sp}$$

$$= \sum_{s'} [\phi(s') - \phi(s)]\, q_{ss'} + \sum_{s''} [\phi(s'') - \phi(s)]\, q_{ss''}$$

$$+ \sum_{s'''} [\phi(s''') - \phi(s)]\, q_{ss'''}$$

where s', s'' and s''' vary over the states belonging to levels $k-1$, k and $k+1$ respectively. Then $\varphi(s) = k$, $\varphi(s') = k-1$, $\varphi(s'') = k$ *and* $\varphi(s''') = k+1$

$$y_s = -\sum_{s'} q_{ss'} + \sum_{s'''} q_{ss'''}$$
$$= \begin{cases} -k\theta + \sum\limits_{s'''} q_{ss'''}, \text{ at least one server is free} \\ -k\gamma + \sum\limits_{s'''} q_{ss'''}\ , \text{ all servers are busy} \end{cases}$$

Since the number of phase is finite, $\sum\limits_{s'''} q_{ss'''}$ is bounded by some fixed constant for any s in level $k \geq 1$. Hence we can find a positive real number K such that $\sum\limits_{s'''} q_{ss'''} < K$ for all s in level $k \geq 1$. Thus, for any $\varepsilon > 0$,we can find K_1 large enough that $y_s < -\varepsilon$ for any s belonging to level $i \geq K_1$. Hence the theorem follows from Tweedie [55]. □

4.3 Steady state distribution

Here the process $\{X(t) : t \geq 0\}$ is a positive recurrent LDQBD and let $x = (x_0, x_1, ...)$ be its steady state distribution. x_i satisfies the relationship $x_{k+1} = x_k R_k$, $k \geq 0$, which gives $x_{k+1} = x_0 \prod\limits_{l=0}^{k} R_l$, where the family of matrices $\{R_k : k \geq 0\}$ are the minimal nonnegative solution of the system of equations

$$A_0 + R_k B_{k+1} + R_k R_{k+1} C_{k+2} = 0 \text{ for } k \geq 0 \tag{4.1}$$

and x_0 is the solution of

$$x_0(B_0 + R_0 C_1) = 0 \tag{4.2}$$

subject to

$$x_0 \left(I + \sum_{k=1}^{\infty} \prod_{l=0}^{k-1} R_l \right) \underline{e} = 1 \tag{4.3}$$

Before we pass on to the numerical computations we construct a dominating process. Here the process under discussion, $\{X(t), t \geq 0\}$, satisfies the condition that for all k$\geq$ 1 and for all i, there exist j such that $(\mathrm{C}_k)_{ij} > 0$. Therefore there exists a dominating process $\bar{X}(t)$ (see Bright and Taylor [13]) on the same state space as X(t) and with generator

$$\bar{Q} = \begin{bmatrix} B_0 & A_0 & & & & \\ 0 & \bar{B}_1 & \bar{A}_0 & & & \\ \cdot & \bar{C}_2 & \bar{B}_2 & \bar{A}_0 & & \\ & & \bar{C}_3 & \bar{B}_3 & \bar{A}_0 & \\ & & & & & \ddots \end{bmatrix},$$

where, $(\bar{A}_0)_{i,j} = \frac{1}{N}\left((A_0\underline{e})_{\max}\right),$

$(\bar{C}_k)_{i,j} = \frac{1}{N}\left((C_{k-1}\underline{e})_{\min}\right), k \geq 2,$

$(\bar{B}_k)_{i,j} = (B_k)_{i,j}, i \neq j$ and $k \geq 1$

in which $N = \sum_{i=1}^{c-1}\sum_{j=0}^{i} m_2^{i-j} m_3^j + (c+1)\sum_{j=0}^{c} m_2^{c-j} m_3^j$, $(A_0\underline{e})_{max}$ is the maximum element of the column vector $A_0\underline{e}$ and $(C_{k-1}\underline{e})_{\min}$ is the minimum element of the column vector $C_{k-1}\underline{e}$

We fix a truncation level K^* from the above method and employ Neuts-Rao [46] procedure in numerical computations. Thus $x_k(K^*)$, $0 \leq k \leq K^*$, is given by $x_k(K^*) = x_0(K^*)\prod_{l=0}^{k-1} R_l$ where $x_0(K^*)$ satisfies $x_0(B_0 + R_0C_1) = 0$.

4.4 System performance measures

We partition each x_k in the steady state probability vector $x = (x_0, x_1, x_2, ...)$ as $x_k = (y_{k0}, y_{k1}, ..., y_{kc})$ in which $y_{ki} = (y_{ki}(0,0), y_{ki}(1,0), ..., y_{ki}(i,0))$, for $k < c$ and $y_{kc} = (y_{kc}(j_1, j_2) : 0 \leq i, j \leq c)$. Here $y_{ki}(j_1, j_2)$ represents the row vector corresponding to $N_2(t) = i$, $N_3(t) = j_1$ and $N_4(t) = j_2$, respectively. We concentrate on the following system performance measures.

- Average number E_1 of customers in the orbit $= \sum_{k=0}^{\infty} k x_i \underline{e}$
- Average number E_2 of successful retrials $= \sum_{k=1}^{\infty} k\theta \left(\sum_{i=0}^{c-1}\sum_{j=0}^{i} y_{ki}(j,0)\right)\underline{e}$
- Average number E_3 of priority generated customers in the system

$$= \sum_{k=0}^{\infty} \left(\sum_{i=1}^{c-1} \sum_{j=1}^{i} j y_{ki}(j,0) + \sum_{i=0}^{c} \sum_{j=1}^{c} (i+j)\, y_{kc}(i,j) \right) \underline{e}$$

- Average number E_4 of priority generated customers waiting

$$= \sum_{k=0}^{\infty} \left(\sum_{i=0}^{c} \sum_{j=1}^{c} j y_{kc}(i,j) \right) \underline{e}$$

- Average number E_5 of priority generated customers lost per unit time

$$= \sum_{k=1}^{\infty} k\gamma \left(\sum_{i=0}^{c} y_{kc}(i,c) \right) \underline{e}$$

- Average number E_6 of idle servers $= \sum_{k=0}^{\infty} \left(\sum_{i=1}^{c-1} (c-i) \sum_{j=0}^{i} y_{ki}(j,0) \right) \underline{e}$.

In order to optimize the number of servers c numerically we construct a cost function as follows. Let

C_1 = Holding cost for each priority generated customer in service.

C_2 = Holding cost per unit of a priority generated customer waiting for service.

C_3 = Loss to the system due to a priority generated customer leaving without getting service.

C_4 = Holding cost of an idle server per unit time.

- The expected total cost, ETC $= (E_3 - E_4)C_1 + E_4C_2 + E_5C_3 + E_6C_4$.

4.5 Numerical examples

Example 1.

Take $D_0 = \begin{bmatrix} -11.0 & 0.50 \\ 0.25 & -0.75 \end{bmatrix}$ and $D_1 = \begin{bmatrix} 10.0 & 0.50 \\ 0.25 & 0.25 \end{bmatrix}$.

Fundamental arrival rate = 3.83333 and correlation = 0.12356

Let $S = \begin{bmatrix} -8.0 & 4.0 \\ 4.0 & -8.0 \end{bmatrix}$, $\quad S_0 = \begin{bmatrix} 4.0 \\ 4.0 \end{bmatrix}$

$$T = \begin{bmatrix} -15.0 & 3.0 \\ 3.0 & -15.0 \end{bmatrix} \text{ and } T_0 = \begin{bmatrix} 12.0 \\ 12.0 \end{bmatrix}$$

with $\alpha = \begin{bmatrix} 0.3 & 0.7 \end{bmatrix}$ and $\beta = \begin{bmatrix} 0.4 & 0.6 \end{bmatrix}$.

Further $\gamma = 10, \vartheta = 5, C_1 = 10, C_2 = 10, C_3 = 200, C_4 = 25$. We then have

Table 1. Number of servers versus expected total cost.

c	E_1	E_2	E_3	E_4	E_5	E_6	ETC
1	0.44208	0.48705	0.22213	0.16043	1.70035	0.64099	358.31605
2	0.13240	0.11774	0.14520	0.06997	0.18007	1.59876	77.43500
3	0.03042	0.10850	0.02080	0.00483	0.00398	2.66702	67.67950
4	0.00560	0.02366	0.00217	0.00027	0.00003	3.67911	92.00545
5	0.00089	0.00401	0.00024	0.00002	0.00000	4.68039	117.01215

Example 2.

Take $D_0 = \begin{bmatrix} -11.0 & 0.50 \\ 0.25 & -3.25 \end{bmatrix}$ and $D_1 = \begin{bmatrix} 10.0 & 0.5 \\ 0.5 & 2.5 \end{bmatrix}$.

Here fundamental arrival rate = 6.21425 and correlation = 0.15475

All other parameters are same as that in Example 1

Table 2. Number of servers versus expected total cost.

c	E_1	E_2	E_3	E_4	E_5	E_6	ETC
1	0.84632	0.98570	0.31515	0.23121	2.71705	0.43764	557.5025.
2	0.20219	0.16648	0.22478	0.11210	0.29429	1.35682	95.0263
3	0.04383	0.15481	0.03060	0.00736	0.00632	2.46227	63.12675
4	0.00793	0.03345	0.00310	0.00039	0.00004	3.48008	87.041
5	0.00126	0.00563	0.00034	0.00003	0.00000	4.48192	112.0514

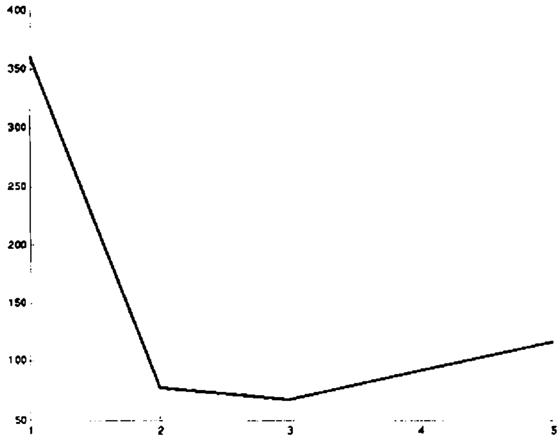

Figure 4.1: No. of servers versus expected total cost. corresponding to Table 1

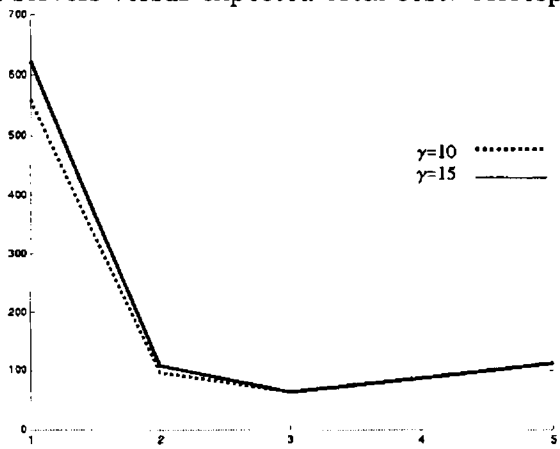

Figure 4.2: No. of servers versus expected total cost. (The performance measures corresponding to $\gamma = 10$ are given in Table 1.)

Table 1 and 2 show that increase in the fundamental arrival rate has its own effect in the system performance measures. Also the effect of variation of self generation of priority on the expected system cost can be seen in Figure 2. Thus numerical experiments indicate that the cost function in the number of servers is convex.

Chapter 5

MAP/PH/1 Multi-priority Retrial Queue

Here we deal with a queueing system with a finite number of priority classes, say m, labelled $1, 2, ..., m-1, m$. Each of the priority class i have a finite waiting space of capacity n_i, $i = 1, \ldots, m-1$, at the service station. Priority class m does not have waiting space in the service station. Hence if the arrival is being at the arrival epoch of a customer of priority m, then it joins an orbit of infinite capacity. These customers try to access the server independently of each other. The inter-retrial times have exponential distribution with parameter θ. If a retrial turns out to be a failure then the customer returns to the orbit and tries again. In addition orbital customers generate priority which we designate as super priority denoted by '$0'$. On priority generation they can get immediately into the service station provided either the server is idle or a waiting space of capacity '$1'$, exclusively for priority 0 customers, is vacant. Else it leaves the system forever. At each service completion epoch the next unit to be taken for service is a super priority customer provided there is one waiting. Customers of priority i, $1 \leq i \leq m-1$, generate into priority j according to an exponentially distributed random time with parameter $\gamma_{i,j}$ for $0 \leq j < i$. At this epoch if the waiting space for priority j is full, such priority generated customer leaves the system forever.

The process under discussion is always stable. We construct a dominating process by Bright and Taylor method and fix the truncation level. Then Neuts-Rao algorithm is employed to obtain the steady state system state probabilities. In 5.1 we formulate the problem mathematically and that the system is always stable is established in 5.2. In 5.3 we provide the steady state system state distribution. 5.4 provides some system performance measures and in 5.5 numerical illustrations are provided.

5.1 Mathematical modelling

Here we consider a single server retrial queueing system with a finite number of priority classes having finite waiting space at the service station and and an orbit of infinite capacity. An arriving customer can directly access the server if the server is free. If the arriving customer is a customer with priority $i = 1, ..., m-1$ and if the server is busy at the time of arrival, join in a priority class according to his priority at the time of arrival, provided there are free spaces. Else they leave the system forever. Let p_i be the probability that the arriving customer belongs to the priority class i. There is one super priority class with priority labelled as 0 in which there is no arrival from outside the system. i.e, $p_0 = 0$. If the arriving customer is one with least priority(this event has probability p_m), find the server busy and then it join the orbit of infinite capacity; where $p_m = 1 - \sum_{i=0}^{m-1} p_i$. The customers in the orbit try independently of each other to access the server at a constant rate θ.

Let n_i be the capacity of the waiting space of the i^{th} priority class, $1 \leq i \leq m-1$. The capacity of the waiting space of the super priority class is 1.i.e, $n_0 = 1$. The service discipline is non-preemptive. i.e, a customer is taken for service according to their priority, only after the service completion of the unit at the service station even when the priority generated customer belongs to the super priority class. Thus the maximum number of i^{th} priority class customer in the system at an epoch is $n_i + 1$, including those in service if it belongs to class i.

A priority class j is defined as a higher priority class than i if $j < i$. Customers in priority class i, $1 \leq i \leq m-1$, generate into higher priority at the rate γ_{ij}, $0 \leq j < i$, come into the j^{th} priority class if there is at least one free waiting space. Else it leaves the system in search of emergency service elsewhere. Priority generation of customers in the orbit is only to the super priority class at the rate γ_{m0}. Thus the generator of the process of priority generation is

$$\Gamma = \begin{bmatrix} 0 & 0 & 0 & & 0 & 0 \\ \gamma_{10} & \gamma_{11} & 0 & & 0 & 0 \\ \vdots & & & & & \\ \gamma_{m-10} & \gamma_{m-11} & \gamma_{m-12} & \cdots & \gamma_{m-1m-1} & 0 \\ \gamma_{m0} & 0 & 0 & & 0 & \gamma_{mm} \end{bmatrix}$$

where $\gamma_{ii} = -\sum\limits_{j=0}^{i-1} \gamma_{ij}, 1 \leq i \leq m.$

Customers arrive according to Markovian Arrival Process(MAP) with representation (D_0, D_1) of order l_1. The service distribution of each customer is phase type with representation (α, S) of order l_2 and $S_0 = -S\underline{e}$ where $\underline{e}$ is a column vector of $1's$ of appropriate order.

Let $\Delta_1(t) = \#$ of customers in the orbit at time t.

$\Delta_2(t) =$ server status at time t.

$$= \begin{cases} 0 \text{ if idle} \\ 1 \text{ if busy with an ordinary customer} \\ i+2 \text{ if busy with the } i^{th} \text{ priority class customer, } 0 \leq i \leq m-1 \end{cases}$$

$J_i(t) = \#$ of customers waiting in the i^{th} priority class at time t, $0 \leq i \leq m-1$.

$\Delta_3(t) =$ phase of arrival process.

$\Delta_4(t) =$ phase of service process.

If $X(t) = (\Delta_1(t), \Delta_2(t), J_{m-1}(t), \ldots.., J_0(t), \Delta_3(t), \Delta_4(t))$ then $\{X(t) : t \geq 0\}$ is a continues time Markov chain with state space

$S = \{(k, 0, ..., 0, v_1) : k \geq 0; 1 \leq v_1 \leq l_1\} \cup$

$\{(k, v_3, j_{m-1}, ..., j_0, v_1, v_2) : k \geq 0; 1 \leq v_3 \leq m+1; 0 \leq j_i \leq n_i;\ 1 \leq v_1 \leq l_1; 1 \leq v_2 \leq l_2\}$

Arranging the state space lexicographically the infinitesimal generator of the Markov chain has the form

$$Q = \begin{bmatrix} B_0 & A_0 & & & \\ C_1 & B_1 & A_0 & & \\ & C_1 & B_1 & A_0 & \\ & & & & \ddots \end{bmatrix}$$

Before describing the block matrices A_0, B_i and C_i we define the products $N_i = \prod_{j=0}^{i} (n_j+1)$ and $M_j = \prod_{i=j}^{m-1} (n_i+1)$. $A_0 = \begin{bmatrix} 0_{l_1 \times l_1} & 0 \\ 0 & A \end{bmatrix}$, where

$A = I_{(m+1)N_{m-1}} \otimes (p_m D_1 \otimes I_{l_2})$

$$B_k = \begin{bmatrix} D_0 - k\theta I_{l_2} & A_{01} & 0 & A_{03} & A_{04} & & A_{0m+1} \\ A_{10} & A_{11} & A_{12} & A_{13} & A_{14} & .. & A_{1m+1} \\ A_{10} & 0 & A_{11}+A_{12} & A_{13} & A_{14} & & A_{1m+1} \\ A_{10} & 0 & A_{12} & A_{11}+A_{13} & A_{14} & & A_{1m+1} \\ \vdots & & & & & & \\ A_{10} & 0 & A_{12} & A_{13} & A_{14} & .. & A_{11}+A_{1m+1} \end{bmatrix},$$

$$A_{0i} = \begin{bmatrix} p_{i-2}D_1 \otimes \alpha & 0 & \dots & 0 \end{bmatrix}_{1\times N_{m-1}}, \ i = 2,3,\dots,m+1, \text{ with } p_2{=}0,$$

$$A_{01} = \begin{bmatrix} p_m D_1 \otimes \alpha & 0 & \dots & 0 \end{bmatrix}_{1\times N_{m-1}} \quad \text{and}$$

$$A_{10} = \begin{bmatrix} Im_1 \otimes S_0 & 0 & \dots & 0 \end{bmatrix}^T_{N_{m-1}\times 1}, \text{ where } T \text{ denotes transpose;}$$

$$A_{12} = I_{M_1} \otimes \begin{bmatrix} 0 & 0 \\ I_{l_1} \otimes S_0\alpha & 0 \end{bmatrix}; \text{ let}$$

$$W_i = \begin{bmatrix} 0 & 0 & & 0 & 0 \\ w_{N_{i-1}} & 0 & & 0 & 0 \\ 0 & w_{N_{i-1}} & & 0 & 0 \\ & & & & .. \\ 0 & 0 & .. & w_{N_{i-1}} & 0 \end{bmatrix}_{(n_i+1)\times(n_i+1)}, \text{ where}$$

$$w_{N_{i-1}} = \begin{bmatrix} I_{l_1} \otimes S_0\alpha & 0 & .. & 0 \\ 0 & 0 & .. & 0 \\ & & .. & .. \\ 0 & 0 & .. & 0 \end{bmatrix}_{N_{i-1}\times N_{i-1}}, \ i = 2,3,\dots,m-1,$$

then $A_{1i+1} = I_{M_i} \otimes W_{i-1}, \ i = 1,2,\dots,m-1$ and $A_{1m+1} = W_{m-1}$;

$A_{11} = A'_{11} - k\gamma_{m0} I_{N_{m-1}l_1l_2}$, in which

$$A'_{11} = \begin{bmatrix} V_{11}^{(m-1)} & V_{12}^{(m-1)} & 0 & 0 & 0 & 0 \\ 1\cdot V_{21}^{(m-1)} & V_{22}^{(m-1)} & V_{12}^{(m-1)} & 0 & 0 & 0 \\ 0 & 2\cdot V_{21}^{(m-1)} & V_{33}^{(m-1)} & V_{12}^{(m-1)} & 0 & 0 \\ & & & & & \\ 0 & 0 & 0 & 0 & V_{n_{m-1}n_{m-1}}^{(m-1)} & V_{12}^{(m-1)} \\ 0 & 0 & 0 & 0 & n_{m-1}\cdot V_{21}^{(m-1)} & V^{(m-1)} \end{bmatrix},$$

$$V^{(m-1)} = V_{n_{m-1}+1,n_{m-1}+1}^{(m-1)} + V_{12}^{(m-1)}, \quad V_{12}^{(m-1)} = I_{N_{m-2}} \otimes (p_{m-1}D_1 \otimes I_{l_2}),$$

$$V_{ii}^{(m-1)} = V_{11}^{(m-1)} + (i-1)\, I_{N_{m-2}} \otimes (\gamma_{m-1,m-1} I_{l_1l_2}), \ i = 2,3,\dots,(n_{m-1}+1),$$

$$V_{11}^{(m-1)} = \begin{bmatrix} V_{11}^{(m-2)} & V_{12}^{(m-2)} & 0 & 0 & 0 & 0 \\ 1\cdot V_{21}^{(m-2)} & V_{22}^{(m-2)} & V_{12}^{(m-2)} & 0 & 0 & 0 \\ 0 & 2\cdot V_{21}^{(m-2)} & V_{33}^{(m-2)} & V_{12}^{(m-2)} & 0 & 0 \\ & & & & .. & \\ 0 & 0 & 0 & 0 & V_{n_{m-2}n_{m-2}}^{(m-2)} & V_{12}^{(m-2)} \\ 0 & 0 & 0 & 0 & n_{m-2}\cdot V_{21}^{(m-2)} & V^{(m-2)} \end{bmatrix},$$

$V^{(m-2)} = V_{n_{m-2}+1,n_{m-2}+1}^{(m-2)} + V_{12}^{(m-2)}, \; V_{12}^{(m-2)} = I_{N_{m-3}} \otimes (p_{m-2}D_1 \otimes I_{l_2}),$

$V_{ii}^{(m-2)} = V_{11}^{(m-2)} + (i-1)\, I_{N_{m-3}} \otimes (\gamma_{m-2,m-2} I_{l_1 l_2}), \; i = 2,3,...,(n_{m-2}+1),$

$$V_{11}^{(2)} = \begin{bmatrix} V_{11}^{(1)} & V_{12}^{(1)} & 0 & 0 & 0 & 0 \\ 1\cdot V_{21}^{(1)} & V_{22}^{(1)} & V_{12}^{(1)} & 0 & 0 & 0 \\ 0 & 2\cdot V_{21}^{(1)} & V_{33}^{(1)} & V_{12}^{(1)} & 0 & 0 \\ & & & & .. & \\ 0 & 0 & 0 & 0 & V_{n_1 n_1}^{(1)} & V_{12}^{(1)} \\ 0 & 0 & 0 & 0 \;\; .. & n_1\cdot V_{21}^{(1)} & V^{(1)} \end{bmatrix},$$

$V^{(1)} = V_{n_1+1,n_1+1}^{(1)} + V_{12}^{(1)}, \; V_{12}^{(1)} = I_{N_0} \otimes (p_1 D_1 \otimes I_{l_2}),$

$V_{ii}^{(1)} = V_{11}^{(1)} + (i-1)\, I_{N_0} \otimes (\gamma_{1,1} I_{l_1 l_2}), \; i = 2,3,...,(n_1+1),$

$$V_{11}^{(1)} = \begin{bmatrix} D_0 \oplus S & 0 \\ 0 & D_0 \oplus S \end{bmatrix}, \quad V_{21}^{(1)} = \begin{bmatrix} 0 & \gamma_{10} I_{l_1 l_2} \\ 0 & \gamma_{10} I_{l_1 l_2} \end{bmatrix};$$

$$V_{21}^{(m-1)} = \begin{bmatrix} U_{11}^{(m-3)} & U_{12}^{(m-2)} & 0 & 0 & 0 \\ 0 & U_{11}^{(m-3)} & U_{12}^{(m-2)} & 0 & 0 \\ & & & & \\ 0 & 0 & 0 & U_{11}^{(m-3)} & U_{12}^{(m-2)} \\ 0 & 0 & 0 & 0 & U_{11}^{(m-3)} + U_{12}^{(m-2)} \end{bmatrix},$$

$U_{12}^{(m-2)} = I_{N_{m-3}} \otimes (\gamma_{m-1,m-2} I_{l_1 l_2}),$

$$U_{11}^{(m-3)} = \begin{bmatrix} U_{11}^{(m-4)} & U_{12}^{(m-3)} & 0 & & 0 & 0 \\ 0 & U_{11}^{(m-4)} & U_{12}^{(m-3)} & & 0 & 0 \\ & & & & .. & .. \\ 0 & 0 & 0 & .. & U_{11}^{(m-4)} & U_{12}^{(m-3)} \\ 0 & 0 & 0 & & 0 & U_{11}^{(m-4)} + U_{12}^{(m-3)} \end{bmatrix},$$

$U_{12}^{(m-3)} = I_{N_{m-4}} \otimes (\gamma_{m-1,m-3} I_{l_1 l_2})$,

$$U_{11}^{(1)} = \begin{bmatrix} U_{11}^{(0)} & U_{12}^{(1)} & 0 & & 0 & 0 \\ 0 & U_{11}^{(0)} & U_{12}^{(1)} & & 0 & 0 \\ & & & & .. & .. \\ 0 & 0 & 0 & .. & U_{11}^{(0)} & U_{12}^{(1)} \\ 0 & 0 & 0 & & 0 & U_{11}^{(0)} + U_{12}^{(1)} \end{bmatrix},$$

$U_{12}^{(1)} = I_{N_0} \otimes (\gamma_{m-1,1} I_{l_1 l_2})$ and

$U_{11}^{(0)} = \begin{bmatrix} 0 & \gamma_{m-1,0} I_{l_1 l_2} \\ 0 & \gamma_{m-1,0} I_{l_1 l_2} \end{bmatrix}$; $C_k = \begin{bmatrix} 0 & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$, $C_{12} = \begin{bmatrix} I_{l_1} \otimes k\theta\alpha & 0 & .. & 0 \end{bmatrix}$,

$C_{21} = \begin{bmatrix} 0 & 0 & .. & 0 \end{bmatrix}^T_{(m+1)N_{m-1} \times 1}$, the 0's in C_{21} are zero matrices of order $l_1 l_2 \times l_1$

$C_{22} = I_{(m=1)M_1} \otimes \begin{bmatrix} 0 & \gamma_{m,0} I_{l_1 l_2} \\ 0 & \gamma_{m,0} I_{l_1 l_2} \end{bmatrix}$, here 0's are zero matrices of order $l_1 l_2 \times l_1 l_2$

5.2 System stability

Theorem 5.1. *The system under discussion is always stable.*

Proof. Consider the Lyapunov test function defined by $\varphi(s) = k$, where 's' is a state in level k. Then the mean drift y_s is given by

$y_s = \sum\limits_{p \neq s} [\phi(p) - \phi(s)] q_{sp}$

$= \sum\limits_{s'} [\phi(s') - \phi(s)]\, q_{ss'} + \sum\limits_{s''} [\phi(s'') - \phi(s)]\, q_{ss''} + \sum\limits_{s'''} [\phi(s''') - \phi(s)]\, q_{ss'''}$

where s', s'' and s''' vary over the states belonging to levels $k-1$, k and $k+1$ respectively. Then $\varphi(s) = k$, $\varphi(s') = k-1$, $\varphi(s'') = k$ *and* $\varphi(s''') = k+1$

$y_s = -\sum\limits_{s'} q_{ss'} + \sum\limits_{s'''} q_{ss'''}$

$= \begin{cases} -k\theta + \sum\limits_{s'''} q_{ss'''}, \text{if tthe server is free} \\ -k\gamma + \sum\limits_{s'''} q_{ss'''} \text{ , if the server is busy} \end{cases}$

Since the number of phase is finite, $\sum\limits_{s'''} q_{ss'''}$ is bounded by some fixed constant for any s in level $k \geq 1$. Hence we can find a positive real number K such that $\sum\limits_{s'''} q_{ss'''} < K$ for all s in level $k \geq 1$. Thus, for any $\varepsilon > 0$,we can find K' large enough that $y_s < -\varepsilon$ for any s belonging to level $i \geq K'$. Hence the theorem follows from Tweedie's [55] result. □

5.3 Steady state distribution

The process under discussion $\{X(t) : t \geq 0\}$ is a positive recurrent LDQBD. Let $x = (x_0, x_1, \ldots..)$ be its steady state distribution, then x_i holds the relationship $x_{k+1} = x_k R_k$, $k \geq 0$, which gives $x_{k+1} = x_0 \prod\limits_{l=0}^{k} R_l$, where the family of matrices $\{R_k : k \geq 0\}$ are the minimal nonnegative solution of the system of equations

$$A_0 + R_k B_{k+1} + R_k R_{k+1} C_{k+2} = 0 \text{ for } k \geq 0 \tag{5.1}$$

and x_0 is the solution of

$$x_0(B_0 + R_0 C_1) = 0 \tag{5.2}$$

subject to

$$x_0\left(I+\sum_{k=1}^{\infty}\prod_{l=0}^{k-1}R_l\right)\underline{e}=1 \tag{5.3}$$

For all $k \geq 1$, the entries in C_k are determined by the rate of successful retrials and priority generation. Therefore there exist at least one nonzero entry in each row of C_k i.e., $(C_k)_{i,j} > 0$. So we can construct a dominating process $\bar{X}(t)$ (see Bright and Taylor [13]) on the same state space as that of the original process $X(t)$. The generator of the dominating process is given by

$$\bar{Q} = \begin{bmatrix} B_0 & A_0 & & & & \\ 0 & \bar{B}_1 & \bar{A}_0 & & & \\ & \bar{C}_2 & \bar{B}_2 & \bar{A}_0 & & \\ & & \bar{C}_3 & \bar{B}_3 & \bar{A}_0 & \\ & & & & & \ddots \end{bmatrix},$$

where, $(\bar{A}_0)_{i,j} = \frac{1}{N}\left((A_0\underline{e})_{\max}\right),$

$(\bar{C}_k)_{i,j} = \frac{1}{N}\left((C_{k-1}\underline{e})_{\min}\right), k \geq 2,$

$(\bar{B}_k)_{i,j} = (B_k)_{i,j}, i \neq j$ and $k \geq 1$

in which N $=(m+1)N_{m-1}l_1l_2$, $(A_0\underline{e})_{max}$ is the maximum element of the column vector $A_0\underline{e}$ and $(C_{k-1}\underline{e})_{\min}$ is the minimum element of the column vector $C_{k-1}\underline{e}$

As explained in Chapter 2 it is possible to fix a truncation level K^* from this dominating process, which will work in Neuts-Rao [46] procedure to determine the steady state system stare distribution numerically.

5.4 System performance measures

We partition each vector x_k of the steady state probability vector $x = (x_0, x_1, ...)$ as $x_k = (y(k,0,...,0), y(k,\nu_1,j_{m-1},...,j_1,j_0))$, $1 \leq \nu_1 \leq m+1$, $0 \leq j_i \leq n_i$ and j_i denotes the number of customers in priority class i, $0 \leq i \leq m-1$. The row vector $y(k, 0, ..., 0)$ for $k \geq 0$ contains l_1 entries and $y(k, \nu_1, j_{m-1}, ..., j_1, j_0)$ contains $l_1 l_2$ entries. We concentrate on the following system performance measures.

- Average number E' of customers in the orbit $= \sum_{k=0}^{\infty} k x_k \underline{e}$
- Average number E'' of successful retrials $= \sum_{k=1}^{\infty} k\theta y(k, 0, ..., 0)\, \underline{e}$
- Average number E_i of customers waiting in priority class i

$= \sum_{j_i=1}^{n_i} j_i \sum_{k=0}^{\infty} \sum_{j_{m-1}=0}^{n_{m-1}} \cdots \sum_{j_{i+1}=0}^{n_{i+1}} \sum_{j_{i-1}=0}^{n_{i-1}} \cdots \sum_{j_0=0}^{n_0} y(k, v_3, j_{m-1}, ... j_0)\underline{e}.$

Let $L_{i,i'} = \sum_{j_i=1}^{n_i} j_i \gamma_{i,i'} \sum_{k=0}^{\infty} \sum_{j_{m-1}=0}^{n_{m-1}} \cdots \sum_{j_{i+1}=0}^{n_{i+1}} \sum_{j_{i-1}=0}^{n_{i-1}} \cdots \sum_{j_0=0}^{n_0} y(k, v_3, j_{m-1}, ... n_{i'}, ..., j_0)\underline{e}$. Then

- Average number L_i of priority generated customers lost from priority class i

$= \sum_{i'=0}^{i-1} L_{i,i'},\ 1 \leq i \leq m-1.$

- Average number L_m of priority generated customers lost from the orbit

$= \sum_{k=1}^{\infty} k\gamma_{m,0} \sum_{j_{m-1}=0}^{n_{m-1}} \cdots \sum_{j_1=0}^{n_1} y(k, v_3, j_{m-1}, ... j_1, 1)\, \underline{e}.$

5.5 Numerical example

Example 1.

Take $D_0 = \begin{bmatrix} -12.0 & 0.25 \\ 0.25 & -3.25 \end{bmatrix}$ and $D_1 = \begin{bmatrix} 11.5 & 0.25 \\ 0.5 & 2.5 \end{bmatrix}$.

Here average arrival rate = 8.25000 and correlation = 0.18064

Let $S = \begin{bmatrix} -8.0 & 4.0 \\ 4.0 & -8.0 \end{bmatrix}$, $\quad S_0 = \begin{bmatrix} 4.0 \\ 4.0 \end{bmatrix}$ with $\alpha = \begin{bmatrix} 0.4 & 0.6 \end{bmatrix}$.

$p_1 = 4.0,\ p_2 = 3.0,\ p_3 = 3.0,\ \theta = 10$ and

Concluding remarks and suggestions for further study

In this thesis we have studied a few models involving self-generation of priorities. Priority queues have been extensively discussed in literature(see Jaiswal(1968), Takagi [36, 52]). However, these are situations involving priority assigned to (or possessed by) customers at the time of their arrival. Nevertheless, customers generating into priority is a common phenomena. Such situations especially arise at a physicians clinic, aircrafts hovering over airport running out of fuel but waiting for clearance to land and in several communication systems. Quantification of these are very little seen in literature except for those cited in some of the work indicated in the introduction. Our attempt is to quantify a few of such problems. In doing so, we have also generalized the classical priority queues by introducing priority generation (going to higher priorities and during waiting). Systematically we have proceeded from single server queue (in Chapter 2) to multi server queue(Chapter 3 and 4). We also introduced customers with repeated attempts (retrial) generating priorities(see page 72). All models that were analyzed in this thesis involve non-preemptive service. Since the models are not analytically tractable, a large number of numerical illustrations were produced in each chapter to get a feel about the working of the systems.

One can extend the models discussed in this thesis to several directions. For example some of the models can be analyzed in the preemptive situation, the results for which are not available till date.

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