Some Problems in Graph Theory

STUDIES ON SOME GRAPH CLASSES

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Certificate

This is to certify that the work reported in the thesis entitled **'Studies on Some Graph Classes'** that is being submitted by Smt. Aparna Lakshmanan S. for the award of Doctor of Philosophy to Cochin University of Science and Technology is based on bonafide research work carried out by her under my supervision in the Department of Mathematics, Cochin University of Science and Technology. The results embodied in this thesis have not been included in any other thesis submitted previously for the award of any degree or diploma.

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Declaration

The thesis entitled **'Studies on Some Graph Classes'** contains no material which had been accepted for any other Degree or Diploma in any University and to the best of my knowledge and belief, it contains no material previously published by any person except where due reference is made in the text of the thesis.

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God gave you a gift of 86,400 seconds today. Have you used one to say "thank you?"

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As we express our gratitude, we must never forget that the highest appreciation is not to utter words, but to live by them.

John Fitzgerald Kennedy

Aparna Lakshmanan S.

STUDIES ON SOME GRAPH CLASSES

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Chapter 1

Introduction

The origin of graph theory dates back to more than two hundred and seventy years when the renowned Swiss Mathematician Leonhard Euler solved the 'Konigsberg Bridge Problem' in his talk 'The solution of a problem relating to geometry of position' presented at St.Petersberg Acadamy on 26th August, 1735. Since then the subject has grown into one of the most inter disciplinary branches in mathematics with a great variety of applications. The first book on this subject was by B.Kõnig [49]. Volumes have been written on the rich theory and the very many applications of graphs ([11], [19], [68], [79]), including the pioneer works of C.Berge [18], F. Harary [43] and O.Ore [61].

The applications of graph theory in operation research, social science, psychology and physics are detailed in C.W.Marshall [56]. J.L.Gross [40] discusses a variety of graph classes with numerous illuminating examples which are of topological relevance. The development of graph theory with its applications to electrical networks, flows and connectivity are included in [20] and [31]. Ramsey theory is an interesting branch of graph theory which relates it to the number theory. R.L.Graham, B.L.Rothschild and J.H.Spencer has written a book [38] in this area which covers all major developments in the subject. In [16], connections of graph theory with other branches of mathematics such as coding theory, algebra etc are discussed.

This thesis entitled '**Studies on Some Graph Classes**' is a humble attempt at making a small addition to the vast ocean of results in graph theory.

By the term ***graph class***, we mean a collection of graphs which satisfies some specific properties.

A graph operator is a mapping $T : \mathcal{G} \to \mathcal{G}'$ where $\mathcal{G}$ and $\mathcal{G}'$ are families of graphs. The most familiar examples of graph operators are the graph complement and the line graph. A variety of graph classes can be obtained by applying suitable graph operators. The study of graph operators initiated with a set of three problems on line graphs posed by O. Ore [61].

- Determine all graphs isomorphic to their line graph.
- When the line graph is given, is the original graph uniquely determined?
- Investigate iterated line graphs.

Graph operators and its dynamics - fixedness, convergence, divergence etc. - are extensively studied in [63]. The Gallai graphs , the anti-Gallai graphs, the cycle graphs and the edge graphs are some of the graph classes obtained by choosing appropriate graph operators.

Another way of identifying graph classes is through finite or infinite collection of forbidden subgraphs. The inclusions between graph classes can be easily identified from the forbidden subgraph characterizations. The cographs , the split graphs , the threshold graphs and the line graphs are some of the interesting graph classes which admit finite forbidden subgraph characterizations. There are other interesting graph classes defined by forbidding an infinite collection of induced subgraphs like the perfect graphs, the distance hereditary graphs, the comparability graphs and the chordal graphs. The famous concept of minors is also an example of forbidden subgraph characterization. Kuratowski's theorem [50] on planar graphs is a striking example of this kind.

Yet another way of defining graph classes is through recursive characterizations. The trees , the cographs and the distance hereditary graphs are some of the graph classes which admit recursive characterizations.

The intersection graph is a very general notion in which objects are assigned to the vertices of a graph and two distinct vertices are adjacent if their objects have non empty intersection. A variety of well studied graph classes including the line graphs, the chordal graphs, the clique graphs and the block graphs are special types of intersection graphs.

Graph classes also arise in connection with various graph parameters such as the clique transversal number, the clique independence number, the chromatic number and the clique number and various sub structures of a graph such as the cliques, the dominating sets etc. The perfect graphs, the clique perfect graphs, the clique irreducible graphs and the weakly clique irreducible graphs are examples of such graph classes.

In any discussion on graph classes, a main source is the classical book by

M.C.Golumbic, *Algorithmic Graph Theory and Perfect Graphs* [37]. A detailed study of about two hundred graph classes with an extensive bibliography is in the book *'Graph Classes : A survey'* by A. Brandstädt, V. B. Le and J. P. Spinrad [14].

This thesis is mainly concerned with the graph classes - the Gallai graphs, the anti-Gallai graphs, the cographs, the clique graphs, the clique irreducible graphs and the weakly clique irreducible graphs.

1.1 Basic definitions and lemmas

The basic notations, terminology and definitions are from [11], [14], [30], [34], [37], [52], [60], [65] and [71].

Definition 1.1.1. A **graph** $G = (V, E)$ consists of a non-empty collection of points, V called its **vertices** and a set of unordered pairs of distinct vertices, E called its **edges**. The unordered pair of vertices $\{u, v\} \in E$ are called the **end vertices** of the edge $e = \{u, v\}$. In that case, the vertex u is said to be **adjacent** to the vertex v. Two edges e and e' are said to be **incident** if they have a common end vertex. $|V|$ is called the **order** of G, denoted by n or $n(G)$ and $|E|$ is called the **size** of G, denoted by m or $m(G)$. A graph G is **trivial** or **empty** if it has no edges.

Definition 1.1.2. A graph $H = (V', E')$ is called a **subgraph** of G if $V' \subseteq V$ and $E' \subseteq E$. A subgraph H is a **spanning subgraph** if $V' = V$. H is called an **induced subgraph** if E' is the collection of all edges in G which has both its end vertices in V'. $< V' >$ denotes the induced subgraph with vertex set V'. A property P of a graph G is **vertex hereditary** if every induced subgraph of G has the property P. A graph H is a **forbidden subgraph** for a property P, if any graph G which satisfies the property P cannot have H as an induced subgraph. A graph G is **H-free** if it does not have H as an induced subgraph.

Definition 1.1.3. The number of vertices adjacent to a vertex v is called the **degree** of the vertex, denoted by $d(v)$. A vertex of degree one is called a **pendant vertex** and a vertex of degree $n-1$ is called a **universal vertex**.

Definition 1.1.4. A graph G is **k-regular** if $d(v) = k$ for every vertex $v \in V(G)$. A spanning 1-regular graph is called a **1-factor** or **perfect matching**.

Definition 1.1.5. The set of all vertices adjacent to a vertex v is called **open neighborhood** of v, denoted by $N(v)$. The open neighborhood of v together with the vertex v is called the **closed neighborhood** of v, denoted by $N[v]$.

Definition 1.1.6. A **false twin** of a vertex u is a vertex v which is adjacent to all the vertices in $N[u]$. A **true twin** of a vertex u is a vertex v which is adjacent to all the vertices in $N(u)$.

Definition 1.1.7. A graph $G = (V, E)$ is **isomorphic** to a graph $H = (V', E')$ if there exists a bijection from V to V' which preserves adjacency. If G is isomorphic to H, we write $G = H$.

Definition 1.1.8. A **path** on n vertices P_n is the graph with vertex set $\{v_1, v_2, ..., v_n\}$ and v_i is adjacent to v_{i+1} for $i = 1, 2, ..., n-1$ are the only edges. If in addition v_n is adjacent to v_1 then it is called a **cycle** of length n, C_n. A path from the vertex u to the vertex v is called a $\mathbf{u-v}$ **path**. A graph G is **connected** if for every $u, v \in V$, there exists a $u-v$ path. If G is not connected then it is **disconnected**.

A maximal connected subgraph of G is called a **component** of G. A component of a graph G is **non-trivial** if it has at least one edge. A graph is **acyclic** if it does not contain cycles. A connected acyclic graph is called a **tree**.

Definition 1.1.9. A graph G is **bipartite** if the vertex set can be partitioned into two non-empty sets U and U' such that every edge of G has one end vertex in U and the other in U'. A bipartite graph in which each vertex of U is adjacent to every vertex of U' is called a **complete bipartite graph**. If $|U| = m$ and $U' = |n|$, then the complete bipartite graph is denoted by $K_{m,n}$. The complete bipartite graph $K_{1,n}$ is called a **star**.

Definition 1.1.10. Let G be a graph. The **complement** of G, denoted by G^c is the graph with vertex set same as that of V and any two vertices are adjacent in G^c if they are not adjacent in G. K_n^c is called **totally disconnected**. A graph G is called **self complementary** if $G = G^c$.

Definition 1.1.11. A subset $I \subseteq V$ of vertices are said to be **independent** if no two vertices of I are adjacent. The maximum cardinality of an independent set is called the **independence number** $\alpha(G)$. A subset $K \subseteq V$ is called a **covering** of G if every edge of G is incident with at least one vertex of K. The number of vertices in a minimum covering is called the **covering number** $\beta(G)$.

Definition 1.1.12. A subgraph H of G is a **complete** if every pair of distinct vertices of G are adjacent. A complete graph on n vertices is denoted by K_n. K_3 is called a **triangle**. A complete is maximal if it is not properly contained in any other complete. A maximal complete subgraph is called a **clique**. The size of the largest clique in G is called the **clique number** $\omega(G)$.

Definition 1.1.13. The **intersection graph** of a graph G is a graph whose vertex set is a collection of objects and any two vertices are adjacent if the corresponding objects intersect. The intersection graph of all cliques of a graph G is called the **clique graph** of G denoted by $K(G)$. If $K(G)$ is complete then G is called **clique complete**.

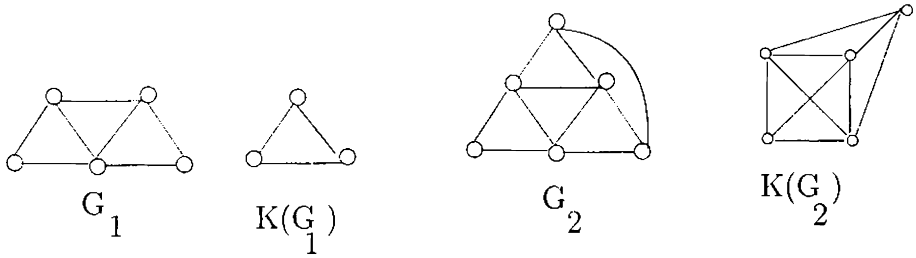


Fig : 1.1

In Fig : 1.1 G_1 is clique complete.

Definition 1.1.14. A collection of objects $\mathcal{E}$ satisfies **Helly property** if for any sub collection $\mathcal{E}' \subseteq \mathcal{E}$, the elements of $\mathcal{E}'$ pair-wise intersect, then $\bigcap_{e\in\mathcal{E}'} e \neq \phi$. If the cliques of a graph G satisfies Helly property then we say that G is **clique-Helly**. If G and all its induced subgraphs are clique-Helly, then G is **hereditary clique-Helly**.

In Fig 1.1 G_1 is clique-Helly, where as G_2 is not.

Definition 1.1.15. Assigning colors to the vertices of a graph is called a **vertex coloring**. If no two adjacent vertices receives the same color, then such a coloring is called a **proper vertex coloring**. The minimum number of colors required for a proper vertex coloring of a graph G is called its **chromatic number** , denoted by $\chi(G)$.

Definition 1.1.16. The **distance** between two vertices u and v of a connected graph G, denoted by $d_G(u, v)$ or $d(u, v)$ is the length of a shortest $u - v$ path. The **eccentricity** of a vertex $e(v) = max\{d(u, v) : v \in V(G)\}$. The **radius** of a graph

$r(G)$ is the minimum of the eccentricities of its vertices and the **diameter** of a graph $d(G)$ is the maximum of the eccentricities of its vertices.

Definition 1.1.17. The **line graph** of a graph G denoted by $L(G)$ has the edges of G as its vertices and any two vertices are adjacent in $L(G)$ if the corresponding edges in G are incident. The **iterated line graphs** of G are defined as $L^k(G) = L(L^{k-1}(G))$ for $k > 1$.

Definition 1.1.18. The **Gallai graph** $\Gamma(G)$ of a graph G has the edges of G as its vertices and any two vertices are adjacent in $\Gamma(G)$ if the corresponding edges are incident in G, but do not span a triangle in G. The **anti-Gallai graph** $\Delta(G)$ of a graph G has the edges of G as its vertices and any two vertices of G are adjacent in $\Delta(G)$ if the corresponding edges are incident in G and lie on a triangle in G. The **iterated Gallai graphs** and the **iterated anti-Gallai graphs** are defined as $\Gamma^k(G) = \Gamma(\Gamma^{k-1}(G)))$ and $\Delta^k(G) = \Delta(\Delta^{k-1}(G))$ respectively for $k > 1$.

Both $\Gamma(G)$ and $\Delta(G)$ are spanning subgraphs of $L(G)$ and their union is $L(G)$.

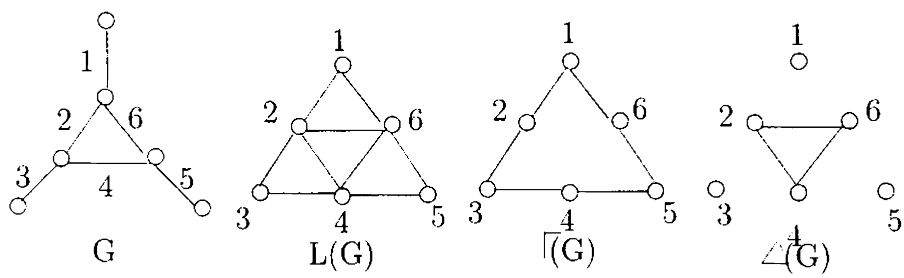


Fig 1.2

Definition 1.1.19. A set $S \subseteq V$ of vertices in a graph G is called a **dominating set** if every vertex $v \in V$ is either an element of S or is adjacent to an element of S. A dominating set S is **minimal dominating** if no proper subset of S is a dominating set. The **domination number** $\gamma(G)$ of a graph G is the minimum cardinality of a dominating set in G. A set $S \subseteq V$ of vertices in a graph G is called a

global dominating set if it dominates both G and G^c. The minimum cardinality of a global dominating set is called the **global domination number** $\gamma_{gcd}(G)$. A set $S \subseteq V$ of vertices in a graph G is called an **independent dominating set** if S is independent and S dominates G. The minimum cardinality of an independent dominating set is called the **independent domination number** $\gamma_i(G)$.

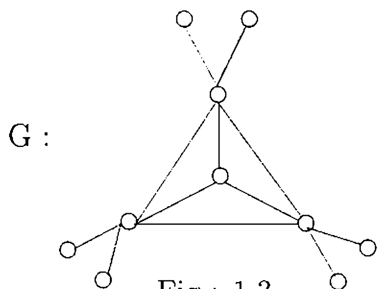


Fig : 1.3

For the graph G in Fig : 1.3, $\gamma(G) = 3$, $\gamma_g(G) = 4$ and $\gamma_i(G) = 5$.

Definition 1.1.20. A graph that can be reduced to edgeless graph by taking complements within components is called a **cograph**.

For example, any graph of order less than or equal to four, except P_4 is a cograph. The complete bipartite graphs and complete graphs are also examples of cographs.

Definition 1.1.21. A **plane representation** of a graph G is an isomorphic copy of G in which any two edges intersect only at the vertices. A graph which admits a plane representation is called a **planar graph**.

Definition 1.1.22. The **union** of two graphs G and H denoted by $G \cup H$ is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

Definition 1.1.23. The **join** of two graphs G and H denoted by $G \vee H$ is the graph with vertex set $V(G) \cup V(H)$ and $E(G \vee H) = E(G) \cup E(H) \cup \{uv : u \in V(G)$ and $v \in V(H)\}$.

Definition 1.1.24. The **tensor product** of two graphs G and H denoted by $G \times H$ is the graph with $V(G \times H) = \{u, v) : u \in V(G_1)$ and $v \in V(G_2)\}$ and any two vertices (u_1, v_1) and (u_2, v_2) are adjacent if $u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)$.

Definition 1.1.25. The **cartesian product** of two graphs G and H denoted by $G\Box H$ is the graph with $V(G\Box H) = \{u, v) : u \in V(G_1)$ and $v \in V(G_2)\}$ and any two vertices (u_1, v_1) and (u_2, v_2) are adjacent if one of the following holds.

(i) $u_1 = u_2$ and $v_1v_2 \in E(G_2)$

(ii) $u_1u_2 \in E(G_1)$ and $v_1 = v_2$.

Definition 1.1.26. The **strong product** of two graphs G and H denoted by $G \otimes H$ is the graph with $V(G \otimes H) = \{u, v) : u \in V(G_1)$ and $v \in V(G_2)\}$ and any two vertices (u_1, v_1) and (u_2, v_2) are adjacent if one of the following holds.

(i) $u_1 = u_2$ and $v_1v_2 \in E(G_2)$

(ii) $u_1u_2 \in E(G_1)$ and $v_1 = v_2$

(iii) $u_1u_2 \in E(G_1)$ and $v_1v_2 \in E(G_2)$.

Definition 1.1.27. A graphical invariant σ is **super multiplicative** with respect to a graph product $\circ$, if given any two graphs G and H, $\sigma(G \circ H) \geqslant \sigma(G)\sigma(H)$ and **sub multiplicative** if $\sigma(G \circ H) \leqslant \sigma(G)\sigma(H)$. A class $\mathcal{C}$ is called a **universal multiplicative** class for σ on $\circ$ if for every graph H, $\sigma(G \circ H) = \sigma(G)\sigma(H)$ whenever $G \in \mathcal{C}$.

Definition 1.1.28. Let $\mathcal{B}$ be a non-empty subset of the collection of all binary n-tuples which does not include $(0, 0, ..., 0)$. **The non-complete extended p-sum** of graphs $G_1, G_2, ..., G_p$ with **basis** $\mathcal{B}$ denoted by NEPS$(G_1, G_2, ..., G_p; \mathcal{B})$, is the graph with vertex set $V(G_1) \times V(G_2) \times ... \times V(G_p)$, in which two vertices $(u_1, u_2, ..., u_p)$ and $(v_1, v_2, ..., v_p)$ are adjacent if and only if there exists $(\beta_1, \beta_2, ..., \beta_p) \in \mathcal{B}$ such that u_i is adjacent to v_i in G_i whenever $\beta_i = 1$ and $u_i = v_i$ whenever $\beta_i = 0$. The graphs $G_1, G_2, ..., G_p$ are called the **factors** of the NEPS.

There are seven possible ways of choosing the basis $\mathcal{B}$ when $p = 2$.

$\mathcal{B}_1 = \{(0,1)\}$

$\mathcal{B}_2 = \{(1,0)\}$

$\mathcal{B}_3 = \{(1,1)\}$

$\mathcal{B}_4 = \{(0,1),(1,0)\}$

$\mathcal{B}_5 = \{(0,1),(1,1)\}$

$\mathcal{B}_6 = \{(1,0),(1,1)\}$

$\mathcal{B}_7 = \{(0,1),(1,0),(1,1)\}$

The NEPS of graphs G_1 and G_2 with basis $\mathcal{B}_3$, $\mathcal{B}_4$ and $\mathcal{B}_7$ are the tensor product, the cartesian product and the strong product respectively.

Definition 1.1.29. A subset V' of V is called a **clique transversal** , if it intersects with every clique of G. The **clique transversal number** $\tau_c(G)$ of a graph G is the minimum cardinality of a clique transversal of G. A collection of mutually non-intersecting cliques is called a **clique independent set.** The maximum cardinality of a clique independent set in a graph G is called the **clique independence number** $\alpha_c(G)$.

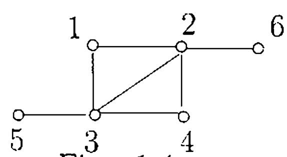


Fig : 1.4

The minimal clique transversal sets of the graph in Fig : 1.4 are $\{1,4,5,6\}$, $\{2,3\}$, $\{2,5\}$ and $\{3,6\}$. Therefore the clique transversal number is two. The maximal clique independent sets are $\{<2,6>,<3,5>\}$, $\{<1,2,3>\}$ and $\{<2,3,4>\}$. Therefore the clique independence number is also two.

Definition 1.1.30. A graph G is **clique perfect** if $\tau_c(H) = \alpha_c(H)$ for every induced subgraph H of G.

The graph in Fig : 1.4 is clique perfect. The smallest example of a graph which is not clique perfect is C_5, since $\tau_c(C_5) = 3$ and $\alpha_c(C_5) = 2$. Note that, by the definition of clique perfect graphs, any graph which contains C_5 as an induced subgraph is also not clique perfect.

Definition 1.1.31. A class $\mathcal{G}$ of graphs satisfies the **$<$ t $>$-property** if $\tau_c(G) \leqslant \frac{n}{t}$ for every $G \in \mathcal{G}_t = \{G \in \mathcal{G}$: every edge of G is contained in a $K_t \subseteq G\}$.

Note that the $< t >$-property does not imply the $< t-1 >$-property. Let $\mathcal{G}$ be the collection of cycles and complete graphs. Then $\mathcal{G}$ does not have the $< 2 >$-property since $\tau_c(C_{2k+1}) = k+1 > \frac{2k+1}{2}$. But, it satisfies the $< 3 >$-property, since $\mathcal{G}_3 = \{K_n : n \geqslant 3\}$ and $\tau_c(K_n) = 1$, for every n..

Definition 1.1.32. A graph G whose vertex set can be partitioned into an independent set and a clique is called a **split graph**.

Fig : 1.5 gives an example of a split graph.

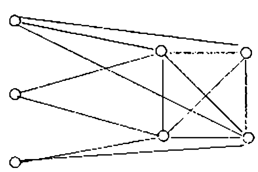


Fig : 1.5

Definition 1.1.33. A graph G is a **threshold graph** if it can be obtained from K_1 by recursively adding isolated vertices and universal vertices.

Definition 1.1.34. A graph G is **perfect** if $\chi(H) = \omega(H)$ for every induced subgraph H of G.

Definition 1.1.35. For a graph G, $T_k(G)$ the **trestled graph of index k** is the graph obtained from G by adding k copies of K_2 for each edge uv of G and joining u and v to the respective end vertices of each K_2.

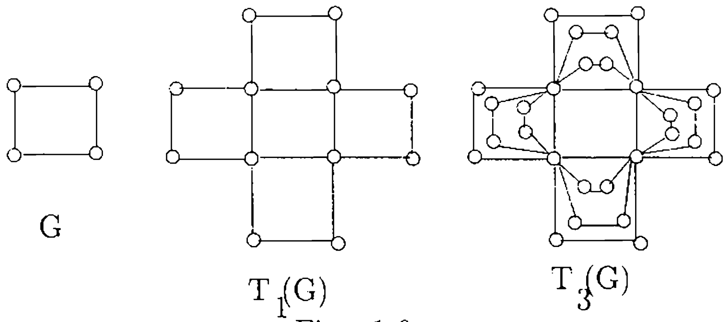


Fig : 1.6

Definition 1.1.36. A graph G is **clique irreducible** if every clique in G has an edge which does not lie in any other clique in G. If G is not clique irreducible then it is **clique reducible**.

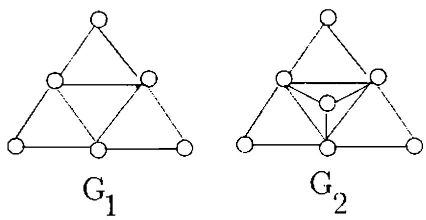


Fig : 1.7

In Fig : 1.7, G_1 is clique reducible and G_2 is clique irreducible.

Definition 1.1.37. A clique is **essential** if it has an edge which does not belong to any other clique in G. A graph G is **weakly clique irreducible** if every edge belongs to at least one essential clique. If G is not weakly clique irreducible then it is **weakly clique reducible**.

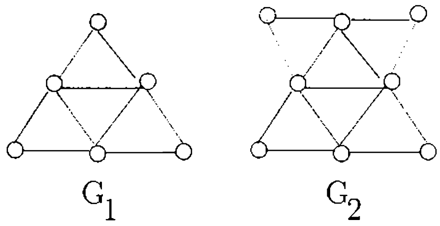


Fig : 1.8

In Fig : 1.8, G_1 is weakly clique irreducible and G_2 is weakly clique reducible. Note that weakly clique irreducible graphs form a super class of clique irreducible graphs. The reverse inclusion does not hold as indicated by the example G_1 in Fig : 1.8.

Definition 1.1.38. A graph G is **distance hereditary** if for every connected induced subgraph H of G, $d_H(u, v) = d_G(u, v)$.

Lemma 1.1.1. *[27] G is a cograph if and only if G is P_4-free.*

Lemma 1.1.2. *[27] Cographs can be recursively characterized as*

(1) K_1 is a cograph.

(2) If G and H are cographs, so is their union $G \cup H$.

(3) If G and H are cographs, so is their join $G \vee H$.

Both the forbidden subgraph characterization and the recursive characterization of cographs are used frequently in this thesis.

Lemma 1.1.3. *[13] The distance hereditary graphs can be recursively characterized as follows.*

(1) K_1 is distance hereditary.

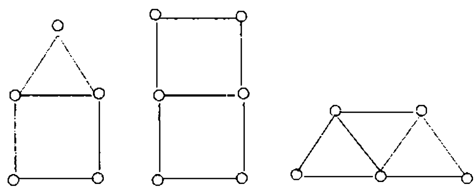
(2) If G distance hereditary then so is the graph obtained by attaching a pendent

vertex to any of the vertices of G.

(3) If G distance hereditary then so is the graph obtained by attaching true twins to any of the vertices of G.

(4) If G distance hereditary then so is the graph obtained by attaching false twins to any of the vertices of G.

Lemma 1.1.4. *[13] A graph G is distance hereditary if and only if it does not contain an induced house, hole, domino or gem, where a hole is a cycle of length greater than five and the other graphs are shown below.*



House Domino Gem

Lemma 1.1.5. *[13] A graph G is a cograph if and only if it is the disjoint union of distance hereditary graphs of diameter at most two.*

Lemma 1.1.6. (Strong perfect graph theorem) *[26] : A graph G is a perfect graph if and only if it does not contain any odd hole or odd anti-hole as an induced subgraph, where an odd hole is a cycle of odd length and an odd anti-hole is the complement of a cycle of odd length.*

Lemma 1.1.7. *[27] Cographs are perfect.*

Lemma 1.1.8. *[54] Cographs are clique perfect.*

Lemma 1.1.9. *[64] If G is hereditary clique-Helly , then it is clique irreducible.*

Lemma 1.1.10. *[25] If a graph G has no induced diamond ($K_4 - e$), then every edge of G belongs to exactly one clique.*

Lemma 1.1.11. *[76] A graph G is hereditary weakly maximal clique irreducible if and only if G does not contain any of the graph $F_1, F_2, ..., F_{19}$ in Fig : 1.9 as an induced subgraph.*

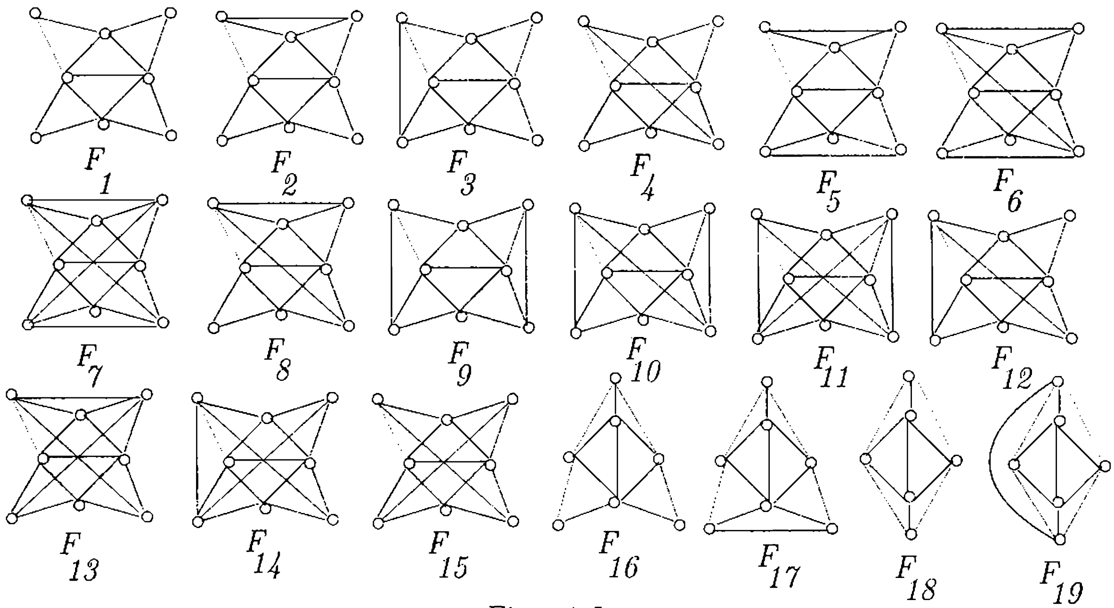
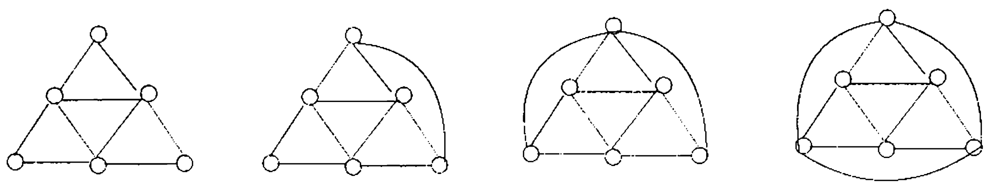


Fig : 1.9

Lemma 1.1.12. *[64] A graph G is hereditary clique-Helly , if it does not contain any of the Hajo's graph as an induced subgraph.*



Hajo's graphs

Lemma 1.1.13. *[11] In a loop less bipartite graph G, the minimum number of vertices that cover all the edges of G is equal to the maximum number of independent edges.*

Lemma 1.1.14. *[36] A graph G is a split graph if and only if it is $(2K_2, P_4, C_4)$-free.*

Lemma 1.1.15. *[29] A graph G is a threshold graph if and only if it is* $(2K_2, C_4, C_5)$*-free.*

1.2 New definitions

Definition 1.2.1. [66] Let $G = (V, E)$ be a graph. A subset V' of V is called a **cographic dominating set** if it dominates G and the subgraph induced by V' is a cograph. The **cographic domination number** $\gamma_{cd}(G)$ is the minimum cardinality of a cographic dominating set.

Definition 1.2.2. [66] Let $G = (V, E)$ be a graph. A subset V' of V is called a **global cographic dominating set** if it dominates both G and G^c and the subgraph induced by V' is a cograph. The **global cographic domination number** $\gamma_{gcd}(G)$ is the minimum cardinality of a global cographic dominating set.

For example, $\gamma_{cd}(K_{1,n}) = 1$ and $\gamma_{gcd}(K_{1,n}) = 2$.

Definition 1.2.3. [5] A graph G is **clique vertex irreducible** if every clique in G has a vertex which does not lie in any other clique in G and it is **clique vertex reducible** if it is not clique vertex irreducible.

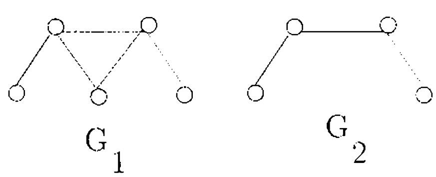


Fig : 1.10

In Fig : 1.10 G_1 is clique vertex irreducible and G_2 is clique vertex reducible. Note that the clique vertex irreducible graphs form a sub class of clique irreducible graphs. The reverse inclusion does not hold as indicated by the example G_2 in Fig : 1.10.

Definition 1.2.4. [6] An edge $e \in E(G)$ is called an **essential edge** if it belongs to exactly one clique in G. A vertex $v \in V(G)$ is called an **essential vertex** if it belongs to exactly one clique in G. A clique C in G is called **vertex essential** , if C has an essential vertex.

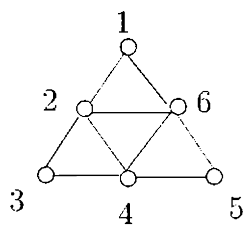


Fig : 1.11

In Fig : 1.11, the essential edges are 12, 23, 34, 45, 56 and 61. The essential vertices are 1, 3 and 5. The vertex essential cliques are $< 1, 2, 6 >$, $< 2, 3, 4 >$ and $< 4, 5, 6 >$.

1.3 A survey of results

The following are some of the fundamental results pertaining to the above said graph classes which we discuss in this thesis.

The Gallai graphs and the anti-Gallai graphs are spanning subgraphs of the well known class of line graphs whose union is the line graph. Though the line graphs admit a forbidden subgraph characterization [17], both the Gallai graphs and the anti-Gallai cannot be characterized using forbidden subgraphs, since it is proved in [52] that given any graph G, both $\Gamma(G^c \vee K_1)$ and $\Delta(G \vee K_1)$ contains G as an induced subgraph. In [52], it has also been proved that the Gallai graph of a graph G is isomorphic to G only for cycles of length greater than three. In [53], the Gallai mortal graphs - graphs whose iterated Gallai graph converges to the trivial graph, are characterized in several ways. In [72] the notion of Gallai perfect graphs - the graphs whose Gallai graphs are perfect, are introduced and discussed.

The class of cographs - complement reducible graphs, were studied by various authors under different names such as D*-graphs, P_4 restricted graphs and HD or hereditary dacey graphs. In [27], eight characterizations of cographs which includes the recursive characterization and the forbidden subgraph characterization (Lemma 1.1.1 and Lemma 1.1.2) are given. A linear recognition algorithm for cographs is given in [28].

An algorithm to solve the Hamiltonian cycle problem - given a graph G, does there exists a cycle that passes through every vertex of G, for the cographs (for the distance hereditary graphs, which form a super class of cographs) is given in [46]. The rank of the adjacency matrix of a graph is bounded by the number of distinct non-zero rows of that matrix. G.F. Royle [70] has proved that in the case of cographs, the rank is equal to the number of distinct non zero rows of its adjacency matrix. In [57] the connection of cographs with chordal graphs, interval graphs and series-parallel graphs are discussed. Cographs are linked with intersection graphs in [58].

The median and the anti-median of cographs are discussed in [67]. It has been proved that any cograph can be expressed as the median graph and the anti-median graph of a cograph that is both Eulerian and Hamiltonian. The cographs which are planar and outer planar are also characterized.

F.Larrión et.al, [51] studied in detail the clique operator on cographs. It has been proved that a cograph is clique convergent if and only if it is clique Helly. A characterization of cographs whose clique graph is a cograph is also given. A cograph G is clique complete if and only if it has a universal vertex.

It is proved in [42] that there are graphs that cannot be the clique graph of any graph. A graph is a clique graph if and only if it admits an edge cover which satisfies the Helly property [69]. In [10] all graphs G for which $d(K(G)) = d(G)-1$, $d(K(G)) = d(G)$ and $d(K(G)) = d(G) + 1$ are characterized and a class of graphs which satisfies $d(K^2(G)) = d(G) + 2$ is obtained. [59] deals with clique divergent graphs and it is proved that $(K(G \vee H))^c = (K(G))^c \Box (K(H))^c$ and $K(G \otimes H) = K(G) \otimes K(H)$. The clique complete graphs are discussed in detail in [55].

J. L. Szwarcfiter has made an excellent survey of the clique graphs [73]. It includes the characterizations of the clique graph, the clique graph of various graph classes, the clique inverse classes, the complexity of recognizing the clique inverse classes, the convergence and the divergence of the clique operator and the diameter of clique graphs. A list of open problems is also included in the survey. One of these problems is settled in [23] by obtaining a counter example and another problem is solved in chapter 4 of this thesis.

As we have already mentioned, the $< t >$-property was introduced to find graph classes which admits a better upper bound for the clique transversal number. The following are some of the upper bounds of the clique transversal number as proved in [33].

(1) $\tau_c(G) \leqslant n - \alpha(G)$.

(2) $\tau_c(G) \leqslant n - \Delta(G)$, where $\Delta(G)$ is the maximum degree of a vertex in G.

(3) $\tau_c(G) \leqslant n + \Delta(G) + 3 - \alpha(G) + \frac{2n}{\alpha(G)}$.

(4) $\tau_c(G) \leqslant n - \sqrt{2n} + \frac{3}{2}$.

(5) If n and k are natural numbers such that $n = k + 1$ and G is a graph on n vertices in which every clique has more than k vertices, then $\tau_c(G) \leqslant n - \sqrt{kn}$, except for C_5.

It is known [33] that every chordal graph satisfies the $< 2 >$-property. In [74], it is proved that the $< 3 >$-property holds for the chordal graphs, the split graphs have the $< 4 >$-property, but do not have the $< 5 >$-property and hence the chordal graphs also do not have the $< 5 >$-property. It is proved [35] that the $< 4 >$-property does not hold for the chordal graphs.

The class of clique perfect graphs were introduced in [41]. The distance hereditary graphs [54], the strongly chordal graphs [24], the dually chordal graphs [15] and the comparability graphs [12] are all subclasses of the rich class of clique perfect graphs. In [23], it is proved that the odd generalized suns are not clique perfect. In [21], the claw-free graphs which are clique perfect are characterized and in [22] diamond-free graphs and Helly-circular arc graphs which are clique perfect are characterized. A characterization of clique prefect graphs is an open problem [73].

Opsut and Roberts [60] introduced the concept of clique irreducible graphs and proved that the interval graphs are clique irreducible. Wallis and Zhang [78] generalized this result and attempted to characterize clique irreducible graphs. In [77], the line graphs which are clique irreducible are characterized using forbidden subgraphs. A characterization of clique irreducible graphs is still an open problem [73].

Tao-Ming Wang [76] introduced the concept of weakly clique irreducible graphs, which form a super class of clique irreducible graphs. In [76] nineteen forbidden subgraphs for a graph to be hereditary weakly clique irreducible is given. The line graphs which are weakly clique irreducible are characterized in [77].

1.4 Summary of the thesis

This thesis entitled **'Studies on Some Graph Classes'** is divided into six chapters. We shall now give a summary of each chapter.

The first chapter is an introduction and contains the literature on various graph classes studied in this thesis. It also includes the basic definitions and terminology.

In the second chapter various properties of the Gallai graphs and the anti-Gallai graphs are studied. The following are some of the results which we have obtained.

- ⋆ There are infinitely many pairs of non-isomorphic graphs of the same order having isomorphic Gallai graphs and anti-Gallai graphs.
- ⋆ There exist a finite family of forbidden subgraphs for the Gallai graphs and the anti-Gallai graphs to be H-free for any finite graph H.
- ⋆ The forbidden subgraph characterizations of G for which the Gallai graphs and the anti-Gallai graphs are cographs, split graphs and threshold graphs.
- ⋆ Characterization of cographs for which the Gallai and anti-Gallai graphs are also cographs.

- ⋆ The relationship between the chromatic number, the radius and the diameter of a graph and its Gallai and anti-Gallai graphs.

In the third chapter we define two new domination parameters, cographic domination number $\gamma_{cd}(G)$ and global cographic domination number $\gamma_{gcd}(G)$ based on cographs. Some of the properties of these domination parameters and results obtained are listed below.

- ✠ There is no tree satisfying the inequality $\gamma(G) < \gamma_{cd}(G) = \gamma_i(G)$.
- ✠ If G is a triangle free graph then $\gamma_{gcd}(G) = \gamma_{cd}(G)$ or $\gamma_{cd}(G) + 1$.
- ✠ If G is a planar graph with $\gamma_{cd}(G) \geqslant 3$, then $\gamma_{gcd}(G) \leqslant \gamma_{cd}(G) + 2$.
- ✠ Two constructions to illustrate the existence of graphs satisfying the inequalities among the various domination parameters.
- ✠ Vizing's type relations of the domination number, the global domination number, the cographic domination number, the global cographic domination number and the independent domination number of NEPS of two graphs.

In the fourth chapter, the clique transversal number and the $< t >$-property of various classes of graphs are studied. The following are some of the results proved.

- ⋈ The domination number is a lower bound for the clique transversal number and that the difference between these two parameters can be arbitrarily large.
- ⋈ The class of clique perfect graphs without isolated vertices satisfies the $< t >$-property for $t = 2$ and 3 and does not satisfy the $< t >$-property for $t \geqslant 4$.

- ⋈ The class of cographs without isolated vertices satisfies the $< t >$-property for $t = 2$ and 3 and does not satisfy the $< t >$-property for $t \geqslant 4$.

- ⋈ The class of planar graphs does not satisfy the $< t >$-property for $t = 2$, 3 and 4 and $\mathcal{G}_t$ is empty for $t \geqslant 5$.

- ⋈ The class of perfect graphs does not satisfy the $< t >$-property for any $t \geqslant 2$.

- ⋈ The class of trestled graphs of index k, $T_k(\mathcal{G})$ satisfies the $< 2 >$-property if and only if $\beta(G) \leqslant \frac{n}{2}\ \forall\ G \in \mathcal{G}$ and $T_k(\mathcal{G})_t$ is empty for $t \geqslant 3$.

- ⋈ The trestled graphs of index k, $T_k(G)$ is clique perfect if and only if G is bipartite.

- ⋈ Also, an open problem on highly clique imperfect graphs posed in [73] is solved.

In the fifth chapter the clique graph of cographs are studied and we obtain the following results.

- ⊕ The diameter of the clique graph of a cograph cannot exceed two.

- ⊕ Any graph on prime number of vertices, other than K_p, cannot be the clique graph of a cograph.

- ⊕ A cograph is clique complete if and only if it has a vertex of full degree.

- ⊕ The number of clique graphs of a cograph with $\chi(K(G)) = s$, where s is a fixed integer is finite.

- ⊕ A realization of cographs and its clique graph which have specific values for the domination number, the clique transversal number and the clique independence number.

The last chapter deals with two graph classes - the clique irreducible graphs and the weakly clique irreducible graphs. A new graph class called the clique vertex irreducible graphs is also defined and the following results are obtained.

- Characterizations of G for which the line graph $L(G)$ and all its iterates to be clique vertex irreducible and clique irreducible.
- Characterizations of G such that the Gallai graph $\Gamma(G)$ is clique vertex irreducible, clique irreducible and weakly clique irreducible.
- Characterizations of G such that the anti-Gallai graph $\Delta(G)$ and all its iterates are clique vertex irreducible, clique irreducible and weakly clique irreducible.
- The clique vertex irreducibility, clique irreducibility and weakly clique irreducibility of graphs which are non-complete extended p-sums (NEPS) of two graphs.
- Necessary and sufficient conditions for the cographs and the distance hereditary graphs to be clique vertex irreducible, clique irreducible and weakly clique irreducible.

1.5 List of publications

Papers presented

(1) Some domination concepts in cographs, International Conference on Discrete Mathematics and its Applications, December 9 - 11, 2004, Amrita Viswa Vidyapeetham, Coimbatore, India.

(2) The clique graph of a cograph, International Conference on Discrete Mathematics, December 15 - 18, 2006, IISc, Bangalore, India.

(3) A note on some domination parameters in graph products, International Conference on Recent Developments in Combinatorics and Graph Theory, June 10 - 14, 2007, Kalasalingam University, Krishnankoil, India.

(4) Characterization of some special classes of Gallai and anti-Gallai graphs, National Seminar on Algebra and Discrete Mathematics, November 14 - 16, 2007, University of Kerala, Thiruvananthapuram, India.

(5) Clique irreducibility and clique vertex irreducibility of graphs, 73rd Annual Conference of Indian Mathematical Society, December 27 - 30, 2007, University of Pune, Pune, India.

(6) On weakly clique irreducible graphs, International Conference on Discrete Mathematics, June 6 - 10, 2008, University of Mysore, Mysore, India.

Papers published / communicated

(1) Aparna Lakshmanan S., S. B. Rao, A. Vijayakumar, Gallai and anti-Gallai graphs of a graph, Math. Bohem., 132(1) (2007), 43 - 54.

(2) Aparna Lakshmanan S., A. Vijayakumar, A note on some domination parameters in graph products, Congr. Numer., (Proceedings of the International Conference on Recent Developments in Combinatorics and Graph Theory, 2007, India), (to appear).

(3) Aparna Lakshmanan S., A. Vijayakumar, Clique irreducibility and clique vertex irreduciblility of graphs, (communicated).

(4) Aparna Lakshmanan S., A. Vijayakumar, Clique irreducibility of some iterative classes of graphs, Discuss. Math. Graph Theory, (to appear).

(5) Aparna Lakshmanan S., A. Vijayakumar, On weakly clique irreducible graphs, (communicated).

(6) Aparna Lakshmanan S., A. Vijayakumar, Some properties of the clique graph of a cograph, Proceedings of the International Conference on Discrete Mathematics, Bangalore, India, (2006), (to appear).

(7) Aparna Lakshmanan S., A. Vijayakumar, The $< t >$-property of some classes of graphs, Discrete Math., (to appear).

(8) S. B. Rao, Aparna Lakshmanan S., A. Vijayakumar, Cographic and global cographic domination number of a graph, Ars Combin. (to appear)

Chapter 2

Gallai and anti-Gallai graphs

This chapter deals with two graph classes the Gallai graphs and the anti-Gallai graphs. We construct infinitely many pairs of graphs G and H such that $\Gamma(G) = \Gamma(H)$. The existence of a finite family of forbidden subgraphs for the Gallai graphs and the anti-Gallai graphs to be H-free, for any finite graph H is proved and the forbidden subgraph characterizations of G for which the Gallai graphs and the anti-Gallai graphs are cographs, split graphs and threshold graphs are discussed in detail. If G is a connected cograph without a universal vertex then $\Gamma(G)$ is a cograph if and only if $G = (pK_2)^c$. The relationships between the radius, the diameter and the chromatic number of a graph and its Gallai (anti-Gallai) graph are also studied in detail.

Some results of this chapter are included in the following paper.

Gallai and anti-Gallai graphs of a graph, Math. Bohem., 132(1) (2007), 43 - 54.

2.1 Gallai and anti-Gallai graphs

It is well known [80] that the only pair of non-isomorphic graphs having the same line graph is $K_{1,3}$ and K_3. But, we first observe that, in the case of both Gallai and anti-Gallai graphs, there are infinitely many pairs of non-isomorphic graphs of the same order having isomorphic Gallai graphs (anti-Gallai graphs).

Theorem 2.1.1. *There are infinitely many pairs of non-isomorphic graphs of the same order having isomorphic Gallai graphs.*

Proof. We prove this theorem by the following two types of constructions.

Type 1 :- Let G be the graph P_4 with n independent vertices joined to both its internal vertices and an end vertex attached to k of these n vertices and H be two copies of $K_{1,n+1}$ with $k+1$ distinct pairs of end vertices made adjacent.

The graph G in type 1 is as follows. Let $v_1v_2v_3v_4$ be an induced P_4. Let v_2 and v_3 be joined to n vertices $u_1, u_2, ..., u_n$. Introduce k end vertices $w_1, w_2, ..., w_k$ such that each w_i is adjacent only to u_i for $i = 1, 2, ..., k$. The edges $v_1v_2, v_2u_1, v_2u_2, ..., v_2u_n$ of G, which are vertices of $\Gamma(G)$ will induce a complete graph on $n+1$ vertices in $\Gamma(G)$. Similarly, $v_3v_4, v_3u_1, v_3u_2, ..., v_3u_n$ will induce another complete graph on $n+1$ vertices in $\Gamma(G)$. The vertex corresponding to the edge v_2v_3 will be adjacent to both the vertices corresponding to v_1v_2 and v_3v_4. The k vertices corresponding to the edges u_iw_i for $i = 1, 2, ..., k$ will be adjacent to the vertices corresponding to the edges u_iv_2 and u_iv_3 for $i = 1, 2, ..., k$ respectively.

The graph H in type 1 is as follows. Let u adjacent to $u_1, u_2, ..., u_{n+1}$ and v adjacent to $v_1, v_2, ..., v_{n+1}$ be the two $K_{1,n+1}$ s in H. Let $u_1v_1, u_2v_2, ..., u_{k+1}v_{k+1}$ be the $k+1$ distinct pairs of adjacent vertices in H. The vertices corresponding to the edges $uu_1, uu_2, ..., uu_{n+1}$ will induce a complete graph on $n+1$ vertices in $\Gamma(H)$. Similarly, the vertices corresponding to $vv_1, vv_2, ..., vv_{n+1}$ will also induce another complete graph on $n+1$ vertices in $\Gamma(H)$. Again, the vertices corresponding to the edges u_iv_i for $i = 1, 2, ..., k+1$ will be adjacent to the vertices corresponding to the edges uu_i and vv_i for $i = 1, 2, ..., k+1$ respectively.

Therefore, both $\Gamma(G)$ and $\Gamma(H)$ are two copies of K_{n+1} together with $k+1$ new vertices made adjacent to $k+1$ distinct vertices of both the copies of K_{n+1}.

Type 2 :- Let G be the graph P_4 with n independent vertices joined to both its internal vertices and an end vertex attached to k of them with $k \geqslant 1$, together with one end vertex each attached to the two end vertices of P_4 and H be two copies of $K_{1,n+1}$ with $k+1$ distinct pairs of end vertices (one from each star) made adjacent and a single pair made adjacent to another vertex.

The graph G in type 2 can be obtained from the graph G in type 1 by attaching two end vertices x and y to v_1 and v_2 respectively. In $\Gamma(G)$ the vertices corresponding to the edges v_1x and v_4y will be adjacent to the vertices corresponding to the edges v_1v_2 and v_3v_4 respectively. The graph H in type 2 can be obtained from the graph H in type 1 by adding a new vertex w and making it adjacent to both u_1 and v_1. In $\Gamma(H)$ the vertices corresponding to the edges wu_1 and wv_1 will be adjacent to the vertices corresponding to the edges uu_1 and vv_1 respectively.

Therefore, both $\Gamma(G)$ and $\Gamma(H)$ are two copies of K_{n+1} together with $k+1$ vertices made adjacent to $k+1$ distinct vertices of both the copies of K_{n+1} and two end vertices made adjacent to one vertex from each of the complete graphs.

The constructions mentioned in type 1 and type 2 are illustrated in Table 2.1.

In both the cases, the graphs G and H have the same Gallai graph. If $n = k$ and $n = k - 1$ in type 1 and type 2 respectively, then the order of G and H is the same.

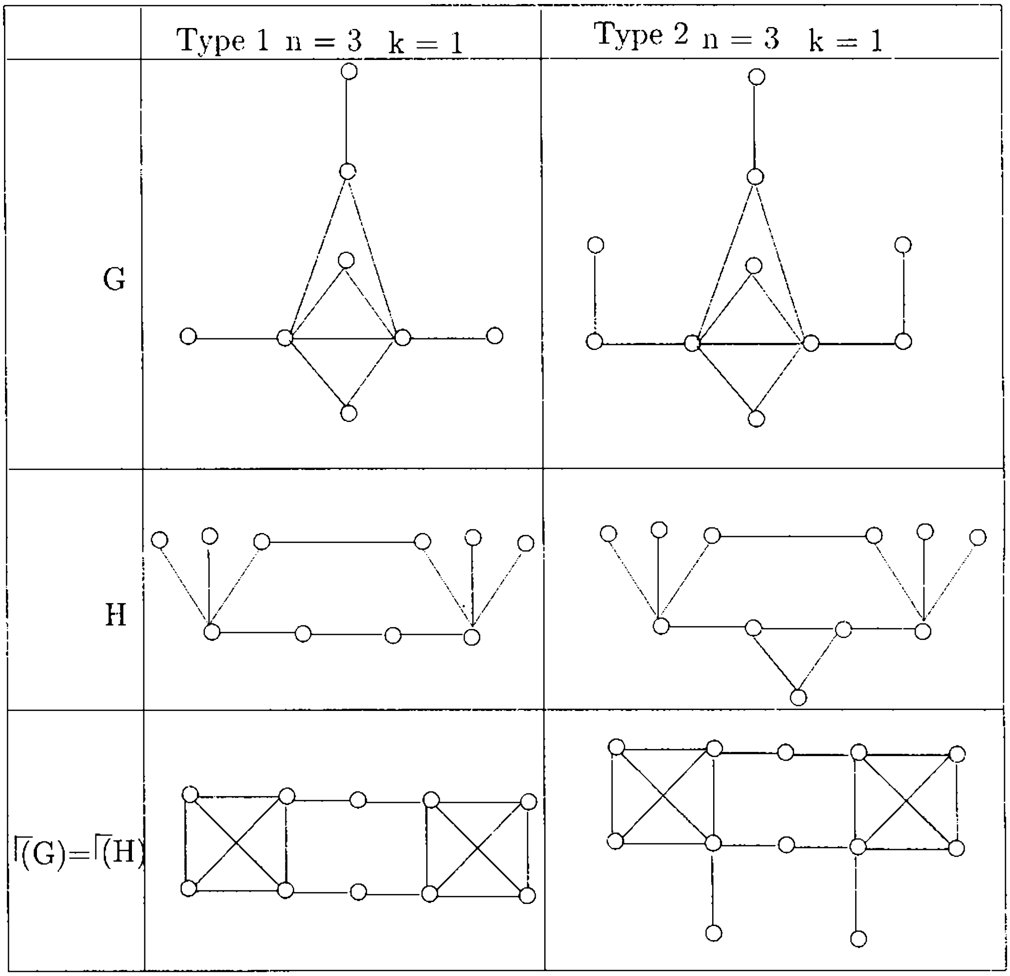


Table 2.1

□

Theorem 2.1.2. *There are infinitely many pairs of non-isomorphic graphs of the same order having isomorphic anti-Gallai graphs.*

Proof. Let G be a graph with vertex set $\{v_1, v_2, ..., v_n\}$ and an edge v_iv_j such that G is not isomorphic to a graph obtained under permutations of the index set of the vertices which interchange i and j and $\Delta(G)$ is connected. Introduce a vertex

u adjacent to v_i and v_j. Let H_1 be the graph obtained by introducing one more vertex u_1 adjacent to u and v_i. Let H_2 be the graph obtained by introducing another vertex u_2 (u_1 is absent here) adjacent to u and v_j. Then by construction H_1 and H_2 are non-isomorphic. $\Delta(H_1)$ is $\Delta(G)$ together with four more vertices corresponding to uv_i, uv_j, uu_1, v_iu_1 in which uv_i and uv_j are adjacent to each other and to v_iv_j, uu_1 and v_iu_1 are adjacent to each other and to uv_i. $\Delta(H_2)$ is $\Delta(G)$ together with four more vertices corresponding to uv_i, uv_j, uu_2, v_ju_2 in which uv_i and uv_j are adjacent to each other and to v_iv_j, uu_2 and v_ju_2 are adjacent to each other and to uv_j. Therefore, $\Delta(H_1)$ is isomorphic to $\Delta(H_2)$.

□

2.2 Forbidden subgraph characterizations

Even though the Gallai and the anti-Gallai graphs cannot be characterized using forbidden subgraphs, in this section we prove the existence of a finite forbidden subgraph characterization for the Gallai graph and the anti-Gallai graph to be H-free and obtain the forbidden subgraph characterizations for the Gallai and the anti-Gallai graphs to be a cograph, a split graph and a threshold graph.

Notation : For a connected graph H, let $\mathcal{G}(H) = \{G : \Gamma(G)$ is H - free$\}$ and $\mathcal{G}^*(H) = \{G : \Delta(G)$ is H - free$\}$.

Theorem 2.2.1. *The properties of being an element of $\mathcal{G}(H)$ and $\mathcal{G}^*(H)$ are vertex hereditary.*

Proof. Let $G \in \mathcal{G}(H)$ and $v \in V(G)$. Consider $G' = G - \{v\}$. It is required to prove that $G' \in \mathcal{G}(H)$. On the contrary assume that $\Gamma(G')$ has H as an induced subgraph. Let $v_1, v_2, ..., v_t$ be neighbors of v. Therefore $\Gamma(G)$ has the vertex set $V(\Gamma(G')) \cup \{vv_1, vv_2, ..., vv_t\}$. In $\Gamma(G)$, vv_i is adjacent to vv_j if v_i is not adjacent to v_j, and vv_i will be adjacent to all edges which have v_i as one end vertex and other end vertex is not v_j for $j = 1, 2, ..., t$. $V(\Gamma(G'))$ induce $\Gamma(G')$ itself. Hence if H is an induced subgraph of $\Gamma(G')$ then H is an induced subgraph of $\Gamma(G)$ also, which is a contradiction.

The case of $\mathcal{G}^*(H)$ follows similarly. □

Corollary 2.2.2. *$\mathcal{G}(H)$ and $\mathcal{G}^*(H)$ have vertex minimal forbidden subgraph characterization.*

Though many well known classes of graphs admit forbidden subgraph characterizations, the number of such forbidden subgraphs need not be finite. However, for $\mathcal{G}(H)$ and $\mathcal{G}^*(H)$ we have

Theorem 2.2.3. *For every vertex minimal forbidden subgraph of $\mathcal{G}(H)$ and $\mathcal{G}^*(H)$, the number of vertices is bounded above by $n(H) + 1$.*

Proof. Let $\mathcal{F}(H)$ be the collection of all vertex minimal forbidden subgraphs of $\mathcal{G}(H)$. Let $L \in \mathcal{F}(H)$. Therefore, $\Gamma(L)$ has H as an induced subgraph. The $n(H)$ vertices of H, which correspond to $n(H)$ edges of L, say $e_1, e_2, ..., e_{n(H)}$, can cover a maximum of $n(H) + 1$ vertices of L, since H is connected.

We have to prove that $n(L) \leqslant n(H) + 1$. On the contrary assume that $n(L) > n(H)+1$. Then there exists at least one vertex $v \in V(L)$ which is not an end vertex of any of $e_1, e_2, ..., e_{n(H)}$. Therefore, $\Gamma(L - v)$ still has H as an induced subgraph, which contradicts that L is a vertex minimal forbidden subgraph of $\mathcal{G}(H)$. Hence,

$n(L) \leqslant n(H) + 1$.

A similar argument holds for $\mathcal{G}^*(H)$ also. □

Corollary 2.2.4. *The number of vertex minimal forbidden subgraphs for $\mathcal{G}(H)$ and $\mathcal{G}^*(H)$ is finite.*

Theorem 2.2.5. *Let G be a graph. Then, $\Gamma(G)$ is a cograph if and only if G does not have the following graphs as induced subgraphs.*

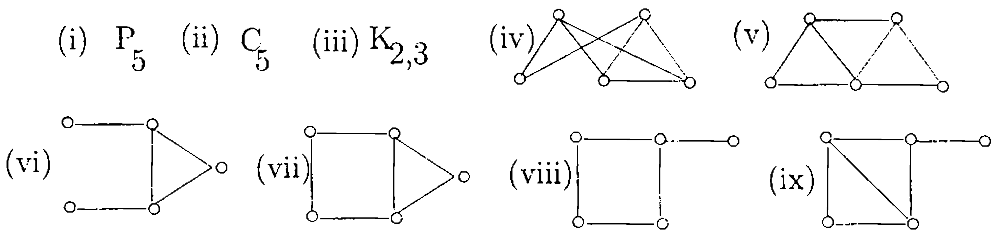


Fig : 2.1

Proof. If $\Gamma(G)$ is not a cograph then there exists an induced P_4 in $\Gamma(G)$, say $e_1e_2e_3e_4$. In G, let $e_1 = u_{11}u_{12}, e_2 = u_{21}u_{22}, e_3 = u_{31}u_{32}$ and $e_4 = u_{41}u_{42}$.

Since e_1 is adjacent to e_2, let $u_{12} = u_{21}$ and let u_{11} be not adjacent to u_{22}. Since e_2 is adjacent to e_3, either $u_{21} = u_{31}$ or $u_{22} = u_{31}$.

If $u_{21} = u_{31}$, then since e_1 is not adjacent to e_3, u_{11} is adjacent to u_{32}. Since e_3 is adjacent to e_4, either $u_{31} = u_{41}$ or $u_{32} = u_{41}$. If $u_{31} = u_{41}$, then since e_1 and e_2 are not adjacent to e_4, both u_{11} and u_{21} are adjacent to u_{42}. If $u_{32} = u_{41}$ then u_{31} is not adjacent to u_{42}.

If $u_{22} = u_{31}$, then u_{21} is not adjacent to u_{32}. Again, since e_3 is adjacent to e_4, either $u_{31} = u_{41}$ or $u_{32} = u_{41}$. If $u_{31} = u_{41}$, then since e_2 and e_4 are not adjacent, u_{21} is adjacent to u_{42}. If $u_{32} = u_{41}$ then u_{31} is not adjacent to u_{42}. The above four resulting graphs are respectively (iv), (vi), (vi) and (i).

In (iv), if we add even a single edge the property of $\Gamma(G)$ not being a cograph will be lost. In (vi), u_{22} adjacent to u_{42} gives (vii), u_{11} adjacent to u_{42} gives (ix) and the combination of both gives iv). The addition of these edges will not change the required property either. In (i), u_{11} adjacent to u_{42} gives (ii), u_{11} adjacent to u_{41} gives (viii) and a combination of both gives (iii). Again, the addition of these edges will not change the required property. However, if we add any other edge then the property will be lost.

Conversely, it can be verified that the Gallai graph will not be a cograph if any of the nine graphs listed above is an induced subgraph of G. □

Theorem 2.2.6. *Let G be a graph. Then $\Delta(G)$ is a cograph if and only if G does not have the following graphs as induced subgraphs.*

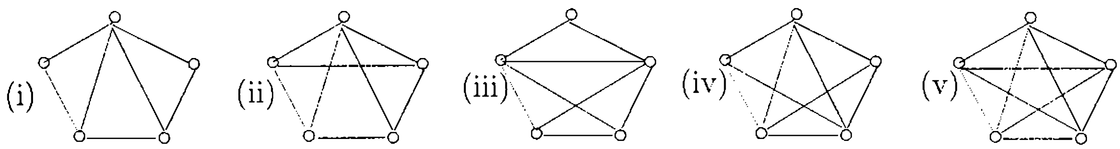


Fig : 2.2

Proof. If $\Delta(G)$ is not a cograph then there exists an induced P_4 in $\Delta(G)$, say $e_1e_2e_3e_4$. In G, let $e_1 = u_{11}u_{12}, e_2 = u_{21}u_{22}, e_3 = u_{31}u_{32}$ and $e_4 = u_{41}u_{42}$.

Since e_1 is adjacent to e_2, let $u_{12} = u_{21}$ and let u_{11} be adjacent to u_{22}. Since e_2 is adjacent to e_3, either $u_{21} = u_{31}$ or $u_{22} = u_{31}$.

If $u_{21} = u_{31}$ then u_{22} is adjacent to u_{32} and u_{11} is not adjacent to u_{31}. Since e_3 is adjacent to e_4, either $u_{31} = u_{41}$ or $u_{32} = u_{41}$. If $u_{31} = u_{41}$, then u_{32} is adjacent to u_{42} and u_{11} and u_{22} are not adjacent to u_{42}. If $u_{32} = u_{41}$ then u_{31} is adjacent to u_{42}.

If $u_{22} = u_{31}$ then u_{12} is adjacent to u_{32}. Again, since e_3 is adjacent to e_4, either $u_{31} = u_{41}$ or $u_{32} = u_{41}$. If $u_{31} = u_{41}$, then u_{32} is adjacent to u_{42} and u_{21} is not adjacent to u_{12}. If $u_{32} = u_{42}$ then u_{31} is adjacent to u_{42}.

All the four resulting graphs are isomorphic to (i) itself. Also, addition of any of the possible edges will leave an induced P_4 in $\Delta(G)$ and hence any graph with five vertices which contains (i) as a (not induced) subgraph are also forbidden. Hence all the above graphs are forbidden.

The converse can be easily proved. □

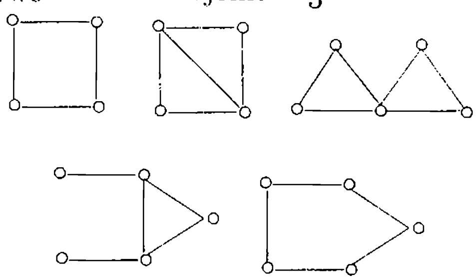
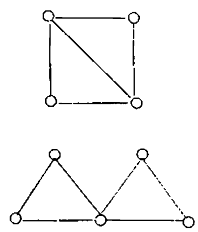
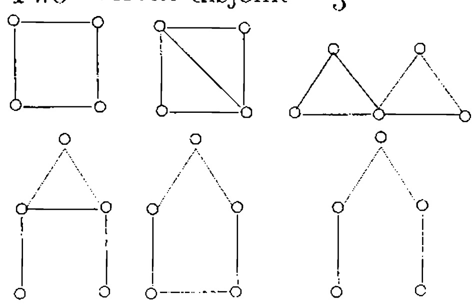
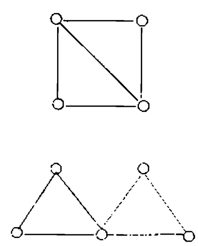
	Gallai graph	anti-Gallai graph
Split graph	Two vertex disjoint P_3 	$2K_3$ 
Threshold graph	Two vertex disjoint P_3 	$2K_3$ 

Table 2.2

If $\mathcal{G}$ is any graph class that admits a finite forbidden subgraph characterization, then using similar arguments as in Theorem 2.2.5 and Theorem 2.2.6, we can obtain forbidden subgraph characterizations for the Gallai graph and the anti-Gallai graph to be in $\mathcal{G}$. In Table 2.2, we list the forbidden subgraphs for $\Gamma(G)$ and $\Delta(G)$ to be a split graph and a threshold graph.

2.3 Applications to cographs

In this section we obtain characterizations for the Gallai graph and the anti-Gallai graph of a cograph to be a cograph.

Theorem 2.3.1. *If G is a connected cograph without a universal vertex then $\Gamma(G)$ is a cograph if and only if $G = (pK_2)^c$.*

Proof. Let $G = (pK_2)^c$. Then the number of vertices of G is $2p$ and the number of edges of G is $2p(p-1)$. Let the vertices of G be $\{v_{11}, v_{12}, ..., v_{1p}, v_{21}, v_{22}, ..., v_{2p}\}$ with v_{1j} and v_{2j} as the only pair of non-adjacent vertices, for $j = 1, 2, ..., p$. Therefore, the vertices of the Gallai graph are of the form $v_{ij}v_{i'j'}$ where $j \neq j'$. By the definition of the Gallai graphs, $v_{ij}v_{i'j'}$ will be adjacent only to $v_{ij}v_{1j'}$ or $v_{ij}v_{2j'}$ and $v_{1j}v_{i'j'}$ or $v_{2j}v_{i'j'}$ according to the value of i and i'. Therefore, $\Gamma(G) = ({}^pC_2)C_4$, which is a cograph.

Conversely, assume that G is a cograph without a universal vertex and $\Gamma(G)$ is also a cograph. For every $u \in V(G)$, there exist at least one $u' \in V(G)$ which is not adjacent to u.

Claim : u' is the only vertex which is not adjacent to u.

On the contrary assume that there exists another vertex u'' which is not adjacent to u. Since G is a connected cograph, $G = G_1 \vee G_2$. Let $u \in V(G_1)$. Since

u is not adjacent to both u' and u'', both of them belong to $V(G_1)$. Since G has no vertex of full degree, G_2 must contain at least two non-adjacent vertices v_1 and v_2. Then the edges $u''v_1, v_1u, uv_2, v_2u'$ will induce a P_4 in $\Gamma(G)$, which is a contradiction.

Therefore $G = (pK_2)^c$, where $2p = n$. □

Notation : Consider the class of graphs which are recursively defined as follows :
$\mathcal{H}_1 = \{G : G = (pK_2)^c \vee (K_q), \text{ where } p, q \geqslant 0\}$.
$\mathcal{H}_i = \{G : G = (\bigcup H_{i-1}) \vee K_r, \text{ where } H_{i-1} \in \mathcal{H}_{i-1} \text{ and } r \geqslant 0\}$ for $i > 1$.
$\mathcal{H} = \bigcup \mathcal{H}_i$

Theorem 2.3.2. *For a connected cograph G, $\Gamma(G)$ is a cograph if and only if $G \in \mathcal{H}$.*

Proof. Let G be a cograph other than K_q with a vertex of full degree. Let V_1 be the collection of all full degree vertices in G. Define $G_1 =< V - V_1 >$. $\Gamma(G_1)$ is an induced subgraph of $\Gamma(G)$. More precisely, $\Gamma(G) = \Gamma(G_1)$ together with some isolated vertices. Therefore, $\Gamma(G)$ is a cograph if and only if $\Gamma(G_1)$ is a cograph. If G_1 is a connected cograph then G_1 has no vertex of full degree and hence $\Gamma(G_1)$ is a cograph if and only if $G_1 = (pK_2)^c$. Therefore, $\Gamma(G)$ is a cograph if and only if $G = (pK_2)^c \vee (K_q) \in \mathcal{H}_1$.

If G_1 is disconnected, then consider each of the connected components of G_1. If the removal of all full degree vertices from each of the components of G_1 preserves connectedness then as above each of these components must be of the form $(pK_2)^c \vee (K_q)$. Therefore, $G = (F_1 \cup F_2 \cup ... \cup F_p) \vee K_q$ where each $F_i \in \mathcal{H}_1$ and $q \geqslant 0$. Consequently, $G \in \mathcal{H}_2$.

If any of the components of G_1, say G_2, is disconnected then repeat the above process to get $G_1 \in \mathcal{H}_2$ and hence $G = (H_1 \cup H_2 \cup ... \cup H_r) \vee K_s$ where each $H_i \in \mathcal{H}_2$ and $r \geqslant 0$. Consequently, $G \in \mathcal{H}_3$.

This process must terminate since the number of vertices of G is finite. Therefore for a connected cograph G, $\Gamma(G)$ is a cograph if and only if $G \in \mathcal{H}$. □

Theorem 2.3.3. *For a connected cograph G, $\Delta(G)$ is a cograph if and only if*
(i) $G = G_1 \vee G_2$, where G_1 is edgeless and G_2 does not contain P_4 as a subgraph (which need not be induced) or
(ii) G is C_4.

Proof. Let G be a connected cograph whose $\Delta(G)$ is also a cograph. Since G is a connected cograph, $G = G_1 \vee G_2$. Let G_1 be an edgeless graph and $u \in V(G_1)$. If G_2 contains a P_4, say $v_1v_2v_3v_4$, then the edges $v_1v_2, v_2u, uv_3, v_3v_4$ of G induce a P_4 in $\Delta(G)$, which is a contradiction. Therefore, if G_1 is edgeless then G_2 does not contain P_4 as a subgraph.

Let $u_1v_1 \in E(G_1)$ and $u_2v_2 \in E(G_2)$. If G_1 contains one more vertex, say v, not adjacent to u_1 and v_1, then the edges $u_1v_1, v_1u_2, u_2v_2, u_2u$ of G induce a P_4 in $\Delta(G)$, which is a contradiction. If v is adjacent to at least one of the vertices, say v_1, then the edges $u_1u_2, u_2v_1, v_1v_2, v_2v$ of G induce a P_4 in $\Delta(G)$, which is a contradiction. A similar argument holds also for the vertex set of G_2. Therefore both G_1 and G_2 are K_2 -s and $G = C_4$.

Conversely, assume that G is a cograph of type (i) or (ii). Then G does not contain any of the graphs in Fig : 2.2 as an induced subgraph and hence $\Delta(G)$ is a cograph by Theorem 2.2.6. □

2.4 Chromatic number

In this section we study the relation between the chromatic numbers of G, $\Gamma(G)$ and $\Delta(G)$.

Theorem 2.4.1. *Given two positive integers a, b, where $a > 1$, there exists a graph G such that $\chi(G) = a$ and $\chi(\Gamma(G)) = b$.*

Proof. If $a = 1$ then G must be a graph without edges, which makes $\Gamma(G)$ empty. So we can assume that $a > 1$.

Let G be the graph K_a together with $b-1$ end vertices attached to any one of the vertices. Then $\Gamma(G)$ is $a-1$ copies of K_b sharing $b-1$ vertices in common together with some isolated vertices. Clearly, $\chi(G) = a$ and $\chi(\Gamma(G)) = b$. □

Lemma 2.4.2. *The anti-Gallai graph of any graph G cannot be bipartite except for the K_3-free graphs.*

Proof. If u_1 is adjacent to u_2 in $\Delta(G)$ then the corresponding edges, say e_1 and e_2, lie in a K_3, say $e_1e_2e_3$. Then the vertex u_3 in $\Delta(G)$ which corresponds to e_3 will be adjacent to both u_1 and u_3. Therefore, $u_1u_2u_3$ induces a cycle of odd length in $\Delta(G)$ and hence $\Delta(G)$ cannot be bipartite. □

Theorem 2.4.3. *Given two positive integers a, b, where $b < a, b \neq 2$, there exists a graph G such that $\chi(G) = a$ and $\chi(\Delta(G)) = b$. Further, for any odd integer a, there exists a graph G such that $\chi(G) = \chi(\Delta(G)) = a$.*

Proof. Since the anti-Gallai graph of a graph G cannot be bipartite except for the triangle free graphs (Lemma 2.4.2), $b = \chi(\Delta(G)) \neq 2$ for any G.

By Myceilski's construction [11] there exists a triangle-free graph H with chromatic number a. If we choose $G = H$, then $\Delta(G)$ is a trivial graph and hence $b = 1$. For $2 < b < a$, there exists an induced subgraph H' of H whose chromatic number is b. Let $v_1, v_2, ..., v_n$ be the vertices of H'. Let G be the graph obtained from H by joining all vertices of H' to a new vertex u. Since $b < a$, $\chi(G) = a$ itself. If v_i and v_j are adjacent (or non-adjacent) in H' then the vertices corresponding to uv_i and uv_j are adjacent (or non-adjacent) in $\Delta(G)$. Therefore, the vertices corresponding to the edges $uv_1, uv_2, ..., uv_n$ induce an H' in $\Delta(G)$. Again for any pair of adjacent vertices, say v_i and v_j in H', the vertices corresponding to the edges uv_i and uv_j are adjacent to the vertex corresponding to v_1v_2. Therefore $\Delta(G)$ is H' together with one vertex each adjacent to both the end vertices of each edge in H'. For $b > 2$, $\chi(\Delta(G)) = \chi(H') = b$.

If a is an odd integer then $\chi(K_a) = a$ and $\chi(\Delta(G)) = \chi(L(G)) = \chi'(K_a) = a$, where χ' is the edge chromatic number. □

The triangle free graph H having chromatic number $a = 4$ obtained using Myceilski's construction, the graph G in the above theorem having $\chi(G) = a = 4$ and its anti-Gallai graph having $\chi(\Delta(G)) = b = 3$ are illustrated in Fig : 2.3.

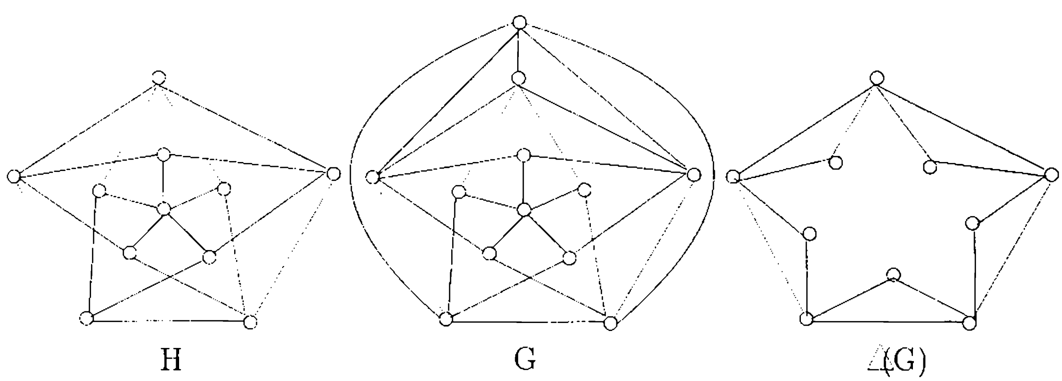


Fig : 2.3

2.5 Radius and diameter

The relation between the radius and the diameter of G with its Gallai and anti-Gallai graphs are studied in this section.

Theorem 2.5.1. *Let G be a graph such that $\Gamma(G)$ is connected. Then $r(\Gamma(G)) \geqslant r(G) - 1$ and $d(\Gamma(G)) \geqslant d(G) - 1$.*

Proof. Let $r(\Gamma(G)) = r$. Then there exists an edge uv in G such that the vertex corresponding uv in $\Gamma(G)$ is at a distance less than or equal to r from every other vertex in $\Gamma(G)$. Hence, any vertex of G is at a distance less than or equal to $r+1$ from both u and v. We have $r(G) \leqslant r+1$, which implies $r(\Gamma(G)) \geqslant r(G) - 1$.

Let $d(G) = d$. There exist two vertices u and v such that $d(u, v) = d$. Let $uu_1u_2...u_{d-1}v$ be a shortest path connecting u and v in G.

Claim:- $d_{\Gamma(G)}(uu_1, u_{d-1}v) = d - 1$.

$uu_1, u_1u_2,..., u_{d-1}v$ is a path of length $d-1$ connecting uu_1 and $u_{d-1}v$ in $\Gamma(G)$. Therefore, $d_{\Gamma(G)}(uu_1, u_{d-1}v) \leqslant d - 1$.

It is required to prove that $d_{\Gamma(G)}(uu_1, u_{d-1}v) = d - 1$. On the contrary assume that there exists an induced path $uu_1, v_1v_1', v_2v_2', , v_{k-1}v_{k-1}', u_{d-1}v$ of length k in $\Gamma(G)$ connecting uu_1 and $u_{d-1}v$, where $k < d - 1$. Then there exists a path of length less than or equal to $d-1$ connecting u and v in G, which contradicts $d(u, v) = d$. Hence, $d_{\Gamma(G)}(uu_1, u_{d-1}v) = d - 1$.

Since there exist two vertices of $\Gamma(G)$ which are at a distance $d-1$, $d(\Gamma(G))$ must be greater than or equal to $d - 1$. Hence, $d(\Gamma(G)) \geqslant d(G) - 1$. □

Remark 2.5.1. If a and b are two positive integers such that $a > 1$ and $b \geqslant a-1$ then there exist graphs G' and G'' whose Gallai graphs are connected and $r(G') = a$, $r(\Gamma(G')) = b$, $d(G'') = a$ and $d(\Gamma(G'')) = b$.

Theorem 2.5.2. *If G is a graph such that $\Delta(G)$ is connected and $r(G) > 1$, $r(\Delta(G)) \geqslant 2(r(G)-1)$ and $d(\Delta(G)) \geqslant 2(d(G)-1)$.*

Proof. Let $r(\Delta(G)) = r > 1$. There exists an edge uv in G such that the vertex corresponding to uv in $\Delta(G)$ is at a distance less than or equal to r from every other vertex in $\Delta(G)$. Let $w \in V(G)$. Since G is connected there exists at least one edge with w as an end vertex, say ww'. There exists a path of length less than or equal to r from ww' to uv in $\Delta(G)$. Any two adjacent edges in $\Delta(G)$ belong to a triangle and hence this path induces a path of length less than or equal to $\frac{r}{2}$ from either u or v to w or w'. Therefore, any vertex is at a distance less than or equal to $\frac{r}{2}+1$ from both u and v. Hence $r(G) \leqslant \frac{r}{2}+1$, which implies that $r(\Delta(G)) \geqslant 2(r(G)-1)$.

Let $d(G) = d$. There exist two vertices u and v such that $d(u,v) = d$. Let $uu_1u_2...u_{d-1}v$ be a shortest path connecting u and v. Consider $d(uu_1, u_{d-1}v)$ in $\Delta(G)$. If it is k, then there exists a path of length less than or equal to $\frac{k}{2}+1$ in G connecting u and v. Therefore, $\frac{k}{2}+1 \geqslant d$, which implies $k \geqslant 2(d-1)$. However, $d(\Delta(G)) \geqslant k$. Hence, $d(\Delta(G)) \geqslant 2(d(G)-1)$. □

Remark 2.5.2. If a and b are two positive integers such that $a > 1$ and $b \geqslant 2(a-1)$ then there exist graphs G' and G'' whose anti-Gallai graphs are connected with $r(G') = a$, $r(\Delta(G')) = b$, $d(G''H) = a$ and $d(\Delta(G'')) = b$.

Chapter 3

Domination in Graph Classes

In [9], Bacsó and Tuza Z. put forward the following problem.

Problem : Let **P** be a property of vertex sets in a graph. Characterize all graphs having a dominating set satisfying the property **P**.

Based on various properties of the vertex set, many domination parameters were introduced and studied. For a detailed study of various domination parameters, the reader may refer to [44].

Inspired by the above problem, in this chapter we define two new domination parameters, cographic domination number $\gamma_{cd}(G)$ and global cographic domination number $\gamma_{gcd}(G)$ based on cographs and some of its properties are discussed.

Some results of this chapter are included in the following paper.

Cographic and global cographic domination number of a graph, Ars Combin., (to appear).

3.1 Cographic domination number

In this section, given any graph G, we prove the existence of a cographic dominating set. The relationship between γ, γ_{cd} and γ_i of a tree is studied.

Theorem 3.1.1. *For any graph G, there exists a dominating induced subgraph which is a cograph.*

Proof. The proof is by induction on n. For $n \leqslant 3$, the theorem can be easily verified. Assume that it is true for all graphs with at most n vertices.

Let G be a graph with $n+1$ vertices. By induction assumption, the graph $G - v$ has a dominating induced subgraph H which is a cograph. If at least one of the vertices in H is adjacent to v, then H is a dominating induced subgraph for G. If not, $H \cup \{v\}$ is a dominating induced subgraph of G which is also a cograph. Therefore by induction, the theorem is true for all graphs. □

Note : For any graph G, $\gamma(G) \leqslant \gamma_{cd}(G) \leqslant \gamma_i(G)$. However, there are graphs with $\gamma(G) < \gamma_{cd}(G) < \gamma_i(G)$. For e.g:-

G :

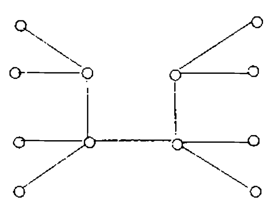



Fig : 3.1

$\gamma(G) = 4$, $\gamma_{cd}(G) = 5$ and $\gamma_i(G) = 6$.

Lemma 3.1.2. *If T is a tree with $\gamma(T) < \gamma_{cd}(T)$, then T must have the graph G in Fig : 3.1 as an induced subgraph.*

Proof. Since $\gamma(T) < \gamma_{cd}(T)$, in every dominating set D with cardinality $\gamma(T)$ there exists an induced $P_4 : u_1u_2u_3u_4$. Since D is minimal dominating and u_i for i = 1,2,3,4 is adjacent to at least one vertex in the dominating set, there exists at least one v_i in the vertex set of T corresponding to each u_i such that v_i is adjacent only to u_i in D for each i = 1,2,3,4. If for one of these 'i', v_i is the only such neighbor of u_i then we can replace u_i by v_i for that i in the dominating set to remove the induced P_4 without changing the cardinality. Therefore, there exists at least one induced P_4 in T such that each of its vertices is adjacent to a pair of vertices. These twelve vertices together induce the required graph. □

Corollary 3.1.3. *For any graph G with less than twelve vertices, $\gamma(G) = \gamma_{cd}(G)$.*

Proof. If G has less than twelve vertices, then G cannot have the graph in Fig : 3.1 as an induced subgraph. Hence, $\gamma(G) = \gamma_{cd}(G)$. □

Lemma 3.1.4. *If T is a tree with $\gamma_{cd}(T) < \gamma_i(T)$, then T has the following graph as an induced subgraph.*

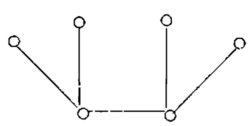


Fig : 3.2

Proof. Since $\gamma_{cd}(T) < \gamma_i(T)$, every cographic dominating set D with cardinality $\gamma_{cd}(T)$ will have at least one pair of adjacent vertices, say uv. Since u and v are mutually dominating, there exist at least two vertices u_1 and v_1 in T which are adjacent only to u and v, respectively. If these are the only such vertices then we can replace u by u_1 or v by v_1 in T to remove the adjacency in D without affecting the cardinality. Therefore, there exist at least one pair of vertices in D which has at least two neighbors of their own. These six vertices induce the required graph. □

Corollary 3.1.5. *For any graph G with less than six vertices, $\gamma_{cd}(G) = \gamma_i(G)$.*

Proof. If G has less than six vertices, then G cannot have the graph in Fig : 3.2 as an induced subgraph. Hence, $\gamma_{cd}(G) = \gamma_i(G)$. □

Theorem 3.1.6. *There is no tree T which satisfies $\gamma(T) < \gamma_{cd}(T) = \gamma_i(T)$.*

Proof. If possible assume that there is a tree T which satisfies $\gamma(T) < \gamma_{cd}(T) = \gamma_i(T)$. Let D be a minimal dominating set of cardinality $\gamma(T)$. Since $\gamma(T) < \gamma_{cd}(T)$, by Lemma 3.1.2, T must contain the graph in Fig : 3.1 as an induced subgraph and the vertices which induce a P_4 in it must be present in D. Also, none of the vertices of this P_4 can be replaced without affecting the domination property and without increasing the cardinality of D. To make D a cographic dominating set, only one vertex is to be replaced, whereas to make D an independent dominating set, two of the vertices are to be replaced. Since D is arbitrary, $\gamma_{cd}(T) < \gamma_i(T)$ which is a contradiction. Hence, the theorem. □

3.2 Global cographic domination number

We prove the existence of a global cographic dominating set for every graph G and study the relation between $\gamma_{cd}(G)$ and $\gamma_{gcd}(G)$ of various special classes of graphs in this section.

Theorem 3.2.1. *Given any graph $G = (V, E)$, there exists a cographic dominating set which dominates G^c also.*

Proof. If D is a cographic dominating set in G which dominates G^c also, then there is nothing to prove. Otherwise, there exists at least one vertex, say v_1 which is not adjacent to any vertex of D in G^c. Adjoin v_1 to D. If $D \cup \{v_1\}$ does not dominate G^c, then there exist a v_2 which is not adjacent to any vertex of $D \cup \{v_1\}$ in G^c. Adjoin v_2 to $D \cup \{v_1\}$. Continue this process until we get a dominating set $D' = D \cup \{v_1, v_2, ..., v_k\}$ which dominates G^c. The process will eventually terminate, since V dominates G^c. The subgraph induced by D' in G is the join of the subgraph induced by D in G with K_p, for some p. Therefore, the subgraph induced by D' is also a cograph by the choice of D and since $D \subseteq D'$, D' dominates G. Therefore, D' is a cographic dominating set in G which dominates G^c also. □

Note : For any graph G, $\gamma_{gcd}(G) \geqslant max\{\gamma_{cd}(G), \gamma_{cd}(G^c)\}$.

Lemma 3.2.2. *For any graph $G \neq K_1$, $\gamma_{gcd}(G) > 1$.*

Proof. If $\gamma_{gcd}(G) = 1$, then $\gamma_{cd}(G) = 1$. Then G has a vertex of full degree and so G^c has an isolated vertex. Therefore, $\gamma_{cd}(G^c) > 1$ and so $\gamma_{gcd}(G) < \gamma_{cd}(G^c)$. This is a contradiction and hence $\gamma_{gcd}(G) > 1$. □

Theorem 3.2.3. *If G is a triangle free graph, then $\gamma_{gcd}(G) = \gamma_{cd}(G)$ or $\gamma_{cd}(G) + 1$.*

Proof. Let $\gamma_{gcd}(G) \neq \gamma_{cd}(G)$. Let D be a minimum cographic dominating set. Since none of the minimum cographic dominating sets dominate G^c, at least one vertex v of G must be adjacent to all the vertices of D. Consider $D \cup \{v\}$. Since the graph is triangle free, none of the neighbors of the vertices of D are adjacent to v. Since D is dominating, every vertex of G is either in D or is adjacent to a vertex of D. Therefore, the only neighbors of v are those present in D. Hence, in G^c, v dominates all the vertices outside D. Also, $D \cup \{v\}$ induces a cograph. Thus, $D \cup \{v\}$ is a cographic dominating set in G as well as G^c, of cardinality $\gamma_{cd}(G) + 1$. □

Remark 3.2.1. There are graphs for which $\gamma_{gcd}(G) = \gamma_{cd}(G)$ and $\gamma_{gcd}(G) = \gamma_{cd}(G)+1$. For example, $\gamma_{cd}(C_4) = \gamma_{gcd}(C_4) = 2$, whereas $\gamma_{cd}(C_5) = 2$ and $\gamma_{gcd}(C_5) = 3$. But, the converse need not be true. In the graphs G_1 and G_2 in Fig : 3.3, $\gamma_{gcd}(G_1) = \gamma_{cd}(G_1) = 3$ and $\gamma_{gcd}(G_2) = 2$ and $\gamma_{cd}(G_2) = 1$.

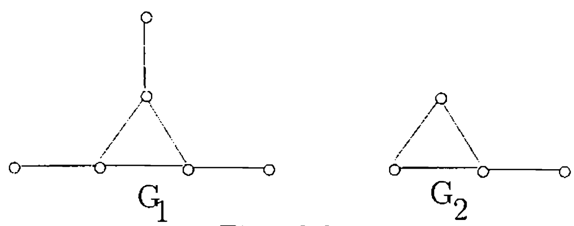


Fig : 3.3

Corollary 3.2.4. *If G is a triangle free graph with $\gamma_{gcd}(G) \neq \gamma_{cd}(G)$, then $\gamma_{cd}(G) = \gamma_i(G)$.*

Proof. Let D be a minimum cographic dominating set of G. Since, none of the minimum cographic dominating sets dominate G^c, at least one vertex v of G must be adjacent to all the vertices of D. Since, G is triangle free, no two vertices of D are adjacent. Therefore, D is an independent dominating set. Hence, $\gamma_{cd}(G) = \gamma_i(G)$. □

Corollary 3.2.5. *Every tree T has $\gamma_{gcd}(T) = \gamma_{cd}(T)$ or $\gamma_{cd}(T) + 1$. Moreover, $\gamma_{gcd}(T) = \gamma_{cd}(T) + 1$ only if T is a rooted tree of depth two in which every vertex (may be except the root) has at least two children.*

Proof. The first statement follows from Theorem 3.2.3, since trees are triangle free. Assume that $\gamma_{gcd}(T) = \gamma_{cd}(T) + 1$ for a tree T. Then as in the proof of corollary 3.2.4, there exists a minimum cographic dominating set D, which is independent and has a common neighbor v. Since D is dominating and T is a tree, v is not adjacent to any other vertex of G. Now, every vertex of D has at least two pendant vertices attached to it. Since, otherwise if $u \in D$ has only one pendant vertex w attached to it, then $(D - \{u\}) \cup \{w\}$ is a global dominating set of cardinality $\gamma_{cd}(T)$, which is a contradiction. Therefore, all the vertices in D have at least two pendant vertices attached to it and so T is a rooted tree of depth two with v as its root in which every vertex has at least two children. □

Fig : 3.4 gives examples of trees with $\gamma_{gcd}(T) = \gamma_{cd}(T) + 1$.

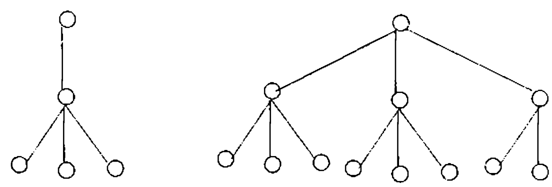


Fig : 3.4

Lemma 3.2.6. *If G is a disconnected graph, then $\gamma_{cd}(G^c) \leqslant 2$ and $\gamma_{gcd}(G) = \gamma_{cd}(G)$.*

Proof. Since G is disconnected, G^c is connected and any two vertices in the two different components of G dominates G^c. So, $\gamma_{cd}(G^c) \leqslant 2$. Also, in any cographic dominating set of G, there will be at least one vertex from each component. Therefore any cographic dominating set of G is a cographic dominating set of G^c also. Hence $\gamma_{gcd}(G) = \gamma_{cd}(G)$. □

Remark 3.2.2. This lemma holds for the domination number and the global domination number of a disconnected graph also.

Theorem 3.2.7. *A cograph G without a universal vertex has $\gamma_{gcd}(G) = \gamma_{cd}(G)$ if and only if there exists two vertices u and v such that $N(u)$ and $N(v)$ partitions $V(G)$ or $V(G) - \{u, v\}$.*

Proof. If $N(u)$ and $N(v)$ partitions $V(G)$ or $V(G) - \{u, v\}$, the cographic domination number of G is 2. In G^c, $\{u, v\}$ itself dominates. Therefore, $\gamma_{gcd}(G) = \gamma_{cd}(G) = 2$.

Conversely, assume that $\gamma_{gcd}(G) = \gamma_{cd}(G)$. Since $\gamma_{gcd}(G) > 1$ and $\gamma_{cd}(G) \leqslant 2$, we have $\gamma_{gcd}(G) = \gamma_{cd}(G) = 2$. Therefore, there exist two vertices u and v such that $\{u, v\}$ dominates both G and G^c. Since, neighbors of u in G will not be adjacent to u in G^c, they must be adjacent to v in G^c. Hence, no vertex in $N(u)$ is adjacent to v in G and vice versa. Also, since $\{u, v\}$ dominates, $N(u) \cup N(v) = V(G)$ or $V(G) - \{u, v\}$. Therefore, $N(u)$ and $N(v)$ partitions $V(G)$ or $V(G) - \{u, v\}$. □

Fig : 3.5 gives an example of a cograph for which $\gamma_{gcd}(G) = \gamma_{cd}(G)$.

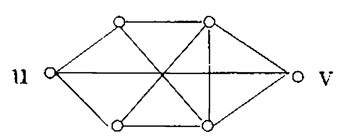


Fig : 3.5

Theorem 3.2.8. *If G is a planar graph with $\gamma_{cd}(G) \geqslant 3$, then $\gamma_{gcd}(G) \leqslant \gamma_{cd}(G)+2$.*

Proof. If possible assume that $\gamma_{gcd}(G) > \gamma_{cd}(G)+2$. Let u_1, u_2, u_3 be three vertices in any γ_{cd}-set D of G. Since $\gamma_{gcd}(G) > \gamma_{cd}(G) + 2$, D cannot dominate G^c and at least three more vertices are to be added to D to make it a global dominating set. Therefore, there exist at least three vertices v_1, v_2, v_3 which are adjacent to each other and to every vertex of D. Then the subgraph induced by these six vertices will be $K_6, K_6 - \{e_1\}, K_6 - \{e_1, e_2\}$ or $K_6 - \{e_1, e_2, e_3\}$ where $e_1, e_2, e_3 \in E(G)$ and are adjacent to each other. Each of the above graph contains $K_{3,3}$ as a subgraph, which is a contradiction to the planarity of G. Hence the theorem. □

Remark 3.2.3. The converse need not be true. For example, in graphs G_1, G_2 and G_3 in Fig : 3.6, $\gamma_{cd}(G_1) = \gamma_{gcd}(G_1) = 2$, $\gamma_{cd}(G_2) = 2$, $\gamma_{gcd}(G_2) = 3$, $\gamma_{cd}(G_3) = 2$ and $\gamma_{gcd}(G_3) = 4$.

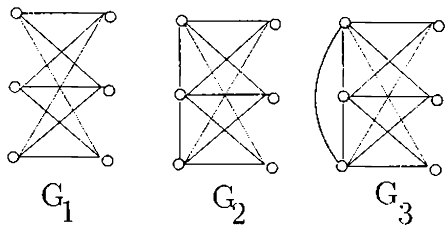


Fig : 3.6

Remark 3.2.4. The bound $\gamma_{gcd}(G) \leqslant \gamma_{cd}(G) + 2$ is strict.

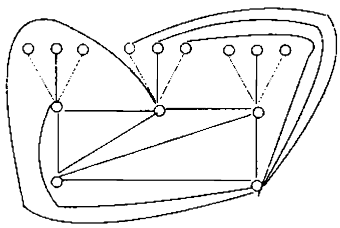


Fig : 3.7

For example, the plane graph in Fig : 3.7 has $\gamma_{cd} = 3$ and $\gamma_{gcd} = 5$.

3.3 Two constructions

Theorem 3.3.1. *Given three positive integers a, b and c satisfying $a \leqslant b \leqslant c$, there is a graph G such that $\gamma(G) = a$, $\gamma_{cd}(G) = b$, $\gamma_{t}(G) = c$.*

Proof. We shall prove the theorem by constructing the required graph and by analyzing the following cases.

Case 1 : a = b = c

Let $G = P_n$ or C_n where $n = 3a$. Then, $\gamma(G) = \gamma_{cd}(G) = \gamma_i(G) = a$.

Case 2 : a = b < c

Let G be the graph P_n where n = 3(a - 1) together with (c - a + 1) pendant vertices each attached to an end vertex of P_n and its neighbor. Then, $\gamma(G) = \gamma_{cd}(G) = a$ and $\gamma_i(G) = c$.

Case 3 : a < b = c

Let G be $P_n : v_1v_2v_3....v_n$, where n = 3a - 7 together with p = b - a + 2 vertices. $u_{i1}, u_{i2}, ..u_{ip}$, made adjacent to each v_i for i = 1,2,3 and 4 and u_{1j} made adjacent to u_{3j} for each j = 1,2,....,p.

Then, the vertices v_1, v_2, v_3 and v_4 dominate all u_{ij} s and v_5. To dominate the remaining (3a - 12) vertices of the path, (a - 4) vertices are required. Therefore, $\gamma(G) = a$. At least one vertex among v_1, v_2, v_3 and v_4 must be replaced to get a cographic dominating set. Remove v_1 and include all the (b - a + 2) vertices. But, then v_3 is also not required in the dominating set so that $\gamma_{cd}(G)$ = a - 2 + b - a + 2 = b. This set is also independent and hence $\gamma_i(G) = b$.

Case 4 : a < b < c

Let G be $P_n : v_1v_2v_3....v_n$, where n = 3a - 7 together with (b - a) vertices made adjacent to v_4, (c - a + 1) vertices made adjacent to v_2 and (c - a + 2) vertices each made adjacent to v_1 and v_3 and to each other.

Then, the vertices v_1, v_2, v_3 and v_4 dominate all pendant vertices attached to them and v_5. To dominate the remaining (3a - 12) vertices of the path, (a - 4) vertices are required. Therefore, $\gamma(G)$ = a. At least one vertex among v_1, v_2, v_3 and v_4 must be replaced to get a cographic dominating set. If we remove v_4, the (b - a) pendant vertices adjacent to it and v_5 are to be adjoined to get a cographic dominating set of cardinality a - 1 + b - a + 1 = b. If we remove v_1, the (c - a + 2) pendant vertices adjacent to it are to be adjoined. But, then v_3 also can be removed from the dominating set to get an independent dominating set of cardinality (a - 2 + c - a + 2) = c. Therefore, $\gamma_{cd}(G)$ = b and $\gamma_i(G)$ = c. □

Illustration

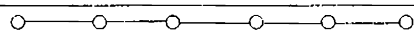
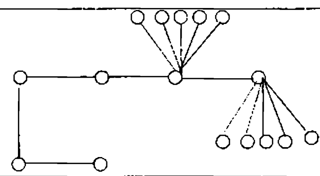
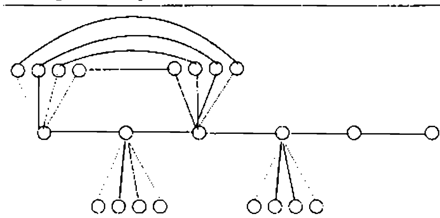
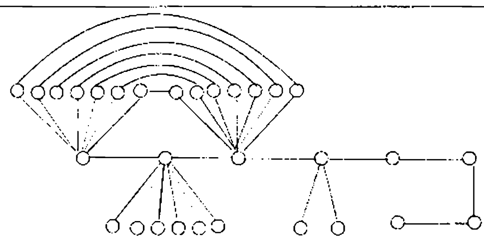
Case 1	a = b = c = 2	
Case 2	a = b = 3, c = 7	
Case 3	a = 5, b = c = 7	
Case 4	a = 5, b = 7, c = 10	

Table 3.1

Theorem 3.3.2. *Given two positive integers a and b satisfying $a \leqslant b$ and $b > 1$, there is a graph G such that $\gamma_{cd}(G) = a$, $\gamma_{gcd}(G) = b$.*

Proof. We shall prove the theorem by constructing the required graph and by analyzing the following cases.

Case 1 : $a = b > 1$.

G is the graph obtained by taking any graph of order a and attaching one pendant vertex to each of the vertices.

Case 2 : $a = 1$ and $a < b$.

$G = K_b$.

Case 3 : $a = 2$ and $a < b$.

G is K_{2b} minus a perfect matching.

Case 4 : $a > 2$ and $a < b$.

The graph G is obtained as per the following constructions based on the Fig : 3.8.

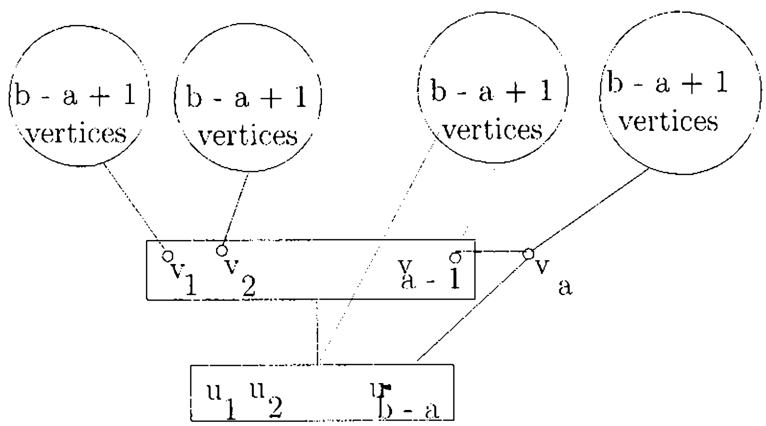


Fig : 3.8

In the Fig : 3.8, the vertices inside each of the circles are independent and the vertices inside both the rectangles form complete graphs. Every vertex v_i for $i = 1, 2, ..., a$ is made adjacent to every vertex inside the circle to which an edge is drawn. All the vertices of the smaller rectangle are made adjacent to, all the vertices in the bigger rectangle, all the vertices inside the circle to which an edge is drawn and to v_a. Further, v_{a-1} is made adjacent to v_a. The graph so obtained is G.

Now, if we choose one vertex from each of the circles, we get an independent set of cardinality a which has no common neighbors. Therefore, any dominating set must contain at least a vertices and $\{v_1, v_2, ..., v_a\}$ is a cographic dominating set. So $\gamma_{cd}(G) = a$.

The set $\{v_1, v_2, ..., v_a\}$ will not dominate u_is in G^c. If we remove any one of the v_is from this cographic dominating set, then all the $b - a + 1$ vertices in the corresponding circle must be included to retain the set as a cographic dominating set. Therefore, the cardinality becomes $a - 1 + b - a + 1 = b$.

If we keep all the v_is, then a vertex from any of the circles, except the one corresponding to v_{a-1} cannot be introduced , since otherwise an induced P_4 will be present. A vertex from the circle corresponding to v_{a-1} cannot dominate u_is in the complement. Also, a u_i cannot dominate u_j for $i \neq j$. Therefore to get a global cographic dominating set all the u_is must be included. Then the cardinality becomes $a + b - a = b$. In any case, $\gamma_{gcd}(G) = b$. □

Illustration of case 4 : $a = 3$, $b = 5$.

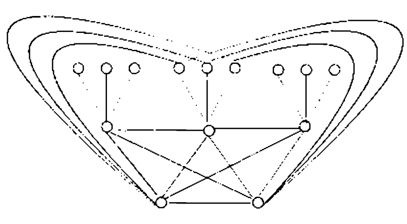


Fig : 3.9

3.4 Complexity aspects

In this section we discuss the complexity aspects of the newly defined parameters.

Theorem 3.4.1. *Determining the cographic domination number of a graph is NP-complete.*

Proof. We prove this by reducing in polynomial time, the dominating set problem in general to the cographic dominating set problem.

Claim: Given a graph G, we can construct a graph H in polynomial time such that G has a dominating set of size k if and only if H has a cographic dominating set of size $k+1$.

Let $G = (V, E)$ where $V = \{v_1, v_2, ..., v_n\}$ be the given graph. Construct H as follows:
Let $V(H) = \{v_1, v_2, ..., v_n\} \cup \{v'_1, v'_2, ..., v'_n\} \cup \{x, y\}$. Make v_i adjacent to v'_j if $v_i v_j \epsilon E(G)$ or $i = j$; x is adjacent to v'_j for every j and x is adjacent to y in H.

Let $\{v_{i_1}, v_{i_2}, ..., v_{i_k}\}$ be a minimal dominating set of cardinality k in G. Then $\{v'_{i_1}, v'_{i_2}, ..., v'_{i_k}, x\}$ is a minimal dominating set in H. Since there is no induced P_4 in this set, it is a minimal cographic dominating set in H of cardinality $k+1$.

Conversely, let $\{u_1, u_2, ..., u_{k+1}\}$ be a cographic dominating in H. (By construction of H, any minimal dominating set is cographic). One of these u_i's must be x or y. Remove that u_i. All other u_i's must be either v_j or v'_k. Keep each v_j unchanged and replace each v'_k by v_k. This new set of cardinality k will be a minimal dominating set of G. Since H can be computed from G in time polynomial in size of G, our claim holds. □

Corollary 3.4.2. *The problem of determining the cographic domination number is NP-complete for the class of bipartite graphs.*

Proof. In the proof above, the graph H constructed from G is bipartite. □

Theorem 3.4.3. *Determining the global cographic domination number of a graph is NP-complete.*

Proof. Claim : Given a graph G, we can construct a graph H in polynomial time such that G has a cographic dominating set of size k if and only if H has a global cographic dominating set of size $k+1$.

Given a graph G define $H = G \cup K_1$. Clearly, a minimum cographic dominating set in G together with the isolated vertex is a minimal global cographic dominating set in H.

Conversely, any minimal global cographic dominating set in H will contain the isolated vertex and the remaining vertices is a minimal cographic dominating set in G. Since H can be computed from G in time polynomial in size of G, our claim holds. □

3.5 Domination in NEPS of two graphs

In this section, we study the relation between the domination parameters γ, γ_g, γ_{cd}, γ_{gcd} and γ_i of G_1 and G_2 with the NEPS of G_1 and G_2 for all possible choices of the basis.

NEPS with basis $\mathcal{B}_1$ and $\mathcal{B}_2$

The value of $\gamma(\text{NEPS}(G_1, G_2; \mathcal{B}_1))$, $\gamma_g(\text{NEPS}(G_1, G_2; \mathcal{B}_1))$, $\gamma_{cd}(\text{NEPS}(G_1, G_2; \mathcal{B}_1))$, $\gamma_{gcd}(\text{NEPS}(G_1, G_2; \mathcal{B}_1))$, $\gamma_i(\text{NEPS}(G_1, G_2; \mathcal{B}_1))$ are $n_1.\gamma(G_2)$, $n_1.\gamma_g(G_2)$, $n_1.\gamma_{cd}(G_2)$, $n_1.\gamma_{gcd}(G_2)$ and $n_1.\gamma_i(G_2)$ respectively and the case of $\text{NEPS}(G_1, G_2; \mathcal{B}_2)$ follows similarly.

NEPS with basis $\mathcal{B}_3$

In [39] it was conjectured that $\gamma(G \times H) \geqslant \gamma(G)\gamma(H)$, where $\times$ denotes the tensor product of two graphs. But, the conjecture was disproved in [48] by giving a realization of a graph G such that $\gamma(G \times G) \leqslant \gamma(G)^2 - k$ for any non-negative integer k.

Theorem 3.5.1. *There exist graphs G_1 and G_2 such that $\sigma(NEPS(G_1, G_2; \mathcal{B}_3)) - \sigma(G_1)\sigma(G_2) = k$ for any positive integer k, where σ denotes any of the domination parameters γ, γ_{cd} or γ_i.*

Proof. Let G_1 be the graph defined as follows. Let $u_{11}u_{12}u_{13}$, $u_{21}u_{22}u_{23}$, ..., $u_{k1}u_{k2}u_{k3}$ be k distinct P_3 s and let u_{j1} be adjacent to $u_{j+1,1}$ for $j = 1, 2, ..., k-1$. Then $\sigma(G_1) = k$. Let G_2 be K_2. Then, $\sigma(G_2) = 1$. Also, $\sigma(\text{NEPS}(G_1, G_2; \mathcal{B}_3)) = 2k$. Therefore, $\sigma(\text{NEPS}(G_1, G_2; \mathcal{B}_3)) - \sigma(G_1)\sigma(G_2) = k$. □

Theorem 3.5.2. *The γ_g and γ_{gcd} are neither sub multiplicative nor super multiplicative with respect to the NEPS with basis $\mathcal{B}_3$. Moreover, given any integer k there exist graphs G_1 and G_2 such that $\sigma(NEPS(G_1, G_2; \mathcal{B}_3)) - \sigma(G_1)\sigma(G_2) = k$, where σ denotes γ_g or γ_{gcd}.*

Proof. Case 1: $k \leqslant 0$ is even

Let $G_1 = K_n$ and $G_2 = K_2$. Then, $\sigma(G_1) = n$ and $\sigma(G_2) = 2$. But, $\sigma(\text{NEPS}(G_1, G_2; \mathcal{B}_3)) = 2$. Therefore, the required difference is $2 - 2n$ which can be zero or any negative even integer.

Case 2: $k < 0$ is odd or $k = 1$

Let $G_3 = P_3$ and G_1 be as in Case 1. Then $\sigma(G_3) = 2$. Also, $\sigma(\text{NEPS}(G_1, G_3; \mathcal{B}_3)) = 3$. Therefore, the required difference is $3 - 2n$ which can be one or any negative odd integer.

Case 3: $k > 1$

Let G_3 be as in Case 2. Let G_4 be the graph defined as follows. Let $u_{11}u_{12}u_{13}$, $u_{21}u_{22}u_{23}$, ..., $u_{k1}u_{k2}u_{k3}$ be k distinct P_3 s and let u_{j1} be adjacent to $u_{j+1,1}$ for $j = 1, 2, ..., k-1$. Then $\sigma(G_4) = k$. Also, $\sigma(\text{NEPS}(G_4, G_3; \mathcal{B}_3)) = 3k$. Therefore, the required difference is k. □

NEPS with basis $\mathcal{B}_4$

Vizing's conjecture [75]: The domination number is super multiplicative with respect to the cartesian product i.e; $\gamma(G \Box H) \geqslant \gamma(G)\gamma(H)$.

Remark 3.5.1. There are infinitely many pairs of graphs for which equality holds in the Vizing's conjecture [62].

Remark 3.5.2. Vizing's type inequality does not hold for cographic, global cographic and independent domination numbers. For example, let G be the graph obtained by attaching k pendant vertices to each vertex of a path on four vertices. Then, $\gamma_{cd}(G) = \gamma_{gcd}(G) = k + 3$ and $\gamma_{cd}(G \Box G) = \gamma_{gcd}(G \Box G) = 16k + 8$. For $k \geqslant 10$, $\gamma_{cd}(G \Box G) \leqslant \gamma_{cd}(G)^2$.

Theorem 3.5.3. *There exist graphs G_1 and G_2 such that $\sigma(NEPS(G_1, G_2; \mathcal{B}_4)) -$*

$\sigma(G_1)\sigma(G_2) = k$ for any positive integer k, where σ denotes any of the domination parameters γ, γ_{cd} or γ_i.

Proof. Let $G_1 = P_n$ and $G_2 = K_2$. Then, $\sigma(G_1) = \lfloor\frac{n+2}{3}\rfloor$ [44] and $\sigma(G_2) = 1$. Also, $\sigma(\text{NEPS}(G_1, G_2; \mathcal{B}_4)) = \lfloor\frac{n+2}{2}\rfloor$ [47]. Therefore, for any positive integer k, if we choose $n = 6k - 2$ the claim follows. ☐

Theorem 3.5.4. *The γ_g and γ_{gcd} are neither sub multiplicative nor super multiplicative with respect to the NEPS with basis $\mathcal{B}_4$. Moreover, given any integer k there exist graphs G_1 and G_2 such that $\sigma(NEPS(G_1, G_2; \mathcal{B}_4)) - \sigma(G_1)\sigma(G_2) = k$, where σ denotes γ_g or γ_{gcd}.*

Proof. Case 1: $k \leqslant 0$ is even.

Let $G_1 = K_n$ and $G_2 = K_2$. Then, $\sigma(G_1) = n$ and $\sigma(G_2) = 2$. But, $\sigma(\text{NEPS}(G_1, G_2; \mathcal{B}_4)) = 2$. Therefore, the required difference is $2 - 2n$ which can be any positive even integer.

Case 2: $k < 0$ is odd.

Let $G_3 = P_3$ and G_1 be as in Case 1. Then $\sigma(G_3) = 2$. Also, $\sigma(\text{NEPS}(G_1, G_3; \mathcal{B}_4)) = 3$. Therefore, the required difference is $3 - 2n$ which can be any negative odd integer.

Case 3: $k \geqslant 1$.

Let $G_4 = P_n$ and $G_5 = P_4$. Then, $\sigma(G_4) = \lfloor\frac{n+2}{3}\rfloor$ and $\sigma(G_5) = 2$. For any positive integer k, if we choose $n = 3k + 4$, then $\sigma(\text{NEPS}(G_4, G_5; \mathcal{B}_4)) = n$. (Note that the value is $n + 1$ only when $n = 1, 2, 3, 5, 6, 9$ [47]). Therefore the required difference is k.

□

NEPS with basis $\mathcal{B}_5$ and $\mathcal{B}_6$

Theorem 3.5.5. *There exist graphs G_1 and G_2 such that $\sigma(NEPS(G_1, G_2; \mathcal{B}_5)) - \sigma(G_1)\sigma(G_2) = k$ for any positive integer k, where σ denotes any of the domination parameters γ, γ_{cd} or γ_i.*

Proof. Let $G_1 = P_n$ and $G_2 = K_2$. Then $\sigma(G_1) = \lfloor\frac{n+2}{3}\rfloor$ and $\sigma(G_2) = 1$. Also, $\sigma(\text{NEPS}(G_1, G_2; \mathcal{B}_5)) = \lfloor\frac{n+2}{2}\rfloor$. For a positive integer k, if we choose $n = 6k - 2$ then the difference is k. Hence, the theorem. □

Theorem 3.5.6. *There exist graphs G_1 and G_2 such that $\sigma(NEPS(G_1, G_2; \mathcal{B}_5)) - \sigma(G_1)\sigma(G_2) = k$ for any negative integer k, where σ denotes γ_g or γ_{gcd}.*

Proof. Let $G_1 = P_n$ and $G_2 = K_2$. Then $\sigma(G_1) = \lfloor\frac{n+2}{3}\rfloor$ and $\sigma(G_2) = 2$. Also, $\sigma(\text{NEPS}(G_1, G_2; \mathcal{B}_5)) = \lfloor\frac{n+2}{2}\rfloor$. Therefore, if we choose $n = 6k - 2$, the required difference is $-k$. □

NEPS with basis $\mathcal{B}_7$

Theorem 3.5.7. *The γ, γ_i and γ_g are sub multiplicative with respect to the NEPS with basis $\mathcal{B}_7$.*

Proof. Let $D_1 = \{u_1, u_2, ..., u_s\}$ be a dominating set of G_1 and $D_2 = \{v_1, v_2, ..., v_t\}$ be a dominating set of G_2. Consider the set $D = \{(u_1, v_1), (u_1, v_2), ..., (u_1, v_t), ..., (u_s, v_1), (u_s, v_2), ..., (u_s, v_t)\}$. Let (u, v) be any vertex in NEPS$(G_1, G_2; \mathcal{B}_7)$. Since D_1 is a γ-set in G_1, there exists at least one $u_i \in D_1$ such that $u = u_i$ or u is adjacent to u_i. Similarly, there exists at least one $v_j \in D_2$ such that $v = v_j$ or v is adjacent to v_j. Therefore, (u_i, v_j) dominates (u, v) in $\mathrm{NEPS}(G_1, G_2; \mathcal{B}_7)$. Hence, $\gamma(\mathrm{NEPS}(G_1, G_2; \mathcal{B}_7)) \leqslant \gamma(G_1)\gamma(G_2)$.

Similar arguments hold for the independent domination and global domination numbers also. □

Remark 3.5.3. The difference between $\gamma(G_1)\gamma(G_2)$ and $\gamma(\mathrm{NEPS}(G_1, G_2; \mathcal{B}_7))$ can be arbitrarily large. Similar is the case for γ_i and γ_g. For, let G_1 be the graph, n copies of C_4 s with exactly one common vertex. Then, $\gamma(G_1) = \gamma_i(G_1) = n + 1$. Also, $\gamma(\mathrm{NEPS}(G_1, G_1; \mathcal{B}_7)) \leqslant n^2 + 3$ and $\gamma_i(\mathrm{NEPS}(G_1, G_1; \mathcal{B}_7)) \leqslant n^2 + 5$. Also, $\gamma_g(K_n) = n$, $\gamma_g(P_3) = 2$ and $\gamma_g(\mathrm{NEPS}(G_2, G_3; \mathcal{B}_7)) = n + 2$, if $n > 1$.

Theorem 3.5.8. *The γ_{cd} and γ_{gcd} are neither sub multiplicative nor super multiplicative with respect to the NEPS with basis $\mathcal{B}_7$. Moreover, for any integer k there exist graphs G_1 and G_2 such that $\sigma(NEPS(G_1, G_2; \mathcal{B}_7)) - \sigma(G_1)\sigma(G_2) = k$, where σ denotes γ_{cd} or γ_{gcd}.*

Proof. Case 1: $k \leqslant 0$

Let G_1 be the graph P_3 with k pendant vertices each attached to all the three vertices of the P_3. Let G_2 be the graph P_4 with k pendant vertices each attached to all the four vertices of the P_4. So, $\sigma(G_1) = 3$ and $\sigma(G_2) = k + 3$. Also, $\sigma\mathrm{NEPS}(G_1, G_2; \mathcal{B}_7)) = 2k + 10$. Therefore, the required difference is $1 - k$.

Case 2: $k \geqslant 0$

Let G_1 be as in Case 1 and G_3 be the graph P_6 with k pendant vertices each attached to all the six vertices of the P_6. So, $\sigma(G_3) = k+5$. Also, $\sigma\mathrm{NEPS}(G_1, G_3; \mathcal{B}_7)) = 4k + 14$. Therefore, the required difference is $k - 1$. □

Chapter 4

The $< t >$-property

The question of determining better upper bounds for the clique transversal number dates back to 1990 when Tuza Z. introduced the concept of the clique transversal number [74]. Erdös et.al. [33] determined various upper bounds for the clique transversal number. In an attempt to find graphs which admit a better upper bound, Tuza Z. [74] introduced the concept of the $< t >$-property. Motivated by the open problems mentioned in [33], we studied the $< t >$-property of the cographs, the clique perfect graphs, the perfect graphs, the planar graphs and the trestled graphs of index k. In the last section, an open problem on highly clique imperfect graphs is solved.

Some results of this chapter are included in the following paper.

The $< t >$-property of some classes of graphs, Discrete Math., (to appear).

4.1 Clique transversal number

In this section we prove that the domination number is a lower bound for the clique transversal number, but the difference can be arbitrarily large.

Theorem 4.1.1. *Every clique transversal set is a dominating set.*

Proof. Let S be a clique transversal set of a graph G and $v \in V(G)$. If $v \in S$ then it is dominated by S. If $v \notin S$ then let C be a clique which contains v. Since, S is a clique transversal set, there exist a vertex $u \in S \cap C$. But then, u dominates v. Therefore, S is a dominating set. □

Corollary 4.1.2. *Let G be a graph. Then, $\gamma(G) \leqslant \tau_c(G)$.*

Theorem 4.1.3. *Let a and b be two positive integers such that $2 \leqslant a \leqslant b$. There exists a clique perfect graph G such that $\gamma(G) = a$ and $\tau_c(G) = b$.*

Proof. Let G be the graph obtained from $K_{b,b}$ by attaching $a-1$ end vertices to $a-1$ distinct vertices in any one of the partitions of G.

To dominate the $a-1$ end vertices, at least $a-1$ vertices are required and those vertices cannot dominate the remaining vertices (there exists at least one such vertex, since $b \geqslant a$) of that partition. Therefore, $\gamma(G)$ is at least a. Again, the $a-1$ distinct neighbors of the $a-1$ end vertices together with one vertex from the other partition of $K_{b,b}$ dominates G. Therefore, $\gamma(G) = a$.

The graph G so constructed is bipartite and hence the only cliques are the edges of G. If we take all the b vertices in the partition of $K_{b,b}$ to which end vertices are attached, then that set forms a clique transversal. Therefore, $\tau_c(G) \leqslant b$. Again, if we take the b independent edges of $K_{b,b}$, it forms a clique independent set of size b. Therefore, $b \leqslant \alpha_c(G) \leqslant \tau_c(G)$. Hence, $\tau_c(G) = b$.

Also, since $\alpha_c(G) = \tau_c(G) = b$, G is clique perfect. □

Illustration

G

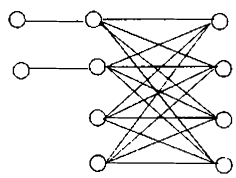


Fig : 4.1

For the graph G is Fig : 4.1, $\gamma(G) = 3$ and $\alpha_c(G) = \tau_c(G) = 4$.

4.2 Cographs and clique perfect graphs

In this section we study the $< t >$-property of cographs and clique perfect graphs. A characterization for cographs and clique perfect graphs which attain maximum value for the clique transversal number is also obtained.

Lemma 4.2.1. *If $G = G_1 \vee G_2$ then $\tau_c(G) = min\{\tau_c(G_1), \tau_c(G_2)\}$.*

Proof. Any clique in G is of the form $H_1 \vee H_2$ where H_1 is a clique in G_1 and H_2 is a clique in G_2. If V' is a clique transversal of G_1 (or G_2), then any clique of G, which contains a clique of G_1 (or G_2), is covered by V' and hence V' is a clique transversal of G also.

Now, let V' be a clique transversal of G. If possible assume that V' does not cover cliques of G_1 and G_2. Let H_1 and H_2 be the cliques of G_1 and G_2 respectively which are not covered by V'. Then $H_1 \vee H_2$ is a clique of G which is not covered by V', which is a contradiction. Hence V' contains a clique transversal of G_1 or G_2.

Therefore, $\tau_c(G) = min\{\tau_c(G_1), \tau_c(G_2)\}$. □

Lemma 4.2.2. *The class of all cographs without isolated vertices does not satisfy the $< t >$-property for $t \geqslant 4$.*

Proof. The proof is by construction.

Case 1 : $t = 4$

Let $G = G_1 \vee G_2$, where $G_1 = (3K_1 \cup K_2) \vee (3K_1 \cup K_2)$ and $G_2 = (3K_1 \cup K_2)$. Then $n = 15, t = 4$ and $\tau_c(G) = 4$ which implies that $\frac{n}{t} < \tau_c(G)$.

Case 2 : $t > 4$

Let $G = G_1 \vee G_2$, where $G_1 = (3K_1 \cup K_{t-3}) \vee (3K_1 \cup K_{t-3})$ and $G_2 = (3K_2 \cup K_{t-2})$.

Then $n(G) = 3t + 4$ and $\tau_c(G) = 4$.

Every edge in G_1 lies in a complete of size t in G since G_2 contains a clique of size $t - 2$. Every edge in G_2 lies in a complete of size t for $t \geqslant 4$ in G since G_1 contains a clique of size $2t - 6$. An edge with one end vertex in G_1 and the other end vertex in G_2 lies in a complete of size t since every vertex in G_1 lies in a complete of size $t - 2$ and every vertex of G_2 lies in a complete of size 2. Hence G is a cograph in which every edge lies in a clique of size t.

Also, $\frac{n}{t} = 3 + \frac{4}{t}$.

Therefore, $\frac{n}{t} < \tau_c(G)$ for $t > 4$. □

Theorem 4.2.3. *The class of clique perfect graphs without isolated vertices satisfies the $< t >$-property for $t = 2$ and 3 and does not satisfy the $< t >$-property for $t \geqslant 4$.*

Proof. Let G be a clique perfect graph in which every edge lies in a complete of size t. G being clique perfect, $\tau_c(G) = \alpha_c(G)$.

Case 1: $t = 2$

Since G is without isolated vertices $\alpha_c(G) \leqslant \frac{n}{2}$. So $\tau_c(G) = \alpha_c(G) \leqslant \frac{n}{2}$ and hence the class of clique perfect graphs satisfies the $< 2 >$-property.

Case 2: $t = 3$

Every edge of G lies in a clique of size 3. So, the size of the smallest clique of G is 3. Therefore, $\alpha_c(G) \leqslant \frac{n}{3}$ and $\tau_c(G) = \alpha_c(G) \leqslant \frac{n}{3}$.

Case 3: $t \geqslant 4$

The class of cographs is a subclass of clique perfect graphs (Lemma 1.1.8). So by Lemma 4.2.2, the claim follows. □

Corollary 4.2.4. *The class of cographs without isolated vertices satisfies the $< t >$-property for $t = 2$ and 3. Moreover, for the class of connected cographs without isolated vertices, $\tau_c(G)$ is maximum if and only if G is the complete bipartite graph $K_{\frac{n}{2},\frac{n}{2}}$.*

Proof. Since the class of cographs is a subclass of clique perfect graphs (Lemma

1.1.8), it satisfies the $< t >$-property for $t = 2$ and 3.

Since the class of cographs satisfy the $< 2 >$-property and $\tau_c(K_{\frac{n}{2},\frac{n}{2}}) = \frac{n}{2}$, the maximum value of $\tau_c(G)$ is $\frac{n}{2}$. Conversely, let G be a connected cograph with $\tau_c(G) = \frac{n}{2}$. Since G is a connected cograph $G = G_1 \vee G_2$. Therefore, $\tau_c(G) = min\{\tau_c(G_1), \tau_c(G_2)\}$. But, both $\tau_c(G_1)$ and $\tau_c(G_2)$ cannot exceed the number of vertices in G_1 and G_2 respectively and hence the number of vertices in G_1 and G_2 must be $\frac{n}{2}$. Again, since $\tau_c(G) = \frac{n}{2}$ all these vertices must be isolated. Therefore, $G = K_{\frac{n}{2},\frac{n}{2}}$. □

Corollary 4.2.5. *For the class of clique perfect graphs without isolated vertices, $\tau_c(G)$ is maximum if and only if there exist a perfect matching in G in which no edge lies in a triangle.*

Proof. The class of clique perfect graphs without isolated vertices satisfies the $< 2 >$-property. Therefore, the maximum value that $\tau_c(G)$ can obtain is $\frac{n}{2}$. Let G be a clique perfect graph with $\tau_c(G) = \frac{n}{2}$. G being clique perfect, $\alpha_c(G) = \tau_c(G) = \frac{n}{2}$. Since each clique must have at least two vertices and there are $\frac{n}{2}$ independent cliques, the cliques are of size exactly two. Again, this independent set of $\frac{n}{2}$ cliques forms a perfect matching of G and a clique being maximal complete, the edges of this perfect matching do not lie in triangles.

Conversely, if there exists a perfect matching in which no edge lies in a triangle, the edges of this perfect matching form an independent set of cliques with cardinality $\frac{n}{2}$. Therefore, $\alpha_c(G) \geqslant \frac{n}{2}$. But, $\alpha_c(G) \leqslant \tau_c(G) \leqslant \frac{n}{2}$ and therefore $\tau_c(G) = \frac{n}{2}$. □

4.3 Planar graphs

Theorem 4.3.1. *The class of planar graphs does not satisfy the* $< t >$*-property for* $t = 2$*, 3 and 4 and* $\mathcal{G}_t$ *is empty for* $t \geqslant 5$.

Proof. Every odd cycle is a planar graph and $\tau_c(C_{2k+1}) = k + 1 > \frac{2k+1}{2}$. Clearly, odd cycles belong to $\mathcal{G}_2$ and hence the class of planar graphs does not satisfy the $< 2 >$-property.

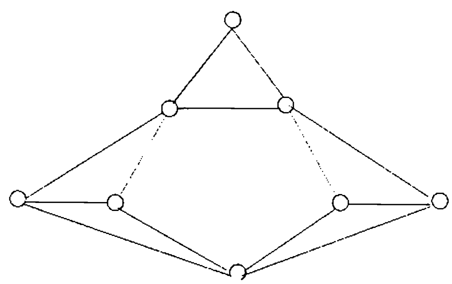


Fig : 4.2

The graph in Fig : 4.2 is planar and every edge lies in a triangle. Here, $n = 8$ and the clique transversal number is 3 which is greater than $\frac{n}{3}$ and hence planar graphs do not satisfy the $< 3 >$-property.

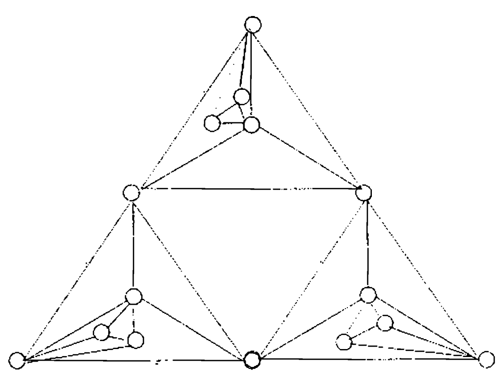


Fig : 4.3

The graph in Fig : 4.3 is planar and every edge lies in a K_4. Here, $n = 15$ and the clique transversal number is 4 which is greater than $\frac{n}{4}$ and hence planar graphs do not satisfy the $< 4 >$-property.

Since K_5 is a forbidden subgraph for planar graphs, there is no planar graph G such that all its edges lie in a K_t for $t \geqslant 5$. Hence, the theorem. □

4.4 Perfect graphs

Theorem 4.4.1. *The class of perfect graphs does not satisfy the $< t >$-property for any $t \geqslant 2$.*

Proof. Let G be the cycle of length $3k$, say $v_1 v_2, \dots ,v_{3k} v_1$ where $k > 2$ is odd, in which the vertices $v_1, v_4, \dots , v_{3k-2}$ are all adjacent to each other. Then G is perfect and $\tau_c(G) = \lceil \frac{3k}{2} \rceil > \frac{3k}{2}$, since $3k$ is odd. Therefore the class of perfect graphs does not satisfy the $< 2 >$-property.

Now, the class of perfect graphs does not satisfy the $< 3 >$-property since $\overline{C_8}$ is a perfect graph (Lemma 1.1.6) in which every edge lies in a triangle and $\tau_c(\overline{C_8}) = 3 > \frac{8}{3}$.

Since the cographs are a subclass of perfect graphs (Lemma 1.1.7) [27], by Lemma 4.2.2, the class of perfect graphs also does not satisfy the $< t >$-property for $t \geqslant 4$. □

4.5 Trestled graph of index k

In this section the clique transversal number and the clique independence number of $T_k(G)$ are determined. A characterization of G for which $T_k(G)$ satisfies the $< 2 >$-property is also given.

Lemma 4.5.1. *For any graph G without isolated vertices, $\tau_c(T_k(G)) = km+\beta(G)$.*

Proof. We shall prove the theorem for the case $k = 1$.

Let $V' = \{v_1, v_2, \ldots, v_\beta\}$ be a vertex cover of G. The cliques of $T_1(G)$ are precisely the cliques of G together with the three K_2 s formed corresponding to each edge of G. Corresponding to each edge uv of G choose the vertex which corresponds to u of the corresponding K_2, if u is not present in V'. If u is present in v' then, choose the vertex corresponding to v, irrespective of v is present in V' or not. Let this new collection together with V' be V''. Then V'' is a clique transversal of $T_1(G)$ of cardinality $m + \beta(G)$. Therefore, $\tau_c(T_1(G)) \leqslant m + \beta(G)$.

Let $V' = \{v_1, v_2, \ldots, v_t\}$, where $t = \tau_c(T_1(G))$ be a clique transversal of $T_1(G)$. Let uv be an edge in G and let $u'v'$ be the K_2 introduced in $T_1(G)$ corresponding to this K_2. At least one vertex from $\{u', v'\}$, say u' must be present in V', since V' is a clique transversal and $u'v'$ is a clique of $T_1(G)$. Remove u' from V'. If V' contains v' also then replace v' by v. If $v' \notin V'$ then $v \in V'$, since V' is a clique transversal and vv' is a clique of $T_1(G)$. In either case, one vertex v of the edge uv is present in the new collection. Repeat the process for each edge in G to get V''. Clearly, V'' is a vertex cover of G with cardinality $\tau_c(T_1(G)) - m$. Hence, $\beta(G) \leqslant \tau_c(T_1(G)) - m$.Thus, $\tau_c(T_1(G)) = m + \beta(G)$.

By a similar argument we can prove that $\tau_c(T_k(G)) = km + \beta(G)$. □

Notation : For a given class $\mathcal{G}$ of graphs, let $T_k(\mathcal{G}) = \{T_k(G) : G \in \mathcal{G}\}$.

Theorem 4.5.2. *The class $T_k(\mathcal{G})$ satisfies the $< 2 >$-property if and only if $\beta(G) \leqslant \frac{n}{2}\ \forall\ G \in \mathcal{G}$ and $(T_k(\mathcal{G}))_t$ is empty for $t \geqslant 3$.*

Proof. Let $G \in \mathcal{G}$. $n(T_k(G)) = n + 2km$ and by Lemma 4.4.1, $\tau_c(T_k(G)) = km + \beta(G)$. Therefore,

$$\tau_c(T_k(G)) \leqslant \tfrac{n(T_k(G))}{2} <=> km + \beta(G) \leqslant \tfrac{n+2km}{2} <=> \beta(G) \leqslant \tfrac{n}{2}.$$

Hence, $T_k(\mathcal{G})$ satisfies $< 2 >$-property if and only if $\beta(G) \leqslant \frac{n}{2}\ \forall\ G \in \mathcal{G}$.

If G contains at least one edge then $T_k(G)$ has a clique of size 2 and hence $(T_k(\mathcal{G}))_t$ is empty for $t \geqslant 3$. □

Lemma 4.5.3. *For any graph G without isolated vertices, $\alpha_c(T_k(G)) = km(G) + \alpha'(G)$.*

Proof. We shall prove the theorem for the case $k = 1$.

Let $E' = \{e_1, e_2, ..., e_{\alpha'}\}$ be a maximum matching of G with cardinality $\alpha'(G)$. Let $C_1 = \{e_{11}, e_{12}, e_{21}, e_{22}, ..., e_{\alpha'1,\alpha'2}\}$ where each e_{i1}, e_{i2} for $i = 1, 2, ..., \alpha'$ are the edges which join e_i to the corresponding K_2 of $T_1(G)$. Note that each e_{ij} is a clique for $i = 1, 2, ..., \alpha'$ and $j = 1, 2$. Let $C_2 = \{f_1, f_2, ..., f_{m-\alpha'}\}$ be the K_2s in $T_1(G)$ corresponding to the edges of $E - E'$. Also, each f_i is a clique in $T_1(G)$ for $i = 1, 2, ..., m - \alpha'$. Therefore, $C_1 \cup C_2$ is a set of independent cliques of $T_1(G)$ with cardinality $2\alpha'(G) + (m(G) - \alpha'(G)) = m(G) + \alpha'(G)$. Hence, $\alpha_c(T_1(G)) \geqslant m(G) + \alpha'(G)$.

Let $S = \{C_1, C_2, ..., C_{\alpha_c}\}$ be a set of independent cliques of $T_1(G)$ with cardinality $\alpha_c(T_1(G))$. Let

$S_1 = \{C_i : V(C_i) \subseteq V(G)\}$,
$S_2 = \{C_i : \exists C_j \text{ with } V(C_i) \cap V(G) = \{u\}, V(C_j) \cap V(G) = \{v\} \text{ where } uv\epsilon E(G)\}$,
$S_3 = S - (S_1 \cup S_2)$

Note that $|S_2|$ is always even and the elements of S_2 can be paired into (C_i, C_j) which satisfy the required property.

Choose one edge from each clique in S_1 and the edge uv corresponding to each pair (C_i, C_j) in S_2 to get an independent set of edges $E' \subseteq E(G)$. Now, $|S_3|$ cannot exceed $m(G)$ and $|S| = \alpha_c(T_1(G))$. Therefore, $|E'| \geqslant \alpha_c(T_1(G)) - m(G)$. Hence, $\alpha'(G) \geqslant \alpha_c(T_1(G)) - m(G)$ and so $\alpha_c(T_1(G)) \leqslant m(G) + \alpha'(G)$. Thus, $\alpha_c(T_1(G)) = m(G) + \alpha'(G)$.

By a similar argument we can prove that $\alpha_c(T_k(G)) = km(G) + \alpha'(G)$. □

Theorem 4.5.4. *$T_k(G)$ is a clique perfect graph if and only if G is a bipartite graph.*

Proof. Let $T_k(G)$ be a clique perfect graph. From Lemma 4.5.1 and Lemma 4.5.3, $\tau_c(T_k(G)) = \alpha_c(T_k(G))$ if and only if $\beta(G) = \alpha'(G)$. If H is an induced subgraph of G then $T_k(H)$ is an induced subgraph of $T_k(G)$ and hence for $T_k(G)$ to be clique-perfect, $\beta(H) = \alpha'(H)$ for every induced subgraph H of G. If G contains an induced odd cycle of length $2k + 1, k \geqslant 1$, then $k + 1 = \beta(C_{2k+1}) \neq \alpha'(C_{2k+1}) = k$, which is a contradiction. Therefore, G is bipartite.

Now, let G be bipartite. Then $T_k(G)$ is bipartite for each k, since $T_k(G)$ contains an odd cycle if and only if G contains an odd cycle. For bipartite graphs, the clique transversal number is same as the minimum number of vertices required to cover all edges and the clique independence number is same as the maximum number of independent edges, since all cliques are of size two. Hence by Lemma 1.1.13 and the fact that each induced subgraph of a bipartite graph is bipartite, it follows that $T_k(G)$ is clique perfect.

□

The $< t >$-property of the various classes of graphs which we have studied in this chapter are summarized in the following table.

Class	Satisfy $< t >$-property	Do not satisfy $< t >$-property
Cographs	2, 3	$\geqslant 4$
Clique perfect graphs	2, 3	$\geqslant 4$
Planar graphs	-	2, 3, 4
Perfect graphs	-	$\geqslant 2$

4.6 Highly clique imperfect graphs

A graph G is **highly clique imperfect** if the difference between $\tau_c(G)$ and $\alpha_c(G)$ is arbitrarily large. In [32], a graph F_t satisfying $\tau_c(F_t)-\alpha_c(F_t) = t$, where t is an arbitrary integer is given where the number of vertices in F_t grows exponentially with t. However, the following problem is open [73] :

Problem : For an arbitrary integer t, are there graphs G such that $\tau_c(G)-\alpha_c(G) = t$ where the number of vertices in G is linear in t.

In this section, this problem is solved by constructing a family of such graphs.

For each positive integer t, define G_t as $K_{1,t+1}$ with 5-cycles attached to t distinct pendant vertices of $K_{1,t+1}$ (Fig : 4.4).

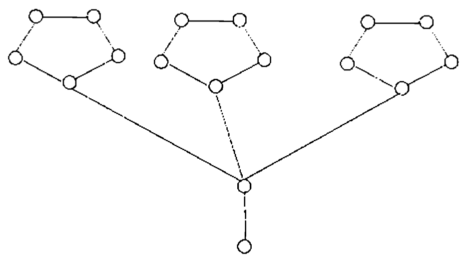


Fig : 4.4

Then $\tau_c(G_t) = 3t + 1$ and $\alpha_c(G_t) = 2t + 1$ so that $\tau_c(G_t) - \alpha_c(G_t) = t$ and the size of G_t is $5t + 2$.

More generally, if $G_{k,t}$ is the graph obtained by replacing the 5-cycles in this example by any odd cycle C_{2k+1}, then $\tau_c(G_{k,t}) = (k+1)t+1, \alpha_c(G_{k,t}) = kt+1$ and the number of vertices in $G_{k,t}$ is $(2k + 1)t + 2$ which is also polynomially bounded in t.

Chapter 5

Clique graphs and cographs

In this chapter the clique graph of cographs are studied and we prove that the diameter of the clique graph of a cograph cannot exceed two. If $n(G) = p$, where p is prime, then G cannot be the clique graph of a cograph except for $G = K_p$. The number of clique graphs of a cograph with $\chi(K(G)) = s$, where s is a fixed integer is finite. A realization of cographs and its clique graph which have specific values for the domination number, the clique transversal number and the clique independence number are given.

5.1 Clique graph of a cograph

Theorem 5.1.1. *If G is a connected cograph then the diameter of $K(G) \leqslant 2$.*

Some results of this chapter are included in the following paper.

Some properties of the clique graph of a cograph, Proceedings of the International Conference on Discrete Mathematics, (2006), Bangalore, India, (to appear).

Proof. Let S_1 and S_2 be any two non-adjacent vertices in $K(G)$. If a vertex in S_1 is adjacent to a vertex in S_2, then there exists a clique S which contains this edge and hence is adjacent to both S_1 and S_2 in $K(G)$. Therefore, $d(S_1, S_2) = 2$. If possible assume that no vertex in S_1 is adjacent to a vertex in S_2. Let $v_1 \in V(S_1)$ and $v_2 \in V(S_2)$. Then, $d(v_1, v_2) = 2$. Hence there exists a vertex v adjacent to both v_1 and v_2. If v_1' is another vertex in $V(S_1)$ then $v_1'v_1vv_2$ should not induce P_4 in G and therefore v_1' is adjacent to v. Since v_1' was arbitrary, every vertex in $V(S_1)$ is adjacent to v. But, this is a contradiction to the maximality of S_1. Hence, for a connected cograph G, diameter of $K(G) \leqslant 2$. □

Theorem 5.1.2. *If G is a connected cograph with prime number of cliques, then G is clique complete.*

Proof. Let $G = G_1 \vee G_2$. The number of cliques in G is the product of the number of cliques in G_1 and G_2. But, the number of cliques in G is prime and hence one of the G_i's must have prime number of cliques and other must be complete. Every clique in G is the join of the cliques of G_1 and G_2. Hence any two cliques in G have a non-empty intersection and therefore the clique graph of G is complete. □

Corollary 5.1.3. *Any graph of prime order, other than K_p, cannot be the clique graph of a cograph.*

Theorem 5.1.4. *A cograph is clique complete if and only if there exists a universal vertex.*

Proof. If there exists a universal vertex in G then that vertex will be present in every clique of G and hence $K(G)$ is complete.

Now, assume that a cograph G is clique complete. Let S be a clique of G with maximum cardinality and ω be its clique number. The proof is by induction on $|V(G) - V(S)|$. If $|V(G) - V(S)| = 0$ then $G(= S)$ itself is complete. If $|V(G) - V(S)| = 1$ then there exist only one vertex v outside S. Since G is connected there exists at least one vertex $u \in S$ which is adjacent to v. Then $\deg(u) = n - 1$. Assume that if $|V(G) - V(S)| = k$ then there exists a vertex of full degree in G.

Now, let $|V(G) - V(S)| = k + 1$ and $v_1, v_2, ..., v_{k+1} \in V(G) - V(S)$. Let G_i be the graph obtained by deleting the vertex v_i for $i \in \{1, 2, .., k + 1\}$. Then $|V(G_i)| = n - 1$ and S is a clique in G_i. Also $|V(G_i) - V(S)| = k$. Therefore by the induction hypothesis, there exists a vertex v_i' of degree $n - 2$ in G_i for all i. Then v_i' belongs to $V(S)$, since it is adjacent to all vertices in G_i and S is maximal complete. If for at least one v_i, v_i is adjacent to v_i', then v_i' will be of full degree in G.

Now, assume that v_i is not adjacent to v_i' for all i and hence $v_i' \neq v_j'$ if $i \neq j$. Consider two arbitrary vertices v_i and v_j where $i \neq j$ and $i, j \in \{1, 2, ..., k+1\}$. If v_i is not adjacent to v_j, then $v_i v_j' v_i' v_j$ is an induced P_4 in G which is a contradiction. Therefore v_i is adjacent to v_j for all $i \neq j$ and $i, j \in \{1, 2, ..., k + 1\}$. Hence $\{v_1, v_2, ..., v_{k+1}\}$ induces a complete graph. So there exists a clique in G which contains all the vertices $v_1, v_2, ..., v_{k+1}$. This clique has non-empty intersection with S, since G is clique complete. Therefore there exists $u \in V(S)$ which in adjacent to v_i for all $i \in 1, 2, ..k + 1$ and hence u will be a vertex of full degree. The proof now follows by the mathematical induction. □

5.2 Chromatic number of the clique graph

Even though the difference between the chromatic numbers of a cograph and its clique graph can be arbitrarily large, the number of clique graphs of a cograph having a fixed chromatic number is finite.

Remark 5.2.1. Given any two positive integers $a, b > 1$, there exists a cograph G such that $\chi(G) = a$ and $\chi(K(G)) = b$. Let $G = K_a$ with $b - 1$ pendant vertices attached to one of its vertices. Therefore, $K(G) = K_b$ and hence $\chi(G) = a$ and $(K(G)) = b$.

Theorem 5.2.1. *The number of clique graphs of a connected cograph G with $\chi(K(G)) = s$ is finite.*

Proof. Let G be a cograph with $\chi(K(G)) = s$. Let $G = G_1 \vee G_2$ be a decomposition of G. Let the number of cliques of G_i be p_i for $i = 1, 2$. If $p_i > s$ for some i, say $i = 1$, then G_1 will have at least $s + 1$ cliques, $H_{11}, H_{12}, \ldots\ldots, H_{1,s+1}$. Let H_2 be a clique of G_2. Then $H_{11} \vee H_2, H_{12} \vee H_2, \ldots\ldots, H_{1,s+1} \vee H_2$ are cliques of G which induce K_{s+1} in $K(G)$. But, then $\chi(K(G)) \geqslant s + 1$ which is a contradiction. Therefore each $p_i \leqslant s$ and hence $|V(K(G))| \leqslant s^2$. Hence, the number of clique graphs of a connected cograph with $\chi(K(G)) = s$ is finite. □

5.3 Some graph parameters

In this section we study the relation between the domination number, the clique transversal number and the clique independence number of a cograph and its clique graph. It is also observed that, though cographs are clique perfect and the clique graph of a cograph satisfies $\tau_c(K(G)) = \alpha_c(K(G))$, they are not clique perfect.

Theorem 5.3.1. *There exists a cograph G such that $\gamma(G) = a$ and $\gamma(K(G)) = b$ if and only if*

(1) $a \leqslant 2$.

(2) $a = 1$ *if and only if* $b = 1$.

(3) $a = 2$ *and* $b \geqslant a$.

Proof. If G is a cograph then $\gamma(G) \leqslant 2$ [66]. Therefore (1) holds. If $\gamma(G) = 1$ then G has a vertex of full degree and hence $K(G)$ is complete. Therefore, $a = 1$ implies that $b = 1$. If $b = 1$ then $K(G)$ has a vertex of full degree. Let C be the clique in G which corresponds to this vertex of full degree in $K(G)$. Let $u_1, u_2, ..., u_p \in V(G) - V(C)$. Every clique in G intersects with C and hence u_i s for $i = 1, 2, ..., p$ must be adjacent to at least one vertex of $V(C)$.

Claim : Every u_i is adjacent to a common vertex $v \in V(G)$.

On the contrary, assume that u_1 and u_2 do not have a common neighbor in C. Let u_1 be adjacent to v_1 and u_2 be adjacent to v_2. But, $u_1v_1v_2u_2$ cannot induce a P_4 in G and hence u_1 is adjacent to u_2. Since, u_1 and u_2 have no common neighbors in C, the clique of G which contains the edge u_1u_2 does not intersect C which is a contradiction. Therefore, our claim holds.

Therefore, v is a vertex of full degree in G and hence $a = \gamma(G) = 1$. Hence, (2) holds.

If $a = 2$ then $b \neq 1$ by (2). Therefore, $b \geqslant a$ and (3) holds.

Conversely, assume that a and b satisfy the given conditions. Let G be the cograph $K_{b,b}$. The clique graph of $K_{b,b}$, $K(K_{b,b}) = K_b \Box K_b$. Therefore, $\gamma(K(K_{b,b})) = b$. If $b > 1$ then $\gamma(G) = 2$ and if $b = 1$ then $\gamma(G) = 1$. Hence, G is the required graph. □

Theorem 5.3.2. *If G is a cograph then $\tau_c(K(G)) = \alpha_c(K(G))$.*

Proof. We use the recursive definition of cographs to prove the theorem. If $G = K_1$, then $K(G) = K_1$ and $\tau_c(K_1) = \alpha_c(K_1) = 1$.

Let G_1 and G_2 be cographs which satisfy $\tau_c(K(G_i)) = \alpha_c(K(G_i))$ for $i = 1, 2$. Let $G = G_1 \cup G_2$. Then, $K(G) = K(G_1) \cup K(G_2)$ and hence $\tau_c(K(G)) = \tau_c(K(G_1)) + \tau_c(K(G_2)) = \alpha_c(K(G_1)) + \alpha_c(K(G_2)) = \alpha_c(K(G))$.

Let $G = G_1 \vee G_2$. Let H_1 be a clique in $K(G_1)$ induced by the vertices corresponding to the cliques $G_{11}, G_{12}, ...G_{1k}$ in G_1. Let $G_{21}, G_{22}, ..., G_{2t}$ be the cliques in G_2. Therefore, $\{G_{1i} \vee G_{2j} : i = 1, 2, ...k$ and $j = 1, 2, ..., t\}$ are cliques in $G_1 \vee G_2$ and the vertices corresponding to these cliques induce a clique in $K(G_1 \vee G_2)$. Let this clique be H_1'. Similarly, if H_2 is a clique in $K(G_1)$, then we can find a clique H_2' in $K(G_1 \vee G_2)$. Moreover, if H_1 and H_2 are independent, then H_1' and H_2' are also independent. Therefore, $\alpha_c(K(G_1 \vee G_2)) \geqslant \alpha_c(K(G_1))$. Similarly we can prove that $\alpha_c(K(G_1 \vee G_2)) \geqslant \alpha_c(K(G_2))$. Therefore, $\alpha_c(K(G_1 \vee G_2)) \geqslant max\{\alpha_c(K(G_1)), \alpha_c(K(G_2))\}$. Using similar arguments, we can prove that $\tau_c(K(G_1 \vee G_2)) \leqslant max\{\tau_c(K(G_1)), \tau_c(K(G_2))\}$. Therefore, $\tau_c(K(G_1) \vee K(G_2)) \leqslant \alpha_c(K(G_1) \vee K(G_2)$. But, by definition, $\tau_c(K(G_1) \vee K(G_2)) \geqslant \alpha_c(K(G_1) \vee K(G_2)$. Therefore, $\tau_c(K(G_1) \vee K(G_2)) = \alpha_c(K(G_1) \vee K(G_2)$.

Hence the theorem. □

Remark 5.3.1. If a and b are any two positive real numbers which satisfies the conditions $a = 1$ if and only if $b = 1$ and $a \leqslant b$ then there exist cographs which satisfies $\tau_c(G) = \alpha_c(G) = a$ and $\tau_c(K(G)) = \alpha_c(K(G)) = b$. For example, consider the cograph $K_{a,b}$. $K(K_{a,b}) = K_a \square K_b$. Therefore, $\tau_c(K_{a,b}) = \alpha_c(K_{a,b}) = a$ and $\tau_c(K(K_{a,b})) = \alpha_c(K(K_{a,b})) = b$.

An interesting observation : Despite Theorem 5.3.2 and Lemma 1.1.8, $K(G)$ of a cograph G need not be clique perfect. For example consider the cograph $G = (K_1 \cup C_4) \vee (2K_1 \cup P_3)$ as in Fig : 5.1.

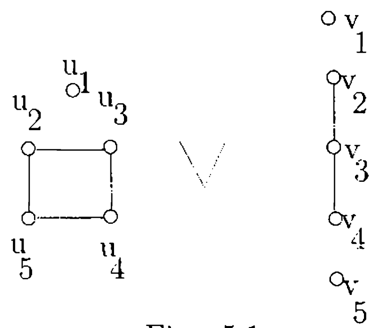


Fig : 5.1

The cliques of G formed by the vertices $\{u_1, v_1\}$, $\{u_1, v_2, v_3\}$, $\{u_4, u_5, v_3, v_4\}$, $\{u_3, u_4, v_5\}$ and $\{u_2, u_3, v_1\}$ induce a C_5 in $K(G)$ and hence $K(G)$ is not clique perfect.

Chapter 6

Clique irreducible and weakly clique irreducible graphs

This chapter deals with two graph classes - the clique irreducible graphs and the weakly clique irreducible graphs. A new graph class called the clique vertex irreducible graphs is also defined. We characterize line graphs and its iterations, Gallai graphs, anti-Gallai graphs and its iterations, cographs and distance hereditary graphs which are clique irreducible, clique vertex irreducible and weakly clique irreducible graphs.

Some results of this chapter are included in the following papers.

(1) Clique irreducibility and clique vertex irreducibility of graphs, (communicated).

(2) Clique irreducibility of some iterative classes of graphs, Discuss. Math. Graph Theory, (to appear).

(3) On weakly clique irreducible graphs, (communicated).

6.1 Iterations of the line graph

In this section the line graphs and all its iterations which are clique irreducible and clique vertex irreducible are characterized.

Theorem 6.1.1. *Let G be a graph. The line graph $L(G)$ is clique vertex irreducible if and only if G satisfies the following conditions.*
(1) Every triangle in G has at least two vertices of degree two.
(2) Every vertex of degree greater than one in G has a pendant vertex attached to it, except for the vertices of degree two lying in a triangle.

Proof. Let G be a graph which satisfies the conditions (1) and (2). The cliques of $L(G)$ are induced by the vertices corresponding to the edges in G which are incident on a vertex of degree at least three, the edges in G which are incident on a vertex of degree two and which do not lie in a triangle and by the edges in G which lie in a triangle. By (2), the cliques in $L(G)$ induced by the vertices corresponding to the edges in G which are incident on a vertex, have a vertex which does not lie in any other clique of $L(G)$. By (1), the cliques in $L(G)$ induced by the vertices which correspond to the edges in G which lie in a triangle, have a vertex which does not lie in any other clique of $L(G)$. Therefore, G is clique vertex irreducible.

Conversely, assume that $L(G)$ is a clique vertex irreducible graph. Let $< u_1, u_2, u_3 >$ be a triangle in G. Let e_1, e_2, e_3 be the vertices in $L(G)$ which correspond to the edges u_1u_2, u_2u_3, u_3u_1 in G. $T =< e_1, e_2, e_3 >$ is a clique in $L(G)$. If $d(u_i) > 2$ for two u_is, u_1 and u_2, then there exist v_1 and v_2 (not necessarily different, but different from u_3) such that u_i is adjacent to v_i for $i = 1, 2$. But then, the vertices e_1 and e_3 will be present in the clique induced by the edges incident on the vertex u_1 and the vertices e_2 and e_3 will be present in the clique induced by the edges incident on the vertex u_2. Therefore, every vertex in T belongs to another clique in $L(G)$ which is a contradiction to the assumption that $L(G)$ is clique vertex irreducible. Hence every triangle in G has at least two vertices of degree two.

Now, let $u \in V(G)$ and $N(u) = \{u_1, u_2, ..., u_p\}$, where $p \geqslant 2$ and if $p = 2$ then u_1 is not adjacent to u_2. Let e_i be the vertex in $L(G)$ corresponding to the edge uu_i in G for $i = 1, 2, ..., p$. Let C be the clique $< e_1, e_2, ..., e_p >$ in $L(G)$. If u has no pendant vertex attached to it then every u_i has a neighbor $v_i \neq u$ for $i = 1, 2, ..., p$. The v_is are not necessarily pairwise different. Moreover, some v_i can be equal to some u_j with $j \neq i$, except in the case $p = 2$. Therefore, for each i, every e_i in $L(G)$ will be present in another clique, either induced by the edges incident on the vertex u_i in G or by the edges in a triangle containing u and u_i in G. But this is a contradiction to the assumption that $L(G)$ is clique vertex irreducible. Hence, every vertex of degree greater than one in G has a pendant vertex attached to it, except for the vertices of degree two which lie in a triangle. □

Fig : 6.1 gives an example of a graph whose line graph is clique vertex irreducible.

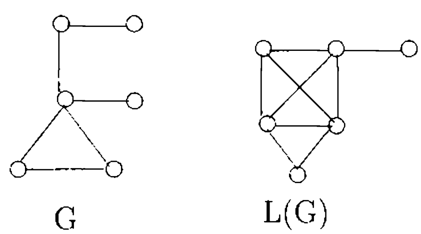
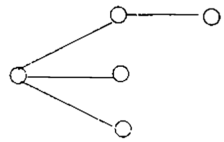


Fig : 6.1

Theorem 6.1.2. *Let G be a connected graph. The second iterated line graph $L^2(G)$ is clique vertex irreducible if and only if G is one of the following graphs.*

(i) K_2 *(ii)* K_3 *(iii)* P_3 *(iv)* P_4 *(v)* P_5 *(vi)* $K_{1,3}$ *(vii)*



Proof. By Theorem 6.1.1, $L^2(G)$ is clique vertex irreducible if and only if
(1) Every triangle in $L(G)$ has at least two vertices of degree two.
(2) Every vertex of degree greater than one in $L(G)$ has a pendant vertex attached to it, except for the vertices of degree two which lie in a triangle.

By (2), every non-pendant edge in G must have a pendant edge attached to it on one end vertex and the degree of that end vertex must be two.

Case 1 : $L(G)$ has a triangle.

A triangle in $L(G)$ corresponds to a triangle or a $K_{1,3}$ (need not be induced) in G. Let it correspond to a triangle in G. If any of the vertices of this triangle has a neighbor outside the triangle, then two vertices in the corresponding triangle in $L(G)$ have neighbors outside the triangle, which is a contradiction. Therefore, since G is connected, in this case G must be K_3.

If the triangle in $L(G)$ corresponds to a $K_{1,3}$ in G, then two of the edges of this $K_{1,3}$ cannot have any other edge incident on any of its end vertices. Therefore. G cannot have a vertex of degree greater than three. Moreover, two vertices of $K_{1,3}$ in G must be pendant vertices. Again, by (2) and since G is connected, we conclude that G is either $K_{1,3}$ or the graph (vii).

Case 2 : $L(G)$ has no triangle.

Since $L(G)$ has no triangle, G cannot have a K_3 or a vertex of degree greater than or equal to 3. Therefore, since G is connected, G must be a path or a cycle of length greater than three. Again, by (2), G cannot be a path of length greater than five or a cycle. Therefore G is K_2, P_3, P_4 or P_5. □

Corollary 6.1.3. *Let G be a connected graph. The n^{th} iterated line graph $L^n(G)$ is clique vertex irreducible if and only if G is $K_3, K_{1,3}$ or P_k where $n+1 \leqslant k \leqslant n+3$, for $n \geqslant 3$.*

Theorem 6.1.4. *The line graph $L(G)$ is clique irreducible if and only if every triangle in G has a vertex of degree two.*

Proof. Let G be a graph such that every triangle in G has a vertex of degree two. Let C be a clique in $L(G)$.

Case 1 : The clique C is induced by the vertices corresponding to the edges in G which are incident on a vertex of degree at least three.

An edge of C can be present in another clique of $L(G)$ if and only if the corresponding pair of edges in G lies in a triangle. Thus, if every edge of C lies in another clique of $L(G)$, then G has an induced K_p, where p is at least four. But, this contradicts the assumption that every triangle in G has a vertex of degree two.

Case 2 : The clique C is induced by the vertices corresponding to the edges in G which are incident on a vertex of degree two and which do not lie in a triangle.

In this case, C is K_2 which always has an edge of its own.

Case 3 : The clique C is induced by the vertices corresponding to the edges which lie in a triangle T in G.

Since T has a vertex v of degree two, the vertices corresponding to the edges which are incident on v induce an edge in C which does not lie in any other clique of $L(G)$.

Therefore, G is clique irreducible.

Conversely, assume that G is a clique irreducible graph. Let $< u_1, u_2, u_3 >$ be a triangle in G. Let e_1, e_2, e_3 be the vertices in $L(G)$ which correspond to the edges u_1u_2, u_2u_3, u_3u_1 of G. $T =< e_1, e_2, e_3 >$ is a clique in $L(G)$. If $d(u_i) > 2$ for each i, there exist v_1, v_2, v_3 such that u_i is adjacent to v_i for $i = 1, 2, 3$ (v_1, v_2 and v_3 are not necessarily different, but they are different from u_1, u_2 and u_3). Then the edges e_1e_2, e_2e_3 and e_3e_1 of $L(G)$ will be present in the cliques induced by edges which are incident on the vertices u_1, u_2 and u_3 respectively. Therefore, every edge in T is in another clique of $L(G)$, which is a contradiction. □

Theorem 6.1.5. *The second iterated line graph $L^2(G)$ is clique irreducible if and only if G satisfies the following conditions.*

(1) Every triangle in G has at least two vertices of degree two.

(2) Every vertex of degree three has at least one pendant vertex attached to it.

(3) G has no vertex of degree greater than or equal to four.

Proof. Let G be a graph such that $L^2(G)$ is clique irreducible. By Theorem 6.1.4, every triangle in $L(G)$ has a vertex of degree two. Then, we have the following cases.

Case 1 : The triangle in $L(G)$ corresponds to a triangle in G.

Let $< u_1, u_2, u_3 >$ be a triangle in G. Let e_1, e_2, e_3 be the vertices in $L(G)$ which correspond to the edges u_1u_2, u_2u_3, u_3u_1 of G. At least one of the vertices of the triangle $< e_1, e_2, e_3 >$ in $L(G)$ must be of degree two. Let e_1 be a vertex of degree two in $L(G)$. Since e_2 and e_3 belong to $N(e_1)$ in $L(G)$, e_1 has no other neighbors in $L(G)$. Therefore, the corresponding end vertices, u_1 and u_2 in G have no other neighbors. Hence (1) holds.

Case 2 : The triangle in $L(G)$ corresponds to a $K_{1,3}$ (need not be induced) in G.

Let e_1, e_2, e_3 be the vertices in $L(G)$ corresponding to the edges uu_1, uu_2, uu_3 in G. At least one of the vertices of the triangle $< e_1, e_2, e_3 >$ in $L(G)$ must be of degree two. Let e_1 be a vertex of degree two in $L(G)$. Vertices e_2 and e_3 belong to $N(e_1)$ in $L(G)$ and hence e_1 has no other neighbors in $L(G)$. Therefore, the corresponding end vertices, u and u_1 in G have no other neighbors. Since u has no other neighbors (3) holds and since u_1 has no other neighbors (2) holds.

Conversely, assume that G is a graph which satisfies all the three conditions. A triangle in $L(G)$ corresponds to a triangle or a $K_{1,3}$ (need not be induced) in G. A triangle in $L(G)$ which corresponds to a triangle in G has at least one vertex of degree two by (1). Again, a triangle in $L(G)$ which corresponds to a $K_{1,3}$ in G has at least one vertex of degree two by (2) and (3). Therefore, every triangle in $L(G)$ has at least one vertex of degree two and by Theorem 6.1.4, $L^2(G)$ is clique irreducible. □

Theorem 6.1.6. *Let G be a connected graph. If $G \neq K_3$ then, $L^3(G)$ is clique irreducible if and only if G satisfies the following conditions.*

(1) G is triangle free.

(2) G has no vertex of degree greater than or equal to four.

(3) At least two of the vertices of every $K_{1,3}$ in G are pendant vertices.

(4) If uv is an edge in G, then either u or v has degree less than or equal to two.

Proof. Let $L^3(G)$ be clique irreducible. By Theorem 6.1.5, $L(G)$ satisfies,

(1') Every triangle in $L(G)$ has at least two vertices of degree 2.

(2') Every vertex of degree three in $L(G)$ has at least one pendant vertex attached to it.

(3') $L(G)$ has no vertex of degree greater than or equal to 4.

A triangle in $L(G)$ corresponds to a triangle or a $K_{1,3}$ (need not be induced) in G. Every triangle in $L(G)$ has at least two vertices of degree two implies that every triangle in G has its three vertices of degree two. i.e: G is a triangle, because G is connected. Since $G \neq K_3$, G must be triangle free. Also, every $K_{1,3}$ in G has at least two pendant vertices and the degree of a vertex cannot exceed three. Therefore (1), (2) and (3) hold. Again (3') implies that no edge in G can have more than three edges incident on its end vertices. Therefore, (4) holds.

Conversely, assume that the given conditions hold. Since G is triangle free, a triangle in $L(G)$ corresponds to a $K_{1,3}$ (need not be induced) in G. Therefore, by (2) and (3) every triangle in $L(G)$ has at least two vertices of degree two.

Let e be a vertex of degree three in $L(G)$ and let uv be the corresponding edge in G. Since e is of degree three in $L(G)$, the number of edges incident on u in G together with the number of edges incident on v in G is three. If u (or v) has three more edges incident on it then u (or v) will be of degree at least four which is a contradiction to the condition (2). Therefore, u has two neighbors and v has one neighbor (or vice versa) in G. Let u_1 and u_2 be the neighbors of u, and let v_1 be the neighbor of v in G. Then $< u, v, u_1, u_2 > = K_{1,3}$ in G and hence at least two of v, u_1 and u_2 must be pendant vertices. Since v is not a pendant vertex, u_1 and u_2 must be pendant vertices. Therefore, e has two pendant vertices attached to it in $L(G)$ corresponding to the edges uu_1 and uu_2 in G. Hence (2') is satisfied.

Again, (2), (3) and (4) together imply (3'). Since the conditions (1'), (2') and (3') are satisfied, by Theorem 6.1.5, $L^3(G)$ is clique irreducible. □

Theorem 6.1.7. *Let G be a connected graph. The fourth iterated line graph $L^4(G)$ is clique irreducible if and only if G is $K_3, K_{1,3}, P_n$ with $n \geqslant 5$ or C_n with $n \geqslant 4$.*

Proof. Let $L^4(G)$ be clique irreducible. Then by Theorem 6.1.6, if $L(G) \neq K_3$ then $L(G)$ must be triangle free. If $L(G) = K_3$ then G is either K_3 or $K_{1,3}$. If $L(G)$ is triangle free then G is triangle free and cannot have vertices of degree greater than or equal to three. Therefore, G is either a path or a cycle of length greater than three.

Conversely, if G is $K_3, K_{1,3}, P_n$ or C_n then $L^4(G)$ is either a triangle, a path or a cycle and all of them are clique irreducible. □

Corollary 6.1.8. *For $n \geqslant 5$, $L^n(G)$ is clique irreducible if and only if it is non-empty and $L^4(G)$ is clique irreducible.*

6.2 Gallai graphs

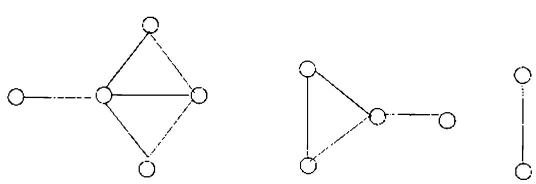
In this section, we give structural and forbidden subgraph characterizations for the Gallai graph to be clique irreducible, clique vertex irreducible and weakly clique irreducible.

Theorem 6.2.1. *The Gallai graph $\Gamma(G)$ is clique vertex irreducible if and only if for every $v \in V(G)$, every maximal independent set I in $N(v)$ with $|I| \geqslant 2$ contains a vertex u such that $N(u) - \{v\} = N(v) - I$.*

Proof. Let G be a graph such that its Gallai graph $\Gamma(G)$ is clique vertex irreducible. A clique C in $\Gamma(G)$ of size at least two is induced by the vertices corresponding to the edges which are incident on a common vertex $v \in V(G)$ whose other end vertices form a maximal independent set I of size at least two in $N(v)$. Let $I = \{v_1, v_2, ..., v_p\}$, where $p \geqslant 2$, be a maximal independent set in $N(v)$. Let e_i be the vertex in $\Gamma(G)$ corresponding to the edge vv_i in G for $i = 1, 2, ..., p$. Let C be the clique $< e_1, e_2, ..., e_p >$ in $\Gamma(G)$. Let e_i be the vertex in C which does not belong to any other clique in G. Therefore, e_i has no neighbors in $\Gamma(G)$ other than those in C. Hence, $N(v_i) - \{v\} = N(v) - I$.

Conversely, assume that for every $v \in V(G)$, every maximal independent set $I = \{v_1, v_2, ..., v_p\}$ in $N(v)$ contains a vertex u such that $N(u) - \{v\} = N(v) - I$. If C is a clique of size one, it contains a vertex of its own. Otherwise, let C be defined as above. By our assumption, there exists a vertex $u = v_i$ such that $N(u) - \{v\} = N(v) - I$. Therefore, e_i has no neighbors outside C. Hence C has a vertex e_i of its own. □

Fig : 6.2 gives an example of a graph whose Gallai graph is clique vertex irreducible.



G Γ(G)

Fig : 6.2

Theorem 6.2.2. *If $\Gamma(G)$ is clique vertex reducible then G contains one of the graphs in Fig : 6.3 as an induced subgraph.*

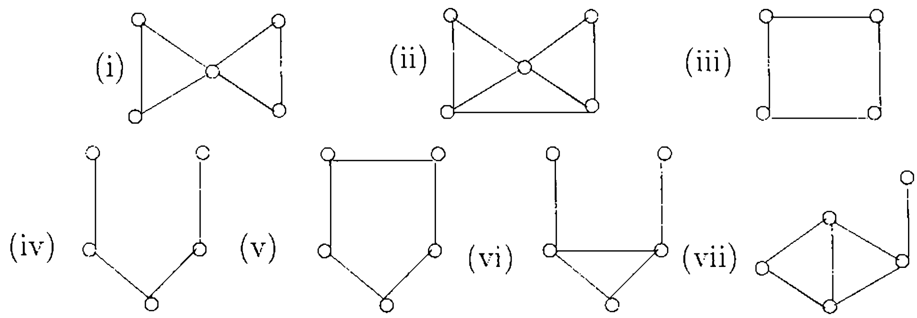


Fig : 6.3

Proof. Let G be a graph such that $\Gamma(G)$ is clique vertex reducible and let C be a clique in $\Gamma(G)$ such that each vertex of C belongs to some other clique in $\Gamma(G)$. Consider the order relation $\preceq$ among the vertices of C where $e \preceq e'$ if $N[e] \preceq N[e']$. If $\preceq$ is a total ordering, then every vertex adjacent to the minimum vertex e is also adjacent to all the vertices in C. Therefore, by maximality of C, e cannot have neighbors outside C. This is a contradiction to the assumption that e belongs to some other clique of $\Gamma(G)$. So, there exist two vertices e_1 and e_2 in C which are not comparable. That is, there exist vertices f_1 and f_2 of $\Gamma(G)$ such that e_i is adjacent to f_j if and only if $i = j$. Let vv_1 and vv_2 be the edges corresponding to e_1 and e_2, respectively. Then v_1 and v_2 are non-adjacent. Let u_1 and u_2 be the end points of f_1 and f_2, respectively, which are both different from v, v_1 and v_2.

Case 1 : Both f_1 and f_2 correspond to the edges incident to v.

In this case, u_1 and u_2 are adjacent to v, u_i is adjacent to v_j if and only if $i \neq j$ and u_1 and u_2 can be either adjacent or not. Therefore $< v, v_1, v_2, u_1, u_2 >$ is the graph (i) or (ii) in Fig : 6.3.

Case 2 : None of f_1 and f_2 correspond to the edges incident to v.

In this case, u_1 and u_2 are adjacent to v_1 and v_2, respectively, and not to v. If

$u_1 = u_2$ then G contains an induced C_4. If $u_1 \neq u_2$ and G does not contain an induced C_4, then $< v, v_1, v_2, u_1, u_2 >$ is either P_5 or C_5.

Case 3 : Exactly one of f_1 and f_2 correspond to the edges incident to v, say f_1.

In this case, u_1 is adjacent to both v and v_2 and is not adjacent to v_1. The vertex u_2 is adjacent to v_2 and is not adjacent to v. If u_2 is adjacent to v_1 then G contains an induced C_4. Otherwise, $< v, v_1, v_2, u_1, u_2 >$ is the graph (vi) or (vii) in Fig : 6.3. □

Remark 6.2.1. The converse need not be true. For example consider the graph G in Fig : 6.4. It contains (iv) in Fig : 6.3 as an induced subgraph. Still $\Gamma(G)$ is clique vertex irreducible.

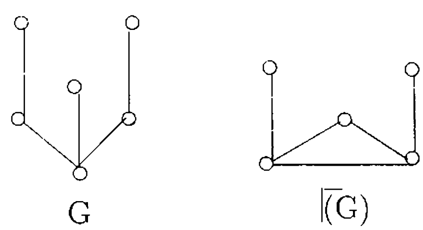


Fig : 6.4

Theorem 6.2.3. *The Gallai graph $\Gamma(G)$ is clique irreducible if and only if for every $v \in V(G)$, $< N(v) >^c$ is clique irreducible.*

Proof. A clique C in $\Gamma(G)$ of size at least two is induced by the vertices corresponding to the edges which are incident on a common vertex $v \in V(G)$ whose other end vertices form a maximal independent set I of size at least two in $N(v)$. Therefore, C has an edge which does not belong to any other clique of $\Gamma(G)$ if and only if I has a pair of vertices both of which together does not belong to any other maximal independent set in $N(v)$. But, this happens if and only if every clique of size at least two in $< N(v) >^c$ has an edge which does not belong to any other clique in $< N(v) >^c$, since a maximal independent set in a graph corresponds to a clique in its complement. □

Theorem 6.2.4. *The second iterated Gallai graph $\Gamma^2(G)$ is clique irreducible if and only if for every $uv \in E(G)$, either $< N(u) - N(v) >$ and $< N(v) - N(u) >$ are clique vertex irreducible or one among them is a clique and the other is clique irreducible.*

Proof. By Theorem 6.2.3, $\Gamma^2(G)$ is clique irreducible if and only if for every $e \in V(\Gamma(G))$, $< N(e) >^c$ is clique irreducible.

Let $e = uv \in E(G)$, $N(u) - N(v) = \{u_1, u_2, ..., u_p\}$ and $N(v) - N(u) = \{v_1, v_2, ..., v_l\}$. Also let $e_i = uu_i$ for $i = 1, 2, ..., p$ and $f_j = vv_j$ for $j = 1, 2, ..., l$. $N_{\Gamma(G)}(e) = \{e_1, e_2, ..., e_p, f_1, f_2, ..., f_l\}$. $< N(e) >^c$ is clique irreducible if and only if every maximal independent set I in $< N(e) >$ has a pair of vertices of its own. e_i is not adjacent to e_j if and only if u_i is adjacent to u_j. Similarly, f_i is not adjacent to f_j if and only if v_i is adjacent to v_j. So, $I = \{e_{i_1}, e_{i_2}, ..., e_{i_k}, f_{j_1}, f_{j_2}, ..., f_{j_l}\}$ if and only if $\{u_{i_1}, u_{i_2}, ..., u_{i_k}\}$ is a clique in $< N(u) - N(v) >$ and $\{v_{j_1}, v_{j_2}, ..., v_{j_l}\}$ is a clique in $N(v) - N(u)$. Therefore, every maximal independent set I in $N_{\Gamma(G)}(e)$ has a pair of vertices of its own if and only if either both $< N(u) - N(v) >$ and $< N(v) - N(u) >$ are clique vertex irreducible or one among them is a clique and the other is clique irreducible. □

Theorem 6.2.5. *If $\Gamma(G)$ is clique reducible then G contains one of the following graphs as an induced subgraph.*

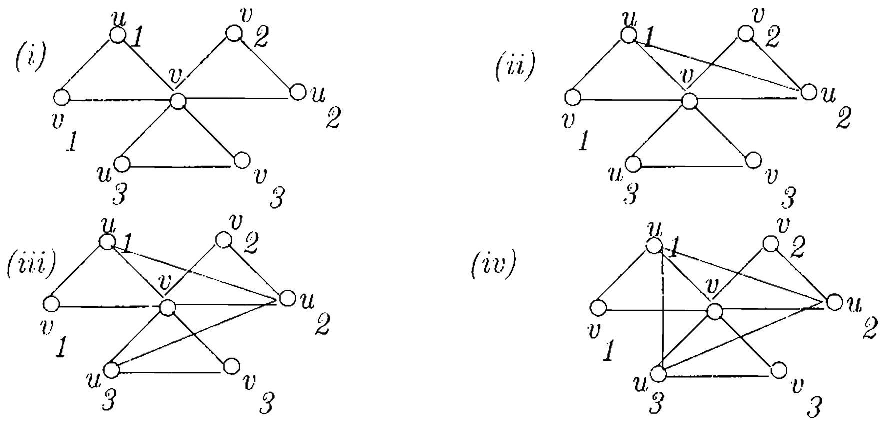


Fig : 6.5

Proof. Let $\Gamma(G)$ be a clique reducible graph. By Lemma 1.1.9 and Lemma 1.1.12, $\Gamma(G)$ contains at least one of the Hajo's graph as an induced subgraph. A Hajo's graph is an induced subgraph of $\Gamma(G)$ if and only if G contains one of the graphs in Fig : 6.5 as an induced subgraph. Hence the theorem. □

Remark 6.2.2. The converse need not be true. Let G be the graph in Fig : 6.6.

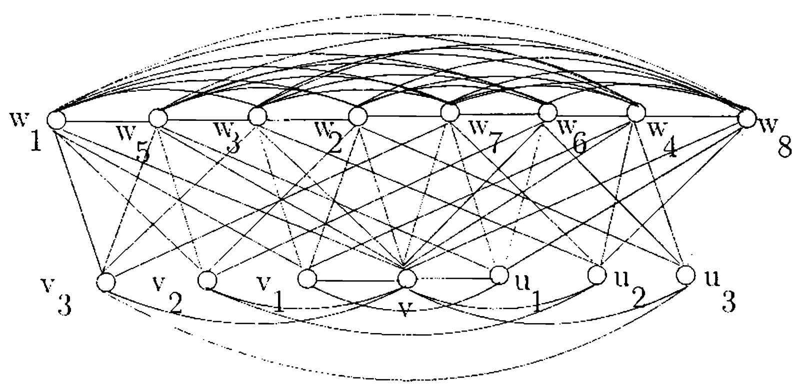


Fig : 6.6

$V(G) = \{v, v_1, v_2, v_3, u_1, u_2, u_3, w_1, w_2, w_3, w_4, w_5, w_7, w_7, w_8\}$. Let $< v, v_1, v_2, v_3, u_1, u_2, u_3 >$ be the graph (i) in Fig : 6.5 and let w_is for $i = 1, 2, ..., 8$ induce a complete graph.

Also, let w_1 be adjacent to $\{v_1, v_2, v_3\}$, w_2 be adjacent to $\{v_1, v_2, u_3\}$, w_3 be adjacent to $\{v_1, u_2, v_3\}$, w_4 be adjacent to $\{v_1, u_2, u_3\}$, w_5 be adjacent to $\{u_1, v_2, v_3\}$, w_6 be adjacent to $\{u_1, v_2, u_3\}$, w_7 be adjacent to $\{u_1, u_2, v_3\}$, w_8 be adjacent to $\{u_1, u_2, u_3\}$ and v adjacent to w_i for $i = 1, 2, ..., 8$.

In $\Gamma(G)$ the vertices corresponding to the edges with one end vertex v induces K_6 minus a perfect matching in which the vertices of each of the eight triangles are adjacent to another vertex each. The remaining vertices induce the graph $H = 4K_{1,8}$. Therefore, $\Gamma(G)$ is clique irreducible.

Theorem 6.2.6. *The Gallai graph of a graph G, $\Gamma(G)$ is weakly clique irreducible if and only if for every vertex $u \in V(G)$, $< N(u) >^c$ is weakly clique irreducible.*

Proof. Let G be a graph such that $\Gamma(G)$ is weakly clique irreducible. Let u_1u_2 be an edge in $< N(u) >^c$ and let e_i be the vertex in $\Gamma(G)$ corresponding to the edge uu_i in G for $i = 1, 2$. Since $\Gamma(G)$ is weakly clique irreducible and e_1e_2 is an edge in $\Gamma(G)$, let $C =< e_1, e_2, ..., e_k >$ be the essential clique in $\Gamma(G)$ which contains the edge e_1e_2. For $i = 3, 4, ..., k$, let uu_i be the edge in G corresponding to the vertex e_i in $\Gamma(G)$. Let e_ie_j be the essential edge in C. Therefore, there exist no independent set in $N(u)$ which contains both the vertices u_i and u_j. Hence, there is no clique in $< N(u) >^c$ which contains the edge u_iu_j, other than the clique $S =< u_1, u_2, ..., u_k >$. Therefore, S is an essential clique in $< N(u) >^c$ which contains the edge u_1u_2. Since the edge u_1u_2 was arbitrary, $< N(u) >^c$ is weakly clique irreducible.

The converse can be proved similarly. □

6.3 Iterations of the anti-Gallai graph

In this section the anti-Gallai graph and all its iterations which are clique irreducible, clique vertex irreducible and weakly clique irreducible are characterized.

Theorem 6.3.1. *The anti-Gallai graph $\Delta(G)$ is clique vertex irreducible if and only if G does neither contain K_4 nor one of the Hajo's graphs as an induced subgraph.*

Proof. Let G be a graph which does neither contain K_4 nor one of the Hajo's graphs as an induced subgraph. The cliques of $\Delta(G)$ are induced by the vertices corresponding to the edges of G incident on a vertex of degree at least 3 whose other end vertices induce a complete graph and by the vertices corresponding to the edges which lie in a triangle. In the first case G contains an induced K_4, which is a contradiction. Therefore, the cliques of $\Delta(G)$ are induced by the edges which lie in a triangle. Let $< u_1, u_2, u_3 >$ be a triangle in G. Let e_1, e_2, e_3 be the vertices in $\Delta(G)$ corresponding to the edges u_1u_2, u_2u_3, u_3u_1 in G. Then $< e_1, e_2, e_3 >$ is a clique in $\Delta(G)$. If a vertex e_i for $i = 1, 2, 3$ lies in another clique of $\Delta(G)$, then the edge corresponding to e_i lies in another triangle. Therefore, the end vertices of the edge corresponding to e_i in G has a neighbor v_i for $i = 1, 2, 3$. $v_i \neq v_j$ if $i \neq j$ and v_1, v_2, v_3 are not adjacent to u_3, u_1, u_2, respectively, since otherwise G contains a K_4, which is a contradiction. Then, $< u_1, u_2, u_3, v_1, v_2, v_3 >$ is one of the Hajo's graphs, a contradiction. Hence, G is clique vertex irreducible.

Conversely, assume that G is clique vertex irreducible. If G contains K_4 or one of the Hajo's graphs as an induced subgraph, then there exists a clique in $\Delta(G)$, corresponding to a triangle in G, which shares each of its vertices with some other clique of $\Delta(G)$. □

Lemma 6.3.2. *If G is K_4-free then $\Delta(G)$ is diamond free.*

Proof. Let G be a graph which does not contain K_4 as an induced subgraph. Therefore, a triangle in $\Delta(G)$ can only be induced by a triangle in G. If two vertices of the triangle in $\Delta(G)$ have a common neighbor, then it forces G to have a K_4, a contradiction. Therefore, $\Delta(G)$ is diamond free. □

Theorem 6.3.3. *The second iterated anti-Gallai graph $\Delta^2(G)$ is clique vertex irreducible if and only if G does not contain K_4 as an induced subgraph.*

Proof. By Theorem 6.3.1, $\Delta^2(G)$ is clique vertex irreducible if and only if $\Delta(G)$ does neither contain K_4 nor one of the Hajo's graphs as an induced subgraph.

Let G be a graph which does not contain K_4 as an induced subgraph. Therefore, G does not contain K_5 as an induced subgraph and hence $\Delta(G)$ does not contain K_4 as an induced subgraph. Again, by Lemma 6.3.2, $\Delta(G)$ cannot have diamond as an induced subgraph and hence it does not contain any of the Hajo's graph as an induced subgraph. Hence, $\Delta^2(G)$ is clique vertex irreducible.

Conversely, assume that $\Delta^2(G)$ is clique vertex irreducible. If G contains K_4 as an induced subgraph then in $\Delta(G)$ the vertices corresponding to the edges of this K_4 induce K_6 minus a perfect matching which is the fourth Hajo's graph, a contradiction. Therefore, G does not contain K_4 as an induced subgraph. □

Theorem 6.3.4. *The n^{th} iterated anti-Gallai graph $\Delta^n(G)$ is clique vertex irreducible if and only if G does not contain K_{n+2} as an induced subgraph.*

Proof. By Theorem 6.3.3, $\Delta^n(G)$ is clique vertex irreducible if and only if $\Delta^{n-2}(G)$

does not contain K_4 as an induced subgraph. $\Delta^{n-2}(G)$ does not contain K_4 as an induced subgraph if and only if $\Delta^{n-3}(G)$ does not contain K_5 as an induced subgraph. Proceeding like this, we get that $\Delta(G)$ does not contain K_{n+1} as an induced subgraph if and only if G does not contain K_{n+2} as an induced subgraph. Therefore, $\Delta^n(G)$ is clique vertex irreducible if and only if G does not contain K_{n+2} as an induced subgraph. □

Theorem 6.3.5. *The anti-Gallai graph $\Delta(G)$ is clique irreducible if and only if G does not contain K_4 as an induced subgraph.*

Proof. Let G be a graph which does not contain K_4 as an induced subgraph. By Lemma 6.3.2 and Lemma 1.1.10, $\Delta(G)$ is clique irreducible.

Conversely, if G contains a $K_4 =< u_1, u_2, u_3, u_4 >$, then it follows that the clique in $\Delta(G)$, corresponding to the triangle $< u_1, u_2, u_3 >$ in G, shares each of its edges with some other clique. Therefore, if $\Delta(G)$ is clique irreducible, then G cannot have K_4 as an induced subgraph. □

Theorem 6.3.6. *The n^{th} iterated anti-Galli graph $\Delta^n(G)$ is clique irreducible if and only if G does not contain an induced K_{n+3}.*

Proof. By Theorem 6.3.5, $\Delta^n(G)$ is clique irreducible if and only if $\Delta^{n-1}(G)$ does not contain an induced K_4. $\Delta^{n-1}(G)$ does not contain an induced K_4 if and only if $\Delta^{n-2}(G)$ does not contain an induced K_5. Proceeding like this, we get, $\Delta(G)$ does not contain an induced K_{n+2} if and only if G does not contain an induced K_{n+3}. Therefore, $\Delta^n(G)$ is clique irreducible if and only if G does not contain an induced K_{n+3}. □

Theorem 6.3.7. *The anti-Gallai graph of a graph G, $\Delta(G)$ is weakly clique irreducible if and only if G is K_4-free.*

Proof. Let $< u_1, u_2, ..., u_k >$ be a clique of size greater than or equal to four in G. Let e_{ij} be the vertex corresponding to the edge u_iu_j in G for $i, j \in \{1, 2, ..., k\}$ and $i \neq j$. (Note that $e_{ij} = e_{ji}$). Consider the edge $e_{12}e_{13}$ in $\Delta(G)$. The cliques in $\Delta(G)$ obtained from the clique $< u_1, u_2, ..., u_k >$ in G, which contains the edge $e_{12}e_{13}$ are $< e_{12}, e_{13}, ..., e_{1k} >$ and $< e_{12}, e_{23}, e_{31} >$. Both these cliques are not essential, since all of their edges are present in at least one of the cliques $< e_{21}, e_{23}, ..., e_{2k} >$, $< e_{31}, e_{32}, ...e_{3k} >$ or $< e_{1i}, e_{ij}, e_{j1} >$ for $i, j \in \{3, 4, ..., k\}$ and $i \neq j$. Similarly, if there is any other clique which contains the vertices u_1, u_2 and u_3 in G, then the corresponding cliques in $\Delta(G)$ will not be essential. Therefore, $\Delta(G)$ is not weakly clique irreducible.

Conversely, assume that G is K_4-free. Then by Theorem 6.3.5, $\Delta(G)$ is clique irreducible and hence is weakly clique irreducible. □

Corollary 6.3.8. *The anti-Gallai graph of a graph G, $\Delta(G)$ is weakly clique irreducible if and only if it is clique irreducible.*

Corollary 6.3.9. *The n^{th} iterated anti-Gallai graph $\Delta^n(G)$ is weakly clique irreducible if and only if it is K_{n+3}-free.*

6.4 Cographs

In this section the cographs which are clique irreducible, clique vertex irreducible and weakly clique irreducible are characterized.

Lemma 6.4.1. *If G^c has at least three non-trivial components then G is clique reducible.*

Proof. Let G be a graph such that G^c has at least three non trivial components. Let $H_1, H_2, ..., H_p$ be the components of G^c. Let $G_i = H_i^c$ for $i = 1, 2, ..., p$. Then, $G = G_1 \vee G_2 \vee ... \vee G_p$. Also, any clique of G is the join of the cliques of G_i s for $i = 1, 2, ..., p$. At least three of the H_i s are non-trivial and hence at least three of the G_i s have more than one clique. Let C_{ij} for $j = 1, 2$ be any two of the cliques of G_i for $i = 1, 2, 3$. Let S_i be a clique of G_i for $i = 4, 5, ..., p$. Consider the clique $C_{11} \vee C_{21} \vee C_{31} \vee S_4 \vee ... \vee S_p$. Every edge of this clique is present in at least one of the cliques $C_{11} \vee C_{21} \vee C_{32} \vee S_4 \vee ... \vee S_p$, $C_{11} \vee C_{22} \vee C_{31} \vee S_4 \vee ... \vee S_p$, $C_{12} \vee C_{21} \vee C_{31} \vee S_4 \vee ... \vee S_p$. Therefore, G is clique reducible. □

Lemma 6.4.2. *If G^c has at least two non-trivial components then G is clique vertex reducible.*

Proof. Let G be a graph whose complement has at least two non trivial components. Let H_i, G_i, C_{ij} for $i = 1, 2, ...p$ and $j = 1, 2$ and S_i for $i = 3, 4, ...p$ be defined as in the proof of Lemma 6.4.1 and consider the clique $C_{11} \vee C_{21} \vee S_3 \vee ... \vee S_p$. Every vertex of this clique is present in at least one of the cliques $C_{11} \vee C_{22} \vee S_3 \vee ... \vee S_p$, $C_{12} \vee C_{21} \vee S_3 \vee ... \vee S_p$. Therefore, G is clique vertex reducible. □

Remark 6.4.1. If G is clique irreducible then G^c is either connected or has exactly two non trivial components and if G is clique vertex irreducible then G^c is either connected or has exactly one non-trivial component.

Lemma 6.4.3. *The clique vertex reducible graphs and the clique reducible graphs are closed for the operations of union and join.*

Theorem 6.4.4. *A cograph G is clique vertex irreducible if and only if it can be reduced to a trivial graph by recursively deleting universal vertices in each of the components.*

Proof. The proof is by induction on $|V| = n$. For $n = 1$ the theorem is trivially true. Assume that the theorem is true for any cograph with less than n vertices. A disconnected graph is clique vertex irreducible if and only if each of its components is clique vertex irreducible. Therefore, we may assume that, G is a connected cograph with n vertices. Then $G = G_1 \vee G_2$. If both G_is are not complete, then G^c will have at least two non trivial components which by Lemma 6.4.2 is a contradiction. Therefore, let G_1 be complete. Every vertex of G_1 is a universal vertex of G. Deleting these vertices we get a cograph G_2 with less than n vertices. Any clique C of G_2 corresponds to a clique $G_1 \vee C$ of G and hence has a vertex which does not lie in any other clique of G_2. Therefore, G_2 is a clique irreducible cograph with less than n vertices and hence by the induction hypothesis G_2 can be reduced to trivial graph by deleting universal vertices. Hence, the theorem. □

Theorem 6.4.5. *A connected cograph G is clique irreducible if and only if $G = G_1 \vee G_2 \vee K_p$ where G_1 and G_2 are clique vertex irreducible cographs such that G_i^c is connected for $i = 1, 2$ and $p \geqslant 0$.*

Proof. Let $G = G_1 \vee G_2 \vee K_p$ where G_1 and G_2 are connected clique vertex irreducible cographs and $p \geqslant 0$. Any clique of G is of the form $H = H_1 \vee H_2 \vee K_p$, where H_1 and H_2 are cliques of G_1 and G_2 respectively. Since, G_1 and G_2 are clique vertex irreducible, there exist vertices $v_1 \in H_1$ and $v_2 \in H_2$ such that they do not lie in any other clique of G. Therefore, the edge v_1v_2 of H does not lie in any other clique of G and hence G is clique irreducible.

Conversely, assume that G is clique irreducible. Since G is a cograph G^c must be disconnected. Therefore by Lemma 6.4.1, G^c has exactly two non trivial components. So, $G = G_1 \vee G_2 \vee K_p$, where G_1^c and G_2^c are both connected. Let H_{11} and H_{12} be any two cliques of G_1 and H_{21} and H_{22} be any two cliques of G_2.

$H = H_{11} \vee H_{21} \vee K_p$ is a clique of G. Every edge in H_{11}, every edge which joins H_{11} to a vertex of K_p and every edge in K_p will be present in the clique $H_{11} \vee H_{22} \vee K_p$. Again, every edge in H_{21}, every edge which joins H_{21} to a vertex of K_p and every edge in K_p will be present in the clique $H_{12} \vee H_{21} \vee K_p$. But, H has an edge which does not lie in any other clique of G. Therefore, that edge must be an edge which joins a vertex of H_{11} to a vertex of H_{21}. Let that edge be u_1u_2. But, then u_1 and u_2 cannot be present in any other clique of G_1 and G_2 respectively. Therefore, G_1 and G_2 are clique vertex irreducible. □

Theorem 6.4.6. *The weakly clique irreducible cographs can be recursively characterized as follows.*

(1) K_1 is a weakly clique irreducible cograph.

(2) If G_1 and G_2 are weakly clique irreducible cographs, then so is their union $G_1 \cup G_2$.

(3) If G_1 is a weakly clique irreducible cograph, then so is $G_1 \vee K_p$.

(4) If G_1 and G_2 are non-complete weakly clique irreducible cographs, then $G_1 \vee G_2$ is a weakly clique irreducible cograph if and only if every edge in G_i belongs to at least one vertex essential clique, for $i = 1, 2$.

Proof. The graph K_1 is weakly clique irreducible and union of any two weakly clique irreducible graphs is weakly clique irreducible. The cliques of $G_1 \vee K_p$ are of the form $H_1 \vee K_p$, where H_1 is a clique in G_1. If H_1 is essential in G_1 then so is $H_1 \vee K_p$ in $G_1 \vee K_p$. If H_1 is an isolated vertex u, then again $H_1 \vee K_p$ is an essential clique in $G_1 \vee K_p$ with all edges with one end vertex u as essential edges. Therefore, $G_1 \vee K_p$ is weakly clique irreducible if G_1 is weakly clique irreducible.

Let G_1 and G_2 be non-complete weakly clique irreducible cographs such that every edge in G_i belongs to at least one vertex essential clique, for $i = 1, 2$. If H_i is a vertex essential clique in G_i where $v_i \in V(H_i)$ is the vertex which does not belong to any other clique in G_i for $i = 1, 2$ then $H_1 \vee H_2$ is an essential clique in $G_1 \vee G_2$ where v_1v_2 is an essential edge. Therefore, every edge in $E(G_i)$ belongs to an essential clique in $G_1 \vee G_2$, since every edge in G_i belongs to at least one vertex essential clique, for $i = 1, 2$. Let $u \in V(G_1)$ and $v \in V(G_2)$. Consider the edge $uv \in E(G_1 \vee G_2)$.

Case 1 : u and v are isolated vertices in G_1 and G_2 respectively.

In this case, uv is a clique and is essential.

Case 2 : u is an isolated vertex in G_1, but v is not an isolated vertex in G_2.

Let $v' \in N(v)$. There exist a vertex essential clique C in G_2 which contains the edge vv'. Let w be the essential vertex in C. Therefore, uw is an essential edge in the clique $\{u\} \vee C$. Hence the edge uv belongs to the essential clique $\{u\} \vee C$ in $G_1 \vee G_2$.

The case where, u is not an isolated vertex in G_1, but v is an isolated vertex in G_2 can be proved similarly.

Case 3 : u and v are not isolated vertices in G_1 and G_2 respectively.

Let $u' \in N(u)$ and $v' \in N(v)$. Let H_1 and H_2 be the vertex essential cliques in G_1 and G_2 respectively, which contains the edges uu' and vv' respectively. Let w_i be the essential vertex in H_i for $i = 1, 2$. Therefore, w_1w_2 is an essential edge in the clique $H_1 \vee H_2$. Hence the edge uv belongs to the essential clique $H_1 \vee H_2$ in $G_1 \vee G_2$.

Therefore, every edge in $G_1 \vee G_2$ belongs to an essential clique and hence it is weakly clique irreducible.

Conversely, assume that G is a weakly clique irreducible cograph. If G is disconnected then it is the union of weakly clique irreducible cographs. If G has universal vertices then it is the join of a weakly clique irreducible graph with K_p, where p is the number of universal vertices.

Therefore, let G be a connected cograph without universal vertices. Hence, $G = G_1 \vee G_2$ where both G_1 and G_2 are not complete. None of the edges in $E(G_1) \cup E(G_2)$ are essential, since both G_1 and G_2 contains more than one clique. Therefore an essential edge in $G_1 \vee G_2$, if it exist, must be of the form uv, where $u \in V(G_1)$ and $v \in V(G_2)$. Then, u and v are essential vertices of G_1 and G_2 respectively. Hence, for $i = 1, 2$, the edges of G_i can be covered by essential cliques if and only if every edge in G_i belongs to at least one vertex essential clique. Therefore, if G_1 and G_2 are non-complete weakly clique irreducible cographs, then $G_1 \vee G_2$ is a weakly clique irreducible cograph if and only if every edge in G_i belongs to at least one vertex essential clique, for $i = 1, 2$.

Hence, the theorem. □

6.5 Distance hereditary graphs

In this section the distance hereditary graphs which are clique irreducible, clique vertex irreducible and weakly clique irreducible are characterized.

Lemma 6.5.1. *The clique vertex reducible (clique reducible) graphs are closed* *under the operations of attaching a pendant vertex, a true twin and a false twin.*

Proof. Let G be a clique vertex reducible (clique reducible) graph and C be a clique in G, all of whose vertices (edges) are present in some other clique in G.

The cliques of the graph obtained by attaching a pendant vertex u to a vertex v of G are the cliques of G together with the clique uv. Therefore C is a clique in this new graph and all of its vertices (edges) are present in some other clique.

The cliques of the graph obtained by attaching a true twin u to the vertex v of G are the cliques of G which does not contain the vertex v and the cliques of G which contains v together with the vertex u. If $v \notin C$, then C is a clique in the new graph and all its vertices (edges) are present in some other clique. If $v \in C$, then all the vertices (edges) in C other than u (the edges with one end vertex u) are already present in some other clique. Since v is (the edges with one end vertex v are) present in some other clique, u (the edges with one end vertex u) also must be present in some other clique.

The cliques of the graph obtained by attaching a false twin u to the vertex v of G are the cliques of G and the cliques of the form $(S \cup \{u\}) - \{v\}$, where S is a clique in G which contains the vertex v. Therefore, C is a clique in this new graph and all of its vertices (edges) are present in some other clique. □

Theorem 6.5.2. *The clique vertex irreducible distance hereditary graphs can be recursively characterized as follows.*

(1) K_1 is a clique vertex irreducible distance hereditary graph.

(2) If G is a clique vertex irreducible distance hereditary graph, then so is the graph obtained by attaching a pendant vertex to a vertex $v \in V(G)$, where v satisfies either $N(v)$ is not complete or there exists $w \in N(v)$ such that $N(w) = N(v)$.

(3) If G is a clique vertex irreducible distance hereditary graph, then so is the graph obtained by attaching a true twin.

(4) If G is a clique vertex irreducible distance hereditary graph, then so is the graph obtained by attaching a false twin to a vertex $v \in V(G)$, where v satisfies $< N(v) >$ is complete.

Proof. The graph K_1 is clique vertex irreducible. Let G be a clique vertex irreducible graph. Let G' be a graph obtained by attaching a pendant vertex u to a vertex v where v satisfies the conditions in theorem. The cliques of G' are precisely, the cliques of G and the edge uv. The clique uv contains the vertex u which does not belong to any other clique of G'. Every clique of G' which does not contain v also has a vertex which does not lie in any other clique of G', since G is clique vertex irreducible. Let C be a clique of G which contains the vertex v. If $N(v)$ is not complete then C contains a vertex $v' \neq v$ which is not present in any other clique of G and hence of G'. If $N(v)$ is complete, then C contains a vertex which does not belong to any other clique of G' if and only if there exist a vertex $w \in V(C)$ which does not belong to any other clique of G. i.e; if and only if $N(w) = N(v)$.

Let G be a clique vertex irreducible graph. Let G' be the graph obtained by attaching a true twin u to a vertex v of G. The cliques of G' are precisely, the cliques of G which does not contain v and the cliques of G which contains v together with the vertex u. Each such clique contains a vertex which does not lie in any other clique of G', since G is clique vertex irreducible and hence G' is also clique vertex irreducible.

Let G' be the graph obtained by attaching a false twin u to a vertex v of G. The cliques of G' are the cliques of G together with the cliques of the form

$(C \cup \{u\}) - \{v\}$ where C is a clique of G which contains v. The cliques of G' which does not contain v will continue to have a vertex which does not lie in any other clique. Let C be a clique of G which contains the vertex v. Every vertex of the clique C other than v will be present in the clique $(C \cup \{u\}) - \{v\}$ also. Therefore, C contains a vertex which does not lie in any other clique of G' if and only if v does not belong to any other clique of G, which happens if and only if $< N(v) >$ is complete.

Also, any distance hereditary graph G can be obtained from K_1 by the operations of attaching pendant vertices, introducing true twins and introducing false twins (Lemma 1.1.3) and by Lemma 6.5.1, the theorem follows.

□

Theorem 6.5.3. *The weakly clique irreducible distance hereditary graphs can be recursively characterized as follows.*

(1) K_2 is a clique irreducible distance hereditary graph.

(2) If G is a clique irreducible distance hereditary graph then so is the graph obtained by attaching a pendant vertex.

(3) If G is a clique irreducible distance hereditary graph then so is the graph obtained by attaching a true twin.

(4) If G is a clique irreducible distance hereditary graph then so is the graph obtained by attaching a false twin to a vertex V if $< N(v) >$ is clique vertex irreducible.

Proof. The graph K_2 is clique irreducible. Let G be a clique irreducible graph. Let G' be the graph obtained by attaching a pendant vertex u to a vertex v of G. The cliques of G' are precisely, the cliques of G and the edge uv. Every clique contains an edge which does not lie in any other clique of G' and hence G' is clique irreducible.

Let G be a clique irreducible graph. Let G' be the graph obtained by attaching a true twin u to a vertex v of G. The cliques of G' are precisely, the cliques of G which does not contain v and the cliques of G which contains v together with the vertex u. Every such clique contains an edge which does not lie in any other clique, since G is clique irreducible and hence G' is also clique irreducible.

Let G' be the graph obtained by attaching a false twin u to a vertex v of G. The cliques of G' are the cliques of G together with the cliques of the form $(C \cup \{u\}) - \{v\}$ where C is a clique of G which contains v. The cliques of G' which does not contain v will continue to have an edge which does not lie in any other clique. Let C be a clique of G which contains the vertex v. Every edge of C which does not contain v will be present in the clique $(C \cup \{u\}) - \{v\}$ also. Therefore, C contains an edge which does not lie in any other clique of G' if and only if there exists an edge vv' which does not lie in any other clique of G. Therefore, the vertex v' is not present in any clique of $< N(v) >$ other than $C - \{v\}$. So, $< N\{v\} >$ is clique vertex irreducible.

The converse follows by Lemma 1.1.3 and by Lemma 6.5.1. □

Lemma 6.5.4. *The class of weakly clique reducible graphs is closed under the operations of attaching pendant vertices, true twins and false twins.*

Proof. Let G be a weakly clique reducible graph and let e be the edge which is not covered by any of the essential cliques in G.

Let G' be the graph obtained from G by attaching a pendant vertex. The essential cliques of G' are the essential cliques of G together with the newly introduced edge. But, these essential cliques will not cover the edge e.

Let G' be the graph obtained from G by attaching a true twin v to a vertex u. The essential cliques of G' are the essential cliques of G which does not contain the vertex u and the cliques of the form $C \cup \{v\}$, where C is an essential clique in G which contains the vertex u. Still, the edge e is not covered by essential cliques.

Let G' be the graph obtained from G by attaching a false twin v to a vertex u. The essential cliques of G' are the essential cliques of G which does not contain the vertex u, the cliques of the form $(C - \{u\}) \cup \{v\}$ and C, where C is an essential clique in G which contains the vertex u and which has an essential edge with one end vertex u. Again, the edge e is not covered by the essential cliques.

Hence the lemma. □

Theorem 6.5.5. *A distance hereditary graph G is weakly clique irreducible if and only if all its induced subgraphs are weakly clique irreducible.*

Theorem 6.5.6. *A distance hereditary graph G is weakly clique irreducible if and only if G does not contain F_{19} in Fig : 1.9 as an induced subgraph.*

Proof. By Theorem 6.5.5, G is weakly clique irreducible if and only if all its induced subgraphs are weakly clique irreducible. But, a graph G is hereditary weakly clique irreducible if and only if G does not contain any of the graphs in Fig : 1.9 as an induced subgraph (Lemma 1.1.11). But, G cannot have any of the graphs $F_1, F_2, ...F_{18}$ as an induced subgraph, since they contain gem as an induced subgraph (Lemma 1.1.4). Hence, the theorem. □

Corollary 6.5.7. *A cograph G is weakly maximal clique irreducible if and only if*

G does not contain F_{19} in 1.1.9 as an induced subgraph.

Proof. Since, cographs are a subclass of distance hereditary graphs (Lemma 1.1.5) and F_{19} in Fig : 1.9 is a cograph, the corollary follows. □

Theorem 6.5.8. *The weakly clique irreducible distance hereditary graphs can be recursively characterized as follows.*

(1) K_2 is a weakly clique irreducible distance hereditary graph.

(2) If G is a weakly clique irreducible distance hereditary graph then so is the graph obtained by attaching pendent vertices to the vertices of G.

(3) If G is a weakly clique irreducible distance hereditary graph then so is the graph obtained by attaching true twins to the vertices of G.

(4) If G is weakly clique irreducible distance hereditary graph then so is the graph obtained by attaching false twins to a vertex u where $< N(u) >$ is C_4-free is also weakly clique irreducible.

Proof. The graph K_2 is weakly clique irreducible. Let G be a weakly clique irreducible distance hereditary graph. If G does not have F_{19} as an induced subgraph then a graph obtained by any of the above operations also cannot have F_{19} as an induced subgraph. Therefore, they are all weakly clique irreducible.

Conversely, by the recursive definition of distance hereditary graphs (Lemma 1.1.3), it is enough if we could prove that, attaching a false twin v to a vertex u which contains a $C_4 =< u_1, u_2, u_3, u_4 >$ in $N(u)$, gives a weakly clique reducible graph. Clearly, $< u, v, u_1, u_2, u_3, u_4 >$ is F_{19}.

Hence the theorem. □

List of some open problems

1. Characterize non-isomorphic graphs of the same order having isomorphic Gallai graphs (anti-Gallai graphs).

2. Characterize graphs G for which the Gallai and the anti-Gallai operators commute.

3. Characterize graphs G for which $\Gamma(G) = \Delta(G)$.

4. Characterize all connected graphs which satisfy $\gamma(G) = \gamma_{cd}(G)$.

5. Characterize all connected graphs which satisfy $\gamma_{cd}(G) = \gamma_{gcd}(G)$.

6. Identify the domination parameters which satisfy Vizing's type relation under any of the graph products.

7. Characterize the clique perfect graphs [73].

8. Identify special classes of clique perfect graphs.

9. Estimate sharp upper bounds for the clique transversal number for special classes of graphs and characterize the graphs which attains this upper bound.

10. Does there exist graph classes which satisfy the $< t >$-property for every t?

11. Characterize the clique irreducible graphs, the clique vertex irreducible graphs and the weakly clique irreducible graphs.

List of symbols

C_n	-	Cycle of length n
$d(v)$	-	Degree of a vertex
$d(G)$	-	Diameter of a graph G
$d(u, v)$ or $d_G(u, v)$	-	Distance between u and v in G
E or $E(G)$	-	Edge set of G
$G \Box H$	-	Cartesian product of G and H
$G \vee H$	-	Join of G and H
$G \otimes H$	-	Strong product of G and H
$G \times H$	-	Tensor product of G and H
$G \cup H$	-	Union of G and H
$K(G)$	-	Clique graph of G
$K_{m,n}$	-	Complete bipartite graph where m and n are the cardinalities of the partitions
K_n	-	Complete graph on n vertices
$L(G)$	-	Line graph of G
$L^k(G)$	-	kth iterated line graph of G
m or $m(G)$	-	Number of edges of G
$N[v]$	-	Closed neighborhood of v
$N(v)$	-	Open neighborhood of v
nG	-	n disjoint copies of G
n or $n(G)$	-	Number of vertices of G
NEPS$(G_1, G_2, \mathcal{B})$	-	Non complete expended p sum of G_1 and G_2 with basis $\mathcal{B}$

P_n	-	Path on n vertices
$r(G)$	-	Radius of G
$< S >$	-	Graph induced by $S \subseteq V$
$T_k(G)$	-	Trestled graph of index k
V or $V(G)$	-	Vertex set of G
$\alpha(G)$	-	Independence number of G
$\alpha_c(G)$	-	Clique independence number of G
$\beta(G)$	-	Covering number of G
$\gamma(G)$	-	Domination number of G
$\gamma_{cd}(G)$	-	Cographic domination number of G
$\gamma_g(G)$	-	Global domination number of G
$\gamma_{gcd}(G)$	-	Global cographic domination number of G
$\gamma_i(G)$	-	Independence domination number of G
$\tau_c(G)$	-	Clique transversal number of G
$\chi(G)$	-	Chromatic number of G
$\omega(G)$	-	Clique number of G
$\Gamma(G)$	-	Gallai graph of G
$\Gamma^k(G)$	-	k^{th} iterated Gallai graph of G
$\Delta(G)$	-	Anti-Gallai graph of G
$\Delta^k(G)$	-	k^{th} iterated anti-Gallai graph of G

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