DEVELOPMENT OF A NEW TRANSFORM: MRT

A THESIS

Submitted by

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Under the guidance of

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CERTIFICATE

This is to certify that the thesis entitled "**DEVELOPMENT OF A NEW TRANSFORM: MRT**" is a bonafide record of the research work carried out by *Rajesh Cherian Roy* under my supervision and guidance in the *Division of Electronics Engineering, School of Engineering, Cochin University of Science and Technology* and that no part thereof has been presented for the award of any other degree.

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DECLARATION

I hereby declare that the work presented in the thesis entitled "**DEVELOPMENT OF A NEW TRANSFORM: MRT**" is based on original research work carried out by me under the supervision and guidance of Dr. R. Gopikakumari in the *Division of Electronics Engineering, School of Engineering, Cochin University of Science and Technology* and that no part thereof has been presented for the award of any other degree.

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ABSTRACT

Fourier transform methods are employed heavily in digital signal processing. Discrete Fourier Transform (DFT) is among the most commonly used digital signal transforms. The exponential kernel of the DFT has the properties of symmetry and periodicity. Fast Fourier Transform (FFT) methods for fast DFT computation exploit these kernel properties in different ways. In this thesis, an approach of grouping data on the basis of the corresponding phase of the exponential kernel of the DFT is exploited to introduce a new digital signal transform, named the M-dimensional Real Transform (MRT), for 1-D and 2-D signals. The new transform is developed using number-theoretic principles as regards its specific features. A few properties of the transform are explored, and an inverse transform presented. A fundamental assumption is that the size of the input signal be even. The transform computation involves only real additions. The MRT is an integer-to-integer transform. There are two kinds of redundancy, complete redundancy & derived redundancy, in MRT. Redundancy is analyzed and removed to arrive at a more compact version called the Unique MRT (UMRT). 1-D UMRT is a non-expansive transform for all signal sizes, while the 2-D UMRT is non-expansive for signal sizes that are powers of 2. The 2-D UMRT is applied in image processing applications like image compression and orientation analysis. The MRT & UMRT, being general transforms, will find potential applications in various fields of signal and image processing.

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SYMBOLS AND DEFINITIONS

Z	-	Set of integers
N	-	Integer, $N \in Z$, mostly used to indicate size of input signal
n	-	Signal time index
x_n	-	1-D signal sample at n
k	-	Frequency index
p	-	Phase index
$Y_k^{(p)}$	-	1-D MRT coefficient at $k_,$ p.
n_1	-	Signal spatial index in the vertical direction
n_2	-	Signal spatial index in the horizontal direction
x_{n_1,n_2}	-	2-D signal sample at n_1, n_2
k_1	-	Frequency index in the vertical direction
k_2	-	Frequency index in the horizontal direction
$Y_{k_1,k_2}^{(p)}$	-	2-D MRT coefficient at k_1, k_2, p.
$g(a,b)$	-	Greatest Common Divisor (GCD) of integers a and b
$a \mid b$	-	a divides b
$((a))_b$	-	Remainder of a/b (Modulo operation)
q	-	Integer
$q \in Z$	-	q is an element of set Z
$[a, b]$	-	$\{q \in Z : a \leq q \leq b\}$
$\lfloor a \rfloor$	-	Smallest integer lesser than or equal to a
$\phi(N)$	-	Totient function of N
$\langle A,B \rangle$	-	Inner product between vectors A and B
$\forall q$	-	For all values of q
$\Rightarrow$	-	Such that

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INTRODUCTION

1.1 Transforms in Digital Signal Processing

A digital signal is a sequence of numbers, real or complex, that represents an information-bearing quantity over discrete co-ordinates of time, space or another variable. Digital signal processing has long come to play a pivotal role in almost all fields of technology. It involves various methods of dealing with digital signals, e.g., analysis and modeling of signals, their coding and transmission, compression, restoration etc. In the process of analyzing signals for further processing, signal processing theory makes use of a fundamental class of operators called signal transforms. Transforms convert the original signal into a form which enables relatively simpler analysis than in the original form. The exact effect of the transform on the signal is unique for each transform. Hence, different signal processing applications may use different transforms, of which many kinds exist.

Transforms can thus be considered to be the fundamental components in digital signal processing, where a majority of operations are performed in the transform-domain. Continuous transforms like Fourier and Laplace transforms were the earliest transforms used in analog signal manipulation. Signal processing operations utilizing these analog transforms were implemented using low pass, high pass, and band pass filters and spectrum analyzers. With the advent of digital computers, signal processing became digital in nature, and digital signal processing has been an area of heavy research activity. New specific applications of digital signal processing have resulted in development of application-specific transforms.

The introduction of the Fast Fourier Transform (FFT), in 1965 by Cooley and Tukey [1], became a turning point in the evolution of digital signal processing. The FFT gave an impetus to all branches of digital signal and image processing and it gave rise to numerous applications for the FFT. The Walsh transform, introduced in 1970, found applications in signal processing. This was followed by the development of a large family of fast transforms. Requirements in data compression led to preferences for transforms that had the property of energy compaction. Transform domain processing has been found suitable for signal and image restoration, enhancement, and feature extraction. The Discrete Cosine Transform (DCT) has the energy compaction property and is the underlying transform in the JPEG and MPEG standards for image, audio, and video compression. The next major step in the evolution of transform-based signal processing was the introduction of the wavelet family of transforms in the 1980s. Wavelets provide a better local representation of signals in contrast to the global representation that is characteristic to Fourier, DCT, and Walsh-Hadamard transforms. Basis functions of the wavelet transform are generated from a primary function, the mother wavelet, by means of coordinate shift and coordinate scaling operations. Since their introduction, wavelets have gained a great popularity. Currently, wavelets constitute a well-established and widely-used part of signal processing transform theory, and have found a wide range of applications.

Main directions of growth in the field of transforms for signal and image processing [2] are
(i) further development of application-specific transforms.
(ii) perfecting numerical representation of natural transforms for new applications
(iii) research aimed at enabling local adaptivity of transform domain processing of signals.
(iv) exploration of the use of digital transforms in new applications, and
(v) development of new practical transform domain processing methods to meet growing demands.

1.2 Linear transforms

A linear transform maps an N-point input vector, x, into a K-point output vector, y, as follows

$$y = A^* x,$$

$$x = \begin{pmatrix} x_0 \\ x_1 \\ \vdots \\ x_{N-1} \end{pmatrix},$$

$$y = \begin{pmatrix} y_0 \\ y_1 \\ \vdots \\ y_{K-1} \end{pmatrix}$$

where A is an N x K matrix of real or complex coefficients and A^* is the K x N conjugate transpose of A. The inverse relationship is given by

$$x = By,$$

A and B being related as,

$$BA^* = I,$$

I being the N x N identity matrix. For the case $N = K$, B and A are related by

$$B^{-1} = A^*$$

Transform coefficients may be expressed as

$$y_q = a_q^* x, \qquad q = 0,1,....,N-1$$

where a_q is the qth column of the N x N matrix, A, also called the kernel of the transform. The inverse transform may be expressed as

$$x = \sum_{q=0}^{N-1} y_q b_q$$

where b_q is the qth column of the N x N matrix, B.

Defining the inner product between two N-dimensional vectors, v and w, as

$$\langle v, w \rangle = \sum_{q=0}^{N-1} v_q w_q^*,$$

The transform is said to be orthogonal if

$$\langle a_i, a_j \rangle = 0, \ \forall i \neq j,$$

and an orthogonal transform is defined to be orthonormal if

$$\langle a_i, a_i \rangle = \|a_i\|^2 = 1, \ \forall i$$

The Karhunen-Loeve Transform (KLT) is an orthonormal transform that has much theoretical significance since it is considered the optimal transform for applications such as signal compression on account of its property of decorrelating the input, providing best energy compaction among all orthonormal transforms. The transform kernel of the KLT is formed from the eigenvectors of the covariance matrix of the input signal. Hence, the KLT is considered a data-dependent transform, which distinguishes it from most other orthonormal transforms that have data-independent kernels.

Transforms can be classified as real transforms or complex transforms. The basis of classification implied here is the presence or absence of complex terms in the kernel of the transform. For example, the DFT is a complex transform since its kernel contains the complex exponential. On the other hand, the kernel of the DCT contains only a cosine term and hence the transform coefficients of the DCT are also real, provided the input data is real, making the DCT a real transform. Further, real transforms may be classified on the basis whether they transform integers to integers. Integer-to-integer transforms have the advantage of ease of implementation.

1.2.1 Complex transforms

1.2.1.1 Discrete Fourier Transform (DFT)

The forward and inverse DFT of an N-point sequence x_n are defined as

$$y_k = \sum_{n=0}^{N-1} x_n e^{\frac{-j2\pi nk}{N}}, \; k = 0,1,\ldots,N-1, \text{ and}$$

$$x_n = \frac{1}{N^2} \sum_{k=0}^{N-1} y_k e^{\frac{j2\pi nk}{N}}, \; n = 0,1,\ldots,N-1, \text{ respectively.}$$

The DFT decomposes the input as a linear combination of complex exponential waveforms of the uniformly spaced frequencies.

1.2.1.2 Number Theoretic Transforms (NTT)

NTTs are discrete Fourier transforms that are defined over finite fields or rings. A forward and inverse NTT pair is defined by

$$y_k = ((\sum_{n=0}^{N-1} x_n \alpha^{nk}))_M, \quad k = 0,1,2,\ldots,N-1$$

and

$$x_n = ((N^{-1} \sum_{n=0}^{N-1} y_k \alpha^{-nk}))_M, \quad n = 0,1,2,\ldots,N-1$$

where N is the transform length, α is a root of unity and M is the modulus. Arithmetic is performed modulo M, where M can be a prime $M = p$, a power of a prime $M = p^m$ or a composite number that results in NTT pairs defined over a Galois field. Only a small subset of NTTs has properties that make them useful for practical purposes.

1.2.1.3 Fractional Fourier Transform (FRFT)

The Fractional Fourier transform (FRFT) is a generalization of the Fourier transform. The FRFT of a function x_t is defined as

$$F^a[x_t] = y_k = \frac{e^{j(\frac{\pi}{4}-\frac{\pi}{2})}}{\sqrt{2\pi \sin\alpha}} e^{\frac{-jk^2 \cot\alpha}{2}} \int_{-\infty}^{\infty} e^{\left(-\frac{jt^2\cot\alpha}{2} - \frac{jkt}{\sin\alpha}\right)} x_t dt$$

The inverse FRFT is given as

$$F^{-a}[y_k] = \frac{e^{-j(\frac{\pi}{4}-\frac{\pi}{2})}}{\sqrt{2\pi \sin\alpha}} e^{\frac{jk^2 \cot\alpha}{2}} \int_{-\infty}^{\infty} e^{\left(\frac{jt^2\cot\alpha}{2} - \frac{jkt}{\sin\alpha}\right)} y_k dk,$$

where $\alpha = a\pi/2$. When $\alpha = \pi/2$, the FRFT reduces to the conventional Fourier transform. Depending on the value of α, called the fractional operator, the actual operation performed on the signal by the FRFT kernel varies. The FRFT has been extended to the discrete domain using many methods.

1.2.1.4 *S* Transform

The *S* transform is a time-frequency representation obtained by extension of the ideas of the wavelet transform, and is based on moving, scalable Gaussian windows. It combines advantages of the Fourier transform and the wavelet transform by providing a frequency-dependent resolution of the time-frequency space along with local phase information. The *S* transform is defined as

$$S(\tau,f)=\int_{-\infty}^{\infty} x_t \frac{|f|}{\sqrt{2\pi}} e^{-\frac{(\tau-t)^2 f^2}{2}} e^{-j2\pi ft} dt \text{, and the inverse as}$$

$$x_t=\int_{-\infty}^{\infty}\left\{\int_{-\infty}^{\infty} S(\tau,f)d\tau\right\} e^{j2\pi ft} df$$

The discrete *S* transform is defined as

$$S(qT,\frac{n}{NT})=\sum_{m=0}^{N-1} H\left[\frac{m+n}{NT}\right] e^{-2\pi^2 m^2/n^2} e^{j2\pi mq/N}, \quad n\neq 0,$$

and the inverse discrete S transform is obtained as

$$h(kT)=\sum_{n=0}^{N-1}\left\{\frac{1}{N}\sum_{q=0}^{N-1} S\left[qT,\frac{n}{NT}\right]\right\} e^{j2\pi nk/N}.$$

1.2.2 Real Transforms

1.2.2.1 Discrete Hartley Transform (DHT)

The Hartley transform is an integral transform that shares some features with the Fourier transform. It produces real output for a real input, and has the duality property. In comparison, a real input is not transformed to a real output by the DFT. Consequently, there is a data redundancy in the transform domain where only one half of the $2N$ real values, representing the real and imaginary parts of the transform coefficients, are carrying any useful information. A modification in the definition of DFT is done in DHT to remove this drawback. The complex kernel of the DFT, given by $e^{-j2\pi nk/N}=\cos(2\pi nk/N)-j\sin(2\pi nk/N)$ is replaced in the discrete Hartley transform by the real kernel $\cos(2\pi nk/N)+\sin(2\pi nk/N)$, also called the *cas* function.

1.2.2.2 Discrete Cosine Transform (DCT)

The 1-D DCT of an N-point sequence x_n is defined as

$$y_k = \alpha(k)\sum_{n=0}^{N-1} x_n \cos\left[\frac{\pi(2n+1)k}{2N}\right], \quad k = 0,1,\ldots,N-1$$

where

$$\alpha(0) = \sqrt{\frac{1}{N}}, \qquad \alpha(k) = \sqrt{\frac{2}{N}}, \qquad k = 1,\ldots,N-1$$

1.2.2.3 Modulated Lapped Transform (MLT)

The 1-D MLT [3] decomposes a signal into over-lapping blocks. The overlap between adjacent blocks is L samples. Hence, each block has $2L$ samples, where each sample belongs to two blocks. The transform is defined by

$$w_{k,n} = h_n \cos\left[\frac{\pi}{L}\left(n - \frac{L-1}{2}\right)\left(k + \frac{1}{2}\right)\right] \qquad k = 0,\ldots,L-1 \qquad n = 0,\ldots,2L-1$$

where h_n is a sine-shaped window function with period $4L$ symmetric about the block centre $(2L-1)/2$.

$$h_n = \sqrt{\frac{2}{L}}\sin\left[\frac{\pi}{2L}\left(n + \frac{1}{2}\right)\right] \qquad n = 0,\ldots,2L-1$$

The basis functions of 2-D MLT $p_{k,l,n,m}$ is obtained by

$$p_{k,l,n,m} = w_{k,n} w_{l,m}$$

The 2-D MLT transform of a signal $s_{n,m}$ is given by

$$S_{k,l} = \sum_{n=0}^{2L-1}\sum_{m=0}^{2L-1} s_{n,m} p_{k,l,n,m} \qquad k,l = 0,\ldots,L-1$$

1.2.2.4 Lapped Directional Transform (LDT)

The LDT [4] is derived from the MLT and a time-inverted and sign-inversed version of the MLT, referred to as MLT'. The basis functions of the 1-D MLT' are given by

$$w'_{k,n} = h_n \sin\left[\frac{\pi}{L}\left(n - \frac{L-1}{2}\right)\left(k + \frac{1}{2}\right)\right] \qquad k = 0,\ldots,L-1 \qquad n = 0,\ldots,2L-1$$

The basis functions of the 2-D MLT' are given by

$$p'_{k,l,n,m} = w'_{k,n}\, w'_{l,m}$$

The 2-D MLT' of signal $s_{n,m}$ is given by

$$S'_{k,l} = \sum_{n=0}^{2L-1}\sum_{m=0}^{2L-1} s_{n,m} p'_{k,l,n,m}$$

The LDT is obtained by using both the MLT and the MLT'

$$Y_{i,j} = \begin{cases} S_{i-1/2,j-1/2} - S'_{i-1/2,j-1/2} & \text{if} \quad i>0, j>0 \\ S_{i-1/2,-j-1/2} + S'_{i-1/2,-j-1/2} & \text{if} \quad i>0, j<0 \\ Y_{-i,-j} & \text{if} \quad i<0 \end{cases}$$

where i,j = $-L + 1/2, -L + 3/2, \ldots, -1/2, 1/2, \ldots, L - 3/2, L-1/2$.

The LDT contains twice as many coefficients as the original image. The advantage of the LDT over the MLT is that it is capable of distinguishing between mirrored orientations in the image.

1.2.3 Wavelet Transform

The Continuous Wavelet Transform (CWT) of 1-D signal x_t is defined as

$$W_{a,b} = \int_{-\infty}^{\infty} x_t \frac{1}{\sqrt{|a|}} \psi^*\left(\frac{t-b}{a}\right) dt$$

where a and b are real and * denotes complex conjugation, and $\psi(t)$ is the mother wavelet. If the mother wavelet satisfies

$$C \equiv \int_{-\infty}^{\infty} \frac{|\Psi(w)|^2}{|w|} dw, \qquad 0 < C < \infty,$$ then the inverse transform is

$$x_t = \frac{1}{C} \int_{a=-\infty}^{\infty} \int_{b=-\infty}^{\infty} \frac{1}{|a|^2} W_{a,b} \frac{1}{\sqrt{|a|}} \psi\left(\frac{t-b}{a}\right) da\, db$$

The Discrete Wavelet Transform (DWT) can be defined in terms of discrete-time multiresolution decomposition, in which the signal is decomposed to yield a low-resolution signal v_n^J and wavelet coefficients w_n^j, $j = 1, 2, \ldots. J$, where J depends on the number of levels of decomposition. The DWT can be computed by the inner products

$$w_k^j = \left\langle x_n, \tilde{h}'^j_{n-2^j k} \right\rangle \qquad j = 1, 2, \ldots. J$$

$$v_k^J = \left\langle x_n, \tilde{g}'^J_{n-2^J k} \right\rangle$$

where $\tilde{g}_n$ is the analysis scaling filter, and $\tilde{h}_n$ is the analysis wavelet filter. The inverse DWT reconstructs the signal as a linear combination of shifted synthesis wavelets weighted by the corresponding wavelet coefficients, plus a very low resolution approximation of the signal, as

$$x_n = \sum_{j=1}^{J}\sum_{k} w_k^j h_{n-2^j k}^j + \sum_k v_k^J g_{n-2^J k}^J$$

where g_n and h_n are the synthesis scaling and wavelet filters respectively. Since shift-variance and other issues like lack of directionality are drawbacks of the real DWT, complex wavelet transforms, having complex wavelet filter coefficients, have been proposed,

1.2.4 Integer Transforms

Integer transforms are discrete transforms which map integer inputs to integer outputs. Some integer transforms use only integer arithmetic, i.e. all the elements of the transform matrix are integers, whereas some other integer transforms use floating-point arithmetic for integer-to-integer transformation. The over-riding advantage of integer transforms is their simpler implementation when compared to non-integer transforms. Transforms such as the Walsh-Hadamard transform and the Slant transform have integer-valued kernels. Integer trigonometric transforms, formed from approximations of the corresponding trigonometric transforms, and integer wavelet transforms, constructed by using lifting techniques, are commonly employed in image processing applications such as lossless compression.

1.2.4.1 Walsh-Hadamard Transform (WHT)

The forward 1-D WHT is given by

$$y_k = \frac{1}{N}\sum_{n=0}^{N-1} x_n (-1)^{\sum_{i=0}^{\log_2 N-1} b_i(n) b_i(k)}, \qquad k = 0,1,2,\dots.N-1$$

where the summation in the exponent is performed in modulo 2 arithmetic, and $b_v(z)$ is the vth bit in the binary representation of z, and N is a power of 2.

The Hadamard matrix is a square array of plus and minus ones whose rows and columns are orthogonal to one another. The product of the Hadamard matrix and its transpose is the identity matrix. The lowest order Hadamard matrix is of order two, and is given by

$$H = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

If $N = 2^n$ (n an integer), and H_N is a Hadamard matrix of order N, then the Hadamard matrix H_{2N} of order $2N$ can be obtained recursively as

$$H_{2N} = \begin{bmatrix} H_N & H_N \\ H_N & -H_N \end{bmatrix}$$

The 1-D WHT 8-point transform kernel is shown in Figure 1.1.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 & -1 & 1 & 1 \\ 1 & 1 & -1 & -1 & 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 1 & -1 & -1 & 1 & -1 & 1 & 1 & -1 \\ 1 & -1 & 1 & -1 & -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \end{bmatrix}$$

Figure 1.1. Transform kernel values for 8-point 1-D WHT

1.2.4.2 Haar Transform

Haar functions are defined as

$$h_0(n) = h_{0,0}(n) = \frac{1}{\sqrt{N}}, \qquad n \in [0,1]$$

$$h_k(n) = h_{p,q}(n) = \frac{1}{\sqrt{N}} \begin{cases} 2^{p/2}, & \dfrac{q-1}{2^p} \le n < \dfrac{q-\frac{1}{2}}{2^p} \\ -2^{p/2}, & \dfrac{q-\frac{1}{2}}{2^p} \le n < \dfrac{q}{2^p} \\ 0, & \text{otherwise, for } n \in [0,1] \end{cases}, \qquad k = 2^p + q - 1$$

For the Haar transform, $n = m/N$, $m = 0,1,\ldots\ldots,N-1$. The Haar transform matrix for $N = 8$ is shown in Figure 1.2.

$$\frac{1}{\sqrt{8}} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} \\ 2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & -2 \end{bmatrix}$$

Figure 1.2: Transform kernel values for 8-point 1-D Haar Transform

1.2.4.3 Integer Trigonometric Transforms

Integer transforms have been obtained for the trigonometric transforms i.e. DWT, DFT, DCT and DST, by replacing the real components in the transform kernels by integers. The transform kernel matrix for one version of 8-point 1-D integer DCT is given in Figure 1.3.

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 5 & 3 & 2 & 1 & -1 & -2 & -3 & -5 \\ 3 & 1 & -1 & -3 & -3 & -1 & 1 & 3 \\ 3 & -1 & -5 & -2 & 2 & 5 & 1 & 3 \\ 1 & -1 & -1 & 1 & 1 & -1 & -1 & 1 \\ 2 & -5 & 1 & 3 & -3 & -1 & 5 & -2 \\ 1 & -3 & 3 & -1 & -1 & 3 & -3 & 1 \\ 1 & -2 & 3 & -5 & 5 & -3 & 2 & -1 \end{bmatrix}$$

Figure 1.3 : Transform kernel values for 8-point 1-D integer DCT

1.2.4.4 Integer Wavelet Transforms

Different approaches to construct integer wavelet transforms exist. An important method is to obtain integer-to-integer wavelet transforms [5] by the use of the lifting scheme, which ensures perfect reconstruction. In this method, the wavelet transform is factored into lifting steps in which the filter output is rounded off to the nearest integer.

1.2.4.5 Modulo Transforms

Modulo transforms [6] offer a new method for construction of orthogonal transforms that map integers to integers, and are an alternative to the lifting scheme. They are based on a theory of quantizing transform coefficients, called critical quantization. Modulo transforms are reversible, have rational coefficients and unit determinant.

1.2.4.6 Radon Transforms

The classical Radon transform is a continuous transform. The Radon transform of a 2-D signal is defined on a family of straight lines. The value of the transform is the integral of the signal along this line. Describing a straight line as $t = mq + \tau$, the Radon transform is defined for this line as

$$R(\tau, m) = \int_{-\infty}^{\infty} x_{\tau+mq,q} \, dq$$

With the advent of computers, the Radon transform was seen to have potential applications in tomography and image processing. This necessitated the introduction of discrete versions of the continuous Radon transform. Various approaches have been proposed for a Discrete Radon Transform (DRT). A DRT proposed by Gertner in [7] on a quadratic array of N x N numbers is described as follows:

Let

$$((ac+bd))_N = 0, \qquad 0 \le c,d \le N-1$$

be a system of linear congruences. Assume that a direction on a square grid N x N is denoted as (c, d). Direction (c', d') is said to be orthogonal to direction (c, d) if

$$((cc'+dd'))_N = 0$$

The solutions to

$$((ac'+bd'))_N = e$$

are said to be orthogonal to the solutions of $((ac + bd))_N = 0$. The solutions for each value of e = 0, 1, 2, … , N-1 can be considered to be N discrete parallel lines.

For a given direction (c, d), the discrete Radon transform is defined as the sum of the data over the corresponding orthogonal directions to (c, d) for each value of e.

$$R_e^{(c,d)} = \sum_i \sum_{j(\forall i,j \Rightarrow ((ic'+jd'))_N = e)} x_{i,j} \qquad e = 0,1,2,\dots,N-1$$

Discrete Periodic Radon Transform (DPRT):

Case 1: N is prime: The DPRT of x_{n_1,n_2} is defined as

$$X_0^b(d) = \sum_{n_2=0}^{N-1} x_{d,n_2}, \qquad 0 \le d \le N-1$$

$$X_m^c(d) = \sum_{n_1=0}^{N-1} x_{n_1,((d+mn_1))_N}, \quad 0 \le d,m \le N-1$$

Case 2: N is a power of 2: The DPRT is defined as

$$X_s^b(d) = \sum_{n_2=0}^{N-1} x_{((d+2sn_2))_N,n_2}, \qquad 0 \le d \le N-1,\ 0 \le s \le (N/2)-1$$

$$X_m^c(d) = \sum_{n_1=0}^{N-1} x_{n_1,((d+mn_1))_N}, \qquad 0 \le d \le N-1,\ 0 \le m \le N-1$$

Inverse DPRT:

Case 1: N is prime: The inverse DPRT is given by

$$x_{n_1,n_2} = \frac{1}{N}\left[\sum_{m=0}^{N-1} X_m^c(((n_2 - mn_1))_N) - \sum_{d=0}^{N-1} X_0^b(d) + X_0^b(n_1)\right]$$

Case 2: N is a power of 2: The inverse DPRT is given by

$$x_{n_1,n_2} = S_{n_1,n_2}^0 - \sum_{i=1}^{\log_2 N-1} S_{n_1,n_2}^i - \frac{2}{N^2}\sum_{d=0}^{N-1} X_0^c(d),$$

where

$$S_{n_1,n_2}^i = \frac{1}{2^i N}\left[\sum_{m=0}^{N/2^i-1}\sum_{j=0}^{2^i-1} X_m^c\left(((n_2 - mn_1))_{N/2^i} + \frac{jN}{2^i}\right) + \sum_{s=0}^{N/2^{i+1}}\sum_{j=0}^{2^i-1} X_s^b\left(((n_1 - 2sn_2))_{N/2^i} + \frac{jN}{2^i}\right)\right]$$

Orthogonal Discrete Periodoc Radon Transform (ODPRT):

The ODPRT of 2-D signal of size N x N, where N is a power of 2, consists of three components, given by,

$$X_m^u(d) = X_m^c(d) - X_m^c(d + N/2), \qquad 0 \le m \le N-1,\ 0 \le d \le N/2-1$$

$$X_s^v(d) = X_s^b(d) - X_s^b(d + N/2), \qquad 0 \le d, s \le N/2-1$$

$$X_{n_1,n_2}^1 = x_{n_1,n_2} + x_{n_1,n_2+N/2} + x_{n_1+N/2,n_2} + x_{n_1+N/2,n_2+N/2}, \qquad 0 \le n_1, n_2 \le N/2-1$$

The inverse ODPRT is given by

$$x_{n_1,n_2} = \frac{1}{4}X^1(((n_1))_{N/2}, ((n_2))_{N/2}) + \frac{1}{2N}\sum_{m=0}^{N-1} X_m^u(((n_2 - mn_1))_N) + \frac{1}{2N}\sum_{s=0}^{N/2-1} X_s^v(((n_1 - 2sn_2))_N)$$

1.3 Motivation

The Discrete Fourier transform (DFT) is important in many applications. In most of the DFT computation techniques, the real data will be converted into complex form, and the computations will be in complex form, which increases the computation time and the memory requirement. One complex multiplication requires 4 real multiplications and 2 real additions. Two memory locations will be required to store one complex data. Also, the computation time will be more for multiplication than addition. Thus, the speed of computation can be improved by reducing the number of complex multiplications. Direct computation of 2-D DFT computation involves N^4 complex multiplications corresponding to an N x N data. An alternative definition for 2-D DFT computation was proposed in [8] as given below.

$$Y_{k_1,k_2} = \sum_{p=0}^{M-1} Y_{k_1,k_2}^{(p)} W_N^p \tag{1.1}$$

where $Y_{k_1,k_2}^{(p)} = \sum_{\forall(n_1,n_2)\Rightarrow z=p} x_{n_1,n_2} - \sum_{\forall(n_1,n_2)\Rightarrow z=p+M} x_{n_1,n_2}$ (1.2)

$$z = ((n_1k_1 + n_2k_2))_N \quad (1.3)$$

$$M = N/2$$

This method involves $N^3/2$ complex multiplications in the transformation. As N increases, the computational complexity will increase exponentially in both the methods. Since $Y_{k_1,k_2}^{(p)}$ is a mapping of the data onto the planes corresponding to twiddle-factor axes of the 2-D DFT, frequency-domain analysis of signals can be made using $Y_{k_1,k_2}^{(p)}$. If a transformation from time domain to frequency domain and vice-versa using $Y_{k_1,k_2}^{(p)}$ in (1.2) is developed, complex arithmetic can be avoided and it would involve real additions only. This motivates the development of the new transform proposed in this thesis.

1.4 Outline of thesis

A survey of the literature on major signal transforms including most of those reviewed in the current chapter is done in Chapter II.

In Chapter III, the 2-D MRT is described in detail. The forward transform is presented in section 3.2 and the method for direct computation of 2-D MRT is presented in section 3.3. The conditions for existence of the 2-D MRT coefficients are given in section 3.4. Also presented in this section are general solutions for indices of data elements that belong in a 2-D MRT coefficient. In section 3.5, a closed-form expression for 2-D MRT computation obtained as a result of the analysis in section 3.4 is presented. A few important properties of the 2-D MRT are presented and proved in section 3.6. Finally, the inverse 2-D MRT for N a power of 2 is detailed in section 3.7.

Chapter IV presents the study of 1-D MRT. The forward 1-D MRT is proposed in section 4.2, in which direct 1-D MRT computation is discussed and a few examples of 1-D MRT are given for various signal sizes. Section 4.3 is dedicated to the analysis of 1-D MRT. Topics treated in this section include phase index of 1-D MRT coefficients, existence of 1-D MRT coefficients, dependence of phase index on frequency index, and a closed-form expression for 1-D MRT. In Chapter III, although redundancy was observed, it was not explored further. A detailed study of redundancy is done in section 4.4. Two types of redundancy – complete redundancy and derived redundancy are described in this section. The redundancy in 1-D MRT is removed to obtain 1-D UMRT in section 4.5, in which UMRT computation and the number of UMRT coefficients also is presented. The important distinction between the 1-D MRT and the 1-D UMRT is that the 1-D UMRT has the same number of coefficients as the data, unlike the expansive 1-D MRT. In section 4.6 the 1-D inverse UMRT is presented. The inverse 1-D MRT can be obtained by using the inverse 1-D UMRT. A method of 1-D signal representation using 1-D MRT is presented in section 4.7.

Complete redundancy in 2-D MRT is analyzed and a redundancy-eliminated representation called UMRT is arrived at in Chapter V. Divisor and non-divisor rows and columns, complete redundancy between rows/columns, number of unique 2-D MRT frequencies and unique 2-D MRT coefficients are the topics described in section 5.2. The 2-D UMRT is obtained from 2-D MRT by removing the complete redundancy present in the latter. The forward and inverse 2-D UMRT for N a power of 2 is presented. An N x N representation for these 2-D UMRT coefficients is also presented in section 5.3.

The thesis concludes with Chapter VI which presents applications, discussion, conclusion and areas for further research. Applications of 2-D MRT in generation of image blocks, and of 2-D UMRT in image compression and orientation estimation are presented in section 6.2. A few important issues relevant to the MRT and UMRT are discussed in section 6.3. The chapter concludes by stating a few possible areas for further research in section 6.4.

Chapter II

REVIEW OF LITERATURE ON SIGNAL TRANSFORMS

2.1 Introduction

A review of the literature on signal transforms would be helpful in realizing the depth and richness of the field of signal transforms. There is a multitude of transforms that have been developed over the years. The DFT is among the most popularly used transforms in signal processing. Many methods have been proposed for efficient computation of the DFT. In Chapter I, some important transforms were described. In this chapter, the literature corresponding to these various transforms is reviewed.

2.2 DFT/FFT

Many FFT methods have been developed to perform fast DFT computation. In [1], a procedure to compute the N-point DFT that requires a number of operations proportional to $N \log N$ rather than N^2 was presented. This paper proved to be a landmark in signal processing, and led to intensified research on other FFT methods. In [9], Yavne proposed a method that requires the least number of additions and multiplications for real and complex data for FFTs of length-2^n. An alternative form of the FFT was proposed in [10]. In contrast to earlier methods that required multiplication by complex constants, this algorithm requires either real or purely imaginary constants for multiplication, hence reducing the number of multiplications. However, this method requires a greater number of additions. A split-radix approach for length-2^n was proposed in [11], in which radix-2 is used for the even part of the transform and radix-4 for the odd part. This method has the same number of multiplications as [10], but much fewer additions. Rader [12] showed that the DFT of a sequence of N points, when N is a prime number, is circular correlation, and proposed an FFT based on this finding. In [13], number theoretic transforms are developed for fast computation of digital convolution. Winograd proposed the Winograd Fourier Transform Algorithm [14]. This algorithm uses Winograd's theorems on computational complexity for small and large values of sequence lengths. For large lengths, the Chinese remainder theorem was proposed to be used as part of the algorithm. Theoretically, this method required lesser number of multiplications than the Cooley-Tukey method. However, Winograd's algorithm was found to be difficult for practical implementation.

The prime factor algorithm for FFT was proposed in [15], in which Rader's idea of conversion of DFT into convolution and results on implementation of short convolutions with minimum number of multiplications was combined to produce efficient algorithms for long sequences. Nussbaumer in [16] introduced polynomial transforms, which have circular convolution property and hence could be used for fast computation of convolutions. In [17] efficient FFT algorithms are derived from polynomial transforms. Granata et al. [18] used the tensor product for modeling and designing FFT algorithms, making use of the strong connection between tensor product constructs and modern computer architectures. For multi-dimensional DFTs, row-column method and vector radix methods have been used [19]. In [20], the split-radix approach is extended for vector-radix FFT to two and higher-dimensions. Saidi [21] presents a new FFT algorithm, the decimation-in-time-frequency algorithm, which is obtained by combining the decimation-in-time and decimation-in-frequency algorithms, thereby reducing the number of real multiplications and additions. This approach can be extended to vector-radix multidimensional FFT algorithms also.

Guo et al. [22] present an approach that uses symmetric properties of the basis function to remove redundancies in the calculation of the DFT. They develop an algorithm called the Quick Fourier Transform which has a simple structure and is well-suited for DFTs on real data. Recently, Lundy and van Buskirk [23] present a new matrix to derive real and complex FFT algorithms for length-2^n. This approach leads to arithmetic operation counts that are better than previously published results of the earlier years. Also, Johnson and Frigo [24] present a recursive modification of the split-radix FFT algorithm that matches the record of lowest number of arithmetic operations in [23]. Polynomial algebra and the Chinese Remainder Theorem were the foundations of many early FFT and fast convolution algorithms, to which Winograd has made considerable contribution. Following on this line, Puschel [25] extends the polynomial algebra framework to include most trigonometric transforms used in signal processing. They have developed a new theory called the algebraic signal processing theory, which relates algebra to linear signal processing. Using this theory, a class of new algorithms including general-radix algorithms for the DCTs can be obtained. In this theory, each unique signal model has its own class of Fourier transforms.

Ansari [26] presents an extension of the DFT defined as a linear combination of the forward and inverse DFTs of a sequence, with the coefficients of the linear combinations forming a new real transform for a real sequence, called the Discrete Combination Fourier Transform (DCFT). The DHT is seen to be a special case of the DCFT.

2.3 FRFT

The fractional Fourier transform (FRFT) is a generalization of the classical Fourier transform. Almeida [27] recognizes the utility of the FRFT to signal processing, presents its properties, interprets the FRFT as a spectrum rotation of the signal in the time-frequency plane, and studies its relationships with time-frequency representations like the short-time Fourier transform and the Wigner distribution. An algorithm for efficient and accurate computation of the FRFT is given in [28]. In [29], a general definition for FRFT is formulated for four signal classes. Soo-Chang Pei [30] proposed the discrete version of the FRFT, the discrete fractional Fourier transform (DFRFT), and established the relationship between FRFT and DFRFT. A closed-form analytic expression for DFRFT is obtained in [31]. However, an alternative definition of the DFRFT is presented in [32]. The DFRFT approach is extended to DCT and DST in [33], yielding corresponding new transforms for each. Relations between fractional operations like fractional convolution, fractional correlation etc and time-frequency distributions are explored in [34]. A new method for DFRFT computation is presented in [35].

2.4 Time-frequency transforms

Mathematically, the two approaches to time-frequency analysis are the linear transform approach and the nonlinear operations approach. The non-linear approach attempts to produce an energy distribution in the time-frequency domain. Gabor proposed linear transforms for time-frequency analysis in [36]. That approach is to characterize a time function in time and frequency simultaneously, known as the Gabor expansion. A time-frequency description of a signal is obtained by performing Fourier analysis on the signal as it appears when seen through a set of identical Gaussian windows translated with respect to each other in time. Difficulty in computing Gabor transform coefficients limited use of the Gabor transform earlier. The proof of existence of the Gabor expansion of a signal into a discrete set of Gaussian elementary signals is given in [37]. The discrete Gabor transform and Gabor expansion for infinite sequences are developed in [38]. A Gabor transform for real, discrete signals is presented in [39], along with a computationally attractive method for computing the transform. A method for calculating the coefficients of the Gabor expansion, in the case of oversampling, is presented in [40]. Undersampled discrete Gabor transforms are investigated in [41]. A related time-frequency transform, the Zak transform was developed in [42]. Discrete Zak transform (DZT) and fast algorithms are presented in [43]. The DZT is shown to be a generalization of the short-time Fourier transform in [44].

2.5 Lapped transforms

A class of unitary transformations, called lapped orthogonal transforms (LOT) is investigated in [45]. An exact derivation of an optimal LOT is presented in [46]. A related transform, the Modulated Lapped transform (MLT) is presented and the LOT has been generalized in [3]. Dietmar and Kunz proposed the Lapped Directional Transform (LDT), a real-valued lapped transform for 2-D signals in [4].

Integer versions of the various lapped transforms, called the Integer Lapped Transforms (ILT) are proposed in [47] and [48]. An orthogonal and orientation selective lapped transform, the Lapped Hartley transform (LHT) is presented and its properties evaluated in [49].

2.6 DHT

The DHT, introduced by Bracewell in [50] is a transform that maps a real-valued sequence to a real-valued spectrum; it is closely related to the DFT. A set of FFT-type algorithms for fast computation of DHT is developed in [51]. Data compression properties of the DHT on a Markov-I signal are studied and compared with those of Fourier transform in [52]. The DHT is generalized into four classes and fast algorithms for the generalized transforms derived in [53]. Uniyal [54] showed that the fast Hartley transform algorithm is not superior to real-valued FFT algorithms.

A split-radix algorithm capable of flexibly computing the DHT of various sequence lengths is presented in [55] and improvements on these algorithms in [56]. A radix-2x2x2 algorithm for 3-D discrete Hartley transform is presented in [57]. A direct method for computation of a length-N type-II generalized DHT from the coefficients of two adjacent length-N/2 generalized DHTs is presented in [58]. Fast algorithms for computing sliding-window generalized DHT are proposed in [59].

2.7 DCT

The discrete cosine transform is defined in [60], where it is computed using the FFT. Eight different types of DCTs exist [61]. [62] presents a fast DCT algorithm that has reduced computational complexity compared to DCT computation using FFT. A fast DCT for 2-D signals is presented by Makhoul [63], making use of complex multiplications. Haque [64] presents 2-D fast DCT algorithms for power-of-2 dimensions that require only real multiplications and additions. A factorization of the DHT is used to obtain fast DCT algorithms in [65]. A DCT-type orthogonal transform, called the Hadamard-structured DCT (HDCT) and its fast algorithms are proposed in [66]. A fast and modular N x N 2-D DCT algorithm using N-point 1-D DCT computations and butterfly stages is presented in [67], a simplification which is presented in [68]. Bi et. al [69] propose a fast algorithm for type-II 2-D DCT that achieves savings on the number of arithmetic operations and decomposes the 2-D DCT into various combinations of cosine and sine sequences. Polynomial transform algorithms for multidimensional DCT (M-D DCT) are proposed in [70]. Fast algorithms for M-D DCT are presented in [71] and [72].

Cham [73] proposed conversion of DCT into integer cosine transforms (ICT) by replacing the real components by integers. The conditions for approximating order-8 sinusoidal transforms by orthogonal integer transforms using integer arithmetic are derived in [74]. General methods have been derived to obtain the integer transforms from a non-integer transform in [75], where integer transforms analogous to DFT, DST and DHT are presented. A method to factor type-II DCT into lifting steps and additions and then obtain a type-II integer DCT (IntDCT-II) free of floating-point multiplications is proposed in [76]. Integer-to-integer DCT algorithms that use floating-point arithmetic are presented in [73]. Cosine transforms are characterized in an algebraic framework in [77].

2.8 Wavelet transforms

Relationships between the Fourier coefficients of a periodic signal and its wavelet coefficients are derived in [78]. The wavelet transform for discrete-time signals leading to the discrete wavelet transform is dealt with in [79], [80] and [81]. Invertible integer-to-integer wavelet transforms are constructed using the lifting scheme in [5]. The representation of the Fourier transform by a wavelet transform that uses a fully scalable modulated window is presented in [82]. Approaches to reducing the problem of shift variance in the DWT are reviewed in [83]. A method for designing translation invariant directional dyadic wavelet transforms is proposed in [84]. The phaselet transform, an approximately shift invariant redundant dyadic wavelet transform, is proposed in [85].

The contourlet transform, proposed in [86] is a two-dimensional transform that can capture the inherent geometrical structure in images. It is hence a purely two-dimensional approach in that it differs from the separability approach of computing 2-D transforms by their corresponding 1-D transforms. The curved wavelet transform [87] applies 1-D filters along curves for efficient image representation. The interval wavelet transform is used to obtain efficient edge representation in [88].

2.9 Radon Transforms

[89] describes the discrete Radon transform (DRT) and its exact inversion algorithm. It is shown that the DRT can be used to compute the classical RT, and that the DRT inversion procedure can be used to invert the classical RT. The DRT is used to generalize the classical RT for straight lines to include curves and weighted curve functions. In [90], a scheme of finite Radon transform on images of prime size is investigated, and image representations using the finite Radon transform are discussed. A discrete periodic Radon transform (DPRT) and its inversion are proposed in [91]. The DPRT preserves important properties of the continuous Radon transform. The inversion formula is developed using the Fourier slice theorem, and is free from multiplications. The DPRT introduced in [91] exhibits redundancy, is an expansive transform, and non-orthogonal. The same authors introduce an orthogonal discrete periodic Radon transform (ODPRT) in [92] using a new decomposition approach that eliminates redundancy. Realization methods and inverse ODPRT are also presented in the same paper. Applications of the ODPRT in 2-D circular convolution and blind image resolution are presented in [93]. A discrete Radon transform based on images of size p^n x p^n where p is prime ($p \geq 2$) is presented in [94], in which corresponding ODPRT for the new size is also developed.

2.10 Two-stage DFT Computation

Two-stage DFT computation methods have been presented in the literature, as in [95]. The Discrete Fourier Pre-processing transform (DFPT), which is part of the strategy used in [95], is a pre-processing transform obtained by decomposing the cosine and sine functions of the DFT in terms of another set of functions.

2.11 Conclusion

The literature review shows that research continues along different lines on the various categories of signal transforms. Faster algorithms and methods to perform transform computation continue to be presented and improved upon. At the same time, newer transforms appear in the literature. These transforms are often extensions of an existing transform. In the case of trigonometric transforms, their corresponding fractional transforms have evolved as a major area of research developed. A recent trend is the development of application-specific transforms, especially for use exclusively in image processing. Integer approximations of transforms and integer-to-integer transforms have been proposed.

Chapter III

DEVELOPMENT OF FORWARD AND INVERSE 2-D MRT

3.1 Introduction

Transforms are used to find an alternative domain where processing of the task at hand is easier to perform. Fourier transform is popular to map a signal from time domain to frequency domain. The Discrete Fourier Transform (DFT) is the digital implementation of the Fourier transform and has many fast implementation algorithms. In the DFT computation, even when the data is real-valued, the DFT coefficients will be complex. In [8], 2-D DFT representation was modified in terms of real additions, which requires $N/2$ complex multiplication in the computation of each of the N^2 DFT coefficients. However, complex multiplications can be avoided if the signal is represented in terms of the signal components which would otherwise be multiplied with the exponential term in the DFT representation developed in [8]. Further, this results in a new way of analyzing signals. Hence a new real transform, named MRT is developed, which can represent signals using real additions and without complex arithmetic, and which offers a different way of signal analysis.

3.2 Forward Transform

The 2-D DFT Y_{k_1,k_2} of a 2-D signal x_{n_1,n_2}, $0 \le n_1,n_2 \le N-1$ is given by

$$Y_{k_1,k_2} = \sum_{n_1=0}^{N-1}\sum_{n_2=0}^{N-1} x_{n_1,n_2} W_N^{n_1k_1+n_2k_2}, \quad 0 \le k_1,k_2 \le N-1 \tag{3.1}$$

where $W_N = e^{\frac{-j2\pi}{N}}$

Since the twiddle factor W_N is periodic, (3.1) can be expressed as

$$Y_{k_1,k_2} = \sum_{n_1=0}^{N-1}\sum_{n_2=0}^{N-1} x_{n_1,n_2} . W_N^{((n_1k_1+n_2k_2))_N} \tag{3.2}$$

The exponent $((n_1k_1 + n_2k_2))_N = p$, $0 \le p \le N-1$ is satisfied by a set of (n_1, n_2) for a given (k_1, k_2). Hence, by grouping such data and applying the property that $W_N^{p+N/2} = -W_N^p$, (3.2) can be expressed as

$$Y_{k_1,k_2} = \sum_{p=0}^{M-1} Y_{k_1,k_2}^{(p)} W_N^p \tag{3.3}$$

where

$$Y_{k_1,k_2}^{(p)} = \sum_{\forall(n_1,n_2)\Rightarrow z=p} x_{n_1,n_2} - \sum_{\forall(n_1,n_2)\Rightarrow z=p+M} x_{n_1,n_2}, \qquad 0 \le p \le M-1 \tag{3.4}$$

$$z = ((n_1k_1 + n_2k_2))_N \tag{3.5}$$

$$M = N/2 \tag{3.6}$$

The computation of the N^2 DFT coefficients Y_{k_1,k_2} using (3.3) and (3.4) involves M complex multiplications each, and thus a total of $N^3/2$ complex multiplications for any even N, given $Y_{k_1,k_2}^{(p)}$. If $Y_{k_1,k_2}^{(p)}$ is developed as a transform, the transformation would involve only real additions and no complex multiplications. Thus, a new transform is proposed, with (3.4) as the forward transform relation. The transformation maps the data x_{n_1,n_2} of size N x N into M matrices $Y_{k_1,k_2}^{(p)}$, for p = 0, 1, ..., M - 1, each of size N x N. Since it maps a matrix of real data into M matrices of real values in the frequency domain, the proposed transform is named as an M-dimensional Real Transform (MRT).

Since the property $W_N^{p+N/2} = -W_N^p$ is used in arriving at the formulation in (3.3) and the definition of the MRT in (3.4), a central condition regarding the value of N in the MRT context is that N needs to be even. In the remainder of the thesis, this condition is assumed.

From (3.4), the forward 2-D MRT can also be expressed as

$$Y_{k_1,k_2}^{(p)} = \sum_{n_1=0}^{N-1}\sum_{n_2=0}^{N-1} A_{k_1,k_2,p,n_1,n_2} x_{n_1,n_2}, \qquad 0 \le k_1,k_2 \le N-1, \quad 0 \le p \le M-1 \tag{3.4a}$$

$$A_{k_1,k_2,p,n_1,n_2} = \begin{cases} 1, & ((n_1k_1 + n_2k_2))_N = p \\ -1, & ((n_1k_1 + n_2k_2))_N = p + M \\ 0, & \text{otherwise} \end{cases} \tag{3.4b}$$

Hence, the transform kernel of 2-D MRT is defined by (3.4b).

3.3 Direct Computation

The direct computation of MRT can be carried out by the following steps, for a given value of k_1, k_2 & p:

1) Initialize the MRT coefficient to zero.
2) For each (n_1, n_2), calculate the term z in (3.5) which involves two multiplications n_1k_1 & n_2k_2, and a modulus operation w.r.t. N.
3) Perform a logical comparison to check the conditions $z = p$, or $z = p + M$.
4) Depending on which condition is satisfied, add or subtract the data element corresponding to indices (n_1, n_2) to or from the current value of MRT coefficient respectively.

$$
x_{n_1,n_2} = \begin{bmatrix}
95 & 82 & 94 & 14 & 45 & 84 & 30 & 38 \\
23 & 44 & 92 & 20 & 93 & 2 & 19 & 86 \\
61 & 62 & 41 & 20 & 47 & 68 & 19 & 85 \\
49 & 79 & 89 & 60 & 42 & 38 & 68 & 59 \\
89 & 92 & 6 & 27 & 85 & 83 & 30 & 50 \\
76 & 74 & 35 & 20 & 53 & 50 & 54 & 90 \\
46 & 18 & 81 & 2 & 20 & 71 & 15 & 82 \\
2 & 41 & 1 & 75 & 67 & 43 & 70 & 64
\end{bmatrix}
$$

$$
Y^{(0)}_{k_1,k_2} = \begin{bmatrix}
3360 & -11 & 149 & -11 & -86 & -11 & 149 & -11 \\
20 & 43 & -142 & -6 & 88 & 137 & -102 & 10 \\
206 & -88 & 125 & 116 & 82 & -88 & 147 & 116 \\
20 & -6 & -102 & 43 & 88 & 10 & -142 & 137 \\
4 & 199 & 195 & 199 & -62 & 199 & 195 & 199 \\
20 & 137 & -142 & 10 & 88 & 43 & -102 & -6 \\
206 & 116 & 147 & -88 & 82 & 116 & 125 & -88 \\
20 & 10 & -102 & 137 & 88 & -6 & -142 & 43
\end{bmatrix}
\quad
Y^{(1)}_{k_1,k_2} = \begin{bmatrix}
0 & 53 & 0 & -316 & 0 & -53 & 0 & 316 \\
-73 & -209 & -88 & -51 & 91 & -157 & 18 & 45 \\
0 & 214 & 0 & 71 & 0 & -214 & 0 & -71 \\
121 & -16 & 10 & 210 & 95 & -24 & -138 & 118 \\
0 & -157 & 0 & -68 & 0 & 157 & 0 & 68 \\
73 & 157 & 88 & -45 & -91 & 209 & -18 & 51 \\
0 & -82 & 0 & 125 & 0 & 82 & 0 & -125 \\
-121 & 24 & -10 & -118 & -95 & 16 & 138 & -210
\end{bmatrix}
$$

$$
Y^{(2)}_{k_1,k_2} = \begin{bmatrix}
0 & 134 & 139 & -134 & 0 & 134 & -139 & -134 \\
68 & 104 & 94 & -139 & -56 & 48 & 62 & -61 \\
-16 & -37 & 295 & 59 & 130 & -37 & -69 & 59 \\
-68 & 139 & -62 & -104 & 56 & 61 & -94 & -48 \\
0 & 122 & 345 & -122 & 0 & 122 & -345 & -122 \\
68 & 48 & 94 & -61 & -56 & 104 & 62 & -139 \\
16 & -59 & 69 & 37 & -130 & -59 & -295 & 37 \\
-68 & 61 & -62 & -48 & 56 & 139 & -94 & -104
\end{bmatrix}
\quad
Y^{(3)}_{k_1,k_2} = \begin{bmatrix}
0 & -316 & 0 & 53 & 0 & 316 & 0 & -53 \\
121 & 210 & -138 & -16 & 95 & 118 & 10 & -24 \\
0 & 125 & 0 & -82 & 0 & -125 & 0 & 82 \\
-73 & -51 & 18 & -209 & 91 & 45 & -88 & -157 \\
0 & -68 & 0 & -157 & 0 & 68 & 0 & 157 \\
-121 & -118 & 138 & 24 & -95 & -210 & -10 & 16 \\
0 & 71 & 0 & 214 & 0 & -71 & 0 & -214 \\
73 & -45 & -18 & 157 & -91 & 51 & 88 & 209
\end{bmatrix}
$$

Figure 3.1: An 8 x 8 signal and its MRT matrices

For a particular value of (k_1, k_2), the number of data elements involved in forming all associated MRT coefficients is N^2. Since there are N^2 pairs (k_1, k_2), there are thus N^4 computational steps 1–4. An example of a 2-D signal of size 8 x 8, and its associated 2-D MRT are shown in Figure 3.1. The following observations can be made from the 2-D MRT presented in Figure 3.1.

1) There are 4 MRT matrices of size 8 x 8 corresponding to the given data of size 8 x 8. Similarly, there will be M MRT matrices of size N x N for a given data of size N x N.

2) The total number of 2-D MRT coefficients will be $N^3/2$ for a data matrix of size N x N. Hence for $N = 8$, there will be 256 coefficients.

3) There are a large number of zero-valued coefficients in the MRT matrices. There are no zero-valued coefficients in $Y^{(0)}_{k_1,k_2}$, 16 such positions in $Y^{(1)}_{k_1,k_2}$ & $Y^{(3)}_{k_1,k_2}$, and 4 in $Y^{(2)}_{k_1,k_2}$, a total of 36 such positions for $N = 8$.

4) The MRT matrices are highly redundant. Many MRT coefficients share the same magnitude, although the sign may be different. e.g. $Y^{(0)}_{1,1} = Y^{(0)}_{5,5}$, $Y^{(1)}_{3,2} = -Y^{(1)}_{7,2}$.

Since there are many MRT coefficients that are zero-valued, it is unnecessary to perform the normal computational steps for such values of (k_1, k_2) & p. There is thus scope for reducing computational complexity involved in MRT computation. Also, if an (n_1, n_2) satisfies $z = p$, then it cannot satisfy $z = p + M$. Hence, the comparison operation need not be performed for both $z = p$ and $z = p + M$. Thus a mathematical analysis of the MRT computation is performed in the following section.

3.4 Analysis

From the preliminary observations made above, there is a need for detailed analysis of the transformation so as to simplify the computation. Thus, there is a need for identifying the trivial MRT coefficients in the computation.

3.4.1 Existence of 2-D MRT Coefficients

In Figure 3.1 many of the MRT coefficients, like $Y_{0,0}^{(1)}$, are zero-valued. The first step involved in the computation of the MRT coefficient is the location of those input data elements (n_1, n_2) that satisfy the congruence $((n_1k_1 + n_2k_2))_N = p$. This becomes $(((n_1 \text{ x } 0) + (n_2 \text{ x } 0)))_N = 1$, for $Y_{0,0}^{(1)}$, $k_1 = 0$, $k_2 = 0$, $p = 1$. No values of (n_1, n_2) satisfy this equation. Hence, no valid MRT coefficient exists for the set of values, $k_1 = 0$, $k_2 = 0$, $p = 1$. Such MRT coefficients, although invalid, are considered zero-valued. There are numerous sets of such invalid coefficients represented as zero as is observed from Figure 3.1. A few other examples of such coefficients in Figure 3.1 are $Y_{0,0}^{(2)}$, $Y_{2,4}^{(3)}$, $Y_{4,4}^{(1)}$, $Y_{6,2}^{(3)}$ etc. These trivial MRT coefficients can be eliminated from the MRT computation. Thus, there is a need for the analysis of existence of MRT coefficients, which is explained below.

From (3.4), the MRT is seen to be comprised of two summations, and a data element x_{n_1,n_2} belongs in either of these summations provided its index (n_1, n_2) is a solution of the congruence equations associated with the summations. The first summation is $\sum_{\forall (n_1,n_2) \Rightarrow z=p} x_{n_1,n_2}$ for which the associated congruence equation is, using (3.5),

$$((n_1k_1 + n_2k_2))_N = p \tag{3.7}$$

The data elements x_{n_1,n_2} corresponding to the solutions (n_1, n_2) of (3.7) are defined as the positive group of the MRT coefficient.

The second summation is $\sum_{\forall (n_1,n_2) \Rightarrow z=p+M} x_{n_1,n_2}$ which has the corresponding congruence equation

$$((n_1k_1 + n_2k_2))_N = p + M \tag{3.8}$$

The data elements x_{n_1,n_2} corresponding to the solutions (n_1, n_2) of (3.8) are defined as the negative group of the MRT coefficient.

Let $q = ((n_1(k_1/g(k_1, k_2)) + n_2(k_2/g(k_1, k_2))))_N$ so that (3.7) becomes

$$((g(k_1, k_2)q))_N = p \tag{3.9}$$

From Appendix A.3, (3.9) has $d = g(g(k_1, k_2), N)$ solutions mod N provided the condition $d \mid p$ is satisfied. q has incongruent solutions of the form

$$q = q_0 + (N/d)t, \quad 0 \le t < d \tag{3.10}$$

where q_0 is a particular solution of (3.9).

Hence, the condition for existence of a positive group is that $d = g(g(k_1, k_2), N) = g(k_1, k_2, N)$ is divisible by p, i.e. $g(k_1, k_2, N) \mid p$.

The condition for existence of positive group is:

A: $g(k_1, k_2, N) \mid p$,

Similarly, the condition for existence of negative group is

B: $g(k_1, k_2, N) \mid (p + M)$.

If $g(k_1, k_2, N) \mid p$ and $g(k_1, k_2, N) \mid M$, then $g(k_1, k_2, N) \mid (p + M)$. If so, then both positive group and negative group exist for the MRT coefficient $Y^{(p)}_{k_1,k_2}$. If condition A is satisfied and if $g(k_1, k_2, N) \mid M$ is not satisfied, then condition B is not satisfied. In this case, only positive group exists for $Y^{(p)}_{k_1,k_2}$. Similarly, if condition A is not satisfied and if $g(k_1, k_2, N) \mid M$ is also not satisfied, then if condition B is satisfied, then only a negative group exists for $Y^{(p)}_{k_1,k_2}$. Hence, it may be concluded that an MRT coefficient $Y^{(p)}_{k_1,k_2}$ has positive and negative groups only if the condition $g(k_1, k_2, N) \mid M$ is true. If this condition is false, then either a positive group or a negative group exists, but both cannot exist together.

The data groups present in the MRT coefficient having been defined, a detailed analysis of the data groups is presented below, which will identify the relations among data elements within a group.

3.4.2 General solutions for elements in a group

Given the particular solution q_0, the solutions to n_1 & n_2 have to be obtained. A general solution to (3.9) can be written as

$$q = Nq_1 + q_0, \; q \in Z \tag{3.11}$$

$$\therefore (k_1/g(k_1, k_2))n_1 + (k_2/g(k_1, k_2))n_2 - Nq_1 = q_0 \tag{3.12}$$

Using Bezout's lemma (Appendix A.1), the minimum value of the quantity $((k_2/a)n_2 - Nq_1)$ in (3.12) is given by $g(k_2 / g(k_1, k_2), N)$. Hence, (3.12) can be expressed as

$$(k_1 / g(k_1,k_2))n_1 + g(k_2 / g(k_1,k_2),N)q_2 = q_0 \, , \; q_2 \in Z \tag{3.13}$$

From (3.13), it is seen that n_1 has $g(k_1 / g(k_1,k_2), k_2 / g(k_1, k_2), N)$ solutions mod $g(k_2 / g(k_1, k_2), N)$, and hence one solution mod $g(k_2 / g(k_1, k_2), N)$, since $g(k_1 / g(k_1, k_2), k_2 / g(k_1, k_2), N) = 1$. This can be justified as follows: $g(k_1 / g(k_1, k_2), k_2 / g(k_1, k_2), N) = g(g(k_1 / g(k_1, k_2), k_2 / g(k_1, k_2)), N)$, and $g(k_1/g(k_1, k_2), k_2 / g(k_1, k_2)) = 1$, from gcd property in Appendix A.4. Hence, $g(g(k_1 / g(k_1, k_2), k_2 / g(k_1, k_2)), N) = g(1, N) = 1$. Thus, $g(k_1 / g(k_1, k_2), k_2 / g(k_1, k_2), N) = 1$.

Hence, n_1 has $N/g(k_2/g(k_1, k_2), N)$ solutions in the range $[0, N - 1]$. Given n_{10} is a solution, n_1 has the general solution

$$n_1 = n_{10} + g(k_2 / g(k_1,k_2),N)t \; , \; 0 \le t < (N / g(k_2 / g(k_1,k_2),N)) - 1 \tag{3.14}$$

The number of solution-pairs (n_1, n_2) in the range $[[0, N-1], [0, N - 1]]$ can be determined only by determining the number of solutions n_2, corresponding to one particular solution n_{10} of n_1 in the range $[0, N - 1]$. Given n_{10}, (3.12) can be written as

$$(k_1 / g(k_1,k_2))n_{10} + (k_2 / g(k_1,k_2))n_2 = Nq_1 + q_0 \tag{3.15}$$

$$\therefore (k_2 / g(k_1,k_2))n_2 - Nq_1 = w_0 - (k_1 / g(k_1,k_2))n_{10} \tag{3.16}$$

From (3.16), n_2 has $g(k_2/g(k_1,k_2),N)$ solutions mod N, for a given n_{10}. Hence, the total number of solution-pairs Tot_{sol} is found by the product of two factors:

i) f_1, the number of solutions of n_1 in the range $[0, N - 1]$, and

ii) f_2, the number of solutions of n_2 that exist for every solution of n_1

As seen already,

$$f_1 = N / g(k_2 / g(k_1,k_2),N) \tag{3.17}$$

$$f_2 = g(k_2 / g(k_1,k_2),N) \tag{3.18}$$

From (3.17) and (3.18),

$$Tot_{sol} = f_1 f_2 = N \tag{3.19}$$

Thus, there are N solution-pairs (n_1, n_2) in the range $[[0, N - 1], [0, N - 1]]$. These solution pairs correspond to a particular solution q_0 of (3.9). If (3.9) has $d = g(g(k_1, k_2), N)$ solutions, then the total number of solution pairs in the range $[[0, N - 1], [0, N - 1]]$ is given by dN.

Similarly, (3.8) becomes

$$((g(k_1,k_2)q))_N = p + M \tag{3.20}$$

(3.20) has $d = g(g(k_1, k_2), N)$ solutions mod N provided the condition $g(k_1, k_2) \mid (p + M)$ is satisfied. q has incongruent solutions of the form

$$q = q_0 + (N / d)t \, , \quad 0 \le t < d \tag{3.21}$$

In (3.21), q_0 is a particular solution of (3.20).

Hence, the condition for existence of a negative group is that $d = g(g(k_1, k_2), N) = g(k_1, k_2, N)$ divides $p + M$, i.e. $g(k_1, k_2, N) | (p + M)$. By using exactly the same analysis that has been done for (3.7), it can be shown that the total number of solution pairs of (3.8) in the range [[0, N - 1], [0, N - 1]] is given by dN, where $d = g(g(k_1, k_2), N)$.

The general solutions for elements in a group are given by:

$$n_1 = n_{10} + g(k_2 / g(k_1,k_2),N)t, \quad 0 \le t < (N / g(k_2 / g(k_1,k_2),N)) - 1, \text{ and}$$

$$n_2 = n_{20} + g(k_1 / g(k_1,k_2),N)t, \quad 0 \le t < (N / g(k_1 / g(k_1,k_2),N)) - 1.$$

where n_{10} and n_{20} are particular solutions, i.e. they are the indices of one of the elements in the group.

3.5 2-D MRT – Closed-form Computation

The analysis shows that the forward MRT computation in (3.4) can be modified so as to eliminate the computation and logical operation associated with (3.5). In this way, MRT computation can be made simpler. Hence, the forward MRT relation is modified in this section.

Recalling from (3.4), (3.5) & (3.6), 2-D MRT has the following definition:

$$Y_{k_1,k_2}^{(p)} = \sum_{\forall (n_1,n_2) \Rightarrow z=p} x_{n_1,n_2} - \sum_{\forall (n_1,n_2) \Rightarrow z=p+M} x_{n_1,n_2}, \qquad 0 \le p \le M-1$$

$$z = ((n_1k_1 + n_2k_2))_N$$

$$M = N/2$$

The matrices $Y_{k_1,k_2}^{(p)}$ for p = 0, 1, …, M - 1 are the MRT matrices.

Recalling (3.7),

$$((n_1k_1 + n_2k_2))_N = p$$

$$n_1k_1 + n_2k_2 - Nq = p, \quad q \in Z$$

Using Bezout's lemma, $n_2k_2 - Nq = q_1 g(k_2, N), \quad q_1 \in Z$

$$n_1k_1 + g(k_2, N)q_1 = p$$

n_1 has $g(k_1, k_2, N)$ solutions mod $g(k_2, N)$. Hence, there are $Ng(k_1,k_2,N)/g(k_2,N)$ solutions mod N. Also, for the same value of n_1 if

$$((n_1k_1 + (n_2 + v)k_2))_N = p \quad v \in Z, \tag{3.22}$$

then, $((vk_2))_N = 0$. (3.23)

v in (3.23) has $g(k_2, N)$ solutions mod N. Hence, n_2 has $g(k_2, N)$ solutions for the same value of n_1. So, given a particular solution (n_{10}, n_{20}), another solution at a different n_1 has the value $n_1 = n_{10} + g(k_2,N)/g(k_1,k_2,N)$. If ($n_1 + u$, $n_2 - v$) is a solution of (3.7), then

$$(((n_1+u)k_1+(n_2-v)k_2))_N = p \qquad u,v \in Z \tag{3.24}$$

From equations (3.7) and (3.24),

$$((uk_1 - vk_2))_N = 0 \tag{3.25}$$

$$((vk_2))_N = ((uk_1))_N \tag{3.26}$$

In the most general case, solutions to (3.26) can be obtained by using congruence tables involving u, v, k_1, k_2 & N. Extended Euclidean algorithm (Appendix A.5) also can be used for its solution. Trial-and-error is another possible approach. However, for special cases for the values of k_1 & k_2, it is possible to obtain the value of v by direct division. The value of u has already been obtained as $g(k_2, N) / g(k_1, k_2, N)$.

Hence, (3.26) becomes

$$((vk_2))_N = ((k_1\, g(k_2,N)/g(k_1,k_2,N)))_N \tag{3.27}$$

In summary, it has been established that for given values of k_1, k_2, p and N, solutions to n_1 occur at a gap of $g(k_2, N) / g(k_1, k_2, N)$. Further, for a given solution n_1, solutions to n_2 occur at a gap of $N/g(k_2, N)$ columns. Finally, for a change in value of n_1 by $g(k_2, N) / g(k_1, k_2, N)$, a corresponding value of n_2 is obtained from the solution to v in (3.27). The same conclusions hold for solutions to the negative group congruence equation

$$((n_1k_1 + n_2k_2))_N = p + M$$

Using the above knowledge, the following closed-form formula is proposed for 2-D MRT.

$$Y_{k_1,k_2}^{(p)} = \sum_{j_1=0}^{N(g(k_1,k_2,N)/g(k_2,N))-1} \sum_{j_2=0}^{g(k_2,N)-1} \Big[x\big(((n_{10}^{+}+j_1 g(k_2,N)/g(k_1,k_2,N)))_N, ((n_{20}^{+}-j_1 v+j_2 N/g(k_2,N)))_N\big) - x\big(((n_{10}^{-}+j_1 g(k_2,N)/g(k_1,k_2,N)))_N, ((n_{20}^{-}-j_1 v+j_2 N/g(k_2,N)))_N\big)\Big] \tag{3.28}$$

In the formula, $n_{10}^{+}, n_{20}^{+}, n_{10}^{-}$ & n_{20}^{-} are particular solutions for the positive and negative groups respectively. The limitations of closed-form formula are

1) There is need for calculation of the parameter v, and this is not always simple.
2) Particular solutions for positive and negative groups need to be available.

Particular solutions

In the closed-formula proposed in (3.28), the particular solutions for positive and negative groups need to be computed. A particular solution for the positive group is one pair of solutions to the positive group congruence equation. Given (n_{10}^{+}, n_{20}^{+}), the negative group particular solution pair (n_{10}^{-}, n_{20}^{-}) can be obtained. Assume $n_{10}^{-} = n_{10}^{+} + e$, $n_{20}^{-} = n_{20}^{+} + f$, $e, f \in Z$

$$((n_{10}^{+}k_1 + n_{20}^{+}k_2))_N = p \tag{3.29}$$

$$(((n_{10}^{+}+e)k_1+(n_{20}^{+}+f)k_2))_N = p+M \tag{3.30}$$

From equations (3.29) & (3.30),

$$((ek_1+fk_2))_N = M \tag{3.31}$$

For a negative group particular solution to exist in the same row as a positive group particular solution, $e = 0$. Hence, (3.31) becomes

$$((fk_2))_N = M \tag{3.32}$$

Using Appendix A.3, f has a solution if $g(k_2, N) \mid M$. Hence, positive and negative group solutions exist in the same row if $g(k_2, N) \mid M$. To obtain a solution for f in (3.32), an evaluation is done if the ratio $k_2/g(k_2, N)$ is always odd, provided $g(k_2,N) \mid M$. Let $q = g(k_2, N)$. Then, $N = qt, t \in Z$

$$\therefore M = (q/2)t$$

Since $q \mid M$, $q \mid (q/2)t$

$$\therefore (q/2)t = qt', t' \in Z$$

$$\therefore t' = t/2$$

For t' to be an integer, t has to be even.

Also, k_2 can be expressed as

$$k_2 = qt_1, \therefore t_1 = k_2/q,\ t_1 \in Z$$

$$g(qt_1, qt) = q,\ \therefore g(t_1, t) = 1$$

Thus, t_1 is odd, and hence the ratio, $k_2 / g(k_2, N)$, is odd. Assume f has the solution,

$$f = (N/2g(k_2,N)) \tag{3.33}$$

Using (3.33), the LHS of equation (3.32) becomes,

$$(((N/2g(k_2,N))k_2))_N = (((N/2)(k_2/g(k_2,N))))_N \tag{3.34}$$

The RHS of (3.34) is equal to 0 if the ratio $k_2 / g(k_2, N)$ is even, and equal to M if the ratio $k_2/g(k_2,N)$ is odd. However, the ratio has already been found to be odd. Hence, the RHS of (3.34) is equal to the RHS of (3.32), and thus, the proposed solution in (3.33) for f is valid. In conclusion, if $g(k_2, N) \mid M$, $n_{10}^{-} = n_{10}^{+}$, and

$$n_{20}^{-} = n_{20}^{+} + f = n_{20}^{+} + (N/2g(k_2,N)) \tag{3.35}$$

Using the same logic, the following can also be concluded: if $g(k_1, N) \mid M$, $n_{20}^{-} = n_{20}^{+}$, and

$$n_{10}^{-} = n_{10}^{+} + [N/(2g(k_1,N))] \tag{3.36}$$

When both $g(k_1, N) \mid M$ and $g(k_2, N) \mid M$ do not hold, $g(k_1, k_2, N) \mid M$ does not hold, and this implies that a negative group and positive group do not exist together. Figure 3.2 gives a flow-chart illustrating a method for finding particular solutions for all phase indices, given (k_1, k_2).

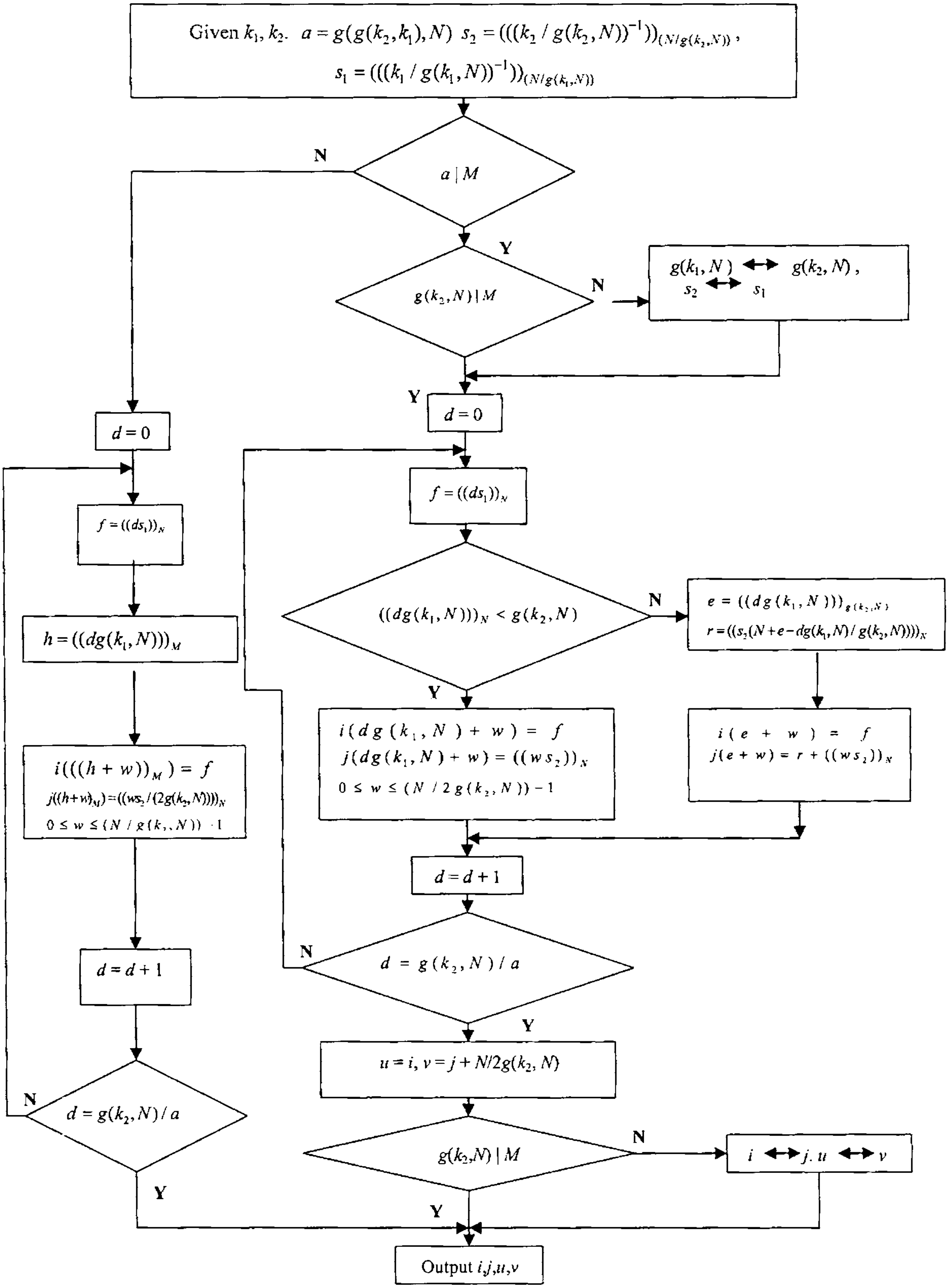


Figure 3.2: Flowchart showing computation method for MRT particular solutions

Algorithm corresponding to flow-chart in Figure 3.2:

Let $a = g(k_1,k_2,N)$. Calculate $s_2 = (((k_2 / g(k_2,N))^{-1}))_{(N/g(k_2,N))}$, $s_1 = (((k_1 / g(k_1,N))^{-1}))_{(N/g(k_1,N))}$

if $a|M$ is satisfied

{

if $g(k_2,N)\ |M$ is not satisfied, interchange ($g(k_1,N)$ & $g(k_2,N)$) and (s_1 & s_2)

for $d = 0$: ($g(k_2,N)/a$) - 1

{

$f = \mathrm{mod}(ds_1,N)$;

if $\mathrm{mod}(d\,g(k_1,N),N) < g(k_2,N)$

{

for $w = 0$: $g(k_2,N)$:M - $g(k_2,N)$

{

$i(d\,g(k_1,N) + w) = f$;

$j(d\,g(k_1,N) + w) = \mathrm{mod}((w/ g(k_2,N))s_2,N)$;

$v(d\,g(k_1,N) + w) = \mathrm{mod}((w/ g(k_2,N))s_2 + N/(2\,g(k_2,N)),N)$;

}

}

else

{

$e = \mathrm{mod}(d\,g(k_1,N), g(k_2,N))$

$r = \mathrm{mod}(s_2(N+e-d\,g(k_1,N))/ g(k_2,N),N)$

for $w = 0$: $g(k_2,N)$:M - $g(k_2,N)$

{

$i(e+w) = f$;

$j(e+w) = \mathrm{mod}(r + (w/ g(k_2,N))s_2,N)$

$v(e+w) = \mathrm{mod}(r + (w/ g(k_2,N))s_2 + N/(2\,g(k_2,N)),N)$

}

}

}

$u = i$

if $g(k_2,N)\ |M$ is not satisfied interchange (i & j) and (u & v)

}

else

{

for $d = 0$: ($g(k_2,N)/a$) - 1

{

$f = \mathrm{mod}(ds_1,N)$

$h = \mathrm{mod}(d\,g(k_1,N),M)$

for $w = 0$: $g(k_2,N)$:(N - $g(k_2,N)$)

{

$i(\mathrm{mod}(h+w,M)) = f$

$j(\mathrm{mod}(h+w,M)) = \mathrm{mod}((w/ g(k_2,N))s_2,N)$;

}

}

$u = i$; $v = j$;

$u(0{:}a{:}M- a/2)=0$; $v(0{:}\ a{:}M- a/2)=0$; $i(a/2{:}\ a{:}M- a)=0$; $j(a/2{:}\ a{:}M- a)=0$;

}

In Figure 3.2, *i, j* represent arrays containing the particular solutions for positive group of all phase indices and *u, v* represent similar arrays for the particular solutions for negative group of all phase indices.

3.5.1 Comparison

In the direct method of computation for 2-D MRT outlined in section 3.3, there is a need to perform a logical comparison for each data index (n_1, n_2). In contrast, the computation using the closed-form formula in (3.28) avoids this computational step. However, closed-form computation requires overhead in the form of calculation of particular solutions. The simulation results of both the methods give exactly same values for the coefficients. A timing comparison is done for the two MRT computation methods and the results, for values of *N* varying upto 128, are plotted in Figure 3.3 and tabulated in Table 3.1, and. For small values of *N*, the direct method is faster. However as the value of *N* increases, it is seen that closed-form method performs exceedingly faster. All computer experiments in this thesis were run on MATLAB 6.5 Release 13, on an HP laptop, with Centrino–duo T2400 CPU running at 1.83 GHz, having 0.99 GB RAM, and running on Windows XP.

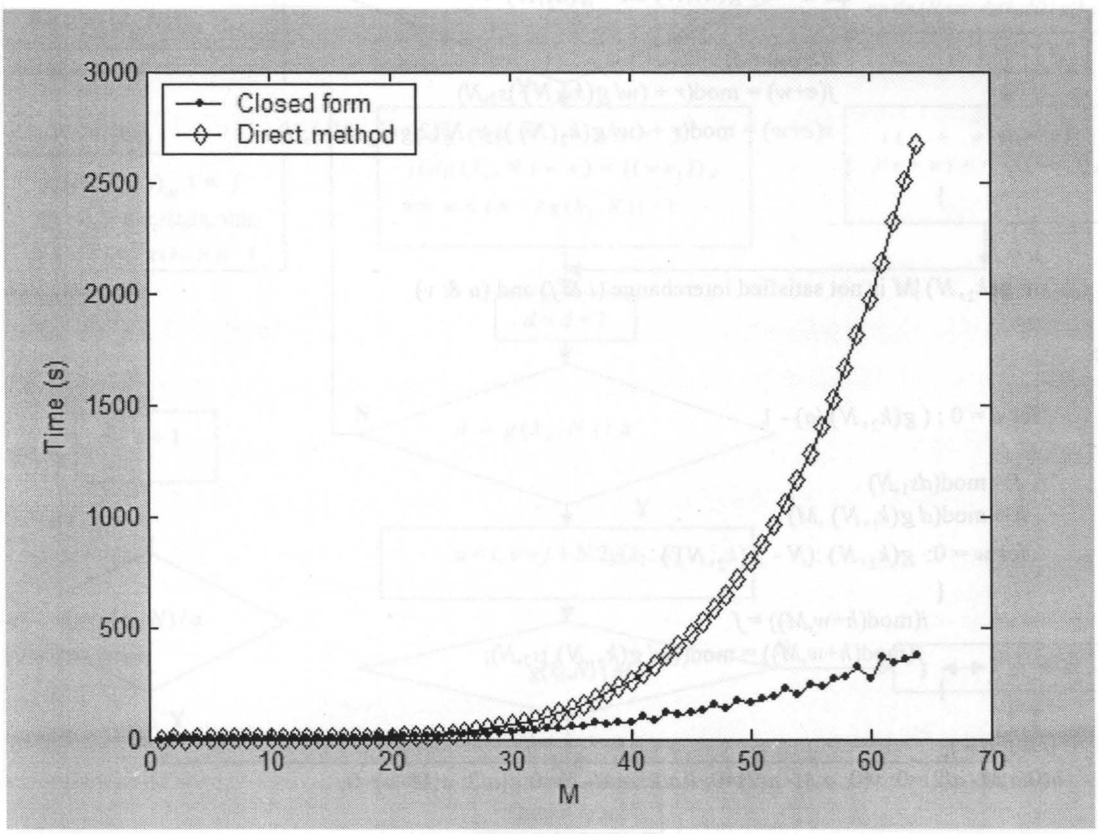


Figure 3.3: Plot showing 2-D MRT computation time using direct and closed form methods

Table 3.1: Time comparison between direct MRT computation and closed-form MRT computation.

N	Direct (seconds)	Closed-form	N	Direct (seconds)	Closed-form
4	0.031	0.031	68	116.672	48.609
6	0	0.078	70	134.734	56.266
8	0.016	0.093	72	155.281	53.063
10	0.015	0.188	74	177.656	71.422
12	0.047	0.266	76	202.782	69.11
14	0.078	0.469	78	230.672	77.25
16	0.094	0.625	80	261.593	77.359
18	0.172	0.922	82	295.891	99.015
20	0.281	1.172	84	333.281	86.672
22	0.453	1.75	86	374.953	115.468
24	0.688	1.875	88	420.078	108.062
26	1	2.859	90	469.735	118.062
28	1.453	3.156	92	524.047	127.766
30	2.047	3.969	94	583.531	153.688
32	2.766	4.719	96	647.609	133.813
34	3.828	6.39	98	717.703	171.656
36	5	6.328	100	793.219	161.406
38	6.547	8.968	102	875.578	183.547
40	8.406	8.937	104	964.313	186.031
42	10.719	11.079	106	1060.89	227.125
44	13.468	12.453	108	1163.516	199.015
46	16.797	16.109	110	1275.406	242.484
48	20.719	15.015	112	1394.922	232.969
50	25.391	20.141	114	1523.688	264.062
52	30.812	20.891	116	1661.109	272.719
54	37.172	24.438	118	1809.718	322.687
56	44.5	25.484	120	1966.032	267.765
58	52.984	33.063	122	2136.594	360.141
60	62.625	29.485	124	2315.875	339.094
62	73.781	40.781	126	2507.406	357.922
64	84.953	39.438	128	2669.11	369.234
66	100.61	45.438			

3.6 Properties

A few properties of the 2-D MRT are analyzed in the following sections.

3.6.1 Linearity

If $x_{n_1,n_2} \xleftrightarrow{MRT} Y^{(p)}_{k_1,k_2}$

and $x'_{n_1,n_2} \xleftrightarrow{MRT} Y'^{(p)}_{k_1,k_2}$

then for any real-valued constants a_1 and a_2,

$$a_1 x_{n_1,n_2} + a_2 x'_{n_1,n_2} \xleftrightarrow{MRT} a_1 Y^{(p)}_{k_1,k_2} + a_2 Y'^{(p)}_{k_1,k_2} \tag{3.37}$$

Proof

Recalling from (3.4), (3.5) & (3.6),

$$\sum_{\forall(n_1,n_2)\Rightarrow z=p} x_{n_1,n_2} - \sum_{\forall(n_1,n_2)\Rightarrow z=p+M} x_{n_1,n_2} = Y^{(p)}_{k_1,k_2}$$

where $z=((n_1k_1+n_2k_2))_N$ and $M=N/2$

$$\therefore \sum_{\forall(n_1,n_2)\Rightarrow z=p} (a_1x_{n_1,n_2}+a_2x'_{n_1,n_2})$$

$$- \sum_{\forall(n_1,n_2)\Rightarrow z=p+M} (a_1x_{n_1,n_2}+a_2x'_{n_1,n_2})$$

$$= \sum_{\forall(n_1,n_2)\Rightarrow z=p} a_1x_{n_1,n_2} + \sum_{\forall(n_1,n_2)\Rightarrow z=p} a_2x'_{n_1,n_2}$$

$$- \sum_{\forall(n_1,n_2)\Rightarrow z=p+M} a_1x_{n_1,n_2} - \sum_{\forall(n_1,n_2)\Rightarrow z=p+M} a_2x'_{n_1,n_2}$$

$$= a_1 \sum_{\forall(n_1,n_2)\Rightarrow(z=p)} x_{n_1,n_2} + a_2 \sum_{\forall(n_1,n_2)\Rightarrow(z=p)} x'_{n_1,n_2}$$

$$-a_1 \sum_{\forall(n_1,n_2)\Rightarrow z=p+M} x_{n_1,n_2} - a_2 \sum_{\forall(n_1,n_2)\Rightarrow z=p+M} x'_{n_1,n_2}$$

$$= a_1\Big(\sum_{\forall(n_1,n_2)\Rightarrow z=p} x_{n_1,n_2} - \sum_{\forall(n_1,n_2)\Rightarrow z=p+M} x_{n_1,n_2}\Big)$$

$$+a_2\Big(\sum_{\forall(n_1,n_2)\Rightarrow z=p} x'_{n_1,n_2} - \sum_{\forall(n_1,n_2)\Rightarrow z=p+M} x'_{n_1,n_2}\Big)$$

$$= a_1Y^{(p)}_{k_1,k_2} + a_2Y'^{(p)}_{k_1,k_2}$$

Thus, the 2-D MRT is a linear transform.

3.6.2 Reversal

By signal reversal, the following operation is meant: Given the signal x_{n_1,n_2}, the reversed signal is given by $x_{((-n_1))_N,((-n_2))_N}$. An example of such an operation done to a 6 x 6 signal is shown in Figure 3.4.

If $x_{n_1,n_2} \xleftrightarrow{MRT} Y^{(p)}_{k_1,k_2}$

and $x_{((-n_1))_N,((-n_2))_N} \xleftrightarrow{MRT} Y'^{(p)}_{k_1,k_2}$, then

$$\therefore Y'^{(p)}_{k_1,k_2} = sY^{(p')}_{k_1,k_2},\quad p'=((M-p))_M,\quad s=\begin{cases}1, & p=0\\ -1, & p\neq 0\end{cases} \tag{3.38}$$

Proof

$$x_{((-n_1))_N,((-n_2))_N} \xleftrightarrow{MRT} Y'^{(p)}_{k_1,k_2}$$

$$Y'^{(p)}_{k_1,k_2} = \sum_{\forall(n_1,n_2)\Rightarrow z=p} x_{((-n_1))_N,((-n_2))_N} - \sum_{\forall(n_1,n_2)\Rightarrow z=p+M} x_{((-n_1))_N,((-n_2))_N}$$

Substituting $n_1' = ((N-n_1))_N, n_2' = ((N-n_2))_N$,

$$Y'^{(p)}_{k_1,k_2} = \sum_{\forall(n_1',n_2')\Rightarrow((((-n_1'))_N k_1+((-n_2'))_N k_2))_N=p} x_{n_1',n_2'} - \sum_{\forall(n_1',n_2')\Rightarrow((((-n_1'))_N k_1+((-n_2'))_N k_2))_N=p+M} x_{n_1',n_2'}$$

$$Y'^{(p)}_{k_1,k_2} = \sum_{\forall(n_1',n_2')\Rightarrow((n_1'k_1+n_2'k_2))_N=((N-p))_N} x_{n_1',n_2'} - \sum_{\forall(n_1',n_2')\Rightarrow((n_1'k_1+n_2'k_2))_N=((N-(p+M)))_N} x_{n_1',n_2'}$$

$$Y'^{(p)}_{k_1,k_2} = \sum_{\forall(n_1',n_2')\Rightarrow((n_1'k_1+n_2'k_2))_N=((M-p+M))_N} x_{n_1',n_2'} - \sum_{\forall(n_1',n_2')\Rightarrow((n_1'k_1+n_2'k_2))_N=((M-p))_N} x_{n_1',n_2'}$$

$$Y'^{(p)}_{k_1,k_2} = -Y^{(M-p)}_{k_1,k_2}$$

$$Y'^{(p)}_{k_1,k_2} = sY^{(((M-p))_M)}_{k_1,k_2}$$

$$\therefore Y'^{(p)}_{k_1,k_2} = sY^{(p')}_{k_1,k_2}, \quad p' = ((M-p))_M, \quad s = \begin{cases} 1, & p=0 \\ -1, & p \neq 0 \end{cases}$$

(a) $$x_{n_1,n_2} = \begin{bmatrix} 95 & 46 & 92 & 41 & 14 & 2 \\ 23 & 2 & 74 & 89 & 20 & 75 \\ 61 & 82 & 18 & 6 & 20 & 45 \\ 49 & 44 & 41 & 35 & 60 & 93 \\ 89 & 62 & 94 & 81 & 27 & 47 \\ 76 & 79 & 92 & 1 & 20 & 42 \end{bmatrix}$$

(b) $$x'_{n_1,n_2} = \begin{bmatrix} 42 & 20 & 1 & 92 & 79 & 76 \\ 47 & 27 & 81 & 94 & 62 & 89 \\ 93 & 60 & 35 & 41 & 44 & 49 \\ 45 & 20 & 6 & 18 & 82 & 61 \\ 75 & 20 & 89 & 74 & 2 & 23 \\ 2 & 14 & 41 & 92 & 46 & 95 \end{bmatrix}$$

(c) $$Y^{(0)}_{k_1,k_2} = \begin{bmatrix} 1837 & 140 & 646 & 93 & 646 & 140 \\ -32 & 85 & 48 & 134 & 47 & -90 \\ 612 & 74 & 711 & 90 & 528 & -7 \\ 7 & 94 & 100 & 103 & 100 & 94 \\ 612 & -7 & 528 & 90 & 711 & 74 \\ -32 & -90 & 47 & 134 & 48 & 85 \end{bmatrix}$$

(d) $$Y'^{(0)}_{k_1,k_2} = \begin{bmatrix} 1837 & -107 & 715 & -93 & 715 & -107 \\ 78 & -51 & -48 & -100 & 57 & -90 \\ 542 & -74 & 643 & -32 & 528 & 51 \\ -7 & 113 & 119 & 103 & 119 & 113 \\ 542 & 51 & 528 & -32 & 643 & -74 \\ 78 & -90 & 57 & -100 & -48 & -5194 \end{bmatrix}$$

(e) $$Y^{(1)}_{k_1,k_2} = \begin{bmatrix} 0 & 154 & -715 & 0 & -476 & -107 \\ -117 & 51 & -21 & -69 & 57 & -213 \\ -542 & -160 & -483 & -32 & -598 & -151 \\ 0 & 104 & 119 & 0 & -26 & -113 \\ -683 & 51 & -711 & 29 & -643 & 141 \\ 78 & 20 & -17 & 100 & 62 & -69 \end{bmatrix}$$

(f) $$Y'^{(1)}_{k_1,k_2} = \begin{bmatrix} 0 & -154 & -646 & 0 & -476 & 140 \\ 117 & -85 & -62 & -69 & 47 & 20 \\ -612 & -141 & -483 & 90 & -711 & 151 \\ 0 & 104 & 100 & 0 & 26 & -94 \\ -683 & -7 & -598 & -29 & -711 & 160 \\ -32 & -213 & 17 & -134 & 21 & -69 \end{bmatrix}$$

(g) $$Y^{(2)}_{k_1,k_2} = \begin{bmatrix} 0 & 107 & 476 & 0 & 715 & -154 \\ -78 & 69 & -62 & -100 & 17 & -20 \\ 683 & -141 & 643 & -29 & 711 & -51 \\ 0 & 113 & 26 & 0 & -119 & -104 \\ 542 & 151 & 598 & 32 & 483 & 160 \\ 117 & 213 & -57 & 69 & 21 & -51 \end{bmatrix}$$

(h) $$Y'^{(2)}_{k_1,k_2} = \begin{bmatrix} 0 & -140 & 476 & 0 & 646 & 154 \\ 32 & 69 & -21 & 134 & -17 & 213 \\ 683 & -160 & 711 & 29 & 598 & 7 \\ 0 & 94 & -26 & 0 & -100 & -104 \\ 612 & -151 & 711 & -90 & 483 & 141 \\ -117 & -20 & -47 & 69 & 62 & 85 \end{bmatrix}$$

Figure 3.4: (a): A 6 x 6 signal; **(b)**: (a) reversed; **(c)**, **(e)** & **(g)**: MRT matrices of **(a)**; **(d)**, **(f)** & **(h)**: MRT matrices of **(b)**.

3.6.3 Circular shift

If the 2-D signal is circularly shifted in either direction, the following is the behaviour of its corresponding $Y^{(p)}_{k_1,k_2}$. Each MRT coefficient can potentially undergo a shift in the value of p to which it belongs, as also a sign-change. The position of the coefficient within the N x N MRT matrix corresponding to the new value of p is retained without any change. This is illustrated using an example in Figure 3.5.

Given $x_{n_1,n_2} \xleftrightarrow{MRT} Y^{(p)}_{k_1,k_2}$,

$$x'_{n_1,n_2} = x_{((n_1-r))_N,((n_2-c))_N}$$

and $x'_{n_1,n_2} \xleftrightarrow{MRT} Y'^{(p)}_{k_1,k_2}$

$$Y'^{(p)}_{k_1,k_2} = sY^{((p-rk_1-ck_2))_M}_{k_1,k_2}, \qquad s = \begin{cases} 1, & ((p-rk_1-ck_2))_N < M \\ -1, & ((p-rk_1-ck_2))_N \geq M \end{cases} \tag{3.39}$$

Proof:

$$Y^{(p)}_{k_1,k_2} = \sum_{\forall(n_1,n_2)\Rightarrow z=p} x_{n_1,n_2} - \sum_{\forall(n_1,n_2)\Rightarrow z=p+M} x_{n_1,n_2}$$

$$Y'^{(p)}_{k_1,k_2} = \sum_{\forall(n_1,n_2)\Rightarrow((n_1k_1+n_2k_2))_N=p} x'_{n_1,n_2} - \sum_{\forall(n_1,n_2)\Rightarrow((n_1k_1+n_2k_2))_N=p+M} x'_{n_1,n_2}$$

$$Y'^{(p)}_{k_1,k_2} = \sum_{\forall(n_1,n_2)\Rightarrow((n_1k_1+n_2k_2))_N=p} x_{((n_1-r))_N,((n_2-c))_N} - \sum_{\forall(n_1,n_2)\Rightarrow((n_1k_1+n_2k_2))_N=p+M} x_{((n_1-r))_N,((n_2-c))_N}$$

Substituting $n'_1 = ((n_1-r))_N, n'_2 = ((n_2-c))_N$

$$Y'^{(p)}_{k_1,k_2} = \sum_{\forall(n'_1,n'_2)\Rightarrow((n'_1k_1+n'_2k_2))_N=((p-rk_1-ck_2))_N} x_{n'_1,n'_2} - \sum_{\forall(n'_1,n'_2)\Rightarrow((n'_1k_1+n'_2k_2))_N=((p-rk_1-ck_2+M))_N} x_{n'_1,n'_2}$$

$$Y'^{(p)}_{k_1,k_2} = \sum_{\forall(n'_1,n'_2)\Rightarrow((n'_1k_1+n'_2k_2))_N=((p-rk_1-ck_2))_N} x_{n'_1,n'_2} - \sum_{\forall(n'_1,n'_2)\Rightarrow((n'_1k_1+n'_2k_2))_N=((p-rk_1-ck_2))_N+M} x_{n'_1,n'_2}$$

$$Y'^{(p)}_{k_1,k_2} = Y^{((p-rk_1-ck_2))_N}_{k_1,k_2}$$

$$Y'^{(p)}_{k_1,k_2} = Y^{((p-rk_1-ck_2))_M}_{k_1,k_2}, \qquad ((p-rk_1-ck_2))_N < M$$

$$Y'^{(p)}_{k_1,k_2} = -Y^{((p-rk_1-ck_2))_M}_{k_1,k_2}, \qquad ((p-rk_1-ck_2))_N \geq M$$

$$\therefore Y'^{(p)}_{k_1,k_2} = sY^{((p-rk_1-ck_2))_M}_{k_1,k_2}, \qquad s = \begin{cases} 1, & ((p-rk_1-ck_2))_N < M \\ -1, & ((p-rk_1-ck_2))_N \geq M \end{cases}$$

3.6.4 Circular convolution

If $x_{n_1,n_2} \xleftrightarrow{MRT} Y^{(p)}_{k_1,k_2}$

and $x'_{n_1,n_2} \xleftrightarrow{MRT} Y'^{(p)}_{k_1,k_2}$

and $x_{n_1,n_2} \otimes x'_{n_1,n_2} \xleftrightarrow{MRT} Y''^{(p)}_{k_1,k_2}$

where $\otimes$ denotes circular convolution, then,

$$Y''^{(p)}_{k_1,k_2} = \sum_{j=0}^{M-1} sY^{(q)}_{k_1,k_2} Y'^{((p-q))_M}_{k_1,k_2}, \qquad s = \begin{cases} 1 & , \quad q < p+1 \\ -1 & , \quad q \geq p+1 \end{cases} \tag{3.40}$$

Proof

Let $x''_{n_1,n_2} = x_{n_1,n_2} \otimes x'_{n_1,n_2}$

$$x''_{n_1,n_2} = \sum_{j_1=0}^{N-1}\sum_{j_2=0}^{N-1} x_{j_1,j_2} x'_{((n_1-j_1))_N,((n_2-j_2))_N}$$

(a) $x_{n_1,n_2} = \begin{bmatrix} 85 & 68 & 30 & 15 & 50 & 34 \\ 53 & 38 & 19 & 70 & 90 & 29 \\ 20 & 83 & 19 & 38 & 82 & 34 \\ 67 & 50 & 68 & 86 & 64 & 53 \\ 84 & 71 & 30 & 85 & 82 & 73 \\ 2 & 43 & 54 & 59 & 66 & 31 \end{bmatrix}$ (b) $x'_{n_1,n_2} = \begin{bmatrix} 31 & 2 & 43 & 54 & 59 & 66 \\ 34 & 85 & 68 & 30 & 15 & 50 \\ 29 & 53 & 38 & 19 & 70 & 90 \\ 34 & 20 & 83 & 19 & 38 & 82 \\ 53 & 67 & 50 & 68 & 86 & 64 \\ 73 & 84 & 71 & 30 & 85 & 82 \end{bmatrix}$

(c) $Y^{(0)}_{k_1,k_2} = \begin{bmatrix} 1925 & -42 & 664 & 5 & 664 & -42 \\ -106 & 32 & -60 & 38 & 58 & 10 \\ 670 & 122 & 678 & 58 & 672 & 20 \\ 41 & 144 & -10 & -43 & -10 & 144 \\ 670 & 20 & 672 & 58 & 678 & 122 \\ -106 & 10 & 58 & 38 & -60 & 32 \end{bmatrix}$ (d) $Y'^{(0)}_{k_1,k_2} = \begin{bmatrix} 1925 & 34 & 474 & -5 & 474 & 34 \\ -21 & -54 & 60 & -23 & -18 & 10 \\ 531 & -122 & 662 & 45 & 672 & 14 \\ -41 & -90 & 34 & -43 & 34 & -90 \\ 531 & 14 & 672 & 45 & 662 & -122 \\ -21 & 10 & -18 & -23 & 60 & -54 \end{bmatrix}$

(e) $Y^{(1)}_{k_1,k_2} = \begin{bmatrix} 0 & -81 & -474 & 0 & -787 & 34 \\ -126 & 54 & -45 & 58 & -18 & 27 \\ -531 & 132 & -585 & 45 & -502 & 1 \\ 0 & 97 & 34 & 0 & -85 & 90 \\ -724 & 14 & -751 & 8 & -662 & -15 \\ -21 & 26 & 35 & 23 & -56 & 21 \end{bmatrix}$ (f) $Y'^{(1)}_{k_1,k_2} = \begin{bmatrix} 0 & -42 & -787 & 0 & -664 & 81 \\ -106 & 21 & 45 & -38 & -35 & 27 \\ -724 & -132 & -678 & -8 & -502 & 20 \\ 0 & -144 & 85 & 0 & -10 & 97 \\ -670 & -1 & -751 & 58 & -585 & 15 \\ 126 & 26 & 58 & 58 & 56 & -32 \end{bmatrix}$

(g) $Y^{(2)}_{k_1,k_2} = \begin{bmatrix} 0 & -34 & 787 & 0 & 474 & 81 \\ 21 & -21 & 56 & -23 & -35 & -26 \\ 724 & 15 & 662 & -8 & 751 & -14 \\ 0 & -90 & 85 & 0 & -34 & -97 \\ 531 & -1 & 502 & -45 & 585 & -132 \\ 126 & -27 & 18 & -58 & 45 & -54 \end{bmatrix}$ (h) $Y'^{(2)}_{k_1,k_2} = \begin{bmatrix} 0 & -81 & 664 & 0 & 787 & 42 \\ -126 & 32 & -56 & -58 & -58 & -26 \\ 670 & -15 & 585 & -58 & 751 & 1 \\ 0 & -97 & 10 & 0 & -85 & 144 \\ 724 & -20 & 502 & 8 & 678 & 132 \\ 106 & -27 & 35 & 38 & -45 & -21 \end{bmatrix}$

Figure 3.5: **(a):** A 6 x 6 signal; **(b):** (a) circularly shifted by 1 row and 1 column; **(c), (e) & (g):** MRT matrices of (a); **(d), (f) & (h):** MRT matrices of (b).

$$
\begin{aligned}
Y''^{(p)}_{k_1,k_2} &= \sum_{\forall(n_1,n_2)\Rightarrow((n_1k_1+n_2k_2))_N=p} x''_{n_1,n_2} - \sum_{\forall(n_1,n_2)\Rightarrow((n_1k_1+n_2k_2))_N=p+M} x''_{n_1,n_2} \\
&= \sum_{\forall(n_1,n_2)\Rightarrow((n_1k_1+n_2k_2))_N=p} \sum_{j_1=0}^{N-1}\sum_{j_2=0}^{N-1} x_{j_1,j_2} x'_{((n_1-j_1))_N,((n_2-j_2))_N} \\
&\quad - \sum_{\forall(n_1,n_2)\Rightarrow((n_1k_1+n_2k_2))_N=p+M} \sum_{j_1=0}^{N-1}\sum_{j_2=0}^{N-1} x_{j_1,j_2} x'_{((n_1-j_1))_N,((n_2-j_2))_N} \\
&= \sum_{j_1=0}^{N-1}\sum_{j_2=0}^{N-1} \Big(\sum_{\forall(n_1,n_2)\Rightarrow((n_1k_1+n_2k_2))_N=p} x_{j_1,j_2} x'_{((n_1-j_1))_N,((n_2-j_2))_N} \\
&\quad - \sum_{\forall(n_1,n_2)\Rightarrow((n_1k_1+n_2k_2))_N=p+M} x_{j_1,j_2} x'_{((n_1-j_1))_N,((n_1-j_1))_N} \Big) \\
&= \sum_{j_1=0}^{N-1}\sum_{j_2=0}^{N-1} \Big(x_{j_1,j_2} \Big(\sum_{\forall(n_1,n_2)\Rightarrow((n_1k_1+n_2k_2))_N=p} x'_{((n_1-j_1))_N,((n_2-j_2))_N} - \sum_{\forall(n_1,n_2)\Rightarrow((n_1k_1+n_2k_2))_N=p+M} x'_{((n_1-j_1))_N,((n_2-j_2))_N} \Big)\Big) \\
&= \sum_{j_1=0}^{N-1}\sum_{j_2=0}^{N-1} x_{j_1,j_2} Y'^{(p-((j_1k_1+j_2k_2))_N)}_{k_1,k_2}
\end{aligned} \tag{3.41}
$$

Let $q = ((j_1k_1 + j_2k_2))_N$. In (3.41), the terms x_{j_1,j_2} will have a common multiplicand $Y'^{(p-q)}_{k_1,k_2}$ if their indices (j_1, j_2) are such that $q = ((j_1k_1 + j_2k_2))_N$. Also, $Y'^{(q+N/2)}_{k_1,k_2} = -Y'^{(q)}_{k_1,k_2}$. Hence,

$$
Y''^{(p)}_{k_1,k_2} = \sum_{j_1=0}^{N-1}\sum_{j_2=0}^{N-1} x_{j_1,j_2} Y'^{(p-q)}_{k_1,k_2}
$$

$$
\begin{aligned}
&= \sum_{q=0}^{N-1} \Big(\sum_{\forall (j_1,j_2) \Rightarrow ((j_1k_1+j_2k_2))_N = q} x_{j_1,j_2} \Big) Y'^{(p-q)}_{k_1,k_2} \\
&= \sum_{q=0}^{M-1} \Big(\sum_{\forall (j_1,j_2) \Rightarrow (((j_1k_1+j_2k_2))_N = q)} x_{j_1,j_2} \Big) Y'^{(p-q)}_{k_1,k_2} + \sum_{q=M}^{N-1} \Big(\sum_{\forall (j_1,j_2) \Rightarrow ((j_1k_1+j_2k_2))_N = q} x_{j_1,j_2} \Big) Y'^{(p-q)}_{k_1,k_2} \\
&= \sum_{q=0}^{M-1} \Big(\sum_{\forall (j_1,j_2) \Rightarrow ((j_1k_1+j_2k_2))_N = q} x_{j_1,j_2} \Big) Y'^{(p-q)}_{k_1,k_2} - \sum_{q=0}^{M-1} \Big(\sum_{\forall (j_1,j_2) \Rightarrow ((j_1k_1+j_2k_2))_N = q+M} x_{j_1,j_2} \Big) Y'^{(p-q)}_{k_1,k_2} \\
&= \sum_{q=0}^{M-1} \Big(\sum_{\forall (j_1,j_2) \Rightarrow ((j_1k_1+j_2k_2))_N = q} x_{j_1,j_2} - \sum_{\forall (j_1,j_2) \Rightarrow ((j_1k_1+j_2k_2))_N = q+M} x_{j_1,j_2} \Big) Y'^{(p-q)}_{k_1,k_2} \\
&= \sum_{q=0}^{M-1} Y^{(q)}_{k_1,k_2} Y'^{(p-q)}_{k_1,k_2}
\end{aligned}
\tag{3.42}
$$

When $(p-q)<0$, $Y'^{(p-q)}_{k_1,k_2} = -Y^{(((p-q))_M)}_{k_1,k_2}$

Thus (3.42) can be written in the following manner,

$$
Y'^{(p)}_{k_1,k_2} = \sum_{j=0}^{M-1} sY^{(q)}_{k_1,k_2} Y'^{(((p-q))_M)}_{k_1,k_2}, \qquad s = \begin{cases} 1, & q < p+1 \\ -1, & q \geq p+1 \end{cases}
$$

3.6.5 Isometric Transformations

There are eight isometric transformations generally used in image processing. Besides the original signal, the seven other transformations are 1) $90°$ rotation, 2) $180°$ rotation, 3) $270°$ rotation, 4) reflection along mid-vertical axis, 5) reflection along mid-horizontal axis, 6) reflection along diagonal, and 7) reflection along cross-diagonal. Bracewell et al [96] have presented a theorem which determines what the 2-D Fourier transform becomes when the signal is subjected to an affine co-ordinate transformation. Using this theorem, the relevant transformations in the 2-D MRT domain can be derived.

If the original block is denoted f and the transformed block is denoted g, and they are related in the following manner,

$$
g(n_1,n_2) = f(xn_1 + yn_2 + z, an_1 + bn_2 + c)
$$

and if F & G stand for the MRT of f and g respectively, then G and F are related in the following manner.

$$
G^{(p)}_{k_1,k_2} = sF^{(p')}_{k_1',k_2'} \tag{3.43}
$$

$$
k_1' = ((xk_1 + yk_2))_N
$$

$$
k_2' = ((ak_1 + bk_2))_N
$$

$$
p' = ((p+w))_M
$$

$$w = k_1((xz)OR(ac)) + k_2((yz)OR(bc)) \quad (3.44)$$

$$s = \begin{cases} 1, & ((p+w))_N < M \\ -1, & ((p+w))_N \geq M \end{cases}$$

In the above mapping, the parameters (x, y, z) and (a, b, c) take the values shown in table 3.2 corresponding to each transformation. The OR function in (3.44) is defined as

$$h_1 \; OR \; h_2 = \begin{cases} 1, & (h_1 \neq 0) \text{ or } (h_2 \neq 0) \\ -1, & h_1 = 0, \; h_2 = 0 \end{cases}$$

Table 3.2: Isometric transformations in MRT domain

Transformation	x	y	z	a	b	c
90°	0	1	0	-1	0	N - 1
180°	-1	0	N- 1	0	-1	N - 1
270°	0	-1	N - 1	1	0	0
mid-horizontal	-1	0	N - 1	0	1	0
mid-vertical	1	0	0	0	-1	N - 1
Diagonal	0	-1	N - 1	-1	0	N - 1
Cross-diagonal	0	1	0	1	0	0

3.7 Inverse Transform (2-D IMRT) for N a power of 2

Since the DFT and the IDFT are duals of each other, and the MRT is related to the DFT, it can be intuitively assumed that the inverse MRT is a dual of the forward MRT. Hence, the following formula is proposed for inverse 2-D MRT for N power of 2:

$$x_{n_1,n_2} = \frac{1}{N^2} \sum_{p=0}^{M-1} \left[\sum_{\forall (k_1,k_2) \Rightarrow z=p} Y_{k_1,k_2}^{(p)} - \sum_{\forall (k_1,k_2) \Rightarrow z=p+M} Y_{k_1,k_2}^{(p)} \right] \qquad 0 \leq n_1, n_2 \leq N-1 \quad (3.45)$$

Proof

The terms in $\sum_{\forall (k_1,k_2) \Rightarrow z=p} Y_{k_1,k_2}^{(p)}$ and in $\sum_{\forall (k_1,k_2) \Rightarrow z=p+M} Y_{k_1,k_2}^{(p)}$ in (3.45) are the MRT coefficients in which x_{n_1,n_2} is present. In other words, these are the values of (k_1, k_2) that satisfy the two equations (3.7) and (3.8), respectively,

$$((n_1k_1 + n_2k_2))_N = p$$
$$((n_1k_1 + n_2k_2))_N = p + M .$$

But any term $Y_{k_1,k_2}^{(p)}$ which contains x_{n_1,n_2} will also have other terms x_{n_a,n_b} such that $((n_ak_1 + n_bk_2))_N = p$. The proof may be divided into two parts: (i) to show the need for division by the factor N^2 in (3.45), and (ii) to demonstrate that all other terms x_{n_a,n_b} besides x_{n_1,n_2} that occur in

$\sum_{\forall(k_1,k_2)\Rightarrow z=p} Y_{k_1,k_2}^{(p)}$ and $\sum_{\forall(k_1,k_2)\Rightarrow z=p+M} Y_{k_1,k_2}^{(p)}$ such that they satisfy the following linear congruence equations $((n_a k_1 + n_b k_2))_N = p$ and $((n_a k_1 + n_b k_2))_N = p + M$ respectively, cancel each other leaving behind the term $N^2 x_{n_1,n_2}$.

Division by N^2:

For any (n_1, n_2), the total number of MRT coefficients $Y_{k_1,k_2}^{(p)}$ whose indices (k_1, k_2, p) solve (3.7) or (3.8) is N^2. This is proved using the following cases.

Case (a): $(n_1 \neq 0, n_2 \neq 0)$

The indices (k_1, k_2) of $Y_{k_1,k_2}^{(p)}$ in which x_{n_1,n_2} is present can be found by solving the linear congruence equations (3.7) and (3.8).

(3.7) can be written as the linear equation

$$n_1 k_1 + n_2 k_2 + Nt = p \tag{3.46}$$

(3.46) can be written as

$$g(n_1, n_2) q + Nt = p \tag{3.47}$$

$$q = \frac{n_1}{g(n_1,n_2)} k_1 + \frac{n_2}{g(n_1,n_2)} k_2,$$

$$q = n_1' k_1 + n_2' k_2$$

$$n_1' = \frac{n_1}{g(n_1,n_2)}, n_2' = \frac{n_2}{g(n_1,n_2)}$$

(3.47) has $g(g(n_1, n_2), N)$ solutions mod N. The general solution is of the form $q = Ns + q_0, s \in Z$, q_0 being a particular solution.

$$n_1' k_1 + n_2' k_2 = Ns + q_0 \tag{3.48}$$

The number of solutions of (k_1, k_2) in the range [[0, N - 1], [0, N - 1]] can be found by multiplying two quantities: (i) the number of solutions of k_1 in the range [0, N - 1], (ii) the number of solutions of k_2 for a given solution of k_1.

Given a particular solution (k_{10}, k_{20}) of (3.48), then

$$n_1'(k_{10} + n_2' t_1) + n_2'(k_{20} - n_1' t_1) = Ns + q_0, t_1 \in Z \tag{3.49}$$

The values that k_1 can have is determined by the relationship between $n_2' t_1$ and N. Using Bezout's lemma,

$$n_2' t_1 + N t_2 = t_3 g(n_2', N), \quad t_2, t_3 \in Z \tag{3.50}$$

Using (3.50), (3.49) can be written as

$$n_1'(k_{10} + t_3 g(n_2', N) - N t_2) + n_2'(k_{20} - n_1' t_1) = Ns + q_0, \quad t_1 \in \mathbf{Z} \tag{3.51}$$

$$n_1'(k_{10}+t_3 g(n_2',N))+n_2'(k_{20}-n_1't_1)=Ns'+q_0,\quad s'=(s+n_1't_2) \tag{3.52}$$

From (3.52), it is seen that k_1 has the general solution $k_{10}+t_3 g(n_2',N)$. Thus, two successive solutions differ by $g(n_2',N)$. The number of solutions of k_1 that lie in the range [0, N - 1] is thus $N / g(n_2',N)$.

The number of solutions of k_2 for the same value of k_1 can be found from the number of solutions for j in the following equation.

$$q=n_1'k_1+n_2'(k_2+j) \tag{3.53}$$

$$n_1'k_1+n_2'(k_2+j)=Nq_1+q_0 \quad j,q_1\in\mathbb{Z} \tag{3.54}$$

From (3.48) and (3.54),

$$n_2'j-N(q_1+s)=0 \tag{3.55}$$

The number of solutions of j is given by $g(n_2',N)$. One of these solutions will be j = 0. The number of solutions of k_2 for the same value of k_1 is thus $g(n_2',N)$.

The number of solutions of (k_1, k_2) in the range [[0, N - 1], [0, N - 1]] is thus N. There are an equal number of solutions to (3.8). Thus, for a given value of p, there are $2N$ solutions for (k_1,k_2). For a given value of p, q in (3.47) has $g(g(n_1, n_2), N)$ solutions. The total number of solutions is thus $2Ng(g(n_1, n_2), N)$.

The number of possible values of p is given by $M/g(g(n_1, n_2), N)$ since p has to be a multiple of $g(g(n_1, n_2), N)$. The total number of solutions over all values of p is thus

$$L_{tot}=2Ng(g(n_1,n_2),N)(M / g(g(n_1,n_2),N))=N^2 \tag{3.56}$$

Example:

Let N = 8, n_1 = 1, n_2 = 1

(3.7) can be written as

$$k_1+k_2+Nt=p,$$
$$q+Nt=p$$
$$q=k_1+k_2$$
$$k_1+k_2=Ns+q_0$$

Let p = 0, hence q_0 = 0, and, (k_1, k_2) = (0,0), (1,7), (2,6), (3,5), (4,4), (5,3), (6,2), (7,1) are solutions. Similarly, there are 8 solutions to (3.8) also, and thus 16 solutions associated with this value of p. The possible values of p = 0, 1, 2, 3. The total number of solutions over all these values of p is thus 64.

Case (b): (n_1 = 0, $n_2 \neq 0$) or ($n_1 \neq 0$, n_2 = 0)

Assume n_1 = 0. Thus, (3.46) becomes

$$n_2k_2 + Nz = p \tag{3.57}$$

Hence, there are $g(n_2, N)$ solutions for k_2 which occur in the range [0, N - 1]. For each solution of k_2, there are N solutions of k_1 since $n_1 = 0$. Accounting for the solutions to (3.8), there are $2Ng(n_2, N)$ solutions for a given value of p. The total number of solutions over all values of p is thus

$$L_{tot} = 2Ng(n_2, N)(N / 2g(n_2, N)) = N^2 \tag{3.58}$$

Case (c): ($n_1 = 0$, $n_2 = 0$)

Here, (3.46) becomes

$$0k_1 + 0k_2 + Nz = p \tag{3.59}$$

Thus all values in $(k_1, k_2) \in \square$ are solutions of (3.59), yielding N^2 solutions in the range [[0, N - 1], [0, N - 1]]. Thus, for all possible values of (n_1, n_2), the total number of MRT values $Y_{k_1,k_2}^{(p)}$ whose indices (k_1, k_2, p) solve the equations (3.7) or (3.8) is N^2. The summation in (3.45) would have the term x_{n_1,n_2} repeating N^2 times. To obtain x_{n_1,n_2} the summation thus needs to be divided by N^2.

(ii) Cancellation of other terms x_{n_a,n_b}

Given a term x_{n_1,n_2} and another term x_{n_a,n_b} that occur with positive signs in the expansion of MRT coefficient $Y_{k_1,k_2}^{(p)}$, there could possibly be other MRT coefficients in which these two terms occur together. The frequency indices of the MRT coefficients in which these two terms occur with positive signs can be found from the solutions of the following equations

$$((n_1k_1 + n_2k_2))_N = p \tag{3.60}$$

$$((n_ak_1 + n_bk_2))_N = p \tag{3.61}$$

Similarly, the terms x_{n_1,n_2} and x_{n_a,n_b} could occur in the same MRT coefficient, but with opposite signs. The frequency indices of such MRT coefficients can be found from

$$((n_1k_x + n_2k_y))_N = p \tag{3.62}$$

$$((n_ak_x + n_bk_y))_N = p + M \tag{3.63}$$

From the method followed for solution of (3.7), it may be recalled that there are $g(g(k_1, k_2), N)$ solutions mod N to (3.60). Assume that (k_{10}, k_{20}) is a particular solution. Thus, k_1 has solution $k_1 = k_{10} + n_2't$ and k_2 has solution $k_2 = k_{20} - n_1't$. Substituting the values of k_{10} and k_{20} in (3.61),

$$((n_a(k_{10} + n_2't) + n_b(k_{20} - n_1't)))_N = p \tag{3.64}$$

(k_{10}, k_{20}) is a solution of (3.61) also.

Hence

$$((n_a k_{10} + n_b k_{20}))_N = p \tag{3.65}$$

From (3.64) and (3.61),

$$(n_a n_2^{'} - n_b n_1^{'})t + Nz = 0 \tag{3.66}$$

The number of solutions of t is $g(n_a n_2^{'} - n_b n_1^{'}, N)$. Thus, the total number of common solutions of (3.60) and (3.61) is $g(g(k_1, k_2), N) g(n_a n_2^{'} - n_b n_1^{'}, N)$.

Similarly, the number of solutions to (3.62) and (3.63) can also be shown to be $g(g(k_1, k_2), N) g(n_a n_2^{'} - n_b n_1^{'}, N)$. Thus the terms cancel each other leaving behind only x_{n_1, n_2}. Two other possibilities are that x_{n_1, n_2} will be both negative for some frequency indices and will be of opposite signs in some other frequency indices. The number of such frequency indices is also same as the two cases above. Thus, the proof of inverse MRT is complete.

3.8 Conclusion

2-D MRT, which has been introduced in this chapter, is a new real transform for 2-D signals. It is formed by grouping data elements on the basis of the twiddle-factor phase to which it corresponds in the context of the DFT definition. Hence, this is a new way of analyzing a signal, from the point of view of phase. The MRT thus carries both frequency and phase information. The MRT is, however, an expansive transform since there are M MRT matrices of size N x N for a data of size N x N. Among these matrices, there are both zero-valued positions as well as redundancies. Removal of these could result in a more compact and efficient form of the MRT. This calls for a detailed study of the phenomenon of redundancy in MRT. Since this analysis would be simpler to do in a 1-D context, the 1-D version of MRT is presented in Chapter IV.

Chapter IV

DEVELOPMENT OF FORWARD AND INVERSE 1- D MRT

4.1 Introduction

2-D MRT has been developed and analyzed in chapter III. A detailed study of redundancy that occurs in 2-D MRT was not taken up in chapter III. Due to the simplification provided by the reduction of dimension, it would be convenient to develop 1-D MRT and study it completely and then use this knowledge in further study of 2-D MRT. Hence, in this chapter, the 1-D MRT is developed, and analyzed in detail. Finally, the 1-D MRT is simplified by removing redundancy to obtain the 1-D UMRT.

4.2 Forward 1-D MRT

The 1-D MRT $Y_k^{(p)}$ of a 1-D sequence x_n, $0 \le n \le N-1$, is defined as

$$Y_k^{(p)} = \sum_{\forall n \Rightarrow ((nk))_N = p} x_n - \sum_{\forall n \Rightarrow ((nk))_N = p+M} x_n, \tag{4.1}$$

$k = 0,1,2,....N-1, \quad p = 0,1,2,....M-1$, and, $M = N/2$.

In (4.1), k is the frequency index, and p is the phase index. Thus 1-D MRT maps an array of length N into M arrays, each of length N. Hence the mapping involves computation of MN coefficients in terms of real additions. The 1-D MRT can also be expressed as

$$Y_k^{(p)} = \sum_{n=0}^{N-1} A_{k,p,n} x_n, \qquad 0 \le k \le N-1, \quad 0 \le p \le M-1 \tag{4.1a}$$

$$A_{k,p,n} = \begin{cases} 1, & ((nk))_N = p \\ -1, & ((nk))_N = p+M \\ 0, & \text{otherwise} \end{cases} \tag{4.1b}$$

Thus, the kernel $A_{k,p,n}$ maps the data x_n into the 1-D MRT $Y_k^{(p)}$.

4.2.1 Direct 1-D MRT Computation

The 1-D MRT $Y_k^{(p)}$, $k = 0,1,2,....N-1$, and $p = 0,1,2,....M-1$, of the given sequence x_n, $0 \le n \le N-1$, is computed as follows, using real additions only.

1) For a given k & p, initialize $Y_k^{(p)} = 0$.
2) For each value of n, $0 \le n \le N - 1$, compute $z = ((nk))_N$

   If $z = p$, $Y_k^{(p)} = Y_k^{(p)} + x_n$, else if $z = p + M$, $Y_k^{(p)} = Y_k^{(p)} - x_n$, else go to the next value of n.

3) For each value of k & p, repeat steps 1-2.

Example 4.1:

Let $x = [95 \quad 23 \quad 61 \quad 49 \quad 89 \quad 76 \quad 46 \quad 2]$, $N = 8$,

Then, $Y_k^{(p)}$, the corresponding MRT of x, is

$$
\begin{aligned}
Y_k^{(0)} &= [441 \quad 6 \quad 77 \quad 6 \quad 141 \quad 6 \quad 77 \quad 6] \\
Y_k^{(1)} &= [\ 0 \quad -53 \quad 0 \quad 47 \quad 0 \quad 53 \quad 0 \quad -47] \\
Y_k^{(2)} &= [\ 0 \quad 15 \quad 48 \quad -15 \quad 0 \quad 15 \quad -48 \quad -15] \\
Y_k^{(3)} &= [\ 0 \quad 47 \quad 0 \quad -53 \quad 0 \quad -47 \quad 0 \quad 53]
\end{aligned}
$$

The direct method requires N computations of z and its logical checking for every MRT coefficient. Computation of all MRT coefficients corresponding to one frequency involves addition of N data. Thus, the total number of additions involved in the MRT computation is $N(N-1)$.

4.2.2 Examples

In this section, the relation between 1-D MRT coefficients and data for values of N = 4, 6, 8 & 10 are presented.

4.2.2.1 1-D MRT for $N = 4$

The relations between 1-D MRT coefficients and corresponding data elements, for N = 4, are given below. The corresponding transform kernel representations $A_{k,p,n}$ are shown in Figure 4.1.

1. $Y_0^{(0)} = x_0 + x_1 + x_2 + x_3$
2. $Y_1^{(0)} = x_0 - x_2$
3. $Y_1^{(1)} = x_1 - x_3$
4. $Y_2^{(0)} = x_0 - x_1 + x_2 - x_3$
5. $Y_3^{(0)} = -x_0 + x_2$
6. $Y_3^{(1)} = -x_1 + x_3$

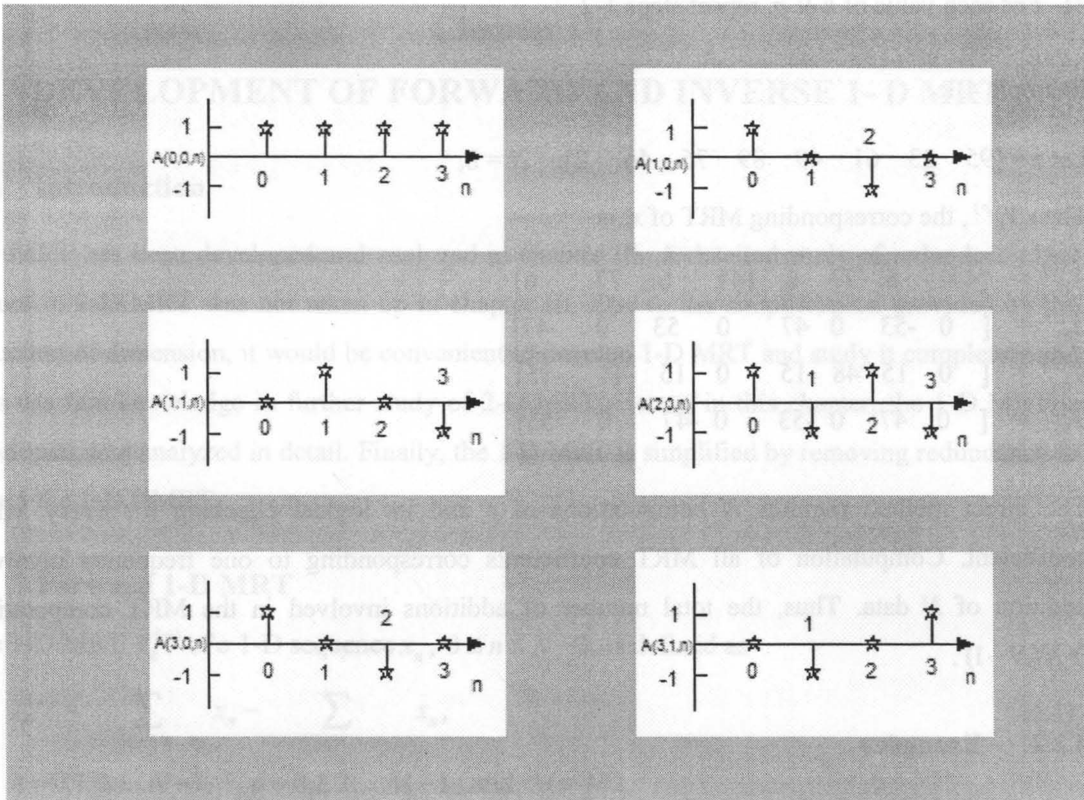


Figure 4.1: Kernel representation $A_{k,p,n}$ of 1-D MRT for $N = 4$

4.2.2.2 1-D MRT for $N = 6$

The relations between MRT coefficients and corresponding data elements, for $N = 6$, are given below. The corresponding transform kernel representations $A_{k,p,n}$ are shown in Figure 4.2.

1. $Y_0^{(0)} = x_0 + x_1 + x_2 + x_3 + x_4 + x_5$
2. $Y_1^{(0)} = x_0 - x_3$
3. $Y_1^{(1)} = x_1 - x_4$
4. $Y_1^{(2)} = x_2 - x_5$
5. $Y_2^{(0)} = x_0 + x_3$
6. $Y_2^{(1)} = -x_2 - x_5$
7. $Y_2^{(2)} = x_1 + x_4$
8. $Y_3^{(0)} = x_0 - x_1 + x_2 - x_3 + x_4 - x_5$
9. $Y_4^{(0)} = x_0 + x_3$
10. $Y_4^{(1)} = -x_1 - x_4$
11. $Y_4^{(2)} = x_2 + x_5$
12. $Y_5^{(0)} = x_0 - x_3$
13. $Y_5^{(1)} = -x_2 + x_5$
14. $Y_5^{(2)} = -x_1 + x_4$

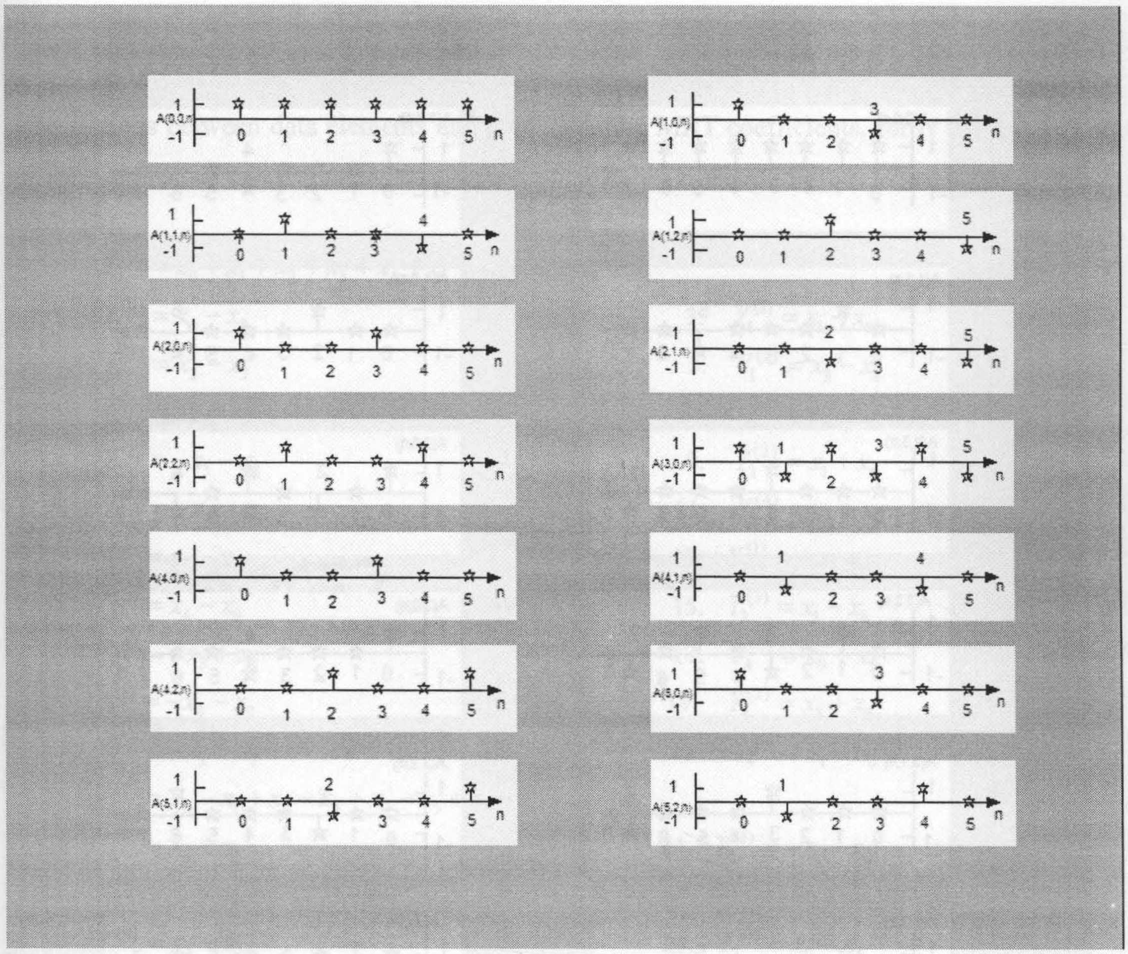


Figure 4.2: Kernel representation $A_{k,p,n}$ of a few 1-D MRT coefficients, $N = 6$

4.2.2.3 1-D MRT for $N = 8$

The relations between data elements and corresponding MRT coefficients, for $N = 8$, are given below. The graphical representation of some the relations are shown in Figure 4.3.

1. $Y_0^{(0)} = x_0 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7$
2. $Y_1^{(0)} = x_0 - x_4$
3. $Y_1^{(1)} = x_1 - x_5$
4. $Y_1^{(2)} = x_2 - x_6$
5. $Y_1^{(3)} = x_3 - x_7$
6. $Y_2^{(0)} = x_0 - x_2 + x_4 - x_6$
7. $Y_2^{(2)} = x_1 - x_3 + x_5 - x_7$
8. $Y_3^{(0)} = x_0 - x_4$
9. $Y_3^{(1)} = x_3 - x_7$
10. $Y_3^{(2)} = -x_2 + x_6$
11. $Y_3^{(3)} = x_1 - x_5$
12. $Y_4^{(0)} = x_0 - x_1 + x_2 - x_3 + x_4 - x_5 + x_6 - x_7$

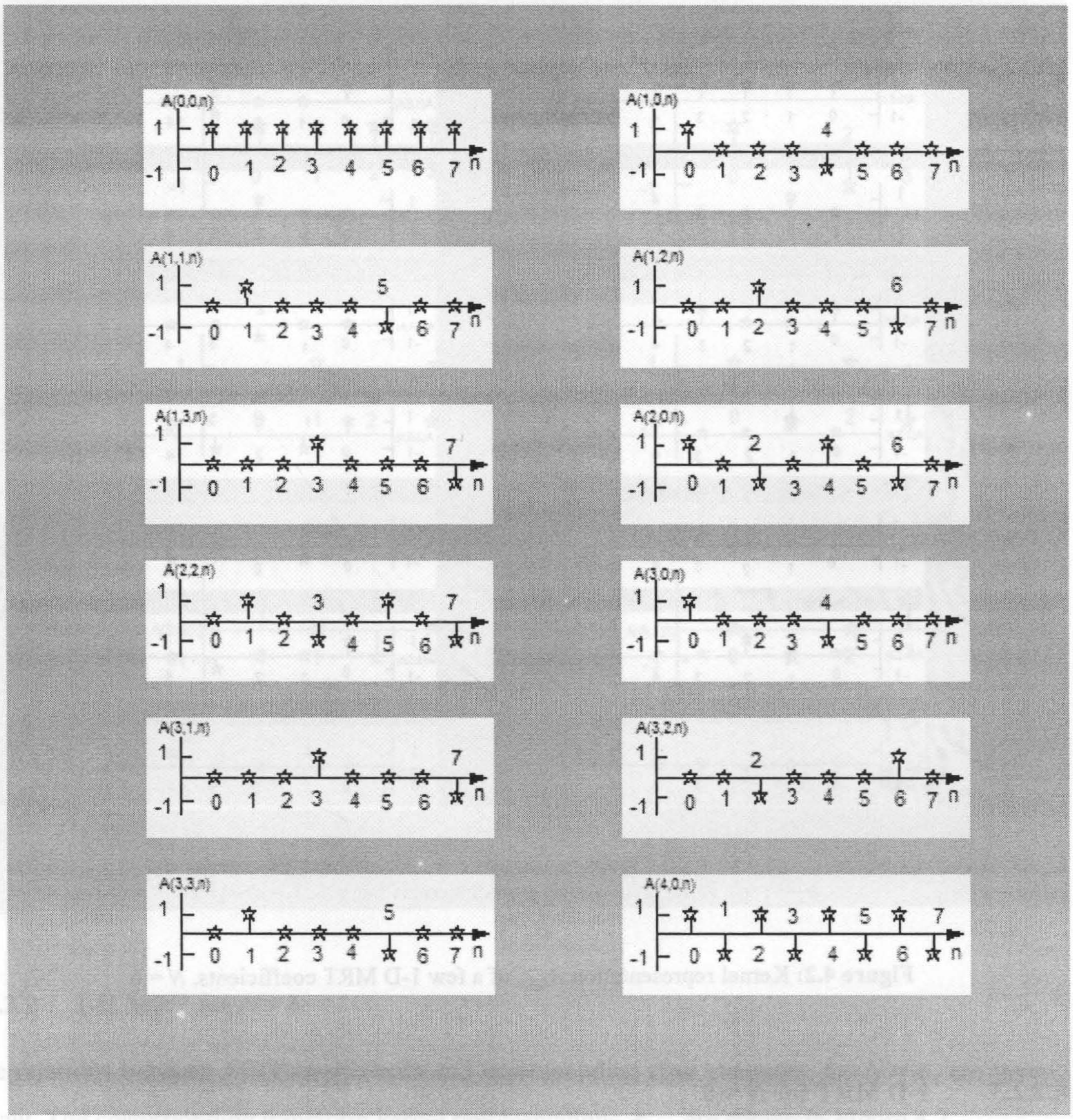


Figure 4.3: Kernel representation $A_{k,p,n}$ of a few 1-D MRT coefficients, $N = 8$

13. $Y_5^{(0)} = x_0 - x_4$

14. $Y_5^{(1)} = -x_1 + x_5$

15. $Y_5^{(2)} = x_2 - x_6$

16. $Y_5^{(3)} = -x_3 + x_7$

17. $Y_6^{(0)} = x_0 - x_2 + x_4 - x_6$

18. $Y_6^{(2)} = -x_1 + x_3 - x_5 + x_7$

19. $Y_7^{(0)} = x_0 - x_4$

20. $Y_7^{(1)} = -x_3 + x_7$

21. $Y_7^{(2)} = -x_2 + x_6$

22. $Y_7^{(3)} = -x_1 + x_5$

4.2.2.4 1-D MRT for $N = 10$

The relations between data elements and corresponding MRT coefficients, for $N = 10$, are given below.

1. $Y_0^{(0)} = x_0 + x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9$
2. $Y_1^{(0)} = x_0 - x_5$
3. $Y_1^{(1)} = x_1 - x_6$
4. $Y_1^{(2)} = x_2 - x_7$
5. $Y_1^{(3)} = x_3 - x_8$
6. $Y_1^{(4)} = x_4 - x_9$
7. $Y_2^{(0)} = x_0 + x_5$
8. $Y_2^{(1)} = -x_3 - x_8$
9. $Y_2^{(2)} = x_1 + x_6$
10. $Y_2^{(3)} = -x_4 - x_9$
11. $Y_2^{(4)} = x_2 + x_7$
12. $Y_3^{(0)} = x_0 - x_5$
13. $Y_3^{(1)} = -x_2 + x_7$
14. $Y_3^{(2)} = x_4 - x_9$
15. $Y_3^{(3)} = x_1 - x_6$
16. $Y_3^{(4)} = -x_3 + x_8$
17. $Y_4^{(0)} = x_0 + x_5$
18. $Y_4^{(1)} = -x_4 - x_9$
19. $Y_4^{(2)} = x_3 + x_8$
20. $Y_4^{(3)} = -x_2 - x_7$
21. $Y_4^{(4)} = x_1 + x_6$
22. $Y_5^{(0)} = x_0 - x_1 + x_2 - x_3 + x_4 - x_5 + x_6 - x_7 + x_8 - x_9$
23. $Y_6^{(0)} = x_0 + x_5$
24. $Y_6^{(1)} = -x_1 - x_6$
25. $Y_6^{(2)} = x_2 + x_7$
26. $Y_6^{(3)} = -x_3 - x_8$
27. $Y_6^{(4)} = x_4 + x_9$
28. $Y_7^{(0)} = x_0 - x_5$
29. $Y_7^{(1)} = x_3 - x_8$
30. $Y_7^{(2)} = -x_1 + x_6$
31. $Y_7^{(3)} = -x_4 + x_9$
32. $Y_7^{(4)} = x_2 - x_7$
33. $Y_8^{(0)} = x_0 + x_5$
34. $Y_8^{(1)} = -x_2 - x_7$
35. $Y_8^{(2)} = x_4 + x_9$
36. $Y_8^{(3)} = -x_1 - x_6$
37. $Y_8^{(4)} = x_3 + x_8$
38. $Y_9^{(0)} = x_0 - x_5$
39. $Y_9^{(1)} = -x_4 + x_9$
40. $Y_9^{(2)} = -x_3 + x_8$
41. $Y_9^{(3)} = -x_2 + x_7$
42. $Y_9^{(4)} = -x_1 + x_6$

4.2.2.5 Observations

The following observations can be made about the relationships between 1-D MRT coefficients and the data, as laid out in sections 4.2.2.1 - 4.2.2.4 for various values of N.

1) For all values of N, MRT coefficient $Y_0^{(0)}$ is made up of the simple addition of all the elements of the data.

2) For all values of N, there is one and only one MRT coefficient which is made up of alternate additions and subtractions among all the elements of the data. Such coefficients are of the form $Y_M^{(0)}$, as seen in coefficient no. 4 corresponding to $N = 4$, coefficient no. 8 corresponding to $N = 6$, and coefficient no. 12 corresponding to $N = 8$.

3) For all values of N, some MRT coefficients are made up of only a subtraction among two elements of the data. Examples are in coefficient nos. 2, 3 & 5 corresponding to $N = 4$.

4) Some MRT coefficients are made up of only one addition among two elements of the data. Examples are in coefficient nos. 5, 7 & 9 corresponding to $N = 6$. Some MRT coefficients are made up of a negated addition among two elements of the data. Examples are in coefficient nos. 6 & 10 corresponding to $N = 6$.

5) There exist MRT coefficients that are exact negations of other MRT coefficients. If the sign of such an MRT coefficient is inverted, another MRT coefficient of the same order N is obtained. Examples for this are the pairs of coefficient nos. 3 & 14 corresponding to $N = 6$, nos. 6 & 11 corresponding to $N = 6$, and nos. 6 & 39 corresponding to $N = 10$.

6) There exist MRT coefficients that are exactly equal to other MRT coefficients. More than one MRT coefficients share the same value for some values of order N. Examples for this are the pairs 2 & 12 corresponding to $N = 6$, 4 & 32 corresponding to $N = 10$, and 10 & 18 corresponding to $N = 10$.

7) For any N, $Y_0^{(0)}$ and $Y_M^{(0)}$ are the only MRT coefficients that involve all the elements of the data.

4.3 Analysis

From the preliminary observations made above, a detailed analysis of the 1-D MRT coefficients is necessary. Data elements form positive and negative groups. The phase index of an MRT coefficient has particular significance. The existence of a 1-D MRT coefficient can be explained on the basis of number theoretic principles. The conditions of existence relate the phase and frequency indices. The index of a data element in positive and negative groups can be found using different methods. It is possible to re-write the forward 1-D MRT in (4.1) in the form of an arithmetic series. The MRT has physical significance. These aspects are discussed in the following sub-sections.

4.3.1 Data Elements in an MRT Coefficient

The first group corresponds to those data elements whose indices satisfy the congruence relation $((nk))_N = p$, and the second group corresponds to the elements with indices that satisfy the congruence relation $((nk))_N = p + M$. Thus there are two congruence relations:

$$((nk))_N = p \tag{4.2}$$

$$((nk))_N = p + M \tag{4.3}$$

(a) Positive Data Group:

The group of data elements whose indices satisfy the congruence relation $((nk))_N = p$ is defined as the positive data group of the 1-D MRT coefficient $Y_k^{(p)}$.

(b) Negative Data Group:

The group of data elements whose indices satisfy the congruence relation $((nk))_N = p + M$ is defined as the negative data group of the 1-D MRT coefficient $Y_k^{(p)}$.

Example 4.2:

From section 4.2.2.3, the data elements x_0, x_2, x_4 and x_6, form the positive group of the MRT coefficient $Y_4^{(0)}$, and data elements x_1, x_3, x_5 and x_7, form the negative group of the MRT coefficient $Y_4^{(0)}$

4.3.2 Phase Index in MRT

An MRT coefficient has two indices, the frequency index and the phase index. By formal definition of the MRT, the phase index has values in the range $[0, M - 1]$.

(a) Valid Phase Index

Although the MRT definition is such that the phase index has values in the range $[0, M - 1]$, the nature of the linear congruence equations involved makes it theoretically possible for the value of phase index p to have values in the range $[0, N - 1]$. Given an MRT coefficient $Y_k^{(p)}$, a value for the phase index p in the range $[0, N - 1]$ is defined to be a valid phase index for a given frequency index k if $k \mid p$.

Example 4.3:

For $N = 6$, if $k = 2$, then $p = 0, 2, \& 4$ satisfy $k \mid p$, and hence these are valid phase indices for this value of k.

(b) Allowable Phase Index

A phase index p is defined to be an allowable phase index if $p < M$. The allowable phase index actually is the phase index that is referred to in the formal definition of MRT.

Theorem 4.1

$$Y_k^{(p)} = -Y_k^{(p+M)}$$

Proof

The elements n_a that are in the positive group of the MRT coefficient $Y_k^{(p)}$ can be found as solutions of

$$((n_a k))_N = p \tag{4.4}$$

The elements n_a' that are in the negative group of the MRT coefficient $Y_k^{(p)}$ can be found as solutions of

$$((n_a' k))_N = p + M \tag{4.5}$$

The elements n_b that are in the positive group of the MRT coefficient $Y_k^{(p+M)}$ can be found as solutions of

$$((n_b k))_N = p + M \tag{4.6}$$

The elements n_b' that are in the negative group of the MRT coefficient $Y_k^{(p+M)}$ can be found as solutions of

$((n_b' k))_N = p + M + M = p + N$, which can be written as

$$((n_b' k))_N = p \tag{4.7}$$

From (4.4) and (4.7), it can be inferred that

$$n_a = n_b' \tag{4.8}$$

From (4.5) and (4.6), it can be inferred that

$$n_a' = n_b \tag{4.9}$$

From (4.8) and (4.9) and the definition of MRT in (4.1),

$$Y_k^{(p)} = \sum n_a - \sum n_b$$

$$= \sum n_{b'} - \sum n_{a'}$$

$$= -\left(\sum n_{a'} - \sum n_{b'}\right)$$

$$= -Y_k^{(p+M)}$$

$$\therefore Y_k^{(p)} = -Y_k^{(p+M)}$$

4.3.3 Existence of 1-D MRT Coefficient

Equations (4.2) and (4.3) that define the 1-D MRT are linear congruences. A basic theorem from number theory, given in Appendix A.3, provides necessary and sufficient condition for a linear congruence to be solvable [97]. It also gives the number of incongruent solutions if the linear congruence is solvable, and also a formula for finding the solutions. Using this theorem, the conditions for existence of MRT coefficient $Y_k^{(p)}$ for data of order N can be stated as a theorem:

Theorem 4.2

An MRT coefficient $Y_k^{(p)}$ exists for data of order N if either of the following two conditions is satisfied:

Condition 1: $g(k,N)|p$

Condition 2: $g(k,N)|(p+M)$

If condition 1 holds, there are elements in the positive data group of the MRT coefficient. If condition 2 holds, there are elements in the negative data group of the MRT coefficient.

Proof

An MRT coefficient $Y_k^{(p)}$ exists for data of order N when there are data elements whose indices satisfy (4.2) and/or (4.3), in other words, when linear congruences (4.2) and/or (4.3) have solutions. From Appendix A.3, the necessary and sufficient condition for (4.2) to be solvable is that $g(k,N)|p$. Similarly, $g(k,N)|(p+M)$ becomes the necessary and sufficient condition for (4.3) to be solvable. Hence, if the two conditions $g(k,N)|p$ and/or $g(k,N)|(p+M)$ is satisfied, an MRT coefficient $Y_k^{(p)}$ exists for data of given order N. If condition 1 is satisfied, there are elements in the positive data group, and if condition 2 is satisfied, there are elements in the negative data group.

From Appendix A.3, a linear congruence $((nk))_N = p$, if solvable, has $g(k,N)$ solutions mod N. Hence, n has $g(k,N)$ solutions in the range [0, N - 1] and thus $g(k,N)$ data elements in the positive group. Also, given n_0 is a member of the positive group (particular solution), the other solutions are given by $n = n_0 + (N / g(k,N))t$, $0 \leq t < g(k,N)$, which are, in other words, the other members of the positive group. The solutions of n are congruent modulo N, i.e. if n_0 is a solution, $n_0 + N$ is also a solution. Hence, if the value of n obtained from the general solution exceeds N, it is understood that this value of n actually corresponds to the data element with index $((n))_N$. It can be assured that the value of n obtained from the general solution does not exceed N by assuring that the particular solution n_0 is the numerically smallest particular solution in the range [0, N - 1]. The same explanation holds for the number of data elements in the negative group and also the general expression for data elements, since the structure of the linear congruence equations corresponding to both groups is the same. In this context, the following can be concluded:

The indices of the data elements in the positive (or negative) data group of an MRT coefficient form an arithmetic progression of the form

$$n_0, n_0 + \frac{N}{g(k,N)}, n_0 + \frac{2N}{g(k,N)}, n_0 + \frac{3N}{g(k,N)} \dots\dots\dots n_0 + \frac{(g(k,N)-1)N}{g(k,N)}, \qquad (4.S.1)$$

given n_0 is the smallest member of the positive (or negative) data group.

Defining $g_k = \dfrac{N}{g(k,N)}$,

(4.S.1) can be written as,

$$n_0, n_0 + g_k, n_0 + 2g_k, n_0 + 3g_k \dots\dots\dots n_0 + (g(k,N)-1)g_k, \text{ or} \qquad (4.S.1a)$$

$$n_0 + jg_k, \qquad j = [0, (g(k,N)-1)] \qquad (4.S.1b)$$

4.3.4 Dependence of Phase Index on Frequency Index

The existence of 1-D MRT coefficients can be studied for their dependence on the frequency index k.

(a) $k = 0$

When $k = 0$, the left-hand side of (4.2) becomes $g(0,N)$, which is equal to N. Thus the condition for existence of positive group becomes $N \mid p$. The only value of p that satisfies this condition is $p = 0$. Hence, for $k = 0$, the positive group exists only for $p = 0$. From (4.3), the condition for existence of negative group becomes $N \mid (p + M)$, which is satisfied only when $p = M$. Hence, the negative group exists only for $p = M$, when $k = 0$. However, from theorem 4.1, this negative group is only a sign-reversal of the positive group for $p = 0$. If the value of p is restricted to the range $[0, M - 1]$ as in (4.1), then the existence condition for negative group is not satisfied by any value of p, which implies the non-existence of a negative group. Thus, an MRT coefficient with $k = 0$ exists only for $p = 0$, and it has only a positive group.

(b) $k = 1$

When $k = 1$, $g(k,N) = 1$. For this case, the existence condition (4.2) becomes $1 \mid p$. This condition is satisfied by all values of p in the range $[0, M - 1]$. Similarly, the negative group condition in (4.3) is also satisfied by all values of p in this range, since the condition is $1|(p+M)$. Hence, when $k = 1$, MRT coefficients exists for all values of p in the range $[0, M-1]$. Also, positive and negative groups exist for all MRT coefficients when $k = 1$.

(c) $k = 2$

Since N is even, $g(k,N) = 2$. The positive group condition here becomes $2 \mid p$. This is satisfied by all even values of p in the range $[0, M - 1]$. The negative group condition is $2 \mid (p + M)$. The solutions of this condition depend on the value of N.

(i) M is even: When M is even, $2 \mid (p + M)$ is satisfied by all even values of p.

(ii) M is odd: When M is odd, $2 \mid (p + M)$ is satisfied by all odd values of p, since the sum of two odd numbers is even.

In summary, when $k = 2$, MRT coefficients have positive groups for even values of p in the range $[0, M - 1]$. Negative groups exist for odd values of p when M is odd and for even values of p when M is even. For M even, MRT coefficients with $k = 2$ have positive and negative groups for even p, and MRT coefficients with odd p do not exist. For M odd, MRT coefficients exist for all values of p, having only positive groups for even p and only negative groups for odd p.

(d) General Value k

(i) k & N are relatively prime: In this case, $g(k,N) = 1$. The condition for existence of positive data group becomes $1 \mid p$, and that for the negative data group becomes $1 \mid (p + M)$. These two conditions are exactly same as those for $k = 1$. Hence, the conclusions are also the same as that for $k = 1$; when $g(k,N) = 1$, MRT coefficients corresponding to k exist for all values of p in the range $[0, M - 1]$. So do positive and negative data groups.

(ii) k is a divisor of N: When k is a divisor of N, $g(k,N) = k$. The condition corresponding to (4.2) here becomes $k \mid p$. This condition is satisfied when p is a multiple of k. The number of such p in the range $[0, N - 1]$ is given by N/k. The values of p are thus $p = 0, k, 2k \ldots N - k$. The condition corresponding to (4.3) is now $k \mid (p + M)$. When $k \mid p$, the condition $k \mid (p + M)$ has solutions only if k is a divisor of M. This implies that an MRT coefficient has both positive and negative groups only if k is a divisor of M. If k is not a divisor of M, only one among the positive and negative groups exists, for a given value of p, as discussed below.

(a) k is a divisor of M: It has been seen in the above paragraph that the valid phase indices for which MRT coefficients have positive groups are given by $p = 0, k, 2k \ldots N - k$. There are N / k such valid phase indices. When $k \mid M$ is satisfied, negative groups also exist for all these MRT coefficients. However, from theorem 4.1, $Y_k^{(p)} = -Y_k^{(p+M)}$. A general term of the series of valid phase indices can be written as dk, where d is an integer in the range $[0, (N / k) - 1]$. Since $k \mid M$, there exists an integer $c < M$ such that $ck = M$. Hence, for a valid phase index p in the range $[0, M-1]$ (allowable phase index), $Y_k^{(p+ck)} = Y_k^{(p+M)} = -Y_k^{(p)}$. This implies that for every valid allowable phase index p, there is a valid non-allowable phase index $p+ck$ such that $Y_k^{(p)} = -Y_k^{(p+ck)}$. MRT coefficients corresponding to these non-allowable phase indices differ only in sign from their allowable phase counterparts. Thus, these non-allowable phase indices may be neglected since they provide MRT coefficients that are sign-reversed versions of MRT coefficients provided by allowable phase indices. There are thus M / k allowable phase indices and hence M / k MRT coefficients, the phase indices being given by the arithmetic series $p = 0, k, 2k, 3k \ldots M - k$. MRT coefficients corresponding to these phase indices have both positive and negative groups simultaneously.

(b) k is not a divisor of M: In this case, for a valid phase index, only one among the positive or negative groups exists. The valid phase indices still form an arithmetic series $p = 0, k, 2k \ldots N - k$. Since k is a non-divisor of M, there does not exist an integer c that satisfies $p + ck = p + M$, and hence, the possibility that $Y_k^{(p)} = -Y_k^{(p+ck)}$ does not exist. Thus, for $p > M$, where p is a valid phase index, there is no valid phase index $p - M$. However, the relation $Y_k^{(p)} = -Y_k^{(p-M)}$ is still valid. Thus, for $p > M$, there exists an allowable but non-valid phase index of value $p - M$. To express the MRT corresponding to a valid phase index $p > M$ in terms of an allowable phase index, the relation $Y_k^{(p)} = -Y_k^{(p-M)}$ may be used. When $k \mid p$, the condition $k \mid (p + M)$ has solutions only if k is a divisor of M. Thus, MRT coefficients formed by valid phase indices have only positive groups. The negative of the MRT coefficient formed by the valid phase index p greater than M, $p > M$ is the MRT coefficient formed by the allowable non-valid phase index $p - M$. Hence, the positive data group of the MRT coefficient with $p > M$ becomes the negative data group of the

MRT coefficient with allowable non-valid phase index $p - M$. Thus, among the M allowable phase indices, a subset of them consists of valid phase indices, while the other subset consists of allowable non-valid phase indices of the form p - M that are obtained from valid phase indices $p > M$.

The distance between two successive valid phase indices is k. Given an allowable and numerically smallest valid phase index $p_1 < M$, assume that the nearest allowable non-valid phase index is p_2. The valid phase index corresponding to p_2 is $p_2^{'} = p_2 + M$. Given q is the smallest integer such that $qk > M$,

$$p_2^{'} = p_1 + qk$$

This equation can be justified along the following lines. p_2 is the nearest allowable non-valid phase index to p_1. p_1 is the smallest allowable valid phase index. There does not exist a valid phase index $p_1^{'} = p_1 + M$. The next valid phase index is thus given by $p_2^{'} = p_1 + qk$, since q is the smallest integer such that $qk > M$

$$\therefore p_2 - p_1 = qk - M$$

Since $k \mid N$, there exists an integer t such that $N = tk$. Hence, $M/k = t/2$. Since k is not a divisor of M, $t/2$ cannot be an integer. For this, t has to be odd. $k = N/t$. Since, division of an even number by an odd number yields an even number, k is even. Since $k \mid N$ and k is even, it follows that $(k/2) \mid M$.

Hence,

$$M = dk/2, \; d \in Z$$

$$\therefore p_2 - p_1 = qk - dk/2$$

$$\therefore p_2 - p_1 = 2q(k/2) - dk/2$$

$$\therefore p_2 - p_1 = (2q - d)(k/2)$$

The value of $(2q - d)$ cannot be greater than 1, since, then in that case,

$$p_2 - p_1 \geq k$$

As assumed earlier, p_1 is the numerically smallest allowable valid phase index and $p_2^{'}$ is the first valid phase index greater than M. Hence, the distance between $p_2^{'}$ and M has to be lesser than k. Thus, the distance between p_1 and p_2 has to be lesser than k. Hence,

$$p_2 - p_1 = k/2 \tag{4.10}$$

The next valid index after p_1 is given by

$$p_3 = p_1 + k \tag{4.11}$$

From (4.10) and (4.11),

$$p_3 - p_2 = k/2$$

If $p_4^{'}$ is the next valid phase index after $p_2^{'}$, then

$$p_4^{'} = p_2^{'} + k \tag{4.12}$$

$p_4^{'}$ has a corresponding non-valid allowable phase index p_4 given by

$$p_4 = p_4^{'} - M \tag{4.13}$$

From (4.12) and (4.13),

$$p_4 = p_2 + k$$

$$\therefore p_4 - p_3 = k/2$$

Hence, there is an allowable non-valid phase index between every successive pair of allowable valid phase indices. The MRT coefficient formed by these allowable non-valid phase indices will have the same magnitude and opposite sign as the MRT coefficient formed by the corresponding non-allowable valid phase indices.

The sequence of allowable phase indices would thus be:

$$p_0, p_0 + \frac{k}{2}, p_0 + k, p_0 + 3\frac{k}{2}, \ldots\ldots\ldots\ldots, p_0 + M - \frac{k}{2}. \tag{4.S.2}$$

When k is not a divisor of M, MRT coefficients exist for these allowable phase indices and they will have either a positive group or a negative group only. There are N/k allowable phase indices in this case, and when k is a non-divisor of M, N/k is odd. Since k is a divisor of N, the condition for existence for solutions is $k \mid p$. The smallest value of p that satisfies this condition is $p = 0$. Hence, the first valid allowable phase index is $p_0 = 0$. This phase index corresponds to an MRT coefficient with a positive group only, and the next allowable phase index is $k/2$, which corresponds to an MRT coefficient with a negative group only. This property alternates till the last allowable phase index $M - (k/2)$ is reached. This last phase index corresponds to an MRT coefficient with only a positive group, since the total number of allowable phase indices is odd. In other words, it can be concluded that MRT coefficients with only a positive group have allowable phase indices that are even multiples of $k/2$, starting from 0 and ending at $(N - k)/2$. The number of such MRT coefficients is hence given by $((N/2k) + (1/2))$. Similarly, the MRT coefficients with only a negative group have allowable phase indices that are odd multiples of $k/2$, starting from $k/2$ and ending at $M - k$. The number of such MRT coefficients is given by

$((N/2k) - (1/2))$. The total number of 1-D MRT coefficients is thus N / k, and the sequence of allowable phase indices is:

$$0, \frac{k}{2}, k, 3\frac{k}{2}, \ldots\ldots\ldots\ldots, M - \frac{k}{2}. \tag{4.S.3}$$

The sequence of allowable phase indices that correspond to MRT coefficients having only positive groups is

$$0, k, 2k, \ldots\ldots\ldots, (N-k)/2, \tag{4.S.4}$$

and the sequence of allowable phase indices that correspond to MRT coefficients having only negative groups is

$$k/2, 3k/2, 5k/2, \ldots\ldots\ldots\ldots, M-k \tag{4.S.5}$$

4.3.5 Particular Solutions for Data Groups

From (4.S.1a), the indices of the elements of the positive (or negative) data group of an 1-D MRT coefficient form an arithmetic series of the form

$$n_0, n_0 + g_k, n_0 + 2g_k, n_0 + 3g_k \ldots\ldots\ldots\ldots n_0 + (g(k,N)-1)g_k,$$

given n_0 is the smallest member of the positive (or negative) data group. Using this arithmetic series, the indices of data elements that are present in the positive and negative data groups of an MRT coefficient can be found out. However, this formula pre-supposes the knowledge of a particular solution n_0. This sub-section deals with the value of this particular solution. The linear congruence equation $((nk))_N = p$ is solvable only if $g(k,N) \mid p$. Two cases for the value of k need to be considered here.

(a) k, a divisor of N

When k is a divisor of N, $g(k,N) = k$. Hence, the linear congruence $((nk))_N = p$ has a particular solution $n_+ = p/k$, so that $((n_0k))_N = (((p/k)k))_N = p$.

Theorem 4.3

(a) When k is a divisor of N, the index n_+ of the first element in the positive data group of an MRT coefficient $Y_k^{(p)}$ is given by $n_+ = p/k$.

(b) For k a divisor of M, if data element with index n occurs in the positive data group of an MRT coefficient $Y_k^{(p)}$, then the data element with index $n + (M/k)$ occurs in the negative data group of the same MRT coefficient.

(c) The index of first data element in the positive group of an MRT coefficient $Y_k^{(p)}$, when k is not a divisor of M is given by $n_+= p / k$. The index of first data element in the negative group of an MRT coefficient $Y_k^{(p)}$, when k is not a divisor of M is given by $n_- = (p+M)/k$.

Proof

(a) The index of the first element of MRT coefficient $Y_k^{(p)}$ is the smallest solution of the congruence equation, (4.2),

$$((nk))_N = p\,.$$

Solutions exist for (4.2) only if $g(k,N)$ divides p, which can be written as $g(k,N) \mid p$, i.e. $k \mid p$, since $g(k,N) = k$.

Since $k \mid p$, the smallest solution to (4.2) is $n_+ = p/k$, and thus $n_+ = p/k$ is the first element in the positive data group of MRT coefficient $Y_k^{(p)}$.

The condition $g(k,M) = k$ is necessary for both positive and negative groups to be present in an MRT coefficient.

(b) Given $((nk))_N = p$.

$$\therefore (((n + \frac{M}{k})k))_N = ((nk))_N + ((\frac{M}{k}k))_N = p + M$$

Thus, if data element with index n occurs in the positive group, data element with index $n + M/k$ occurs in the negative group since $(((n + M/k)\,k))_N = p + M$.

(c) From (4.S.3), when k is not a divisor of M, the sequence of allowable phase indices is

$$0, \frac{k}{2}, k, \frac{3k}{2}, ..., M - \frac{k}{2}.$$

From (4.S.4), the sequence of allowable phase indices that correspond to MRT coefficients having only positive groups is

$$0, k, 2k, ..., M - \frac{k}{2}.$$

From (4.S.5), the sequence of allowable phase indices that correspond to MRT coefficients having only negative groups is

$$\frac{k}{2}, \frac{3k}{2}, \frac{5k}{2}, ..., M - k.$$

The allowable phase indices that provide MRT coefficients with only positive groups are valid phase indices. Hence, the first data element that satisfies $((nk))_N = p$ is obtained by the straight-forward division p / k since $k \mid p$ is satisfied, p being a valid phase index. Hence, index of the first data element in the positive group is $n_+ = p / k$.

Also, the allowable phase indices that provide MRT coefficients with only negative groups are actually non-valid allowable phase indices. Given a non-valid phase index p_-, the valid phase index p_+ corresponding to p_- is given by $p_+ = p_- + M$. Hence, index of the first data element in the negative group is $n_- = (p + M)/k$.

(b) k, a non-divisor of N

In this case, $n_+ = p / g(k,N)$ cannot be a solution since $g(k,N) \neq k$. The extended Euclidean algorithm (Appendix A.5) can be used to find a particular solution for this case.

Example 4.4:

If the existence of MRT coefficient $Y_2^{(2)}$ needs to be checked for $N = 8$, using theorem 4.2, the sufficient conditions are $g(2,8) \mid 2$ and/or $g(2,8) \mid 6$. Both of these conditions are satisfied, since $g(2,8) = 2$, and both $2 \mid 2$ and $2 \mid 6$ hold. Hence, the MRT coefficient $Y_2^{(2)}$ exists for $N = 8$. Checking existence of MRT coefficient $Y_2^{(1)}$ for $N = 8$, this MRT coefficient does not exist since both the necessary conditions $g(2,8) \mid 1$ and $g(2,8) \mid 5$ do not hold. Similarly, (4.S.1) gives the indices of the data elements that make up the MRT coefficient. The number of coefficients in the positive group of MRT coefficient $Y_2^{(2)}$ for order 8 is equal to $g(2,8) = 2$, and in the negative group of the same MRT coefficient is also $g(2,8) = 2$. Given n_0 is an index present in the positive group (n_0 can be obtained by finding a particular solution for (4.2), other indices are given by $n = n_0 + (8 / g(2,8))t$, where $0 \leq t < g(2,8)$. Thus, $n = n_0 + 4t$, $0 \leq t < 2$.

4.3.6 1-D MRT: Closed-form Expression

From the definition of MRT, and from the arithmetic series of indices of members in the positive and negative data groups in (4.S.1a), and, given n_+ is a member of the positive group, and n_- is a member of the negative group, a 1-D MRT coefficient can be expressed in the following manner:

$$Y_k^{(p)} = \left[x_{n_+} + x_{n_+ + g_k} + x_{n_+ + 2g_k} + x_{n_+ + 3g_k} \cdots\cdots\cdots + x_{n_+ + (g(k,N)-1)g_k} \right] - \left[x_{n_-} + x_{n_- + g_k} + x_{n_- + 2g_k} + x_{n_- + 3g_k} + \cdots\cdots\cdots + x_{n_- + (g(k,N)-1)g_k} \right] \quad (4.14)$$

This can be further simplified as

$$Y_k^{(p)} = \sum_{j=0}^{g(k,N)-1} \left[x_{n_+ + jg_k} - x_{n_- - jg_k} \right] \tag{4.14a}$$

(4.14a) is a closed-form formula for computing 1-D MRT coefficients. However, an important requirement in (4.14) is that the index of one member each of the positive and negative groups respectively needs to be known beforehand. As discussed earlier, the extended Euclidean algorithm provides these indices.

When k is a divisor of N, theorem 4.3(a) specifies the value of the index of the first element in the positive group, $n_+ = p / k$. Also, from theorem 4.3(b), there exists a relation between an index in the positive group and an index in the negative group, given k is a factor of M. Hence, given n_+, $n_- = n_+ + (M / k)$. When k is not a divisor of M, theorem 4.3(c) gives the value of phase indices that correspond to MRT coefficients that have only positive groups, and MRT coefficients that have only negative groups.

Case 1: Structure of 1-D MRT Coefficient When k is a Divisor of M

When k is a divisor of M, $n_+ = p / k$, and $n_- = n_+ + (M / k)$. Also, $g(k,N) = k$. Using these relations to rewrite (4.14),

$$Y_k^{(p)} = \left[x_{\frac{p}{k}} + x_{\frac{p}{k}+\frac{N}{k}} + x_{\frac{p}{k}+\frac{2N}{k}} + x_{\frac{p}{k}+\frac{3N}{k}} \cdots\cdots\cdots + x_{\frac{p}{k}+\frac{(k-1)N}{k}} \right] - \left[x_{\frac{p}{k}+\frac{N}{2k}} + x_{\frac{p}{k}+\frac{3N}{2k}} + x_{\frac{p}{k}+\frac{5N}{2k}} + x_{\frac{p}{k}+\frac{7N}{2k}} + \cdots\cdots\cdots + x_{\frac{p}{k}+\frac{(2k-1)N}{2k}} \right] \tag{4.15}$$

On further simplification,

$$Y_k^{(p)} = \sum_{j=0}^{k-1} \left[x_{\frac{jN+p}{k}} - x_{\frac{(2j+1)N+2p}{2k}} \right] \tag{4.15a}$$

$p = 0, k, 2k, \ldots M - k$

Case 2: Structure of MRT Coefficient when k is a non-divisor of M

When k is not a divisor of M, it has been found in section 4.3.4.4 that positive and negative groups cannot exist together for the same MRT coefficient. For certain values of p, only positive groups exist. For other values of p, only negative groups exist. As seen, for positive groups, $n^+ = p / k$, and for negative groups, $n^- = (p + M)/k$. An MRT coefficient with only a positive group has the following form:

$$Y_k^{(p)} = \left[x_{\frac{p}{k}} + x_{\frac{p}{k}+\frac{N}{k}} + x_{\frac{p}{k}+\frac{2N}{k}} + x_{\frac{p}{k}+\frac{3N}{k}} \cdots\cdots\cdots + x_{\frac{p}{k}+\frac{(k-1)N}{k}} \right] \tag{4.16}$$

which, when simplified, becomes

$$Y_k^{(p)} = \sum_{j=0}^{k-1} \left[x_{\frac{jN+p}{k}} \right] \tag{4.16a}$$

$p = 0, k, 2k, \ldots M - k/2$

Similarly, an MRT coefficient with only a negative group has the following form:

$$Y_k^{(p)} = -\left[x_{\frac{p+M}{k}} + x_{\frac{p+M}{k}+\frac{N}{k}} + x_{\frac{p+M}{k}+\frac{2N}{k}} + \cdots\cdots\cdots + x_{\frac{p}{k}+\frac{(k-1)N}{k}} \right] \tag{4.17}$$

which when simplified, becomes

$$Y_k^{(p)} = -\sum_{j=0}^{k-1} \left[x_{\frac{jN+p+M}{k}} \right] \tag{4.17a}$$

$p = k/2, 3k/2, \ldots M - k$

Case 3: Structure of MRT Coefficient when k is co-prime to N

When k is co-prime to N, $g(k,N) = 1$. Thus, there is only one element in the positive group and similarly only one element in the negative group. The values of n_+ and n_- need to be found out using the Euclidean algorithm or by trial-and-error method. The MRT coefficient will have the form

$$Y_k^{(p)} = x_{n_+} - x_{n_-} \tag{4.18}$$

$p = 0, 1, 2, 3, \ldots, M - 1.$

4.3.7 Physical Significance of MRT

An MRT coefficient has both frequency and phase indices. In the MRT coefficient $Y_k^{(p)}$, k is a frequency index, and p is a phase index. In comparison, a DFT coefficient has only one index, and that is the frequency index. Hence, a distinguishing feature of the MRT coefficient is the presence of an extra index, the phase index. The presence of this index provides some information regarding phase of the signal. This point can be best examined from Figure 4.3, which shows the basis vectors of an MRT coefficient for $N = 8$. Consider the basis vector for MRT coefficient $Y_1^{(0)}$. The value of the frequency index of this MRT coefficient indicates that the coefficient provides information about presence of frequency content $k = 1$ in the signal. Hence, there is a positive peak and a negative peak in the basis vector which indicates one frequency cycle. The location of the positive peak of the frequency cycle is dependent on the value of this phase index. The distance of the negative peak from the positive peak depends on the frequency. The physical significance of an MRT coefficient is this: the phase index specifies the starting time of the frequency cycle. The MRT can hence be considered a time-frequency representation of the input signal. In contrast to the DFT which is a frequency transform, the MRT, though directly related to the DFT, has localization in time as well as in frequency. Also, MRT coefficients can be considered to be constituent parts of the DFT; parts which, if weighted by the exponential kernel, would yield the DFT of the corresponding frequency. For $N = 8$, DFT coefficient Y_k can be expressed in terms of corresponding MRTs as

$$Y_k = Y_k^{(0)} W_8^0 + Y_k^{(1)} W_8^1 + Y_k^{(2)} W_8^2 + Y_k^{(3)} W_8^3$$

4.4 Redundancy in MRT

In section 4.2.2.5, one of the observations made regarding 1-D MRT is that the coefficients are sometimes exact negations of other coefficients, i.e. a number of 1-D MRT coefficients can be obtained by reversing the sign of an MRT coefficient. Also, MRT coefficients are sometimes exactly equal to other MRT coefficients, i.e. more than one MRT coefficients share the same values. In general, for certain values of N, a group of MRT coefficients having different frequency and phase indices, have the same magnitude. The sign of the coefficients may or may not be the same. This phenomenon can be looked at in different ways. First, it means that the same information is represented by various MRT coefficients. This implies that such MRT coefficients share some common properties. MRT is a measure of signal content at a specific frequency and pertaining to a specific phase/temporal location. Thus, when many MRT coefficients have the same value, this means that the information conveyed is the same for all these combinations of frequency and phase indices. Hence, there is an element of predictability involved between different combinations of frequency and phase on the basis of their associated MRT coefficients. Secondly, predictability leads to the next aspect of redundancy. The presence of the same value at different frequency/phase indices indicates an element of redundancy in the transform. If this repetition can be accurately predicted, then a simpler transform structure can be evolved by removing the redundant MRT coefficients to yield a transform that has no redundancy. Hence, a detailed analysis of different redundancies present in MRT coefficients is performed as follows.

4.4.1 Complete Redundancy

By complete redundancy, it is meant that another MRT coefficient has exactly the same magnitude as $Y_k^{(p)}$ except for a possible difference in sign.

From a basic theorem [98] in number theory,

if

$$((q))_N = d \tag{4.19}$$

and h is a multiplication factor, then

$$((hq))_{\left(\frac{N}{\gcd(h,N)}\right)} = hd \tag{4.20}$$

If $g(h,N) = 1$, (4.20) becomes

$$((hq))_N = hd \tag{4.21}$$

Given $Y_k^{(p)}$, n satisfies

$$((nk))_N = p \tag{4.22}$$

and n' such that

$$((n'k))_N = p + M \tag{4.23}$$

If there is h such that $g(h,N) = 1$, using (4.19), (4.20) & (4.21),

$$((n(hk)))_N = hp \tag{4.24}$$

and

$$((n'(hk)))_N = h(p+M) \tag{4.25}$$

$$((n'(hk)))_N = hp + hM \tag{4.26}$$

Since $g(h,N) = 1$, h is odd, and hence

$$hp + hM \equiv hp + M \tag{4.27}$$

Using (4.27), (4.26) may be written as

$$((n'(hk)))_N = hp + M \tag{4.28}$$

From the definition of 1-D MRT, and (4.24) & (4.28), it is seen that n and n' form the MRT coefficient $Y_{hk}^{(hp)}$. Using (4.22), (4.23), (4.24) & (4.28),

$$Y_k^{(p)} = Y_{((hk))_N}^{((hp))_N}$$

If $hp \geq M$, then

$$Y_{((hk))_N}^{(((hp))_N)} = -Y_{((hk))_N}^{(hp-M)} \tag{4.29}$$

since

$$Y_k^{(p)} = -Y_k^{(p-M)} = -Y_k^{(p+M)}$$

Hence, the following theorem on redundancy can be stated:

Theorem 4.6

Given $Y_k^{(p)}$, for all h such that $g(h,N) = 1$, $Y_{((hk))_N}^{((hp))_N} = Y_k^{(p)}$ for $((hp))_N < M$, and,

$$Y_{((hk))_N}^{(((hp))_N - M)} = -Y_k^{(p)} \text{ for } ((hp))_N \geq M. \tag{4.30}$$

Theorem 4.6 shows that redundancy can be predicted from the values of N and the values of the frequency and phase indices. Given a pair of frequency index and phase index (k, p), the frequency and phase index pairs of all other MRT coefficients that are redundant with respect to the MRT coefficient with frequency and phase index pair (k, p) can be found out from theorem 4.6, which states that the condition for redundancy is that the multiplication factor that relates the frequency indices of two redundant MRT coefficients is co-prime to N.

Example 4.5:

Let $N = 6$. From section 4.2.2.2, it is seen from coefficients nos. 2, 3 & 4 and from 12, 13 & 14 that MRT coefficients corresponding to $k = 5$ are completely redundant with those corresponding to $k = 1$. Relations of complete redundancy exist also between MRT coefficients with $k = 2$ and $k = 4$. The only integer h, other than 1, in [0, 5], that satisfies $g(h,6) = 1$, is $h = 5$. Since $5 = ((5 \times 1))_6$ and $4 = ((5 \times 2))_6$, this explains redundancy between $k = 1$ & $k = 5$, and $k = 2$ & $k = 4$, respectively.

4.4.1.1 Significance of Frequency Index

The value of the frequency index k is significant with regard to the predictability of other MRT coefficients k' from knowledge of MRT coefficient with frequency index k. If $k' = hk$ and $g(h,N) = 1$, then there exists redundancy between MRT coefficients of the two frequency indices k & k'.

(a) Frequency index co-prime to N

When $k = 1$, all those values of k' that are co-prime to N can be obtained from MRT coefficients with frequency index $k = 1$.

(b) Frequency index, Divisor of N

If k' is a divisor of N, then k' cannot be obtained by multiplying an integer k with an integer h, $k' = hk$, such that $g(h,N) = 1$ and $k \neq k'$. This can be explained as follows: Assume that k' can be expressed as $k' = hk$, and $h \neq 1$, and $k \neq 1$. Then, both h and k are divisors of N too. Hence, $g(h,N) = h$, and $g(k,N) = k$. Thus, if either $h = 1$ and $k \neq 1$, or $h \neq 1$ and $k = 1$, then $h = k'$, or $k = k'$, which is a trivial solution. Thus, there cannot be an h such that $g(h,N) = 1$, and MRT coefficients with frequency indices that are divisors of N cannot be redundant to each other.

(c) Frequency index, Non-prime Non-divisor of N

Assume a frequency index k' exists such that $k' = hk$, such that $g(h,N) = 1$ and $k \neq k'$. If k' is non-prime, it has a non-unity gcd with N, i.e. it has a non-trivial common divisor with N. Hence, even if it is not a divisor of N, it can be obtained by complete redundancy from the common divisor.

Theorem 4.7

MRT coefficients with frequency indices that have common gcd w.r.t. N are all completely redundant to each other.

Proof

Assume k_1 and k_2 are two frequency indices that have common gcd w.r.t. N.

$$g(k_1, N) = k \tag{4.31}$$

$$g(k_2, N) = k \tag{4.32}$$

Assume h exists such that

$$g(h, N) = 1 \tag{4.33}$$

From (4.33), h and N have only one common divisor, and it is one. The extra divisors of the product hk_1 when compared with divisors of k_1 will be the divisors of h. Since none of these extra divisors (except divisor 1) are also present in N, the common divisors among hk_1 and N are the same as the common divisors among k_1 and N, which implies that,

$$g(hk_1, N) = k \tag{4.34}$$

$$((hk_1))_N = hk_1 - Nq \text{, if } 0 \leq hk_1 - Nq < N \text{, } q \in \square \tag{4.35}$$

Using (4.35) and gcd property

$$g(((hk_1))_N, N) = g(hk_1 - Nq, N) = g(hk_1, N) = k \tag{4.36}$$

From (4.32) and (4.36),

$$k_2 = ((hk_1))_N \tag{4.37}$$

From (4.33) & (4.37), and using theorem 4.6, it can be concluded that frequency indices k_1 and k_2 are completely redundant. Hence, theorem is proved.

Example 4.6:

Let $N = 8$. From section 4.2.2.3, the following relations can be verified:

$Y_1^{(0)} = Y_3^{(0)} = Y_5^{(0)} = Y_7^{(0)}$,

$Y_1^{(1)} = Y_3^{(3)} = -Y_5^{(1)} = -Y_7^{(3)}$,

$Y_1^{(2)} = -Y_3^{(2)} = Y_5^{(2)} = -Y_7^{(2)}$, and,

$Y_1^{(3)} = Y_3^{(1)} = -Y_5^{(3)} = -Y_7^{(1)}$.

Since k = 1, 3, 5 & 7 share the same gcd of 1 w.r.t. N, there is redundancy among MRT coefficients of these frequencies. Similarly, since $g(2,8) = g(6,8) = 2$, $Y_2^{(0)} = Y_6^{(0)}$, and $Y_2^{(2)} = -Y_6^{(2)}$.

4.4.1.2 Redundant Frequency Groups

Theorem 4.7 implies that frequency indices can be grouped on the basis of their gcd w.r.t. N. All non-divisor frequency indices are related to divisor frequency indices through multiplication factors h such that $g(h,N) = 1$. Thus the multiplicative factors, that are co-prime to N, are at the heart of the phenomenon of complete redundancy. The number of possible multiplication factors that are involved in complete redundancy is given by Euler's totient function $\phi(N)$, defined as the number of positive integers $\leq N$ that are co-prime to N, where 1 is counted as being co-prime to all numbers. Given 1-D MRT coefficients of frequency $k = 1$, there are $\phi(N)$-1 other frequency indices whose associated MRT coefficients can be derived from the MRT coefficient with $k = 1$. Along with $k = 1$, these $\phi(N)$ frequency indices thus form the set of frequency indices that have $g(k,N) = 1$. There are similar sets of frequency indices that have common gcds w.r.t. N, and each set is associated with a particular divisor of N. The size of this set of frequency indices can be derived for a divisor k of N. There are $\phi(N)$ possible multiplicative factors that can be used to generate other members of the set of frequency indices corresponding to k. From theorem 4.6, the equation for complete redundancy is $k' = ((hk))_N$, where $g(h,N) = 1$. Also,

$$((hk))_N = (((h + N/k)k))_N \tag{4.38}$$

(4.38) implies that the set of k' that is generated from divisor k is unique only for multiplicative factors in the set [0, (N / k)-1], and repeats thereafter for the remaining sets of the same length.

Hence the problem now gets reduced to $k' = ((hk))_{N/k}$ where $g(h, N/k) = 1$. The number of such multiplicative factors h is $\phi(N/k)$, and these factors are the totatives (Appendix A.6) of N/k. Hence, the number of frequency indices that are related by complete redundancy to a frequency index k is given by $\phi(N/k)$ and they are obtained by $k' = ((hk))_N$ where $g(h, N/k) = 1$.

From number theory [99]

$$\sum_{d|N} \phi(d) = N, \qquad \text{if } N \geq 1, \tag{4.39}$$

From (4.39), the sum of the terms $\phi(N/k)$ over all divisors of N is given by N, since $\sum_{d|N} \phi(d) = \sum_{d|N} \phi(N/d)$. Hence, all the N frequency indices $k = [0, N-1]$ have been mapped.

Thus, all frequency indices of 1-D MRT can be classified on the basis of their gcd w.r.t. N.

Theorem 4.8

There exists one-to-one mapping between phase indices of 1-D MRT coefficients of frequencies k and k' related through complete redundancy.

Proof

For two MRT coefficients of frequencies k and k' that are related through complete redundancy, the number of phase indices corresponding to each frequency is the same since $g(k, N) = g(k', N)$, and the number of phase indices is given by $N/g(k, N)$. The phase indices are in the range [0, M - 1]. From theorem 4.6 on complete redundancy, the relation between phase is given by $p' = ((hp))_N$, $g(h, N) = 1$. Using theorem on reduced residue systems in Appendix A.8, on multiplication with h, the resultant set of phase indices p' also will have the same composition as the original set. Multiplication of the phase indices in [0, M - 1] by h and then performing modulus w.r.t. M reproduces the set, but with the order of the elements in the set possibly altered. In this way, the phase indices of MRT coefficients of two completely redundant frequencies are completely mapped to each other.

Example 4.7:

Let $N = 8$. From section 4.2.2.3,

$Y_1^{(0)} = Y_3^{(0)}$,

$Y_1^{(1)} = Y_3^{(3)}$,

$Y_1^{(2)} = -Y_3^{(2)}$, and,

$Y_1^{(3)} = Y_3^{(1)}$.

Since $k=1$ and $k=3$ are redundant through the co-prime h = 3, the set of phase indices of k = 1, $p=\{0,1,2,3\}$, when subjected to the operation $p'=((hp))_N$, would result in $p'=\{0,3,6,1\}$, which reduces to $p'=\{0,3,2,1\}$ after the condition $((hp))_N < M$ is checked and relevant sign change. Hence, $p=\{0,1,2,3\}$ maps to $p'=\{0,3,2,1\}$.

4.4.2 Derived redundancy

The concept of complete redundancy has been presented in section 4.4.1. For $N=6$, from 2, 3, 4 & 8 in section 4.2.2.2

$$Y_3^{(0)} = Y_1^{(0)} - Y_1^{(1)} + Y_1^{(2)} \tag{4.40}$$

Also, from 1, 5, 6 & 7 in section 4.2.2.2,

$$Y_0^{(0)} = Y_2^{(0)} - Y_2^{(1)} + Y_2^{(2)} \tag{4.41}$$

For N = 6, all MRT coefficients with $k=1$ combine to re-appear in the MRT coefficient with $k=3$. The same happens between MRT coefficients of frequency $k=2$ and $k=0$. Thus, although MRT coefficients with $k=3$ and $k=0$ do not exhibit complete redundancy; they are actually formed by combinations of unique MRT coefficients of frequency $k=1$ and $k=2$ respectively. These MRT coefficients can thus be derived from combinations of other unique MRT coefficients. Hence, there exists an element of redundancy in these coefficients, but it is not complete redundancy. These may thus be called derived redundant coefficients and the phenomenon may be called derived redundancy.

An MRT coefficient is called a derived MRT coefficient if it can be obtained by a combination of other MRT coefficients. The main feature in derived redundancy is that more than one MRT coefficient, all of the same frequency, is completely present in another MRT coefficient of a different frequency. It is a relationship in which one MRT coefficient 'A' completely contains another MRT coefficient 'B', but 'B' only partially contains 'A'. 'A' is a combination of other MRT coefficients, with each of which 'A' has the same nature of relationship that it has with 'B'.

Let $Y_{k'}^{(p)}$ be an MRT coefficient of type 'A', and $Y_k^{(p_a)}$ be an MRT coefficient of type 'B'. The congruence relations for $Y_k^{(p_a)}$ are $((nk))_N = p_a$ and $((nk))_N = p_a + M$. The congruence relations for $Y_{k'}^{(p)}$ are $((nk'))_N = p$ and $((nk'))_N = p+M$. Assume that a relation $k'=dk$ exists between k and k'. Hence, $((dnk))_N = p$ and $((dnk))_N = p+M$. $((dnk))_N = p$ may be written as

$((nk))_N = p/d$ if $g(d,N)=d$, and $d \mid p$. In other words, d should be a divisor of N. From $((nk))_N = p/d$ and $((nk))_N = p_\alpha$, $p_\alpha = p/d$. Multiplying both sides of $((nk))_N = p_\alpha + M$ by d, $((dnk))_N = dp_\alpha + dM$. If d is odd, this can be written as $((dnk))_N = dp_\alpha + M$, which becomes $((nk'))_N = p + M$. Hence, if there is an odd divisor d of N, $k' = dk$, then derived redundancy exists between $Y_k^{(p_\alpha)}$ and $Y_{k'}^{(p)}$. Hence, it can be concluded that derived redundancy cannot occur when N is a power of 2 since N has no odd divisors. For all other even values of N, derived redundancy occurs since N would have odd divisors. The smallest value of p that satisfies $p_\alpha = p/d$ is obtained when $p < N$. Let the lowest phase index among the group of MRT coefficients of frequency k will be $p_0 = p/d$. Other values of the phase indices will be $p_\alpha = (p+N)/d, (p+2N)/d$ etc.

From (4.15), given a frequency index k, and a phase p_i, MRT coefficient $Y_k^{(p_i)}$ has the following structure:

$$Y_k^{(p_i)} = \left[x_{\frac{p_i}{k}} + x_{\frac{p_i}{k}+\frac{N}{k}} + x_{\frac{p_i}{k}+\frac{2N}{k}} + x_{\frac{p_i}{k}+\frac{3N}{k}} \cdots\cdots\cdots + x_{\frac{p_i}{k}+\frac{(k-1)N}{k}} \right] - \left[x_{\frac{p_i}{k}+\frac{N}{2k}} + x_{\frac{p_i}{k}+\frac{3N}{2k}} + x_{\frac{p_i}{k}+\frac{5N}{2k}} + x_{\frac{p_i}{k}+\frac{7N}{2k}} \cdots\cdots\cdots + x_{\frac{p_i}{k}+\frac{(2k-1)N}{2k}} \right] \tag{4.42}$$

An MRT coefficient of the same frequency k, but another phase p_j, $Y_k^{(p_j)}$ has,

$$Y_k^{(p_j)} = \left[x_{\frac{p_j}{k}} + x_{\frac{p_j}{k}+\frac{N}{k}} + x_{\frac{p_j}{k}+\frac{2N}{k}} + x_{\frac{p_j}{k}+\frac{3N}{k}} \cdots\cdots\cdots + x_{\frac{p_j}{k}+\frac{(k-1)N}{k}} \right] - \left[x_{\frac{p_j}{k}+\frac{M}{k}} + x_{\frac{p_j}{k}+\frac{3M}{k}} + x_{\frac{p_j}{k}+\frac{5M}{k}} + x_{\frac{p_j}{k}+\frac{7M}{k}} \cdots\cdots\cdots + x_{\frac{p_j}{k}+\frac{(2k-1)M}{k}} \right] \tag{4.43}$$

Another frequency k' will have an MRT coefficient $Y_{k'}^{(p)}$ with the phase p, and will have the form

$$Y_{k'}^{(p)} = \left[x_{\frac{p}{k'}} + x_{\frac{p}{k'}+\frac{N}{k'}} + x_{\frac{p}{k'}+\frac{2N}{k'}} + x_{\frac{p}{k'}+\frac{3N}{k'}} \cdots\cdots\cdots + x_{\frac{p}{k'}+\frac{(k-1)N}{k'}} \right] - \left[x_{\frac{p}{k'}+\frac{M}{k'}} + x_{\frac{p}{k'}+\frac{3M}{k'}} + x_{\frac{p}{k'}+\frac{5M}{k'}} + x_{\frac{p}{k'}+\frac{7M}{k'}} \cdots\cdots\cdots + x_{\frac{p}{k'}+\frac{(2k-1)M}{k'}} \right] \tag{4.44}$$

By derived redundancy,

$$Y_{k'}^{(p)} = Y_k^{(p_i)} + Y_k^{(p_j)} + \ldots + Y_k^{(p_\alpha)} \tag{4.45}$$

The difference between the index of the first element in the positive group and the index of the second element in the positive group of MRT coefficient $Y_{k'}^{(p)}$ is $q' = N/k'$. The difference between the first index of the positive group of MRT coefficient $Y_k^{(p_i)}$ and the first index of positive group of MRT coefficient $Y_k^{(p_j)}$ is $q = (p_j - p_i)/k$. From (4.42), (4.43), (4.44) & (4.45), derived redundancy occurs when $q = q'$ and $p_i / k = p / k'$.

$$\therefore \frac{N}{k'} = \frac{(p_j - p_i)}{k}$$

$$\therefore (p_j - p_i) = \frac{Nk}{k'} \tag{4.46}$$

Also, $p = \frac{p_i k'}{k}$

The number of valid phase indices for MRT coefficient $Y_{k'}^{(p)}$ is N/k'. Similarly, the number of valid phase indices for MRT coefficient $Y_k^{(p_i)}$ is N/k. There are k' elements in positive group of $Y_{k'}^{(p)}$, and k elements in positive group of $Y_k^{(p_i)}$. When derived redundancy exists between MRT coefficient of higher frequency $Y_{k'}^{(p)}$ and MRT coefficients of lower frequency $Y_k^{(p_i)}$, $Y_k^{(p_j)}$ etc., the number of MRT coefficients of the lower frequency k that should combine to form one MRT coefficient of a higher frequency is k'/k. There are N/k' MRT coefficients of frequency k', and N/k MRT coefficients of frequency k. For N/k coefficients of frequency k to combine to form N/k' coefficients of frequency k', k'/k coefficients of frequency k need to combine to form one coefficient of frequency k'. Given p_0 is the smallest valid phase index of the k'/k MRT coefficients of lower frequency k that combine to form the MRT coefficient of higher frequency k', using (4.46), the valid phase indices of the k'/k MRT coefficients form the following sequence: $p_0, p_0 + \frac{Nk}{k'}, p_0 + 2\frac{Nk}{k'}, p_0 + 3\frac{Nk}{k'}, \ldots\ldots\ldots\ldots, p_0 + N - \frac{Nk}{k'}$.

Assume there is an integer q such that

$$q\frac{Nk}{k'} = M \ ,$$

which when simplified becomes,

$$k' = 2qk$$

Hence, if $k' = 2qk$, there will be a term $p_0 + qNk/k' = p_0 + M$ in the sequence of valid phase indices, and hence any term p such that $p < M$ will have another term $p + qNk/k' = p + M$ in the sequence. Thus, MRT coefficient $Y_{k'}^{(p)}$ will be a sum of MRT coefficients with valid phase indices that form a sequence which contains elements of the form p and $p+M$. From theorem 4.1,

$$Y_k^{(p)} = -Y_k^{(p+M)}$$

Hence, a sum of MRT coefficients having valid phase indices in this sequence will have zero elements, which implies the non-existence of the MRT coefficient. Hence, for a lower frequency k and a higher frequency k' that satisfy $k' = 2qk$, it is not possible for MRT coefficients of the lower frequency to combine to form an MRT coefficient of the higher frequency. The relation $Y_k^{(p)} = -Y_k^{(p+M)}$ cannot exist between any two valid phase indices in the above sequence if there is no integer q such that $qNk/k' = M$. Hence, there would be non-allowable valid phase indices that are greater than M and these would have corresponding allowable non-valid phase indices.

The distance between successive valid phase indices is Nk/k'. Given an allowable and numerically smallest valid phase index $p_1 < M$, assume that the nearest allowable non-valid phase index is p_2. The valid phase index corresponding to p_2 is $p_2' = p_2 + M$. Given q is the smallest integer such that $qNk/k' > M$,

$$p_2' = p_1 + q\frac{Nk}{k'}$$

$$\therefore p_2 - p_1 = q\frac{Nk}{k'} - M \tag{4.47}$$

For derived redundancy, it is required that

$$k' = dk$$

where d is an odd integer.

$$\frac{Nk}{k'} = \frac{N}{d}.$$

$$\frac{N}{2} = \frac{dNk}{2k'}$$

Using (4.47),

$$q\frac{Nk}{k'} - \frac{N}{2} = q\frac{Nk}{k'} - \frac{dNk}{2k'} = (2q-d)\frac{Nk}{2k'}$$

$$p_2 - p_1 = (2q-d)\frac{Nk}{2k'}$$

The value of $(2q - d)$ cannot be even since it is a difference between an even integer $2q$ and an odd integer d. The value of $(2q - d)$ cannot be greater than 1 since then

$$\therefore p_2 - p_1 \geq \frac{Nk}{k'}$$

As assumed, p_1 is the numerically smallest allowable valid phase index and $p_2^{'}$ is the first valid phase index greater than M. Hence, the difference between $p_2^{'}$ and M has to be lesser than Nk/k'. Thus, the distance between p_1 and p_2 has to be lesser than Nk/k'. Hence,

$$p_2 - p_1 = \frac{Nk}{2k'}$$

There is thus an allowable non-valid phase index between every couple of allowable valid phase indices. The MRT coefficient formed by these allowable non-valid phase indices will have the same magnitude and opposite sign as the MRT coefficient formed by the corresponding non-allowable valid phase indices.

The sequence of allowable phase indices would thus be:

$$p_0, p_0 + \frac{Nk}{2k'}, p_0 + \frac{Nk}{k'}, p_0 + \frac{3Nk}{2k'}, \ldots\ldots\ldots\ldots, p_0 + \frac{N}{2} - \frac{Nk}{2k'}.$$

Hence, the equation for derived redundancy can be stated as:

$$Y_{k'}^{(p)} = Y_k^{p_0} - Y_k^{(p_0 + \frac{Nk}{2k'})} + Y_k^{(p_0 + \frac{Nk}{k'})} - Y_k^{(p_0 + \frac{3Nk}{2k'})} \ldots\ldots\ldots\ldots Y_k^{(p_0 + \frac{N}{2} - \frac{Nk}{2k'})}$$

There are N/k' groups of MRT coefficients of lower frequency k that combine to form N/k' MRT coefficients of higher frequency k'. There are k'/k MRT coefficients in each of these groups. Also, the structure of each of these groups is the same. The smallest possible phase index for MRT coefficient of frequency k' is given by $p = 0$. The phase index of the first MRT coefficient of lower frequency k that are related through derived redundancy to the MRT coefficient of frequency k' is $p_i = p/d = 0$. The next higher phase index of MRT coefficient of frequency k' is given by $p = k'$. The phase index of the first MRT coefficient of lower frequency k related through derived redundancy to this MRT coefficient of frequency k' and phase k' is given by $p_i = k'/d = k$. Hence, the MRT coefficient of frequency k' and phase p is given by

$$Y_{k'}^{(p)} = Y_k^{(p/d)} - Y_k^{(\frac{p}{d} + \frac{Nk}{2k'})} + Y_k^{(\frac{p}{d} + \frac{Nk}{k'})} - Y_k^{(\frac{p}{d} + \frac{3Nk}{2k'})} \ldots\ldots\ldots\ldots Y_k^{(\frac{p}{d} + M - \frac{Nk}{2k'})},$$

$$p = 0, k', 2k', 3k', \ldots\ldots, M - k'$$

In the discussion so far, the assumption has been that $k \mid M$. In case this condition does not hold, the number of MRT coefficients of frequency k is given by N/k. If $k \mid M$ does not hold, then $k' \mid M$ does not hold either. Hence, the number of MRT coefficients of higher frequency k' is given by

N/k'. For a frequency index k' that is not a divisor of M, from section 4.3.4.4, the sequence of values of allowable phase indices is given by

$$p_0, p_0+\frac{k'}{2}, p_0+k', p_0+3\frac{k'}{2}, \ldots\ldots\ldots\ldots, p_0+M-\frac{k'}{2}., \text{ where } p_0 = 0.$$

Hence, the MRT coefficient of frequency k' and phase p is given by

$$Y_{k'}^{(p)} = Y_k^{(p/d)} - Y_k^{(\frac{p}{d}+\frac{Nk}{2k'})} + Y_k^{(\frac{p}{d}+\frac{Nk}{k'})} - Y_k^{(\frac{p}{d}+\frac{3Nk}{2k'})} \ldots\ldots\ldots + Y_k^{(\frac{p}{d}+M-\frac{Nk}{2k'})},$$

$$p = 0, k'/2, k', 3k'/2, \ldots.., M-k'/2$$

In summary,

$$Y_{k_r}^{(p)} = Y_k^{(\frac{pk}{k_r})} - Y_k^{(\frac{k}{k_r}(p+M))} + Y_k^{(\frac{k}{k_r}(p+2M))} - Y_k^{(\frac{k}{k_r}(p+3M))} + \ldots\ldots\ldots Y_k^{(\frac{k}{k_r}(p+(\frac{k_r}{k}-1)M))}, \tag{4.48}$$

$p = 0, k_r, 2k_r, 3k_r, \ldots.., M-k_r,$ if $k \mid M, k_r \mid M$

$p = 0, k_r/2, k_r, 3k_r/2, \ldots.., M-k_r/2$, otherwise.

$$\text{i.e. } Y_{k_r}^{(p)} = \sum_{j=0}^{k_r-1} (-1)^j Y_k^{(\frac{k}{k_r}(p+jM))}, \tag{4.48a}$$

$p = 0, k_r, 2k_r, 3k_r, \ldots..M-k_r$, if $k \mid M, k_r \mid M$

$p = 0, k_r/2, k_r, 3k_r/2, \ldots..M-k_r/2$, otherwise

Thus, a new representation of the MRT can be derived by removing the various types of redundancies present in the MRT, as explained in the next section.

Tables 4.1:(a)-(e) show the mapping between divisors, odd divisors and non-divisors corresponding to complete redundancy and derived redundancy for various values of N. The divisors of N are in the first column. The co-primes and odd-divisors of N are in the first row. Non-divisors are obtained from divisors through multiplication with co-primes and modulus w.r.t. N. The entire set $[0, N-1]$ is accounted for in this way, as seen in the tables. Also, there is derived redundancy relationship through multiplication with odd divisors.

Table 4.1(a): Complete redundancy and derived redundancy relations for $N = 6$

$N = 6$	Co-prime 5	Odd divisor 3
1	5	3
2	4	6
3		
6		

Table 4.1(b): Complete redundancy and derived redundancy relations for $N = 12$

$N = 12$	Co-primes			Odd divisors
	5	7	11	3
1	5	7	11	3
2	10			6
3		9		
4	8			12
6				
12				

Table 4.1(c): Complete redundancy and derived redundancy relations for $N = 18$

$N = 18$	Co-primes					Odd divisors	
	5	7	11	13	17	3	9
1	5	7	11	13	17	3	9
2	10	14	4	8	16	6	18
3	15						
6	12						
9							
18							

Table 4.1(d): Complete redundancy and derived redundancy relations for $N = 24$

$N = 24$	Co-primes							Odd divisors
	5	7	11	13	17	19	23	3
1	5	7	11	13	17	19	23	3
2	10	14	22					6
3	15	21	9					
4	20							12
6		18						
8	16							24
12								
24								

Table 4.1(e): Complete redundancy and derived redundancy relations for $N = 30$

$N = 30$	Co-primes							Odd divisors		
	7	11	13	17	19	23	29	3	5	15
1	7	11	13	17	19	23	29	3	5	15
2	14	22	26	4	8	16	28	6	10	30
3	21		9		27					
5		25								
6	12		18		24					
10		20								
15										
30										

4.5 1-D Unique MRT (1-D UMRT)

On the basis of the concepts of complete redundancy and derived redundancy, 1-D MRT coefficients can be classified as unique and relatively unique. It is seen that MRT coefficients that are of divisor frequencies, cannot be obtained from other divisors through complete redundancy. Hence they are called unique MRT coefficients. However, if such divisors are related to other divisors through multiplication by an odd divisor, then they exhibit derived redundancy. For example, for $N = 6$, although $k = 3$ is a divisor frequency and cannot be obtained through complete redundancy from any other divisor frequency, it exhibits derived redundancy since it can be obtained through derived redundancy from $k = 1$ using the odd divisor 3. Such divisors are thus only relatively unique. Removing the relatively unique divisors frequencies from the group of unique divisor frequencies, only those divisor frequencies remain which are not related to other divisor frequencies through multiplication by an odd divisor. These divisors can be considered absolutely unique. For $N = 6$, the list of divisor frequencies is given by the set $\{1, 2, 3, 6\}$. Removing relatively unique divisor frequencies 3 & 6 from the set would result in the absolutely unique divisor set $\{1, 2\}$. Absolutely unique divisors should not be expressible in form $k' = dk$, where d is an odd integer. Among the divisors of N, the only divisors that satisfy this requirement are those that are powers of 2, as observed in the above example. If $k' = 2^a$, then for any value of k and odd values of d other than $d = 1$, $k' \neq dk$. Hence, the set of unique divisors of N consists only of those divisors that are powers of 2. MRT coefficients having unique divisor frequencies are called unique MRT (UMRT) coefficients. Thus, 1-D UMRT is composed of all MRT coefficients that have frequencies that are powers of 2.

4.5.1 Number of Unique Coefficients

There are totally MN MRT coefficients, as in section 4.2, for a 1-D signal of length N. It is sought to determine the exact number of UMRT coefficients. A frequency that is a power of 2 yields unique coefficients. Since $k = 0 = ((N))_N$, and N is a power of 2, then $k = 0$ is a frequency that yields a unique coefficient. Similarly, if N is not a power of 2, $k = 0$ does not yield UMRT coefficients. Hence, the set of unique coefficients vary depending on whether N is a power of 2 or not.

Case 1: N a power of 2

The frequency indices that produce unique coefficients for N power of 2 are themselves powers of 2. The first frequency index is $k = 1$, and followed by $k = 2, 4, \ldots, N$. The number of unique coefficients produced by each frequency index k is given by the number of allowable phase indices for each frequency index k. From section 4.3.4, for an MRT coefficient $Y_k^{(p)}$ to exist, p should be divisible by k. When $k = N$, this condition has only one solution for p, $p = 0$. All other frequency indices have M / k coefficients each. Hence, the total number of UMRT coefficients is given by

$$Tot = 1 + \sum_{t=0}^{\log_2 M} \frac{M}{2^t}$$

$$= 1 + M \sum_{t=0}^{\log_2 M} \frac{1}{2^t}$$

$$= 1 + M(\frac{1-(1/2)^{(\log_2 M+1)}}{(1/2)}$$

$$= 1 + N(1 - 2^{-(\log_2 M+1)})$$

$$= 1 + N(1 - \frac{1}{N})$$

$$= N$$

Hence, the number of UMRT coefficients, when N is a power of 2, is N.

Case 2: N not a power of 2

Assume k' is the frequency index that is the highest power of 2 and also a divisor of N. Let $N = dk'$. Here, d has to be an odd integer; otherwise, if d is even, then k' cannot be the highest power of 2 that is a divisor of N. Since d is odd, k' cannot be a divisor of M since $d/2$ is not an integer. There are N/k' valid phase indices and thus N/k' MRT coefficients when k' is not a divisor of M. All divisors of N that are powers of 2 and lesser than k' are divisors of M. These are k = 1,2,4, ..., $k'/2$. There are M/k MRT coefficients for all these frequency indices. For N not a power of 2, $k = 0$ is a frequency that can be obtained by derived redundancy. A frequency $k = 0$ can also be considered to be $k = N$, since $((N))_N = 0$. This frequency N is related to k' by $N = dk'$, where d is odd. This is a sufficient condition for derived redundancy. Hence, unlike the case when N is a power of 2, $k = 0$ does not produce an absolutely unique MRT coefficient. Hence, the total number of UMRT coefficients over all frequencies is given by

$$Tot = \frac{N}{k'} + \sum_{t=0}^{\log_2(k'/2)} \frac{M}{2^t}$$

$$= \frac{N}{k'} + M\frac{(1 - 2^{(-\log_2(k'/2)-1)})}{(1/2)}$$

$$= \frac{N}{k'} + N(1 - \frac{2}{2k'})$$

$$= N$$

It can thus be concluded that the number of unique MRT coefficients, needed to represent a 1-D sequence of size N, is N irrespective of the type of N, corresponding to frequencies that are powers of 2 and divisors of N, starting from $k = 1$.

4.5.2 1-D UMRT Computation

The 1-D UMRT coefficients can be computed as below:

(i) N a power of 2

$$Y_0^{(0)} = \sum_{n=0}^{N-1} x_n$$

$$Y_k^{(p)} = \sum_{j=0}^{k-1}\left(x_{\frac{jN+p}{k}} - x_{\frac{(2j+1)N+2p}{2k}} \right), \quad k=2^t,\ 0 \le t \le \log_2 M, \quad p=tk,\ 0 \le t \le \frac{M}{k}-1$$

(ii) N not a power of 2

$$Y_k^{(p)} = \sum_{j=0}^{k-1}\left(x_{\frac{jN+p}{k}} - x_{\frac{(2j+1)N+2p}{2k}} \right), \quad k=2^t,\ 0 \le t \le (\log_2 k')-1, \qquad p=tk,\ 0 \le t \le \frac{M}{k}-1$$

$$Y_{k'}^{(p)} = \sum_{j=0}^{k'-1}\left(x_{\frac{jN+p}{k'}} \right) \qquad p=tk',\ 0 \le t \le \frac{M}{k'}-\frac{1}{2}$$

$$Y_{k'}^{(p)} = -\sum_{j=0}^{k'-1}\left(x_{\frac{jN+p+M}{k'}} \right) \qquad p=tk'+\frac{k'}{2},\ 0 \le t \le \frac{M}{k'}-\frac{3}{2}$$

where k' is the highest frequency index that is a power of 2 and also a divisor of N.

4.6 1-D Inverse UMRT

Since there are N 1-D UMRT coefficients, it can be expected that a 1-D signal can be completely reconstructed from its UMRT representation. Since the MRT is a many-to-many mapping, there would be many corresponding UMRT coefficients in which x_n is present. Hence, the inverse procedure for recovering x_n from UMRT coefficients would logically involve only those UMRT coefficients in which x_n are present. Since the forward MRT is a process of subtraction between summations of two groups of data elements depending on the value of $z = ((nk))_N$, the inverse process would also proceed along similar lines. The phase indices of UMRT coefficients in which x_n are present can be found by multiplying n with each unique frequency. An inverse formula that is based on these arguments is presented below along with the relevant proof.

4.6.1 *N*, a power of 2

Theorem 4.9

Given the UMRT of a 1-D signal of size N, N being a power of 2, the 1-D signal can be reconstructed from its UMRT by the following formula

$$x_n = \frac{1}{N} Y_0^{(0)} + \sum_{t=0}^{\log_2 M} \frac{1}{2^{t+1}} Y_{2^t}^{((2^t n))_N} \quad , 0 \le n \le N-1 \tag{4.49}$$

Proof

The data element that needs to be recovered from the UMRT is given by x_n. For any frequency index k, the value of the phase index p of the UMRT coefficient $Y_k^{(p)}$ that contains x_n is given by $((nk))_N = p$. Thus for a frequency that is a power of 2, $k = 2^a$, the UMRT coefficient that contains x_n is $Y_{2^a}^{((2^a n))_N}$ .The UMRT coefficient $Y_0^{(0)}$ contains all the elements of the data including x_n since $((nk))_N = 0$, $\forall n$, when $k = 0$. In (4.49), $Y_0^{(0)}$ is multiplied by $(1/N)$, and the other UMRT coefficients by $(1/2^{t+1})$. As a result, the x_n that is present in these coefficients is multiplied by the corresponding factors. The resultant factor f that multiplies x_n as a result of the summation can thus be found by adding up these individual multiplication factors.

$$f = \frac{1}{N} + \sum_{t=0}^{\log_2 M} \frac{1}{2^{t+1}}$$

$$f = \frac{1}{N} + \sum_{t=0}^{\log_2 M} 2^{-(t+1)}$$

$$f = \frac{1}{N} + \frac{1}{2}\frac{(1-2^{-(\log_2 M+1)})}{\frac{1}{2}}$$

$$f = \frac{1}{N} + 1 - \frac{1}{N} = 1.$$

Hence, as a result of the summation, one of the components of the result is the data x_n.
A UMRT coefficient $Y_{2^a}^{((2^a n))_N}$ contains other terms besides x_n. For the summation formula to be correct, these other terms that occur in the various UMRT coefficients $Y_{2^a}^{((2^a n))_N}$ need to vanish. To prove that they do, the first observation is regarding the smallest frequency index k where any of the other data elements occur along with x_n. Here, $Y_0^{(0)}$ is excluded since it contains all data elements and all these elements have a positive sign. Excluding $Y_0^{(0)}$, another element occurs along with x_n first when frequency index $k = 1$. For example, for $N = 8$, $Y_1^{(0)} = x_0 - x_4$. Hence, if x_0 is the data to be found, it is seen that x_4 occurs with an opposite sign along with x_0 in the MRT coefficient corresponding to $k = 1$, given by $Y_1^{(0)}$. From (4.S.1), when $k = 1$, $g(k, N) = 1$, and hence there is only one data element in both positive and negative groups. Also, from theorem 4.3(b), the distance between an element in the positive group and a corresponding element in the

negative group is given by M / k. Hence, the data elements x_n and x_{n+M}, occur with opposite signs, in any MRT coefficient $Y_1^{(p)}$, i.e. having frequency $k = 1$. Similarly, for frequency index $k = 2$, the data element x_{n+M} occurs with positive sign since $Y_2^{(p)} = x_n - x_{n+N/4} + x_{n+N/2} - x_{n+3N/4}$, given $((nk))_N = p$. From section 4.3, the distance between two successive data elements in a positive or negative group is given by $N / g(k, N)$. Since k is a divisor of N, this distance becomes N / k. From theorem 4.3(b), the distance between the first data element in a positive group and the first data element in a negative group is given by M / k. Hence, if an element $x_n^{'}$ occurs along with x_n in a UMRT coefficient $Y_k^{(p)}$ but with opposite sign as x_n, then,

$$n' = n + q_{odd}\frac{N}{2k} \tag{4.50}$$

where q_{odd} is an odd integer. At the next higher frequency $k' = 2k$, (4.50) becomes

$$n' = n + q_{odd}\frac{N}{k'} \tag{4.51}$$

From (4.S.1b), the general form for a data element x_w of same sign present along with element x_n in the MRT coefficient $Y_{k'}^{(p)}$ is given by (since k' is a divisor of N, $g(k',N) = k'$)

$$n' = n + j\frac{N}{k'} \tag{4.52}$$

where $j = 0,1,2,3,\ldots\ k' - 1$

It is seen that (4.51) is a special case of (4.52). The same holds for any higher frequency of the form $k' = 2^k$. Hence, given k is the smallest frequency index at which any element $x_n^{'}$ occurs along with x_n with opposite sign as x_n, for all higher frequencies $k' = 2^k$, the element $x_n^{'}$ occurs along with x_n, however, with the same sign as x_n. An example can be used from section 4.2.2.3; for $N = 8$, $Y_1^{(0)} = x_0 - x_4$, $Y_2^{(0)} = x_0 - x_2 + x_4 - x_6$, and $Y_4^{(0)} = x_0 - x_1 + x_2 - x_3 + x_4 - x_5 + x_6 - x_7$. Considering data element x_4, it is seen that it occurs with opposite sign as x_0 in $Y_1^{(0)}$, and with same sign as x_0 in coefficients of higher frequencies, $k = 2$ & $k = 4$. Conversely, it can also be concluded that any element $x_n^{'}$ that occurs with element x_n in a UMRT coefficient of a frequency k' and has the same sign as x_n also occurs along with x_n in a UMRT coefficient k such that $k' = 2^k$, but having opposite sign as x_n.

Given k is the smallest frequency index at which any element $x_n^{'}$ occurs along with x_n with opposite sign as x_n, the multiplication factor associated with $x_n^{'}$ from the inverse formula is

(-1/2k). For all higher frequencies up to M, the multiplication factor is $1/2k'$, $k' = 2k, 4k...M$. Thus the sum of the series

$$f = \frac{1}{N} - \frac{1}{2k} + \frac{1}{4k} + \frac{1}{8k} + \ldots\ldots \frac{1}{2M} \tag{4.53}$$

will provide the value of the multiplication factor f associated with element $x_n^{'}$.

Assume $k = N/q$. First the sum of the following series can be found

$$\frac{1}{4k} + \frac{1}{8k} + \ldots\ldots \frac{1}{2M} = f_a$$

$$f_a = \sum_{j=\log_2 2N/q}^{\log_2 M} \frac{1}{2^{j+1}}$$

The number of terms in this summation is $\log_2 q - 1$.

$$f_a = \frac{q}{4N} \frac{(1 - 2^{-(\log_2 q - 1)})}{1/2} = \frac{q}{2N} \frac{q-2}{q} = \frac{q-2}{2N}$$

From (4.53),

$$f = \frac{1}{N} - \frac{1}{2k} + f_a = \frac{1}{N} - \frac{1}{2k} + \frac{q-2}{2N} = \frac{q}{2N} - \frac{1}{2k} = 0$$

Thus, all other data elements $x_n^{'}$ that occur along with x_n in the various MRT coefficients in the summation of the inverse formula cancel out, leaving behind only the desired data element x_n. Hence, the formula for inverse UMRT is proved.

Example 4.8

Let $N = 4$, and let x_1 be the data element to be determined. The unique frequencies are $k = 0, 1, 2$. The corresponding phase indices $((nk))_4$ are $p = 0, 1, 2$. $Y_0^{(0)} = x_0 + x_1 + x_2 + x_3$, $Y_1^{(1)} = x_1 - x_3$, $Y_2^{(2)} = -Y_2^{(0)} = -x_0 + x_1 - x_2 + x_3$. The associated multiplication factors in (4.49) for these three MRT coefficients are 1/4, 1/2, and 1/4 respectively. First, the effect of the multiplication factors on x_1 when using (4.49) can be calculated. It is seen that the sum of these factors gives 1, thus ensuring x_1 is available as the result of (4.49). The sum of the multiplication factors for x_0 gives 1/4 – 1/4 = 0. For x_2, this is 1/4 – 1/4 = 0. For x_3, this becomes 1/4 – 1/2 + 1/4 = 0. Thus, x_0, x_2 and x_3 cancel out, leaving behind only x_1.

4.6.2 N, Not a power of 2

Theorem 4.10

Given the UMRT of a 1-D signal of size N, N not being a power of 2, the 1-D signal can be reconstructed from its UMRT by the following formula

$$x_n = \frac{1}{k'} Y_{k'}^{((nk'))_N} + \sum_{t=0}^{\log_2 k'-1} \frac{1}{2^{t+1}} Y_{2^t}^{((2^t n))_N} \tag{4.54}$$

where k' is the highest frequency index that is a power of 2 and also a divisor of N.

Proof

The data element that needs to be recovered from the UMRT is given by x_n. For any frequency index k, the value of the phase index p of the UMRT coefficient $Y_k^{(p)}$ that contains x_n is given by $((nk))_N = p$. Thus for a frequency index that is a power of 2, $k = 2^a$, the UMRT coefficient that contains x_n is $Y_{2^a}^{((2^a n))_N}$. The UMRT coefficient $Y_{k'}^{((nk'))_N}$ is multiplied by $(1/k')$, and the other UMRT coefficients are multiplied by $(1/2^{t+1})$. The resultant factor f that multiplies x_n as a result of the summation can be proved to be using the same method used in section 4.6.1 for case 1. Similarly, using the method adopted for case 1 in section 4.6.1, it can also be shown that other data elements other than x_n cancel out in the summation, leaving behind only the desired data element x_n. Hence, the proposed formula is proved.

4.6.3 Any even N

From (4.49) and (4.54), for the two categories of values of N, it can be observed that the structure of both formulae is similar. In both, there is a summation term and a second term that contains only one UMRT coefficient. First, a comparison can be made between the two single-coefficient terms of both equations. The denominator in (4.49) for this term is N, while the denominator in (4.54) for the same term is k'. N is the order of the data, while k' is the highest power-of-2 divisor of N. When N is a power of 2, the value of k' is actually equal to N. Hence, the denominator for these terms in (4.49) and (4.54) can be generalized as k', the highest power-of-2 divisor of N, irrespective of whether N is a power of 2 or otherwise. However, the frequency index of the first term in (4.49) is zero, while that of the corresponding term in (4.54) is k'. A generalizing term for both these values is $((k'))_N$. This term becomes zero for values of N that are powers of 2, and it remains k' for other values of N. Next, the summation terms in (4.49) and (4.54) may be compared with each other. The upper limit of the summation in (4.49) is $\log_2 M$

while it is $\log_2(k')$-1 in (4.54). As discussed above, for N a power of 2, $k' = N$. Hence the term $\log_2(k')$-1 becomes $\log_2(N)$-1 which is equal to $\log_2 M$. Thus the upper limit can be generalized to the term $\log_2(k')$-1. There is no further difference between (4.49) and (4.54). Thus, in light of these observations, (4.54) can be generalized to be applicable to any even value of N. Hence, the following equation can be used for signal reconstruction from UMRT, for a signal of size N, N being any even.

$$x_n = \frac{1}{k'} Y_{((k'))_N}^{((nk'))_N} + \sum_{t=0}^{\log_2 k'-1} \frac{1}{2^{t+1}} Y_{2^t}^{((2^t n))_N} \tag{4.55}$$

where k' is the highest power-of-2 divisor of N.

(4.55) makes use of the relation $Y_k^{(p)} = -Y_k^{(p+M)}$, given by theorem 4.1. (4.55) can also be expressed in a way that shows the duality in the inverse transform relation with the forward transform. Then there would be a need for checking if the value of $((nk))_M$ exceeds M or not.

As an example, (4.55) can be used to reconstruct the 8-point 1-D sequence used as an example in section (4.2.1). Since $N = 8$, $k' = 8$, and t in the summation in (4.55) takes the values 0, 1, & 2.

Table 4.2: Example showing reconstruction of 8-point 1-D sequence using inverse 1-D UMRT.

N	0	1	2	3	4	5	6	7
$(k' = 8)$, $Y_0^{(0)}/8$	55.125	55.125	55.125	55.125	55.125	55.125	55.125	55.125
$(t = 0)$, $Y_1^{(((n))_8)}/2$	3	-26.5	7.5	23.5	-3	26.5	-7.5	-23.5
$(t = 1)$, $Y_2^{(((2n))_8)}/4$	19.25	12	-19.25	-12	19.25	12	-19.25	-12
$(t = 2)$, $Y_4^{(((4n))_8)}/8$	17.625	-17.625	17.625	-17.625	17.625	-17.625	17.625	-17.625
x_n	95	23	61	49	89	76	46	2

Number of computations:

From (4.55), it is seen that in order to reconstruct a data element, an MRT coefficient of each power-of-2 divisor frequency is required. The total number of MRT coefficients required is hence given by the number of power-of-2 divisors of N. Given k' is the highest power-of-2 divisor of N, the number of power-of-2 divisors of N is given by $\log_2 k' + 1$, which thus gives the number of MRT coefficients involved in the computation of each data element. Also, each of the corresponding MRT coefficients needs to be multiplied by a scaling factor. Table 4.3 shows the number of UMRT coefficients required for computing 1-D inverse UMRT for different values of N.

Table 4.3: Number of UMRT coefficients involved in inverse 1-D UMRT for a few values of N.

N	k'	$\log_2 k' + 1$
4	4	3
6	2	2
8	8	4
10	2	2
12	4	3
14	2	2
16	16	5
18	2	2
20	4	3

4.7 1-D Signal Representation

In section 4.5, it is shown that a 1-D signal of size N could be represented by using 1-D UMRT coefficients with frequencies that are powers of 2. It is also shown that the number of such UMRT coefficients is equal to N. However, there are applications which need non-power-of-2 frequencies in the signal representation. Thus some of the UMRT coefficients can be replaced by exploiting the derived redundancy relationship between the UMRT coefficients and MRT coefficients having non-power-of-2 frequencies. Hence, from the UMRT coefficients spread over M arrays of the MRT, it is required to form a single array of N UMRT coefficients to represent the 1-D signal in the UMRT domain. Both of these representations are proposed in the following sub-sections.

4.7.1 Representation using 1-D UMRT

MRT coefficients of a 1-D signal have two indices, the frequency index k and the phase index p. It is sought to arrive at a 1-D array comprising UMRT coefficients. Hence, there is a need for a mapping from the indices (k, p) of a UMRT coefficient to the position v of that coefficient in the 1-D transform array. This mapping needs to be known in the reverse direction also, from the position index to the frequency and phase indices. The following mapping is proposed to this effect. k_{hf} is the largest divisor of N that is also a power of 2. The mapping is done in such a way that MRT coefficients appear in the 1-D array in ascending order of the frequency. Since $k = 0$ corresponds to $k = N$, MRT coefficients with frequency $k = 0$ appears last.

$(k, p) \rightarrow v:$

$$\left.\begin{aligned} k' &= k_{hf}/2 \\ v_1 &= N - M/k' \end{aligned}\right\} \quad \textit{for } k = ((k_{hf}))_N$$

$$\left.\begin{aligned} k' &= k \\ v_1 &= N - N/k' \end{aligned}\right\} \quad \textit{for } k \neq ((k_{hf}))_N$$

$$v = v_1 + \lfloor p/k' \rfloor$$

Similarly, the reverse mapping relations are given by the following relations:

$v \rightarrow (k, p)$

$$\left.\begin{aligned} N - N/k \le v &< N - N/2k \\ k &= \{1, 2, 4, 8 \ldots k_{hf}/2\} \\ p &= k(v - N + N/k) \end{aligned}\right\} \quad v < N - N/k_{hf}$$

$$\left.\begin{aligned} k &= ((k_{hf}))_N \\ p &= k_{hf}(v - N + N/k_{hf})/2 \end{aligned}\right\} \quad v \ge N - N/k_{hf}$$

Tables 4.4 and 4.5 shows the mappings for $N = 8$ and 12 respectively.

Table 4.4: Proposed mapping between 1-D UMRT indices and array indices, for $N = 8$

(k,p)	(1,0)	(1,1)	(1,2)	(1,3)	(2,0)	(2,2)	(4,0)	(0,0)
v	0	1	2	3	4	5	6	7

Table 4.5: Proposed mapping between 1-D UMRT indices and array indices, for $N = 12$

(k,p)	(1,0)	(1,1)	(1,2)	(1,3)	(1,4)	(1,5)	(2,0)	(2,2)	(2,4)	(4,0)	(4,2)	(4,4)
v	0	1	2	3	4	5	6	7	8	9	10	11

4.7.2 Representation associating derived redundancy to UMRT

In sec. 4.7.1, a 1-D signal was represented in the UMRT domain in terms of power-of-2 frequencies. However, the derived redundancy that exists between these UMRT frequencies and other frequencies allows for alternate representations of the 1-D signal using MRT coefficients. Consider an example, $N = 12$. The set of UMRT frequencies, defined here as the basic set of frequencies, corresponding to this value of N, is $k = \{1, 2, 4\}$. There exists one odd divisor of N, $d = 3$, which gives rise to derived redundancy between each frequency in the basic set and a corresponding frequency given by $((dk))_N$. The relevant equations are given below.

$$Y_3^{(0)} = Y_1^{(0)} - Y_1^{(2)} + Y_1^{(4)}$$

$$Y_3^{(3)} = Y_1^{(1)} - Y_1^{(3)} + Y_1^{(5)}$$

$$Y_6^{(0)} = Y_2^{(0)} - Y_2^{(2)} + Y_2^{(4)}$$

$$Y_0^{(0)} = Y_4^{(0)} - Y_4^{(2)} + Y_4^{(4)}$$

In the signal representation proposed in section 4.7.1, the set of UMRT coefficients on the RHS of the equations above could be used for a UMRT representation of the signal since they correspond to the power-of-2 frequencies. Now, a possible alternative method of representation is to replace one element each from the RHS of these equations with the corresponding derived MRT coefficient on the LHS in the UMRT representation of the 1-D signal. In the method proposed earlier, the output would be the set of UMRT coefficients $\{Y_1^{(0)}\ Y_1^{(1)}\ Y_1^{(2)}\ Y_1^{(3)}\ Y_1^{(4)}\ Y_1^{(5)}\ Y_2^{(0)}\ Y_2^{(2)}\ Y_2^{(4)}\ Y_4^{(0)}\ Y_4^{(2)}\ Y_4^{(4)}\}$. In the presently proposed method, $Y_1^{(2)}$ would be replaced by its corresponding derived MRT coefficient $Y_3^{(0)}$, and similarly $Y_1^{(3)}$ by $Y_3^{(3)}$, $Y_2^{(2)}$ by $Y_6^{(0)}$, and $Y_4^{(2)}$ by $Y_0^{(0)}$. The output set is now given by $\{Y_1^{(0)}\ Y_1^{(1)}\ Y_3^{(0)}\ Y_3^{(3)}\ Y_1^{(4)}\ Y_1^{(5)}\ Y_2^{(0)}\ Y_6^{(0)}\ Y_2^{(4)}\ Y_4^{(0)}\ Y_0^{(0)}\ Y_4^{(4)}\}$. The inverse transform developed in section 4.6 requires all the MRT coefficients of the basic frequencies. Hence, in order to obtain the 1-D signal from the present set of MRT coefficients, coefficients corresponding to all the UMRT frequencies are required. However, the entire set of coefficients corresponding to the UMRT frequencies is not available in this method, since some of them have been replaced by derived MRT coefficients. However, using the derived redundancy equations, the replaced UMRT frequency coefficients can be obtained from the coefficients actually available in the set of output coefficients. Once the entire set of UMRT coefficients is available, the 1-D signal can be obtained by performing the inverse UMRT on this set of coefficients.

Formalizing the above, let N have n odd divisors. Hence, all of these divisors would produce derived redundancy. Also, let k' be the highest power-of-2 divisor of N. Hence, the UMRT set of frequencies is given by $\{1\ 2\ 4\ 8\ \dots\ k'\}$. Each of these frequencies would have corresponding derived frequencies obtained from products with the odd divisors. Let the set of odd divisors be given by $\{x_1\ x_2\ \dots\ x_n\}$. Let k_r be a frequency related to a UMRT frequency k through derived redundancy. The derived redundancy equations relating MRT coefficients of frequencies k_r and k are recalled here from (4.48):

$$Y_{k_r}^{(p)} = Y_k^{(\frac{pk}{k_r})} - Y_k^{(\frac{k}{k_r}(p+M))} + Y_k^{(\frac{k}{k_r}(p+2M))} - Y_k^{(\frac{k}{k_r}(p+3M))} + \ldots + Y_k^{(\frac{k}{k_r}(p+(\frac{k_r}{k}-1)M))},$$

$p = 0, k_r, 2k_r, 3k_r, \ldots. M - k_r$, if $k \mid M, k_r \mid M$

$p = 0, k_r/2, k_r, 3k_r/2, \ldots. M - k_r/2$, otherwise

The proposed method is to replace one of the MRT coefficients on the RHS by the MRT coefficient on the LHS, i.e. $Y_{k_r}^{(p)}$. There are k_r/k MRT coefficients on the RHS. One among these has to be chosen to be replaced by $Y_{k_r}^{(p)}$ in the output set. A logical choice is to choose the coefficient in the centre among the RHS coefficients. The serial position of this coefficient is given by $(\frac{k_r}{k}+1)/2$. The MRT coefficient corresponding to this position is $Y_k^{(\frac{k}{k_r}(p+((\frac{k_r}{k}-1)/2)M))}$. Hence, the coefficient $Y_k^{(\frac{k}{k_r}(p+((\frac{k_r}{k}-1)/2)M))}$ among the set of UMRT coefficients of frequency k is replaced by the derived coefficient $Y_{k_r}^{(p)}$. Since, there are M/k_r derived-frequency coefficients, M/k_r UMRT coefficients are replaced in this manner by the corresponding derived frequency MRT coefficients. After this replacement, there are only $(M/k - M/k_r)$ UMRT coefficients of frequency k remaining in the output set. Half of them occur serially before the M/k_r derived frequency replacement coefficients in the output set, and half of them after. The sequence of phase indices of the first half of these UMRT frequency coefficients is given by $p = 0, k, 2k, \ldots.((1/2)(M/k - M/k_r)-1)k$. The sequence of the latter half is given by $p = (M-(1/2)(M/k - M/k_r)-1)k$, $(M-(1/2)(M/k - M/k_r))k$, $(M-(1/2)(M/k - M/k_r)+1)k$, …. , $M-k$. Thus, the proposed placement of the output coefficients has been arrived at. But this has been done only for one derived frequency. However, in the general case, there can be more than one odd divisor of N, and hence more than one derived frequency for a UMRT frequency. Here arises the problem of forming the output set of coefficients containing coefficients from all the UMRT frequencies and the derived frequencies. Thus, in the placement method exchange the UMRT frequency with the corresponding derived frequency coefficient. The following procedure is proposed for $k = 1$ and its derived frequencies. By using relevant values for the starting and ending serial positions for each frequency in a 1-D array, the following steps can be used for all UMRT frequencies and their associated derived redundancy frequencies:

1) All odd divisors are sorted in ascending order.

2) The value (1/2) $(M/k - M/k_r) + 1$ is calculated for each k_r. This is the serial position of the first derived frequency coefficient of frequency k_r in the output set. Also, the value M-(1/2) $(M/k - M/k_r)$ is calculated. This is the serial position of the last derived frequency coefficient of frequency k_r in the output set.

3) For each k_r, starting from the smallest value, coefficients of frequency k_r are placed at locations starting from the above-calculated first serial position until a first serial position corresponding to the next higher value of k_r is reached.

4) For each k_r, starting from the smallest value, coefficients of frequency k_r are placed at locations starting from the above-calculated last serial position until a last serial position corresponding to the next higher value of k_r is reached.

5) For the highest value of k_r, all MRT coefficients of k_r will be present in the output set.

Figure 4.4(a) shows the array of UMRT coefficients of $N = 12$ which is modified according to the proposed placement method for derived and basic frequency MRT coefficients to yield the new array shown in Figure 4.4(b).

Also, it is possible to replace any of the UMRT coefficients of power-of-2 frequencies with UMRT coefficients that are completely redundant with the former.

$Y_1^{(0)}$	$Y_1^{(1)}$	$Y_1^{(2)}$	$Y_1^{(3)}$	$Y_1^{(4)}$	$Y_1^{(5)}$	$Y_2^{(0)}$	$Y_2^{(2)}$	$Y_2^{(4)}$	$Y_4^{(0)}$	$Y_4^{(2)}$	$Y_4^{(4)}$

(a)

$Y_1^{(0)}$	$Y_1^{(1)}$	$Y_3^{(0)}$	$Y_3^{(3)}$	$Y_1^{(4)}$	$Y_1^{(5)}$	$Y_2^{(0)}$	$Y_6^{(0)}$	$Y_2^{(4)}$	$Y_4^{(0)}$	$Y_0^{(0)}$	$Y_4^{(4)}$

(b)

Figure 4.4: (a) - Placement of UMRT coefficients for $N = 12$, **(b)** - Proposed placement of MRT and UMRT coefficients for $N = 12$

The 1-D signal can be reconstructed from the above UMRT representation, including non power-of-2 frequencies, by the following method. The signal reconstruction using the inverse UMRT, all the coefficients corresponding to the power-of-2 frequencies are required. Since in the present representation consists of coefficients corresponding to power-of-2 frequencies as well as derived non-power-of-2 frequencies, the preliminary task is to re-convert the derived frequency coefficients into their corresponding power-of-2 frequency coefficients. From (4.48a), relating a derived frequency coefficient k_r with its power-of-2 frequency counterpart k:

$$Y_{k_r}^{(p)} = \sum_{j=0}^{\frac{k_r}{k}-1} (-1)^j Y_k^{(\frac{k}{k_r}(p+jM))},$$

$$p = 0, k_r, 2k_r, 3k_r, \ldots M - k_r, \text{ if } k \mid M, k_r \mid M$$

$$p = 0, k_r/2, k_r, 3k_r/2, \ldots M - k_r/2, \text{ otherwise}$$

In the alternate representation using non-power-of-2 frequencies, the power-of-2 frequency coefficient $Y_k^{(\frac{k}{k_r}(p+((\frac{k_r}{k}-1)/2)M))}$ is replaced by the derived frequency coefficient $Y_{k_r}^{(p)}$. Hence, during the reconstruction, the reverse process needs to be performed, i.e. the coefficient $Y_{k_r}^{(p)}$ is to be replaced by the coefficient $Y_k^{(\frac{k}{k_r}(p+((\frac{k_r}{k}-1)/2)M))}$. The reconstruction formula may be written as

$$Y_k^{(\frac{k}{k_r}(p+\frac{1}{2}(\frac{k_r}{k}-1)M))} = (-1)^{\frac{1}{2}(\frac{k_r}{k}-1)} \left[Y_{k_r}^{(p)} - \sum_{j=0}^{\frac{1}{2}(\frac{k_r}{k}-3)} (-1)^j Y_k^{(\frac{k}{k_r}(p+jM))} - \sum_{j=\frac{1}{2}(\frac{k_r}{k}+1)}^{(\frac{k_r}{k}-1)} (-1)^j Y_k^{(\frac{k}{k_r}(p+jM))} \right] \quad (4.56)$$

From (4.56), it is seen that higher the value of k_r, the number of coefficients that are in the first half of the set of power-of-2 frequency coefficients in the output is higher. The same is true for the number of coefficients in the latter half of the set of UMRT frequency coefficients. This also implies that the number of UMRT frequency coefficients replaced by derived frequency coefficients is lesser, higher the value of k_r. Hence, lower the value of k_r, more the number of replaced UMRT frequency coefficients.

4.8 Conclusion

1-D MRT is a new representation of 1-D signals and involves only real additions. However, the MRT is expansive and redundant. The 1-D UMRT removes these features of MRT to give a real, invertible, non-expansive 1-D signal transform for any even value of N. The derived redundancy property of the transform ensures flexibility in signal representation by allowing for inter-frequency conversion. A 2-D counterpart for the 1-D UMRT, sharing the inversion and non-expansion properties, could be a useful tool for 2-D signal processing. The study of complete redundancy in 2-D MRT is performed to derive 2-D UMRT and is presented in Chapter V.

Chapter V

DEVELOPMENT OF FORWARD AND INVERSE 2-D UMRT

5.1 Introduction

The occurrence of redundancy in 2-D MRT is observed in section 3.2, and a detailed study of redundancy in 1-D MRT is performed in section 4.4. In this chapter, the study of redundancy performed in Chapter IV is extended to the 2-D MRT. Complete redundancy in 2-D MRT is investigated, and 2-D UMRT is developed by eliminating complete redundancy in 2-D MRT.

5.2 Redundancy in 2-D MRT

The 2-D MRT is developed with the assumption that the data is a square matrix of size N x N, where N is even. It is observed in sec. 4.4.1.2 that the divisors of N form groups comprising of the divisor and associated non-divisors, where all members of the group have a common gcd w.r.t. N. Extending this result to 2-D signals, rows and columns can be classified into divisor rows/columns and non-divisor rows/columns. The mapping between each divisor and related non-divisors in the case of 1-D translates to mapping between rows/columns and non-divisor rows/columns. Complete redundancy can exist between divisor columns and non-divisor columns, and also within divisor columns. These issues are studied and expressions for the number of unique frequencies and the number of UMRT coefficients derived in the following sections.

5.2.1 Mapping of Divisors

It can be recalled from section 4.4.1.2 that integers in the range [0, N - 1] form groups on the basis of their respective gcd w.r.t. N. For eg, for $N = 8$, each element in the set of integers {1, 3, 5, 7} share the common gcd of 1 w.r.t. N. Similarly, the common gcd is 2 for the set {2, 6}, 4 for {4}, and 8 for {0}. In each group, one element is a divisor of N, and the other elements can be derived from this divisor element by multiplication with integers that are co-prime to N. For example, in the set {2, 6}, 6 can be obtained by multiplying 2 with 3, 3 being co-prime to 8. These divisors, 0, 1, 2, 4 thus form the generators of the non-divisors for $N = 8$. In this manner, all non-divisors can be mapped to the divisors of any even N. Hence, each divisor along with its set of associated non-divisors can be considered to be a group identified by the common gcd of all the elements in the group, and this gcd is equal to the value of the divisor. Since each divisor has an associated gcd group, the number of such groups is equal to the number of divisors of N.

This property is used in analyzing the redundancy in 2-D MRT. The analysis can the done in terms of rows or columns. In the following sections, although a column-wise approach is used, the concepts developed apply equally to rows as well.

5.2.2 Divisor Columns

A divisor column is defined as the set of frequencies (k_1, k_2) where k_1 = [0, N - 1], and k_2 is a divisor of N. Consider the operation $(((hk_1))_N,((hk_2))_N)$, where $g(h,N)=1$. Since k_2 is a divisor, the product $((hk_2))_N$ will produce another element in the group of integers that share a common value of k_2 for the gcd w.r.t. N. Also, $((hk_2))_N$ is a non-divisor of N. The product $((hk_1))_N$ will similarly produce a value that has the same gcd w.r.t. N as k_1 has. Since k_1 = [0, N - 1], k_1 can belong to any one of the groups corresponding to the common gcd w.r.t. N. Multiplication of each k_1 in the range [0, N - 1] with a co-prime to N implies multiplication of each gcd group in the same range with a co-prime to N. The result of each product is the same group, however with the position of the elements within the group altered. For example, when N = 8, multiplying the gcd group corresponding to gcd 1, {1, 3, 5, 7}, with the co-prime 5 and taking the modulus w.r.t. N = 8 results in the group {5, 7, 1, 3}, and the similar operation with the gcd group corresponding to gcd 2, {2, 6}, results in the same group {2, 6}. Thus, the union of all these product groups would still form the set of integers in the range [0, N - 1]. Hence, the product $((hk_1))_N$ for k_1=[0, N - 1] maps in a one-to-one manner to integers in the range [0, N - 1]. In conclusion, the operation $(((hk_1))_N,((hk_2))_N)$ signifies another column, and since $((hk_2))_N$ is a non-divisor of N, the new column is a non-divisor column. In this way, the N columns of the N x N matrix can be classified into divisor columns and non-divisor columns. Similarly, the N rows can also be classified into divisor rows and non-divisor rows. Since the mapping between divisor columns and non-divisor columns takes place through multiplication by a co-prime to N, using theorem 4.6, there is complete redundancy between the divisor columns and corresponding non-divisor columns. This is studied in section 5.2.3

5.2.3 Complete Redundancy between Columns

Since non-divisor columns are completely redundant with divisor columns, the divisor columns can be considered unique columns, and the number of unique columns is thus equal to the number of divisors of N. Since there is complete redundancy between the divisor columns and corresponding non-divisor columns, from the knowledge of the MRT coefficients corresponding to the divisor columns, the MRT coefficients corresponding to all the non-divisor columns can be found. The number of non-divisor columns that are redundant with a divisor column can be obtained by extending the results of section 4.4.1.2. From that section, the number of non-divisors that are related by complete redundancy to a divisor k is given by $\phi(N/k)$ and they are obtained by $k' = ((hk))_{N/k}$ where $g(h, N/k) = 1$. Hence, given a divisor column ([0, N - 1], k_2), the number of columns completely redundant with this divisor column is given by $\phi(N/k_2)$. For example, when $N = 8$, there are $\phi(N/k_2) = \phi(4) = 2$ redundant columns when $k_2 = 2$, as can be observed in Figure 3.1. Complete redundancy can exist within divisor columns also. From a certain number of unique frequencies within a divisor column, all other indices of the divisor column can be obtained through complete redundancy. This is discussed in section 5.2.4, where an expression for the number of unique frequencies (k, k_2) where k satisfies $g(k, N) = k_1$ for a given column k_2 is obtained.

5.2.4 Complete Redundancy within Divisor Columns

Given a frequency (k_1, k_2), complete redundancy exists within the column k_2 if $((hk_2))_N = k_2$, given $g(h, N) = 1$. $((hk_2))_N = k_2$ being a congruence equation, from Appendix A.3, the general solution for h is given by $h = h_0 + Nt / g(k_2, N)$, $0 \leq t \leq N - g(k_2, N)$, $h_0 = 1$. From section 4.4.1.2, the number of integers that share a common gcd k_2 w.r.t. N, is given by $\phi(N/k_2)$. Hence, the number of co-primes in the set of co-primes to N that are sufficient to generate all elements in the gcd group corresponding to k_2 is also given by $\phi(N/k_2)$. The number of co-primes to N is given by $\phi(N)$. From among these $\phi(N)$ co-primes, since only $\phi(N/k_2)$ are needed to generate the gcd group, more than one co-prime among the set of $\phi(N)$ co-primes maps to the same element in the gcd group, and thus more than one co-prime maps k_2 to itself. The number of co-primes that maps an element in a gcd group to itself is thus given by $\phi(N)/\phi(N/k_2) = l(k_2)$. Thus, the number of co-primes h that satisfy $((hk_2))_N = k_2$ and $g(h, N) = 1$ is also $l(k_2)$.

Let N be

$$N = \prod_{i=1}^{q} r_i^{a_i} \tag{5.1}$$

The totient function (Appendix A.7), is defined as

$$\phi(N) = N \frac{\prod_{i=1}^{q} (r_i - 1)}{\prod_{i=1}^{q} r_i} \tag{5.2}$$

Let $k_2 = \prod_{i=1}^{q} r_i^{w_i}$ (5.3)

$$N / k_2 = \prod_{i=1}^{q} r_i^{a_i - w_i} \tag{5.4}$$

$$\phi(N / k_2) = (N / k_2) \frac{\prod_{i=1, \forall i \Rightarrow a_i \neq w_i}^{q} (r_i - 1)}{\prod_{\forall i \Rightarrow a_i \neq w_i}^{q} r_i} \tag{5.5}$$

$$\therefore l(k_2) = \phi(N) / \phi(N / k_2) = k_2 \frac{\prod_{i=1, \forall i \Rightarrow a_i = w_i}^{q} (r_i - 1)}{\prod_{i=1, \forall i \Rightarrow a_i = w_i}^{q} r_i} \tag{5.6}$$

$l(k_2)$ is the number of co-primes that cause k_2 to be mapped to itself by $((hk_2))_N$. If the same co-prime h maps k_1 such that $((hk_1))_N \neq k_1$, then this is an example of redundancy within divisor column k_2. However, if $((hk_1))_N = k_1$ & $((hk_2))_N = k_2$, then $(((hk_1))_N, ((hk_2))_N) = (k_1, k_2)$. This is only a trivial form of redundancy. Within a column, the k_1 indices from $[0, N - 1]$ can be grouped on the basis of their gcd w.r.t. N. $l(k)$ is the number of co-primes that map an element of gcd group k to itself.

Example 5-1:

Let $N = 24$ and $(k_1, k_2) = (2,3)$.

Totatives of $N = \{1, 5, 7, 11, 13, 17, 19, 23\}$

$$\phi(N) = \phi(24) = 8$$

$$\phi(N / k_2) = \phi(8) = 4$$

$$l(k_2) = \phi(N) / \phi(N / k_2) = 2$$

Let elements counted by $l(k_2)$ form the set of co-primes $L(k_2)$. In example 5.1, $L(k_2) = \{1, 17\}$. The totatives of N can be divided into two sets of 4 co-primes each for the case $k_2 = 3$. A pair of co-primes from each group have the property that they map a member of the gcd set corresponding to $k_2 = 3$ to the same member itself. The co-prime pair $\{1, 17\}$ has this property. If any element of divisor column 3 is multiplied with either of the co-primes 1 or 17, the resultant product also belongs in the same divisor column, since $((1 \times 3))_{24} = ((17 \times 3))_{24} = 3$. Hence, given any element in divisor column k_2 whose first index belongs to a gcd set, multiplication with co-primes 1 and 17 ensures that the product element also is in the same divisor column with a first index that is in the same gcd set. This resultant element is thus redundant with the original element, and this is an example of redundancy within the same column, provided the resultant product of the first index is not equal to the first index of the original element. In that case, the original element has simply mapped to itself, thus it is a case of trivial redundancy, i.e. $(((hk_1))_N, ((hk_2))_N) = (k_1, k_2)$. Hence, two conditions must be satisfied for redundancy within the same column:

1) The second index should map to itself ($((hk_2))_N = k_2$), and,
2) The first index should not map to itself ($((hk_1))_N \neq k_1$).

In other words, the set of co-primes that map k_2 to itself (1 & 17 in the example above) and the set of co-primes that map the first index k_1 to itself should not have any common elements (other than 1, which corresponds to trivial redundancy). From Appendix A.3, a co-prime that maps a divisor to itself has the general equation

$$h = h_0 + Nt/g(k_2, N),\ 0 \leq t < g(k_2, N),\ h_0 = 1 \tag{5.7}$$

Since divisors are under consideration here, $g(k_2, N) = k_2$, and (5.7) can be written as

$$h_2 = 1 + Nt_2/k_2,\ 0 \leq t_2 < k_2 \tag{5.8}$$

Similarly, corresponding co-primes for k_1 have the form

$$h_1 = 1 + Nt_1/k_1,\ 0 \leq t_1 < k_1 \tag{5.9}$$

If there are common elements among h_1 and h_2, then, for some value of t_1 and t_2,

$$h_1 = h_2 = h. \tag{5.10}$$

For such values, from (5.8) - (5.10),

$$Nt_1/k_1 = Nt_2/k_2 \tag{5.11}$$

$$t_1k_2 = t_2k_1 \tag{5.12}$$

$$t_1k_2 - t_2k_1 = 0 \tag{5.13}$$

From (5.13) and Appendix A.2, t_1 has $g(k_1, k_2)$ solutions mod k_1, and t_2 has $g(k_1, k_2)$ solutions mod k_2. Hence, the number of common elements among h_1 and h_2 is given by $g(k_1, k_2)$.

A particular solution for (5.13) is $t_1 = 0$, $t_2 = 0$. General solutions are given by

$$t_1 = q_1k_1/g(k_1, k_2),\ t_2 = q_2k_2/g(k_1, k_2) \quad , \qquad q_1, q_2 \in Z \tag{5.14}$$

From (5.8) & (5.14),

$$h_2 = 1 + N(q_2k_2/g(k_1, k_2))/k_2 \tag{5.15}$$

$$= 1 + N(q_2/g(k_1, k_2)) \tag{5.16}$$

Similarly,

$$h_1 = 1 + N(q_1/g(k_1, k_2)) \tag{5.17}$$

(5.16) and (5.17) can be combined as

$$h = 1 + Nq/g(k_1, k_2), \tag{5.18}$$

(5.18) is the equation for the integers that map the pair (k_1,k_2) to itself. But there is the additional requirement that h is a co-prime number. The number of such co-primes h is hence given by $l(g(k_1,k_2))$. Hence, the number of co-primes that satisfy the condition $(((hk_1))_N,((hk_2))_N)$ = (k_1,k_2) is given by $l(g(k_1,k_2))$. This implies that the number of frequency indices completely redundant with a given frequency index (k_1, k_2) is given by $\phi(N)/l(g(k_1,k_2))$, which, using (5.6), becomes $\phi(N/g(k_1,k_2))$.

The number of elements in the set of integers that share a common gcd k_1 w.r.t. N is given by $\phi(N/k_1)$. The number of elements in this group that are unique in column k_2 is hence given by $\phi(N/k_1)/l(k_2)$, assuming there are no common elements among $L(k_1)$ and $L(k_2)$. For example 5.1, if $k_1 = 2$, then the number of elements in the gcd set corresponding to this value of k_1 is $\phi(N/2)=4$, given by {2, 6, 10, 14}. If $k_2 = 2$, then the number of co-primes that map 2 to itself, given by $l(k_2) = 2$, is {1, 13}. Hence, each frequency (k,k_2), where k is an element in the gcd group k_1, is completely redundant with another frequency (k',k_2), where $k' \neq k$, besides being trivially redundant with itself. The number of unique frequencies among the group (k,k_2) is thus assumed to be $\phi(N/k_1)/l(k_2)$. However, the number of common co-primes among $L(k_1)$ & $L(k_2)$ is given by $l(g(k_1, k_2))$, and these co-primes are {1, 13}. Since $l(g(k_1,k_2)) = 2$, these common co-primes cause trivial redundancy and thus do not cause complete redundancy. Hence, to account for the presence of these common co-primes, the expression presently obtained, $\phi(N/k_1)/l(k_2)$, needs to be multiplied by the factor $l(g(k_1, k_2))$. Thus, the number of unique frequencies (k,k_2) in column k_2 such that all k has the common gcd k_1 w.r.t. N, i.e. $g(k, N) = k_1$, is given by

$$U_{k_1,k_2} = \frac{\phi(N/k_1)l(\mathrm{g}(k_1,k_2))}{l(k_2)} \tag{5.19}$$

$$= \frac{\phi(N)l(\mathrm{g}(k_1,k_2))}{l(k_1)l(k_2)} \tag{5.20}$$

$$= \frac{\phi(N/k_1)\phi(N/k_2)}{\phi(N/\mathrm{g}(k_1,k_2))} \tag{5.21}$$

Example 5.2:

Table 5.1 shows the values of $U_{k_1k_2}$ evaluated for all possible values of k_1, k_2 for $N = 8$. Given $N = 8$, $\phi(N)=4$, $l(1) = 1$, $l(2) = 2$, $l(4) = 4$, and $l(8) = 4$.

Table 5.1: Values of U_{k_1,k_2} evaluated for divisor frequencies k_1, k_2 for $N = 8$.

k_2 / k_1	1	2	4	8
1	4	2	1	1
2	2	2	1	1
4	1	1	1	1
8	1	1	1	1

(5.19) - (5.21) give the number of unique frequencies in column k_2 for gcd k_1 w.r.t. N. The next objective is to compute the number of unique frequencies over all divisor columns.

5.2.5 Number of unique frequencies

Since analysis needs to be done in terms of divisors, it is suitable to express N in terms of its prime divisors as given below.

Let $N = \prod_{i=1}^{q} r_i^{a_i}$ (5.22)

The divisors of N are given by the different powers of prime divisor r.

The number of unique 2-D MRT frequencies for any given divisor can be considered first. For any given value of $g(k_1, k_2)$, the number of k_1 that share this gcd with k_2 can be calculated.

Let $k_2 = \prod_{i=1}^{q} r_i^{w_i}$, and $k_1 = \prod_{i=1}^{q} r_i^{v_i}$ (5.23)

$$g(k_1,k_2) = \prod_{i=1}^{q} r_i^{\min(v_i,w_i)} = \prod_{i=1}^{q} r_i^{u_i} \qquad (5.24)$$

From (5.23) and (5.24), the following cases arise:

i) If $u_i < w_i$, the possible values of v_i are given by,

$v_i = u_i$ (5.25)

ii) If $u_i = w_i$, the possible values of v_i are,

$v_i = u_i, u_i + 1, u_i + 2, u_i + 3, \ldots, a_i$ (5.26)

iii) If $u_i = w_i = a_i$, the possible values of v_i are,

$v_i = a_i = u_i$ (5.27)

Hence, from (5.23), (5.24) and (5.25), if $u_i < w_i$ for all i, then

$k_1 = g(k_1, k_2)$ (5.28)

$\therefore l(k_1) = l(g(k_1,k_2)) \quad if\ \forall i, u_i < w_i$

If $u_i = w_i$ for some i,

$$k_1 = \left(\prod_{\forall i \Rightarrow u_i = w_i} r_i^{b_i} \right) g(k_1, k_2), \quad 0 \le b_i \le a_i - u_i \tag{5.29}$$

Hence, from (5.29) and (B.2) in Appendix B.2,

$$l(k_1) = \left(\prod_{\forall i \Rightarrow u_i = w_i} r_i^{b_i} \right) l(g(k_1, k_2)), \quad 0 \le b_i \le a_i - u_i - 1 \qquad if \ \forall i, \ v_i \ne a_i \tag{5.30}$$

$$= \left(\prod_{\forall i \Rightarrow v_i = a_i} (r_i - 1) \right) \left(\prod_{\forall i \Rightarrow v_i \ne a_i} r_i^{b_i} \right) l(g(k_1, k_2)), \ 0 \le b_i \le a_i - u_i - 1, \ if \ \ v_i = a_i \tag{5.31}$$

The number of unique frequency indices for a given value of (k_1, k_2) is given by (5.20). For a given value of $g(k_1, k_2)$, the values of k_1 corresponding to $g(k_1, k_2)$ are known from (5.25) – (5.27). If the number of unique frequencies corresponding to each such value of k_1 is calculated using (5.20) and these are summed, the number of unique frequencies $U(g(k_1, k_2), k_2)$ corresponding to the given values of $g(k_1, k_2)$ and k_2 are obtained.

$$\therefore U(g(k_1, k_2), k_2) = \sum_{\forall k_1 \Rightarrow g(k_1, k_2)} \frac{\phi(N) l(g(k_1, k_2))}{l(k_1) l(k_2)} \tag{5.32}$$

In (5.32), the notation $\forall k_1 \Rightarrow g(k_1, k_2)$ implies that the summation is done for all values of k_1 that satisfies the specific value of $g(k_1, k_2)$.

Considering the case when N has only one prime divisor r, for $u = w$,

since $g(k_1, k_2)$ and k_2 are constant,

$$U(g(k_1, k_2), k_2) = \frac{\phi(N) l(g(k_1, k_2))}{l(k_2)} \sum_{\forall k_1 \Rightarrow g(k_1, k_2)} \frac{1}{l(k_1)}$$

$$= \frac{\phi(N) l(g(k_1, k_2))}{l(k_2)} \left(\frac{1}{l(g(k_1, k_2))} + \frac{1}{r l(g(k_1, k_2))} + \frac{1}{r^2 l(g(k_1, k_2))} + \ldots\ldots \right.$$

$$\left. + \frac{1}{r^{(a_n - 1)} l(g(k_1, k_2))} + \frac{1}{r^{(a_n - 1)} (r - 1) l(g(k_1, k_2))} \right)$$

$$= \frac{\phi(N)}{l(k_2)} \left(1 + \frac{1}{r} + \frac{1}{r^2} + \ldots\ldots \frac{1}{r^{(a_n - 1)}} + \frac{1}{r^{(a_n - 1)} (r - 1)} \right) \tag{5.33}$$

Simplifying,

$$\left(1 + \frac{1}{r} + \frac{1}{r^2} + \ldots\ldots \frac{1}{r^{(a_n - 1)}} \right) = \frac{(r^a - 1)}{r^{(a-1)} (r - 1)} \tag{5.34}$$

$$\therefore \left(\frac{(r^a - 1)}{r^{(a_n - 1)} (r - 1)} + \frac{1}{r^{(a_n - 1)} (r - 1)} \right) = \frac{r}{r - 1} \tag{5.35}$$

$$\therefore U(g(k_1,k_2),k_2) = \frac{\phi(N)}{l(k_2)}\frac{r}{(r-1)} \tag{5.36}$$

Example 5.3:

When $N = 8$, $r = 2$. If $k_2 = 2$ and $g(k_1,k_2) = 2$, the condition $u = w$ is satisfied, and hence from (5.36), $U(2, 2) = 4$. These four unique frequencies are (0,2), (2,2), (4,2) and (6,2). It can be verified that all these 4 frequencies are such that $g(k_1,k_2) = 2$.

If $u < w$, $k_1 = g(k_1, k_2)$, $l(k_1) = l(g(k_1, k_2))$, hence

$$U(g(k_1,k_2),k_2) = \frac{\phi(N)l(g(k_1,k_2))}{l(k_1)l(k_2)} = \frac{\phi(N)}{l(k_2)} \tag{5.37}$$

Example 5.4:

For $N = 8$, if $k_2 = 4$ and $g(k_1,k_2) = 2$, the condition $u < w$ is satisfied, and hence from (5.37), $U(g(k_1,k_2), k_2) = 1$. This unique frequency is (2,4), and $g(2,4) = 2$.

If $w = a$, & $u = w$, $k_1 = g(k_1,k_2)$, $l(k_1) = l(g(k_1,k_2))$, hence

$$U(g(k_1,k_2),k_2) = \frac{\phi(N)l(g(k_1,k_2))}{l(k_1)l(k_2)} = \frac{\phi(N)}{l(k_2)} \tag{5.37a}$$

If N has two divisors, r_1 & r_2,

when $u_1 = w_1$, and $u_2 = w_2$,

$$k_1 = g(k_1,k_2),\ r_1 g(k_1,k_2),\ r_1^2 g(k_1,k_2),\ r_1^3 g(k_1,k_2),\ldots\ldots$$

$$\ldots r_1^{a_1} g(k_1,k_2),\ r_2 g(k_1,k_2),\ r_1 r_2 g(k_1,k_2),\ r_1^2 r_2 g(k_1,k_2),\ldots$$

$$\ldots r_1^{a_1} r_2 g(k_1,k_2),\ r_2^{a_2} g(k_1,k_2),\ r_1 r_2^{a_2} g(k_1,k_2),\ r_1^2 r_2^{a_2} g(k_1,k_2),\ldots\ldots r_1^{a_1} r_2^{a_2} g(k_1,k_2)$$

$$U(g(k_1,k_2),k_2) = \frac{\phi(N)l(g(k_1,k_2))}{l(k_2)} \sum_{\forall k_1 | g(k_1,k_2)} \frac{1}{l(k_1)} =$$

$$= \frac{\phi(N)l(g(k_1,k_2))}{l(k_2)}\left(\frac{1}{l(g(k_1,k_2))} + \frac{1}{r_1 l(g(k_1,k_2))} + \frac{1}{r_1^2 l(g(k_1,k_2))} + \ldots\right.$$

$$+ \frac{1}{r_1^{(a_1-1)} l(g(k_1,k_2))} + \frac{1}{r_1^{(a_1-1)}(r_1-1) l(g(k_1,k_2))}$$

$$+ \frac{1}{r_2 l(g(k_1,k_2))} + \frac{1}{r_1 r_2 l(g(k_1,k_2))} + \frac{1}{r_1^2 r_2 l(g(k_1,k_2))} + \ldots$$

$$+ \frac{1}{r_1^{(a_1-1)} r_2 l(g(k_1,k_2))} + \frac{1}{r_1^{(a_1-1)}(r_1-1) r_2 l(g(k_1,k_2))}$$

$$+\frac{1}{r_2^2 l(g(k_1,k_2))}+\frac{1}{r_1 r_2^2 l(g(k_1,k_2))}+\frac{1}{r_1^2 r_2^2 l(g(k_1,k_2))}+\dots$$

$$\frac{1}{r_1^{(a_1-1)} r_2^2 l(g(k_1,k_2))}+\frac{1}{r_1^{(a_1-1)}(r_1-1) r_2^2 l(g(k_1,k_2))}+\dots.$$

$$+\frac{1}{r_2^{(a_2-1)}(r_2-1) l(g(k_1,k_2))}+\frac{1}{r_1 r_2^{(a_2-1)}(r_2-1) l(g(k_1,k_2))}+\dots\dots\frac{1}{r_1^{(a_1-1)} r_2^{(a_2-1)}(r_2-1) l(g(k_1,k_2))}$$

$$+\frac{1}{r_1^{(a_1-1)}(r_1-1) r_2^{(a_2-1)}(r_2-1) l(g(k_1,k_2))})$$

$$=\frac{\phi(N)}{l(k_2)}(1+\frac{1}{r_1}+\frac{1}{r_1^2}+\dots\dots\frac{1}{r_1^{(a_1-1)}}+\frac{1}{r_1^{(a_1-1)}(r_1-1)}$$

$$+\frac{1}{r_2}+\frac{1}{r_1 r_2}+\frac{1}{r_1^2 r_2}+\dots\dots\frac{1}{r_1^{(a_1-1)} r_2}+\frac{1}{r_1^{(a_1-1)}(r_1-1) r_2}$$

$$+\frac{1}{r_2^2}+\frac{1}{r_1 r_2^2}+\frac{1}{r_1^2 r_2^2}+\dots\dots\frac{1}{r_1^{(a_1-1)} r_2^2}+\frac{1}{r_1^{(a_1-1)}(r_1-1) r_2^2}+\dots.$$

$$+\frac{1}{r_2^{(a_2-1)}(r_2-1)}+\frac{1}{r_1 r_2^{(a_2-1)}(r_2-1)}+\dots\dots\frac{1}{r_1^{(a_1-1)} r_2^{(a_2-1)}(r_2-1)}+\frac{1}{r_1^{(a_1-1)}(r_1-1) r_2^{(a_2-1)}(r_2-1)})$$

$$=\frac{\phi(N)}{l(k_2)}(1+\frac{1}{r_1}+\frac{1}{r_1^2}+\dots\dots\frac{1}{r_1^{(a_1-1)}}+\frac{1}{r_1^{(a_1-1)}(r_1-1)})(1+\frac{1}{r_2}+\frac{1}{r_2^2}+\dots\dots\frac{1}{r_2^{(a_2-1)}}+\frac{1}{r_2^{(a_2-1)}(r_2-1)})$$

$$=\frac{\phi(N)}{l(k_2)}\frac{r_1}{(r_1-1)}\frac{r_2}{(r_2-1)} \tag{5.38}$$

Similarly, if $u_1 = w_1$, and $u_2 < w_2$,

$$\sum_{\forall k_1 \Rightarrow g(k_1,k_2)} \frac{l(g(k_1,k_2))}{l(k_1)}=\frac{r_1}{r_1-1}, \quad U(g(k_1,k_2),k_2)=\frac{\phi(N)}{l(k_2)}\frac{r_1}{r_1-1} \tag{5.38a}$$

and if $u_1 < w_1$, and $u_2 = w_2$,

$$\sum_{\forall k_1 \Rightarrow g(k_1,k_2)} \frac{l(g(k_1,k_2))}{l(k_1)}=\frac{r_2}{r_2-1}, \quad U(g(k_1,k_2),k_2)=\frac{\phi(N)}{l(k_2)}\frac{r_2}{r_2-1} \tag{5.38b}$$

If $u_1 < w_1$ and $u_2 < w_2$, then $k_1 = g(k_1,k_2)$, hence $l(k_1) = l(g(k_1,k_2))$, and,

$$U(g(k_1,k_2),k_2)=\frac{\phi(N) l(g(k_1,k_2))}{l(k_2) l(k_1)}=\frac{\phi(N)}{l(k_2)} \tag{5.39}$$

The number of unique frequencies for a given pair $(g(k_1,k_2),k_2)$ has been obtained and the next logical step is to obtain the total number of unique frequencies for N by summing the term $U(g(k_1,k_2),k_2)$ over all values of divisor k_2 of N.

Let N have only one prime divisor. In this case, k_2 can have the values $1, r, r^2, r^3, \ldots, r^a$.

As seen earlier, when $u = w$, $U(g(k_1,k_2),k_2)$ is given by (5.36) and , when $u < w$, $U(g(k_1,k_2),k_2)$ is given by (5.37). The total number of unique frequencies for a given value of k_2, defined by $B(k_2)$, is thus obtained by summing $U(g(k_1,k_2),k_2)$ over all possible values of $g(k_1,k_2)$ corresponding to this value of k_2. For any given value of $k_2 = r^w$, the possible values of $g(k_1,k_2)$ are, $g(k_1,k_2) = r^u$, $u = 0, 1, 2, 3, \ldots. w$. For $k_2 < N$, using (5.36) and (5.37), $B(k_2)$ is thus given by

$$B(k_2) = \frac{\phi(N)}{l(k_2)}\left[(\sum_{t=0}^{w-1} 1) + \frac{r}{r-1}\right] \tag{5.40}$$

$$= \frac{\phi(N)}{l(k_2)}\left[w + \frac{r}{r-1}\right] \tag{5.41}$$

From Appendix B.1,

if $k_2 < N$, $l(k_2) = k_2 = r^w$, else $l(k_2) = r^{a-1}(r-1)$ (5.42)

Hence, for $k_2 = 1, r, r^2, \ldots r^{a-1}$,

$$B(k_2) = \frac{\phi(N)}{r^w}\left[w + \frac{r}{r-1}\right], \tag{5.43}$$

And, for $k_2 = N = r^a$, $((k_2))_N = 0$), and using (5.37a),

$$B(k_2) = \frac{\phi(N)}{r^{a-1}(r-1)}[a+1]. \tag{5.44}$$

Example 5.5:

When $N = 8$, and $k_2 = 4$, the possible values of $g(k_1,k_2)$ are $g(k_1,k_2) = 1, 2$ & 4. Using (5.37), $U(1,4) = 1$, $U(2,4) = 1$, and using (5.36), $U(4,4) = 2$. Summing these values gives $B(4) = 4$. Verifying this value of $B(4)$ using (5.43),

$$B(4) = \frac{\phi(N)}{4}\left[2 + \frac{2}{2-1}\right] = 4.$$

Continuing with $N = 8$, when $k_2 = 8$, the possible values of $g(k_1, k_2)$ are $g(k_1, k_2) = 1, 2, 4$ & 8. Using (5.37), $U(1,8) = 1$, $U(2,8) = 1$, $U(4,8) = 1$ and using (5.37a), $U(8,8) = 1$. Summing these values gives $B(8) = 4$. Verifying this value of $B(8)$ using (5.44),

$$B(8) = \frac{\phi(N)}{4(2-1)}[3+1] = 4.$$

To obtain the total number of unique frequencies for a given N, defined as U, the terms $B(k_2)$ needs to be summed for all divisors k_2 of N.

$$U = \sum_{\forall k_2} B(k_2)$$

$$= \sum_{w=0}^{a-1} \frac{\phi(N)}{r^w}\left[w + \frac{r}{r-1}\right] + \frac{\phi(N)}{r^{a-1}(r-1)}[a+1]$$

$$= \phi(N)\left[\sum_{w=0}^{a-1}\left[\frac{w}{r^w} + \left(\frac{r}{r-1}\right)\left(\frac{1}{r^w}\right)\right] + \frac{a+1}{r^{a-1}(r-1)}\right] \tag{5.45}$$

Simplifying,

$$\sum_{w=0}^{a-1}\left[\frac{w}{r^w}\right] = \frac{r^a - 1 - ar + a}{(r-1)^2 r^{a-1}} \tag{5.46}$$

$$\sum_{w=0}^{a-1}\left[\left(\frac{r}{r-1}\right)\left(\frac{1}{r^w}\right)\right] = \left(\frac{r}{r-1}\right)\frac{(r^a-1)}{(r-1)r^{a-1}} \tag{5.47}$$

From (5.46) and (5.47),

$$\sum_{w=0}^{a-1}\left[\frac{w}{r^w} + \left(\frac{r}{r-1}\right)\left(\frac{1}{r^w}\right)\right] = \frac{(r^a-1)(r+1) - a(r-1)}{(r-1)^2 r^{a-1}} \tag{5.48}$$

From (5.48), (5.45) becomes,

$$U = \phi(N)\left[\frac{(r^a-1)(r+1) - a(r-1)}{(r-1)^2 r^{a-1}} + \frac{(a+1)}{r^{a-1}(r-1)}\right] \tag{5.49}$$

$$= \phi(N)\left[\frac{(r^a-1)(r+1) - a(r-1) + (r-1)(a+1)}{(r-1)^2 r^{a-1}}\right]$$

$$= \phi(N)\left[\frac{(r^a-1)(r+1) - (r-1)(a-a-1)}{(r-1)^2 r^{a-1}}\right]$$

$$= \phi(N)\left[\frac{(r^a-1)(r+1) + (r-1)}{(r-1)^2 r^{a-1}}\right] \tag{5.49a}$$

From Appendix A.7,

$$\phi(N) = N\frac{r-1}{r} \tag{5.49b}$$

$$U = N\left(\frac{r-1}{r}\right)\left(\frac{(r^a-1)(r+1) + (r-1)}{(r-1)^2 r^{a-1}}\right)$$

$$= \left(\frac{N}{r}\right)\left(\frac{(r^a-1)(r+1) + (r-1)}{(r-1)r^{a-1}}\right)$$

$$=\left(\frac{r^a}{r}\right)\left(\frac{(r^a-1)(r+1)+(r-1)}{(r-1)r^{a-1}}\right)$$

$$=\left(\frac{(r^a-1)(r+1)+(r-1)}{(r-1)}\right)$$

$$=\frac{(r^a-1)(r+1)}{(r-1)}+1 \tag{5.50}$$

Example 5.6:

For $N = 8$, the possible values of k_2 are $k_2 = 1, 2, 4$ & 8. Using (5.43), $B(1) = 8$, $B(2) = 6$, $B(4) = 4$ and using (5.44), $B(8) = 4$. Summing these values gives $U = 22$. Calculating U from (5.50) to verify the value obtained above,

$$U=\frac{(2^3-1)(2+1)}{(2-1)}+1=22\,.$$

The total number of unique frequencies when N has one prime divisor is given by (5.50). Considering the case when N has two prime divisors r_1 and r_2,

let $N = r_1^{a_1} r_2^{a_2}$ (5.51)

Number of divisors = $(a_1 + 1)(a_2 + 1)$

Let $k_2 = r_1^{w_1} r_2^{w_2}$ (5.52)

Let $g(k_1,k_2) = r_1^{u_1} r_2^{u_2}$ (5.53)

For a given value of k_2, number of possible values of $g(k_1, k_2) = (w_1 + 1)(w_2 + 1)$, since $u_1 = 0,1,2,\ldots,w_1$, $u_2 = 0,1,2,\ldots, w_2$.

(i) When $w_1 < a_1$, $w_2 < a_2$

From Appendix B.1, $l(k_2) = k_2 = r_1^{w_1} r_2^{w_2}$

(a) When $u_1 < w_1$, $u_2 < w_2$,

$u_1 = 0,1,2,\ldots,w_1-1$, $u_2 = 0,1,2,\ldots, w_2-1$

The number of such values of $g(k_1,k_2)$ is $w_1 w_2$.

From (5.28), $k_1 = g(k_1,k_2)$, hence $l(k_1) = l(g(k_1,k_2))$. From (5.39) and Appendix B.1, the number of unique frequencies for a given value of $g(k_1,k_2)$ & k_2 is hence given by

$$U(g(k_1,k_2),k_2)=\frac{\phi(N)}{r_1^{w_1} r_2^{w_2}} \tag{5.54}$$

(b) When $u_1 = w_1$, $u_2 < w_2$,

$v_1 = w_1, w_1 + 1, \ldots, a_1$, $v_2 = u_2$

From (5.38a), $\sum_{\forall k_1 \Rightarrow g(k_1,k_2)} \frac{l(g(k_1,k_2))}{l(k_1)} = \frac{r_1}{r_1 - 1}$

The number of possible values of $g(k_1,k_2)$ is w_2, since $u_2 = 0,1,2,\ldots, w_2 - 1$. Hence, from (5.32) and Appendix B.1,

$$U(g(k_1,k_2),k_2) = \frac{\phi(N)}{r_1^{w_1} r_2^{w_2}} \frac{r_1}{r_1 - 1} \tag{5.55}$$

(c) Similarly, when $u_1 < w_1$, $u_2 = w_2$,

$v_1 = u_1, \quad v_2 = w_2, w_2 + 1, \ldots, a_2$

From (5.38b), $\sum_{\forall k_1 \Rightarrow g(k_1,k_2)} \frac{l(g(k_1,k_2))}{l(k_1)} = \frac{r_2}{r_2 - 1}$

The number of possible values of $g(k_1,k_2)$ is w_1, since $u_1 = 0,1,2,\ldots, w_1 - 1$. From (5.32),

$$U(g(k_1,k_2),k_2) = \frac{\phi(N)}{r_1^{w_1} r_2^{w_2}} \frac{r_2}{r_2 - 1} \tag{5.56}$$

(d) When $u_1 = w_1$, $u_2 = w_2$,

$v_1 = w_1, w_1 + 1, \ldots, a1, \quad v_2 = w_2, w_2 + 1, \ldots, a_2$

From (5.38), $\sum_{\forall k_1 \Rightarrow g(k_1,k_2)} \frac{l(g(k_1,k_2))}{l(k_1)} = \frac{r_1}{r_1 - 1} \frac{r_2}{r_2 - 1}$

There is only one possible value of $g(k_1,k_2)$, since $u_1 = w_1$, $u_2 = w_2$.

$$U(g(k_1,k_2),k_2) = \frac{\phi(N)}{r_1^{w_1} r_2^{w_2}} \frac{r_1}{r_1 - 1} \frac{r_2}{r_2 - 1} \tag{5.57}$$

The sum of (5.54), (5.55), (5.56) & (5.57) each scaled appropriately by the corresponding number of possible values of $g(k_1,k_2)$ gives the number of unique frequencies over all possible values of $g(k_1,k_2)$ for a given value of k_2, for the case $w_1 < a_1$, $w_2 < a_2$.

$$B(k_2) = \frac{\phi(N)}{r_1^{w_1} r_2^{w_2}} \left[w_1 w_2 + w_2 \left(\frac{r_1}{r_1 - 1} \right) + w_1 \left(\frac{r_2}{r_2 - 1} \right) + \left(\frac{r_1}{r_1 - 1} \right) \left(\frac{r_2}{r_2 - 1} \right) \right] \tag{5.58}$$

(ii) $w_1 = a_1$, $w_2 < a_2$

From Appendix B.1, $l(k_2) = r_1^{w_1 - 1}(r_1 - 1) r_2^{w_2}$

(a) When $u_1 < w_1$, $u_2 < w_2$,

$u_1 = 0,1,2,\ldots,w_1\text{-}1, \quad u_2 = 0,1,2,\ldots, w_2\text{-}1$

No values of u_1 and u_2 that satisfy this condition.

From (5.28), $k_1 = g(k_1,k_2)$, hence $l(k_1) = l(g(k_1,k_2))$. From (5.39),

$$U(g(k_1,k_2),k_2)=\frac{\phi(N)}{r_1^{w_1-1}(r_1-1)r_2^{w_2}} \tag{5.59}$$

(b) When $u_1 = w_1$, $u_2 < w_2$,

$v_1 = w_1, w_1 + 1, \dots, a_1, \quad v_2 = u_2$

Since $u_1 = w_1 = a_1$, $v_1 = a_1$, and $k_1 = g(k_1,k_2)$, $l(k_1) = l(g(k_1,k_2))$. Hence, $\sum_{\forall k_1 \Rightarrow g(k_1,k_2)} \frac{l(g(k_1,k_2))}{l(k_1)} = 1$

The number of possible values of $g(k_1,k_2)$ is w_2.

$$U(g(k_1,k_2),k_2)=\frac{\phi(N)}{r_1^{w_1-1}(r_1-1)r_2^{w_2}} \tag{5.60}$$

(c) Similarly, when $u_1 < w_1$, $u_2 = w_2$,

$v_1 = u_1, \quad v_2 = w_2, w_2 + 1, \dots, a_2$

From (5.38b), $\sum_{\forall k_1 \Rightarrow g(k_1,k_2)} \frac{l(g(k_1,k_2))}{l(k_1)} = \frac{r_2}{r_2-1}$

The number of possible values of $g(k_1,k_2)$ is w_1.

$$U(g(k_1,k_2),k_2)=\frac{\phi(N)}{r_1^{w_1-1}(r_1-1)r_2^{w_2}}\frac{r_2}{r_2-1} \tag{5.61}$$

(d) When $u_1 = w_1$, $u_2 = w_2$,

$v_1 = w_1, w_1 + 1, \dots, a_1, \quad v_2 = w_2, w_2 + 1, \dots, a_2$

Since $u_1 = w_1 = a_1$, and using the method used to obtain (5.38),

$$\sum_{\forall k_1 \Rightarrow g(k_1,k_2)} \frac{l(g(k_1,k_2))}{l(k_1)} = \frac{r_2}{r_2-1}$$

There is only one possible value of $g(k_1,k_2)$

$$U(g(k_1,k_2),k_2)=\frac{\phi(N)}{r_1^{w_1-1}(r_1-1)r_2^{w_2}}\frac{r_2}{r_2-1} \tag{5.62}$$

From (5.59), (5.60), (5.61) & (5.62) and appropriate scaling, the number of unique frequencies over all possible values of $g(k_1,k_2)$ for a given value of k_2 for the case $w_1 = a_1$, $w_2 < a_2$ is given by,

$$B(k_2)=\frac{\phi(N)}{r_1^{w_1-1}(r_1-1)r_2^{w_2}}\left[w_1w_2+w_2+w_1\left(\frac{r_2}{r_2-1}\right)+\left(\frac{r_2}{r_2-1}\right)\right] \tag{5.63}$$

(iii) Similarly, when $w_1 < a_1$, $w_2 = a_2$,

$$B(k_2)=\frac{\phi(N)}{r_1^{w_1}r_2^{w_2-1}(r_2-1)}\left[w_1w_2+w_1+w_2\left(\frac{r_1}{r_1-1}\right)+\left(\frac{r_1}{r_1-1}\right)\right] \tag{5.64}$$

(iv) Finally, when $w_1 = a_1$, $w_2 = a_2$,

From Appendix B.1, $l(k_2) = r_1^{w_1-1}(r_1-1)r_2^{w_2-1}(r_2-1)$

(a) When $u_1 < w_1$, $u_2 < w_2$,

$u_1 = 0,1,2,\ldots,w_1\text{-}1$, $\quad u_2 = 0,1,2,\ldots, w_2\text{-}1$

The number of possible values of $g(k_1,k_2)$ is w_1w_2.

$k_1 = g(k_1,k_2)$, hence $l(k_1) = l(g(k_1,k_2))$, and $\sum_{\forall k_1 \Rightarrow g(k_1,k_2)} \frac{l(g(k_1,k_2))}{l(k_1)} = 1$

$$U(g(k_1,k_2),k_2) = \frac{\phi(N)}{r_1^{a_1-1}(r_1-1)r_2^{a_2-1}(r_2-1)} \tag{5.65}$$

(b) When $u_1 = w_1$, $u_2 < w_2$,

$v_1 = w_1, w_1+1, \ldots, a_1$, $\quad v_2 = u_2$

Since $u_1 = w_1 = a_1$, $v_1 = a_1$, and $k_1 = g(k_1,k_2)$, $l(k_1) = l(g(k_1,k_2))$. Hence, $\sum_{\forall k_1 \Rightarrow g(k_1,k_2)} \frac{l(g(k_1,k_2))}{l(k_1)} = 1$

The number of possible values of $g(k_1,k_2)$ is w_2.

$$U(g(k_1,k_2),k_2) = \frac{\phi(N)}{r_1^{a_1-1}(r_1-1)r_2^{a_2-1}(r_2-1)} \tag{5.66}$$

(c) Similarly, when $u_1 < w_1$, $u_2 = w_2$,

$v_1 = u_1$, $\quad v_2 = w_2, w_2+1, \ldots, a_2$

Since $u_2 = w_2 = a_2$, $v_2 = a_2$, and $k_1 = g(k_1,k_2)$, $l(k_1) = l(g(k_1,k_2))$. Hence, $\sum_{\forall k_1 \Rightarrow g(k_1,k_2)} \frac{l(g(k_1,k_2))}{l(k_1)} = 1$

The number of possible values of $g(k_1,k_2)$ is w_1.

$$U(g(k_1,k_2),k_2) = \frac{\phi(N)}{r_1^{a_1-1}(r_1-1)r_2^{a_2-1}(r_2-1)} \tag{5.67}$$

(d) When $u_1 = w_1$, $u_2 = w_2$,

$v_1 = w_1, w_1+1, \ldots, a_1$, $\quad v_2 = w_2, w_2+1, \ldots, a_2$

since $u_1 = a_1$, $u_2 = a_2$, $\sum_{\forall k_1 \Rightarrow g(k_1,k_2)} \frac{l(g(k_1,k_2))}{l(k_1)} = 1$

There is only one possible value of $g(k_1,k_2)$. From (5.32) and Appendix B.1,

$$U(g(k_1,k_2),k_2) = \frac{\phi(N)}{r_1^{a_1-1}(r_1-1)r_2^{a_2-1}(r_2-1)} \tag{5.68}$$

From (5.65), (5.66), (5.67) & (5.68) and doing appropriate scaling,

$$B(k_2) = \frac{\phi(N)}{r_1^{a_1-1}(r_1-1)r_2^{a_2-1}(r_2-1)}\left[w_1w_2 + w_2 + w_1 + 1\right]$$

$$= \frac{\phi(N)}{r_1^{a_1-1}(r_1-1)r_2^{a_2-1}(r_2-1)}\left[(a_1+1)(a_2+1)\right] \tag{5.68a}$$

From (5.58), (5.63), (5.64) & (5.68a), the total number of unique frequencies of N can be calculated as

$$U = \phi(N)\left[\sum_{w_1=0}^{a_1-1}\sum_{w_2=0}^{a_2-1}\frac{w_1w_2}{r_1^{w_1}r_2^{w_2}} + \sum_{w_1=0}^{a_1-1}\sum_{w_2=0}^{a_2-1}\frac{w_2}{r_1^{w_1}r_2^{w_2}}\frac{r_1}{r_1-1} + \sum_{w_1=0}^{a_1-1}\sum_{w_2=0}^{a_2-1}\frac{w_1}{r_1^{w_1}r_2^{w_2}}\frac{r_2}{r_2-1}\right.$$

$$+\sum_{w_1=0}^{a_1-1}\sum_{w_2=0}^{a_2-1}\frac{r_1r_2}{r_1^{w_1}r_2^{w_2}(r_1-1)(r_2-1)} + \sum_{w_2=0}^{a_2-1}\frac{w_2(a_1+1)}{r_1^{a_1-1}(r_1-1)r_2^{w_2}} + \sum_{w_2=0}^{a_2-1}\frac{r_2(a_1+1)}{r_1^{a_1-1}(r_1-1)r_2^{w_2}(r_2-1)}$$

$$\left.+\sum_{w_1=0}^{a_1-1}\frac{w_1(a_2+1)}{r_2^{a_2-1}(r_2-1)r_1^{w_1}} + \sum_{w_1=0}^{a_1-1}\frac{r_1(a_2+1)}{r_2^{a_2-1}(r_2-1)r_1^{w_1}(r_1-1)} + \frac{(a_1+1)(a_2+1)}{r_1^{a_1-1}r_2^{a_2-1}(r_1-1)(r_2-1)}\right]$$

$$= \phi(N)\left[\sum_{w_1=0}^{a_1-1}\frac{w_1}{r_1^{w_1}}\sum_{w_2=0}^{a_2-1}\frac{w_2}{r_2^{w_2}} + \sum_{w_1=0}^{a_1-1}\frac{1}{r_1^{w_1}}\frac{r_1}{r_1-1}\sum_{w_2=0}^{a_2-1}\frac{w_2}{r_2^{w_2}} + \sum_{w_1=0}^{a_1-1}\frac{w_1}{r_1^{w_1}}\sum_{w_2=0}^{a_2-1}\frac{1}{r_2^{w_2}}\frac{r_2}{r_2-1}\right.$$

$$+\sum_{w_1=0}^{a_1-1}\frac{1}{r_1^{w_1}}\frac{r_1}{r_1-1}\sum_{w_2=0}^{a_2-1}\frac{1}{r_2^{w_2}}\frac{r_2}{r_2-1} + \frac{a_1+1}{r_1^{a_1-1}(r_1-1)}\sum_{w_2=0}^{a_2-1}\frac{w_2}{r_2^{w_2}} + \frac{a_1+1}{r_1^{a_1-1}(r_1-1)}\sum_{w_2=0}^{a_2-1}\frac{r_2}{r_2^{w_2}(r_2-1)}$$

$$\left.+\frac{a_2+1}{r_2^{a_2-1}(r_2-1)}\sum_{w_1=0}^{a_1-1}\frac{w_1}{r_1^{w_1}} + \frac{a_2+1}{r_2^{a_2-1}(r_2-1)}\sum_{w_1=0}^{a_1-1}\frac{r_1}{r_1^{w_1}(r_1-1)} + \frac{a_1+1}{r_1^{a_1-1}(r_1-1)}\frac{a_2+1}{r_2^{a_2-1}(r_2-1)}\right]$$

$$= \phi(N)\left[\sum_{w_1=0}^{a_1-1}\frac{w_1}{r_1^{w_1}}\left[\sum_{w_2=0}^{a_2-1}\frac{w_2}{r_2^{w_2}} + \frac{r_2}{r_2-1}\sum_{w_2=0}^{a_2-1}\frac{1}{r_2^{w_2}} + \frac{(a_2+1)}{r_2^{a_2-1}(r_2-1)}\right]\right.$$

$$+\frac{r_1}{r_1-1}\sum_{w_1=0}^{a_1-1}\frac{1}{r_1^{w_1}}\left[\sum_{w_2=0}^{a_2-1}\frac{w_2}{r_2^{w_2}} + \frac{r_2}{r_2-1}\sum_{w_2=0}^{a_2-1}\frac{1}{r_2^{w_2}} + \frac{(a_2+1)}{r_2^{a_2-1}(r_2-1)}\right]$$

$$\left.+\frac{a_1+1}{r_1^{a_1-1}(r_1-1)}\sum_{w_1=0}^{a_1-1}\frac{1}{r_1^{w_1}}\left[\sum_{w_2=0}^{a_2-1}\frac{w_2}{r_2^{w_2}} + \frac{r_2}{r_2-1}\sum_{w_2=0}^{a_2-1}\frac{1}{r_2^{w_2}} + \frac{(a_2+1)}{r_2^{a_2-1}(r_2-1)}\right]\right]$$

$$= \phi(N)\left[\sum_{w_1=0}^{a_1-1}\frac{w_1}{r_1^{w_1}} + \frac{r_1}{r_1-1}\sum_{w_1=0}^{a_1-1}\frac{1}{r_1^{w_1}} + \frac{(a_1+1)}{r_1^{a_1-1}(r_1-1)}\right]\left[\sum_{w_2=0}^{a_2-1}\frac{w_2}{r_2^{w_2}} + \frac{r_2}{r_2-1}\sum_{w_2=0}^{a_2-1}\frac{1}{r_2^{w_2}} + \frac{(a_2+1)}{r_2^{a_2-1}(r_2-1)}\right] \tag{5.69}$$

From (5.49a),

$$\left[\sum_{w=0}^{a-1}\frac{w}{r^w} + \frac{r}{r-1}\sum_{w=0}^{a-1}\frac{1}{r^w} + \frac{(a+1)}{r^{a-1}(r-1)}\right] = \frac{(r^a-1)(r+1)+(r-1)}{(r-1)^2r^{a-1}}$$

$$\therefore U = \phi(N)\left[\frac{(r_1^{a_1}-1)(r_1+1)+(r_1-1)}{(r_1-1)^2r_1^{a_1-1}}\right]\left[\frac{(r_2^{a_2}-1)(r_2+1)+(r_2-1)}{(r_2-1)^2r_2^{a_2-1}}\right]$$

$$= r_1^{a_1} r_2^{a_2} \frac{(r_1-1)(r_2-1)}{r_1 r_2}\left[\frac{(r_1^{a_1}-1)(r_1+1)+(r_1-1)}{(r_1-1)^2 r_1^{a_1-1}}\right]\left[\frac{(r_2^{a_2}-1)(r_2+1)+(r_2-1)}{(r_2-1)^2 r_2^{a_2-1}}\right]$$

$$= \left[\frac{(r_1^{a_1}-1)(r_1+1)+(r_1-1)}{(r_1-1)}\right]\left[\frac{(r_2^{a_2}-1)(r_2+1)+(r_2-1)}{(r_2-1)}\right]$$

$$= \left[\frac{(r_1^{a_1}-1)(r_1+1)}{(r_1-1)}+1\right]\left[\frac{(r_2^{a_2}-1)(r_2+1)}{(r_2-1)}+1\right] \tag{5.70}$$

Thus, an expression for the number of unique frequencies has been obtained for N that has two prime divisors.

Example 5.7:

Let $N = 6$, so that $r_1 = 2$, $r_2 = 3$, $a_1 = 1$, $a_2 = 1$, $\phi(6) = 6\frac{2-1}{2}\frac{3-1}{3} = 2$.

(i) When $w_1 < a_1$, $w_2 < a_2$

$w_1 = 0$, $w_2 = 0$.

(a) $u_1 < w_1$, $u_2 < w_2$,

No values of u_1 and u_2 that satisfy this condition.

(b) $u_1 = w_1$, $u_2 < w_2$,

No values of u_1 and u_2 that satisfy this condition.

(c) $u_1 < w_1$, $u_2 = w_2$,

No values of u_1 and u_2 that satisfy this condition.

(d) $u_1 = w_1$, $u_2 = w_2$,

$u_1 = 0$, $u_2 = 0$.

$v_1 = 0, 1$, $v_2 = 0, 1$.

From (5.57),

$$U(1,1) = \frac{2}{1}\frac{2}{2-1}\frac{3}{3-1} = 6$$

From (a), (b), (c) & (d),

$$B(1) = U(1,1) = 6$$

$$B(1) = \frac{2}{1}\left[\left(\frac{2}{2-1}\right)\left(\frac{3}{3-1}\right)\right] = 6$$

(ii) $w_1 = a_1$, $w_2 < a_2$

$w_1 = 1$, $w_2 = 0$.

(a) $u_1 < w_1$, $u_2 < w_2$,

No values of u_1 and u_2 that satisfy this condition.

(b) $u_1 = w_1$, $u_2 < w_2$,

No values of u_1 and u_2 that satisfy this condition.

(c) $u_1 < w_1$, $u_2 = w_2$,

$u_1 = 0, \quad u_2 = 0.$

$v_1 = 0, \quad v_2 = 0, 1.$

From (5.61), $U(1,2) = \frac{2}{1}\frac{3}{3-1} = 3$

(d) $u_1 = w_1$, $u_2 = w_2$,

$u_1 = 1, \quad u_2 = 0.$

$v_1 = 1, \quad v_2 = 0, 1.$

From (5.61),

$$U(2,2) = \frac{2}{1}\frac{3}{3-1} = 3$$

From (a), (b), (c) & (d),

$$B(2) = U(1,2) + U(2,2) = 6$$

$$B(2) = \frac{2}{1}\left[1\left(\frac{3}{3-1}\right) + \left(\frac{3}{3-1}\right)\right] = 6$$

(iii) $w_1 < a_1$, $w_2 = a_2$

$w_1 = 0, \quad w_2 = 1.$

$$B(3) = \frac{2}{(3-1)}\left[\left(\frac{2}{2-1}\right) + \left(\frac{2}{2-1}\right)\right] = 4$$

(iv) $w_1 = a_1$, $w_2 = a_2$

$w_1 = 1, \quad w_2 = 1.$

(a) $u_1 < w_1$, $u_2 < w_2$,

$u_1 = 0, \quad u_2 = 0$

The number of possible values of $g(k_1; k_2)$ is 1.

$$U(1,6) = \frac{2}{(3-1)} = 1$$

(b) $u_1 = w_1$, $u_2 < w_2$,

$u_1 = 1, \quad u_2 = 0$

$v_1 = 1, \quad v_2 = 0$

The number of possible values of $g(k_1, k_2)$ is 1.

$$U(2,6) = \frac{2}{(3-1)} = 1$$

(c) $u_1 < w_1$, $u_2 = w_2$,

$u_1 = 0, \quad u_2 = 1$

$v_1 = 0, \quad v_2 = 1$

The number of possible values of $g(k_1,k_2)$ is 1.

$$U(3,6) = \frac{2}{(3-1)} = 1$$

(d) $u_1 = w_1$, $u_2 = w_2$,

$u_1 = 1, \quad u_2 = 1$

$v_1 = 1, \quad v_2 = 1$

There is only one possible value of $g(k_1,k_2)$.

$$U(6,6) = 1$$

From (a), (b), (c) & (d),

$$B(6) = \frac{2}{(3-1)}\left[(1+1)(1+1)\right] = 4$$

$$U = B(1) + B(2) + B(3) + B(6) = 20$$

Verifying this value using (5.70),

$$U = \left[\frac{(2-1)(2+1)}{(2-1)} + 1\right]\left[\frac{(3-1)(3+1)}{(3-1)} + 1\right] = (4)(5) = 20.$$

If $N_1 = r_1^{a_1}$, and $N_2 = r_2^{a_2}$, from (5.50),

$$U_{N_1} = \left[\frac{(r_1^{a_1} - 1)(r_1 + 1)}{(r_1 - 1)} + 1\right] \tag{5.70a}$$

and,

$$U_{N_2} = \left[\frac{(r_2^{a_2} - 1)(r_2 + 1)}{(r_2 - 1)} + 1\right], \tag{5.70b}$$

given $g(r_1,r_2) = 1$.

If $N = N_1 N_2 = r_1^{a_1} r_2^{a_2}$, from (5.70), (5.70a), and (5.70b), $U_N = U_{N_1} U_{N_2}$. Hence, it can be concluded that the number of unique 2-D MRT frequencies is a multiplicative function. By mathematical induction, for N that has q prime divisors, $N = \prod_{i=1}^{q} r_i^{a_i}$, the number of unique frequencies is hence given by

$$U = \prod_{i=1}^{q}\left[\frac{(r_i^{a_i} - 1)(r_i + 1)}{(r_i - 1)} + 1\right] \tag{5.71}$$

The number of unique frequencies to be used in a representation of 2-D MRT using only the MRT coefficients corresponding to these unique frequencies is given by (5.71). Table 5.2 shows the number of unique frequencies calculated using (5.71) for various values of N. Now, the number of unique MRT coefficients corresponding to these unique frequencies is to be found out, as described below.

Table 5.2: Number of unique frequencies for various values of N

N	Representation of N in terms of prime divisors	No. of unique frequencies	N	Representation of N in terms of prime divisors	No. of unique frequencies
4	2^2	10	42	2 3 7	180
6	2 3	20	60	2^2 3 5	350
8	2^3	22	64	2^6	190
10	2 5	28	80	2^4 5	322
12	2^2 3	50	100	2^2 5^2	370
14	2 7	36	128	2^7	382
16	2^4	46	144	2^3 3^2	782
18	2 3^2	68	180	2^2 3^2 5	1190
20	2^2 5	70	200	2^3 5^2	814
24	2^3 3	110	210	2 3 5 7	1260
30	2 3 5	140	256	2^8	766
32	2^5	94	324	2^2 3^4	1610
36	2^2 3^2	170	500	2^2 5^3	1870
40	2^3 5	154	512	2^9	1534

5.2.6 Number of Unique Coefficients

If the number of unique 2-D MRT frequencies is available, it is possible to determine the number of unique 2-D MRT coefficients. The case when N has only one prime divisor is considered first. If that prime divisor is assumed to be 2, then N is a power of 2. The next case is when N has two prime divisors. Since there is a prime divisor other than 2, such values of N are not powers of 2. The expressions obtained for these two cases can be generalized for the case when N has any number of prime divisors. From section 3.4.1, the existence of a 2-D MRT coefficient is dependent on whether $g(k_1,k_2,N)\ |\ p$ and $g(k_1,k_2,N)\ |\ (p + M)$. Also, an MRT coefficient has positive and negative groups only if the condition $g(k_1,k_2,N)\ |\ M$ is true. Similarly, using theorem 4.2 and the explanation given in 3.4.1, a 1-D MRT coefficient has positive and negative groups only if the condition $g(k,N) \mid M$ is true. For k a divisor of N, $g(k,N) = k$. From section 4.3.4, the number of 1-D MRT coefficients associated with divisor frequency k is M / k, if $k \mid M$, and N / k if $k \mid M$ is not satisfied. In the 2-D case, for unique frequencies corresponding to divisor columns, $g(k_2,N) = k_2$, and hence, $g(k_1,k_2,N) = g(k_1,k_2)$. Thus, for a given value of $g(k_1,k_2)$, the number of MRT coefficients is $M / g(k_1,k_2)$, if $g(k_1,k_2) \mid M$, and $N / g(k_1,k_2)$ if $g(k_1,k_2) \mid M$ is not satisfied.

Given the number of unique frequencies corresponding to the given values of $g(k_1,k_2)$ and k_2, $U(g(k_1,k_2),k_2)$, the number of unique MRT coefficients corresponding to these values of $g(k_1,k_2)$ and k_2, defined by $A(g(k_1,k_2),k_2)$, is given by

$$A(g(k_1,k_2),k_2) = \begin{cases} U(g(k_1,k_2),k_2)\dfrac{M}{g(k_1,k_2)}, & g(k_1,k_2) \mid M \\ U(g(k_1,k_2),k_2)\dfrac{N}{g(k_1,k_2)}, & \text{otherwise} \end{cases} \tag{5.72}$$

The total number of unique 2-D MRT coefficients corresponding to $g(k_1,k_2)$, defined by $C_{g(k_1,k_2)}$, can be found by summing the LHS of (5.72) over all possible values of k_2. Finally, the total number of unique MRT coefficients for a given value of N, defined by C, can be found by summing $C_{g(k_1,k_2)}$ over all possible values of $g(k_1,k_2)$. The analysis presented below proceeds along these lines.

Case (i) $N = r^a$, $r = 2$

The possible values of $g(k_1,k_2)$ are $g(k_1,k_2) = r^u, 0 \le u \le a$

Given $g(k_1,k_2)$, k_2 has to be greater than or equal to $g(k_1,k_2)$, hence $k_2 = r^w$, $u \le w \le a$

The number of unique frequencies for a given value of $g(k_1, k_2)$ and k_2, is given by (5.36), (5.37) & (5.37a). When $g(k_1,k_2) = 1$, & $k_2 = 1$, and since $l(1) = 1$, the number of unique frequencies is given, from (5.36), by

$$U(1,1) = \frac{\phi(N)r}{r-1},$$

$$\therefore A(1,1) = U(1,1)\frac{N}{2} = \frac{\phi(N)r}{r-1}\frac{N}{2} \tag{5.72a}$$

Similarly, when $g(k_1,k_2) = 1$, other possible values of $k_2 = r, r^2, \ldots, r^a$, then the number of unique frequencies is, from (5.37),

$$U(1,k_2)=\frac{\phi(N)}{l(k_2)}$$

$$\therefore A(1,k_2)==\begin{cases}\frac{\phi(N)}{r^w}\frac{N}{2}, & 1\le w\le a-1\\ \frac{\phi(N)}{r^{a-1}(r-1)}\frac{N}{2}, & w=a\end{cases} \tag{5.72b}$$

Let $C_{g(k_1,k_2)}$ be the total number of unique MRT coefficients corresponding to $g(k_1,k_2)$.

$$\therefore C_{g(k_1,k_2)}=\sum_{i=u}^{a}A(g(k_1,k_2),r^i) \tag{5.73}$$

Hence, from (5.72a) and (5.72b) and (5.73),

$$C_1=\phi(N)\left[\frac{r}{r-1}+\sum_{w=1}^{a-1}\frac{1}{r^w}+\frac{1}{r^{a-1}(r-1)}\right]\frac{N}{2}$$

Simplifying the terms in RHS,

$$\sum_{w=b}^{a-1}\frac{1}{r^w}+\frac{1}{r^{a-1}(r-1)}=\frac{1}{r^{b-1}(r-1)} \tag{5.74}$$

$$\therefore\sum_{w=1}^{a-1}\frac{1}{r^w}+\frac{1}{r^{a-1}(r-1)}=\frac{1}{(r-1)} \tag{5.75}$$

Hence,

$$\frac{r}{r-1}+\sum_{w=1}^{a-1}\frac{1}{r^w}+\frac{1}{r^{a-1}(r-1)}=\frac{(r+1)}{(r-1)} \tag{5.76}$$

$$\therefore C_1=\phi(N)\left[\frac{r+1}{r-1}\right]\frac{N}{2} \tag{5.77}$$

Considering the case of $g(k_1,k_2)=r$, the possible values of k_2 are $k_2=r, r^2,\ldots r^a$. When $k_2=r$, $l(r)=r$; the number of unique frequencies is given from (5.36) as

$$U(r,r)=\frac{\phi(N)r}{r(r-1)}.$$

$$A(r,r)=\frac{\phi(N)r}{r(r-1)}\frac{N}{2r}$$

And for $k_2=r^2$, (5.37) gives

$$U(r,r^2)=\frac{\phi(N)}{r^2}.$$

$$A(r,r^2)=\frac{\phi(N)}{r^2}\frac{N}{2r}$$

$$\therefore A(r,k_2)=\begin{cases}\dfrac{\phi(N)}{r^w}\dfrac{N}{2r}, & 1\le w\le a-1\\ \dfrac{\phi(N)}{r^{a-1}(r-1)}\dfrac{N}{2r}, & w=a\end{cases} \tag{5.78}$$

Summing all such terms, the number of unique MRT coefficients for $g(k_1,k_2)=r$ is given by

$$C_r=\frac{\phi(N)}{r}\left[\frac{r}{r-1}+\sum_{w=1}^{a-1}\frac{1}{r^w}+\frac{1}{r^{a-1}(r-1)}\right]\frac{N}{2r}=\frac{\phi(N)}{r}\frac{r+1}{r-1}\frac{N}{2r}=\phi(N)\frac{r+1}{r-1}\frac{N}{2}\frac{1}{r^2} \tag{5.79}$$

Generalizing from (5.79), for $g(k_1,k_2)=r^d$, $0\le d\le a-1$, the number of unique 2-D MRT coefficients is given by

$$C_{r^d}=\phi(N)\frac{r+1}{r-1}\frac{N}{2}\frac{1}{(r^d)^2},\quad 0\le d\le a-1 \tag{5.80}$$

When $g(k_1,k_2)=r^a$, $g(k_1,k_2)\mid M$ is not satisfied, hence there is $N/g(k_1,k_2)=N/r^a=1$ MRT coefficient for each unique frequency. Also, the only possible value of k_2 is $k_2=r^a$. Also, $l(r^a)=r^{a-1}(r-1)$. Hence, from (5.37a), the number of unique MRT coefficients for $g(k_1,k_2)=r^a$ is given by

$$C_{r^a}=\frac{\phi(N)}{r^{a-1}(r-1)} \tag{5.81}$$

(5.80) and (5.81) give the number of unique MRT coefficients for each possible value of $g(k_1,k_2)$. The total number of unique MRT coefficients for a given N can be obtained by adding the number of unique MRT coefficients corresponding to each value of $g(k_1,k_2)$ and is given by

$$C=\sum_{i=0}^{a}C_{r^i}=\phi(N)\frac{(r+1)}{(r-1)}\frac{N}{2}\left[\sum_{w=0}^{a-1}\frac{1}{r^{2w}}\right]+\frac{\phi(N)}{r^{a-1}(r-1)} \tag{5.82}$$

$$=\phi(N)\frac{(r+1)}{(r-1)}\frac{N}{2}\frac{(r^{2a}-1)}{(r^2-1)}\frac{1}{r^{2(a-1)}}+\frac{\phi(N)}{r^{a-1}(r-1)}$$

$$=\frac{\phi(N)}{(r-1)r^{a-1}}\left[\frac{N}{2}\frac{r}{r-1}\frac{(r^{2a}-1)}{r^a}+1\right]$$

Using (5.49b),

$$C=\left[\frac{N(r-1)}{r}\frac{1}{(r-1)r^{a-1}}\right]\left[\frac{N}{2}\frac{r}{r-1}\frac{(r^{2a}-1)}{r^a}+1\right] \tag{5.83}$$

$\left[\dfrac{N(r-1)}{r}\dfrac{1}{(r-1)r^{a-1}}\right]=1$, and, since $r=2$,

$$\left[\frac{N}{2}\frac{r}{r-1}\frac{(r^{2a}-1)}{r^{a}}+1\right]=r^{2a}$$

$$\therefore C=[1]\left[r^{2a}\right]=N^{2} \quad (5.84)$$

Hence, for N a power of 2, the number of unique MRT coefficients is N^2.

Example 5.8:

Let $N = 8$.

The possible values of $g(k_1,k_2)$ = 1, 2, 4, 8. k_2 has to be greater than or equal to $g(k_1,k_2)$ for a given $g(k_1,k_2)$. When $g(k_1,k_2)$ = 1, & k_2 = 1, and since $l(1)$ = 1, the number of unique frequencies is given, from (5.36), by

$$U(1,1)=\frac{\phi(N)r}{r-1}=8.$$

Similarly, when $g(k_1,k_2)$ = 1, other possible values of k_2 = 2, 4, 8, then the number of unique frequencies is, from (5.37), $U(1,2)$ = 2 , $U(1,4)$ = 1, $U(1,8)$ = 1. Since there are 4 MRT coefficients associated with $g(k_1,k_2)$ = 1, from (5.72), $A(1,1) = 32$, $A(1,2) = 8$, $A(1,4) = 4$, $A(1,8) = 4$. The total number of MRT coefficients for $g(k_1, k_2) = 1$, and all possible values of k_2 = 1, 2, 4, 8 is given by

$$C_1 = A(1,1)+A(1,2)+A(1,4)+A(1,8)=48$$

Similarly, when $g(k_1,k_2) = 2$, the possible values of k_2 are k_2 = 2, 4, 8. When $k_2 = 2$, the number of unique frequencies is given from (5.36) as

$$U(2,2)=\frac{\phi(8)2}{2(2-1)}=4.$$

And for $k_2 = 4$, (5.37) gives $U(2,4) = 1$, $U(2,8) = 1$. $A(2,2) = 8$, $A(2,4) = 2$, $A(2,8) = 2$. Summing, the number of unique MRT coefficients for $g(k_1,k_2) = 2$ is

$$C_2 = A(2,2)+A(2,4)+A(2,8)=12$$

Using similar approach, $U(4,4) = 2$, $U(4,8) = 1$, $A(4,4) = 2$, $A(4,8) = 1$, $U(8,8) = 1$, $A(8,8) = 1$ and $C_4 = A(4,4) + A(4,8) = 3$. Also, $C_8 = A_8 = 1$.

The total number of unique MRT coefficients for $N = 8$ is thus $C = C_1 + C_2 + C_4 + C_8 = 64$.

Hence, for $N = 8$, the number of unique MRT coefficients is $N^2 = 64$.

Case (ii): $N = r_1^{a_1} r_2^{a_2}$

The possible values of $g(k_1,k_2)=r_1^{u_1}r_2^{u_2}$, $0 \le u_1 \le a_1, 0 \le u_2 \le a_2$. Given $g(k_1,k_2)$, k_2 has to be greater than or equal to $g(k_1,k_2)$, hence $k_2 = r_1^{w_1}r_2^{w_2}$, $u_1 \le w_1 \le a_1, u_2 \le w_2 \le a_2$.

The total number of unique 2-D MRT coefficients corresponding to a specific value of $g(k_1,k_2)$, defined by $C_{g(k_1,k_2)}$, is

$$C_{g(k_1,k_2)} = \sum_{i_1=u_1}^{a_1} \sum_{i_2=u_2}^{a_2} A(g(k_1,k_2), r_1^{i_1} r_2^{i_2}) \tag{5.85}$$

In the following, only a few of the possible values of $g(k_1,k_2)$ are considered in detail since the analysis is similar for the remaining values. Similarly, for a specific value of $g(k_1,k_2)$, analysis is done only for a few possible values of k_2 in order to avoid repeating similar steps.

To begin with, let $g(k_1,k_2) = 1$.

When $k_2 = 1$, from (5.57), the number of unique MRT coefficients is given by

$$A(1,1) = U(1,1)\frac{N}{2} = \phi(N)\frac{r_1}{r_1-1}\frac{r_2}{r_2-1}\frac{N}{2} \tag{5.86}$$

For $k_2 = r_2$, from (5.55),

$$A(1,r_2) = U(1,r_2)\frac{N}{2} = \frac{\phi(N)}{r_2}\frac{r_1}{r_1-1}\frac{N}{2}$$

Thus, in general, for $k_2 = r_2^{w_2}$,

$$A(1,r_2^{w_2}) = U(1,r_2^{w_2})\frac{N}{2} = \frac{\phi(N)}{r_2^{w_2}}\frac{r_1}{r_1-1}\frac{N}{2}, \qquad 1 \le w_2 \le a_2 - 1$$

And for $k_2 = r_2^{a_2}$,

$$A(1,r_2^{a_2}) = U(1,r_2^{a_2})\frac{N}{2} = \frac{\phi(N)}{r_2^{a_2-1}(r_2-1)}\frac{r_1}{r_1-1}\frac{N}{2} \tag{5.87}$$

Similarly, when $k_2 = r_1$,

$$A(1,r_1) = U(1,r_1)\frac{N}{2} = \frac{\phi(N)}{r_1}\frac{r_2}{r_2-1}\frac{N}{2} \tag{5.88}$$

$$A(1,r_1^{w_1}) = U(1,r_1^{w_1})\frac{N}{2} = \frac{\phi(N)}{r_1^{w_1}}\frac{r_2}{r_2-1}\frac{N}{2}, \qquad 1 \le w_1 \le a_1 - 1$$

$$A(1,r_1^{a_1}) = U(1,r_1^{a_1})\frac{N}{2} = \frac{\phi(N)}{r_1^{a_1-1}(r_1-1)}\frac{r_2}{r_2-1}\frac{N}{2}$$

For $k_2 = r_1 r_2$,

$$A(1,r_1r_2) = U(1,r_1r_2)\frac{N}{2} = \frac{\phi(N)}{r_1r_2}\frac{N}{2} \tag{5.89}$$

$$\therefore A(1,r_1^{w_1}r_2^{w_2}) = U(1,r_1^{w_1}r_2^{w_2})\frac{N}{2} = \frac{\phi(N)}{r_1^{w_1}r_2^{w_2}}\frac{N}{2}, \qquad 1 \le w_1 \le a_1 - 1, \quad 1 \le w_2 \le a_2 - 1$$

$$\therefore A(1,r_1^{w_1}r_2^{a_2}) = U(1,r_1^{w_1}r_2^{a_2})\frac{N}{2} = \frac{\phi(N)}{r_1^{w_1}r_2^{a_2-1}(r_2-1)}\frac{N}{2}, \qquad 1 \le w_1 \le a_1 - 1$$

$$\therefore A(1,r_1^{a_1}r_2^{w_2}) = U(1,r_1^{a_1}r_2^{w_2})\frac{N}{2} = \frac{\phi(N)}{r_1^{a_1-1}(r_1-1)r_2^{w_2}}\frac{N}{2}, \qquad 1 \le w_2 \le a_2 - 1$$

$$A(1,r_1^{a_1}r_2^{a_2}) = U(1,r_1^{a_1}r_2^{a_2})\frac{N}{2} = \frac{\phi(N)}{r_1^{a_1-1}(r_1-1)r_2^{a_2-1}(r_2-1)}\frac{N}{2}$$

The summation of $A(1, k_2)$ over all possible values of k_2 gives

$$C_1 = \phi(N)\frac{N}{2}\left[\left[\frac{r_1}{r_1-1}\frac{r_2}{r_2-1} + \frac{1}{r_2}\frac{r_1}{r_1-1} + \frac{1}{r_2^2}\frac{r_1}{r_1-1} + \ldots\frac{1}{r_2^{a_2-1}(r_2-1)}\frac{r_1}{r_1-1}\right]\right.$$

$$+\left[\frac{1}{r_1}\frac{r_2}{r_2-1} + \frac{1}{r_1}\frac{1}{r_2} + \frac{1}{r_1}\frac{1}{r_2^2} + \ldots\frac{1}{r_1}\frac{1}{r_2^{a_2-1}(r_2-1)}\right] + \left[\frac{1}{r_1^2}\frac{r_2}{r_2-1} + \frac{1}{r_1^2}\frac{1}{r_2} + \frac{1}{r_1^2}\frac{1}{r_2^2} + \ldots\frac{1}{r_1^2}\frac{1}{r_2^{a_2-1}(r_2-1)}\right]$$

$$+\ldots\ldots\left[\frac{1}{r_1^{a_1-1}}\frac{r_2}{r_2-1} + \frac{1}{r_1^{a_1-1}}\frac{1}{r_2} + \frac{1}{r_1^{a_1-1}}\frac{1}{r_2^2} + \ldots\frac{1}{r_1^{a_1-1}}\frac{1}{r_2^{a_2-1}(r_2-1)}\right]$$

$$\left.+\left[\frac{1}{r_1^{a_1-1}(r_1-1)}\frac{r_2}{r_2-1} + \frac{1}{r_1^{a_1-1}(r_1-1)}\frac{1}{r_2} + \frac{1}{r_1^{a_1-1}(r_1-1)}\frac{1}{r_2^2} + \ldots\frac{1}{r_1^{a_1-1}(r_1-1)}\frac{1}{r_2^{a_2-1}(r_2-1)}\right]\right] \tag{5.90}$$

$$= \phi(N)\frac{N}{2}\left[\frac{r_1}{r_1-1} + \frac{1}{r_1} + \frac{1}{r_1^2} + \ldots\frac{1}{r_1^{a_1-1}(r_1-1)}\right]\left[\frac{r_2}{r_2-1} + \frac{1}{r_2} + \frac{1}{r_2^2} + \ldots\frac{1}{r_2^{a_2-1}(r_2-1)}\right]$$

Using (5.76),

$$C_1 = \phi(N)\frac{N}{2}\left[\frac{r_1+1}{r_1-1}\right]\left[\frac{r_2+1}{r_2-1}\right] \tag{5.91}$$

Next, the case of $g(k_1,k_2) = r_2$ is considered. For $k_2 = r_2$, the number of unique MRT coefficients is

$$A(r_2,r_2) = U(r_2,r_2)\frac{N}{2r_2} = \frac{\phi(N)}{r_2}\frac{r_1}{r_1-1}\frac{r_2}{r_2-1}\frac{N}{2r_2} \tag{5.92}$$

For $k_2 = r_2^2$,

$$A(r_2,r_2^2) = U(r_2,r_2^2)\frac{N}{2r_2} = \frac{\phi(N)}{r_2^2}\frac{r_1}{r_1-1}\frac{N}{2r_2} \tag{5.93}$$

Generalizing,

$$A(r_2, r_2^{w_2}) = U(r_2, r_2^{w_2})\frac{N}{2r_2} = \frac{\phi(N)}{r_2^{w_2}}\frac{r_1}{r_1-1}\frac{N}{2r_2}, \quad 1 \le w_2 \le a_2 - 1 \tag{5.93a}$$

For $k_2 = r_2^{a_2}$,

$$A(r_2, r_2^{a_2}) = \frac{\phi(N)}{r_2^{a_2-1}(r_2-1)}\frac{r_1}{r_1-1}\frac{N}{2r_2} \tag{5.94}$$

For $k_2 = r_1 r_2$,

$$A(r_2, r_1 r_2) = \frac{\phi(N)}{r_1 r_2}\frac{r_2}{r_2-1}\frac{N}{2r_2} \tag{5.95}$$

Generalizing,

$$A(r_2, r_1^{w_1} r_2) = \frac{\phi(N)}{r_1^{w_1} r_2}\frac{r_2}{r_2-1}\frac{N}{2r_2}, \quad 1 \le w_1 \le a_1 - 1$$

$$A(r_2, r_1^{a_1} r_2) = \frac{\phi(N)}{r_1^{a_1-1}(r_1-1) r_2}\frac{r_2}{r_2-1}\frac{N}{2r_2}$$

For $k_2 = r_1 r_2^2$,

$$A(r_2, r_1 r_2^2) = \frac{\phi(N)}{r_1 r_2^2}\frac{N}{2r_2} \tag{5.96}$$

Generalizing,

$$A(r_2, r_1^{w_1} r_2^{w_2}) = \frac{\phi(N)}{r_1^{w_1} r_2^{w_2}}\frac{N}{2r_2}, \quad 1 \le w_1 \le a_1 - 1, \ \ 2 \le w_2 \le a_2 - 1$$

$$A(r_2, r_1^{a_1} r_2^{w_2}) = \frac{\phi(N)}{r_1^{a_1-1}(r_1-1) r_2^{w_2}}\frac{N}{2r_2}, \quad 2 \le w_2 \le a_2 - 1$$

$$A(r_2, r_1^{w_1} r_2^{a_2}) = \frac{\phi(N)}{r_1^{w_1} r_2^{a_2-1}(r_2-1)}\frac{N}{2r_2}, \quad 1 \le w_1 \le a_1 - 1$$

$$A(r_2, r_1^{a_1} r_2^{a_2}) = \frac{\phi(N)}{r_1^{a_1-1}(r_1-1) r_2^{a_2-1}(r_2-1)}\frac{N}{2r_2}$$

Summing the terms $A(r_2, k_2)$ for all possible values of k_2,

$$C_{r_2} = \phi(N)\frac{N}{2r_2}\left[\left[\frac{r_1}{r_1-1}\frac{r_2}{r_2-1}\frac{1}{r_2} + \frac{1}{r_2^2}\frac{r_1}{r_1-1} + \frac{1}{r_2^3}\frac{r_1}{r_1-1} + \dots + \frac{1}{r_2^{a_2-1}(r_2-1)}\frac{r_1}{r_1-1}\right]\right.$$

$$+\left[\frac{1}{r_1 r_2}\frac{r_2}{r_2-1} + \frac{1}{r_1 r_2^2} + \frac{1}{r_1 r_2^3} + \dots + \frac{1}{r_1}\frac{1}{r_2^{a_2-1}(r_2-1)}\right]$$

$$+\left[\frac{1}{r_1^2 r_2}\frac{r_2}{r_2-1} + \frac{1}{r_1^2 r_2^2} + \frac{1}{r_1^2 r_2^3} + \dots + \frac{1}{r_1^2}\frac{1}{r_2^{a_2-1}(r_2-1)}\right]$$

$$+\ldots\ldots\left[\frac{1}{r_1^{a_1-1}}\frac{r_2}{r_2-1}\frac{1}{r_2}+\frac{1}{r_1^{a_1-1}}\frac{1}{r_2^2}+\frac{1}{r_1^{a_1-1}}\frac{1}{r_2^3}+\ldots+\frac{1}{r_1^{a_1-1}}\frac{1}{r_2^{a_2-1}(r_2-1)}\right]$$

$$+\left[\frac{1}{r_1^{a_1-1}(r_1-1)}\frac{r_2}{r_2-1}\frac{1}{r_2}+\frac{1}{r_1^{a_1-1}(r_1-1)}\frac{1}{r_2^2}+\frac{1}{r_1^{a_1-1}(r_1-1)}\frac{1}{r_2^3}+\ldots+\frac{1}{r_1^{a_1-1}(r_1-1)}\frac{1}{r_2^{a_2-1}(r_2-1)}\right]\Bigg] \tag{5.97}$$

$$=\phi(N)\frac{N}{2r_2}\frac{1}{r_2}\left[\frac{r_1}{r_1-1}+\frac{1}{r_1}+\frac{1}{r_1^2}+\ldots+\frac{1}{r_1^{a_1-1}(r_1-1)}\right]\left[\frac{r_2}{r_2-1}+\frac{1}{r_2}+\frac{1}{r_2^2}+\ldots+\frac{1}{r_2^{a_2-1}(r_2-1)}\right]$$

$$=\phi(N)\frac{N}{2r_2}\frac{1}{r_2}\left[\frac{r_1+1}{r_1-1}\right]\left[\frac{r_2+1}{r_2-1}\right] \tag{5.98}$$

In (5.98), the term $N/2r_2$ corresponds to the value of $g(k_1,k_2)$ and the term $(1/r_2)$ corresponds to $l(g(k_1, k_2))$. Thus, in general, if $g(k_1,k_2)=r_1^{u_1}r_2^{u_2}$ and $N=r_1^{a_1}r_2^{a_2}$, and if $u_1 < a_1$, $u_2 < a_2$, the number of unique MRT coefficients corresponding to this value of $g(k_1,k_2)$ is given by

$$C_{r_1^{u_1}r_2^{u_2}}=\phi(N)\frac{N}{2r_1^{u_1}r_2^{u_2}}\frac{1}{r_1^{u_1}r_2^{u_2}}\frac{r_1+1}{r_1-1}\frac{r_2+1}{r_2-1},\quad u_1<a_1,u_2<a_2 \tag{5.99}$$

Similarly, when $g(k_1,k_2)=r_2^{a_2}$, the number of unique MRT coefficients is

$$C_{r_2^{a_2}}=\phi(N)\frac{N}{2r_2^{a_2}}\frac{1}{r_2^{a_2-1}(r_2-1)}\frac{r_1+1}{r_1-1} \tag{5.100}$$

And, when $g(k_1,k_2)=r_1r_2^{a_2}$, this is given by

$$C_{r_1r_2^{a_2}}=\phi(N)\frac{N}{2r_1r_2^{a_2}}\frac{1}{r_1r_2^{a_2-1}(r_2-1)}\frac{r_1+1}{r_1-1} \tag{5.101}$$

Since N is even, either $r_1 = 2$, or $r_2 = 2$. Assume $r_1 = 2$.

When $g(k_1,k_2)=r_1^{a_1}r_2^{w_2},\ 0\le w_2\le a_2$, $g(k_1,k_2)\,|\,M$ is not satisfied.

Hence, when $g(k_1,k_2)=r_1^{a_1}r_2^{w_2},\quad 0\le w_2\le a_2-1$,

$$C_{r_1^{a_1}r_2^{w_2}}=\phi(N)\frac{N}{r_1^{a_1}r_2^{w_2}}\frac{1}{r_1^{a_1-1}(r_1-1)r_2^{w_2}}\frac{(r_2+1)}{(r_2-1)}$$

Finally, when $g(k_1,k_2)=r_1^{a_1}r_2^{a_2}$,

$$C_{r_1^{a_1}r_2^{a_2}}=\phi(N)\frac{N}{r_1^{a_1}r_2^{a_2}}\frac{1}{r_1^{a_1-1}(r_1-1)r_2^{a_2-1}(r_2-1)} \tag{5.102}$$

Summing the terms corresponding to all the possible values of $g(k_1,k_2)$, the total number of unique MRT coefficients, defined by C, can be obtained as

$$C=\sum_{i_1=0}^{a_1}\sum_{i_2=0}^{a_2}C_{r_1^{i_1}r_2^{i_2}}$$

$$\therefore C=\phi(N)\frac{N}{2}\frac{r_1+1}{r_1-1}\frac{r_2+1}{r_2-1}+\phi(N)\frac{N}{2r_2^2}\frac{r_1+1}{r_1-1}\frac{r_2+1}{r_2-1}+\phi(N)\frac{N}{2r_2^4}\frac{r_1+1}{r_1-1}\frac{r_2+1}{r_2-1} \tag{5.103}$$

$$+\ldots..\phi(N)\frac{N}{2r_2^{(a_2-1)^2}}\frac{r_1+1}{r_1-1}\frac{r_2+1}{r_2-1}+\phi(N)\frac{N}{2r_2^{a_2}}\frac{1}{r_2^{(a_2-1)}r_2-1}\frac{r_1+1}{r_1-1}$$

$$+\phi(N)\frac{N}{2r_1^2}\frac{r_1+1}{r_1-1}\frac{r_2+1}{r_2-1}+\phi(N)\frac{N}{2r_1^2r_2^2}\frac{r_1+1}{r_1-1}\frac{r_2+1}{r_2-1}+\ldots\ldots.\phi(N)\frac{N}{2r_1^2}\frac{1}{r_2^{a_2}r_2^{a_2-1}(r_2-1)}\frac{r_1+1}{r_1-1}$$

$$+\phi(N)\frac{N}{2r_1^4}\frac{r_1+1}{r_1-1}\frac{r_2+1}{r_2-1}+\phi(N)\frac{N}{2r_1^4r_2^2}\frac{r_1+1}{r_1-1}\frac{r_2+1}{r_2-1}+\ldots\ldots.\phi(N)\frac{N}{2r_1^4}\frac{1}{r_2^{a_2}r_2^{a_2-1}(r_2-1)}\frac{r_1+1}{r_1-1}$$

$$+\ldots\ldots\ldots\phi(N)\frac{N}{2r_1^{(a_1-1)^2}}\frac{r_1+1}{r_1-1}\frac{r_2+1}{r_2-1}+\phi(N)\frac{N}{2r_1^{(a_1-1)^2}r_2^2}\frac{r_1+1}{r_1-1}\frac{r_2+1}{r_2-1}$$

$$+\ldots.\phi(N)\frac{N}{2r_1^{(a_1-1)^2}}\frac{1}{r_2^{a_2}r_2^{a_2-1}(r_2-1)}\frac{r_1+1}{r_1-1}$$

$$+\phi(N)\frac{N}{r_1^{a_1}r_1^{(a_1-1)}(r_1-1)}\frac{r_2+1}{r_2-1}+\phi(N)\frac{N}{r_1^{a_1}r_1^{(a_1-1)}(r_1-1)r_2^2}\frac{r_2+1}{r_2-1}$$

$$+\ldots\ldots.\phi(N)\frac{N}{r_1^{a_1}r_1^{(a_1-1)}(r_1-1)}\frac{1}{r_2^{a_2}r_2^{a_2-1}(r_2-1)}$$

$$=\phi(N)\frac{N}{2}\Bigg[\frac{r_1+1}{r_1-1}\Bigg[\frac{r_2+1}{r_2-1}\Bigg[1+\frac{1}{r_2^2}+\frac{1}{r_2^4}+\ldots..\frac{1}{r_2^{(a_2-1)^2}}\Bigg]+\frac{r_2}{r_2-1}\frac{1}{r_2^{a_2^2}}\Bigg]$$

$$+\frac{1}{r_1^2}\frac{r_1+1}{r_1-1}\Bigg[\frac{r_2+1}{r_2-1}\Bigg[1+\frac{1}{r_2^2}+\frac{1}{r_2^4}+\ldots..\frac{1}{r_2^{(a_2-1)^2}}\Bigg]+\frac{r_2}{r_2-1}\frac{1}{r_2^{a_2^2}}\Bigg]$$

$$+\frac{1}{r_1^4}\frac{r_1+1}{r_1-1}\Bigg[\frac{r_2+1}{r_2-1}\Bigg[1+\frac{1}{r_2^2}+\frac{1}{r_2^4}+\ldots..\frac{1}{r_2^{(a_2-1)^2}}\Bigg]+\frac{r_2}{r_2-1}\frac{1}{r_2^{a_2^2}}\Bigg]$$

$$+\ldots..\frac{1}{r_1^{(a_1-1)^2}}\frac{r_1+1}{r_1-1}\Bigg[\frac{r_2+1}{r_2-1}\Bigg[1+\frac{1}{r_2^2}+\frac{1}{r_2^4}+\ldots..\frac{1}{r_2^{(a_2-1)^2}}\Bigg]+\frac{r_2}{r_2-1}\frac{1}{r_2^{a_2^2}}\Bigg]\Bigg]$$

$$+\phi(N)\frac{N}{r_1^{a_1^2}}\frac{r_1}{r_1-1}\Bigg[\frac{r_2+1}{r_2-1}\Bigg[1+\frac{1}{r_2^2}+\frac{1}{r_2^4}+\ldots..\frac{1}{r_2^{(a_2-1)^2}}\Bigg]+\frac{r_2}{r_2-1}\frac{1}{r_2^{a_2^2}}\Bigg] \tag{5.104}$$

$$\frac{r_2+1}{r_2-1}\left[1+\frac{1}{r_2^2}+\frac{1}{r_2^4}+\ldots..\frac{1}{r_2^{(a_2-1)^2}}\right]+\frac{r_2}{r_2-1}\frac{1}{r_2^{a_2^2}}=\frac{r_2+1}{r_2-1}\frac{r_2^{2a_2}-1}{r_2^2-1}\frac{1}{r_2^{2a_2-2}}+\frac{r_2}{r_2-1}\frac{1}{r_2^{a_2^2}}$$

$$=\frac{r_2^{2a_2}-1}{(r_2-1)^2}\frac{r_2^2}{r_2^{2a_2}}+\frac{r_2}{r_2-1}\frac{1}{r_2^{a_2^2}}$$

$$=\frac{r_2}{(r_2-1)r_2^{a_2^2}}\left[\frac{(r_2^{2a_2}-1)r_2}{(r_2-1)}+1\right] \tag{5.105}$$

$$C=\phi(N)\frac{N}{2}\left[\frac{r_1+1}{r_1-1}\left[1+\frac{1}{r_1^2}+\frac{1}{r_1^4}+\ldots..\frac{1}{r_1^{(a_1-1)^2}}\right]\left[\frac{r_2}{(r_2-1)r_2^{a_2^2}}\left[\frac{(r_2^{2a_2}-1)r_2}{(r_2-1)}+1\right]\right]\right]$$

$$+\phi(N)N\frac{1}{r_1^{a_1^2}}\frac{r_1}{r_1-1}\left[\frac{r_2}{(r_2-1)r_2^{a_2^2}}\left[\frac{(r_2^{2a_2}-1)r_2}{(r_2-1)}+1\right]\right] \tag{5.106}$$

$$C=\phi(N)N\left[\left[\frac{1}{2}\frac{r_1+1}{r_1-1}\left[1+\frac{1}{r_1^2}+\frac{1}{r_1^4}+\ldots..\frac{1}{r_1^{(a_1-1)^2}}\right]+\frac{1}{r_1^{a_1^2}}\frac{r_1}{r_1-1}\right]\left[\frac{r_2}{(r_2-1)r_2^{a_2^2}}\left[\frac{(r_2^{2a_2}-1)r_2}{(r_2-1)}+1\right]\right]\right]$$

$$=\phi(N)N\left[\left[\frac{r_1}{(r_1-1)r_1^{a_1^2}}\left[\frac{(r_1^{2a_1}-1)r_1}{2(r_1-1)}+1\right]\right]\left[\frac{r_2}{(r_2-1)r_2^{a_2^2}}\left[\frac{(r_2^{2a_2}-1)r_2}{(r_2-1)}+1\right]\right]\right]$$

$$=\phi(N)N\left[\left[\frac{r_1}{(r_1-1)r_1^{a_1^2}}\left[\frac{(r_1^{2a_1}-1)r_1}{2(r_1-1)}+1\right]\right]\left[\frac{r_2}{(r_2-1)r_2^{a_2^2}}\left[\frac{(r_2^{2a_2}-1)r_2}{(r_2-1)}+1\right]\right]\right] \tag{5.107}$$

$$\phi(N)N=\frac{N(r_1-1)(r_2-1)}{r_1r_2}N=\frac{r_1^{a_1^2}r_2^{a_2^2}(r_1-1)(r_2-1)}{r_1r_2} \tag{5.108}$$

$$\therefore C=\frac{r_1^{a_1^2}r_2^{a_2^2}(r_1-1)(r_2-1)}{r_1r_2}\frac{r_1}{(r_1-1)r_1^{a_1^2}}\frac{r_2}{(r_2-1)r_2^{a_2^2}}\left[\frac{(r_1^{2a_1}-1)r_1}{2(r_1-1)}+1\right]\left[\frac{(r_2^{2a_2}-1)r_2}{(r_2-1)}+1\right] \tag{5.109}$$

$$=\left[\frac{(r_1^{2a_1}-1)r_1}{2(r_1-1)}+1\right]\left[\frac{(r_2^{2a_2}-1)r_2}{(r_2-1)}+1\right] \tag{5.110}$$

This analysis, extended along similar lines, to N that has q prime factors would give, for $N=\prod_{i=1}^{q}r_i^{a_i}$,

$$C=\left[\frac{(r_1^{2a_1}-1)r_1}{2(r_1-1)}+1\right]\prod_{i=2}^{q}\left[\frac{(r_i^{2a_i}-1)r_i}{(r_i-1)}+1\right] \tag{5.111}$$

Since, $r_1 = 2$, the term $\left[\frac{(r_1^{2a_1}-1)r_1}{2(r_1-1)}+1\right]$ reduces to 2^{2a_1}, so that (5.111) can be re-written as

$$C = 2^{2a_1}\prod_{i=2}^{q}\left[\frac{(r_i^{2a_i}-1)r_i}{(r_i-1)}+1\right], \text{ given } N = 2^{a_1}\prod_{i=2}^{q}r_i^{a_i} \tag{5.112}$$

Example 5.9:

This example illustrates how to calculate the number of unique MRT coefficients for a given value of N, and N is chosen as 6. Since the divisors of 6 are 1, 2, 3 & 6, the possible values of $g(k_1, k_2) = 1, 2, 3, 6$.

For $g(k_1,k_2) = 1$, the possible values of $k_2 = 1, 2, 3, 6$.

From (5.36) & (5.72a), $U(1,1) = 6$, $A(1,1) = (U(1,1))(3) = 18$

From (5.37) & (5.72b), $U(1,2) = 3$, $A(1,2) = (U(1,2))(3) = 9$

$U(1,3) = 2$, $A(1,3) = (U(1,3))(3) = 6$

$U(1,6) = 1$, $A(1,6) = (U(1,6))(3) = 3$

Thus,

$C_1 = A(1,1) + A(1,2) + A(1,3) + A(1,6) = 36$, which can be verified using (5.91) or (5.99).

Similarly,

$U(2,2) = 3$, $A(2,2) = (U(2,2))(3) = 9$

$U(2,6) = 1$, $A(2,6) = (U(2,6))(3) = 3$

$C_2 = A(2,2) + A(2,6) = 12$

$U(3,3) = 2$, $A(3,3) = (U(3,3))(1) = 2$

$U(3,6) = 1$, $A(3,6) = (U(3,6))(1) = 1$

$C_3 = A(3,3) + A(3,6) = 3$

$U(6,6) = 1$, $A(6,6) = (U(6,6))(1) = 1$

$C_6 = A(6,6) = 1$

$C = C_1 + C_2 + C_3 + C_6 = 52$, as can be verified using (5.112).

Table 5.3 shows the total number of MRT coefficients and the corresponding number of unique MRT coefficients calculated using (5.112) for various values of N. For N a power of 2, there are N^2 UMRT coefficients. An expansion factor can be calculated by taking the ratio of the RHS of (5.112) to N^2. This is defined as E to be

$$E = \frac{C}{N^2} = \frac{2^{2a_1}\prod_{i=2}^{q}\left[\frac{(r_i^{2a_i}-1)r_i}{(r_i-1)}+1\right]}{2^{2a_1}\prod_{i=2}^{q}r_i^{2a_i}} = \prod_{i=2}^{q}\left[\frac{(r_i^{2a_i+1}-1)}{r_i^{2a_i}(r_i-1)}\right]$$

$$=\prod_{i=2}^{q}\left[\frac{(r_i^{2a_i}+r_i^{2a_i-1}+r_i^{2a_i-2}+\ldots 1)}{r_i^{2a_i}}\right]=\prod_{i=2}^{q}\left[1+\frac{1}{r}+\frac{1}{r^2}+\ldots\frac{1}{r^{2a_i}}\right] \tag{5.112a}$$

Table 5.3: Number of unique 2-D MRT coefficients for various values of N

N	N in terms of prime divisors	Number of MRT coefficients	Number of UMRT coefficients	E	N	N in terms of prime divisors	Number of MRT coefficients	Number of UMRT coefficients	E
4	2^2	32	16	1	42	2 3 7	37044	2964	1.68
6	2 3	108	52	1.44	60	2^2 3 5	108000	6448	1.79
8	2^3	256	64	1	64	2^6	131072	4096	1
10	2 5	500	124	1.24	80	2^4 5	256000	7936	1.24
12	$2^2$3	864	208	1.44	100	2^2 5^2	500000	12496	1.25
14	2 7	1372	228	1.16	128	2^7	1048576	16384	1
16	2^4	2048	256	1	144	2^3 3^2	1492992	30976	1.49
18	2 3^2	2916	484	1.49	180	2^2 3^2 5	2916000	60016	1.85
20	2^2 5	4000	496	1.24	200	2^3 5^2	4000000	49984	1.25
24	2^3 3	6912	832	1.44	210	2 3 5 7	4630500	91884	1.98
30	2 3 5	13500	1612	1.79	256	2^8	8388608	65536	1
32	2^5	16384	1024	1	324	2^2 3^4	17006112	157456	1.5
36	2^2 3^2	23328	1936	1.49	500	2^2 5^3	62500000	312496	1.25
40	2^3 5	32000	1984	1.24	512	2^9	67108864	262144	1

From Table 5.3, it can be inferred that the expansion factor increases in proportion to the number of distinct prime divisors of N. For a given number of prime divisors, there is also an increase in expansion factor in proportion to the exponents of the prime divisors, although this increase is relatively less rapid when compared to increase of number of prime divisors.

Thus the number of unique frequencies and the number of unique MRT coefficients corresponding to the unique frequencies are identified in section 5.2. 2-D signal can be represented in terms of the UMRT coefficients corresponding to the set of unique frequencies by removing redundancy and is explained in Section 5.3.

5.3 Signal representation using 2-D UMRT coefficients

The number of unique MRT coefficients corresponding to a 2-D signal has been presented in section 5.2.6. It was found that when N is a power of 2, the number of unique MRT coefficients is

N^2 and is same as the number of signal samples. All other MRT coefficients can be derived from this set of N^2 coefficients. These coefficients may hence be considered as unique MRT (UMRT) representation for 2-D signals, yielding a new, real non-expansive 2-D signal transform.

5.3.1 Choosing Unique Frequencies

The number of unique frequencies and the number of UMRT coefficients have been obtained in (5.71) and (5.112) respectively. One possible method to obtain the set of unique frequencies is to choose all powers of 2 as values for k_2 and to find out the possible values of k_1 for each k_2. Let $N = 2^a$, $k_2 = 2^w$, $k_1 = 2^v$. To identify the unique frequencies (k_1,k_2), the following cases are considered.

(1) $\boldsymbol{k_2 < N, g(k_1, k_2) < k_2}$: Since $g(k_1,k_2) < k_2$, $k_1 = g(k_1,k_2)$, hence $l(k_1) = l(g(k_1, k_2))$; since $k_2 < N$, $l(k_2) = k_2$. Hence, from (5.20), the number of unique frequencies is given by $\phi(N)/k_2 = N/2k_2$, since $\phi(N) = N/2$, for N a power of 2. Thus, out of the $\phi(N/k_1)$ elements in the totative set of N/k_1, only $N/2k_2$ are required to generate the unique frequencies. Hence, there is redundancy relation between elements in totative set of $\phi(N/k_1)$ w.r.t. column k_2. If $Z_{N/k1}$ and Z_N are defined as the totative sets of N/k_1 and N respectively, h_1 & h_2 are two elements in $Z_{N/k1}$, and h is an element of Z_N, and if $(((h_1k_1h))_N, ((k_2h))_N) = (((h_2k_1))_N, k_2)$, then the elements h_1k_1 and h_2k_1 are redundant w.r.t. column k_2. Hence, the condition in which $(((h_1k_1h))_N, ((k_2h))_N) = (((h_2k_1))_N, k_2)$ occurs needs to be explored. The element h needs to be a solution of $((hk_2))_N = k_2$. Solutions are of the form $h = 1 + tN/k_2$, $0 \le t < k_2$. The other relevant equation is $((h_1k_1h))_N = ((h_2k_1))_N$. Let $h_2 = h_1 + d$.

$$((h_1k_1(1 + tN/k_2)))_N = (((h_1 + d)k_1))_N$$
$$((h_1k_1 + h_1k_1tN/k_2))_N = ((h_1k_1 + dk_1))_N$$
$$((h_1k_1tN/k_2))_N = ((dk_1))_N$$
$$((k_1(d - h_1tN/k_2)))_N = 0$$
$$d - h_1tN/k_2 = vN/k_1$$
$$d = h_1tN/k_2 + vN/k_1$$

Using Bezout's lemma,

$$d = g(N/k_2, N/k_1) \tag{5.113}$$

Hence, elements h_1k_1 and h_2k_1 are redundant w.r.t. k_2 if d satisfies (5.113).

The first element in $Z_{N/k1}$ is 1. Hence, if $h_1 = 1$, then $h_2 = h_1 + d = 1 + g(N/k_2, N/k_1)$.

In the present case, $g(N / k_2, N / k_1) = N / k_2$. Hence, if $h_2 = 1 + (N / k_2)$, then there is redundancy between k_1 and $((h_2k_1))_N$ w.r.t. k_2. The serial position of $h_2 = 1 + (N / k_2)$ among the elements of $Z_{N/k1}$ can be found from the distribution of co-primes of N that are powers of 2. As seen from Table 5.4, when N is a power of 2, every odd integer is co-prime to N, and hence it is possible to accurately predict the serial position of any element in Z_N.

Table 5.4: Totatives for a few integers that are powers of 2

N	$\phi(N)$	M	Totatives of N
4	2	2	1,3
6	2	3	1,5
8	4	4	1,3,5,7
10	4	5	1,3,7,9
12	4	6	1,5,7,11
16	8	8	1,3,5,7,9,11,13,15
24	8	12	1,5,7,11,13,17,19,23
32	16	16	1,3,5,7,9,11,13,15,17,19,21,23,25,27,29,31

The first co-prime element is 1, and from then on, every second integer belongs in Z_N. The element at any serial position q_1 is thus given by $(2q_1 - 1)$. Conversely, an element q_2 will have a serial position $(q_2 + 1)/2$ in Z_N. Hence, the serial position of element $h_2 = 1 + (N / k_2)$ is thus $(1 + (N / k_2) + 1)/2 = 1 + (N / 2k_2)$. The number of unique frequencies (k, k_2) in column k_2 such that $g(k, N) = k_1$, is given by $N / 2k_2$. Since the co-prime $h_2 = 1 + (N / k_2)$, at position $1 + (N / 2k_2)$, is redundant w.r.t. $h_1 = 1$, none of the co-primes lesser than this value of h_2 have a corresponding value of h_1 that is lesser than $1 + (N / k_2)$. Hence none of them are redundant w.r.t. k_2. Also, the number of these co-primes is $N / 2k_2$, which is equal to the number of unique frequencies. Hence, these $N / 2k_2$ co-primes can be used to generate the $N / 2k_2$ elements of gcd k_1 w.r.t. N that are unique in column k_2 by multiplying each of them with k_1.

(2) $k_2 < N, g(k_1, k_2) = k_2$: Since $k_2 < N$, $l(k_2) = k_2$. Also, $l(g(k_1, k_2)) = l(k_2)$. When $g(k_1, k_2) = k_2$, the possible values for k_1 are, $k_1 = k_2, 2k_2, 4k_2, \ldots, M, N$. Hence, corresponding values for $l(k_1)$ are, $l(k_1) = k_2, 2k_2, \ldots M, M$, since $l(N) = l(M) = M$. The number of unique frequencies, from (5.20), corresponding to each value of k_1 is hence given by $(N / 2k_2), (N / 4k_2), \ldots, 1, 1$. Using the same arguments used in case 1 above, the unique frequencies can be found by multiplying k_1 with the first $M / l(k_1)$ elements in Z_N.

(3) $\boldsymbol{k_2 = N, g(k_1, k_2) < k_2}$: $k_1 = g(k_1, k_2)$, $l(k_1) = l(g(k_1, k_2))$. $l(k_2) = l(N) = M$. From (5.20), the number of unique frequencies is 1. Hence, the unique frequency can be chosen to be k_1.

(4) $\boldsymbol{k_2 = N, g(k_1, k_2) = k_2 = N}$: $k_1 = N$. $l(k_1) = l(g(k_1, k_2))$. The number of unique frequencies, from (5.20), is 1 and the unique frequency is thus $k_1 = N$.

Example 5.10:

In this example, cases 1-4 are considered for $N = 8$.

(1) $k_2 < N$, $g(k_1,k_2) < k_2$:

$k_2 = 2$, $g(k_1,k_2) = 1$, hence $k_1 = 1$, $l(k_1) = l(g(k_1, k_2)) = 1$, $l(k_2) = 2$. From (5.20), the number of unique frequencies (k,k_2) in column k_2 such that $g(k, N) = k_1$ is 2. The number of elements in gcd set of $k_1 = 1$ is given by $\phi(N/k_1) = \phi(8) = 4$. Hence, out of these 4 elements, only 2 are unique in column 2. $Z_8 = \{1, 3, 5, 7\}$. h_1, h_2 & h are elements in Z_8, and if $(((h_1k_1h))_N, ((k_2h))_N) = (((h_2k_1))_N, k_2)$, then the elements h_1k_1 and h_2k_1 are redundant w.r.t. column k_2. Hence, the condition in which $(((h_1h))_8,((2h))_8) = (((h_2))_8,2)$ occurs needs to be explored. The element h needs to be a solution of $((2h))_8 = 2$. Solutions are of the form $h = 1 + 4t$, $0 \le t < 2$, i.e. $h = 1, 5$. The other relevant equation is $((h_1h))_N = ((h_2))_N$. Let $h_2 = h_1 + d$.

$$((h_1(1 + 4t)))_8 = (((h_1 + d)))_8$$

$$((h_1 + 4h_1t))_8 = ((h_1 + d))_8$$

$$((4h_1t))_8 = ((d))_8$$

$$(((d - 4h_1t)))_8 = 0$$

$$d - 4h_1t = 8v$$

$$d = 4h_1t + 8v$$

Using Bezout's lemma, the smallest value of d is

$$d = g(4, 8) = 4$$

Hence, elements h_1 and h_2 are redundant w.r.t. 2 if $d = 4$.

The first element in Z_8 is 1. Hence, if $h_1 = 1$, then $h_2 = h_1 + d = 1 + 4 = 5$.

Hence, there is redundancy between 1 and 5 w.r.t. 2. The serial position of $h_2 = 5$ among the elements of Z_8 is position 3. The number of unique frequencies (k,k_2) in column k_2 such that $g(k,N) = k_1$, is 2. Since the co-prime $h_2 = 5$, at position 3, is redundant w.r.t. $h_1 = 1$, none of the co-primes lesser than this 5 have a corresponding value of h_1 that is lesser than 5. Hence none of them are redundant w.r.t. k_2. Also, the number of the co-primes is leeser than 5 is 2, which is equal to the number of unique frequencies. Hence, these 2 co-primes, viz. 1 & 3, can be used to generate the 2 elements of gcd 1 w.r.t. N that are unique in column 2 by multiplying each of them with $k_1 = 1$.

(2) $k_2 = 2$, $g(k_1,k_2) = 2$: Since $k_2 < 8$, $l(k_2) = 2$. Also, $l(g(k_1, k_2)) = l(k_2) = 2$. When $g(k_1,k_2) = k_2$, the possible values for k_1 are, $k_1 = 2, 4, 8$. Hence, corresponding values for $l(k_1)$ are, $l(k_1) = 2, 4, 4$, since $l(8) = l(4) = 4$. The number of unique frequencies, from (5.20), corresponding to each value of k_1 is hence given by 2, 1, 1. Using the same arguments used in case 1 above, the unique frequencies can be found by multiplying k_1 with the first $4 / l(k_1)$ elements in Z_8.

(3) $k_2 = 0$, $g(k_1,k_2) < 2$: $k_1 = 2$, $l(k_1) = l(g(k_1,k_2)) = 2$. $l(k_2) = 4$. From (5.20), the number of unique frequencies is 1. Hence, the unique frequency can be chosen to be k_1.

(4) $k_2 = 0$, $g(k_1,k_2) = 0$: $k_1 = 0$. $l(k_1) = l(k_2) = l(g(k_1,k_2)) = 4$. The number of unique frequencies, from (5.20), is 1 and the unique frequency is thus $k_1 = 0$.

From cases 1-4 studied above, the following procedure can be used to generate the UMRT frequencies and the corresponding UMRT coefficients for N a power of 2:

1) For $k_2 = 1$ to N,
2) For $g(k_1, k_2) = 1$ to $k_2/2$, $k_1 = g(k_1,k_2)$ and for $g(k_1,k_2) = k_2$, $k_1 = k_2$ to N
3) Calculate number of frequencies $w = \phi(N)l(g(k_1,k_2)) / l(k_1)l(k_2)$.
4) Multiply the first w elements of Z_N by k_1 to obtain the UMRT frequencies (k,k_2) in column k_2, such that $g(k,N) = k_1$.
5) For $g(k_1,k_2) < N$, the number of valid phase indices is given by $M / g(k_1,k_2)$, and these are given by $p = 0$, $g(k_1,k_2)$, $2g(k_1,k_2)$, $\ldots M - g(k_1,k_2)$ and for $g(k_1,k_2) = N, p = 0$.

Table 5.5 gives the UMRT frequencies and the corresponding number of UMRT coefficients derived using the above procedure for $N = 8$. k2 takes the values 1, 2, 4 & 8. For a particular value of k2, the possible values of g(k1,k2) are enumerated. For each such value of g(k1,k2), the corresponding values of k1 are also shown. The UMRT frequencies corresponding to these values of k1 and k2 are calculated. The total number of unique MRT coefficients is found to be 64. The number of redundancies are also found. The total number of all coefficients taking into account each UMRT coefficient along with its corresponding redundant MRT coefficients (this can be found by summing all elements in the right-most column of Table 5.5) gives a total equal of 216, which when added to the number of zero-valued positions (non-existent MRT coefficients), which is 36 for $N = 8$ (as seen in section 3.3) gives the total number of 2-D MRT coefficients for $N = 8$ as 256, which equals $N^3 / 2$.

Table 5.5: Unique frequencies for $N = 8$

k_2	$g(k_1,k_2)$	k_1	No. of unique frequencies	UMRT frequencies	No. of unique coefficients	No. of redundancies $\phi\,(N/g(k_1,k_2))$	No. of coefficients including redundancies
1	1	1	4	(1,1), (3,1), (5,1), (7,1)	16	4	64
		2	2	(2,1), (6,1)	8		32
		4	1	(4,1)	4		16
		8	1	(0,1)	4		16
2	1	1	2	(1,2), (3,2)	8	4	32
	2	2	2	(2,2), (6,2)	4	2	8
		4	1	(4,2)	2		4
		8	1	(0,2)	2		4
4	1	1	1	(1,4)	4	4	16
	2	2	1	(2,4)	2	2	4
	4	4	1	(4,4)	1	1	1
		8	1	(0,4)	1		1
8	1	1	1	(1,0)	4	4	16
	2	2	1	(2,0)	2	2	4
	4	4	1	(4,0)	1	1	1
	8	8	1	(0,0)	1	1	1

The basis images corresponding to the UMRT coefficients are shown in Figure 5.1, (a) – (c). The corresponding UMRT coefficient is obtained by addition of data elements in white-shaded cells and subtraction of data elements in black-shaded cells in the basis images. The data elements corresponding to grey-shaded cells are not part of the particular coefficient. The UMRT basis images have a spatial-filtering nature, being composed of 1, -1, and 0.

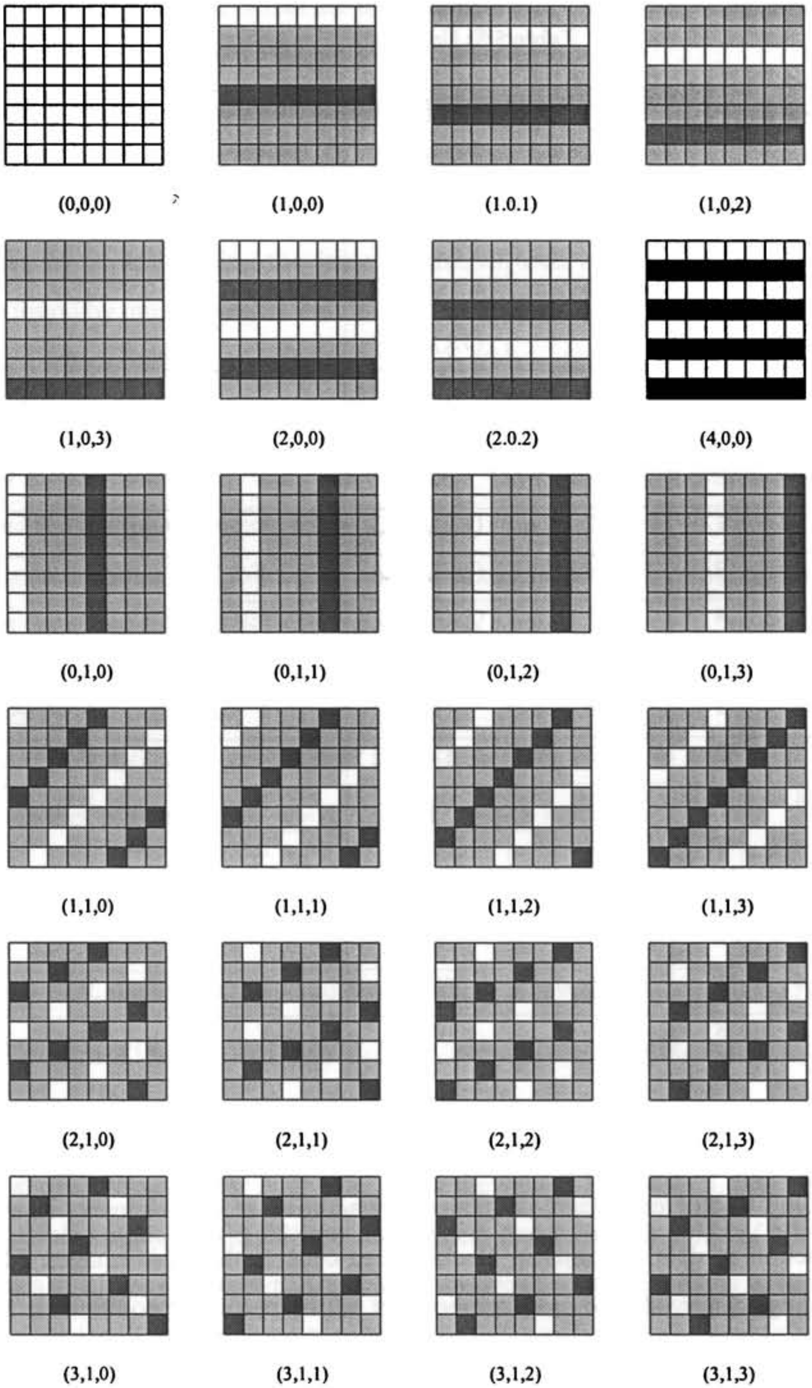


Figure 5.1 (a): Basis images corresponding to the UMRT coefficients for $N = 8$, identified by (k_1, k_2, p)

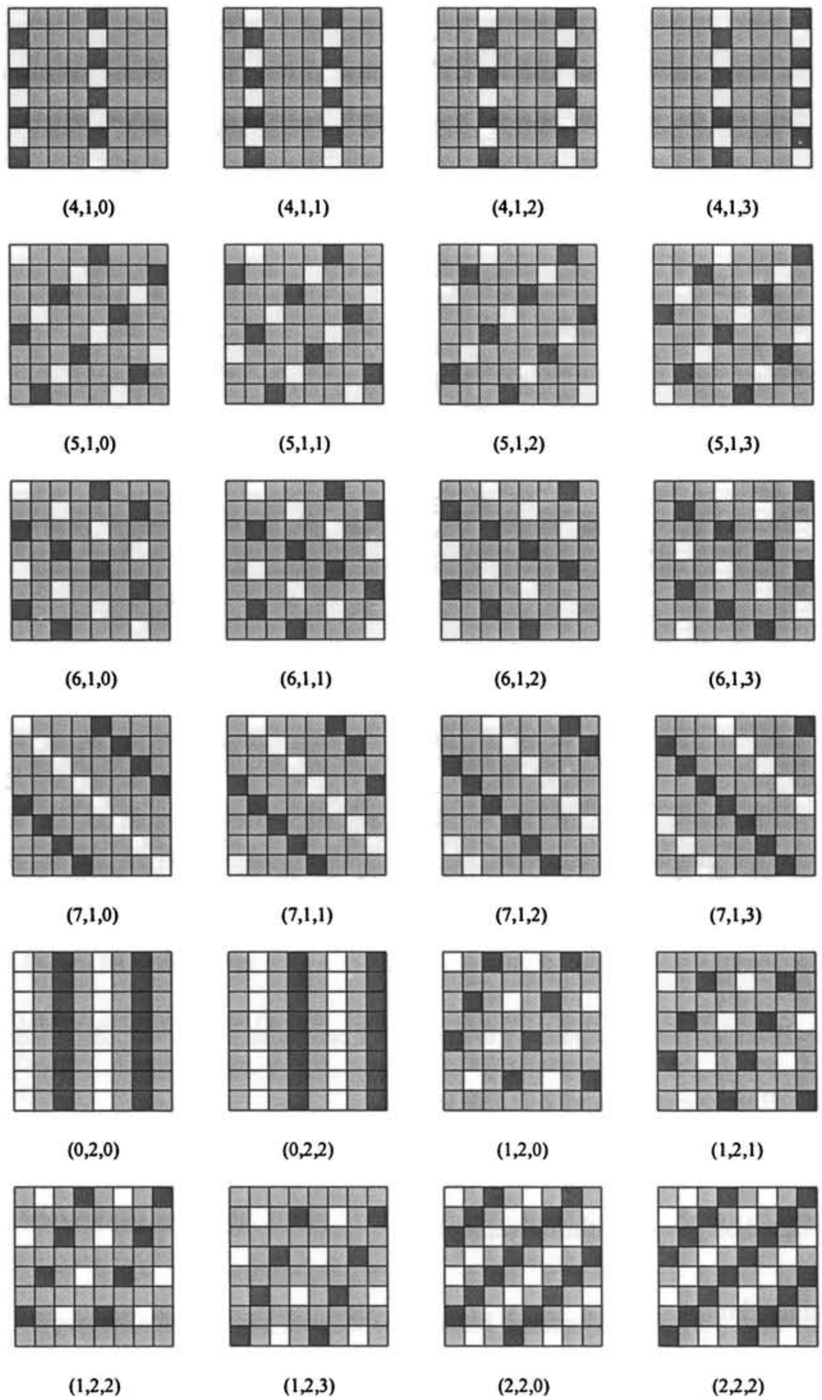


Figure 5.1(b): Basis images corresponding to the UMRT coefficients for $N = 8$, identified by (k_1, k_2, p)

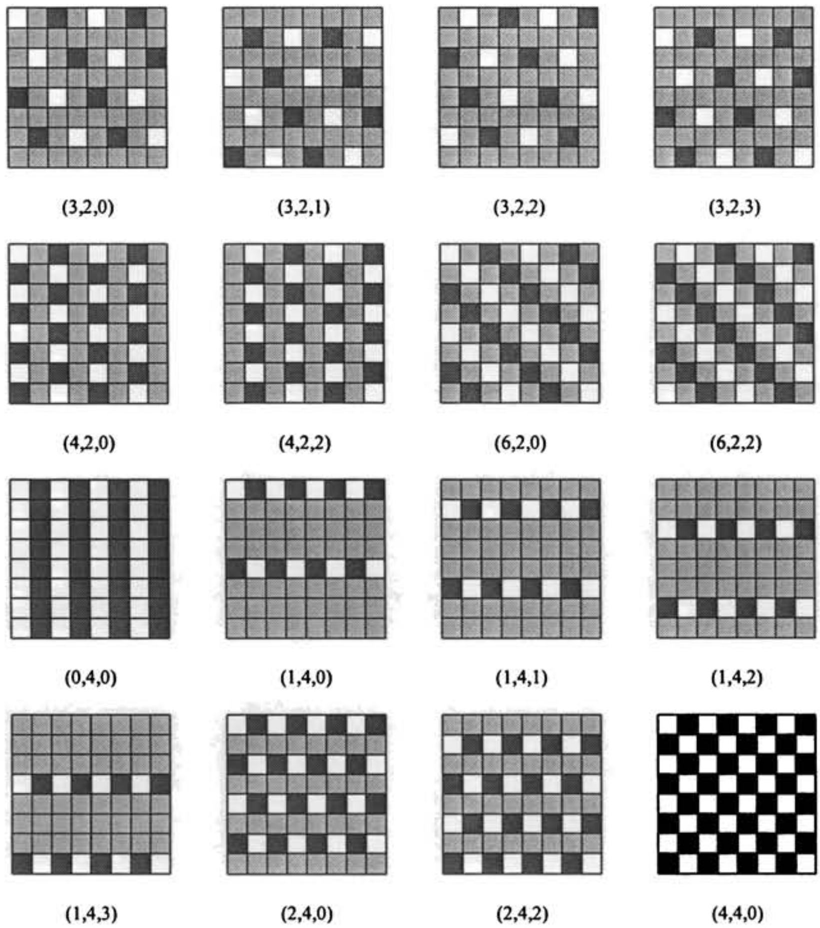


Figure 5.1(c): Basis images corresponding to the UMRT coefficients for $N = 8$, identified by (k_1, k_2, p)

5.3.2 N x N Representation of 2-D UMRT

Two approaches were used to represent the N^2 UMRT coefficients in an N x N matrix. The first approach is based on the algorithm given below.

Placement algorithm:

Initialize $y1\ (N, \log_2 N, M) = 0$, $y(N,N) = 0$,
$k = [0,\ 1,\ 2,\ 4,\ 8\ ,\ ..M\ ]$, $q = [1\ ,\ 2\ ,\ 4\ ..N/4\ ]$, $i = 1$
For $j_2 = 0$ to $\log_2 N$, increment by $\log_2 N$
For $j_1 = 0$ to $\log_2 N$
For $p = 0$ to $M - 1$
$include = 0$, $count = 0$
For $n_1 = 0$ to $N - 1$
For $n_2 = 0$ to $N - 1$
$z = ((n_1 k(j_1) + n_2 k(j_2)))_N$
If $z = p$

$include = 1$, $count = count + 1$
$y1(k(j_1), j_2, p) = y1(k(j_1), j_2, p) + A(n_1, n_2)$
elseif $z = p + M$
$include = 1$, $count = count + 1$
$y1(k(j_1), j_2, p) = y1(k(j_1), j_2, p) - A(n_1, n_2)$
If $include = 1$
$i_1 = \text{round}((i - 1)/N)$, $i_2 = i - i_1(N - 1)$
$y(i_1, i_2) = y1(k(j1), j_2, p)$
$k_1index(i_1, i_2) = k(j_1)$
$k_2index(i_1, i_2) = k(j_2)$, $pindex(i_1, i_2) = p$
$elementcount(i_1, i_2) = count$
$i = i + 1$
For $j_2 = 0$ to $\log_2 N - 2$
$maxrange = \log_2 q(j_2)$, $bottomofrange = 0$
For $numberofranges = 0$ to $maxrange$
$topofrange = N/(2maxrange-numberofranges)$,
$k_1step = 2(numberofranges)$
For $k_1 = bottomofrange$ to
$topofrange - k_1step$, increment by k_1step
For $p = 0$ to $M - 1$
$include = 0$, $count = 0$
For $n_1 = 0$ to $N - 1$
For $n_2 = 0$ to $N - 1$
$z = ((n_1k_1 + n_2q(j_2)))_N$
If $z = p$
$include = 1$, $count = count + 1$
$y1(k_1, j_2, p) = y1(k_1, j_2, p) + A(n_1, n_2)$
elseif $z = p + M$
$include = 1$, $count = count + 1$
$y1(k_1, j_2, p) = y1(k_1, j_2, p) - A(n_1, n_2)$
If $include = 1$
$i_1 = round((i - 1)/N)$, $i_2 = i - i_1(N - 1)$
$y(i_1, i_2) = y1(k_1, j_2, p)$
$k_1index(i_1, i_2) = k_1$
$k_2index(i_1, i_2) = q(j_2)$, $pindex(i_1, i_2) = p$
$elementcount(i_1, i_2) = count$
$i = i + 1$
$y(i_1, i_2) = y(i_1, i_2)/elementcount(i_1, i_2)$

In the algorithm above, $k_1index\ (i_1, i_2)$, $k_2index\ (i_1, i_2)$ and $pindex\ (i_1, i_2)\,(i_1,i_2)$ give the values of k_1, k_2 and p of the UMRT coefficient that will be placed in the $N \times N$ array at position (i_1,i_2). The placement of UMRT coefficients for $N = 8$ in an 8 x 8 array according to the algorithm is shown in Figure 5.2.

i_2

i_1

0,0,0	1,0,0	1,0,1	1,0,2	1,0,3	2,0,0	2,0,2	4,0,0
0,4,0	1,4,0	1,4,1	1,4,2	1,4,3	2,4,0	2,4,2	4,4,0
0,1,0	0,1,1	0,1,2	0,1,3	1,1,0	1,1,1	1,1,2	1,1,3
2,1,0	2,1,1	2,1,2	2,1,3	3,1,0	3,1,1	3,1,2	3,1,3
4,1,0	4,1,1	4,1,2	4,1,3	5,10	5,1,1	5,1,2	5,1,3
6,1,0	6,1,1	6,1,2	6,1,3	7,1,0	7,1,1	7,1,2	7,1,3
0,2,0	0,2,2	1,2,0	1,2,1	1,2,2	1,2,3	2,2,0	2,2,2
3,2,0	3,2,1	3,2,2	3,2,3	4,2,0	4,2,2	6,2,0	6,2,2

Figure 5.2: Positional details of 8 x 8 2-D UMRT matrix formed by placement algorithm, specified by values of (k_1, k_2, p)

From Figure 5.2, it is seen that the algorithm accommodates all UMRT coefficients corresponding to $k_2 = 0$, & 4 in the topmost 2 rows. The next 4 rows hold coefficients corresponding to $k_2 = 1$, and the last 2 rows containing $k_2 = 2$. This approach is is somewhat arbitrary and mechanical since there are no appropriate mathematical relations linking the positions with the UMRT indices. To remove this shortcoming, the following method is proposed for representing the N^2 UMRT coefficients in an N x N matrix and vice versa when N is a power of 2. (u, v) represents the index of the UMRT coefficient $Y_{k_1,k_2}^{(p)}$ in the proposed N x N matrix.

1) $(k_1, k_2, p) \rightarrow (u, v)$

If $k_2 = 0$

$$u = k_1 + 2p \tag{5.114a}$$

$$v = N - 1 \tag{5.114b}$$

If $k_2 \neq 0$

$$u = k_1 + N((p))_{k_2} / k_2 \tag{5.115}$$

$$v = N - N / k_2 + \lfloor p / k_2 \rfloor \tag{5.116}$$

2) $(u, v) \rightarrow (k_1, k_2, p)$

If $v = N - 1$

$$k_1 = ((u))_{2g(u,N)} \tag{5.117}$$

$$k_2 = 0 \tag{5.118}$$

$$p = (u - k_1)/2 \tag{5.119}$$

If $v \neq N-1$

$$k_1 = ((u))_{N\,g(u,N)/k_2} \tag{5.120}$$

$$k = \{1, 2, 4, 8, 16, ...M\}$$

$$k_2 = k \text{ if } N-(N/k) \leq v < N-(N/2k) \tag{5.121}$$

$$p = (u - k_1)(k_2/N) + k_2(v - N + (N/k_2)) \tag{5.122}$$

The placement of UMRT coefficients for $N = 8$ in an 8 x 8 array, according to (5.114) – (5.116) is shown in Figure 5.3.

v →

u ↓

0,1,0	0,1,1	0,1,2	0,1,3	0,2,0	0,2,2	0,4,0	0,0,0
1,1,0	1,1,1	1,1,2	1,1,3	1,2,0	1,2,2	1,4,0	1,0,0
2,1,0	2,1,1	2,1,2	2,1,3	2,2,0	2,2,2	2,4,0	2,0,0
3,1,0	3,1,1	3,1,2	3,1,3	3,2,0	3,2,2	1,4,1	1,0,1
4,1,0	4,1,1	4,1,2	4,1,3	4,20	4,2,2	4,4,0	4,0,0
5,1,0	5,1,1	5,1,2	5,1,3	1,2,1	1,2,3	1,4,2	1,0,2
6,1,0	6,1,1	6,1,2	6,1,3	6,2,0	6,2,2	2,4,2	2,0,2
7,1,0	7,1,1	7,1,2	7,1,3	3,2,1	3,2,3	1,4,3	1,0,3

Figure 5.3: Positional details of 8 x 8 2-D UMRT matrix formed from (5.114) – (5.116), specified by values of (k_1, k_2, p)

5.3.3 Inverse 2-D UMRT

From (3-45) in section 3.7, the formula for inverse 2-D MRT is

$$x_{n_1,n_2} = \frac{1}{N^2}\sum_{p=0}^{M-1}\left[\sum_{\forall(k_1,k_2)\ni z=p} Y_{k_1,k_2}^{(p)} - \sum_{\forall(k_1,k_2)\ni z=p+M} Y_{k_1,k_2}^{(p)}\right] \qquad 0 \leq n_1, n_2 \leq N-1$$

Every data element x_{n_1,n_2} is involved in formation of N^2 MRT coefficients, and these N^2 MRT coefficients are used to obtain the data element in the inverse process. However, from section 5.2.4, an MRT coefficient $Y_{k_1,k_2}^{(p)}$ is completely redundant with $\phi(N/g(k_1,k_2,N))$ MRT coefficients. In other words, there are $\phi(N/g(k_1,k_2,N))$ MRT coefficients that have the same composition of data elements. Hence, in the inverse transform for any of these common data elements, all these $\phi(N/g(k_1,k_2,N))$ MRT coefficients need to be used. Hence, when these $\phi(N/g(k_1,k_2,N))$ MRT coefficients are added together, the sum is equal to $\phi(N/g(k_1,k_2,N))Y^{(p)}_{k_1,k_2}$. Hence, since there are only N^2 UMRT coefficients, only these N^2 coefficients need be used in the inverse transform, provided they are weighted by the term $\phi(N/g(k_1,k_2,N))$.

Hence, the inverse 2-D UMRT can be obtained as follows:

1) For each data element x_{n_1,n_2}
2) Initialize, $s = 0$.
3) For each UMRT frequency (k_1, k_2),
4) Calculate $q = \phi(N/g(k_1,k_2,N))$
5) Calculate $z = ((n_1k_1 + n_2k_2))_N$.
6) If $0 \le z < M$, $s = s + q\,Y^{(z)}_{k_1,k_2}$, else $s = s - q\,Y^{(z-M)}_{k_1,k_2}$
7) $x_{n_1,n_2} = s / N^2$

In this way, the methods for generating the N^2 UMRT coefficients of an N x N image and also for reconstructing the N x N image from the N^2 UMRT coefficients have been presented. In this context, a notable feature is that while the input is two-dimensional, the output is three-dimensional since the MRT coefficient has three variables. This could be an impediment in practical use of the 2-D UMRT. Hence, if a method can be developed to represent the N^2 UMRT coefficients in the form of an N x N matrix just like the input image is represented, then this would make use of the MRT representation easier.

5.4 Derived Redundancy in 2-D MRT

Derived redundancy, which was explained for 1-D signals in section 4.4.2, has its counterpart in 2-D signals also. For example, for $N = 6$,

$Y^{(0)}_{3,3} = Y^{(0)}_{1,1} - Y^{(1)}_{1,1} + Y^{(2)}_{1,1}$, and

$Y^{(0)}_{3,0} = Y^{(0)}_{1,2} - Y^{(1)}_{1,2} + Y^{(2)}_{1,2}$

From these examples, though it can be intuitively stated that odd divisors are responsible for derived redundancy, a detailed study of derived redundancy in 2-D MRT and inferences of possible differences from the 1-D case remain to be explored. For N a power of 2, derived redundancy does not exist, which is proved by the fact that there are N^2 UMRT coefficients for such N. However, for N not a power of 2, number of UMRT coefficients is greater than N^2. This results in an expansive UMRT as shown in table 5.3. However, exploiting derived redundancy, it might still be possible to express the 2-D signal using only N^2 UMRT coefficients, though the remaining coefficients may have to be re-calculated for performing the inverse transform for signal reconstruction.

5.5 Conclusion

In this chapter, complete redundancy in 2-D MRT is analyzed in detail. Redundancy between divisor columns and also within divisor columns are analyzed. An expression is obtained for the number of frequencies of the form (k, k_2) in a column k_2 such that $g(k,N) = k_1$. The total number of unique frequencies, for any even value of N, is calculated over all divisor columns. The number of UMRT coefficients, for any even value of N, is also derived using the expressions for the number of unique frequencies. The number of UMRT coefficients is found to be N^2 when N is a power of 2. Hence, when complete redundancy is removed from 2-D MRT, the resulting 2-D UMRT is non-expansive when N is a power of 2, but expansive when N is not a power of 2. A method is presented to arrive at the frequency and phase of the N^2 UMRT coefficients when N is a power of 2. A method to perform the inverse 2-D UMRT, when N is a power of 2, is also presented. Hence, a real, non-expansive transform has been developed through UMRT, when N is a power of 2. Derived redundancy has not been removed from 2-D MRT yet. Once this is done, there is scope for obtaining a non-expansive, non-redundant UMRT when N is not a power of 2.

Chapter VI

APPLICATIONS, DISCUSSION AND CONCLUSION

6.1 Introduction

The MRT and the UMRT being general-purpose transforms, they can be applied for signal processing tasks of 1-D and 2-D signals in general. Also, if extended to higher dimensions, they could have applications for higher dimension signals as well. A few applications of 2-D MRT and 2-D UMRT are attempted to justify its applicability. One application of 2-D MRT – generation of image blocks, and two applications of 2-D UMRT – image compression and orientation estimation, are presented in this chapter. A few aspects of particular interest related to the MRT and UMRT are also discussed in the chapter. Finally, the conclusion is presented along with a few directions for further research.

The 2-D MRT can be used to generate a variety of image blocks from a given image block. The idea is to manipulate the MRT matrices of an image block in different ways.

The main steps involved in image compression are (i) creating of image sub-blocks, (ii) transform coding of each sub-block, (iii) quantization and (iv) entropy coding. The 2-D UMRT can be used as the inherent transform for image compression problem.

Many methods have been applied to the task of describing directionality in images. Oriented patterns have been described by using a flow coordinate system in [101]. Lapped Directional Transforms (LDT) unambiguously detect spatial orientations from spatial energy [4]. A directional filter bank whose pass band regions provide directional information has been proposed in [102]. Directional wavelets like curvelets [103] and contourlets [86] have basis functions that localize in space and angle. Fingerprints are widely used as biometric identification tools and fingerprint recognition is popularly used for automatic personal identification. The pattern of ridges and valleys on the surface of a fingertip define the fingerprint. The orientation field of a fingerprint is defined as the local orientation of the ridges of the fingerprint. Estimation of the orientation field is an important step in the fingerprint identification process. Applications requiring knowledge of ridge orientation include enhancement, singular points detection, ridge detection, and fingerprint pattern classification. Techniques proposed for orientation detection include gradient-based methods [104-105], spatial convolution, polynomial model [106] and directional Fourier filtering [107].

6.2 Applications of 2-D MRT/UMRT

Since a new representation of 2-D signals using 2-D MRT and 2-D UMRT coefficients have been obtained, three application scenarios are considered.

6.2.1 Generation of Image Blocks

As already seen, the 2-D MRT of an N x N image block has M MRT matrices of size N x N. If these M matrices are manipulated suitably and the inverse 2-D MRT is performed, the resulting N x N signal is an image block that is newly generated from the original block. An assumption here is that M is prime. The following are a few of the manipulations that were considered.

1) Sign change: All MRT matrices except $Y_{k_1,k_2}^{(0)}$ are multiplied by -1, and the inverse MRT is done to obtain a new block.
2) Constant scaling: All MRT matrices except $Y_{k_1,k_2}^{(0)}$ are multiplied by a constant scaling factor, and the inverse MRT gives the new block. Different choices of the scaling factor can be used.
3) Variable scaling: All MRT matrices except $Y_{k_1,k_2}^{(0)}$ are multiplied by different scaling factors keeping $Y_{k_1,k_2}^{(0)}$ unchanged and the inverse is taken.
4) Total variable scaling: All MRT matrices are multiplied by different scaling factors, and the inverse is taken.
5) Adjacent exchange: The adjacent MRT matrices, except $Y_{k_1,k_2}^{(0)}$, are interchanged. For example, $Y_{k_1,k_2}^{(1)}$ is interchanged with $Y_{k_1,k_2}^{(2)}$, and $Y_{k_1,k_2}^{(3)}$ is interchanged with $Y_{k_1,k_2}^{(4)}$.
6) Reverse exchange: The MRT matrices except $Y_{k_1,k_2}^{(0)}$ are exchanged in a reverse order. For example, if $N = 10$, $Y_{k_1,k_2}^{(1)}$ is exchanged with $Y_{k_1,k_2}^{(4)}$, and $Y_{k_1,k_2}^{(2)}$ is exchanged with $Y_{k_1,k_2}^{(3)}$.
7) Middle exchange: The MRT matrices except $Y_{k_1,k_2}^{(0)}$ are exchanged along the middle element. $Y_{k_1,k_2}^{(d)}$ is exchanged with $Y_{k_1,k_2}^{(d+(M-1)/2)}$. For example, if $n = 14$, $Y_{k_1,k_2}^{(1)}$ is exchanged with $Y_{k_1,k_2}^{(4)}$, and $Y_{k_1,k_2}^{(2)}$ is exchanged with $Y_{k_1,k_2}^{(5)}$.

The image blocks generated from a single image block using the manipulations 1-7 separately and in combination are shown in Figure 6.1. The numbers above each block identify the associated manipulations performed to obtain the block. The top-left block with number 1 is the original block. For example, a block with label '26' has been obtained by computing the MRT of the original block and then using a particular value for scaling factor in operation 2 followed by operation 6 on the computed MRT, and finally taking the inverse MRT of the manipulated MRT.

6.2.2 Image Compression

An image compression technique using UMRT is proposed in the following steps:

1. The image to be compressed is divided into sub-images of size 8 x 8
2. Forward 2-D UMRT is applied to each sub-image block

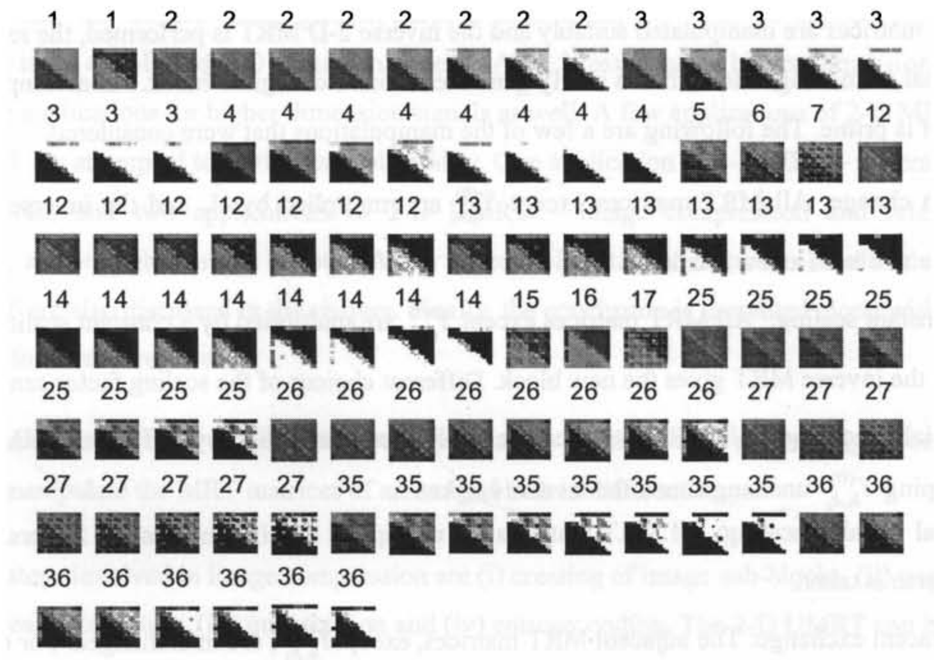


Figure 6.1: Image blocks generated using MRT manipulation from a single image block.

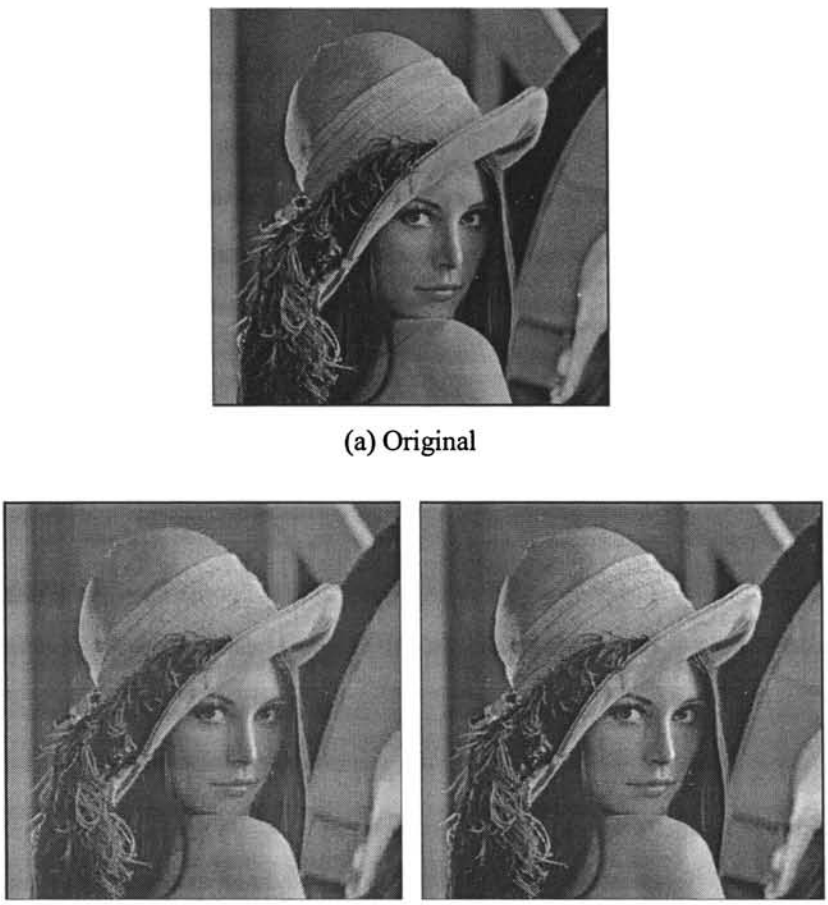
3. Quantization is applied to the UMRT coefficients. Many quantization methods are possible. The approach used here is to divide all UMRT coefficients by a positive integer and to round off the result.

4. UMRT coefficients $Y_{0,0}^{(0)}, Y_{0,1}^{(0)}, Y_{0,1}^{(1)}, Y_{0,1}^{(2)}, Y_{0,1}^{(3)}, Y_{0,2}^{(0)}$ & $Y_{0,2}^{(2)}$ are divided by half the quantization factor used to divide all other coefficients. The mean intensity of the image is represented by $Y_{0,0}^{(0)}$. The remaining coefficients cited above correspond to vertical structure which is strong in most natural images. Typical values for this quantization factor lie in the range from 5 to 50.

5. The above coefficients are scanned first, followed by the remaining coefficients in sequential order from the top row. Long runs of zeros will result for each sub-block as a result of this scanning order.

6. The coefficient $Y_{0,0}^{(0)}$ is a DC coefficient. Single-delay DPCM differences are taken for DC coefficients. Combinations of non-zero AC coefficients and zero run-lengths are grouped. These are then coded using Huffman tables specified in the sequential mode of the JPEG standard [100].
7. At the decoder, the compressed bit-stream is decoded and the coefficients are then re-arranged using the same scanning order and multiplied by the same quantization matrix used at the encoder.
8. Lastly, inverse MRT is applied to each block to obtain the reconstructed image.

The results of image compression using UMRT on five commonly used test images is shown in Table 6.1. Two examples of a reconstructed image are shown in Figure 6.2.



(a) Original

(b) 0.5 bpp, 31.54 dB

c) 1 bpp, 35.55 dB

Figure 6.2: Original and reconstructed images, 'Lenna'

Table 6.1: Compression results obtained using UMRT on 5 test images

	PSNR(dB)			
Image	0.25 bpp	0.50 bpp	0.75 bpp	1.0 bpp
Lenna	28.34	31.54	33.72	35.55
Barbara	22.84	25.96	28.27	30.59
Mandrill	22.28	24.22	25.93	27.41
Goldhill	26.81	29.18	30.41	32.11
Cameraman	27.98	32.06	34.79	38.14

6.2.3 Orientation Estimation

By orientation of a pattern (assuming that the grey-cells indicate a UMRT pattern) is meant the angle θ between the horizontal axis and the UMRT pattern, as shown in Figure 6.3. Recalling from section 3.5, for a given solution n_1, solutions to n_2 occur at a gap of $N / g(k_2, N)$ columns. Also, if a data element has the index (n_1, n_2) and is present in $Y_{k_1,k_2}^{(p)}$, then $Y_{k_1,k_2}^{(p)}$ will contain a data element in the row given by $n_1 + g(k_2, N)/g(k_1, k_2, N)$.

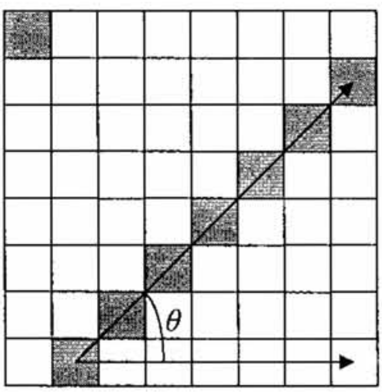


Figure 6.3: Analysis of UMRT pattern orientation

The column number of this data element is given by $n_2 - v$, where v satisfies (3.27). In other words, (n_1, n_2) and $(n_1 + g(k_2, N)/g(k_1, k_2, N), n_2 - v)$ belong in $Y_{k_1,k_2}^{(p)}$. Using this knowledge, the following analysis may be done regarding directionality of certain UMRT patterns. Assume $k_2 = 1$. Hence, $k_2 = g(k_2, N) = 1$ and, $v = k_1$ & $g(k_2, N)/g(k_1, k_2, N) = 1$. Thus, given (n_1, n_2), $(n_1 + 1, n_2 - v)$ also belongs in $Y_{k_1,k_2}^{(p)}$. Also, there is only one data element in a row. The angle θ is calculated as $\tan^{-1} 1/v$. If $k_1 = q$, then $\tan(\theta) = 1/q$. Also, $((N - q))_N = ((-q))_N$. Hence, if $k_1 = q$, $v = q$, and if $k_1 = N - q$, $v = -q$. The magnitude of v is the same, but the direction of the change represented by v is opposite to each other. Hence, if θ is the angle between the pattern and the horizontal axis in the positive direction for $k_1 = q$, then if $k_1 = N - q$, θ is the angle between the corresponding pattern and the horizontal axis in the negative direction. Hence, the angle between this pattern and the horizontal axis in the positive direction is $(180^{\circ} - \theta)$. The pattern angles corresponding to a few UMRT frequencies when $N = 16$ are shown in Table 6.2.

Table 6.2: Pattern orientations for a few UMRT frequencies, $N = 16$.

Frequency index of UMRT coefficient	Direction of pattern (degrees)	Frequency index of UMRT coefficient	Direction of pattern (degrees)
(1,0)	0	(3,1)	18
(2,0)	0	(4,1)	14
(0,1)	90	(5,1)	11
(1,1)	45	(12,1)	166
(2,1)	27	(13,1)	162
(14,1)	153	(15,1)	135
(0,2)	90	(1,2)	63
(7,2)	117		

MRT coefficients form well-defined patterns in the 2-D signal domain that are unique for each coefficient. Basis images can be plotted that show how pixel locations in an image combine to form each MRT coefficient. Figure 6.4(a) shows basis images corresponding to MRT coefficients $Y_{1,1}^{(0)}$, $Y_{1,1}^{(1)}$, $Y_{1,1}^{(2)}$ and $Y_{1,1}^{(3)}$ respectively for $N = 8$. The basis images formed by MRT coefficients $Y_{1,0}^{(0)}$, $Y_{1,0}^{(1)}$, $Y_{1,0}^{(2)}$ and $Y_{1,0}^{(3)}$ respectively are shown in Figure 6.4(b). The corresponding MRT coefficient is obtained by addition of data elements in white-shaded cells and subtraction of data elements in black-shaded cells in the basis images. The grey-shaded cells do not involve in the computation. It is evident from the basis images that MRT coefficients can be viewed as the results of performing a spatial-filtering operation on the image block using different spatial masks. The nature of each mask is determined by the basis image corresponding to each MRT coefficient.

From (4.S.1), since incongruent solutions to a linear congruence equation form an arithmetic progression, solutions to the two linear congruence equations relevant to MRT form 2-D patterns in the spatial domain. The constant difference of the arithmetic progression depends on the values

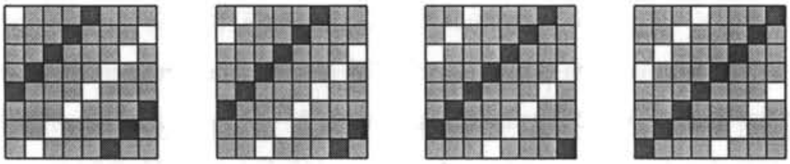


Figure 6.4 (a): Basis images corresponding to MRT coefficients $Y_{1,1}^{(0)}$, $Y_{1,1}^{(1)}$, $Y_{1,1}^{(2)}$ and $Y_{1,1}^{(3)}$

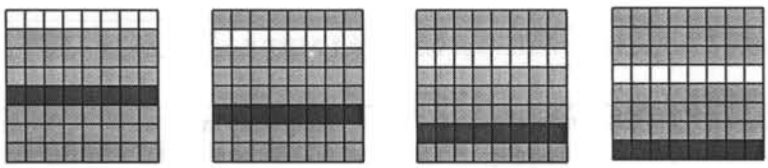


Figure 6.4(b): Basis images corresponding to MRT coefficients $Y_{1,0}^{(0)}$, $Y_{1,0}^{(1)}$, $Y_{1,0}^{(2)}$ and $Y_{1,0}^{(3)}$

(a) (b)

Figure 6.5: Global patterns formed by union of: (a) MRT coefficients with frequency index (1,1), (b) MRT coefficients with frequency index (1,0).

of k_1 and k_2. Hence, given k_1 and k_2, the basic structure of the spatial pattern associated with all MRT coefficients having these frequency indices are the same, irrespective of the value of the phase index p. The exact location of the pattern on the image lattice depends on p. Thus, spatial patterns of MRT coefficients having common frequency indices k_1 and k_2 but different phase indices p are all parallel to each other. Further, the union of all these spatial patterns covers the entire image block. This set of all patterns is a pattern in itself and may be called a global pattern for the frequency index (k_1, k_2). The global pattern is a filter mask that spatially filters the entire image block. *Figure 6.5(a) shows the global patterns formed by the union of MRT coefficients* with frequency index (1,1). Thus, the basis images in Figure 6.5(a) form the components that make up the global pattern in Figure 6.4(a) associated with an 8 x 8 MRT. Similarly, the global pattern

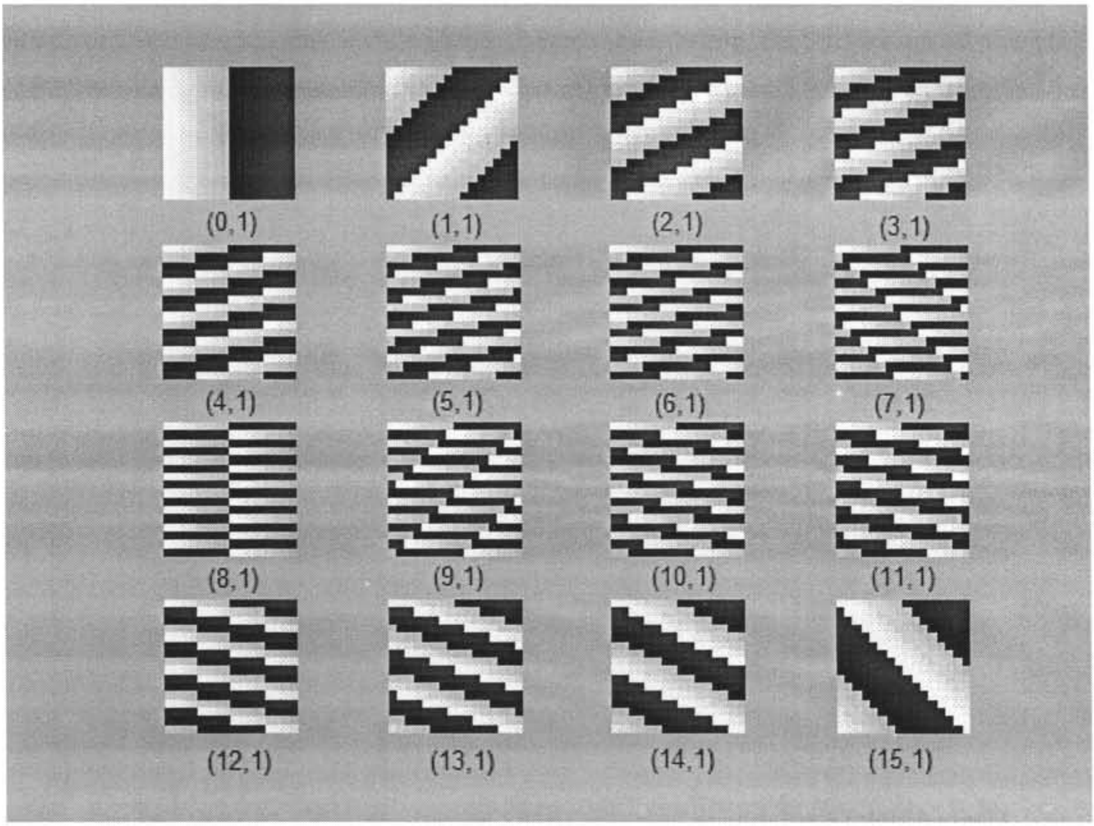


Figure 6.6 (a): Global patterns formed by UMRT coefficients for $N = 16$, (k_2=1)

corresponding to frequency index (1,0), shown in Figure 6.5(b), have the basis images shown in Figure 6.4(b) as its components.

UMRT coefficients can be used to estimate fingerprint orientation by the following steps:

1. The input fingerprint image is divided into non-overlapping square blocks of size 16 x 16, which is a commonly-used block-size in fingerprint image processing.

2. The UMRT of each block is computed.

3. Figure 6.6 shows the global patterns associated with UMRT coefficients for $N = 16$. 15 sets of UMRT coefficients shown in Table 6.2 are used for direction analysis. These coefficients are chosen on account of the strong directionality of their associated patterns, and likeness of these patterns to the ridge flow in a fingerprint image. For a frequency index (k_1, k_2), the sum of

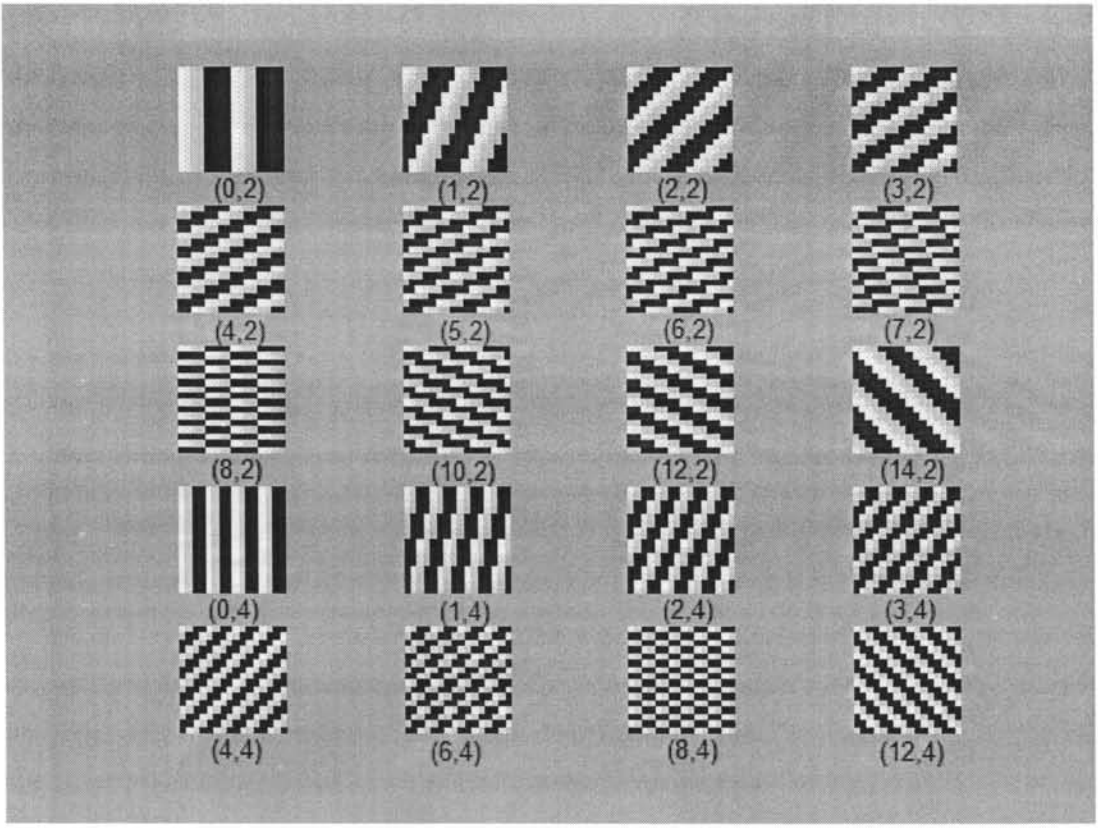


Figure 6.6(b): Global patterns formed by UMRT coefficients for $N = 16$, ($k_2 = 2, 4$)

absolute values of each UMRT coefficient with index (k_1, k_2, p) for all applicable values of phase index p is calculated. This value may be called a pattern strength indicator associated with a given frequency index (k_1, k_2).

$$P_{SI}(k_1,k_2) = \sum_p \left| Y^p_{k_1,k_2} \right|$$

4. Because of the highly directional nature of ridges in a fingerprint image, blocks other than singular points will have a single dominant direction. This dominant direction is identified by the frequency index (k_1,k_2) which gives maximum value for the pattern strength indicator defined above. The absolute value of the UMRT coefficients is taken since the present objective is only to estimate the strength of a pattern in a particular direction in an image block. The sign of the UMRT coefficient gives information about the relative location of the dark and light areas of the pattern and hence it is not relevant to the present application.

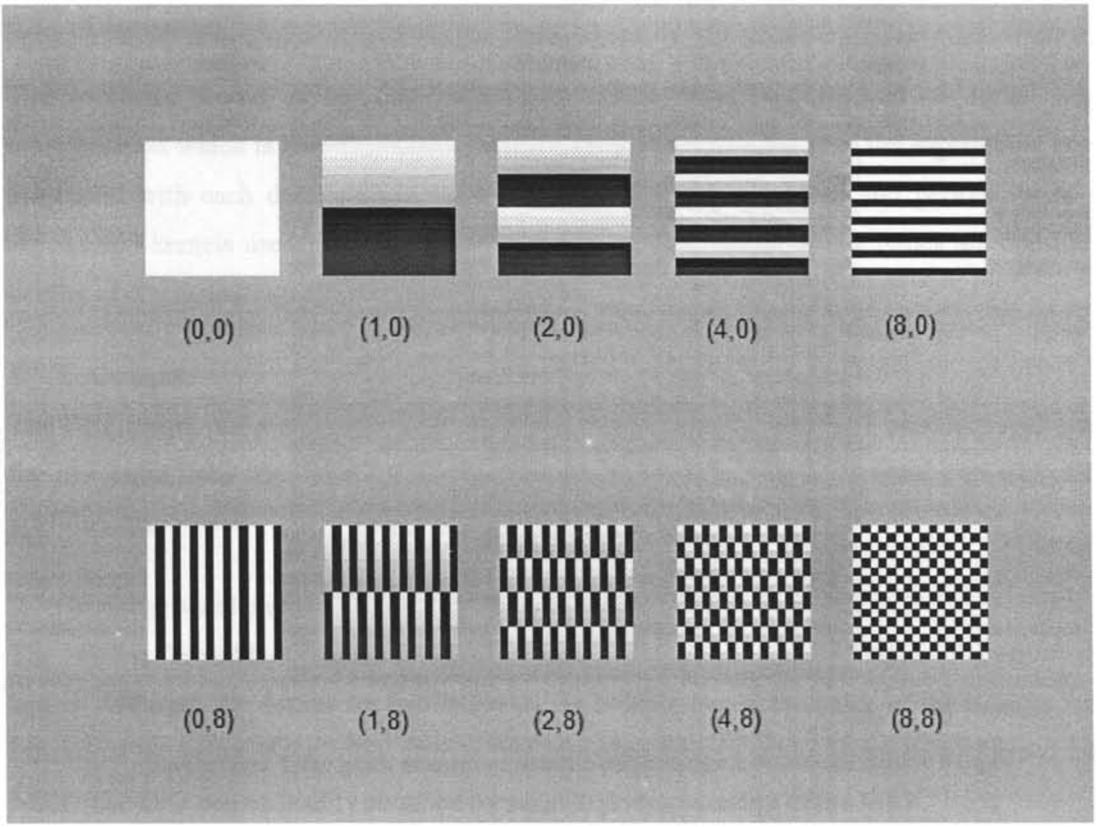


Figure 6.6(c): Global patterns formed by UMRT coefficients for $N = 16$, ($k_2 = 0, 8$)

This method has been used to estimate orientation field of some fingerprint images to study the accuracy of orientation estimation. Figure 6.7(a) and Figure 6.7(b) show two examples of the results on fingerprint images taken from database DB2 of [108]. The estimated orientation of each 16 x 16 block is indicated by the mark superimposed on each block in the fingerprint image. Since it is difficult to obtain an objective error measure, the performance of the method has to be judged by manual inspection. On observation of the results, it is seen that MRT coefficients can be used to obtain a fairly accurate estimate of the orientation field of the fingerprint image. When the fingerprint image is of good quality, the method will have very high accuracy.

Figure 6.7(a): Results of fingerprint orientation estimation using MRT coefficients.

Figure 6.7(b): Results of fingerprint orientation estimation using MRT coefficients.

6.3 Discussion

The overlying theme of the thesis has been to present a new method of digital signal representation which is based on a grouping of data based on the value of the exponential kernel associated with each data in the DFT context. Further, the symmetry and periodicity of the exponential kernels used in the DFT were also exploited. The following issues are considered worthy of discussion regarding the new transforms.

1. Real output:

The DFT transforms a real input to a complex output. However, the MRT produces real output for real input data. This is seen as an advantage in the context of the numerous real-valued applications.

2. Grouping based on angle:

The MRT is formed by grouping data on the basis of the angular values taken by the exponential kernel. Although the data is not multiplied by the twiddle factor, the value of the twiddle factor influences the value of the MRT. Due to this, complex-domain analysis can be performed using MRT. The DFT can be readily obtained by suitable post-processing of the MRT.

3. New approach:

The DFT maps signals from time-domain to frequency-domain. However, it requires N^4 and $(3/4)N^2 \log N$ complex multiplications in direct DFT and radix-2 vector radix FFT respectively for an N x N data. In [8], a modified DFT relation was introduced that requires $N^3/2$ complex multiplications at the final stage. The 2-D MRT presented in Chapter III, obtained from the modified DFT relation, requires only real additions. The direct computation of 2-D MRT requires computation of z for each coefficient and a logical checking which adds complexity to the computation. Hence, a closed form representation is derived exploiting the patterns present in the MRT. The 2-D MRT computation maps an N x N data into M matrices of size N x N, and has considerable redundant coefficients. Thus, the 2-D UMRT is developed removing the redundancy present in 2-D MRT. Hence, the development of 2-D UMRT gives an N x N matrix from an N x N data using only real additions. DFT gives only global information whereas MRT can be used for extracting both global and localized information. Thus, the 2-D UMRT is a compact representation of 2-D signals in the frequency domain using only simple real arithmetic.

4. Computation:

The computation of 2-D UMRT involves identification of the frequency and phase indices followed by computation of the coefficients using either the direct method or the closed form computation. In applications like generation of images, 2-D MRT will be advantageous and can be derived from 2-D UMRT by adding the redundancy. 2-D DFT can be derived from 2-D MRT by multiplying with the twiddle factors associated with each MRT coefficient. Thus, three methods are possible for 2-D MRT computation: (i) Direct computation, (ii) Closed form computation, and (iii) Computation from 2-D UMRT coefficients. A plot comparing the computation times for the three methods is shown in Figure 6.8. In the direct method, the computation time is highest. It is reduced when the closed form computation is used. In the method using UMRT coefficients, only the UMRT coefficients need to be computed. All the remaining MRT coefficients can be computed through complete redundancy from the UMRT coefficients. The computation of UMRT coefficients requires knowledge of particular solution which could be an overhead in computation.

5. Frequency indices:

In the DFT, the frequency indices are directly given by $k_1 = [0, 1, 2, N - 1]$, $k_2 = [0, 1, 2, N - 1]$. In the UMRT, for N a power of 2, the frequency indices (k_1, k_2) are arrived at using methods based on number theory principles. Using the mapping between the frequency and phase indices (k_1, k_2, p) and indices (u, v), the indexing of the UMRT coefficients also can be similar to that of the DFT.

6. Orientation:

In the Fourier domain, the power spectrum of a directional pattern clusters along a line through the origin, and the orientation of the pattern is perpendicular to this line. The MRT has orientation properties too. The MRT's orientation property provides information about the orientation of a pattern as well as the spatial location of the pattern. MRT coefficients are formed by sums and differences of data elements along well-defined patterns, and hence can be considered to be strength-indicators of spatial patterns in an image. A few MRT coefficients represent highly directional patterns and these can be used to identify directional features in images.

7. Space-frequency transform, Separability, Orthogonality:

The DFT is a frequency transform. The MRT, in contrast, can be considered to be a space/time-frequency transform, since it has both frequency and phase indices. 2-D MRT can not be implemented by row-column decomposition. The UMRT is an orthogonal transform and can be verified using examples. It can be made orthonormal by suitably adjusting the scaling factors in the forward and inverse transforms.

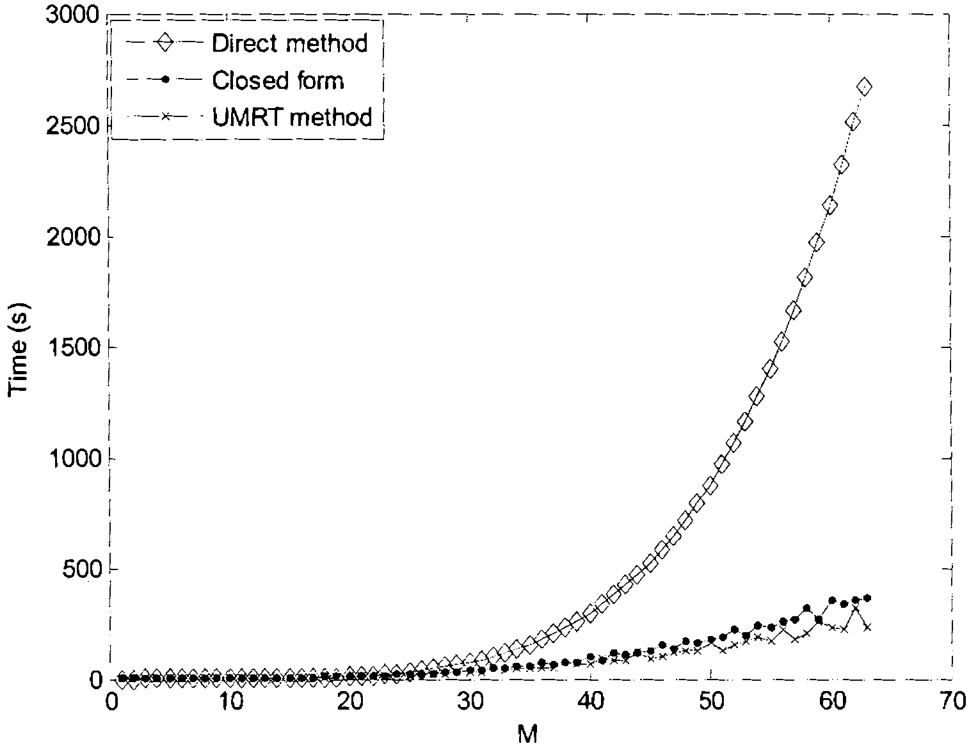


Figure 6.8: Plot showing 2-D MRT computation time using direct method, closed form method, and computation from 2-D UMRT coefficients

8. Expansive transform:

As observed in Chapter III, the MRT is an expansive transform, requiring more memory space than the original signal. Although the MRT can be converted into the non-expansive UMRT, the indexing of the UMRT is not straight-forward and hence there is an issue of complexity here. The presence of the third dimension introduces a certain level of complexity in MRT/UMRT representation. The 2-D MRT and 2-D UMRT are inherently three-dimensional, although the UMRT can be mapped into an N x N matrix. Also, the size of the signal N is required to be even.

9. Relation with other transforms:

The relationship of MRT with the DFT is inherent in the very definition of the MRT. Considering other transforms, there is a connection between the 1-D Haar transform and the 1-D UMRT, for N a power of 2. The basis vector of the 1-D Haar transform can be obtained by a modification of the basis vector of the 1-D UMRT for N a power of 2.. The 2-D MRT and 2-D UMRT have connections with Radon transforms like the DPRT and the ODPRT that are based on linear congruences. A subset of the ODPRT coefficients is seen to have values equal to a subset of 2-D UMRT coefficients, i.e. those corresponding to $\gcd(k_1, k_2) = 1$. Two-stage methods being composed of a pre-processing stage and a post-processing stage, the MRT can be considered to be a pre-processing transform. The DFT can be computed by using the MRT as the first stage of computation by using suitable post-processing stages. The basis image of the DFPT also is composed of 1, -1, and 0, similar to that of the MRT and UMRT. Like the DCT, the UMRT is a real and non-expansive transform. However, the DCT has the important property of energy compaction which has made it a popular transform.

6.4 Conclusion and Further Work

This thesis has presented work that has been done on the development of a new transform, named MRT and its simplified version called the UMRT. The motivation for the new transform is to process data on the basis of associated kernel values while avoiding the multiplication with the kernel, and to utilize the symmetry and periodicity properties of the exponential transform kernel of the DFT, while maintaining the capability for frequency-domain analysis. The MRT expresses the data in terms of simple additions among various data elements. If required, the DFT can be obtained from the MRT. Hence, the MRT is a new and simple way of expressing and analyzing 1-D and 2-D signals. While both the 1-D MRT and 2-D MRT are expansive and redundant; the 1-D UMRT is non-expansive and non-redundant for all even values of N. The 2-D UMRT is non-expansive and non-redundant for N a power of 2, and expansive and non-redundant for N not a power of 2. However, there are computational requirements to arrive at the unique frequency indices, and indexing issues to represent the UMRT in a matrix. The idea of utilizing transform kernel symmetries to arrive at simpler signal representations appears to have further potential. A few possible directions for further study are presented below:

1. The concept of derived redundancy has been studied for 1-D signals only. Since derived redundancy is an important aspect of MRT theory, it is necessary to explore the occurrence of derived redundancy in 2-D signals also.

2. 2-D UMRT for N a power of 2 has been described in Chapter V. The study of UMRT for N that is not a power of 2 remains to be performed. It has been found that the number of unique coefficients is greater than N^2 for N not a power of 2. However, using derived redundancy, an attempt could be made to obtain an $N \times N$ representation when N is not a power of 2.

3. The work on inverse MRT for N not a power of 2, though started, remains to be completed. The inverse for the case when M is prime has been obtained.

4. In Chapter III, the properties of the 2-D MRT are described. Further properties of the MRT remain to be explored. Also, properties of the 2-D UMRT offer another possible direction for further investigation.

5. New applications can be found for the 2-D UMRT by studying its properties in detail. A few potential applications are in image de-noising, image filtering, and pattern analysis etc. Methods could be found for optimum use of the transform in applications like image compression and orientation analysis.

6. The directional feature of MRT and its pattern structure has potential applications in areas which require a simple tool to describe directionality of images. Although global MRT patterns were made use of in this thesis and sign of coefficients ignored, the sign of individual MRT coefficients too conveys important information regarding relative location of light and dark areas in images. Applications that could take advantage of this feature of the MRT need to be identified. The use of MRT in the broad areas of pattern analysis and computer vision requires to be further explored.

7. The MRT approach could be applied to other transform kernels.

8. The applicability of MRT in analyzing time-varying signals is to be explored.

APPENDIX A

A.1 Bezout's Lemma:

For all integers a and b there exist integers s and t such that

$$\gcd(a,b) = sa + tb$$

A.2 Linear Diophantine Equation

Diophantine equations are equations that require integer solutions. The linear Diophantine equation $ax + by = c$ has a solution only if $\gcd(a,b)$ divides c. In that case, there are infinite number of solutions given by: $x = x_0 + (b / \gcd(a,b))t$, $y = y_0 - (a / \gcd(a,b))t$, where (x_0,y_0) is a solution, $t \in Z$.

A.3 Theorem on Linear Congruence

The linear congruence $((nk))_N \equiv p$ is solvable if and only if $d \mid p$, where $d = \gcd(k, N)$. If $d \mid p$ then it has d incongruent solutions. The linear congruence, when solvable, has the general solution $n = n_0 + (N/d)t$, where $0 \leq t < d$, and n_0 is a particular solution.

A.4 Greatest Common Divisor (gcd).

Definition: Let $a,b,c \in Z$. If $a \neq 0$ or $b \neq 0$, $\gcd(a,b)$ is defined to be the largest integer d such that $d \mid a$ and $d \mid b$ and is denoted as $g(a,b)$.

gcd properties:

1. If $e \mid a$ then $-e \mid a$..

2. If $a \neq 0$, then the largest positive integer that divides a is $|a|$.

3. $\gcd(a,b) = \gcd(|a|,|b|)$.

4. $\gcd(a,b) = \gcd(b,a)$.

5. If $a \neq 0$ or $b \neq 0$, then $\gcd(a,b)$ exists and satisfies

$$0 < \gcd(a,b) \leq \min\{|a|,|b|\}.$$

6. $\gcd(a,b,c) = \gcd(\gcd(a,b),c)$.

A.5 Euclidean Algorithm, Extended Euclidean Algorithm.

The Euclidean algorithm is used to determine the gcd of any two integers.

Let $a,b \in Z$ be such that $a \geq b > 0$. Set $r_0 = a$ and $r_1 = b$. Suppose that

$$
\begin{aligned}
r_0 &= r_1 q_1 + r_2, 0 \le r_2 < r_1 \\
r_1 &= r_2 q_2 + r_3, 0 \le r_3 < r_2 \\
&\;\;\ldots\ldots \\
r_{n-2} &= r_{n-1} q_{n-1} + r_n, 0 \le r_n < r_{n-1} \\
r_{n-1} &= r_n q_n.
\end{aligned}
$$

Then, $\gcd(a,b) = r_n =$ (the last non-zero remainder).

Proof:

Since $r_0 \ge r_1 > r_2 > \ldots$, there is an n such that $r_n = 0$. Therefore, the algorithm does terminate eventually.

$$\gcd(a,b) = \gcd(r_0, r_1) = \gcd(r_0 - q_1 r_1, r_1) = \gcd(r_2, r_1).$$

Similarly, $\gcd(r_2, r_1) = \gcd(r_1, r_2) = \gcd(r_1 - q_2 r_2, r_2) = \gcd(r_3, r_2)$, etc.

Therefore, $\gcd(a,b) = \gcd(r_2, r_1) = \ldots = \gcd(r_n, r_{n-1}) = r_n$.

Given positive numbers a and b, the extended Euclidean algorithm computes *(d, u, v)* such that $d = \gcd(a,b) = au + bv$.

1. Set $a_1 = a$, $a_2 = b$; $x_1 = 1$, $x_2 = 0$; $y_1 = 0$, $y_2 = 1$.
2. Let $q = \lfloor a_1 / a_2 \rfloor$
3. Set $a_1 = a_2$, $a_2 = a_1 - q a_2$; $x_1 = x_2$, $x_2 = x_1 + q x_2$; $y_1 = y_2$, $y_2 = y_1 + q y_2$.
4. If $a_2 > 0$ go back to Step 2.
5. If $a x_1 - b y_1 > 0$ return $(d,u,v) = (a_1, x_1, -y_1)$, else return $(d,u,v) = (a_1, -x_1, y_1)$.

A.6 Totative.

Definition: A totative of N is a positive integer less than or equal to a number N which is also relatively prime to N, where 1 is counted as being relatively prime to all numbers. For example, there are eight totatives of 24, {1, 5, 7, 11, 13, 17, 19, 23}.

A.7 Totient Function.

Definition: The totient function $\phi(N)$, also called Euler's totient function, is defined as the number of positive integers $\le N$, that are co-prime to (i.e., do not contain any factor other than 1 in common with) N, where 1 is counted as being co-prime to all numbers. The totient function $\phi(N)$ can be simply defined as the number of totatives of N. For example, $\varphi(24) = 8$. It is mathematically expressed as

$$\phi(N) = N\prod_{r|N}\left(1-\frac{1}{r}\right)$$

A.8 Residue Systems

Complete residue system: A set of m integers such that each element of the set A_i, $1 < i < m$, produces a unique value for mod(A_i, m) is called a complete residue system modulo m. For example, the set {0, 1, 2, 3, 4, 5} is a complete residue system modulo 6.

Reduced residue system: Given a complete residue system modulo m, the subset of a such that gcd(A_i, m) = 1 is called a reduced residue system modulo m. The size of this set is given by $\varphi(m)$.

Theorem: If $\{A_1, \ldots, A_{\varphi(m)}\}$ is a reduced residue system modulo m and if gcd(q,m) = 1, then $\{qA_1, \ldots, qA_{\varphi(m)}\}$ also is a reduced residue system modulo m.

APPENDIX B

B.1 Values of function $l(k)$:

Let $N = \prod_{i=1}^{q} r_i^{a_i}$,

and $k = \prod_{i=1}^{q} r_i^{c_i}$,

where N and k are integers, and r_i are prime divisors.

$$\therefore l(k) = \frac{\phi(N)}{\phi(N/k)} = k \frac{\prod\limits_{\forall i \Rightarrow a_i = c_i}^{q} (r_i - 1)}{\prod\limits_{\forall i \Rightarrow a_i = c_i}^{q} r_i} = \prod_{i=1}^{q} r_i^{c_i} \frac{\prod\limits_{\forall i \Rightarrow a_i = c_i}^{q} (r_i - 1)}{\prod\limits_{\forall i \Rightarrow a_i = c_i}^{q} r_i}$$

Case (i): N has one prime divisor:

$$N = r^a$$
$$\therefore k = r^c, \quad c = 0,1,2,\dots,a$$

If $c < a$,

$$\therefore l(k) = r^c \frac{\prod\limits_{\text{if } a=c} (r-1)}{\prod\limits_{\text{if } a=c} r} = r^c = k, \quad \because a \neq c$$

If $c = a$,

$$\therefore l(k) = r^c \frac{\prod\limits_{\text{if } a=c} (r-1)}{\prod\limits_{\text{if } a=c} r} = r^{c-1}(r-1) = N\frac{r-1}{r} = \varphi(N)$$

$$\therefore l(k) = \begin{cases} k, & k < N \\ r^{c-1}(r-1), & k = N \end{cases}$$

Case (ii): N has two prime divisors:

$$N = r_1^{a_1} r_2^{a_2}$$
$$\therefore k = r_1^{c_1} r_2^{c_2}, \quad c_1 = 0,1,2,\dots,a_1, \quad c_2 = 0,1,2,\dots,a_2$$

If $c_1 < a_1$, $c_2 < a_2$,

$$\therefore l(k) = r_1^{c_1} r_2^{c_2} \frac{\prod\limits_{\forall i \Rightarrow a_i = c_i} (r_i - 1)}{\prod\limits_{\forall i \Rightarrow a_i = c_i} r_i} = r_1^{c_1} r_2^{c_2} = k, \quad \because a_1 \neq c_1, a_2 \neq c_2$$

If $c_1 = a_1$, $c_2 < a_2$,

$$\therefore l(k) = r_1^{c_1} r_2^{c_2} \frac{\prod_{\forall i \Rightarrow a_i = c_i} (r_i - 1)}{\prod_{\forall i \Rightarrow a_i = c_i} r_i} = r_1^{c_1} r_2^{c_2} \frac{r_1 - 1}{r_1} = r_1^{c_1 - 1}(r_1 - 1) r_2^{c_2}, \quad \because a_1 = c_1, a_2 \neq c_2$$

If $c_1 < a_1$, $c_2 = a_2$,

$$\therefore l(k) = r_1^{c_1} r_2^{c_2} \frac{\prod_{\forall i \Rightarrow a_i = c_i} (r_i - 1)}{\prod_{\forall i \Rightarrow a_i = c_i} r_i} = r_1^{c_1} r_2^{c_2} \frac{r_2 - 1}{r_2} = r_1^{c_1} r_2^{c_2 - 1}(r_2 - 1), \quad \because a_1 \neq c_1, a_2 = c_2$$

If $c_1 = a_1$, $c_2 = a_2$,

$$\therefore l(k) = r_1^{c_1} r_2^{c_2} \frac{\prod_{\forall i \Rightarrow a_i = c_i} (r_i - 1)}{\prod_{\forall i \Rightarrow a_i = c_i} r_i} = r_1^{c_1} r_2^{c_2} \frac{r_1 - 1}{r_1} \frac{r_2 - 1}{r_2} = r_1^{c_1 - 1} r_2^{c_2 - 1}(r_1 - 1)(r_2 - 1), \quad \because a_1 = c_1, a_2 = c_2$$

$$\therefore l(k) = \begin{cases} k, & c_1 < a_1,\ c_2 < a_2 \\ r_1^{c_1 - 1}(r_1 - 1) r_2^{c_2}, & c_1 = a_1,\ c_2 < a_2 \\ r_1^{c_1} r_2^{c_2 - 1}(r_2 - 1), & c_1 < a_1,\ c_2 = a_2 \\ r_1^{c_1 - 1} r_2^{c_2 - 1}(r_1 - 1)(r_2 - 1), & c_1 = a_1,\ c_2 = a_2 \end{cases}$$

B.2 Multiplication Property of function $l(k)$:

Let $N = \prod_{i=1}^{q} r_i^{a_i}$,

and $k = \prod_{i=1}^{q} r_i^{c_i}$,

$$\therefore l(k) = k \frac{\prod_{\forall i \Rightarrow a_i = c_i}^{q} (r_i - 1)}{\prod_{\forall i \Rightarrow a_i = c_i}^{q} r_i} = \prod_{i=1}^{q} r_i^{c_i} \frac{\prod_{\forall i \Rightarrow a_i = c_i}^{q} (r_i - 1)}{\prod_{\forall i \Rightarrow a_i = c_i}^{q} r_i}$$

Let $t = \prod_{i=1}^{q} r_i^{d_i}$, $\quad d_i = 0$ if $a_i = c_i$,

$$k' = tk = \prod_{i=1}^{q} r_i^{d_i} \prod_{i=1}^{q} r_i^{c_i} = \prod_{i=1}^{q} r_i^{c_i + d_i} = \prod_{i=1}^{q} r_i^{e_i}$$

$$\therefore l(k') = k' \frac{\prod_{\forall i \Rightarrow a_i = e_i}^{q} (r_i - 1)}{\prod_{\forall i \Rightarrow a_i = e_i}^{q} r_i} = tk \frac{\prod_{\forall i \Rightarrow a_i = e_i}^{q} (r_i - 1)}{\prod_{\forall i \Rightarrow a_i = e_i}^{q} r_i}$$

$$= \prod_{i=1}^{q} r_i^{c_i+d_i} \frac{\prod_{\forall i \Rightarrow a_i = c_i + d_i}^{q} (r_i - 1)}{\prod_{\forall i \Rightarrow a_i = c_i + d_i}^{q} r_i}$$

$$= \prod_{i=1}^{q} r_i^{c_i} \frac{\prod_{\forall i \Rightarrow a_i = c_i}^{q} (r_i - 1)}{\prod_{\forall i \Rightarrow a_i = c_i}^{q} r_i} \prod_{i=1}^{q} r_i^{d_i} \frac{\prod_{\forall i \Rightarrow a_i \neq c_i, a_i = c_i + d_i}^{q} (r_i - 1)}{\prod_{\forall i \Rightarrow a_i \neq c_i, a_i = c_i + d_i}^{q} r_i}$$

$$= \prod_{i=1}^{q} r_i^{d_i} \frac{\prod_{\forall i \Rightarrow a_i \neq c_i, a_i = c_i + d_i}^{q} (r_i - 1)}{\prod_{\forall i \Rightarrow a_i \neq c_i, a_i = c_i + d_i}^{q} r_i} l(k)$$

$$= \left[\prod_{\forall i \Rightarrow a_i \neq c_i, a_i \neq c_i + d_i}^{q} r_i^{d_i} \prod_{\forall i \Rightarrow a_i \neq c_i, a_i = c_i + d_i}^{q} r_i^{d_i - 1} \prod_{\forall i \Rightarrow a_i \neq c_i, a_i = c_i + d_i}^{q} (r_i - 1) \right] l(k)$$

$$= \left[\prod_{\forall i \Rightarrow a_i \neq c_i, a_i \neq e_i}^{q} r_i^{d_i} \prod_{\forall i \Rightarrow a_i \neq c_i, a_i = e_i}^{q} r_i^{d_i - 1} \prod_{\forall i \Rightarrow a_i \neq c_i, a_i = e_i}^{q} (r_i - 1) \right] l(k) \tag{B.1}$$

If $\forall i, a_i \neq e_i$, then (B.1) becomes

$$l(k') = \left[\prod_{\forall i \Rightarrow a_i \neq c_i, a_i \neq e_i}^{q} r_i^{d_i} \right] l(k) == \left[\prod_{i=1}^{q} r_i^{d_i} \right] l(k) = t l(k) \tag{B.2}$$

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