EVIDENCE FOR THE EXISTENCE OF SARKOVSKII ORDERING IN A TWO-DIMENSIONAL MAP

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We analyse numerically the bifurcation structure of a two-dimensional noninvertible map and show that different periodic cycles are arranged in it exactly in the same order as in the case of the logistic map. We also show that this map satisfies the general criteria for the existence of Sarkovskii ordering, which supports our numerical result. Incidently, this is the first report of the existence of Sarkovskii ordering in a two-dimensional map.

"Deterministic chaos" is an active area of research in the present decade [1]. Analytical methods combined with numerical studies have revealed a number of interesting features regarding the onset of chaos in simple nonlinear deterministic systems. The recent interest in such systems is mostly due to the discovery of universal metric properties in unimodal mappings by Feigenbaum [2,3]. However, it should be noted that most of the studies are done on one-dimensional (1D) noninvertible maps and on 2D invertible maps [4,5]. Comparatively, studies on 2D noninvertible maps are very few and their properties are less well understood today. It has been pointed out that 1D maps are not useful in understanding the chaotic behaviour in many of the systems known at present. For example, Braun et al. [6] have recently discovered chaotic behaviour in electrical discharges and they have stressed the importance of a 2D map as a useful model for this system.

Recently, Ruelle [7] has recommended the study of time evolutions with "adiabatically fluctuating parameters" (AFPs). In other words, evolutions of the form

$$X_{t+1}=f(X,\lambda(t)) \tag{1}$$

for discrete time. Here, the evolution of λ might itself be determined by a dynamical system. In this Letter we present some interesting properties of a 2D noninvertible map which is of the form suggested by Ruelle:

$$X_{t+1}=4\lambda_t X_t(1-X_t)\,,$$
$$\lambda_{t+1}=4\mu\lambda_t(1-\lambda_t)\,. \tag{2}$$

Note that, for $0\leqslant\mu\leqslant 1$, (2) maps a square interval $I\subset R^2$, defined by $0\leqslant X_t\leqslant 1$ and $0\leqslant\lambda_t\leqslant 1$, into itself and is continuous on the interval. We have already shown that [8,9] this "modulated" logistic map (MLM) is a very interesting dynamical system and, in fact, shows universal behaviour at the onset of chaos.

For many values of μ in the range $0.75<\mu<1$, we have a stable n-cycle for λ_t. So, at first sight, one might expect that the corresponding X_t values will vary randomly without being attracted to any stable cycle. However, as we see below, this is no longer true. We have shown [9] that, as μ is increased, MLM undergoes a cascade of period doubling bifurcations and turns chaotic at exactly the same parameter value μ_∞ at which the logistic map turns chaotic, with the "universal" bifurcation ratio δ of Feigenbaum. Here, we mainly concentrate on the parameter range $\mu_\infty<\mu<1$. We compute the bifurcation structure of MLM in the chaotic region and compare it with that of the logistic map.

We first consider a small range of the parameter $0.89\leqslant\mu\leqslant 0.91$. Computing the asymptotic values of

X_t and λ_t for μ in this range, we plot them against μ separately (see figs. 1A and 1B). The complete bifurcation structure of MLM is three dimensional and figs. 1A and 1B are the projections of this 3D diagram onto the (X_t, μ) plane and (λ_t, μ) plane respectively. Moreover, fig. 1B is just the bifurcation structure of the logistic map for the specified range of the parameter. In figs. 1A and 1B, we denote by $\mu^{(n)}$ the value of μ at which the n-cycle becomes stable, by $\mu'^{(n)}$ at which it becomes unstable and by $\mu_\infty^{(n)}$ at which the "window" of the basic n-cycle disappears. Periodic windows with basic periods $n=12$, $n=10$ and $n=6$ are clearly seen in that order. A comparison of figs. 1A and 1B shows that $\mu^{(n)}$, $\mu'^{(n)}$ and $\mu_\infty^{(n)}$ are exactly the same for both of them. In other words, if λ_t has a stable n-cycle in a particular interval along the μ-axis, then X_t also has a stable n-cycle in the same interval.

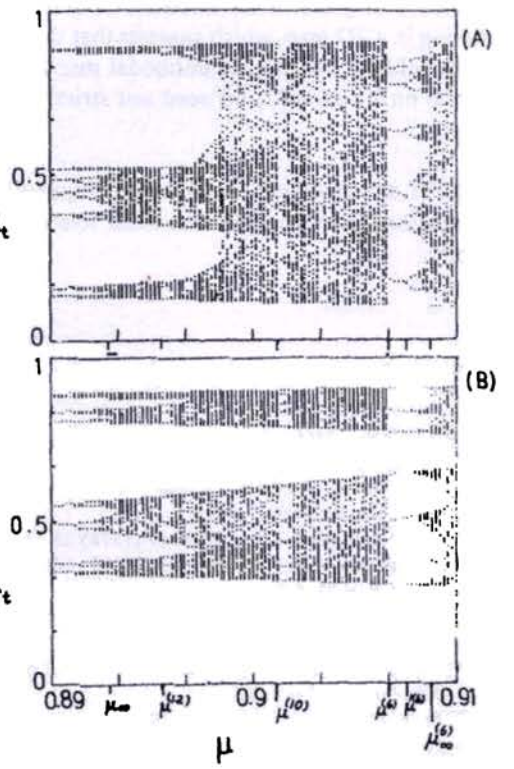

Fig. 1. Bifurcation structure of MLM for the parameter range $0.89<\mu<0.91$ projected (A) onto the (X_t, μ) plane and (B) onto the (λ_t, μ) plane which is just the bifurcation structure of the logistic map for the specified range of the parameter.

To confirm this, we have repeated the above procedure for another range of the parameter μ from 0.95 to 0.97. The projection of the diagram in the (X_t, μ) and (λ_t, μ) planes are shown in figs. 2A and 2B respectively. From the figures, we see that X_t and λ_t have a window of basic period 3 for exactly the same range of μ values.

The above result has the important consequence that the complete 3D bifurcation diagram of MLM has a stable n-cycle for the same range of μ in which the logistic map has a stable n-cycle. Now, there are an uncountable number of periodic cycles in the chaotic region and, as noted by May [10], most of them occupy an extremely narrow window of parameter values. This fact, coupled with the long time it is likely to take for the transients associated with the

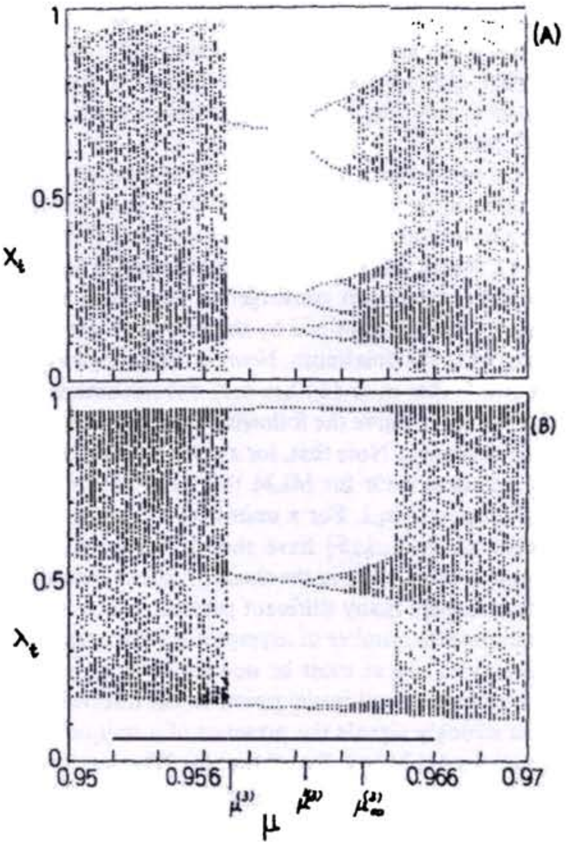

Fig. 2. Bifurcation structure of MLM for the parameter range $0.95<\mu<0.97$ projected (A) onto the (X_t, μ) plane and (B) onto the (λ_t, μ) plane.

initial conditions to damp out, implies that in practice the unique cycle is unlikely to be unmasked, excepting the prominent ones. Since the prominent cycles occupy exactly the same interval for the logistic map and the MLM, it is reasonable to expect that the order in which different periodic cycles are arranged along the μ-axis is the same for both of them. It is also interesting to note that, even though the band structure in the (X_t, μ) plane is different from that in the (λ_t, μ) plane, this ordering is left unaffected.

Now, the order in which different periodic cycles appear in the bifurcation structure of the logistic map can be explained in terms of a rigourous mathematical theorem due to Sarkovskii [11,12]. This theorem was independently obtained by Li and Yorke [13] also, who showed that period 3 implies the existence of all other periods. Interestingly, the Sarkovskii theorem is rigourously proved only for 1D maps of an interval on the real line [12]. Since no higher dimensional maps are known to obey Sarkovskii ordering, it is widely believed to be only a 1D result.

Let us now look into the general criteria for a mapping to show the Sarkovskii ordering. It is known to be a universal structural property of one hump maps [14]. Recall that the universal metric properties such as the Feigenbaum convergence rate δ and scaling factor α are determined by the analytic form of the map near the maximum. Now, it can be rigourously shown [#1] that map (2) has only one maximum in I. Here, we only give the following evidence to support our statement. Note that, for a given value of μ there is a unique cycle for MLM that attracts almost all initial points in I. For a unimodal mapping, Guckenheimer et al. [15] have shown that for each parameter value, even in the chaotic regime where there are infinitely many different periodic orbits and an uncountable number of asymptotically aperiodic orbits, there can at most be one stable attractor that attracts almost all initial points in the interval. This fact strongly signals the presence of a unique maximum for MLM in I. Recall that for 2D maps, in general, more than one attractor, each with its own basin of attraction, can coexist at a given parameter value. For example, Grebogi et al. [16] have shown this for the Hénon map, which is considered as the 2D analogue of the quadratic map. It has been shown that the Sarkovskii theorem does not hold for the Hénon map [12]. Note that the existence of a unique maximum for MLM is also supported by our result [9] that it obeys the universal behaviour of Feigenbaum near the transition to chaos.

We conclude this note with the following remarks. Sarkovskii ordering is simply the order in which different periodic cycles are arranged along the parameter axis. It says nothing about the observability or measure of these periods. For a detailed study of the total number of stable cycles and their arrangement, we need the powerful tool of symbolic dynamics. This has been done for a wide class of unimodal mappings using the so called MSS sequences [17,18] and the arrangement was again found to be universal.

Finally, it should be noted that we have not given any formal proof of the Sarkovskii theorem in 2D. We have only shown the existence of Sarkovskii ordering in a 2D map, which suggests that the theorem may hold for continuous unimodal maps of an interval on $\mathbb{R}^2$ as well, and need not strictly be a 1D result.

[#1] We plan to incorporate this result in a separate communication.

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