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Solitons and their resonances on two-dimensional superfluid films

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Abstract. The dynamics of saturated two-dimensional superfluid ^{4}He films is shown to be governed by the Kadomtsev-Petviashvili equation with negative dispersion. It is established that the phenomena of soliton resonance could be observed in such films. Under the lowest order nonlinearity, such resonance would happen only if two dimensional effects are taken into account. The amplitude and velocity of the resonant soliton are obtained.

Keywords. Solitons; superfluid ^{4}He; soliton resonance; Kadomtsev-Petviashvili equation.

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1. Introduction

The concept of solitons, their propagation and interaction have been playing increasingly important roles in different branches of physics. In superfluid ^{4}He films, small finite amplitude localised density fluctuations can lead to the existence of solitons made up of superfluid condensate. This arises essentially due to the balance between dispersion and the nonlinearity arising from the Van der Waals potential of the substrate. Huberman (1978) showed that the nonlinear local density fluctuations in very thin ^{4}He films ($\sim 10^{-7}$ cm) may travel unattenuated for large times. These superfluid density fluctuations were shown to be governed by the Korteweg-de Vries (KdV) equation (Korteweg and de Vries 1895). Nakajima *et al* (1980a) examined the problem from a hydrodynamical point of view and applied the reductive perturbation scheme to obtain the KdV equation.

Biswas and Warke (1980) confirmed theoretically the predictions about the existence of superfluid solitons by deriving a KdV equation from the phenomenological Hamiltonian suggested by Rutledge *et al* (1978). This KdV equation had a nonlinearity different from that proposed by Huberman. Recently it has been shown by Radha Balakrishnan *et al* (1989) that the equation of motion of the superfluid condensate could be obtained by starting from a microscopic theory of nonlinear dynamics in superfluid ^{4}He, formulated using a model in which a system of bosons with hard-cores plus nearest neighbour interaction is described by a pseudospin Hamiltonian on a lattice.

These results were later generalized to quasi-two-dimensional wave propagation (where essentially the direction of propagation was chosen to be along the x-axis and the y-dependence was assumed to be weak) by Biswas and Warke (1983) who obtained the Kadomtsev-Petviashvili (K-P) equation (Kadomtsev and Petviashvili 1970). Based on these results, we have (Sreekumar and Nandakumaran 1985) studied the phenomenon of two soliton resonances of the K-P equation for the superfluid surface density fluctuations and obtained the amplitude and velocity of the resonant solitons. The existence of large amplitude solitons in the very thin two dimensional superfluids has also been established recently by us (Sreekumar and Nandakumaran 1989), by numerically integrating the fully nonlinear Schrödinger equation representing the superfluid surface density fluctuation.

All these have been studied on very thin ^{4}He films. Nakajima *et al* (1980b) extended this analysis to the so-called saturated film of superfluid ^{4}He, whose thickness is of the order of 10^{-6} cm. In such films, the surface tension plays a decisive role in the dynamics of the system, which was totally ignored for very thin films. The effect of surface tension (Nakajima *et al* 1980b) is to increase the characteristic length of the soliton, and to reduce the soliton velocity. This makes the detection and generation of solitons using conventional third sound apparatus easier. The analysis done by Nakajima *et al* was restricted to one dimension. It seems natural, therefore, to search for quasi two dimensional solitons in such systems.

In this paper we investigate the nonlinear waves propagating on a two-dimensional saturated film of superfluid ^{4}He. Here we are concerned with the temporal evolution of the fluctuations in thickness of the superfluid. This is in contrast to the earlier investigations by Huberman (1978), Biswas and Warke (1980, 1983), Nakajima *et al* (1980a) and Sreekumar and Nandakumaran (1985), where the surface deformation was negligible and only density fluctuations were present, due to the very small thickness of the superfluid film. In the small amplitude regime our system reduces to the K-P equation. It is shown that resonance of solitons can be observed in such films as against the one-dimensional case studied by Nakajima *et al* (1980b).

In the next section we derive the governing equations for the surface displacement. In §3, we consider the small amplitude regime, and obtain the K-P equation. It is shown in §4 that in the lowest order nonlinearity, soliton resonance could be obtained only if two dimensional effects are taken into account.

2. Finite amplitude surface waves

When saturated films of superfluids are considered one has to include the effects of surface tension, which is generally ignored for very thin films. The Van der Waals force is the nonlinear force acting on the superfluid film. The acceleration of the superfluid due to a temperature gradient, which acts as a very small correction factor (Rutledge *et al* 1978) in our low temperature film is neglected in this paper. We consider the x and y axis to be lying on the substrate on which the superfluid of equilibrium depth d exists. Geometrical configuration of the system is shown in figure 1.

Since the superflow is irrotational we can describe it by the velocity potential $\Phi_{(x,y,z,t)}$. If we treat the system to be incompressible we can write the equation of continuity in bulk as

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}=0. \tag{1}$$

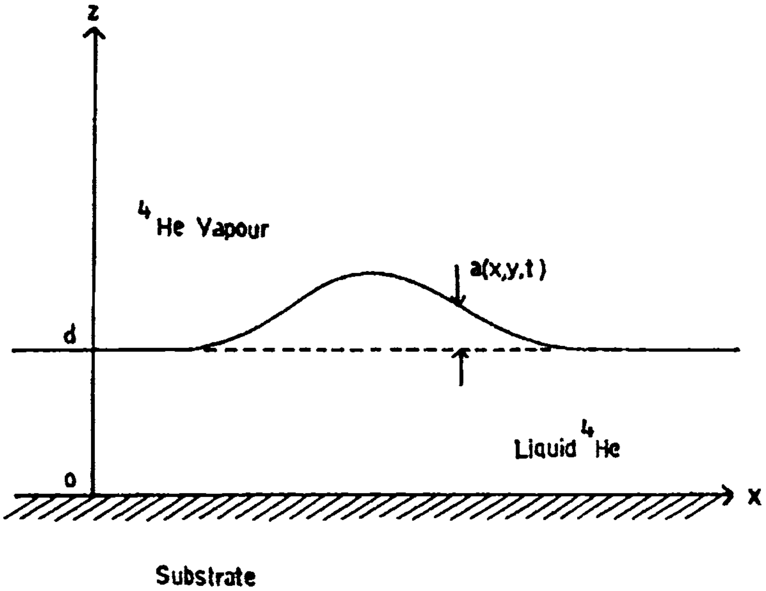

Figure 1. Geometrical configuration of the system. The x and y axes lie on the plane surface of the substrate.

There is the additional condition that the superfluid would not flow into the substrate.

$$\left.\frac{\partial\Phi}{\partial z}\right|_{z=0}=0. \tag{2}$$

The continuity equation at the film-vapour interface takes the form

$$\frac{\partial z_1}{\partial t}+\left(\frac{\partial\Phi}{\partial x}\right)_1\frac{\partial z_1}{\partial x}+\left(\frac{\partial\Phi}{\partial y}\right)_1\frac{\partial z_1}{\partial y}-\left(\frac{\partial\Phi}{\partial z}\right)_1=0. \tag{3}$$

The index 1 refers to the film-vapour interface $z_1=d+a(x,y,t)$, where $a(x,y,t)$ is the departure of the film surface from its equilibrium position. The equation of motion at the surface is (Nakajima *et al* 1980b)

$$\left(\frac{\partial\Phi}{\partial t}\right)_1+\frac{1}{2}\left[\left(\frac{\partial\Phi}{\partial x}\right)_1^2+\left(\frac{\partial\Phi}{\partial y}\right)_1^2+\left(\frac{\partial\Phi}{\partial z}\right)_1^2\right]-\frac{\sigma}{\bar\rho}\left(\frac{\partial^2 a}{\partial x^2}+\frac{\partial^2 a}{\partial y^2}\right)$$
$$+g_1 a-\frac{1}{2}\frac{g_2}{d}a^2=0. \tag{4}$$

The last two terms appearing in (4) represent the leading terms in the expansion of the Van der Waals force term. $g_1=3\alpha/d^4$ and $g_2=12\alpha/d^4$. α is the Van der Waals constant, $\bar\rho$ is the density of the superfluid and σ its surface tension. Equations (3) and (4) represent finite amplitude surface waves propagating on the superfluid film. We expand $\Phi(x,y,z,t)$ as (Lamb 1980)

$$\Phi(x,y,z,t)=\sum_{n=0}^{\infty}z^n\phi_n(x,y,t). \tag{5}$$

Now by using (1) and (2), we get comparing like powers of z,

$$\Phi(x, y, z, t) = \cos(z\nabla)\phi_0(x, y, t), \tag{6}$$

where ∇ is the two dimensional gradient having components $\partial/\partial x$ and $\partial/\partial y$ along x and y axes respectively. Equation (4) can be rewritten after operating once with ∇ as

$$\frac{\partial \mathbf{V}}{\partial t} + \tfrac{1}{2}\nabla V_1^2 - \frac{\sigma}{\bar{\rho}}\nabla\nabla^2 a + g_1\nabla a - \frac{g_2}{2d}\nabla a^2 = 0, \tag{7}$$

where $\mathbf{V}$ is the velocity in two dimensions.

3. Solitary waves

Using (3) and (7) we have studied the dynamics of localised disturbances, of long wavelength and small amplitude, in the superfluid film thickness. We orient the horizontal coordinate system such that the principal direction of propagation is chosen as the x-axis. We make the following coordinate transformation.

$$X \to x + C_3 t, \quad t \to t, \tag{8}$$

where C_3 is velocity of the moving frame.

To transform the equations (3) and (7) into a wave equation with respect to the superfluid surface displacement, the reductive perturbation method by Tanuiti and Wei (1968) can be applied using the scaling transformation

$$\bar{x} = \varepsilon^{1/2}X, \quad \bar{y} = \varepsilon y, \quad \bar{t} = \varepsilon^{3/2}t. \tag{9}$$

We regard ε as an infinitesimal, however it disappears in the final equation. We are essentially looking for fluctuations in the thickness of the film which travel with long wavelength along the x-direction and we assume that the y-coordinate dependence of the wave is weak. We expand a and $\mathbf{V}$ in powers of ε.

$$a = a_0 + \varepsilon a_1(x, y, t) + \varepsilon^2 a_2(x, y, t), \tag{10a}$$

$$u = \varepsilon u_1 + \varepsilon^2 u_2 + \cdots, \tag{10b}$$

$$v = \varepsilon^{1/2}(\varepsilon v_1 + \varepsilon^2 v_2 + \cdots), \tag{10c}$$

where the expansions of the x and y components of the velocity, u and v, are made consistent with $\Phi_{xy} = \Phi_{yx}$. Using equations (3) and (6)–(10) and comparing coefficients of $\varepsilon^{3/2}$ and $\varepsilon^{5/2}$ we get

$$C_3\frac{\partial a_1}{\partial \bar{x}} + d\frac{\partial u_1}{\partial \bar{x}} = 0 \tag{11}$$

$$C_3\frac{\partial u_1}{\partial \bar{x}} + g_1\frac{\partial a_1}{\partial \bar{x}} = 0 \tag{12}$$

$$\frac{\partial u_1}{\partial \bar{t}} + C_3\frac{\partial u_2}{\partial \bar{x}} + u_1\frac{\partial u_1}{\partial \bar{x}} - \frac{\sigma}{\bar{\rho}}\frac{\partial^3 a_1}{\partial \bar{x}^3} + g_1\frac{\partial a_2}{\partial \bar{x}} - \frac{g_2}{d}a_1\frac{\partial a_1}{\partial \bar{x}} - \frac{C_3 d^2}{2}\frac{\partial^3 u_1}{\partial \bar{x}^3} = 0 \tag{13}$$

$$\frac{\partial a_1}{\partial \bar{t}} + C_3 \frac{\partial a_2}{\partial \bar{x}} + u_1 \frac{\partial a_1}{\partial \bar{x}} + a_1 \frac{\partial u_1}{\partial \bar{x}} + d \frac{\partial v_1}{\partial y} + d \frac{\partial u_2}{\partial \bar{x}} - \frac{1}{6} d^3 \frac{\partial^3 u_1}{\partial \bar{x}^3} = 0. \tag{14}$$

Using the boundary conditions that a_1 and u_1 goes to zero as $x \to \infty$, (11) and (12) can be solved to get

$$u_1 = \frac{-C_3}{d} a_1 \quad \text{and} \quad u_1 = \frac{-g_1}{C_3} a_1 \quad \text{i.e. } C_3^2 = g_1 d. \tag{15}$$

Now eliminating u_2 between (13) and (14),

$$\frac{\partial}{\partial \bar{x}} \left[2C_3 \frac{\partial a_1}{\partial \bar{t}} + \left(g_2 - \frac{3C_3^2}{d} \right) a_1 \frac{\partial a_1}{\partial \bar{x}} + \left(\frac{\sigma d}{\bar{\rho}} - \frac{C_3^2 d^2}{3} \right) \frac{\partial^3 a_1}{\partial \bar{x}^3} \right] - C_3^2 \frac{\partial^2 a_1}{\partial \bar{y}^2} = 0. \tag{16}$$

Equation (16) is the K-P equation, which can be expressed in the more familiar form by the following transformations.

$$a_1 = -6\gamma\rho, \quad \frac{1}{\gamma} = \left(g_2 - \frac{3C_3^2}{d} \right) \frac{1}{2C_3}, \quad \xi = -k_0 \bar{x}$$

$$-\frac{1}{k_0^2} = \left(\frac{\sigma d}{\rho} - \frac{C_3^2 d^2}{3} \right) \frac{1}{2C_3}, \quad \eta = \sqrt{\frac{2k_0^2}{C_3}} \bar{y}, \quad \tau = k_0 \bar{t}. \tag{17}$$

So (16) would become

$$\frac{\partial}{\partial \xi} \left[\frac{\partial \rho}{\partial \tau} + 6\rho \frac{\partial \rho}{\partial \xi} + \frac{\partial^3 \rho}{\partial \xi^3} \right] + \frac{\partial^2 \rho}{\partial \eta^2} = 0. \tag{18}$$

The K-P equation represented by (18) is the one with negative dispersion and it is known to possess N-soliton solutions (Satsuma 1976). This has been discussed in great detail by Ohkuma and Wadati (1983). The one soliton can be written as (Satsuma and Ablowitz 1979)

$$\rho = \tfrac{1}{2} k^2 \operatorname{sec} h^2 \zeta \tag{19}$$

where

$$\zeta = \tfrac{1}{2} k [\xi + p\eta - (k^2 + p^2)\tau] + \zeta^{(0)}$$

and k and kp are the components of the linear momentum along the ξ and η directions respectively. Equation (19) describes a soliton propagating with velocity $(k^2 + p^2)/\sqrt{1+p^2}$ in the direction making an angle $\tan^{-1}(p)$ with the x-axis. This angle should be small because the K-P equation holds under the assumption that the two-dimensional effect is small.

4. Soliton resonance

The two soliton solution for equation (18) is obtained from

$$\rho = 2(\log f_2)_{\xi\xi} \tag{20}$$

where

$$f_2 = 1 + e^{2\zeta_1} + e^{2\zeta_2} + A_{12}\exp 2(\zeta_1 + \zeta_2)$$

$$\zeta_i = \tfrac{1}{2}k_i[\xi + p_i\eta - (k_i^2 + p_i^2)\tau] + \zeta^{(0)}$$

and

$$A_{12} = \frac{3(k_1 - k_2)^2 - (p_1 - p_2)^2}{3(k_1 + k_2)^2 - (p_1 - p_2)^2}. \tag{20a}$$

Soliton resonance occurs (Hirota and Ito 1983) when $A_{12} = 0$ or ∞, i.e. for $3(k_1 \pm k_2)^2 - (p_1 - p_2)^2 = 0$. The plus sign refers to the case $A_{12} = \infty$ and is called plus resonance and the other case ($A_{12} = 0$) is called minus resonance.

The resonant soliton in general can be written (Sreekumar and Nandakumaran 1985) in the form

$$\rho^{(1\pm 2)} = \tfrac{1}{2}(k_1 \pm k_2)^2 \operatorname{sec}h^2(\zeta_1 \pm \zeta_2). \tag{21}$$

The amplitude and velocity of the resonant soliton, in the original coordinate system, can be written as

$$A_r = \frac{6C_3 d}{(g_2 d - 3C_3^2)}(k_1 \pm k_2)^2 = \frac{6(g_1 d)^{1/2}}{(g_2 - 3g_1)}(k_1 \pm k_2)^2 \tag{22a}$$

$$V_r = \frac{[6\sigma d + C_3^2 d^2 \bar{\rho}]^{1/2}[k_1(k_1^2 + p_1^2) \pm k_2(k_2^2 + p_2^2)]}{\left\{12C_3^2\bar{\rho}(k_1 \pm k_2)^2 + \frac{d}{2}(k_1 p_1 \pm k_2 p_2)^2(6\sigma d + C_3^2 d^2 \bar{\rho})\right\}^{1/2}} - (g_1 d)^{1/2}. \tag{22b}$$

If the resonance is to be observed in actual experimental set up, the resonance conditions given by (20a) should be consistent with the conservation laws. Tajiri and Nishitani (1982a) showed that this condition is satisfied for the K-P equation (18) in the following way. First a similarity transformation is applied to the K-P equation. Then the resonance conditions of the resulting equation are shown to satisfy the corresponding conservation laws. The similarity transformation has the form (Tajiri and Nishitani 1982b)

$$t' = \frac{\xi}{P^{3/2}} + \frac{2}{3}\int \frac{Q}{P^{5/3}}\mathrm{d}\tau \tag{23a}$$

$$x' = \frac{\eta}{P^{1/3}} + \frac{P'}{6P^{4/3}}\xi^2 - \frac{Q}{3P^{4/3}}\xi - \frac{2}{9}\int \frac{Q^2}{P^{7/3}}\mathrm{d}\tau - \frac{1}{3}\int \frac{R}{P^{4/3}}\mathrm{d}\tau \tag{23b}$$

$$\rho = \frac{P'}{6\rho}\eta + \frac{1}{6}\left\{\frac{1}{3}\left(\frac{P'}{P}\right)^2 - \frac{P''}{2P}\right\}\xi^2 + \frac{1}{6}\left(\frac{Q'}{P} - \frac{2P'Q}{3P^2}\right)\xi$$
$$+ \frac{R}{6P} + \frac{Q^2}{18P^2} - \frac{1}{2P^{2/3}} + \frac{1}{P^{2/3}}u(x', t'), \tag{23c}$$

where P, Q and R are function of τ, $p' = \mathrm{d}p/\mathrm{d}\tau$, $p'' = \mathrm{d}^2 p/\mathrm{d}\tau^2$ and $Q' = \mathrm{d}Q/\mathrm{d}\tau$. Using (23) in (12) we get the Boussinesq type equation

$$u_{t't'} - u_{x'x'} + (u^2)_{x'x'} + u_{x'x'x'x'} = 0. \tag{24}$$

Tajiri and Nishitani (1982a) showed that this equation exhibits soliton resonance and the resonance conditions do satisfy its conservation laws. This suggests that soliton resonance may be observed in two-dimensional saturated films of superfluid ^{4}He.

Now we turn our attention to one dimensional wave propagation in saturated superfluid films. The governing equations (Nakajima *et al* 1980b) in this case, under weak nonlinearity, is the KdV equation.

$$u_t + 6uu_x + u_{xxx} = 0. \tag{25}$$

The one soliton solution for KdV equation is

$$u = 2k^2 \operatorname{sech}^2 (kx - \Omega t)$$

and its resonance conditions are given by

$$(k_1 \mp k_2)[\Omega_1 - \Omega_2 + (k_1 - k_2)^3] = 0. \tag{26}$$

The first two conserved quantities of the KdV equation are $\int u \, dx$ and $\int u^2 \, dx$. For two soliton resonant interaction, these conservation laws give

$$k_1 \pm k_2 = K \tag{27a}$$

$$k_1^3 \pm k_2^3 = K^3, \tag{27b}$$

where k_1 and k_2 corresponds to the initial solitons and K to the final resonant soliton. The plus and minus signs corresponds to the two different types of resonance. Equations (27a & b) are not satisfied for any k_1 and k_2 except for the trivial cases $k_1 = 0$, $k_2 = 0$ or $k_1 = -k_2$. Hence we can say that it is not possible to have soliton resonance in one-dimensional saturated superfluid films, under weak nonlinearity.

5. Discussion

We have reduced the hydrodynamics equations for saturated superfluid films to the K-P equation with negative dispersion in the small amplitude regime. This is to be compared with the result of Biswas and Warke, who obtained K-P equation with a positive dispersion. The two problems are, however entirely different. They consider superfluid density fluctuations in thin films whereas in the present paper we are discussing the fluctuations in thickness of the superfluid films.

In the saturated films one is able to observe the phenomenon of soliton resonances when two-dimensional wave propagation is considered. The amplitude and velocity of the resonant soliton are given by (22). We have shown explicitly that under lowest order of nonlinearity soliton resonance is observable only when two-dimensional wave propagation is taken into consideration. In the actual experimental situation, this would mean that, resonance can be observed only when the initial profile has a small decay along the direction perpendicular to its direction of motion.

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