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The quantum effects in quadratically damped systems

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Abstract

A dynamical system with a damping that is quadratic in velocity is converted into the Hamiltonian format using a nonlinear transformation. Its quantum mechanical behaviour is then analysed by invoking the Gaussian effective potential technique. The method is worked out explicitly for the Duffing oscillator potential.

1. Introduction

During the last two decades, a number of ingenious techniques has been developed to incorporate damped or dissipative systems into the framework of quantum mechanics [1]. The problem of the classical damped oscillator has been studied extensively, considering either an enlarged phase space that includes the bath, or allowing the Hamiltonian to be explicitly time dependent [2,3]. Since quantum mechanics is essentially a Hamiltonian theory, to incorporate friction or damping one usually introduces a complex Hamiltonian with a symplectic structure [4,5]. All these works relate to systems with linear damping, i.e. a damping which is proportional to the velocity.

In this paper we consider a class of systems with a particular type of quadratic damping. Such a quadratic damping has physical significance as it arises in the context of Josephson tunnel junctions at finite temperatures. The dynamics of these systems are governed by equations of the form

$$\ddot{x}+k|\dot{x}|\dot{x}+f(x)=0\,. \tag{1}$$

Here x is the independent variable and k, the normalised damping coefficient. $f(x)=-\partial V(x)/\partial x$ is in general a nonlinear function of x. It is easy to see that for such systems, the phase space area A contracts at a rate given by

$$\frac{1}{A}\frac{dA}{dt}\approx -2k|\dot{x}|\,, \tag{2}$$

while it is $-k$, for linearly damped systems. The nonlinear nature of the equations in this case gives rise to chaotic behaviour when acted by a periodic rf field [6,7]. To understand the quantum mechanics of the situation one has to first give a Hamiltonian structure to the equations. A few isolated works reported earlier, relating to such systems, bring out the existence of an integral of motion for the system when $\dot{x}$ is either >0 or <0 [8]. Moreover for systems with a general quadratic term in velocity, such as $g(x)\dot{x}^2$, the first integral of motion and its quantum mechanics have been worked out in detail by Lakshmanan et al. [9,10].

We start by defining the Lagrangian and the Hamiltonian for systems governed by (1), separately for

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$\dot{x}>0$ and $\dot{x}<0$ and then connect them by means of a transformation. Such a transformation has been introduced by one of us [11]. The quantum mechanical effects in the transformed system can then be analysed using variational techniques. In particular, for the system that we work out in detail in this paper, we invoke the idea of the Gaussian effective potential (GEP) technique due to Stevenson [12], and obtain the ground state and the first few excited states of the system. The quantum effects seem to provide corrections to the ground state energy. We also find that, even when the classical potential has a double-well nature, the GEP reveals a single-well behaviour with a bound state in one of the wells.

The paper is organised as follows. In Section 2, we construct the Hamiltonian for the system based on the classical formalism. The basic concepts behind the GEP technique are given briefly in Section 3, while the details of its application to a Duffing oscillator with quadratic damping are included in Section 4. Section 5 contains our concluding remarks and comments.

2. The classical formalism

In this section, we derive the Hamiltonian for systems governed by (1). The integral of motion for $\dot{x}<0$ or $\dot{x}>0$ can be shown to be [11]

$$I_1=\frac{e^{\pm 2kx}\dot{x}^2}{2}+\int f(x)e^{\pm 2kx}\,dx\,, \tag{3}$$

with plus sign for $\dot{x}>0$ and minus sign for $\dot{x}<0$. The Lagrangian that reproduces (1) is

$$L=\tfrac{1}{2}\dot{x}^2e^{\pm 2kx}-\int e^{\pm 2kx}f(x)\,dx\,. \tag{4}$$

The integral in (4) can be evaluated once the form of $f(x)$ (usually a polynomial) in the equation of motion is known. The conjugate momentum is

$$p_x=\dot{x}e^{\pm 2kx} \tag{5}$$

and the Hamiltonian is

$$H=\tfrac{1}{2}p_x^2e^{\pm 2kx}+\int e^{\pm 2kx}f(x)\,dx\,. \tag{6}$$

Such a Hamiltonian would represent physically a system or particle whose mass increases exponentially with x for $\dot{x}>0$ and decreases for $\dot{x}<0$. But there is a discontinuity about $\dot{x}=0$, i.e. at the turning point in H or L. This discontinuity can be taken care of for all computational purposes by using $\tan^{-1}$ or tanh functions. Thus (6) can be written as,

$$H=\tfrac{1}{2}p_x^2\exp\left(-2kx\frac{2}{\pi}\tan^{-1}(\gamma\dot{x})\right)+\int f(x)\exp\left(2kx\frac{2}{\pi}\tan^{-1}(\gamma\dot{x})\right)dx\,. \tag{7}$$

In the limit $\gamma\to\infty$, (7) gives the equation of motion.

We introduce a novel way of looking at such damped systems. The idea is to transform to a new set of coordinates (X, P) through the equations

$$X=\frac{1}{k}e^{\pm kx},\quad P=\pm p_xe^{\pm kx}\,. \tag{8}$$

Then

$$P=\dot{X},\quad L=\tfrac{1}{2}\dot{X}^2-V(X)\,, \tag{9}$$

where $V(X)$ is obtained by performing the integration in (4) and then using (8),

$$H=\tfrac{1}{2}P^2+V(X)\,. \tag{10}$$

We make the following observations.

(i) The transformation in (8) maps the phase plane (x, p_x) to the half-plane $(X, P|X>0)$.

(ii) In the transformed system (X, P), the equation of motion as well as the potential $V(X)$ have the same form for both $x>0$ and $x<0$ provided the original potential $V(x)$ is symmetric with respect to x. So the discontinuity in H in (6) is not explicit in (10).

(iii) It is obvious that the original system is dissipative, but in (X, P), the coordinates, considered as functions of (x, p_x), change during every oscillation in such a way as to compensate for the damping in (x, p_x).

Therefore the asymptotic states in the original system should correspond to steady states in the new system.

3. The GEP technique

In general, the potential $V(X)$ in (10) is fairly complicated with a nonpolynomial form and so an analytic solution of the Schrödinger equation is practically difficult. So to understand the quantum be-

haviour of the system, at least semiclassically, we have to consider certain approximation techniques. We choose the GEP technique for this purpose, since this has given excellent results for quartic and other anharmonic potentials [12]. Moreover it is a powerful and versatile technique that is computationally easier to handle.

For a system with Hamiltonian H, the expectation value of energy is calculated as $\langle F|H|F\rangle$ where the normalised wave function $|F\rangle$ is Gaussian in form. To calculate the ground state, we take

$$|0\rangle_\Omega = |F\rangle = (\Omega/\hbar\pi)^{1/4} \exp\left(-\frac{\Omega}{2\hbar}(X-X_0)^2\right). \tag{11}$$

Here the width of the Gaussian, Ω, is the adjustable parameter and the GEP is defined as

$$V_G = \min_\Omega \langle F|H|F\rangle . \tag{12}$$

To evaluate (12) we start by writing

$$X = X_0 + U ,$$

where U can be expressed using creation and annihilation operators as [13]

$$U = \hbar(2\hbar\Omega)^{-1/2}(a_\Omega + a_\Omega^+) \tag{13}$$

and

$$P = -\tfrac{1}{2}i(2\hbar\Omega)^{1/2}(a_\Omega - a_\Omega^+) . \tag{14}$$

The potential $V(X)$ can be expanded about X_0 as

$$V(X) = V(X_0) + UV'(X_0) + \frac{U^2}{2!}V''(X_0) + \frac{U^3}{3!}V'''(X_0) + \frac{U^4}{4!}V''''(X_0) + \ldots , \tag{15}$$

where the primes indicate derivatives w.r.t. X. In our calculations, we retain terms up to the fourth derivative of V. Using Eqs. (11)–(15), it can be shown that

$$V_G = \tfrac{1}{4}\hbar\Omega + V(X_0) + \tfrac{1}{4}\hbar\Omega V''(X_0) + \frac{\hbar^2}{32\Omega^2}V''''(X_0) . \tag{16}$$

This is to be minimised w.r.t. Ω, which gives the equation for $\Omega_{\min}$,

$$\Omega_{\min}^3 - \Omega_{\min}V''(X_0) - \tfrac{1}{4}\hbar V''''(X_0) = 0 . \tag{17}$$

Using (17) in (16)

$$V_G = V(X_0) = \tfrac{1}{2}\hbar\Omega_{\min} - \frac{\hbar^2}{32\Omega_{\min}^2}V''''(X_0) . \tag{18}$$

For each X_0, V_G is calculated and the minimum value gives the ground state energy E_G^0 of the system in this approximation.

To get the excited states, $|F\rangle$ in (11) is replaced by $|n\rangle_\Omega$ obtained by applying a^+ on $|0\rangle_\Omega$ n times. Then

$$V_G^n = V(X_0) + \hbar\Omega_{\min}(n+\tfrac{1}{2}) - \frac{\hbar^2}{32\Omega_{\min}^2}V''''(X_0)(2n^2+2n+1) , \tag{19}$$

with $\Omega_{\min}$ given by

$$\Omega_{\min}^3 - V''(X_0)\Omega_{\min} - \frac{\hbar}{4(2n+1)}V''''(X_0)(2n^2+2n+1) = 0 . \tag{20}$$

The minimum of V_G^n gives the excited state energies E_G^n.

In the following section, the quantum corrections for a quadratically damped Duffing oscillator are highlighted using the above formalism.

4. The quadratically damped Duffing oscillator

We take the equation of motion of the Duffing oscillator in the following form,

$$\ddot{x} + k|\dot{x}|\dot{x} - \alpha x + \beta x^3 = 0 . \tag{21}$$

The potential of this nonlinear oscillator has a double-well nature with minima at $x = \pm\sqrt{\alpha/\beta}$ and a maximum at $x=0$. The details of the potential for the parameter values studied are shown in Table 1. The effect of quadratic damping is incorporated into the potential by transforming to variables (X, P) where the potential has the form,

$$V(X) = \frac{X^2}{2k^2}\left[\beta(\ln kX)^3 - \tfrac{3}{2}\beta(\ln kX)^2 + \tfrac{3}{2}\beta\ln kX - \tfrac{3}{4}\beta - \alpha k^2\ln kX + \tfrac{1}{2}\alpha k^2\right] . \tag{22}$$

For $\alpha > 0$, $V(X)$ has one maximum at $\bar{X}_0$ and two minima at $\bar{X}_{1,2}$ given by

Table 1
The characteristics of the potential for the Duffing oscillator for the parameter values chosen

k	β	X_b	V_b	X_{01}	$V_{X_{01}}$	X_{02}	$V_{X_{02}}$
0.9	0.6	1.111	−0.034294	0.347658	−0.172023	3.55109	−1.668090
0.9	3.0	1.111	−1.406036	0.66083	−1.455198	1.868201	−1.554918
0.6	3.5	1.667	−9.432870	1.209385	−9.4841198	2.296852	−9.5344437

$$\bar{X}_0=\frac{1}{k},\quad \ln k\bar{X}_{1,2}=\pm(\alpha/\beta)^{1/2}k\,. \tag{23}$$

The classical equation of motion is

$$\ddot{X}=-\frac{\beta X}{k^2}(\ln kX)^3+\alpha X\ln kX\,. \tag{24}$$

Considering motion near the extrema $\bar{X}$ of the potential, where

$$X=\bar{X}+Y\,, \tag{25}$$

with $Y/\bar{X}\ll 1$, we can show that,

$$\ddot{Y}=\alpha Y,\quad \text{for } \bar{X}_0=1/k\,,$$

$$\ddot{Y}=-2\alpha Y,\quad \text{for } \bar{X}_{1,2}\,. \tag{26}$$

This would mean that oscillatory motion is to be expected classically near the equilibrium points $\bar{X}_{1,2}$.

The quantum mechanical ground state energy is calculated using (18) as

$$V_G=V(X_0)+\tfrac{1}{2}\hbar\Omega_{\min}+\frac{\hbar^2 f_2}{32X_0^2\Omega_{\min}^2}\,, \tag{27}$$

with $\Omega_{\min}$ given by

$$\Omega_{\min}^3-\Omega_{\min}f_1+\frac{\hbar}{4X_0^2}f_2=0\,. \tag{28}$$

Here

$$f_1=\frac{\beta}{k^2}(\ln kX_0)^3+\frac{3\beta}{k^2}(\ln kX_0)^2-\alpha\ln kX_0-\alpha\,, \tag{29}$$

$$f_2=\frac{3\beta}{k^2}(\ln kX_0)^2-\frac{6\beta}{k^2}-\alpha\,. \tag{30}$$

The excited states are

$$V_G^n=V(X_0)+\hbar\Omega_{\min}(n+\tfrac{1}{2})+\frac{\hbar^2 f_2}{32X_0^2\Omega_{\min}^2}(2n^2+2n+1)\,, \tag{31}$$

with

$$\Omega_{\min}^3-f_1\Omega_{\min}+\frac{\hbar}{4X_0^2}\frac{2n^2+2n+1}{2n+1}f_2=0\,. \tag{32}$$

In our calculations, we keep $\alpha=1$ and $\hbar=1$. We study the quantum effects using the GEP expressions given above. Our results are presented in Table 2.

Table 2
The ground state E_0^0 and the first two excited state energies E_0^n calculated using the GEP technique

k	β	E_0^n	n	X_0	Ω
0.9	0.6	−1.150296	0	3.330998	0.9606712
		−0.7985667	1	2.174987	0.3987142
		−0.3884149	2	2.160996	0.5388581
0.9	3.0	−1.091304	0	1.423011	1.151759
		−0.02626151	1	1.407	1.426401
		1.471493	2	1.447002	1.678426
0.6	3.5	−9.274662	0	2.018999	0.8659412
		−8.512424	1	1.984037	1.109861
		−7.340580	2	2.015	1.339981

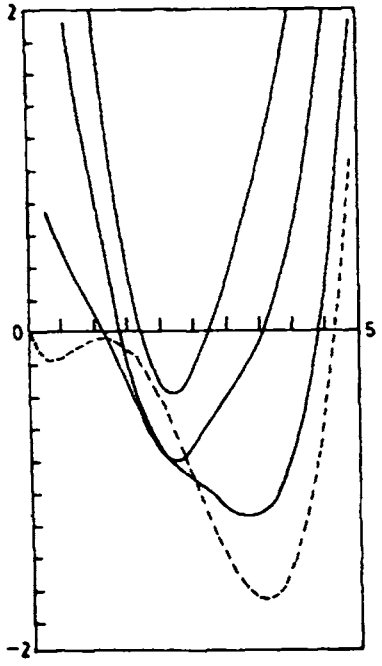

Fig. 1. The Gaussian effective potential for the double-well potential of the Duffing oscillator in (21) with $\alpha=1$, $k=0.9$ and $\beta=0.6$. The dashed curve shows the potential while the solid curves indicate the ground state and the first two excited states, successively from below.

In all cases, we have studied, we find that V_G shows only a single-well behaviour, even though the corresponding classical situation is a double-well potential. For large values of k, the potential is an asymmetric double-well, the second well being much deeper than the first. Then for $\beta<1$, V_G has a minimum inside the deeper well. Thus the ground state corresponds to a bound state inside the well. The excited states have their minima shifted towards the left. This is clear from Fig. 1, which shows the potential as well as the ground state and the first two excited states calculated using GEP. However as β is increased to 3, the ground state occurs above the classical barrier. The ground state motion is then an oscillation between the classical minima (Fig. 2). Similar behaviour occurs for $k=0.6$ and $\beta=3.5$ as shown in Fig. 3.

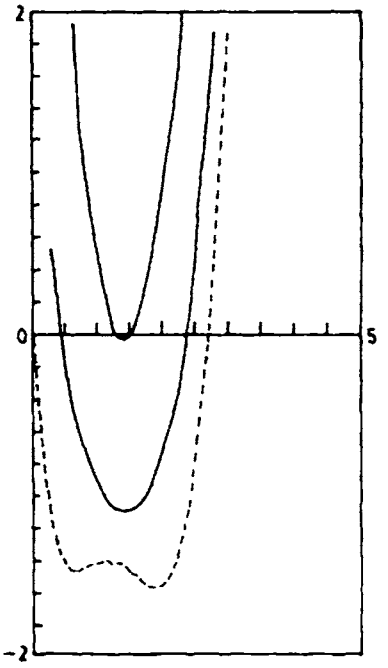

Fig. 2. The same as Fig. 1 but with $\beta=3$. Only the ground state and the first excited state are shown.

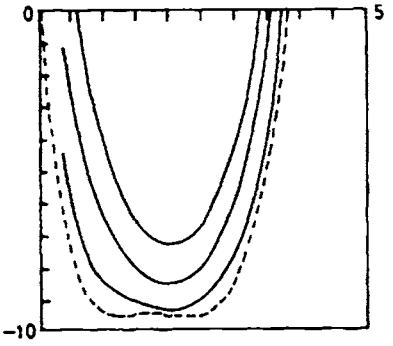

Fig. 3. The Gaussian effective potential for $k=0.6$ and $\beta=3.5$.

5. Concluding remarks

In this paper, we have introduced a method to deal with quadratically damped systems. By transforming them to the Hamiltonian format, we make an attempt to study their quantum mechanical behaviour using the GEP algorithm. Our results for the Duffing oscillator, reveal bound state motion inside the classical well or above the barrier depending on the val-

ues of k and β. The absence of a barrier in V_G rules out the possibility of tunnelling phenomenon in the system considered. However for very small values of β, the excited states reveal a double-well nature. But tunnelling cannot be dealt with by the static GEP approach used in this work. For this purpose explicitly time dependent GEP should be used. This and other related studies will be reported separately.

Since our potential has logarithmic terms, certain regions (near $X=0$) remain opaque to the technique. Here the equation for Ω_{min} fails to give a positive definite real solution. This is certainly a pathology of the potential and not of the GEP technique. However, this does not affect our conclusions much, since interesting regions like the classical minima and barrier occur away from this region for potentials of the type discussed in this paper.

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