DYNAMICS OF NONLINEAR WAVES ON SUPERFLUID HELIUM FILMS

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CERTIFICATE

Certified that the work reported in the present thesis is based on the bonafide work done by Mr. J.Sreekumar under my guidance in the Department of Physics, Cochin University of Science and Technology and has not been included in any other thesis submitted previously for award of any degree.

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PREFACE

This thesis is a report of the investigations on the dynamics of ^{4}He films carried out by the author under the guidance of Dr. V.M. Nandakumaran in the Department of Physics, Cochin University of Science and Technology, during the year 1984-89. The wave propagation on thin and saturated films of superfluid ^{4}He under varying degrees of nonlinearity is studied.

The rapid developments that have taken place in the field of nonlinear dynamics provide deep insight into the properties of many physical systems. They have significantly increased the number of exactly solvable physical problems and have made possible a clear understanding of certain other nonlinear systems, which do not have an exact solution. Of all the ideas that revolutionised 'nonlinear dynamics', the concept of solitons and their interactions have been playing increasingly important roles.

In superfluid ^{4}He films, small finite amplitude localised density fluctuations can lead to the existence of solitons made up of superfluid condensate. This arises essentially due to the balance between dispersion and the nonlinearity arising from the Van der Waals potential of the substrate. These nonlinear local density fluctuations may travel unattenuated for large times, and they are governed by the Korteweg de Vries (KdV) equation when the propagation was confined to a single direction.

In this thesis we study the propagation and interaction of solitons on thin superfluid ^{4}He films. First the study is done on monolayer films of ^{4}He. Later we take up the case of thicker ($\gtrsim 10^{-6}$ cm) films, which is known to support solitary waves of fluctuations in thickness.

The thesis contains six chapters. An introduction to the recent developments in the fields of superfluid films and soliton dynamics are given briefly in the first chapter. The first half of the chapter is devoted to a discussion of the properties of helium films and the propagation of linear waves on such films. The nonlinear Schrödinger equation representing the dynamics of monolayer films is also introduced. The latter half of the chapter discusses the propagation and interaction of solitons on superfluid films, after introducing the basic concepts of soliton dynamics. The stability of solitons in two space dimension and the chaos caused by the collision of solitons are also examined.

In chapter II we present studies on the propagation of weakly two dimensional solitons on monolayer superfluid films under the lowest order of nonlinearity. The properties of the Kadomtsev-Petvishvili (K-P) solitons and also the phenomena of soliton resonances are discussed . Following these results we study in this chapter the phenomenon of "two soliton resonance" of the K-P equation for the superfluid density fluctuations and obtain the velocity of the resonant soliton.

The dynamics of large amplitude local density fluctuations on a two dimensional superfluid film is considered in chapter III. The nonlinear Schrödinger equation representing the superfluid density fluctuations is reduced to a dimensionless form and solved numerically for various arbitrary initial profiles. It is shown that the initial profile would split into two "quasi-solitons" travelling in opposite directions, which have particle like stability and keep their shape unchanged during interactions with each other. The "quasi-solitons" are asymmetric in shape, unlike the KdV or K-P solitons.

In chapter IV, the chaos induced by the collision of large amplitude one dimensional quasi solitons on a very thin superfluid film is studied numerically. After defining a suitable phase space for the system we have shown that two initially close trajectories in this phase space of the system separates exponentially in time with the collision of the quasi-solitons. The instability at the collision spot propagates spatially.

When the thickness of the superfluid film is increased, the dynamics of the system is altered. This is dealt with in chapter V. In this chapter we study the dynamics of the thickness fluctuations on a saturated two dimensional superfluid ^{4}He film and show that the equation governing the system is the K-P equation with negative dispersion. It is established that the phenomenon of soliton resonance could be observed in such films. Under the lowest order of nonlinearity, such resonances take place only if two dimensional effects are taken into account. The amplitude and velocity of the resonant solitons are obtained explicitly.

In the last chapter we present a summary of the investigations presented in the preceding chapters. Important conclusions are highlighted and scope for future investigations are indicated.

Chapter VI is followed by an appendix where all the numerical codes used in evaluating different problems in the thesis are given. The programme for the three dimensional plot of temporal development of solitons is also discussed.

Part of the investigations presented in the thesis has provided materials for the following publications.

1. Soliton Resonances in Helium films, Phys. Lett. 112A, No.3,4 (1985) 168.

2. Solitons and their resonances on two dimensional superfluid films, Pramana- J. Phys. Vol.33, No.6, (1989) 697.

3. Two dimensional large amplitude quasi solitons in thin Helium films, Modern Phys. Lett.B, Vol.4, No.1 (1990) 47.

4. Dynamics of solitons on ^{4}He Films, Proceedings of the workshop on *Symmetries and singularity structure aspects on nonlinear dynamical systems*, held at Trichy, Nov.29 to Dec3, 1989 (to be published).

5. Chaos induced by soliton-soliton collisions in thin superfluid films, submitted to J. Phys.C

CONTENTS

CHAPTER 1

SUPERFLUIDITY AND NONLINEAR WAVES: AN INTRODUCTION

SUPERFLUIDITY AND NONLINEAR WAVES : AN INTRODUCTION

Recently, there has been an upsurge of renewed interest in the study of superfluid helium, especially in the dynamics of thin and thick superfluid films. This has been boosted up by the exciting developments in the field of nonlinear dynamics. In this chapter we give a brief introduction to the basic concepts of superfluid films and also to the dynamics of nonlinear waves. The phenomenon of third sound, observed in superfluid films, is discussed thoroughly. The possible nonlinear waves that can exist on such films are examined in succeeding sections of this chapter. The propagation and interaction of solitary waves are studied in detail. Behaviour of these solitons in two space dimensions and also under varying degrees of nonlinearity are also discussed.

The scheme of presentation is as follows. This chapter is divided into eleven sections. A brief discussion of the properties of liquid helium in bulk is given in the first section. The linear modes of wave propagation that can exist on superfluid films are examined in section 2. In section 3 the dynamics of very thin superfluid films is considered. The pseudospin model, which gave a theoretical basis to the phenomenological equation of motion of thin superfluid films, is discussed in section 4. We discuss the basic concepts of solitons in sections 6 and 7, so as to begin a study of the nonlinear waves on superfluid films. Section 8 introduces the Reductive Perturbation method, which provides a systematic procedure for reducing a given nonlinear evolution equation into one which has soliton solutions. The propagation of solitons on superfluid films is discussed in section 9. The Kadomtsev-Petviashvili equation and the behaviour of solitons in two space dimensions are discussed in section 10. Chaos induced by soliton-soliton interaction is studied in the last section of this chapter.

1.1 SUPERFLUID ^{4}He

Most of the diverse application of superfluid helium have been due its peculiar nature, it displays quantum effects on the macroscopic level and obeys hydrodynamics on the microscopic level[1]. A striking feature in the hydrodynamics of superfluid ^{4}He is its ability to exhibit new types of wave propagations, not found in other systems. We shall explore this and other features of superfluid ^{4}He in the present and subsequent sections. Before going into the details of these, we shall first examine some fundamental properties of liquid ^4He.

Immediately below the boiling point, ^4He behaves like ordinary liquids with small viscosity. However at 2.17 K, called the 'λ-point', liquid ^4He undergoes a phase transition. The λ-point, characterized by an anomaly in the specific heat, marks the transition between two different forms of ^4He, known conventionally as He I above the λ-point and He II below it. Under normal pressure, He II remains a liquid even if the temperature is lowered to absolute zero. The reluctance of helium atoms to condense stems from its low atomic mass and the extremely weak forces between them[2].

He II has many remarkable properties. The most prominent one being its flow past obstacles with zero resistance. The persistent current experiments of Reppy and Depatie[3] suggests that the viscosity of He II is virtually zero for flow through a capillary. On the other hand, experiments using oscillating disks[4], vibrating wires[5] and rotation viscometer[6] shows the existence of a viscous drag. These results show that HE II can be viscous and nonviscous at the same time. An explanation for these apparently contradicting results was given by Tisza[6] and Landau[7] through the phenomenological theory of He II - known as the 'two-fluid model', in terms of which many hydrodynamical properties of He II can be explained.

According to the two-fluid model, He II is to be considered as an intimate mixture of two liquids - the normal fluid and the superfluid. The normal fluid is like an ordinary liquid having viscosity. But on the contrary, the superfluid part is capable of moving without friction. It has zero-point energy, momentum and entropy. However, it is to be noted that the two fluids cannot be separated physically. That is, one can say that He II is capable of two motions at the same time, each of which has its own local velocity and mass density. If ρ_n and ρ_s represent the mass densities of the normal and superfluid parts respectively, then the total density ρ of He II is

$$\rho = \rho_n + \rho_s \qquad (1.1.1)$$

and the total current density is

$$\vec{j} = \rho_n \vec{v}_n + \rho_s \vec{v}_s \qquad (1.1.2)$$

where $\vec{v}_n$ and $\vec{v}_s$ represent the local velocity corresponding to the normal and superfluid parts respectively.

Because of zero viscosity, the superfluid part can flow through very narrow channels, called 'superleaks', without any resistance. If we choose the state at absolute zero as a standard one for the purpose of understanding the dynamics of He II and if we assign zero entropy to this state, then we can say that the flow of the superfluid part does not carry any entropy along with it. The flow of a zero entropy, zero viscosity fluid explains the very high heat conductivity of He II.

Andronikashvili[8] demonstrated experimentally the validity of two fluid model by measuring the period of oscillations, and thereby the drag, of a pile of equally spaced plates suspended by a torsion fiber into He II, maintained at a desired temperature. As the period of oscillation would depend on the amount of He II that would also oscillate with the disks, his experiments gave a direct method of measuring variation of ρ_n/ρ with temperature. It was shown that He II is almost entirely superfluid below 1K.

This approach, where the two fluids are treated independently, is very useful in developing the complete set of hydrodynamic equations of He II when the velocities are small. At higher velocities, the superfluid flow becomes dissipative, the normal fluid exhibits turbulence, and there is the possibility of interaction between the two. When such interactions are to be taken care of, the two - fluid equations become rather complicated.

The phase transition at the λ-point and the formation of the superfluid phase can be clearly understood from the point of view of the Bose-Einstein condensation. An ideal Bose gas at the density of liquid helium will undergo a sharp transition at 3.2K[9], which is close to the λ-point. This suggests that the λ-point marks the onset of Bose-Einstein condensation in liquid ^{4}He . However, the specific heat anomalies are quite different for liquid helium and the ideal Bose gas.

Liquid helium is obviously a system in which the attractive forces between the atoms play an essential part. Hence, it is not surprising that the ideal Bose gas model does not give the correct value of the λ-transition. The effect of the interatomic interactions is to reduce the number of particles condensed into the lowest energy level and to alter the nature of the excited levels of the system[2]. Thus, at absolute zero, although a finite fraction of the particles still occupy the lowest level, some occupy slightly higher levels caused by the interaction. The experimental evidence is that He II is a pure superfluid at absolute zero. Hence, it appears that the superfluid fraction comprises of both the condensate and the particles occupying the levels caused by interaction.

At all temperatures above absolute zero the thermally excited levels of the system are occupied to some extent. These states do not correspond to single-particle states. They are elementary excitations of the whole system, which in the lowest order could be treated as non-interacting quasi-particles. We can identify the normal fluid part with these thermal excitations.

He II can be described by a condensate wave function. This is a complex quantity, which gives the superfluid density when the absolute value of its square is taken. Under steady state conditions, if we can write the condensate wave function ψ in the form

$$\psi(\vec{r}) = \psi_o \exp\left[i S(\vec{r}) \right] \tag{1.1.3}$$

Then the condensate momentum $\vec{P}$ can be obtained by operating the momentum operator on ψ . Thus

$$\vec{p} = \hbar \nabla S \tag{1.1.4}$$

Also, if the superfluid velocity is $\vec{V}_s$, we can write

$$\vec{p} = m_4 \vec{v}_s \tag{1.1.5}$$

Equations (1.1.5) and (1.1.4) gives

$$\vec{v}_s = \frac{\hbar}{m_4} \nabla S \tag{1.1.6}$$

On taking the curl, we get the relation,

$$\text{curl } \vec{v}_s = 0 \tag{1.1.7}$$

which has immense use in developing the hydrodynamic relations.

He II, like other fluids, admits the propagation of sound waves. Here, the normal and superfluid parts vibrate together to produce the ordinary longitudinal pressure waves involving fluctuations in the total density at constant temperature. This is called 'first sound'. If the normal and superfluid parts vibrate out of phase then we get what is called the second sound. In this mode, the total density remains uniform throughout the liquid. However, the local value of the ratio ρ_s/ρ , and consequently the temperature, undergo oscillations.

First sound can be generated by any oscillating material body immersed in the He II bath. On the other hand, a heat source with its temperature undergoing oscillations can generate second sound. Waves of second sound are propagated with almost zero attenuation, provided that the rate of heat supply is not too large and the frequency is not too high.

1.2 THIRD SOUND

In the early days of superfluid research it was found that when two vessels were arranged one inside the other and were filled with He II at two different levels, some mysterious process led to a quick equalization of levels[10]. It was Daunt and Mendelssohn[11], who showed that the equalization took place by a flow of the superfluid through an adsorbed film of He II bridging the two vessels.

The formation of the film could be discussed in terms of the forces of attraction between the helium atoms and the wall. Any solid body in contact with the vapour of a substance has an adsorbed film on it due to the short ranged Van der Waals force of attraction that exists between the molecules of the solid body and the substance. The Van der Waals potential that exists between a helium atom and an atom of the wall can be expressed as

$$\phi_v = \frac{m\alpha}{z^3} \tag{1.2.1}$$

where m is mass of helium atom and α is a constant depending on the strength of interatomic forces. It is the contribution due to this term in the equation of motion, that provides many interesting nonlinear effects to the dynamics of the superfluid films.

The thick film studied in this type of experiment has been called the 'saturated film', because it exists in equilibrium with its saturated vapour[12]. Adsorption on a surface in contact with any liquid or its saturated vapour is very common, but He II films are unusually thick. Optical measurements have revealed[13] that the saturated films of He II would have a thickness of the order of a few hundred Å. It is also possible to study a different case, where the film is in equilibrium with gas at a pressure less than the vapour pressure. In both the cases, the film thickness is small enough to prevent the flow of the normal fluid, with the result that the film acts as a kind of superleak.

For saturated films, with which we are primarily concerned in this section, the two fluid hydrodynamics of Landau could be employed to study the dynamics in the linear regime. A considerable amount of simplification occurs to these equations due to the fact that the normal component of He II is practically immobile in such a film.

The problem is essentially similar to that of capillary waves occurring on a fluid surface, but with the exception that the fluid thickness is small. Since only the superfluid part can flow, the resulting wave is different from the usual capillary waves. The wave would be a composite one of the superfluid thickness and an associated temperature variation. The temperature variation occurs due to the fact that the superfluid part does not carry any entropy along with it. Such a wave, known as the 'third sound', was first generated and detected using a chopped infrared beam and a polarimeter by Everitt et al[14] . The most accurate measurements of these waves, which are similar to shallow water waves in nature, are made through the use of thin aluminium films evaporated onto the substrate. By imposing an external magnetic field and biasing currents these strips could be operated near their superconducting transition. Thus a small change in temperature leads to a huge change in resistance. In this way third sound is detected through its associated temperature variations.

The linearised version of the two-fluid hydrodynamics gives the velocity of third sound u_3 as[15]

$$u_3^2 = \frac{\bar{\rho}_s d}{\rho} \frac{\partial \Omega_v(d)}{\partial d} \left(1 + \frac{ST}{L} \right) \qquad (1.2.2)$$

where T is the temperature, d is the equilibrium thickness of helium film, L is the latent heat per gram, Ω_v is the potential due to Van der Waals attraction and $\vec{\rho}_s$ is the equilibrium superfluid density.

The experimentally determined values of the third sound for very thin films differs much from that given above. This disagreement is primarily due to the fact that in the derivation of equation (1.2.2), Van der Waals force of attraction was not given due consideration[16]. In such films Van der Waals force causes pressure which is more than sufficient for the solidification of the helium atoms close to the substrate[15]. Thus the film is separated from the substrate by a solid layer, one or two atomic layers thick.

The second reason for the difference in the theoretical and experimental values of the third sound could be qualitatively understood as follows. The superfluid wave function must vanish at a boundary. Hence $\rho_s \rightarrow 0$ smoothly as the boundary wall is approached. A sudden change in ρ_s at the wall is forbidden, since it would violate the requirement that ψ should be single valued everywhere. At a free surface the amplitude of the wave function should also tend to zero. The distance over which ρ_s falls to zero from its bulk value is called the 'healing length' . Though the value of the healing length is very small, of the order of atomic spacing, it plays a crucial role in the dynamics of very thin films.

A recalculation of the third sound velocity, after incorporating the above modifications, would yield[17]

$$C_3^2 = \left[F(D) \frac{\rho_s}{\rho} D + \frac{k^2 \beta D}{\rho} \right] \frac{\tanh(kD)}{kD} \qquad (1.2.3)$$

where k is the wave vector, D is the film thickness, F(D) is the Van der Waals force and β is the surface tension. In equation (1.2.3), the first term dominates for kd << 1 . The second term, which is proportional to β, dominates for kd >> 1. In the second case, the surface wave goes over to the bulk surface wave, and the elementary excitations are called ripplons[18].

The third-sound is present only in the linear approximation - that is when only small amplitude waves are propagating on the helium film. When comparatively large amplitude disturbances occur on the film, the nonlinearities come into play. Many interesting properties are seen when these nonlinearities are taken into consideration. For example, in the lowest order of nonlinearity, the film supports localized travelling waves which have particle like stability[19] and keeps their shape even after collision with other such waves. In this thesis we are primarily interested in such nonlinear waves on helium films.

We have seen that the thickness of the films considered in this section is typically of the order of a few hundred Å. There would be a radical change in the dynamics of the film if the thickness is reduced to the order of a few atomic layers. Such films have recently been studied experimentally[17] and will be dealt with in detail in the next section.

1.3 VERY THIN FILMS

Experiments on third-sound gave valuable information on the properties of helium films[20,1]. Scholtz et al[1] made measurements of third-sound velocities for film coverages down to 2.1 atomic layer thickness and at temperatures as low as 0.1K. Their measurements were interpreted within the frame work of the Ginzburg - Pitaevskii (G.P) theory[21] . These measurements were made for thick superfluid films. For very thin films the G.P. theory, which predicts a temperature dependant healing length, would fail. This is because for a monolayer film the concept of healing length is not applicable. For such very thin films, even the concept of film thickness is not an exact one. As the superfluid surface density falls below a monolayer the thickness clearly fails to describe the coverage.

Third-sound excitations have been investigated with great accuracy in monolayer films by Rutledge et al[17]. They found positive dispersion for the surface modes. The dispersion relation and the temperature dependence of these modes were then derived in the linear regime using a two dimensional quantum hydrodynamics for the superfluid condensate. This is essentially an extension of the Landau's superfluid hydrodynamics[7], the proper approach when dealing with such thin and inhomogeneous films. The essence of this phenomenological theory, which aims in arriving at an equation of motion for the macroscopic fluid flow, would be elaborated in this section.

The macroscopic variables used in the discussion of the usual thick superfluid films are superfluid density and velocity as well as the thickness. But, for very thin films, we have seen that the thickness fails to describe the surface coverage of superfluid atoms. Hence, for such films, the only macroscopic variables definable are the superfluid surface density $\rho(\vec{x})$, defined in atoms per unit area, and the tangential superfluid velocity $\vec{v}_s(x)$. Note that $\vec{x}$ is a two dimensional vector describing the position on the surface. Here we assume that a condensate wave function exists, and write down a phenomenological equation of motion for the macroscopic superfluid flow using surface quantities.

Let $\psi(\vec{x})$ be a complex order parameter, which is proportional to the condensate wave function. So at absolute zero of temperature we can write

$$\rho_s(\vec{x}) = | \psi (\vec{x}) |^2 \tag{1.3.1}$$

The energy of this quantum state consists of four contributions. One kinetic term and three potential terms, H_{VW}, H_{CP} and H_{SE}, defined by equations (1.3.3)-(1.3.5). The kinetic energy of the superfluid film is

$$H_{KE} = \int d^2x \ \frac{\hbar^2}{2m} \ | \vec{\nabla} \psi |^2 \tag{1.3.2}$$

where m is the bare ^{4}He mass. The integral is taken over the physical surface areas. The Van der Waals binding of the film to the substrate would be of the form

$$H_{VW} = \int d^2x \ \frac{A}{2\ (a+\rho_s)^2} \ , \tag{1.3.3}$$

where a and A are constants. The chemical potential term, which controls the equilibrium value of $|\psi|^2$, is

$$H_{CP} = -\int d^2x \ \mu\rho_s \qquad (1.3.4)$$

The third term contributing to the potential energy is one similar to the surface energy, which is written in the form

$$H_{SE} = \int d^2x \ \frac{1}{2} \ B(\rho_s) \ (\vec{\nabla}\rho_s)^2 \quad , \qquad (1.3.5)$$

where B if some function of the surface density. In the limit of bulk helium, $\beta_o = B(\infty)\rho_o^2$ would represent the surface tension, where ρ_o is the bulk superfluid density.

The total energy is the sum of the above four terms. As ψ is a wave function, the natural equation of motion for the condensate wave function would be the nonlinear Schrödinger equation,

$$i\hbar \ \frac{\partial\psi \ (\vec{x},t)}{\partial t} = \frac{\delta H}{\delta\psi^*(x,t)} \quad , \qquad (1.3.6)$$

where the variational derivative is taken treating ψ and ψ^* as independent quantities. Thus

$$i\hbar \frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi - \frac{A\psi}{(a + |\psi|^2)^3} - \mu\psi - B\psi \ \nabla^2|\psi|^2$$

.......(1.3.7)

where B is treated as a constant.

If we search for a solution of the form

$$\psi(\vec{x},t) = \rho_s^{1/2}(\vec{x},t)\; e^{i\phi(\vec{x},t)} \quad ,$$

we would get the two dimensional continuity equation

$$\frac{\partial \rho_s}{\partial t} + \vec{\nabla} \cdot \vec{J}_s = 0 \quad , \qquad (1.3.8)$$

where $\vec{J}_s(\vec{x},t) = \mathrm{Re}\ \left[(\hbar/im)\psi^* \vec{\nabla}\psi\ \right]$ is the quantum mechanical current density. This expression is a consequence of the fact that we have not included any dissipative processes. During the actual experimental situation, one has to give an initial localized excitation to the two dimensional system to observe the dynamics. After this is done, equation (1.3.8) would be valid.

Rutledge et al[17] evaluated the solutions of the linearised version of (1.3.7) and obtained the dispersion relation. They were also able to calculate the third-sound velocity as

$$C_s^2(T) = C^2 \left\{ 1 - \frac{1.202 \ T^3}{2\pi m\hbar^2 \ C^4 \rho_o} \left[\frac{3}{2} - \frac{\rho_o^2}{C} \frac{\partial^2 C}{\partial \rho_o^2} \right] \right\}$$

......(1.3.9)

where C is the velocity at absolute zero temperature.

Equation (1.3.7) would display many interesting phenomena when terms of different degrees of nonlinearity are retained. These would be dealt with in detail in later chapters.

Though Rutledge et al[17] derived equation (1.3.7) in a phenomenological manner, they were able to describe satisfactorily the results of their experiments on very thin films on the basis of this equation. One interesting experimental result which they obtained was that the value of B was zero for monolayer films.

Recently, Radha Balakrishnan et al[23] obtained the same equation for two dimensional superfluids, by retaining the nonlinear terms in the pseudospin model of Matsubara and Matsuda[24]. The details of this work, which provided an insight into the microscopic origin of the terms appearing in equation (1.3.7), is the theme for the next section.

1.4 THE PSEUDOSPIN MODEL

The phenomenological theory of Rutledge et al[17] was quite successful in describing the dynamics of two dimensional superfluid films. The experimentally observed third - sound velocity and dispersion relation were quite satisfactorily explained by the nonlinear Schrödinger equation (equation 1.3.7) representing the time evolution of the condensate wave function. Recently, Radha Balakrishnan et al[22] have derived this equation from a microscopic theory of nonlinear dynamics in superfluid ^{4}He. This theory, which has provided a firm footing for the phenomenological equation of motion proposed by Rutledge et al, would be discussed in this section.

Matsubara and Matsuda[23] described the dynamics of superfluid ^{4}He using a pseudospin (quantum-lattice gas) model. This in essence is a treatment of a system of bosons with hard cores plus nearest-neighbour interactions being described by a pseudospin Hamiltonian on a lattice. The hard core in the potential is incorporated by using Fermi-like anticommutation relations for the field operators at the same site and Bose-like commutation relations for operators belonging to different sites. Only the linear terms in the formulation were taken into consideration. A study[24] of the ground state thermodynamic properties and the nature of the elementary excitations using this linearised equation in the random phase approximation was able to reproduce the well known low-density-limit results[25]. The value of T_λ and the quasi particle spectrum at T=0, evaluated using this model have also been shown[26] to be in agreement with experimental results.

Radha Balakrishnan et al[22] studied the problem by retaining the nonlinear terms in the above formalism. They were able to derive, using the spin coherent representation, an evolution equation for the superfluid order parameter for all $T \leq T_\lambda$ without using a Hartee approximation. The phenomenological GP equation[27] for bulk helium and the equation of Rutledge et al[17] emerge as special cases of this formalism when certain terms are neglected. They were able to show that the formalism supports a travelling wave solution, as should be expected. For a specific choice of parameters, a static kink solution was also obtained.

1.5 SUPERFLUIDS - A PLAY GROUND FOR NONLINEAR PHENOM

Several experiments on third-sound propagation have revealed finite amplitude effects that could not be explained by a linearised theory[28]. This suggests the need for extending the theories so as to include finite amplitude effects and thereby study different nonlinear phenomena taking place in superfluid films.

There is experimental evidence[29] for the existence of an ordered pair of localized nonlinear waves. These nonlinear waves, known as 'solitary waves', show many unique and interesting properties. An introduction to the properties of these waves is given in the following section.

1.6 SOLITARY WAVES

It is customary to start any elementary discussion on solitons with the classic description by John Scott Russel [30] of a new type of water wave, travelling along a narrow channel, which continued its course many miles down the channel without any apparent change of form or speed. After this discovery in 1834, Russell did many experiments on such 'solitary waves', which led him to conclude[30] that the momentum transfer during the motion of a solitary wave remains local and the velocity of propagation depends on amplitude.

In 1895 Korteweg and de Vries (KdV)[31] derived an equation for the propagation of waves in one direction on the surface of a shallow canal. The equation of motion in dimensionless form is

$$u_\tau + u_\xi + 12\, u\, u_\xi + u_{\xi\xi\xi} = 0 \quad , \qquad (1.6.1)$$

where u denotes deviation from equilibrium surface and subscripts denote partial derivatives.

This equation has a solution

$$u = \frac{1}{4} a^2 \, \mathrm{Sech}^2 \frac{1}{2} \left[a\xi - (a+a^3)\tau + \delta \right] ,$$

$$......(1.6.2)$$

where a and δ are arbitrary constants. This equation represents a hump, exactly as described by Scott Russel, which moves without change of shape and has an amplitude-dependent velocity.

Now we shall slightly drift away form the topic of solitons for a while and consider a problem investigated by Fermi, Pasta and Ulam (FPU) in 1955. They studied[32] the behaviour of certain equations which were primarily linear but in which nonlinearity was added as a perturbation. In particular, the equation which they took as a model was

$$\ddot{Q} = f\,(\,Q_{n+1} - Q_n\,) - f\,(\,Q_n - Q_{n-1}\,) \;, \qquad (1.6.3)$$

where f(Q) is a nonlinear function. Two cases which FPU considered where

$$f(Q) = \gamma Q + \alpha Q^2 \qquad (1.6.4)$$

and

$$f(Q) = \gamma Q + \beta Q^3 \quad , \qquad (1.6.5)$$

where α and β were chosen such that the effect of the nonlinearity is small. In the absence of nonlinearity(i.e. α=o and β=o), the energy in each of the normal modes of the system would be constant. It was expected that the introduction of the nonlinearity would lead to the energy of the system being evenly distributed throughout all possible modes, in accordance with the equipartition theorem. But a numerical evaluation of the equation showed a quite different result. FPU integrated equation(1.6.3) numerically with a sine-wave as initial data and found that the energy does not spread through all the normal modes but remains in the initial mode and a few nearby modes. Over a large number of oscillations, the energy in each normal mode was shown to be almost periodic in time with no loss of energy to higher modes as time increases.

For a decade the FPU problem remained as one unrelated to solitary waves. In 1965, using high speed computers Zabusky and Kruskal[33] studied the KdV equation as a model for the FPU problem and reconfirmed the recurrence phenomena. They chose a periodic wave as the initial condition and a periodic boundary condition to solve the problem and noted that the initial wave evolved into solitary waves which travelled in opposite directions and, due to the periodic boundary conditions, collided with each other, while keeping their identity during travel and after collisions. They coined the name 'soliton' for these waves with particle like properties.

The fact that the numerical solutions of the KdV equation were composed of solitons was so exciting that the analytic method of integration was investigated. Thus in 1967 Gardner, Green, Kruskal and Miura[34] were able to solve the KdV equation on the real line for solitons that tended sufficiently fast to a constant value as $|x| \to \infty$. The method of solution has now come to be known as the method of Inverse Scattering Transform (IST).

After the discovery by Zabusky and Kruskal, a number of equations were searched for soliton solution. This showed that there are equations other than the KdV which posses soliton solutions, some of which have a functional form different from that obtained for the KdV case. It is of interest to note that the method of IST was extended by Ablowitz et al[35] so as to include some of these nonlinear evolution equations. These equations characterized different physical systems.

The one soliton solution is easy to find by simple integration. But, this won't work if we wish to have an analytical formula to describe the interaction of two or more solitons. One way to do this would be the IST method, which works in precisely the same way as Fourier Transform does in linear problems. It transforms the dependent variable which satisfies the given partial differential equation to a set of new independent variables whose evolution in time is described by an infinite sequence of ordinary differential equations. The success of IST method rests in those class of partial differential equations, for which these infinite sequence of ordinary differential equations are separable and hence trivially integrable.

Hirota[36,37] developed a more direct and systematic way of finding exact solutions of a certain class of nonlinear evolution equations. In this method the nonlinear evolution equation in question is transformed, by changing the dependent variable(s), into bilinear differential equations of the form

$$F\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}, \frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right) f(x,t)\, f(x',t') \Big|_{\substack{t=t' \\ x=x'}} = 0$$

.......(1.6.7)

These bilinear differential equations were solved using a perturbational approach, with the use of the D operators defined by[37]

$$D_t^m\, D_x^n\, f.g = \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial t'}\right)^m \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n f(x,t)g(x',t')\Bigg|_{\substack{x=x' \\ t=t'}}$$

.......(1.6.8)

The bilinear differential equation (1.6.7) can either be obtained from the original nonlinear differential equation by a Cole-Hopf transformation[38,39] or by the use of [N/M] Pade' approximants[37]. Hirota[40] solved the KdV equation using this method and obtained the N-soliton solutions.

For a given set of nonlinear evolution equation representing a particular physical problem, it may not always be possible to use the above methods to search for N soliton solutions. However, Su and Gardner[41] have shown that for a wide class of nonlinear Galilean-invariant systems, if the nonlinearity is weak and if one makes the long-wavelength approximation, the governing equation can be reduced to either of the following equations.

$$n_\tau + n\, n_\xi + \delta\, n_{\xi\xi\xi} = 0 \qquad (1.6.9)$$

$$n_\tau + n\, n_\xi - \nu\, n_{\xi\xi} = 0 \qquad (1.6.10)$$

More about this method shall be discussed in section 1.8.

The different nonlinear evolution equations presented in this section were restricted to one dimension (1 space dimension + 1 time dimension). The nature and form of such nonlinear equations in two dimensions (2 space + 1 time) would be dealt with in section 1.10.

1.7 SOLITONS - A BALANCE BETWEEN DISPERSION AND NONLINEARITY

We have seen in the last section that the properties of the solitary waves having the form $Sech^2(k.x-\omega t)$ are different compared to those of linear harmonic waves. Other than being a non oscillatory motion, the solitary waves show the interesting character of amplitude-dependent velocity. To understand why the solitons behave like this, let us take a closer look at the nonlinear and dispersion effects taking place, with KdV equation as a specific example[98].

$$v_t + v\,v_x + v_{xxx} = 0 \qquad (1.7.1)$$

Here the second term is the nonlinear term and the third one is the dispersion term. Now taking $v = u+1$ and neglecting the dispersion term, we get

$$u_t + (u+1)\,u_x = 0 \qquad (1.7.2)$$

The linearised version of (1.7.2), viz., $u_t+u_o u_x=o$ has the solution of the form $u(x,t)=f(x-u_o t)$. Hence, by analogy, we try a solution of the form

$$u = f\,(\,x - (u+1)t\,) \qquad (1.7.3)$$

To show that this solution is possible, we can check by direct calculation of u_t and u_x .

$$\begin{aligned} u_t &= -\left[\,t\,u_t + u + 1\,\right] f' \\ u_x &= (\,1 - u_x\,t\,)\,f' \end{aligned} \qquad (1.7.4)$$

which gives

$$(u_t + (u+1)u_x)\,(\,1 + tf'\,) = 0 \qquad (1.7.5)$$

Equation (1.7.5) is satisfied by solutions of (1.7.3).

It is easy to solve (1.7.2) with the help of (1.7.3) for a piece-wise linear initial data, in the form of a triangle,

$$f(x) = \begin{cases} u_o\, x & 0 < x < 1 \\ u_o(2-x) & 1 < x < 2 \\ 0 & \text{otherwise} \end{cases} \qquad (1.7.6)$$

So (1.7.3) gives, after putting $\eta=x-t$,

$$u(\eta,t) = \begin{cases} u_o(\eta-ut) & 0 \le \eta - ut \le 1 \\ u_o(2 - \eta + ut) & 1 \le \eta - ut \le 2 \\ 0 & \text{otherwise} \end{cases} \qquad (1.7.7)$$

Now solving the linear equations in (1.7.7) gives

$$u = \begin{cases} \dfrac{u_0 \eta}{1 + u_0 t} & 0 \le \eta - ut \le 1 \\ \dfrac{u_0(2-\eta)}{1-u_0 t} & 1 \le \eta - ut \le 2 \\ 0 & \text{otherwise} \end{cases} \tag{1.7.8}$$

$$u_\eta = \begin{cases} \dfrac{u_0}{1+u_0 t} & 0 \le \eta - ut \le 1 \\ \dfrac{-u_0}{1-u_0 t} & 1 \le \eta - ut \le 2 \\ 0 & \text{otherwise} \end{cases} \tag{1.7.9}$$

From (1.7.9) it can be seen that the gradient of the right hand part of the triangle is slowly increasing with time while that of the left hand side is decreasing. This is shown schematically in Fig.1.1. It can be seen that gradient of the right hand side of the triangle changes from negative to positive as time increases. At $t=1/u_0$ this side is vertical, after which the triangle turns over.

It is clear from the above discussion that the action of the nonlinear term in the equation is to steepen any initial waveform. Now to study the effect of dispersion, we can neglect the nonlinear term in (1.7.1). This would give

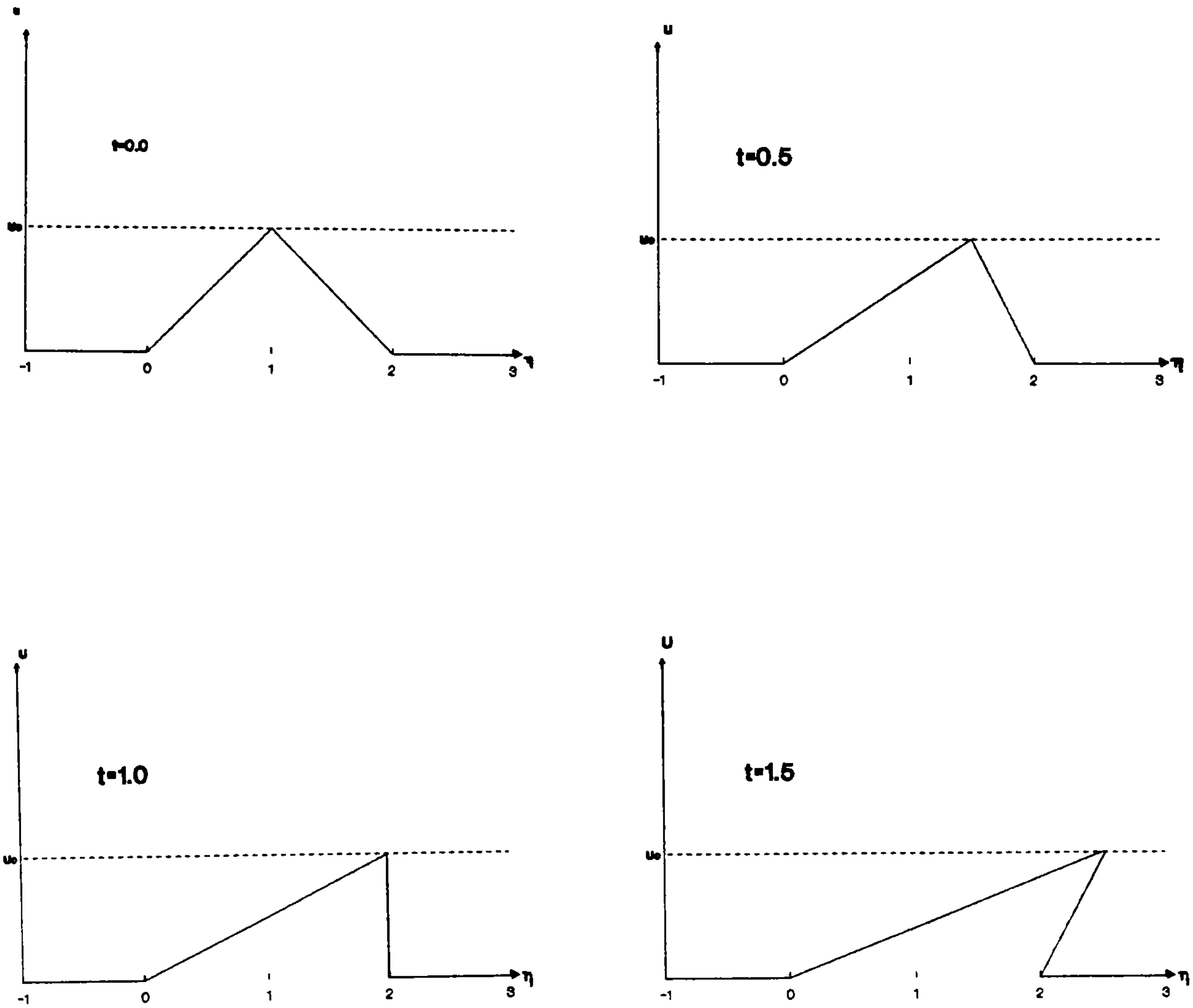

Fig 1.1

$$u_t + u_x + u_{xxx} = 0 \qquad (1.7.10)$$

which can be solved straight away. The result would be the harmonic waves.

The soliton solution of equation (1.7.1) shows a different character from the two cases considered above. As we have seen, in the numerical experiments of Zabusky and Kruskal[33] smooth initial data in the form of a sine wave evolved into a set of solitons. This can be explained as follows. For very short times the first two terms of equation (1.7.1) dominate and a steepening of profile occurs at those points with negative gradient. But this steepening won't lead to a discontinuity, because as the steepening progresses the third term becomes important and stabilizes the sharp edge.

It is clear from the foregoing discussions that the solitary wave is a manifestation of both dispersive effects and nonlinear effects. For the KdV equation one can find[42] a connecting link between solitary waves and the linear harmonic waves. By setting v=12f in (1.7.1) and writing the steady state solution in the form f(x−ct), two integrations are possible. The resulting expression is

$$(f_x)^2 = -4f^3 + cf^2 + af + b \qquad (1.7.11)$$

which can be written in the factored form

$$(f_x)^2 = -4(f-\alpha_1)(f-\alpha_2)(f-\alpha_3) \qquad (1.7.12)$$

This equation has a solution[42] in terms of the Jacobian elliptic function Sn(z,k) as

$$f(x-ct) = \alpha_3 - (\alpha_3 - \alpha_2)\,Sn^2\left[\sqrt{\alpha_3 - \alpha_1}\,(x-ct), k\right]$$

$$\ldots\ldots (1.7.13)$$

The function has two limiting forms[42]

$$Sn(z,o) = Sin\ z \qquad (1.7.14)$$

and $$Sn(z,1) = \tanh z$$

Thus when $\alpha_2 \to \alpha_1$

$$f(x-ct) = \alpha_2 + (\alpha_3 - \alpha_2)\,Sech^2\left[\sqrt{\alpha_3-\alpha_2}\,(x-ct)\right] \qquad (1.7.15)$$

and when $\alpha_3 \to \alpha_2$

$$f(x-ct) = (\alpha_2-\alpha_3)\, \mathrm{Sin}^2 \left[\sqrt{\alpha_3-\alpha_1}\ (x-ct) \right] + \alpha_3$$

.......(1.7.16)

The nonlinear oscillatory solutions represented by equation (1.7.13) forms a bridge between purely linear oscillations and solitary wave motions.

1.8 REDUCTIVE PERTURBATION METHOD

All nonlinear evolution equations representing different physical problems may not always be analysed using the methods discussed in section 1.7. At times it may be possible to reduce the equation, using suitable methods, for the leading order of nonlinearity to some known equation having soliton solutions. One such method known as 'Reductive Perturbation Method' was devised and applied to a wide class of nonlinear systems by Taniuti and Wei[43,44]. A similar method was developed by Su and Gardner[41] to derive the KdV or Burgers equation for a wide class of nonlinear Galilean-invariant systems under the weak nonlinearity and long wavelength approximations. In this section we discuss the method developed by Su and Gardner.

We consider the governing equation for a set of state variables n,u and f in the form

$$n_t + (nu)_x = 0 \quad , \qquad (1.8.1)$$

$$(nu)_t + (nu^2 + P)_x = 0 \quad , \qquad (1.8.2)$$

$$P = P\,(\,f, n\,, u\,, f_x\,, n_x\,, u_x\,, f_{xt}\,, n_{xt}\,, u_{xt}\,, \ldots\ldots)\;, \qquad \ldots\ldots(1.8.3)$$

$$F\,(f, n, u, f_x\,, n_x\,, u_x\,, f_{xt}\,, n_{xt}\,, u_{xt}\,, \ldots\ldots) = 0\;, \qquad \ldots\ldots(1.8.4)$$

where we assume that both P and F can be expanded as Taylor series around a uniform state. By a proper choice[41] of P and F different physical systems can be considered. Equations (1.8.1) and (1.8.2) can be combined to give

$$nu_t + nuu_x + (P)_x = 0 \qquad (1.8.5)$$

At equilibrium, all the derivatives in P and F will drop out and we leave out the dependence of P and F on u to preserve Galilean invariance.

$$P = P(f,n) \quad \text{and} \quad F(f,n) = 0 \qquad (1.8.6)$$

Hence equation (1.8.2) gives

$$u_t + uu_x + \frac{1}{n} a^2 n_x = 0 \quad , \qquad (1.8.7)$$

where

$$a^2 = \left[P_n - (F_n / F_f) P_f \right]$$

In the limit of infinitesimal perturbations around a uniform state, equations (1.8.1) and (1.8.7) can be reduced to

$$u_{tt} - a_o^2 u_{xx} = 0 \quad , \qquad (1.8.8)$$

which is the linear wave equation with the constant speed of propagation a_o. Equation (1.8.8) gives solutions which consists of two oppositely travelling form-preserving waves. We change to a frame of reference which moves with one of these waves. To account for this and the slow variation of the waveform, we introduce a scale transformation of the independent variables;

$$\xi \;=\; \epsilon^{\alpha}\,(x-a_o t) \qquad , \qquad (1.8.9)$$

$$\tau \;=\; \epsilon^{\beta} t \qquad , \qquad (1.8.10)$$

where we assume $0 < \varepsilon < 1$ and $\alpha > 0$. The value of α is to be chosen later in such a way that the time variation of a state variable is balanced by both nonlinear and dispersive or dissipative effects.

The derivatives are then related by

$$\frac{\partial}{\partial x} \rightarrow \epsilon^{\alpha}\frac{\partial}{\partial \xi}\,, \quad \frac{\partial}{\partial t} \rightarrow \epsilon^{\beta}\frac{\partial}{\partial \tau} - a_o\epsilon^{\alpha}\frac{\partial}{\partial \xi} \qquad (1.8.11)$$

In view of this, we can rewrite equations (1.8.1) and (1.8.7) as

$$\epsilon^{\beta-\alpha}\, n_{\tau} + (u-a_o)n_{\xi} + n u_{\xi} \;=\; 0 \qquad (1.8.12)$$

and

$$\epsilon^{\beta-\alpha}\, u_{\tau} \quad (u-\,a_o)u_{\xi} + \frac{1}{n}\,P_{\xi} \;=\; 0 \qquad (1.8.13)$$

If we set $\beta-\alpha = 1$ then the perturbation series can be made to proceed in integral powers of ϵ. We will choose the value of α in such a way that the evolution equation, under the lowest order of nonlinearity, is independent of ϵ .

If we can expand n, f and u asymptotically as a series in powers of ϵ about an equilibrium state represented by $n=n_o$, $f=f_o$ and $u=0$; i.e.,

$$n = n_o + \epsilon n^{(1)} + \epsilon^2 n^{(2)} + \qquad .....(1.8.14)$$

$$f = f_o + \epsilon f^{(1)} + \epsilon^2 f^{(2)} + \qquad .....(1.8.15)$$

$$u = 0 + \epsilon u^{(1)} + \epsilon^2 u^{(2)} + \qquad .....(1.8.16)$$

We expand P and F about the equilibrium state with equation (1.8.9) and (1.8.10) in mind; i.e.,

$$P = P + P_{f_o}(f - f_o) + P_{n_o}(n - n_o) + P_{u_o}(u - u_o) + O(\epsilon^2) \qquad ....(1.8.17)$$

$$F = F_o + F_{f_o}(f - f_o) + F_{n_o}(n - n_o) + F_{u_o}(u - u_o) + O(\epsilon^2) \qquad ....(1.8.18)$$

As stated earlier, to ensure Galilean invariance we must have $P_{u_o}=0$ and $F_{u_o}=0$. Hence equation (1.8.17) and

(1.8.18) gives

$$\frac{\partial P^{(1)}}{\partial \xi} = \left[P_{n_o} - \frac{F_{f_o}}{F_{n_o}} P_{f_o} \right] \frac{\partial n^{(1)}}{\partial \xi} \equiv a_o^2 \frac{\partial n^{(1)}}{\partial \xi}$$

Hence the leading approximation to (1.8.12) and (1.8.13) can be written in view of the above expansions as

$$a_o \, n_\xi^{(1)} = n_o u_\xi^{(1)} \qquad (1.8.19)$$

$$a_o \, u_\xi^{(1)} = (a_o^2 \,/\, n_o)\, n_\xi^{(1)} \qquad (1.8.20)$$

These two equations can be integrated with the boundary conditions noted at $\xi \to \pm\infty$ to obtain

$$u^{(1)} = \frac{a_o}{n_o} n^{(1)} \qquad (1.8.21)$$

This reduces the leading-order problem to that of one variable. Within the order of our approximation we can write

$$P_\xi^{(2)} \approx a_o^2 \, n_\xi^{(2)} + A n^{(1)} n_\xi^{(1)} + \epsilon^{\alpha-1} B n_{\xi\xi}^{(1)} + \epsilon^{2\alpha-1} C \, n_{\xi\xi\xi}^{(1)} ,$$

.......(1.8.22)

where A,B and C are constants depending on the partial derivatives P and F evaluated at the equilibrium. In the next order of approximation, (1.8.12) and (1.8.13) gives

$$n_{\tau}^{(1)} + 2\,\frac{a_o}{no}\,n^{(1)}\,n_{\xi}^{(1)} - a_o n_{\xi}^{(2)} + n_o\,u_{\xi}^{(2)} = 0 \qquad (1.8.23)$$

and

$$\frac{a_o}{n_o} n_{\tau}^{(1)} + \frac{A}{n_o}\,n^{(1)}\,n_{\xi}^{(1)} + \epsilon^{\alpha-1}\,\frac{B}{n_o}\,n_{\xi\xi}^{(1)} + \epsilon^{2\alpha-1}\,\frac{C}{n_o}\,n_{\xi\xi\xi}^{(1)}$$

$$- a_o\,u_{\xi}^{(2)} + \frac{a_o^2}{n_o}\,n_{\xi}^{(2)} = 0 \qquad (1.8.24)$$

An evolution equation for $n^{(1)}$ can be obtained by eliminating $n_{\xi}^{(2)}$ and $u_{\xi}^{(2)}$ in the above two equations; i.e,

$$n_{\tau}^{(1)} + \left[\frac{A}{2a_o} + \frac{a_o}{n_o}\right] n^{(1)}\,n_{\xi}^{(1)} + \epsilon^{\alpha-1}\,\frac{B}{2a_o}\,n_{\xi\xi}^{(1)}$$

$$+ \epsilon^{2\alpha-1}\,\frac{C}{2a_o}\,n_{\xi\xi\xi}^{(1)} = 0 \qquad (1.8.25)$$

If B ≠ o then the last term is neglected because it is of higher order. Then equation (1.8.25) reduces to the Burgers equation (1.6.9) with a choice of α=1. Although this equation does not possess soliton solutions, it is a nonlinear equation of great importance.

If B = o; i.e, if the system is not dissipative, we can set $\alpha = 1/2$ so that (1.8.25) reduces to the KdV equation.

This procedure has been applied[41,44] to gas dynamics, waves in shallow water, hydromagnetic waves, ion acoustic waves in cold plasma, etc. Recently[45,46,47], this method has been successfully applied to nonlinear wave propagation on superfluid films. We shall discuss this in the next section.

Finally, we note that a more general matrix version of the above procedure has been developed by Taniuti and Wei[43]. Taniuti and Yajima developed[48,49] a reductive perturbation method suitable for nonlinear wave modulation.

1.9 SOLITONS ON SUPERFLUID FILMS

During several experiments on third-sound propagation in superfluid ^{4}He, finite amplitude effects that cannot be explained within the frame work of a linearised theory have been observed[28]. In particular, some evidence for the existence of an ordered pair of solitons has been observed by Kono et al[29]. These effects point to the need for a study of the superfluid condensate in its nonlinear form.

Huberman[51] was the first to study the problem of nonlinear wave propagation on a monolayer film of superfluid ^{4}He . It was shown that in addition to third-sound modes, small amplitude effects can lead to the existence of gapless solitons made up of superfluid condensate. Starting from the phenomenological equation of motion for superfluid condensate developed by Rutledge et al[17], he was able to derive the KdV equation with positive dispersion in a heuristic manner. He showed that it should be possible to create superfluid solitons by applying heating or cooling pulses to a localized region of the film, thereby altering the superfluid density locally. In the formalism he argued that in the thick film limit, the KdV equation may still hold but with a negative dispersion relation.

A systematic derivation of the nonlinear evolution equation for the superfluid density fluctuations on monolayer He II films was given by Biswas and Warke[45]. They applied the reductive perturbation method to the phenomenological equation of Rutledge et al[17] and obtained the following KdV equation

$$\frac{\partial \rho_1}{\partial t} + \frac{C_3(\rho_0 - 3a)}{2\rho_0(a + \rho_0)} \rho_1 \frac{\partial \rho_1}{\partial x} + \left[\frac{\hbar^2 + 4mB\rho_0}{8m^2 C_3} \right] \frac{\partial^3 \rho_1}{\partial x^3} = 0 \qquad \ldots\ldots (1.9.1)$$

where ρ_o is the equilibrium superfluid density, C_3 is the third sound velocity, m is the mass of ^{4}He atom, a=1.2 atomic layers and ρ_1 represents the leading term in the expansion of the density ρ as a series in ϵ. Note that ρ_1 is measured in units of atomic layers. Though this equation was similar to the one investigated by Huberman, the coefficient of the nonlinear term was different. From the condition that the width of a soliton should be real, they derived the inequality,

$$A_o (\rho_o - 3a) > 0 \qquad (1.9.2)$$

where A_o is the amplitude of the soliton. Thus if ρ_o>3a, equation (1.9.2) would imply $A_o > 0$. This describes the propagation of a local compression of the superfluid density relative to its average density ρ_o. While if ρ_o<3a (i.e. A_o<0) the corresponding solutions of (1.9.1) describe the propagation of a local rarefaction of the superfluid density relative to ρ_o.

Biswas and Warke predicted that when ρ_o >3a , it is possible to create solitary waves by applying a cooling pulse to a localized region of the film and if ρ_o <3a atomic layers, the application of a localized heating pulse will generate solitary waves. It is clear from (1.9.1) that when $\rho_o \approx 3.6$ atomic layers the nonlinear term in the KdV equation vanishes and hence it is not possible to create solitary waves either through a heating or a cooling pulse, and only third sound waves can be generated.

Nakajima et al[46] also obtained the KdV equation for nonlinear wave propagation on very thin superfluid films. The derivation was based on the Landau two-fluid hydrodynamics. They have discussed the conditions for the generation and detection of such solitons.

Since the usual generators and detectors of third-sound have much greater spatial and temporal scales than the single soliton solutions obtained according to the above theories[51], only a train of solitons can be studied in an actual experimental situation. This led Nakajima et al[51] to study the nonlinear wave propagation on saturated films of superfluid ^{4}He, whose thickness is of the order of 10^{-6} cms. In such films the surface tension, which can be neglected for the monolayer films[17], plays a decisive role. An estimation of parameters shows[51] that each solitary wave could be studied separately.

In all the works cited above, only the lowest order nonlinearity was taken into account. Kurihara[52] considered the effect of full nonlinearity on the propagation of solitons on thin superfluid films. He studied the phenomenological equation of Rutledge et al[17] in its fully nonlinear form. The analysis was done numerically. He observed that an arbitrary initial wave profile splits into oppositely travelling waves, which keeps its shape even after collision with each other. These stable waves, or 'quasi - solitons'[52] were the bound - states of localized excitations in amplitude and phase of the condensate wave function.

A close examination of these quasi solitons revealed[52] that they are asymmetric in shape, as against the solitons observed in the weak nonlinearity limit. Also at very large times the waves were not quite stable. The finite life time and asymmetry in shape of these quasi-solitons were essentially due to the effects of higher order nonlinearity. Later, Kurihara obtained analytically a travelling wave solution[53] for these large amplitude fluctuations.

The studies of nonlinear wave propagation on superfluid films discussed in this section are confined to one dimension. That is, the wave was assumed to travel on the superfluid films with a decay of wave profile along the direction of propagation, but with no change along the direction perpendicular to it. The question, whether such waves can be detected experimentally or whether they are stable or not needs a detailed discussion. The next section throws light into the more general problem of the stability of the solitary wave solutions of KdV and other such equations with respect to a transverse perturbation.

1.10 SOLITONS IN TWO DIMENSIONS

A soliton in one dimension (one space + one time dimension) is fully stable by virtue of its one dimensionality[54]. It was observed in the study of solitary waves in one dimension that a smooth initial wave form would evolve into a sharp soliton profile due to the nonlinearities present in the system. In the real world, such solitons represent different physical phenomena like, for example, the surface wave propagation in shallow water. In such a wave propagation it is assumed that there is no change in the wave profile along the direction (Y) perpendicular to its direction of motion (X). A natural question that arises at this level is that whether the one dimensional solitons would be stable if the pulse sharpening effects due to nonlinearities take place along the Y-direction also. In this section we study the stability of solitons in two space dimensions. The propagation of two dimensional solitons on thin superfluid films is also considered.

The stability of solitons in two dimensions (2 space+1 time dimension) was first studied by Kadomtsev and Petviashvili[54]. They considered the case of weak two dimensionality - that is the dependence on y-coordinate was assumed to be weak. For this purpose they studied the KdV equation with an additional term

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = \frac{\partial \phi}{\partial y} \qquad (1.10.1)$$

The term on the right hand side is the extra term which represents the weak y-dependence, where ϕ can be shown to satisfy the following relation for positive and negative dispersion.

$$\frac{\partial \phi}{\partial x} = \mp \frac{C}{2}\frac{\partial u}{\partial y}, \qquad (1.10.2)$$

where C is the velocity of motion of the coordinate frame along the x-direction and the signs - and + correspond to negative and positive dispersions respectively. Thus combining (1.10.1) and (1.10.2) we get

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}\right) = \mp \frac{C}{2}\frac{\partial^2 u}{\partial y^2}, \qquad (1.10.3)$$

which is known as the Kadomtsev-Petviashvili (K-P) equation.

Equations (1.10.1) and (1.10.2) were solved by the Krylov-Bogolyubov method, i.e, by introducing slowly varying parameters. Kadomtsev and Petviashvili came to the conclusion that the solitons represented by equation (1.10.1) were stable with respect to long wave perturbations in the direction transversal to their motion for negative dispersion and unstable for positive dispersion.

We can write equation (1.10.3) in the standard form as

$$(u_t + 6\, u\, u_x + u_{xxx})_x + \alpha\, u_{yy} = 0 \quad , \qquad (1.10.4)$$

where we have chosen $u \to 6u$ and $c = 2$. α take the values +1 and -1 and determines the dispersive property of the system. Using Hirota's method, Satsuma[55] obtained the N soliton solution of the K-P equation (1.10.4). The one soliton solution is[55]

$$u = \frac{1}{2}\; k_x^2\, \mathrm{Sech}^2\, (k_x\, x + k_y\, y - \omega\, t\,) \qquad (1.10.5)$$

where k_x, k_y, and ω are real constants satisfying

$$-k_x\, \omega + k_x^{\,4} + \alpha\, k_y^{\,2} = 0 \qquad (1.10.5a)$$

Equation (1.10.5) describes a wave propagating with the velocity $\omega/\sqrt{k_x^2 + k_y^2}$ in the direction making an angle of $\tan^{-1}(k_y / k_x)$ with the x-axis. The N-soliton solution has also been obtained by Ohkuma and Wadati[56], using the trace method. The two soliton solution has the form

$$u = 2\,(\log f_2)_{xx} \tag{1.10.6}$$

where

$$f_2 = 1 + \exp 2\eta_1 + \exp 2\eta_2 + A_{12} \exp\left[2(\eta_1+\eta_2) \right]$$

.......(1.10.7)

$$\eta_i = \frac{1}{2} k_i \left[x + p_i y - (k_i^2 + \alpha p_i^2)t \right] + \eta_i^{(0)} \tag{1.10.8}$$

$$A_{12} = \frac{3(k_1 - k_2)^2 - \alpha (p_1 - p_2)^2}{3(k_1 + k_2)^2 - \alpha (p_1 - p_2)^2} \tag{1.10.9}$$

and k_i and $k_i p_i$ are components of the linear momentum along x and y directions respectively. The value of A_{12} gives the nature of interaction. For $A_{12}=0$ or ∞ we have the phenomenon of 'Soliton Resonance'. When this happens, two interacting solitons create a third soliton. These three solitons resonantly couple each other such that a Y-shaped structure is formed by the three wave crests. Details of soliton resonance of K-P equation would be dealt with in Chapter II, when we study the soliton resonance in thin superfluid films.

Soliton resonance is not a phenomenon restricted to just the K-P equation. This can be observed in the two dimensional nonlinear Schrödinger equation[57], the three dimensional ion acoustic system[58], etc.

The discovery of soliton resonance in two and higher dimensional space led many in search of the phenomena in one dimensional systems. Tajiri and Nishitani[59] showed that resonance is displayed by a Boussinesq type equation,

$$u_{tt} - u_{xx} + (u^2)_{xx} + u_{xxxx} = 0 \qquad (1.10.10)$$

Later, Hirota and Ito[60] were able to demonstrate that the Sawada Kotera equation and the model equation for shallow water waves also exhibited soliton resonance.

There is a noticeable difference in the properties of soliton resonance occurring in one dimension. As is obvious, a Y-shaped structure cannot form in one dimensional space. Instead, two solitons interact each other to form a single soliton or a soliton splits into two[61].

The concept of soliton resonance was successfully used to explain 'mach reflection' of soliton[62,63]. Soliton resonance has been observed experimentally in the interaction of waves on shallow beaches[64] and in the interaction of ion acoustic waves in unmagnetized plasma[65,66,67].

The result that the solitons of the K-P equation with positive dispersion (- sign in equation (1.10.4)) are unstable is due to the fact that in the derivation, Kadomtsev and Petviashvili[54] restricted the transverse perturbations to the small k limit (i.e. long wavelength). Katyshev and Makhankov[68] approached the problem from a different point of view. They considered the K-P equation in the form

$$\phi_{tx} + (\phi \, \phi_x)_x + \phi_{xxxx} - \phi_{yy} = 0 \qquad (1.10.11)$$

Using the variation of Action Method, they were able to establish that there is a threshold behaviour in the instability. It was shown that the solitons are unstable for transverse perturbations with wave number k, satisfying the inequality

$$k < k_c \qquad (1.10.12)$$

where $k_c = 6\eta^2$ and the parameter η follows from the soliton solution for (1.10.11) in the form

$$\phi = 12\, \eta^2 \, \mathrm{Sech}^2 \, \eta \, (x - x_o - 4\eta^2 t) .$$

Zakharov[69] has presented a sufficient condition for instability in the region

$$k < (3)^{1/2} \eta^2 \qquad (1.10.13)$$

It was later shown by Laedke and Spatschek[70] that the K-P solitons are indeed stable between the limits governed by (1.10.12) and (1.10.13). Thus the K-P soliton with positive dispersion is stable outside the region governed by (1.10.13). Hence, as described by Ablowitz and Segur[71], in the context of nonlinear water waves in tanks , experimental observation of such solitons is possible if the width of the system is small enough to avoid the perturbations having wavelengths larger than the critical value.

There are equations other than the K-P equation which have two dimensional solitons. The two dimensional cubic nonlinear Schrödinger equation is one example

$$i A_t + A_{xx} + A_{yy} + \sigma |A|^2 A = 0 \qquad (1.10.14)$$

The weakly two dimensional form of this equation was obtained by Benney and Roskes[72] for surface water waves, where the effects of gravity, surface tension and arbitrary depth are included.

$$
\left.\begin{aligned}
iA_t + \sigma_1 A_{xx} + A_{yy} &= \sigma_2 |A|^2 A + \Phi_x A , \\
a \Phi_{xx} + \Phi_{yy} &= -b \left(|A|^2\right)_x
\end{aligned}\right\} \quad (1.10.15) ,
$$

where a, b , σ_1, and σ_2 depends on the depth and surface tension. Oblique interaction of solitons on a two dimensional surface has been observed experimentally, a typical example being the one photographed by Toedtemeier off the Oregon coast[73]. Recently, there are indications[74] that two dimensional solitons could be observed in thin superfluid films.

Biswas and warke[74] extended their earlier results of one dimensional soliton propagation to include weak two dimensionality and obtained the K-P equation. We shall discuss their work in detail in Chapter II, where we have extended their results to show that K-P equation can exhibit soliton resonance in thin superfluid films.

1.11 CHAOS AND SOLITONS

We have seen in the earlier sections of this chapter that there are systems with infinite degrees of freedom which are completely integrable and whose time evolution is completely described by nonlinear modes including solitons. On the other hand, there are many simple systems, with hardly a few degrees of freedom, showing chaotic behaviour. When one knows that even very simple systems show chaotic behaviour, it seems not so surprising if most of nonlinear systems with infinite degrees of freedom show chaotic behaviour. Apparent exceptions are the completely integrable systems.

By the term Chaos we mean 'sensitive dependence on initial conditions'. That is, the dynamics of the nonlinear system can be completely altered by an infinitesimal change in the initial conditions. This can also be viewed as a large (or exponential) increase in the distance between nearby trajectories in the phase space of the system. Such an exponential divergence of nearby trajectories would mean a 'non-predictable' behaviour of the system.

In the recent developments of nonlinear physics, the generation and characterization of chaos has been receiving considerable attention. Several routes to chaos have been found in some simple dynamical systems with only a few degrees of freedom[75]. One of the most important, and the most widely used quantitative measure of chaos is the Lyapunov characteristic exponent[76]. It gives the mean exponential rate of divergence of nearby trajectories of a chaotic dynamical system in phase space. The chaos is defined[77] as a state of exponentially growing separation distance, that is the state where the maximum Lyapunov exponent is positive.

In energy conserved systems ergodicity is closely related to the problem of chaos. Kolmogorov-Arnold-Moser (KAM) theorem[78] gives a very important understanding in the ergodicity theory. Roughly speaking, KAM theorem discusses the fate of trajectories of systems with an integrable Hamiltonian, but with a small perturbation added. To be precise, let the system be described by the Hamiltonian $\mathcal{H}_o$ and let a small perturbation be added to $\mathcal{H}_o$, so that the new Hamiltonian is

$$\mathcal{H} = \mathcal{H}_o + \lambda \mathcal{H}_1 \quad , \qquad (1.11.1)$$

where λ determines the strength of perturbation. Then KAM theorem states that[79] "provided λ is sufficiently small and $\mathcal{H}_1$ is analytic in a given domain, the phase space can be separated into two regions of nonvanishing volume, one of which is small compared to the other and shrinks to zero as $\lambda \to o$". The larger region has a structure similar to those of an integrable system.

The effect of perturbation on many completely integrable systems has revealed[80,81,82] many interesting properties of soliton propagation in such systems. Imada[82] studied the dynamics of solitons under an energy preserving perturbation to the usual Sine - Gordon equation. The Hamiltonian chosen was

$$\mathcal{H} = \mathcal{H}_o + \mathcal{H}_1 \quad , \tag{1.11.2}$$

where

$$\mathcal{H}_o = \frac{1}{2} (\phi_t^2 + c_o^2 \phi_x^2) + 1 - \text{Cos}\phi \tag{1.11.3}$$

and

$$\mathcal{H}_1 = (1-R)^2 \frac{1 - \text{Cos}\phi}{1 + R^2 + 2 R\text{Cos}\phi} - (1 - \text{Cos}\phi) \tag{1.11.4}$$

C_o^2 is a constant and R is a parameter which determines the deviation from complete integrability. When R=0 the Hamiltonian $\mathcal{H}$ reduces to Sine-Gordon Hamiltonian. Imada investigated numerically the separation distance between two trajectories in which the initial conditions are slightly different. The local separation distance between two trajectories in the phase space was defined as[82]

$$\delta s(x,t) = \sqrt{\left[\delta \phi(x,t) \right]^2 + \left[\delta \phi_t(x,t) \right]^2} \tag{1.11.5}$$

where $\delta\Phi$ and $\delta\Phi_t$ are the difference of Φ and Φ_t between the two trajectories. The initial conditions were chosen in such a way that the initial values of $\delta\Phi$ and $\delta\Phi_t$, i.e.,

$\delta\Phi(x,o)$ and $\delta\Phi_t(x,o)$ were very small. The total separation distance has the form[82]

$$\delta S(t) = \sqrt{\frac{1}{L}\int_o^L [\ \delta s(x,t)\ ]^2\, dx} \qquad (1.11.6)$$

where L is the system size. Imada imposed a periodic boundary condition for the system and chose the soliton-antisoliton pair solution as the initial condition. During the numerical evaluation of the system, it was observed that during each collision of the soliton-antisoliton pair, the value of $\log(\delta S(t))$ increased significantly. Also, the effect of the collision was observed to propagate spatially so as to increase $\log(\delta S(t))$ almost linearly with time. Imada calculated the maximum Lyapunov exponent[82,77]

$$\lambda = \lim_{t\to\infty} \frac{1}{t} \ln \left[\frac{\delta S(t)}{\delta S(0)} \right] \qquad (1.11.7)$$

and found that it is positive for positive R. The value of λ was zero, as expected, when the perturbation was made zero (i.e., R=0).

A study of the quasi soliton propagation in a nonlinear wave equation with fifth order dispersion has also shown chaotic behaviour[83]. The equation had the form

$$u_t + u\, u_x - \gamma^2 u_{xxxxx} = 0 \qquad (1.11.8)$$

Similar studies have also been done for energy nonconserving perturbations[82,84,85].

In chapter II we present the studies of the phenomenon of two-soliton resonance on thin ^{4}He films.

CHAPTER 2

SOLITON RESONANCES IN THIN SUPERFLUID FILMS

SOLITON RESONANCES IN THIN SUPERFLUID FILMS

Of the rich variety of nonlinear phenomena exhibited by very thin superfluid ^{4}He films, one of the most interesting one is the phenomenon of 'Soliton resonance'. In this chapter we extend the results of Biswas and Warke[74] and study the two soliton resonances of the Kadomtsev-Petviashvili equation for the superfluid surface density fluctuations in thin superfluid ^{4}He films. The final form of the resonant soliton is obtained and the expressions for their amplitude and velocity are derived.

This chapter is divided into three sections. The nonlinear evolution equation describing superfluid density fluctuations in the lowest order of nonlinearity, as obtained by Biswas and Warke, is discussed in section 2.1. We examine the phenomenon of soliton resonance in section 2.2. In the last section we study the soliton resonance in monolayer superfluid films.

2.1 K-P EQUATION AND TWO DIMENSIONAL SUPERFLUID FILMS

The phenomenological equation of motion for the monolayer superfluid density fluctuations, proposed by Rutledge et al[17], is given by

$$i\hbar \frac{\partial \psi}{\partial t} = - \frac{\hbar^2}{2m} \nabla^2 \psi - \frac{A\psi}{(a + |\psi|^2)^3} - \mu\psi - B\psi \nabla^2 |\psi|^2 , \qquad \ldots\ldots(2.1.1)$$

where $\psi(x,t)$ is the condensate wave function, m is the mass of the helium atom, A and a (A=14k and a=1.2 atomic layers) are constants of Van der Waals interaction, μ is the chemical potential and B is the surface tension. The form of ψ was chosen as

$$\psi(x,t) = \left[\rho(\vec{x},t) \right]^{1/2} \exp \left[i \phi(\vec{x},t) \right] \qquad (2.1.2)$$

so that the superfluid density is

$$\rho(\vec{x},t) = | \psi(\vec{x},t) |^2 \qquad (2.1.3)$$

where ρ and ϕ are real functions. The reductive perturbation method is applied to the resulting set of equations. Then the solution that is being looked for corresponds to a characteristic collective density oscillation mode propagating along the x direction in the two dimensional superfluid. The variation in density along the y direction is made very small by choosing the expansion parameters in such a way that the power of the parameter is greater for the y-coordinate transformation compared to that for the x-coordinate. But this choice of parameters is made consistent with the equality

$$\frac{\partial^2 \phi}{\partial x \partial y} = \frac{\partial^2 \phi}{\partial y \partial x}$$

The analysis gives the velocity of third sound as

$$C_3^2 = \frac{3A\ \rho_o}{m(a+\rho_o)^4} , \quad (2.1.4)$$

where ρ is the equilibrium density.

The final equation which they arrived at was

$$\frac{\partial}{\partial \tilde{x}} = \left[\frac{\partial \rho_1}{\partial \tilde{t}} + \frac{C_3(\rho_o - 3a)}{2\rho_o(a + \rho_o)}\rho_1 \frac{\partial \rho_1}{\partial \tilde{x}} + \frac{\hbar^2 + 4mB\rho_o}{8m^2 C_3} \frac{\partial^3 \rho_1}{\partial \tilde{x}^3}\right]$$

$$- \frac{C_3}{2}\frac{\partial^2 \rho_1}{\partial \tilde{y}^2} = 0 \quad (2.1.5)$$

where $\tilde{x},\tilde{y}$ and $\tilde{t}$ are the scaled coordinates. Equation (2.1.5) is the K-P equation. It was argued that in the general case the solitons represented by the system would be unstable because the equation (2.1.5) is the K-P equation with positive dispersion. But as has been discussed in section 1.10, this K-P equation would have stable soliton solutions if the width of the system is chosen to be less than the critical wavelength. However, the system admits stable 'lump solitons' in the general case. These lumps have properties similar to solitons, but for the algebraic decay of its tails in x and y directions.

2.2 RESONANACE OF SOLITONS

We have seen in the Chapter I that most of the stable solitons are one-dimensional entities. Such one dimensional solitons can interact two-dimensionally with each other. This is because in any isotropic media, solitons can propagate in all directions and interact obliquely with each other. Such interactions could be described[86] by the K-P equation, for weakly nonlinear, weakly dispersive and almost unidirectional wave propagations.

We consider the interactions of the K-P solitons, which have the form

$$u_i = \frac{1}{2} k_i^2 \, \mathrm{Sech}^2 \, k_i \left[x + p_i \, y - \omega_i \, (k_i \, , \, p_i) t \right] \qquad (2.2.1)$$

when two such solitons interact a phase shift δ_{12}, which depends on the interaction angle[55], is caused due to the mutual collision. Hence the structure of the solution of the K-P equation during interaction of the solitons would involve two wedges formed by the pair of lines[57] $\psi_1=C_1$, $\psi_2=C_2+\delta_{12}$ and $\psi_1=C_1+\delta_{12}$, $\psi_2=C_2$, where C_1 and C_2 are constants. This is illustrated in Fig.2.1. If the phase shift is zero then the vertices of the two wedges are coincident and the arms of each wedges are simply continuation of the opposite arms of the other (Fig.2.2). For a finite phase shift, the linear crest joining the two vertices is given[57] by the equation.

$$(k_1 + k_2)\, x + (k_1 p_1 + k_2 p_2)\, y + (\omega_1 + \omega_2)t = C_1 + C_2 + \delta_{12} \qquad ....(2.2.2)$$

Miles[62,63] noted that the two soliton of the K-P equation breaks down at a certain critical angle. Based on the frame work of the Zakharov-Shabat theory[87] of integrable systems with more than one spatial dimension, Newell and Redekopp[57] studied a general criterion for strongly interacting solitons and found that the two soliton solution breaks down at the critical angle which corresponds to the condition

$$\omega(\vec{k}_1 + \vec{k}_2) = \omega(\vec{k}_1) + \omega(\vec{k}_2) \qquad (2.2.3)$$

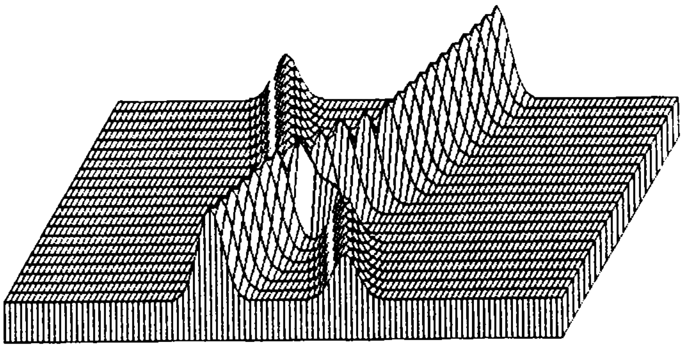

Fig 2.1

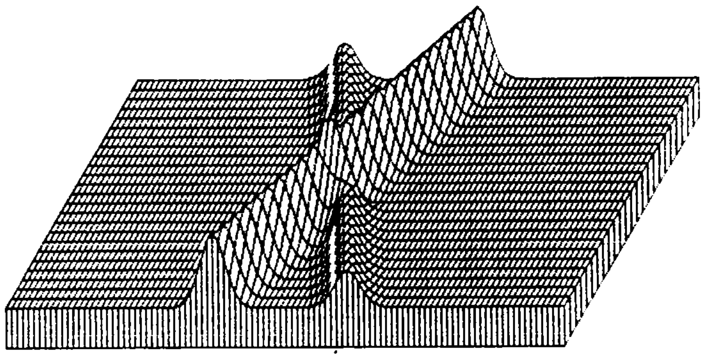

Fig 2.2

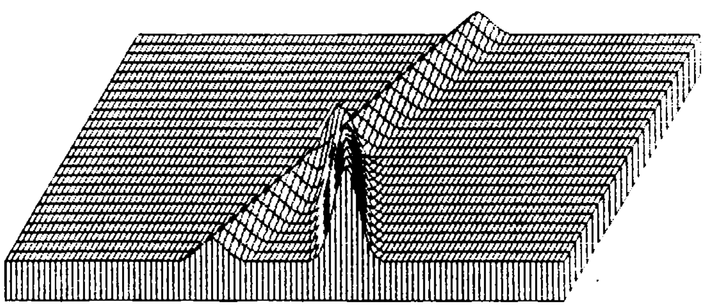

Fig 2.3

During such an interaction, the phase shift tends to infinity[57] and the crest described by (2.2.2), which joins the vertices of the interacting waves, becomes infinitely long and it satisfies the soliton dispersion relation. Hence in this process, called the 'Soliton resonance', a third soliton is created by the collision of two obliquely interacting solitons; the three solitons resonantly couple each other, such that a Y-shaped structure is formed by the three wave crests.

Ohkuma and Wadati[56] showed that the phase shift δ_{12} would become infinite when A_{12} (given by equation (1.10.9)) takes the values zero or infinity. When this condition is satisfied, resonance is possible among three solitons; a coupled system of these solitons thus make a phase locked system. When the amplitude of the third wave is a function of the sum of the amplitudes of the colliding waves, it is called plus resonance[58]. On the other hand, when it is a difference it is called minus resonance.

2.3 SOLITON RESONANCES ON SUPERFLUID FILMS

We make the following substitutions in equation (2.1.5) to obtain it in the more familiar form as given by equation (2.3.2)

$$u = -6\rho_1/\gamma \quad , \quad \gamma = 2\rho_o(a+\rho_o)/(3a-\rho_o) \quad ,$$

$$\bar{x} = k_o\tilde{x} \quad , \quad \bar{y} = \sqrt{2}\, k_o\, \tilde{y} \; , \quad \bar{t} = C_3\, k_o\, \tilde{t} \quad ,$$

$$k_o^2 = 8\; m^2\; C_3^2 \,/\, (\hbar^2 + 4\; m\; B\; \rho_o)\; x \qquad (2.3.1)$$

Thus,

$$\left[\, u_{\bar{t}} + 6\; u\; u_{\bar{x}} + u_{\bar{x}\bar{x}\bar{x}} \,\right]_{\bar{x}} - u_{\bar{y}\bar{y}} = 0 \qquad (2.3.2)$$

The K-P equation in the form given above has been discussed in great detail in sections 1.10. The one soliton solution has the form[()]

$$u = \frac{1}{2}\, k^2\; \mathrm{Sech}^2\; \eta \qquad (2.3.3)$$

where

$$\eta = \frac{1}{2}\, k \left[\, \bar{x} + p\bar{y} - (k^2-p^2)t \,\right] \qquad (2.3.4)$$

and k and kp are components of linear momentum along the x and y direction respectively.

The two soliton solution of equation (2.3.2) is given by

$$U = 2(\log\; f_2)_{\bar{x}\bar{x}} \qquad (2.3.5)$$

where

$$f_2 = 1 + \exp 2\eta_1 + \exp 2\eta_2 + A_{12} \exp\left[2(\eta_1 + \eta_2)\right] \qquad (2.3.6a)$$

$$\eta_i = \frac{1}{2} k_i \left[\bar{x} + p_i \bar{y} - (k_i^2 - p_i^2) \bar{t} \right] \eta_i^{(o)} \qquad (2.3.6b)$$

$$A_{12} = \frac{3(k_1 - k_2)^2 + (p_1 - p_2)^2}{3(k_1 + k_2)^2 + (p_1 - p_2)^2} \qquad (2.3.6c)$$

As we have already explained in section 2.2, there is an intermediate region during the interaction of the solitons. It was shown that soliton resonance is the special case under which the intermediate regime tends to infinity. This would happen when A_{12}=o or $A_{12}=\infty$.

i.e. for

$$3(k_1 \pm k_2) + (p_1 - p_2)^2 = 0 \qquad (2.3.7)$$

The plus sign refers to the case $A_{12}=\infty$ and corresponds to plus resonance and the other case (A_{12}=0) is the minus resonace. The resonance phenomena can be best understood by the asymptotic behaviour of the two solitons under the above conditions.

$$U = U^{(1)} + U^{(2)} = \frac{1}{2} k_1^2 \operatorname{Sech}^2 \eta_1 + \frac{1}{2} k_2^2 \operatorname{Sech}^2 \eta_2 , \qquad y \to -\infty$$

$$= U^{(1+2)} = \frac{1}{2} (k_1 + k_2)^2 \operatorname{Sech}^2(\eta_1 + \eta_2) , \qquad y \to +\infty \qquad (2.3.8a)$$

$$U = U^{(1+2)} = \frac{1}{2} (k_1 + k_2)^2 \operatorname{Sech}^2(\eta_1 + \eta_2) , \qquad y \to -\infty$$

$$= U^{(1)} + U^{(2)} = \frac{1}{2} k_1^2 \operatorname{Sech}^2 \eta_1 + \frac{1}{2} k_2^2 \operatorname{Sech}^2 \eta_2 , \qquad y \to +\infty \qquad (2.3.8b)$$

The above two equations represents plus resonance and is illustrated in Fig.2.3.

The minus resonances are given by

$$U = U^{(1)} = \frac{1}{2} k^2{}_1 \operatorname{Sech}^2 \eta_1 , \qquad y \to -\infty ,$$

$$= U^{(2)} = \frac{1}{2} k_2^2 \operatorname{Sech}^2 \eta_2 , \qquad y \to +\infty ,$$

$$= U^{(1-2)} = \frac{1}{2} (k_1 - k_2)^2 \operatorname{Sech}^2(\eta_1 - \eta_2) , \qquad x \to +\infty \qquad (2.3.9a)$$

$$U = U^{(2)} = \frac{1}{2} k_2^2 \; \mathrm{Sech}^2 \eta_2 \;, \qquad y \to -\infty \,,$$

$$= U^{(1)} = \frac{1}{2} k_1^2 \; \mathrm{Sech}^2 \eta_1 \;, \qquad y \to +\infty \,,$$

$$= U^{(1-2)} = \frac{1}{2} (k_1 - k_2)^2 \; \mathrm{Sech}^2 (\eta_1 - \eta_2) \;, \qquad x \to -\infty \qquad (2.3.9b)$$

In general, the resonant soliton can be written in the form,

$$U = U^{(1 \pm 2)} = \frac{1}{2} (k_1 \pm k_2)^2 \; \mathrm{Sech}^2 (\eta_1 \pm \eta_2) \qquad (2.3.10)$$

From the form of the above solution we can calculate the amplitude and velocity of the resonant soliton in the laboratory frame as given below.

$$A_r = \left[6\rho_o \, (a + \rho_o)/(\rho_o - 3a) \right] (k_1 \pm k_2)^2 \qquad (2.3.11)$$

$$V_r = C_3 \frac{\left[k_1 (k_1^2 - p_1^2) \pm k_2 (k_2^2 - p_2^2) \right] - (k_1 \pm k_2)}{\left\{ (k_1 \pm k_2)^2 + 2(k_1 p_1 \pm k_2 p_2)^2 \right\}^{1/2}} \qquad (2.3.12)$$

The interacting solitons and the resonant soliton form a coupled system with a Y-shaped structure and travels with the above velocity. We have shown that the phenomenon of soliton resonances could be observed in very thin superfluid ^{4}He films.

We can visualize equation (2.3.8a) as the formation of a single soliton as the result of a collision of two solitons. Equation (2.8b) could be considered as the splitting of a soliton into two solitons. The process of formation of a single soliton from two solitons, as depicted by equation (2.8a) is shown pictorially in Fig.2.3.

Minus resonance could also be viewed in a manner similar to the plus resonance - i.e. equation (2.3.9a) could be viewed as the formation of a single soliton from two solitons and (2.3.9b) as the reverse process.

It should be possible to observe these resonant solitons by measuring the velocities in the asymptotic limits using suitable detecting devices. But, it should be borne in mind that as explained in section 1.10, the K-P equation with positive dispersion relation is unstable with respect to transverse perturbations having wavelengths greater than a critical value (obtainable from equation (1.10.13). Hence the width of the film should be limited to a value less than the critical wavelength as discussed by Ablowitz and Segur[71] in the context of water wave experiments in tanks.

As has been suggested by Huberman[50], the dispersion relation could change when the thickness of the film is increased. Then it might be possible to observe the solitons without any restriction to the film width. This problem is taken up in chapter.5.

Since the whole formalism presented in this chapter is based on the K-P equation, which holds for the superfluid only when the lowest order nonlinearity is taken into account, the amplitude of the solitons during an experimental run should be carefully chosen. The dynamics of the waves are different when higher nonlinearities are taken into consideration.

CHAPTER 3

LARGE AMPLITUDE QUASI SOLITONS IN THIN SUPERFLUID FILMS

LARGE AMPLITUDE QUASI SOLITONS IN THIN SUPERFLUID FILMS

The soliton dynamics on super fluid ^{4}He films are not just confined to the small amplitude regime. It is possible to obtain localized waves even under full nonlinearity, though the KdV and K-P equations are able to describe the dynamics of superfluid density fluctuations only in the weak nonlinearity limit. In this chapter we study the propagation of large amplitude quasi solitons in thin two dimensional superfluid films. We have seen in section (1.9) that such localized waves have asymptotic temporal stability[52,53] when one dimensional wave propagation is considered. The fate of these large amplitude waves when a weakly two dimensional wave propagation is considered is studied in this chapter. This work is done numerically for a monolayer superfluid film.

The chapter is divided into two sections. The equation governing the large amplitude density fluctuations on thin superfluid films is reduced to a dimensionless form in section 3.1. In the second section we numerically study the large amplitude soliton propagation on such films.

3.1 LARGE AMPLITUDE SOLITONS

We begin the analysis by starting from the equation of motion for the superfluid density fluctuations (equation (1.3.7)), reproduced below for convenience.

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi - \frac{A\,\psi}{(a+|\psi|^2)^3} - \mu\psi - B\,\psi\,\nabla^2|\psi|^2 \qquad (3.1.1)$$

where m,A,a,μ and B have their usual meanings. For the monolayer films we are going to consider, we can take B=0[17]. If we search for a solution of the form $\psi(x,y,t) = \rho_s^{1/2}(x,y,t)\ e^{i\phi(x,y,t)}$, where $\rho_s(x,y,t)$ is the superfluid density, one would get the two dimensional continuity equation[17]

$$\frac{\partial\rho_s}{\partial t} + \nabla\cdot\vec{j}_s = 0 \qquad (3.1.2)$$

where $\vec{j}_s(x,y,t) = \mathrm{Re}[(\hbar/im)\ \psi^*\ \vec{\nabla}\psi]$ is the quantum-mechanical current density.

For the purpose of numerical analysis we transform equation (3.1.1) to a dimensionless form. We assume that ψ depends only on the time coordinate t and space coordinates x and y. The scale for ψ is chosen as its equilibrium value ψ_o, obtained from the relation

$$\mu + \frac{A}{(a+|\psi_o|^2)^3} = 0 \qquad (3.1.3)$$

We can fix the equilibrium value of the superfluid thickness as

$$d_o = \psi_o^2 / a \qquad (3.1.4)$$

Scales for the space coordinates and time coordinate are fixed by the characteristic wave vector k and frequency ω :

$$k = \left[\frac{2mW}{\hbar^2} \right]^{1/2} , \quad \omega = W/\hbar \qquad (3.1.5)$$

where $W = \dfrac{A}{a^3(1+d_o)^3}$ is the Van der Waals energy.

Now we can rewrite equation (3.1.1) in the normalized form

$$i \frac{\partial\chi}{\partial\tau} = - \frac{\partial^2\chi}{\partial\xi^2} - \frac{\partial^2\chi}{\partial\eta^2} - \left\{ \left[\frac{1 + d_o}{1 + d_o|\chi|^2} \right]^3 - 1 \right\} \chi \qquad (3.1.6)$$

where $\chi = \psi / \psi_o$, $\xi = kx$, $\eta = ky$, and $\tau = \omega t$. Since for such a monolayer superfluid film we cannot have surface deformations, we will be studying the superfluid density fluctuations occurring in the two dimensional film. We assume that initially the superfluid density is locally altered. For example, this could be done by heating the film locally. After this is done equation (3.1.2) would hold.

3.2 NUMERICAL STUDIES

In this section we undertake the numerical study of equation (3.1.6). For the sake of simplicity we look for solutions propagating along ξ-axis. The size of the superfluid film (= 100 along the ξ-direction) is chosen arbitrarily in such a way as to be larger than the characteristic size of the localized excitations. Equation (3.1.6) is treated as an initial value problem. It is assumed that at t=o the whole superfluid is at rest - that is we choose the initial value of the phase of the wave to be constant through out the film. The dynamics of the system is independent of the actual value of this constant.

We study the time evolution of the superfluid density fluctuation $\rho(\xi,\eta,\tau) = |\psi(\xi,\eta,\tau)|^2 - 1$. This study is done for two different cases of the initial profile, $\rho(\xi,\eta,o)$. First we study the dynamics when a squared secant hyperbolic profile is used as the initial condition. This has special importance - we are essentially studying the fate of the K-P solitons under the full nonlinearity. As a second case we take an initial profile which is very much different from the squared secant hyperbolic profile. The rounded rectangular form is chosen as a suitable wave form. For both the above cases we impose the periodic boundary condition,

$$\rho(100,\eta,\tau) = \rho(o,\eta,\tau) \qquad (3.2.1)$$

Since there is no surface deformation to the monolayer film, we do not have to take the kinematic boundary conditions.

The numerical results for the first case are shown in figures in Fig.3.1 and Fig.3.2. Fig. 3.1 shows the time evolution of the superfluid density fluctuations for the initial profile

$$\rho(\xi,\eta,o) = \rho_o \operatorname{sech}^2 \left[[(\xi - 25)/3] + [(\eta - 50)/55] \right] \qquad \ldots.(3.2.2)$$

The parameters chosen are $d_o=1$ and $\rho_o=0.2$. The superfluid velocity corresponding to the density fluctuations are plotted in Fig. 3.2. The numerical experiments carried out for several other values of the parameters showed similar results.

Two solitons emerge from the single peak of the initial profile and travel in opposite directions. These solitons preserve their identity after interaction with each other and are quite stable. Under close examination these peaks are found to be asymmetric. These solitons, "quasi-solitons"[52] are not completely stable as the large time behaviour might suggest. The finite life time of the solitons as well as the asymmetry arise essentially

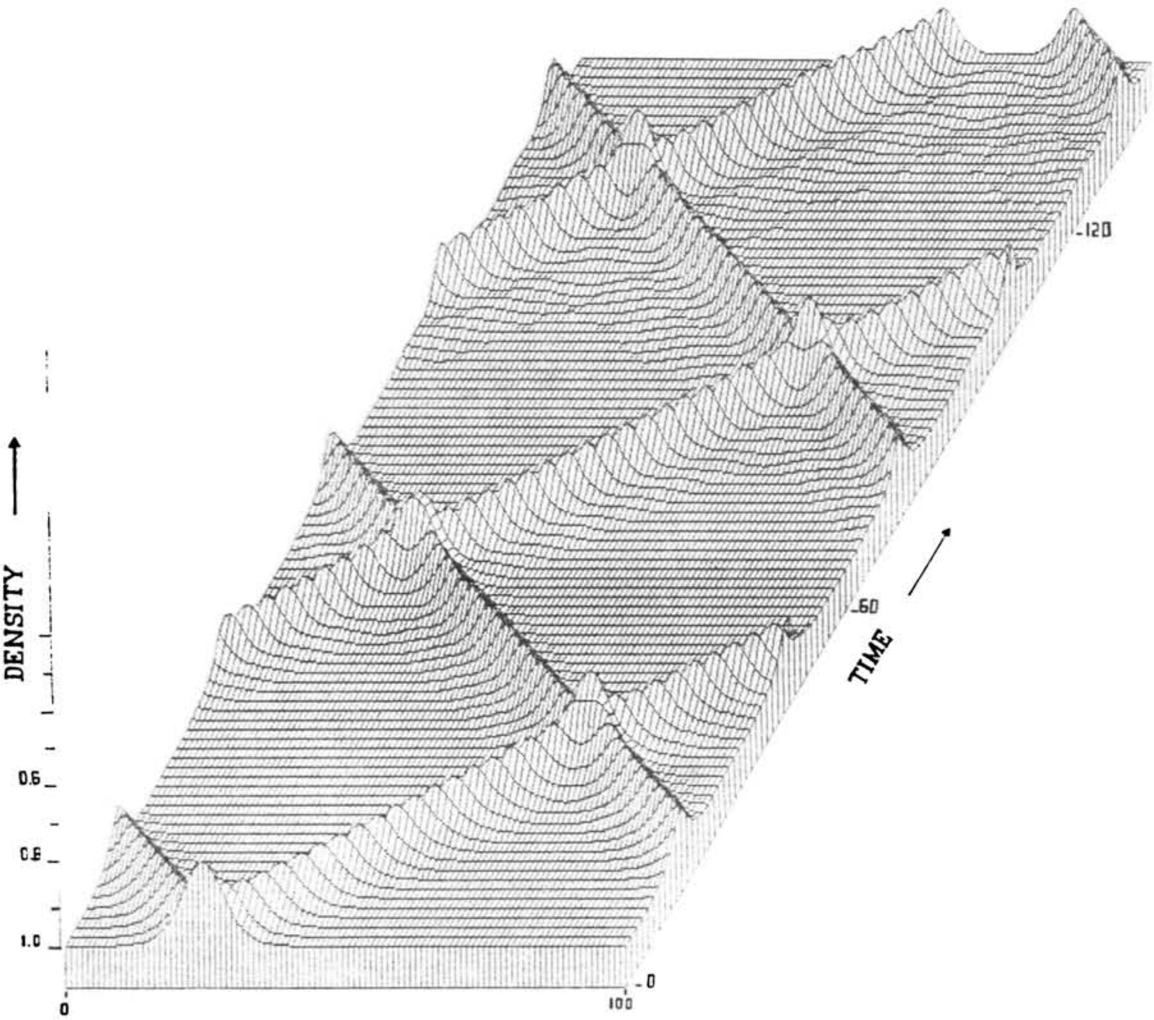

Fig 3.1. Time evolution of the amplitude of the superfluid density fluctuations occurring along the ξ-direction, for the squared secant hyperbolic initial profile.

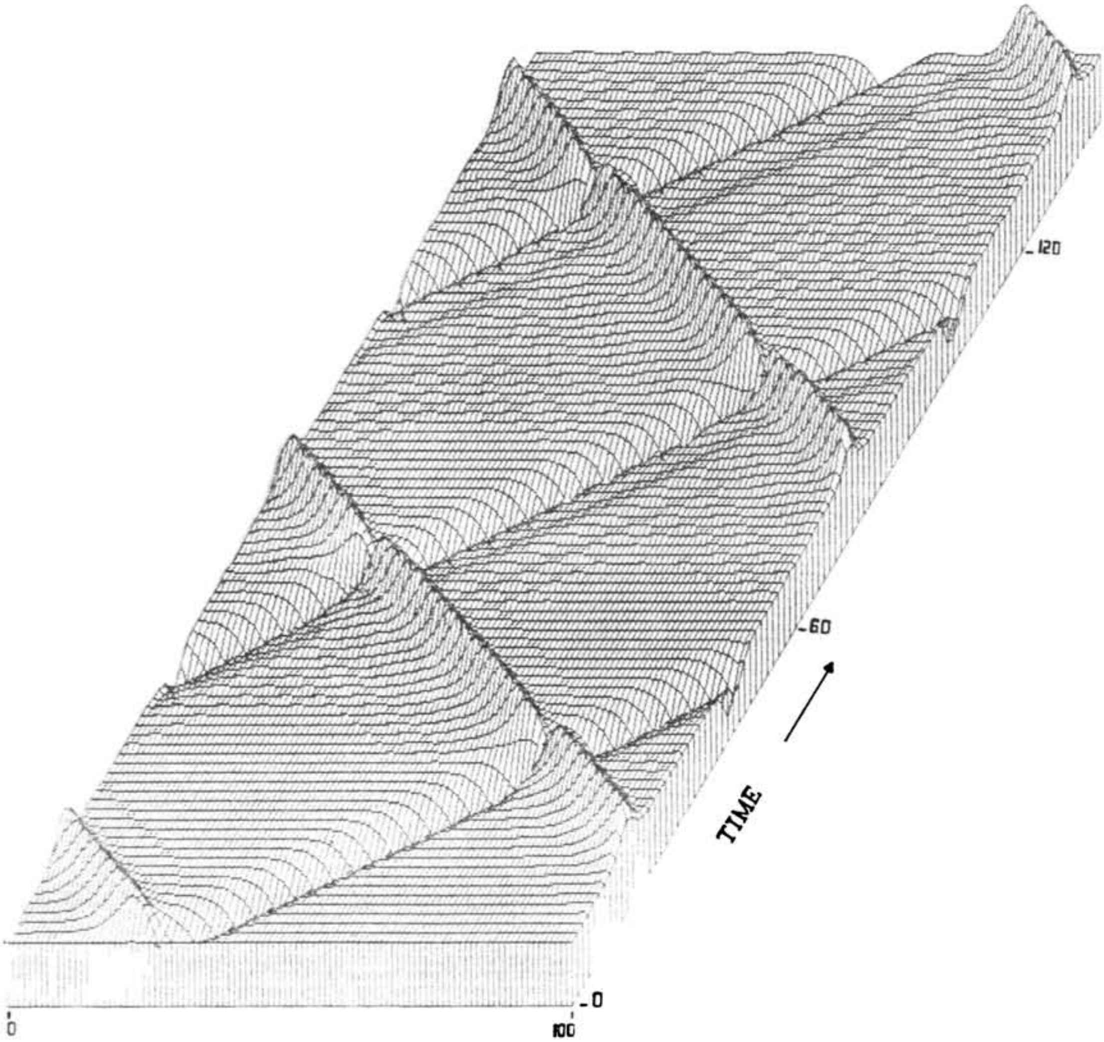

Fig 3.2. - The superfluid velocity, corresponding to the density fluctuations ploted in Fig.3.1, in arbitrary units.

from the higher order nonlinearity.

Now we turn our attention to the second case. Here we use as the initial profile the rounded rectangular form,

$$\rho(\xi,\eta,o) = \rho_o \frac{1}{2} \left\{1-\tanh\left[\left[(\xi - 25)/1.5\right]^2+\left[(\eta-50)/55\right]^2-4\right]\right\} \qquad \ldots(3.2.3)$$

Again, we choose the parameters as $d_o=1$ and $\rho_o=0.2$. The resulting temporal developments in amplitude and velocity are plotted if Fig.3.3 and Fig.3.4.

As in the previous case, the initial profile splits into two. But as the resulting pair of waves travel in opposite directions, they continuously emit secondary waves. Even after a few interaction with each other, the resultant waves keep their original identity - of a basic wave profile continuously emitting secondary waves.

The emission of secondary waves could be viewed as follows. As the original wave shape is quite different from the actual travelling wave solution, which the completely nonlinear system would admit, it emits its excess energy in the form of secondary waves. This can also be seen in the first case - the pair of oppositely travelling waves emit secondary waves at large times. The

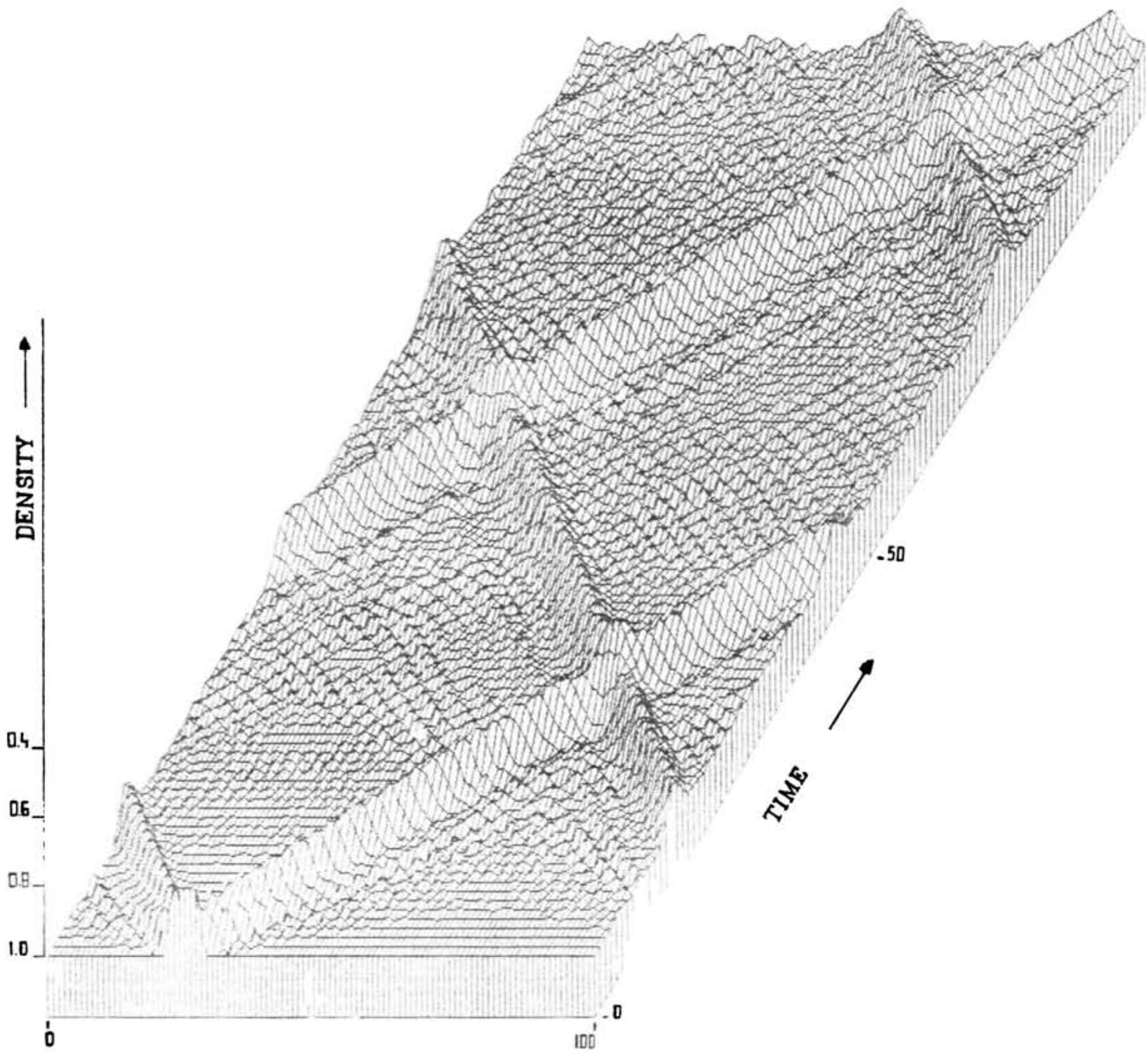

Fig 3.3. Time evolution of the amplitude of the superfluid density fluctuations occurring along the ζ-direction, for the rounded rectangular initial profile.

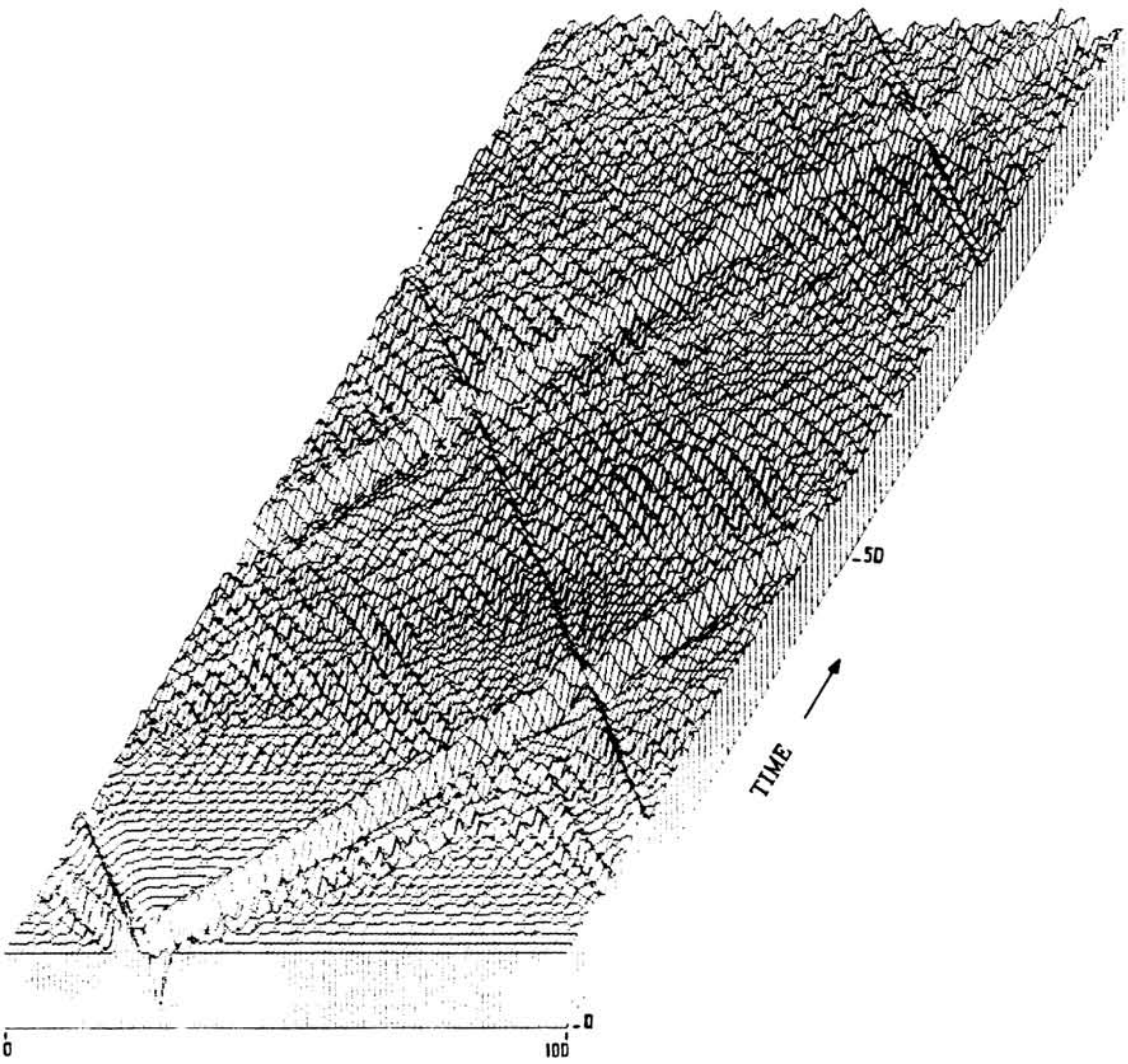

Fig 3.4. The superfluid velocity, corresponding to the density fluctuations plotted in Fig.3.3, in arbitrary units.

reason for such a slow emission of secondary waves in the first case is the fact that the initial wave itself is only slightly different from an exact travelling wave solution. These results have been generalized by taking as initial profile other wave shapes like, for example, a narrow gaussian profile.

We have shown numerically that even under strong nonlinearity the two-dimensional ^{4}He films admit stable composite quasi-solitons of the superfluid density fluctuations and the superfluid velocity. These solitons are quite different from those obtained in the case of K-P equation or the two-dimensional nonlinear schrödinger equation.

We have also shown that any arbitrary initial profile would decay into a stable soliton solution by the emission of the excess energy as secondary waves. Hence the two emerging solitons could be viewed in their early stages of development as two dissipative waves. An interesting property to discuss at this point would be the dynamics of such a system in the phase space, as to the predictability of the soliton solutions with respect to the amplitude of the initial profile and the effect of collision of the two waves on the phase space trajectory.

CHAPTER 4

CHAOS CAUSED BY SOLITON - SOLITON INTERACTION

CHAOS CAUSED BY SOLITON-SOLITON INTERACTION

The propagation of large amplitude quasi solitons in thin two dimensional superfluid films was investigated in the last chapter. The dynamics of these quasi solitons would change noticeably with variations in the initial profile.

In this Chapter we investigate in detail the effect of changing the initial shape of the wave profile. For this purpose we study the effect of very small changes in the initial conditions of the wave profile on the dynamics of the quasi solitons. In particular, we consider the sensitivity to initial conditions on the propagation of these large amplitude waves. For this purpose we try the squared secant hyperbolic, the gaussian and the rounded rectangular waveforms as different initial profiles.

This chapter is divided into two sections. In section 4.1 the nonlinear evolution equation representing the large amplitude local density fluctuations in a one dimensional monolayer superfluid ^{4}He film is discussed.

After defining a suitable phase space, we study in section 4.2 the dynamics of the system in the phase space.

4.1 NUMERICAL STUDIES

The equation of motion is described by the phenomenological equation of Rutledge et al.[17]

$$i\hbar \frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi - \frac{A\,\psi}{(a+|\psi|^2)^3} - \mu\psi - B\,\psi\,\nabla^2|\psi|^2 \qquad (4.1.1)$$

Here also we can take B=0, since only monolayer films are considered.

For the purpose of numerical analysis we transform the equation (4.1.1) to a dimensionless form as was done in Chapter 3, and consider only one space direction for simplicity.

That is,

$$i\,\frac{\partial\chi}{\partial t} = -\frac{\partial^2\chi}{\partial\xi^2} - \left\{\left[\frac{1+d_o}{1+d_o|\chi|^2}\right]^3 - 1\right\}\chi \qquad (4.1.2)$$

where $\chi = \psi/\psi_o$, $\xi = kx$, $\tau = \omega t$ and we assume that initially the superfluid density is locally altered.

Equation (4.1.2) is integrated numerically, treating it as an initial value problem. The size of the superfluid film (= 100) is chosen arbitrarily in such a way as to be larger than the characteristic size of the localized excitations. We assume that at t=o the whole superfluid is at rest - that is we choose the initial value of the phase of the wave to be constant throughout the film.

The time evolution of the superfluid density fluctuation $\rho(\xi,\tau) = |\chi(\xi,\tau)|^2 - 1$ and the superfluid phase $\phi(\xi,\tau)=\arg(\chi(\xi,\tau))$ are studied for the rounded rectangular form as the initial profile.

$$\rho(\xi,\eta,o)=\rho_o\frac{1}{2}\left\{1-\tanh\left[\left[(\xi-25)/1.5\right]^2-4\right]\right\} \qquad (4.1.3)$$

with $\phi(\xi,o)$ = constant.

The actual value of this constant is not important in the dynamics of the system. We impose the periodic boundary condition $a(100,\tau)=a(0,\tau)$.

The numerical experiment is performed for two initial profiles differing only in the value of ρ_o and obtain a pair of data concerning the temporal development of the amplitude and also the phase at each point on the superfluid film.

Fig.4.1 and Fig.4.2 show the temporal development of the amplitude and velocity of the superfluid density fluctuations for the values of the parameters ρ_o=0.2 and d_o=1 . The second set of numerical experiment is done for ρ_o=0.20001 and with the same value for d_o. The initial profile can be seen to split into two and travel in opposite directions. Fig.4.1 shows that these waves continuously emit waves of higher velocity as they propagate along the medium.

4.2 DYNAMICS IN PHASE SPACE

In this section we study temporal development of the initial separation distance in the phase space. For this we define a phase space with variables ρ,ϕ,ρ_t and ϕ_t. In this phase space the local separation between two points $(\rho,\phi,\rho_t,\phi_t)$ and $(\rho+\delta\rho,\phi+\delta\phi,\rho_t+\delta\rho_t,\phi_t+\delta\phi_t)$ is given by[82]

$$\delta\ s(\xi,\tau) = \sqrt{(\delta\rho(\xi,\tau))^2+(\delta\rho_\tau(\xi,\tau))^2+(\delta\phi(\xi,\tau))^2+(\delta\phi_\tau(\xi,\tau))^2} \qquad \ldots\ldots(4.2.1)$$

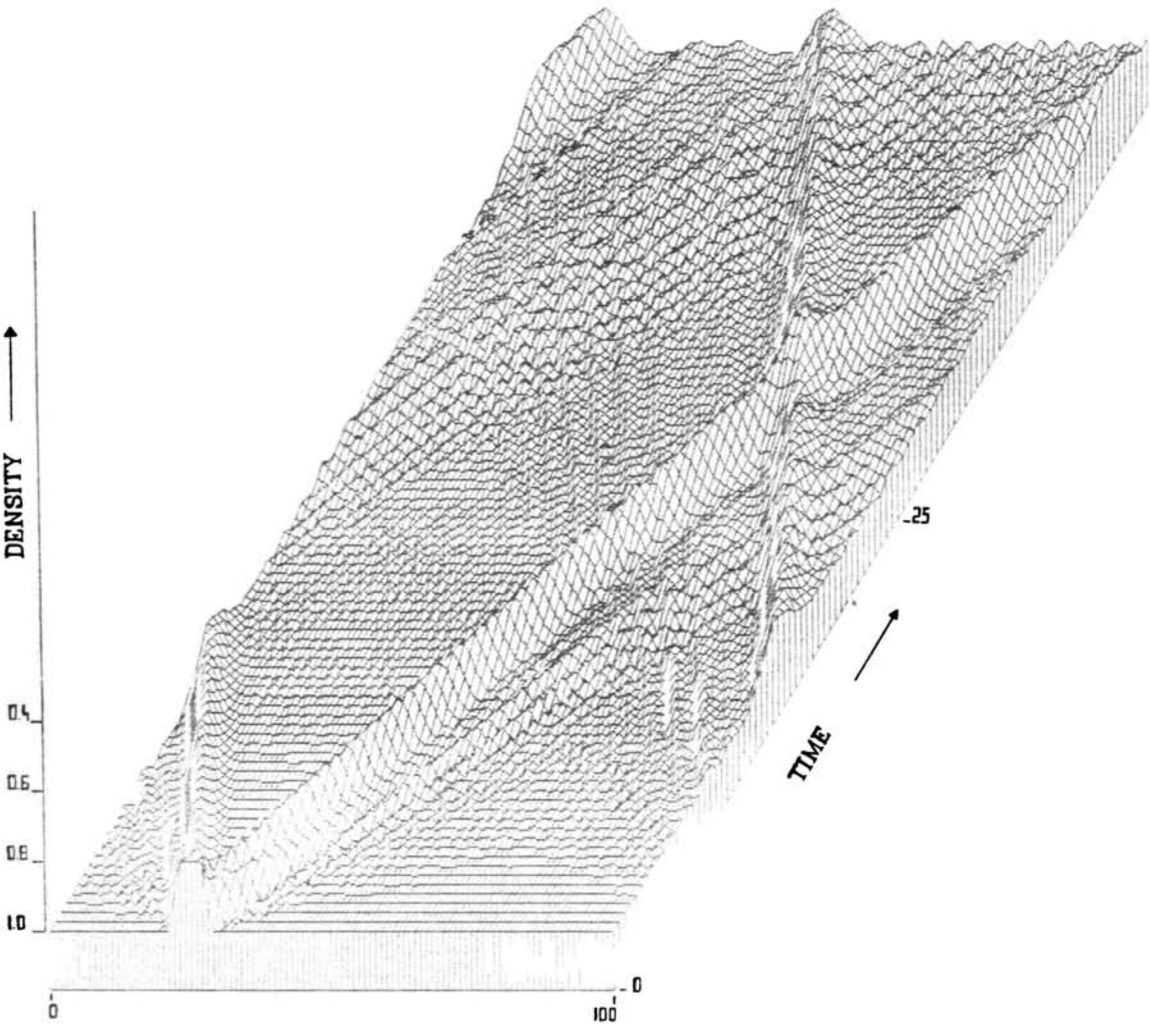

Fig 4.1. Time evolution of the amplitude of the superfluid density fluctuations occurring along the ξ-direction, for the squared secant hyperbolic initial profile.

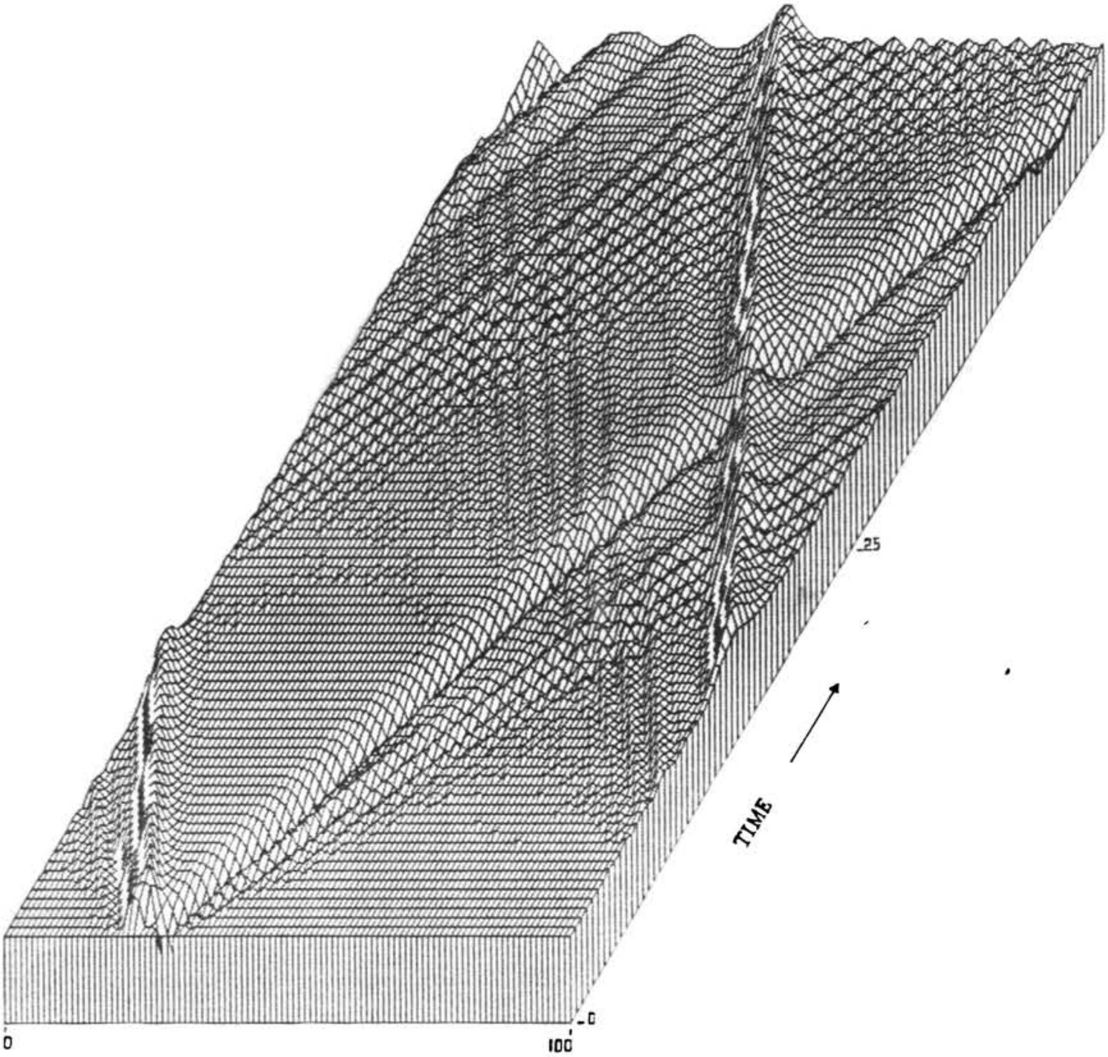

Fig 4.2. The superfluid velocity, corresponding to the density fluctuations plotted in Fig.4.1, in arbitrary units.

As the system evolves in time, the point $(\rho,\phi,\rho_t,\phi_t)$ in the phase space traces a trajectory. Different initial profiles would give different trajectories. We study the time evolution of the separation distance between two such trajectories corresponding to slightly different initial profiles.

The total separation distance of the trajectories in the phase space has the form[82]

$$\delta\, S(\tau) = \sqrt{\frac{1}{L} \int_0^L (\,\delta\, s(\xi,\tau))^2\, d\xi} \qquad (4.2.2)$$

where L is the length of the film. We choose the initial values of the parameters for the two sets of numerical experiments as follows: ρ_o=0.2 and $\phi(\xi,0)$=0 for the first set and ρ_o=0.20001 and $\phi(\xi,o)$=0 for the second.

The temporal development of $\delta S(\tau)$ is plotted in Fig.4.3. It can be seen that the total separation distance is increasing with time, and eventually saturating to a maximum value after a collision. But the next collision of the solitons would again increase the separation distance exponentially (the points of collision are shown by arrows in Fig.4.3). The collision points would be less evident from the figure after the second

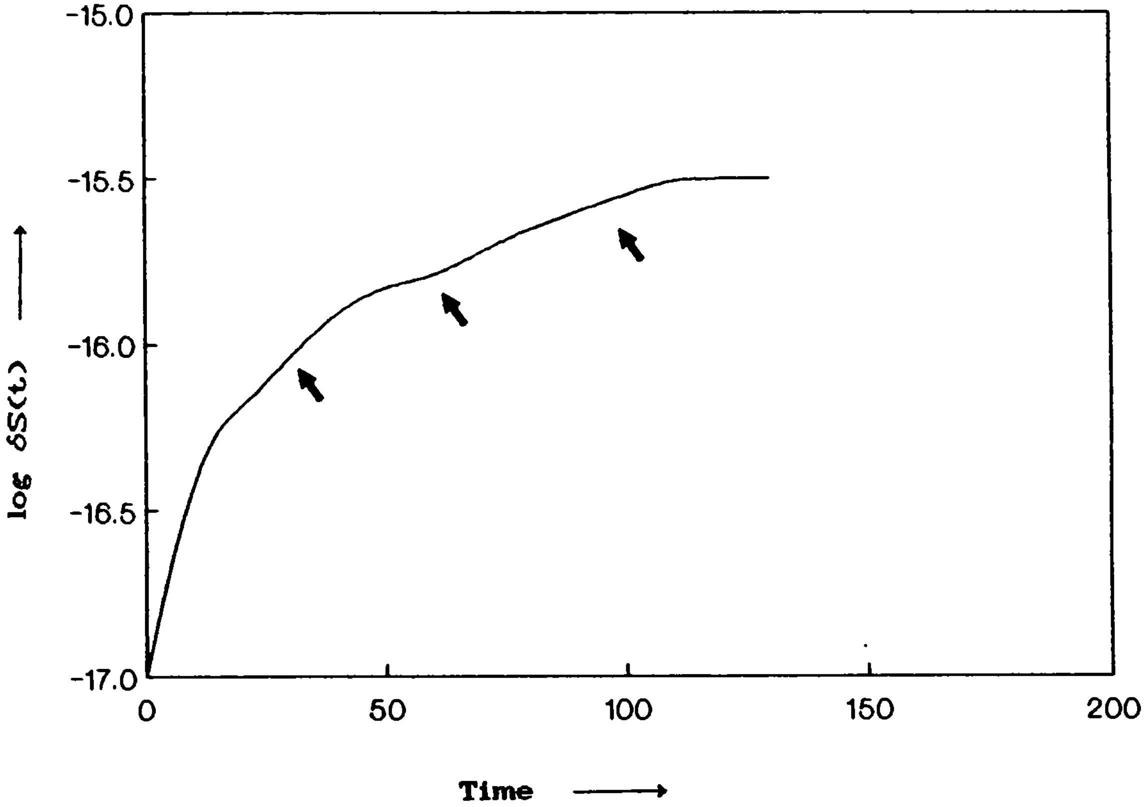

Fig 4.3 Temporal development of the total separation distance, $\delta S(t)$, of the trajectories in phase space. The collision of solitons are indicated by arrows.

collision, because of the emission of waves from the soliton.

To understand the increase in separation distance caused by soliton-soliton collision we plot in Fig.4.4 the temporal development of $(\delta S)_{max}$, which is the maximum value of the local separation distance at a given time.

It is seen that $(\delta S)_{max}$ would increase slightly due to the initial splitting of solitons. Then it remains almost constant till the collision of solitons, at which instant there is a sharp increase in $(\delta S)_{max}$. $(\delta S)_{max}$ is plotted for three successive collisions. For each collision there is a marked increase in $(\delta S)_{max}$. This increase becomes smaller and smaller after each collision, because of the emission of waves by the "quasi-solitons". It should also be noted that emitted waves collide with each other much before the solitons, but this does not produce any change in the phase separation, as can be seen clearly from Fig.4.4. Thus it is clear that the collision of the quasi-solitons are responsible for the increase in the separation distance and this increase in local separation distance propagates spatially with the solitons, which explains the nature of the total separation distance.

Next we undertake a similar study with the

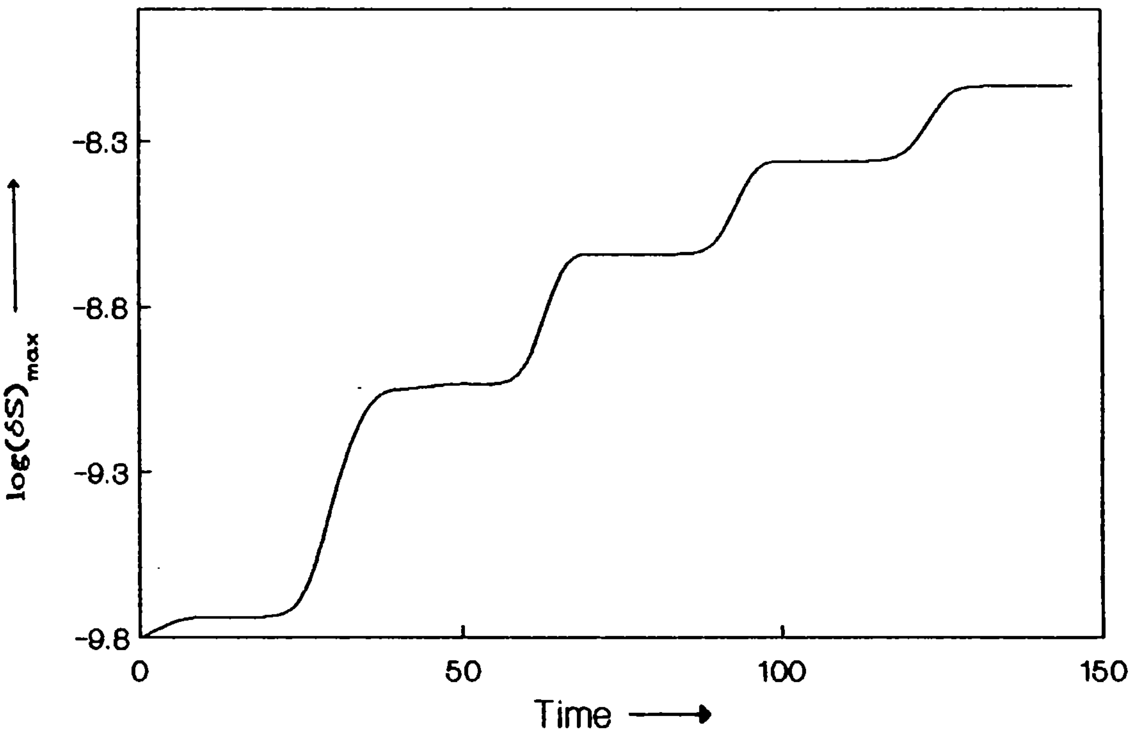

Fig 4.4 Temporal development of $(\delta s)_{max}$, the maximum value of the local separation distance at any given time. Each 'step' in the curve corresponds to a collision of the solitons.

squared secant hyperbolic profile for the initial wave form.

$$\rho(\xi,\eta,o) = \rho_o \operatorname{sech}^2 \left[(\xi-25)/3\right] \qquad \text{......(4.2.3)}$$

The results are plotted in Figures 4.5, 4.6, 4.7 and 4.8. It is evident from Fig.4.7 and 4.8 that the collision of the quasi solitons do not produce any noticeable and permanent increase in the phase space separation of trajectories. This is because the initial wave profile has a form almost similar to an exact soliton solution under full nonlinearity.

A similar study was also taken up with a very narrow Gaussian profile. It showed results similar to that of the rounded rectangular form. However, when the width of the Gaussian profile was increased, the dynamics of the phase space trajectories showed characteristics similar to that of the squared secant hyperbolic profile, but there were small jumps in the $(\delta S)_{max}$ curve at the instants of collision. This is because there is only a weak emission of waves.

To conclude we have shown that two initially close trajectories in the phase space increases exponentially with time. This separation is enhanced due

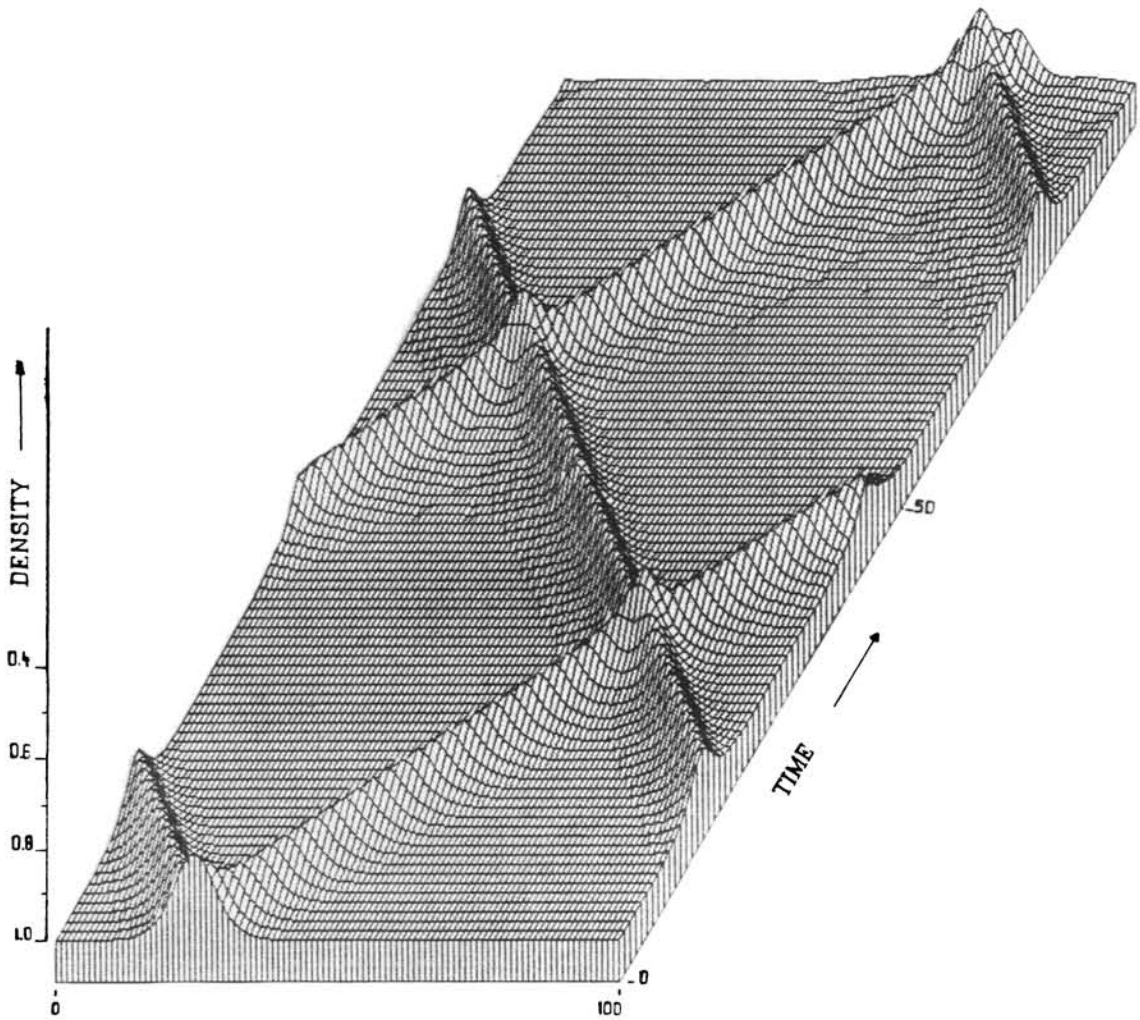

Fig 4.5. Time evolution of the amplitude of the superfluid density fluctuations occurring along the ζ-direction, for the rounded rectangular initial profile.

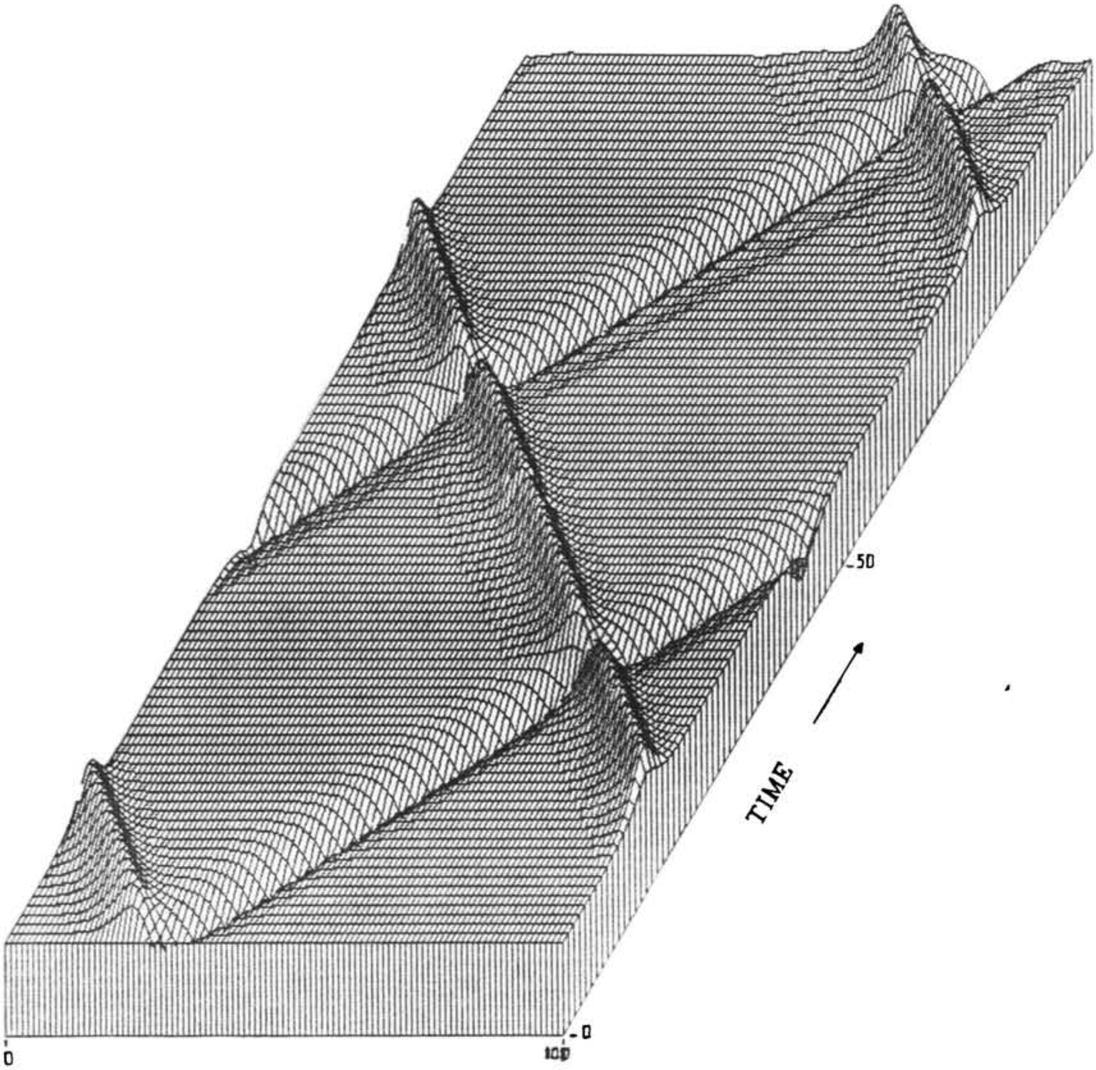

Fig 4.6. The superfluid velocity, corresponding to the density fluctuations plotted in Fig.4.5, in arbitrary units.

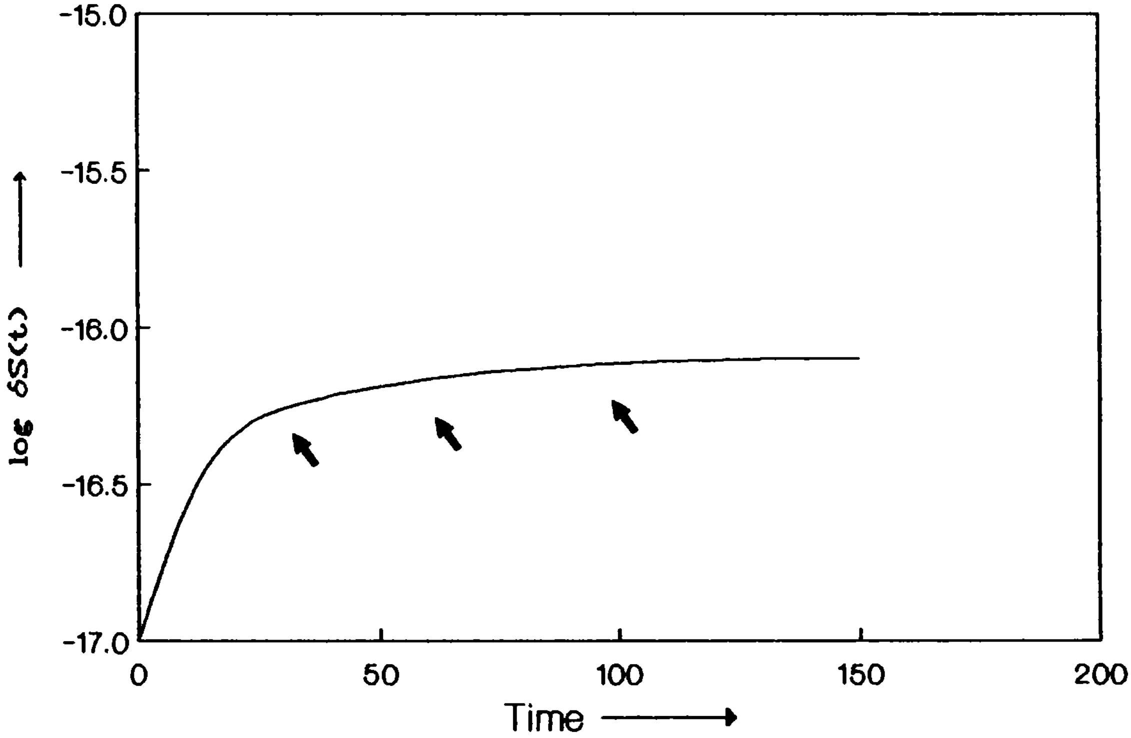

Fig 4.7 Temporal development of the total separation distance, $\delta S(t)$, of the trajectories in phase space. The collision of soliton are indicated by arrows. Here a squared secant hyperbolic initial profile is used.

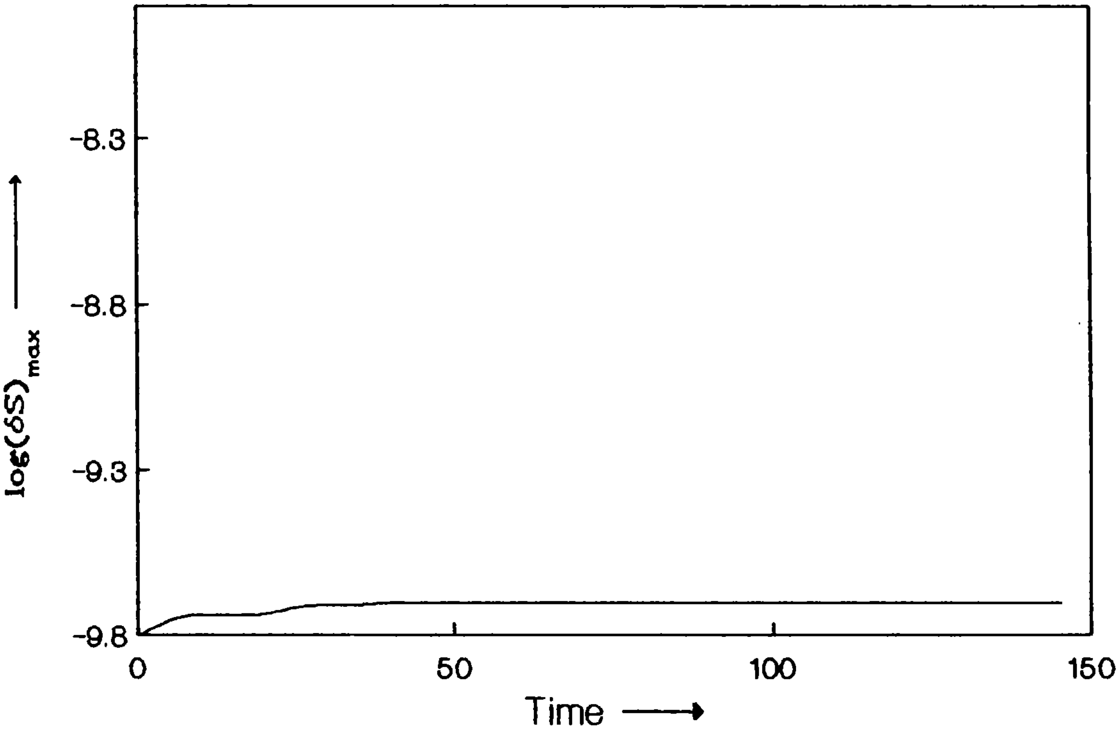

Fig 4.8 Temporal development of $(\delta s)_{max}$, the maximum value of the local separation distance at any given time. Each 'step' in the curve corresponds to a collision of the solitons.

to the collision of solitons taking place on the film. Finally, we note that one should take into consideration these chaotic phenomena before attempting to explain experiments connected with large amplitude soliton propagations in very thin ^4He films.

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CHAPTER 5

SOLITONS AND SOLITON RESONANCES ON SATURATED SUPERFLUID HELIUM FILMS

SOLITONS AND SOLITON RESONANCES ON SATURATED SUPERFLUID HELIUM FILMS

In the earlier chapters we had focused our attention mainly on the propagation and interaction of localized density fluctuations on very thin superfluid films. In this chapter we consider the localized thickness fluctuations on the so called saturated films.

As we have discussed in section 1.9 , Nakajima et al.[51] were able to extend the analysis of one dimensional surface wave propagation on very thin films ($\sim 10^{-7}$ Cm) to films whose thickness is of the order of 10^{-6} cm. In such films, the surface tension plays a decisive role in the dynamics of the system, which was totally ignored for very thin films. The effect of surface tension[51] is to increase the characteristic length of the soliton, and to reduce the soliton velocity. This makes the detection and generation of solitons using conventional third sound apparatus easier. The analysis done by Nakajima et al was restricted to one dimension. It seems natural, therefore, to search for quasi two dimensional solitons in such systems.

In this chapter we investigate the nonlinear waves propagating on a two dimensional saturated film of superfluid ^{4}He. Here we are concerned with the temporal evolution of the fluctuations in thickness of the superfluid. This is in contrast to the investigations cited in the earlier chapters, where the surface deformation was negligible and only the density fluctuations were present, due to the very small thickness of the superfluid film. In the small amplitude regime we have been able to derive the K-P equation with negative dispersion. It is shown that resonance of solitons can be observed in such films as against the one dimensional case studied by Nakajima et al.[51]

In the next section we derive the equations governing the surface displacement. In section 5.2, we consider the small amplitude regime, and obtain the K-P equation. It is shown in section 5.3 that in the lowest order nonlinearity, soliton resonances could be obtained only if two dimensional effects are taken into account. Last section is devoted to a discussion of the results.

5.1.FINITE AMPLITUDE SURFACE WAVES

When saturated films of superfluids are considered one has to include the effects of surface tension, which is generally ignored for very thin films. The Van der

Waals force is the nonlinear force acting on the superfluid film. The acceleration of the superfluid due to a temperature gradient, which acts as a very small correction factor[17] in our low temperature film is neglected in this work. We consider the x and y axes to be lying on the substrate on which the superfluid of equilibrium depth d exists. Geometrical configuration of the system is shown in Fig.5.1.

Since the superflow is irrotational, we can describe it by the velocity potential $\Phi(x,y,z,t)$. If we treat the system to be incompressible we can write the equation of continuity in bulk as

$$\frac{\partial^2\Phi}{\partial x^2} + \frac{\partial^2\Phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} = 0 \qquad (5.1.1)$$

There is the additional condition that the superfluid would not flow into the substrate.

$$\left.\frac{\partial\Phi}{\partial z}\right|_{z=0} = 0 \qquad (5.1.2)$$

The continuity equation at the film-vapour interface takes the form

$$\frac{\partial z_1}{\partial t} + \left(\frac{\partial\Phi}{\partial x}\right)_1 \frac{\partial z_1}{\partial x} + \left(\frac{\partial\Phi}{\partial y}\right)_1 \frac{\partial z_1}{\partial y} - \left(\frac{\partial\Phi}{\partial z}\right)_1 = 0$$

...... (5.1.3)

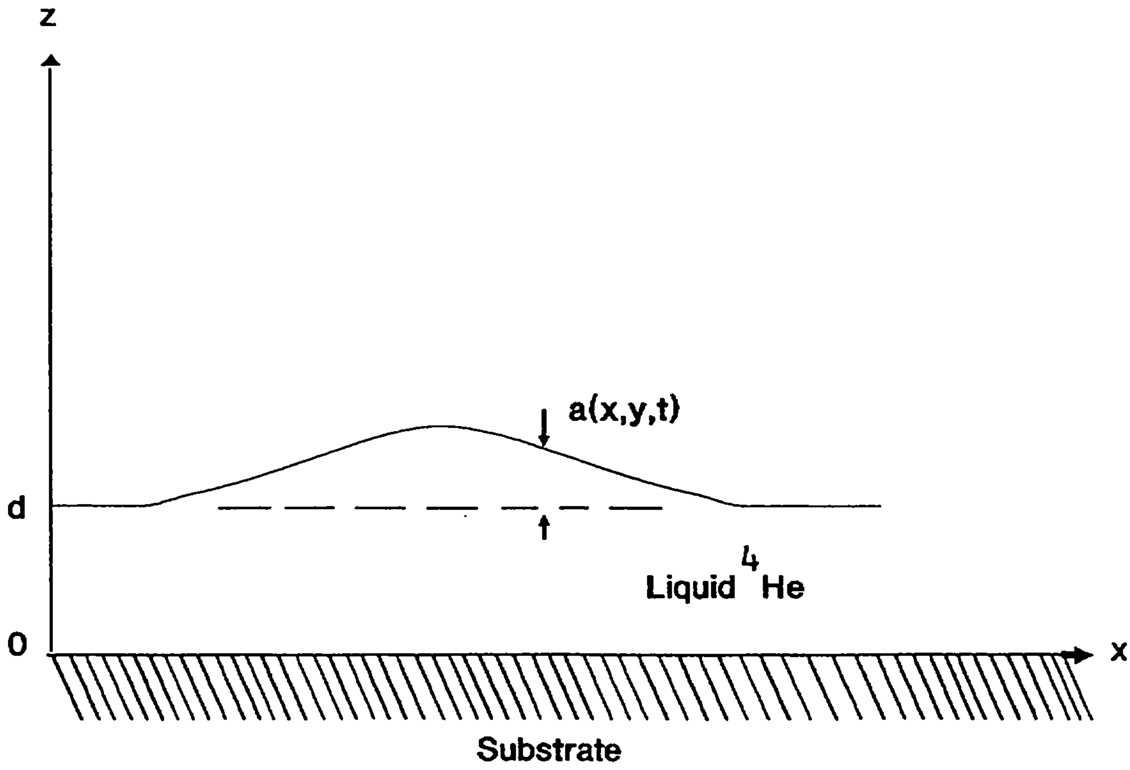

Fig 5.1 Geometrical configuration of the saturated superfluid film system. The x and y axes lie on the plane surface of the substrate

The index 1 refers to the film - vapour interface $z_1=d+a(x,y,t)$, where $a(x,y,t)$ is the departure of the film surface from its equilibrium position. The equation of motion at the surface is[51]

$$\left(\frac{\partial\Phi}{\partial t}\right)_1+\frac{1}{2}\left[\left(\frac{\partial\Phi}{\partial x}\right)_1^2+\left(\frac{\partial\Phi}{\partial y}\right)_1^2+\left(\frac{\partial\Phi}{\partial z}\right)_1^2\right]-\frac{\sigma}{\bar{\rho}}\left[\frac{\partial^2 a}{\partial x^2}+\frac{\partial^2 a}{\partial y^2}\right]$$

$$+\varepsilon_1 a-\frac{1}{2}\frac{\varepsilon_2}{d}a^2=0 \qquad (5.1.4)$$

The last two terms appearing in equation (5.1.4) represent the leading terms in the expansion of the Van der Waals force term. $\varepsilon_1= 3\alpha/d^4$ and $\varepsilon_2=12\alpha/d^4$ α is the Van der Waals constant, $\bar{\rho}$ is the density of the superfluid and σ its surface tension. Equations (5.1.3) and (5.1.4) represent finite amplitude surface waves propagating on the superfluid film. We expand $\Phi(x,y,z,t)$ as[42]

$$\Phi(x,y,z,t)=\sum_{n=0}^{\infty} z^n\,\phi_n(x,y,t) \qquad (5.1.5)$$

Now by using equations (5.1.1) and (5.1.2), we get comparing like powers of z,

$$\Phi(x,y,z,t)=\mathrm{Cos}(z\,\vec{\nabla})\,\phi_0(x,y,t) \qquad (5.1.6)$$

where $\vec{\nabla}$ is the two dimensional gradient having components

$\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ along x and y axes respectively.

5.2. SOLITARY WAVES

Using equations (5.1.3) and (5.1.4) we have studied the dynamics of localized disturbances, of long wavelength and small amplitude, in the superfluid film thickness. We orient the horizontal coordinate system such that the principal direction of propagation is chosen as the x-axis. We make the following coordinate transformation.

$$X \rightarrow x + C_3 t \quad , \quad t \rightarrow t \qquad (5.2.1)$$

where C_3 is velocity of the moving frame.
To transform the equations (5.1.3) and (5.1.4) into a wave equation with respect to the superfluid surface displacement, the reductive perturbation method by Taniuti and Wei[43] can be applied using the scaling transformation

$$\bar{x} = \varepsilon^{1/2} X \ , \ \bar{y} = \varepsilon y \quad , \ \bar{t} = \varepsilon^{3/2} t \qquad (5.2.2)$$

We regard ε as an infinitesimal, however it disappears in the final equation. We are essentially looking for fluctuations in the thickness of the film which travel with long wavelength along the x-direction and we assume that the y-coordinate dependence of the wave is weak. We expand a and ϕ_o in powers of ε.

$$a = a_o + \varepsilon\, a_1(x,y,t) + \varepsilon^2\, a_2(x,y,t) + \ldots \qquad (5.2.3a)$$

$$\phi_o = \varepsilon^{1/2}\, \phi_o^{(1)}(x,y,t) + \varepsilon^{3/2}\phi_o^{(2)}(x,y,t) + \ldots \qquad (5.2.3b)$$

Using equations (5.1.3), (5.1.6)and (5.2.1) - (5.2.3) and comparing coefficients of $\varepsilon^{3/2}$ and $\varepsilon^{5/2}$, we get the following equations.

$$C_3\, \frac{\partial a_1}{\partial \bar{x}} + d\, \frac{\partial^2 \phi_o^{(1)}}{\partial \bar{x}^2} = 0 \qquad (5.2.4)$$

$$C_3\, \frac{\partial^2 \phi_o^{(1)}}{\partial \bar{x}^2} + g_1 \frac{\partial a_1}{\partial \bar{x}} = 0 \qquad (5.2.5)$$

$$C_3\, \frac{\partial^2 \phi_o^{(2)}}{\partial \bar{x}^2} - \frac{1}{2}\, C_3\, d^2\, \frac{\partial^2 \phi_o^{(1)}}{\partial \bar{x}^4} + \frac{\partial}{\partial \bar{t}}\left[\frac{\partial \phi_o^{(1)}}{\partial \bar{x}}\right] + \frac{\partial \phi_o^{(1)}}{\partial \bar{x}}\, \frac{\partial^2 \phi_o^{(1)}}{\partial \bar{x}^2}$$

$$- \frac{\sigma}{\bar{\rho}}\, \frac{\partial^3 a_1}{\partial \bar{x}^3} + g_1\, \frac{\partial a_2}{\partial \bar{x}} - \frac{g_2}{d}\, a_1\, \frac{\partial a_1}{\partial\, \bar{x}} = 0 \qquad (5.2.6)$$

$$\frac{\partial a_1}{\partial \bar{t}} + C_3 \frac{\partial a_2}{\partial \bar{x}} + \frac{\partial \phi_o^{(1)}}{\partial \bar{x}} \frac{\partial a_1}{\partial \bar{x}} + a_1 \frac{\partial^2 \phi_o^{(1)}}{\partial \bar{x}^2} + d \frac{\partial^2 \phi_o^{(2)}}{\partial \bar{x}^2} + d \frac{\partial^2 \phi_o^{(1)}}{\partial \bar{y}^2}$$

$$- \frac{d^3}{6} \frac{\partial^4 \phi_o^{(1)}}{\partial \bar{x}^4} = 0 \qquad (5.2.7)$$

Using the boundary conditions that a_1 and $\phi_c^{(1)}$ goes to zero as $x \to \infty$, equations (5.2.4) and (5.2.5) can be solved to get

$$\frac{\partial \phi_o^{(1)}}{\partial \bar{x}} = \frac{-C_3}{d} \qquad a_1 = \frac{-\varepsilon_1}{C_3} a_1$$

$$\therefore \qquad C_3^2 = \varepsilon_1 d \qquad (5.2.8)$$

Now eliminating $\dfrac{\partial \phi_o^{(2)}}{\partial \bar{x}}$ between (5.2.6) and (5.2.7) we get

$$\frac{\partial}{\partial \bar{x}} \left[2C_3 \frac{\partial a_1}{\partial \bar{t}} + \left(\varepsilon_2 - \frac{3C_3^2}{d} \right) a_1 \frac{\partial a_1}{\partial \bar{x}} + \left(\frac{\sigma d}{\bar{\rho}} - \frac{C_3^2 d^2}{3} \right) \frac{\partial^3 a_1}{\partial \bar{x}^3} \right]$$

$$- C_3^2 \frac{\partial^2 a_1}{\partial \bar{y}^2} = 0 \qquad (5.2.9)$$

Equation (5.2.9) is the K-P equation, which can be expressed in the more familiar form by the following transformations.

$$a_1 = -6\gamma\rho \quad , \quad \frac{1}{\gamma} = \left(\varepsilon_2 - \frac{3C_3^2}{d} \right) \frac{1}{2C_3} \quad , \quad \xi = -k_o \bar{x} \quad ,$$

(5.2.10)

$$-\frac{1}{k_o^2} = \left(\frac{\sigma d}{\bar{\rho}} - \frac{C_3^2 \, d^2}{3} \right) \frac{1}{2C_3} \quad , \quad \eta = \sqrt{\frac{2k_o^2}{C_3}} \; \bar{y} \quad , \quad \tau = k_o \bar{t}$$

So equation (5.2.9) would become

$$\frac{\partial}{\partial\xi}\left[\frac{\partial\rho}{\partial\tau} + 6\rho\frac{\partial\rho}{\partial\xi} + \frac{\partial^3\rho}{\partial\xi^3} \right] + \frac{\partial^2\rho}{\partial\eta^2} = 0 \qquad (5.2.11)$$

The K-P equation represented by (5.2.11) is the one with a negative dispersion and it is known to possess N-soliton solutions[55,56].. The one soliton solution can be written as[88,89]

$$\rho = \frac{1}{2}\, k^2 \; \text{Sech}^2 \, \zeta \qquad (5.2.12)$$

where

$$\zeta = \frac{1}{2}k \left[\xi + p\eta - (k^2 + p^2)\tau \right] + \zeta^{(0)}$$

and k and kp are the components of the linear momentum along the ξ and η directions respectively. Equation (5.2.12) describes a soliton propagating with velocity $(k^2+p^2)/\sqrt{1+p^2}$ in the direction making an angle $\tan^{-1}(p)$ with the x-axis. This angle should be small because the K-P equation holds under the assumption that the two dimensional effect is small.

5.3 SOLITON RESONANCES

The two soliton solution for equation (5.2.11) is obtained from

$$\rho = 2(\log f_2)_{\xi\xi} \qquad (5.3.1)$$

where $$f = 1 + e^{2\zeta_1} + e^{2\zeta_2} + A_{12}\, e^{2(\zeta_1+\zeta_2)}$$

$$\zeta_i = \frac{1}{2} k_i \left[\xi + p_i\eta - (k_i^2 + p_i^2)\tau\right] + \zeta^{(0)}$$

and
$$A_{12} = \frac{3(k_1-k_2)^2 - (p_1-p_2)^2}{3(k_1+k_2)^2 - (p_1-p_2)^2} \qquad (5.3.1a)$$

Soliton resonance occurs[60] when $A_{12} = 0$ or ∞, i.e. for $3(k_1 \pm k_2)^2 - (p_1-p_2)^2 = 0$. The plus sign refers to plus resonance and the other case is the minus resonance.

As we have seen in Chapter2, the resonant soliton in general can be written in the form

$$\rho^{(1\pm2)} = \frac{1}{2}(k_1 \pm k_2)^2 \, \mathrm{Sech}^2(\zeta_1 \pm \zeta_2) \qquad (5.3.2)$$

The amplitude and velocity of the resonant soliton, in the original coordinate system, can be written as

$$A_r = \frac{6C_3 d}{(\varepsilon_2 d - 3C_3^2)}(k_1 \pm k_2)^2 = \frac{6\sqrt{\varepsilon_1 d}}{(\varepsilon_2 - 3\varepsilon_1)}(k_1 \pm k_2)^2 \qquad (5.3.2a)$$

$$V_r = \frac{\left[6\sigma d + C_3^2 d^2 \bar{\rho}\right]^{1/2} \left[k_1(k_1^2+p_1^2) \pm k_2(k_2^2+p_2^2)\right]}{\left\{12C_3^2 \bar{\rho}(k_1 \pm k_2)^2 + \frac{d}{2}(k_1 p_1 \pm k_2 p_2)^2 (6\sigma d + C_3^2 d^2 \bar{\rho})\right\}^{1/2}} - \sqrt{\varepsilon_1 d}$$

.......(5.3.2b)

If the resonance is to be observed in actual experimental set up, the resonance conditions given by equations(5.3.1a) should be consistent with the conservation laws. Tajiri and Nishitani[59] showed that this condition is satisfied for the K-P equation (5.2.11) in the following sense. First a similarity transformation is applied to the K-P equation. Then the resonance conditions of the resulting equation are shown to satisfy the corresponding conservation laws. The similarity transformation has the form[90]

$$t' = \frac{\xi}{P^{3/2}} + \frac{2}{3}\int \frac{Q}{P^{5/3}} d\tau \qquad (5.3.3a)$$

$$x' = \frac{\eta}{P^{1/3}} + \frac{P'}{6P^{4/3}}\xi^2 - \frac{Q}{3P^{4/3}}\xi - \frac{2}{9}\int \frac{Q^2}{P^{7/3}} d\tau - \frac{1}{3}\int \frac{R}{P^{4/3}} d\tau$$

$$\ldots\ldots\ldots (5.3.3b)$$

$$\rho = \frac{P'}{6\rho}\eta + \frac{1}{6}\left\{\frac{1}{3}\left(\frac{P'}{p}\right)^2 - \frac{P'}{2P}\right\}\xi^2 + \frac{1}{6}\left[\frac{Q'}{P} - \frac{2P'Q}{3P^2}\right]\xi$$

$$+ \frac{R}{6p} + \frac{Q^2}{18P^2} - \frac{1}{2P^{1/3}} + \frac{1}{P^{2/3}}u(x',t') \qquad (5.3.3c)$$

where P,Q and R are function of τ , $P' = dp/d\tau$, $P'' =$

$d^2P/d\tau^2$ and $Q' = dQ/d\tau$. Using equation (5.3.3) in (5.2.11) we get the Boussinesq type equation.

$$u_{t't'} - u_{x'x'} + (u^2)_{x'x'} + u_{x'x'x'x'} = 0 \qquad (5.3.4)$$

Tajiri and Nishitani[59] showed that this equation exhibits soliton resonance and the resonance conditions do satisfy its conservation laws. This suggests that soliton resonance may be observed in two dimensional saturated films of superfluid ^{4}He.

Now we turn our attention to one dimensional wave propagation in saturated superfluid films. The governing equations[51] in this case, under weak nonlinearity, is the KdV equation.

$$u_t + 6uu_x + u_{xxx} = 0 \qquad (5.3.5)$$

The one soliton solution for K-dV equation is $u = 2k^2 \, \mathrm{Sech}^2(kx-\Omega t)$ and its resonance conditions are given by

$$(k_1 \mp k_2) \left[\Omega_1 - \Omega_2 + (k_1 - k_2)^3 \right] = 0 \qquad (5.3.6)$$

The first two conserved quantities of the KdV equation are

$\int u dx$ and $\int u^2 dx$. For two soliton resonant interaction, these conservation laws give

$$k_1 \pm k_2 = K \qquad (5.3.7a)$$

$$k_1^3 \pm k_2^3 = K^3 \qquad (5.3.7b)$$

where k_1 and k_2 corresponds to the initial solitons and K to the final resonant soliton. The plus and minus signs corresponds to the two different types of resonance. Equations (5.3.7 a&b) are not satisfied for any k_1 and k_2 except for the trivial cases $k_1 = o, k_2 = o$ or $k_1 = -k_2$. Hence we can say that it is not possible to have soliton resonance in one dimensional saturated superfluid films, under weak nonlinearity.

5.4 DISCUSSIONS

We have reduced the hydrodynamics equations for saturated superfluid films to the K-P equation with negative dispersion in the small amplitude regime. This is to be contrasted with the result of Biswas and Warke[74] , who obtained K-P equation with a positive dispersion for wave propagation on a monolayer superfluid film. The two problems are, however, entirely different. In ref [74], the superfluid density fluctuations in thin films is considered, where as here we are discussing the fluctuations in thickness of the superfluid films.

In the saturated films one is able to observe the phenomenon of soliton resonances when two dimensional wave propagation is considered. The amplitude and velocity of the resonant soliton are given by equation (5.3.2). We have shown explicitly that under the lowest order of nonlinearity soliton resonance is observable only when two dimensional wave propagation is taken into consideration.

CHAPTER 6

CONCLUDING REMARKS

CONCLUDING REMARKS

Before concluding the thesis, we would like to highlight some of the results obtained in the previous chapters. In addition, we would also like to indicate the scope for future work in the field.

In the study of two soliton interactions on monolayer superfluid films, we have obtained in chapter II the resonance of solitons on monolayer superfluid films. These solitons are the localized density fluctuations occurring on such films. The amplitude and velocity of the resonant soliton are also obtained explicitly. The whole analysis was done in the small amplitude regime.

The effect of higher order nonlinearity on the propagation of large amplitude waves in two dimensional monolayer superfluid films is studied in chapter III. The nonlinear Schrödinger equation representing the superfluid density fluctuations is reduced to a dimensionless form and solved numerically for arbitrary initial profiles. It is shown that the initial profile would split into two "quasi-solitons" travelling in opposite directions, which have particle like stability and keep their shape unchanged during interactions with each other. The "quasi-solitons" are asymmetric in shape, unlike the KdV or K-P solitons.

It was observed in chapter III that the propagation of large amplitude waves on superfluid films depended critically on the initial wave profile. This led us to study the chaos caused by the propagation and interaction of such waves. In chapter IV, the chaos induced by the collision of large amplitude one dimensional quasi solitons on very thin superfluid film is studied numerically. It is shown that two initially close trajectories in the phase space of the system separates exponentially with the collision of the quasi-solitons. The instability at the collision spot propagates spatially.

Following the arguments of Huberman[50] that the dispersion relation of the superfluid dynamics would change from positive to negative as the thickness of the film is increased, we have studied the dynamics of saturated superfluid films under the lowest order of nonlinearity in chapter V. Our calculations show that the equation governing the system is the K-P equation with negative dispersion. It is established that the phenomenon of soliton resonance could be observed in such films. Under the lowest order of nonlinearity, such resonances take place only if two dimensional effects are taken into account. The amplitude and velocity of the resonant soliton are obtained explicitly.

The investigations presented in this thesis has opened up a whole lot of new problems for future research. Some of the important ones are discussed below.

Layered helium films - i.e, spin polarised hydrogen, ^{3}He or a layer of electrons on superfluid ^{4}He films - would provide interesting nonlinear systems for studying soliton behaviour. The surface charged superfluid films may provide a new mechanism for charge transfer across such films, which could be used for practical applications in delay lines.

A second problem which could be studied analytically is the interaction of solitons on monolayer superfluid films when the second order of nonlinearity is taken into consideration.

A third, and perhaps a more interesting problem would be to study the dynamics of the electrons distributed on a superfluid film. This forms a very good two dimensional electron system and the phase transitions occurring on such two dimensional films could be studied using the methods of nonequilibrium statistical mechanics as employed by Pratap and Sreekumar [93] in the study of Hall conductivity.

APPENDIX

All the programmes used in this thesis are discussed briefly in this appendix.

A.1. NUMERICAL INTEGRATION

For the numerical evaluation of equation (3.1.6) and (4.1.2) we used the following method.

First the initial profile for the required size of the superfluid film was generated using a straight forward programme implemented in FORTRAN. The data thus obtained was integrated according to equation (3.1.6) or (4.1.2) as follows.

a) i) All the space derivatives were evaluated using central difference scheme.

ii) Forward difference scheme was used to evaluate the time derivative.

iii) The space step was given the value 0.1 and the time step was fixed as 0.00002.

iv) Hundred iterations were done on the initial profile. The data so obtained was stored along with the original data set for the next procedure.

b) i) Using the two data sets, we integrate as in (a) but with a central difference scheme for time coordinate integration. The value of the time step is chosen as 0.002.

ii) The data set is stored at regular time intervals (t=0,0.5,1.0,1.5,...).

All the calculations were done in double precision. To test the accuracy of the results, we checked two conserved quantities at regular intervals of time. The two conserved quantities chosen were the total mass and the linear momentum. These quantities underwent only negligible changes during the entire period of the numerical experiment. This shows the accuracy of the results.

A.2. THE THREE DIMENSIONAL PLOT.

In the thesis we have used three dimensional graphics to show the time evolution of large amplitude solitons. The programme for this, which is written in BASIC, uses the following algorithm.

i) The first data set is plotted along the x-axis, with the y-axis showing the amplitude of the quantity of interest. The spacing along the x and y axes are chosen as 5 units. All the successive points are connected by straight lines.

ii) The next data set is drawn above the previous one, but with a displacement of 5 units along the x and y axes. Drawing the curve is done as in (i). But curves are not drawn at those points where a curve already exists.

iii) This procedure is repeated till all the data sets are plotted.

iv) The curves so obtained are used to generate a 'Printer data set', to drive the printer. A separate programme is used to take 'hard copies' of the three dimensional figures using this printer data

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